① Question 1: The point $A(-1, 1)$ is a stationary point of the function $f(x) = x^2|x^3| + 3ax^2 + b$. Find the value of $\frac{a}{b}$

Solution:

The function is:

Given: $f(x) = x^2|x^3| + 3ax^2 + b \rightarrow$ ①

Here, $|x^3| = x^3$ for $x \geq 0$

Also $|x^3| = -x^3$ for $x \leq 0$

As $|x^3|$ depends on the sign of x.

for $x = -1$, then substitute this in $|x^3|$

$$|(-1)^3| = |-1| = 1$$

Thus the function becomes:

$$f(x) = x^2(-x^3) + 3ax^2 + b \quad \text{for } x < 0$$

$$f(x) = -x^5 + 3ax^2 + b \quad \text{for } x < 0$$

Eq. ②

① To find the stationary point:

A stationary point occurs when $f'(x) = 0$

(ie) $f'(x)$ is the first derivative of function $f(x)$. for $x < 0$.

$f(x) = -x^5 + 3ax^2 + b$

$$\boxed{f'(x) = -5x^4 + 6ax}$$

$\to$ 3

Equation 3 represents the first derivative of the function $f(x)$.

Step 2: To Evaluate at $x = -1$

Given. $A(-1, 1)$ is a stationary point

$f'(-1) = 0$

Substitute $x = -1$ at Eq. 3

$f'(-1) = -5(-1)^4 + 6(a)(-1)$

$f'(-1) = -5 - 6a = 0$

$\Rightarrow -5 - 6a = 0$

$-5 = 6a$

$\Rightarrow \boxed{a = \frac{-5}{6}} \to$ 4

) To find b using a value:

Given $f(-1) = 1$:

$f(-1) = -(-1)^3 + 3a(-1)^2 + b$ (From Eq. 2 sub $x = -1$)

$f(-1) = -(-1) + 3a + b = 1$

$1 = -1 + 3a + b$

$\Rightarrow 2 = 3a + b$

$2 = \cancel{3}\left(\frac{-5}{\cancel{6}_2}\right) + b$

$\Rightarrow 2 = \frac{-5}{2} + b$

$b = 1 + 1.5 = 2.5 = \frac{5}{2}$

4 To calculate $\frac{a}{b}$:

Where $\frac{-5}{6} = a$

$b = \frac{5}{2}$

$$\frac{a}{b} = \frac{\frac{-5}{6}}{\frac{5}{2}}$$

$$\frac{a}{b} = \frac{-5}{6} \times \frac{2}{5} = \frac{-10}{30} = \frac{-1}{3}$$

Thus the value of $\frac{a}{b}$ is $\frac{-1}{3}$.

② For the matrix :

$$A = \begin{bmatrix} 2 & -1 & b \\ -1 & 2 & -1 \\ b & -1 & 2 \end{bmatrix}.$$

(i) Discuss the rank of the matrix based on the value of b.

(ii) For what values of b the matrix has non-negative eigen values?

Solution :

Given : $A = \begin{bmatrix} 2 & -1 & b \\ -1 & 2 & -1 \\ b & -1 & 2 \end{bmatrix}$

(i) To find the rank of the matrix, we need to calulate the determinant of the matrix :

$$\det(A) = 2\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ b & 2 \end{vmatrix}$$

$$+ b\begin{vmatrix} -1 & b \\ 2 & -1 \end{vmatrix}$$

①

Calculating the 2x2 determinants in Eq. ①.

$$\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - (-1)(-1) = 4 - 1 = \boxed{3}$$

$$\begin{vmatrix} -1 & -1 \\ b & 2 \end{vmatrix} = -2 - (-1)(b) = -2 + b \Rightarrow \boxed{b - 2}$$

$$\begin{vmatrix} -1 & b \\ 2 & -1 \end{vmatrix} = 1 - 2(b) = 1 - 2b \Rightarrow \boxed{2b - 1}$$

