SIG787 - Mathematics for AI

Assignment - 1

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Question 1:

Consider the following function:

$$f(x) = (\sqrt{x})^{\log_5(x) - 1}$$

(i) Find the domain of $f(x)$

(ii) Solve the equation of $f(x) = 5$

Hint: You need to use the rules of logarithms to solve the problem. In general, for the function:

$y = \log_b(x)$ where base $b > 0$ and $b \neq 1$

Solution 1:

Since $\sqrt{x} = (x)^{1/2}$, so we can rewrite the function now,

$$f(x) = x^{1/2(\log_5(x) - 1)} \rightarrow ①$$

We know that:

x^a is defined except for $x = a = 0$.

The logarithmic function is defined for all $x > 0$

(i) Here $\log_5(x)$ is defined for all $x > 0$.

Hence the domain for $f(x)$ is $(0, \infty)$

(ii) To solve the equation for $f(x) = 5$

Implementing $f(x) = 5$ in (1)

$$x^{\frac{1}{2}(\log_5(x) - 1)} = 5$$

Now we have to take log on both sides,

$$\log_5\left(x^{\frac{1}{2}(\log_5(x) - 1)}\right) = \log_5 5$$

Here base is taken as 5.

With the help of Logarithm properities,

$$\left[\frac{1}{2}\left(\log_5(x) - 1\right)\right] \log_5(x) = 1 \rightarrow (2)$$

$$\because \log_5(a^b) = b \log_5(a) \rightarrow (3)$$

Also, $\log_5 5 = 1 \rightarrow (4)$

On simplifying the above Eq. (2) using

Eq. (3) + (4).

$$[\log_5(x)]^2 - \log_5(x) = 2 \quad \rightarrow (5)$$

On taking, $y = \log_5(x)$. The above Eq. (5) is replaced with this.

$$\boxed{y^2 - y = 2} \rightarrow (6)$$

Solving this eq. (6) for y:

$$y^2 - y = 2$$

$$y^2 - y - 2 = 0$$

-2

-2 +1

$\cancel{-2} \times +1$

$-2+1 \Downarrow -1$

$\Downarrow -2$

On comparing this above equation with:

$$ax^2 + bx + c = 0$$

where $b = -1$

$c = -2$

$$(y-2)(y+1) = 0$$

On equating the above Equation to 0.

$$y = 2; \quad y = -1$$

When $y = 2$:

$$\log_5(x) = 2$$

$$\Rightarrow \log_5(5)^2$$

Since $\log_5(a^b) = b \log_5(a)$

Cancelling logarithm both sides,

$$x = 5^2 = 25$$

When $x = -1$:

$$\log_5(x) = -1$$
$$= \log_5(5^{-1})$$

The solution of Equation $f(x) = 5$:

are $x = 25, \frac{1}{5}$

(i) Domain of $f(x)$ is $(0, \infty)$

(ii) Solution for $f(x) = 5$ are $25, \frac{1}{5}$

Q2. Consider the following function:

$$f(x) = \frac{x}{1 + x|x|}$$

(i) Find the domain of $f(x)$

(ii) Find all x & y intercepts

(iii) Rewrite the above functions as piecewise defined functions

(iv) Find all the stationary points + classify them

(v) Determine the intervals for which the function is increasing and the intervals for which the function is decreasing.

(vi) Find the second derivative of the function and identify all the intervals that the function is convex (or) concave.

(vii) Sketch the function by hand based on the information you gained through (i) to (vi) steps.

Label all the important points on the graph of function:

Solution:

Given: $f(x) = \dfrac{x}{1+x|x|} \rightarrow ①$

(i) Domain of $f(x)$:

The function is defined for all real numbers except where the denomination becomes zero.

(ie) where $1+x|x| = 0$.

Since the denominator has an absolute value, this occurs when x is either negative (or) positive, that leads to $|x| = -x$ (or) $|x| = x$ respectively.

For $x<0$, $|x| = -x$, so the eq. becomes $1-x^2 = 0$ and has no real solutions.

For $x \geq 0$, $|x| = x$, the equation becomes $1+x^2=0$, This also has no real solutions

The domain for $f(x)$ is real numbers.

(ie) $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

(ii) Find the x & y intercepts:

→ we have to set $f(x)=0$ to find x-intercept:

(i.e) $\frac{x}{1+x|x|}$ at $x=0$

The numerator x can be zero. Now the intercept is at $x=0$

→ Now $f(0) = \frac{0}{1+0} = 0$

Now $y=0$ (that is y intercept is 0) is also got.