Now det(A) becomes:

$$\det(A) = 2(3) - (-1)(b-2) + b(1-2b)$$

$$\det(A) = 6 + b - 2 + b - 2b^2 = 4 + 2b - 2b^2$$

$$\boxed{\det(A) = -2b^2 + 2b + 4} \quad \text{Eq. ②}$$

Here the determinant is a quadratic function, Rank depends on whether the determinant is zero (or) non zero.

$$-2b^2 + 2b + 4 = 0$$

$$\boxed{b^2 - b - 2 = 0} \quad \text{Eq. ③}$$

Solving the quadratic equation ③:

$$b = \frac{1 \pm \sqrt{b^2 - 4ac}}{2a}$$

where $a = 1$
$b = 1$
$c = 2$

$$b = \frac{1 \pm \sqrt{(1)^2 - 4(1)(2)}}{2(1)}$$

$$b = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm \sqrt{9}}{2} = \frac{1 \pm 3}{2}$$

$$b = \frac{1+3}{2} \text{ (or) } b = \frac{1-3}{2}$$

$$b = \frac{4}{2} \text{ (or) } \frac{-2}{2} \Rightarrow \boxed{b = 2 \text{ or } b = -1}$$

The determinant is zero for $b = 2$ & $b = -1$

(i) For the values of $b = 2, -1$; Rank is less than

Other values of $b \Rightarrow$ Rank is 3

(ii) Non-Negative Eigen Values:

To find the value of b for which matrix has non negative eigen values, find the characteristic polynomial.

The characteristic polynomial of A is:

$$\det(A - \lambda I) = 0$$

$$\det\begin{bmatrix} 2-\lambda & -1 & b \\ -1 & 2-\lambda & -1 \\ b & -1 & 2-\lambda \end{bmatrix} = 0$$

$\rightarrow$ (4)

Now Again, calculate the determinant:

$$\det(A-\lambda I) = 2-\lambda\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} - (-1)\begin{vmatrix} -1 & b \\ -1 & 2-\lambda \end{vmatrix}$$

$$+b\begin{vmatrix} -1 & b \\ 2-\lambda & -1 \end{vmatrix}$$

$\rightarrow$ (5)

Now calculating the 2x 2 determinants.

$$\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = (2-\lambda)(2-\lambda) - (-1)(-1)$$

$$= 4 - 2\lambda - 2\lambda + \lambda^2 - 1$$

$$= \lambda^2 - 4\lambda + 3 \Rightarrow (2-\lambda)^2 - 1$$

$$\begin{vmatrix} -1 & b \\ -1 & 2-\lambda \end{vmatrix} = (2-\lambda)(-1) - (b)(-1)$$

$$= -2 + \lambda + b$$

$$\begin{vmatrix} -1 & b \\ 2-\lambda & -1 \end{vmatrix} = (-1)(-1) - (2-\lambda)(b)$$
$$= 1 - b(2-\lambda) = 1 - 2b + \lambda b$$

Use the values in Eq. (5).

$$\det(A - \lambda I) = (2-\lambda)((2-\lambda)^2 - 1) + (2-\lambda-1b)$$
$$+ b(1 - 2b + \lambda b)$$

$$\det(A - \lambda I) = (\lambda - 2)^3 - (\lambda - 2) + (\lambda - 2 + b) +$$
$$b(\lambda b - 2b + 1)$$

(ii) The matrix A is symmetric. Thus eigenvalues are real.

We use properities of symmetric matrices & Gershgorin Circle Theorem to ensure non-negative eigen values in Matrix A.

Here $a_{11} = 2 > 0$

Also det(A) of top left matrix > 0

(ie) $\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - (1) = 3 > 0$

Also $\det(A) = 4 - 1b^2 + 2b$ From Eq. (2)

For A to be positive, the determinant of A must be positive:

We have to solve for $-2b^2+2b+4>0$

It becomes

$$2b^2-2b-4<0$$

$$\Rightarrow \boxed{b^2-b-2<0}$$

The roots for the above equation are $b=2$ and $b=-1$, Thus value of b lies between:

$$\boxed{-1<b<2}$$

Thus Matrix A has non negative eigenvalues for b in the range $-1\le b\le 2$; But with strictly positive eigenvalues,

It is: $-1<b<2$ for the given Matrix.

③ Question ③:

Given:

$$Ax=\begin{bmatrix}2\\4\\2\end{bmatrix} \text{ is } x=\begin{bmatrix}2\\0\\0\end{bmatrix}+t\begin{bmatrix}1\\1\\0\end{bmatrix}+s\begin{bmatrix}0\\0\\1\end{bmatrix}$$

Where $s,t\in R$

(i) What is the dimension of row space of A?

The Equation:

$$Ax = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$$

where $x = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

where $s, t \in \mathbb{R}$.

(i) Dimension of rowspace of A:

The solution of x is:

$$x = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

where x is a combination (linear) of vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Thus the null space of A is said to be 2-dimensional.

Given that A maps x to a 3-dimensional vector and the nullspace of A has dimension 2, by the rank-nullity theorem:

$$\text{rank}(A) + \text{nullity}(A) = n$$

where n is the number of columns of A.

Since $n = 3$;

$$\boxed{\text{rank}(A) + 2 = 3}$$

Thus, $\boxed{\text{rank}(A) = 1}$

Since the rank of A is 1, thus the dimension of the row space of A is also said to be 1.

(II) To find matrix A:

We know that:

$$A \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$$

$$\text{Let } A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \rightarrow (1)$$

From the equation $A\begin{bmatrix}2\\0\\0\end{bmatrix}=\begin{bmatrix}2\\4\\2\end{bmatrix}$, we get

$2g=2$
$2d=4$
$2a=2$

$\because \left[A\begin{bmatrix}a\\d\\g\end{bmatrix}=\begin{bmatrix}2\\4\\2\end{bmatrix} \right]$

Thus: $a=1,\ d=2,\ g=1$

Thus Eq. 1 becomes:

$$A=\begin{bmatrix}1 & b & c\\ 2 & e & f\\ 1 & h & i\end{bmatrix}$$

Using the vectors $\begin{bmatrix}1\\1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\0\\1\end{bmatrix}$, we can find the values of b, c, e, f, h and i respectively.

These $\begin{bmatrix}1\\1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\0\\1\end{bmatrix}$ are in null space of A.

$$A\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}1+b\\2+e\\1+h\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ f \\ i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus we get equations:

$c = 0$

$f = 0$

$i = 0$

$1 + b = 0 ; \quad b = -1$

$2 + e = 0 ; \quad e = -2$

$1 + h = 0 ; \quad h = -1$

Thus the matrix A becomes:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

Answers:

(i) The dimension of row space of A is 1.

(ii) The matrix A is $\begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix}$

To find the stationary points, we have to set $f_x(x,y)=0$ + $f_y(x,y)=0$:

$$-\frac{1}{x^2}+y=0 \,; \quad \boxed{y=\frac{1}{x^2}} \rightarrow (4)$$

$$x-\frac{1}{y^2}=0$$

$$\boxed{x=\frac{1}{y^2}} \rightarrow (5)$$

Substitute (4) in (5):

$$x=\frac{1}{\left(\frac{1}{x^2}\right)^2}=\frac{1}{\left(\frac{1}{x^4}\right)}=x^4$$

$$x^4=x \Rightarrow x^4-x=0$$

$$\Rightarrow x(x^3-1)=0$$

Thus we have 2 solutions:

$$x=0\,; \quad x^3=1\,; \quad \boxed{x=1}$$

If $x=1$, then $y=\frac{1}{(1)^2}=1$

If $x=0$, then $y=\frac{1}{(0)^2}=$ undefined;

↳ Not a valid stationary point.