The intercepts x & y are $(0,0)$.

(iii) Piecewise defined functions:

At $x=1$, Eq① becomes,

$$f(1) = \frac{1}{1+(1)|1|} = \frac{1}{2}$$

$$f(-1) = \frac{-1}{1+(-1)|-1|} = \frac{-1}{1+(-1)(1)} = \frac{-1}{0}$$

$$f(2) = \frac{2}{1+2|2|} = \frac{2}{5}$$

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On rewriting we get:

$$f(x) = \frac{x}{1+x^2} ; x \geq 0$$

$$f(x) = \frac{x}{1-x^2} ; x < 0$$

The domain is for all real numbers except $x = -1, 1$.

x intercept is $(0,0)$

y intercept is $(0,0)$

(iv) Find all the stationary points & classify them

For $x \geq 0$:

Differentiating : $$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x) \frac{d}{dx}[f(x)] - f(x)\left(\frac{d}{dx}[g(x)]\right)}{g(x)^2}$$ → ②

(ie) $f(x) = x$
$g(x) = 1 + x^2$ as $x \geq 0$

Applying the $f(x)$ & $g(x)$ in Eq. (2), It becomes

$$\frac{(1+x^2)\frac{d}{dx}(x) - x\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

Differentiating:

$$f'(x) = \frac{1+x^2 -2x}{(1+x^2)^2}$$

x is raised to power of 1

$$f'(x) = \frac{1+x^2 + 2x^1 x^1}{(1+x^2)^2}$$

Combing the powers using the power rule;

$$a^m . a^n = a^{m+n}$$

$$f'(x) = \frac{1+x^2 - 2x^2}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$$

↳ (3)

To find the stationary points we set $f'(x)=0$ on Eq. ③

$$\frac{-x^2+1}{(x^2+1)^2} = 0$$

$$\Rightarrow \quad -x^2 = -1$$

$$x^2 = 1$$

$$\boxed{x=1}$$

for $x < 1$, $f'(x) > 0$. This inturn implies that function is increasing.

for $x > 1$, $f'(x) < 0$, this implies that function is decreasing.

For $x < 0$;

$$f'(x) = \frac{x^2+1}{(1-x^2)^2}$$

$$f'(x) \neq 0 \quad \text{for} \quad x<0$$

At $x=0$, the function has stationary point.

This is the local maximum point.

(v) Determine the intervals for which the function is increasing and decreasing.

Now differentiating with respect to x

For $x \geq 0$;

$$f'(x) = \frac{-x^2+1}{(x^2+1)^2}$$

we get $x = 1$, for $f'(x) = 0$

For $f'(x) > 0$, function is increasing

For $f'(x) < 0$, its decreasing.

$f(x)$ is not defined at $-1, 1$

So the critical points are $x = -1, 0, 1$

For $x \geq 0$ Intervals: $[0, 1), (1, \infty)$

$[0, 1) \Rightarrow f'(x) > 0$; $[1, \infty) \Rightarrow f'(x) < 0$

For $x < 0$:

$f'(x) > 0$ for all $x < 0$

$f'(x) > 0$ for $(-\infty, 0)$

(vi) Find the second derivative of function + identity all the intervals that function is concave or convex.

Using the Quotient rule:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

Using Eq. ③ we get:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{(x^2+1)^2(-2x) - (-x^2+1)\frac{d}{dx}(x^2+1)^2}{(x^2+1)^4}$$

$$f''(x) = \frac{(x^2+1)^2(-2x) - (-x^2+1)(2(x^2+1)\frac{d}{dx}(x^2+1))}{(x^2+1)^4}$$

On simplifying, we get:

$$f''(x) = \frac{(x^2+1)^2(-2x) - 4(-x^2+1)((x^2+1)x)}{(x^2+1)^4}$$

for $f''(x) = 0$; $f''(x) = \dfrac{2x(x^2-3)}{(x^2+1)^3}$

And values of x are : $0, \sqrt{3}$

If $x = 1$, $f''(1) = \frac{-4}{8} < 0$

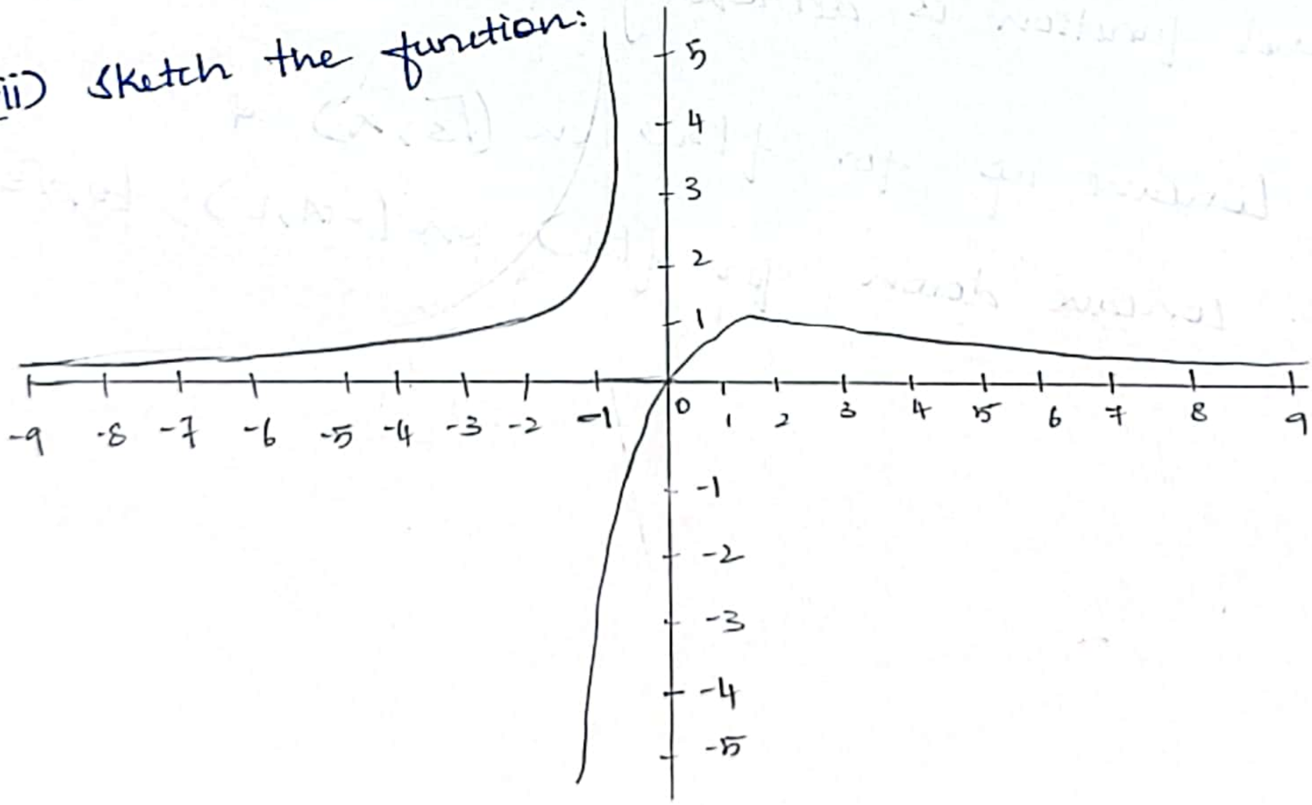
If $x = \sqrt{3}+1$ $f''(\sqrt{3}+1) > 0$

So; $f''(x) > 0$ for $x \in (\sqrt{3}, \infty)$
$f''(x) < 0$ for $x \in (0, \sqrt{3})$

$f(x)$ is concave up for $(\sqrt{3}, \infty)$ & concave down for $(0, \sqrt{3})$.

Also $f(x)$ is concave down on $(-\infty, 0)$

(vii) Sketch the function:



Final answers / conclusions :

(i) The domain of the function is all real numbers except $x = -1, 1$

(ii) x intercept is $(0, 0)$ & y intercept is $(0, 0)$

(iii) Piece wise function :

$$f(x) = \begin{cases} \frac{x}{1+x^2} & x \geq 0 \\ \frac{x}{1-x^2} & x < 0 \end{cases}$$

(iv) At $x = 0$, the function has stationary point. This point is local maximum.

(v) Function is increasing on $(-\infty, 0)$, $[0, 1)$ for $f'(x) > 0$ and function is decreasing on $(1, \infty)$ for $f'(x) < 0$

(vi) Concave up for $f(x)$ on $(\sqrt{3}, \infty)$ & concave down for $f(x)$ on $(-\infty, 0)$, $(0, \sqrt{3})$