4) Find and classify all the stationary points of:

$$f(x,y) = \frac{1}{x} + xy + \frac{1}{y}$$

<u>Solution:</u>

<u>Given:</u>

$$f(x,y) = \frac{1}{x} + xy + \frac{1}{y} \rightarrow (1)$$

To find first partial derivatives:

With respect to:

x:

$$f_x(x,y) = \frac{\partial}{\partial x}\left(\frac{1}{x} + xy + \frac{1}{y}\right)$$

$$\boxed{f_x(x,y) = \frac{-1}{x^2} + y} \rightarrow (2)$$

y:

$$f_y(x,y) = \frac{\partial}{\partial y}\left(\frac{1}{x} + xy + \frac{1}{y}\right)$$

$$\boxed{f_y(x,y) = x - \frac{1}{y^2}} \rightarrow (3)$$

(i) Thus the only valid stationary point ; (x, y) is $(1, 1)$.

(ii) To compute second derivatives of $f(x, y)$ in order to classify the stationary points.

$$f_{xx}(x, y) = \frac{\partial}{\partial x}\left(\frac{-1}{x^2} + y\right) = \frac{2}{x^3}$$

$$f_{yy}(x, y) = \frac{\partial}{\partial y}\left(x - \frac{1}{y^2}\right) = \frac{2}{y^3}$$

$$f_{xy}(x, y) = \frac{\partial}{\partial x \partial y}\left(\frac{-1}{x^2} + y\right) = 1$$

At point $(1, 1)$:

$$f_{xx}(x, y) = \frac{2}{(1)^3} = 2$$

$$f_{yy}(x, y) = \frac{2}{(1)^3} = 2$$

$$f_{xy}(x, y) = 1$$

The Determinant D : $f_{xx}(1, 1)\, f_{yy}(1, 1) - (f_{xy}(1, 1))^2$
✓ Hessian

$$D = (2)(2) - (1)^2 = 4 - 1 = 3$$

$\because \Delta > 0$, and $f_{xx}(1,1) > 0$;

The function $f(x,y)$ has local minimum at stationary point $(1,1)$.

⑮ Find the maximum value of $f(x,y) = x^{\frac{2}{3}} y^{1/3}$ subject to constraint $x + 4y = 96$.

<u>Solution:</u>

$$f(x,y) = x^{2/3} y^{1/3}.$$

where $x + 4y = 96$.

By the method of Lagrange Multipliers,

(i) The Lagrangian function $\mathcal{L}$ is given by:

$$\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda(g(x,y) - c)$$

where $g(x,y) = x + 4y$ and $c = 96$

Thus:

$$\mathcal{L}(x,y,\lambda) = x^{2/3} y^{1/3} - \lambda(x + 4y - 96)$$

(ii) Finding the Partial Derivatives of $\mathcal{L}$ with respect to x, y, λ and set them to zero.

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{2}{3} x^{-1/3} y^{1/3} - \lambda = 0,$$

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{1}{3} x^{2/3} \cdot y^{1/3} - 4\lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -(x + 4y - 96) = 0$$

(iii) Solving the system of Equations:

from $\frac{\partial \mathcal{L}}{\partial x} = 0$;

$$\boxed{\lambda = \frac{2}{3} x^{-1/3} \cdot y^{1/3}} \rightarrow ①$$

From $\frac{\partial \mathcal{L}}{\partial y} = 0$,

$$\boxed{\lambda = \frac{1}{12} x^{2/3} y^{-2/3}} \rightarrow ②$$

Equating ① & ②:

$$\frac{2}{3} x^{-1/3} y^{1/3} = \frac{1}{12} x^{2/3} y^{-2/3}$$

$$8x^{-1/3} y^{1/3} = x^{2/3} \cdot y^{-2/3}, \quad 8y = x$$

$$x = 8y$$

Now substituting $x = 8y$ in $x + 4y = 96$.

$$8y + 4y = 96$$

$$12y = 96$$

$$\boxed{y = 8}$$

Substituting $y = 8$ in $x = 8y$;

$$x = 8 \cdot 8 = 64$$

$$\boxed{x = 64}$$

When $x = 64$, $y = 8$;

$$x + 4y = 96$$

$$64 + 4(8) = 64 + 32 \Rightarrow 96$$

Thus these x, y values satisfy the constraint.

$$f(x,y) = f(64, 8) = 64^{2/3} \cdot 8^{1/3}$$

$$64^{2/3} = (4^3)^{2/3} = 4^2 = 16$$

$$8^{1/3} = (2^3)^{1/3} = 2 = 2$$

$$f(64, 8) = 16 \cdot 2 = 32$$

$$\boxed{f(64, 8) = 32}$$

Thus the maximum value of $f(x,y) = x^{2/3} \cdot y^{1/3}$ subjecting to constraint $x + 4y = 96$ is 32.

⑥. Question 6:

x	1	2	3	4	5	6
$P(X=x)$	a	a	a	b	b	$1/3$

(i) Suppose $E[X] = \frac{25}{6}$. Find values of $a + b$.

Solution:

$$E[X] = \sum_{x=1}^{6} x.P(X=x) \rightarrow ①$$

Eq ①. shows the formula for expected value $E[x]$.

$P[X=1) = a$ $\quad P(X=3) = a$ $\quad P(X=5) = b$

$P(X=2) = a$ $\quad P(X=4) = b$ $\quad P(X=6) = 1/3$

All probabilities sum up to 1:

$$3a + 2b + \frac{1}{3} = 1$$

$$3a + 2b = 1 - \frac{1}{3} = \frac{2}{3}$$

$$\boxed{3a + 2b = \frac{2}{3}} \rightarrow ②$$

The expected value is:

$$E[X] = 1.a + 2.a + 3.a + 4 \cdot b + 5.b + 6.\left(\frac{1}{3}\right) = \frac{25}{6}$$

$$6a + 9b + 2 = \frac{25}{6}$$

$$6a + 9b = \frac{25}{6} - 2 = \frac{25 - 12}{6} = \frac{13}{6}$$

$$E[x] = 6a + 9b = \frac{13}{6} \rightarrow (3)$$

From (2) & (3)

$$3a + 2b = \frac{2}{3}$$

$$6a + 9b = \frac{13}{6}$$

Multiply Eq. (2) $\times$ 2

$$6a + 4b = \frac{4}{3}$$

$$(-)\, 6a\, (-)\, 9b = (-)\, \frac{13}{6}$$

$$-9b + 4b = \frac{-13}{6} + \frac{4}{3} \times 2$$

$$-5b = \frac{-13 + 8}{6}$$

$$-5b = \frac{-5}{6}$$

$$b = \frac{5}{30} = \frac{1}{6}$$

$$\boxed{b = \frac{1}{6}} \rightarrow (4)$$

Put ④ in ②

$3a + 2\left(\frac{1}{6}\right) = \frac{2}{3}$

$3a + \frac{1}{3} = \frac{2}{3}$

$3a = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$

$\boxed{a = \frac{1}{9}}$

The values of $[a, b]$ for $E[X] = \frac{25}{6}$ are $\left[\frac{1}{9}, \frac{1}{6}\right]$

(ii) Find k:

y	1	2	3	4	5
F(y)	$\frac{1}{10}$	$\frac{2}{10}$	3k	4K	5K

Given the CDF of Y, F(y) is:

CDF means Cummulative Distribution Function,

Since F(y) is a CDF, Then $F(5) = 1$

$5k = 1$

$\boxed{k = \frac{1}{5}}$

(iii) Find the probability mass function of Y:

y	$F(y)$	$P(Y=y)$
1	$\frac{1}{10}$	$\frac{1}{10}$
2	$\frac{2}{10}$	$\frac{2}{10} - \frac{1}{10} = \frac{1}{10}$
3	$\frac{3}{10}$	$3K - \frac{2}{10} = \frac{3}{10} - \frac{2}{10} = \frac{1}{10}$
4	$\frac{4}{10}$	$4K - 3K = \frac{4}{10} - \frac{3}{10} = \frac{1}{10}$
5	1	$5K - 4K = 1 - \frac{4}{10} = \frac{1}{10}$

$$P(Y=y) = \begin{cases} \frac{1}{10} & \text{for } y \in \{1, 2, 3, 4, 5\} \\ 0 & , \text{otherwise.} \end{cases}$$