SIG718 - Real World Analytics

End-Term Assignment

1. A garment factory produces shirts and pants for Kmart chain. The contract is such that quality control is done before shipping and all products supplied to Kmart satisfying the quality requirements would be accepted by the chain. The factory employs 20 workers in the cutting department, 50 workers in the sewing department, and 14 workers in the packaging department. The garment factory works 8 productive hours a day (no idle time during these 8 hours). There is a daily demand for at most 180 shirts. The demand for pants is unlimited. Each worker can participate only in one activity- the activity to which they are assigned. The table below gives the time requirements (in minutes) and profit per unit for the two garments.

	Amount (minutes) per operation			
	Cutting	Sewing	Packaging	Profit per unit($)
Shirts	40	40	20	10
Pants	20	100	20	8

a) Explain why a Linear Programming (LP) model would be suitable for this case study. [5 marks]

Answer:

The core idea is to build a model based on the data and scenarios to make better decisions. This factory problem involves maximizing profits by producing shirts and pants with some constraints like labour hours and market demands. This can be solved by forming linear equations of shirts and pants and optimize the linear objective function (i.e, Maximizing the Profit).

Let us try to understand the elements of Linear Programming Model:

- Here, the decision variables which are of certain quantity of a material are shirts and pants.
- The objective function here is to maximize the profits.
- There may be multiple objectives taken as single objective they are called as constraints which itself may impose restrictions on the demand, supply and resource allocation.
- For example, here the daily demand of production of shirts is not more than 180.
- The demand for pants is unlimited. Each worker can participate only in one activity- the activity to which they are assigned.

Since Linear Programming technique is said to be solving problems that has so many constraints in the problem itself.

b) Formulate a LP model to help the factory management to determine the optimal daily production schedule that maximises the profit while satisfying all constraints. [10 marks]

Answer:

Let x be the number of shirts produced per day.

Let y be the number of pants produced per day.

Its mandatory to calculate the total minutes available for each department per day.

- Total minutes available for cutting department per day:

20 workers x 8 hours / day x 60 minutes / hour = 9600 minutes

- Total minutes available for sewing department per day:

50 workers x 8 hours / day x 60 minutes / hour = 24000 minutes

- Total minutes available for packaging department per day:

14 workers x 8 hours / day x 60 minutes / hour = 6720 minutes

The objective function is to maximize the profits Max_z , the linear equation is given by:

Max_z = ((Profit per unit) * (Number of shirts) + (Profit per unit) * (Number of Pants))

$$Max_z = 10x + 8y$$

The above equation matches with the given table.

The following below are the constraint equations:

- Constraint equation for Cutting department:

$$40x + 20y <= 9600$$

- Constraint equation for sewing department:

  40x + 100y <= 24000

- Constraint equation for packaging department:

  20x + 20y <= 6720

- Demand constraints for shirts:

  x <= 180

- Non- Negativity Constraints, since we cannot have negative number of shirts and pants:

  x >= 0, y>= 0

Our LP model combines all the equations together, giving us:

- 40x + 20y <= 9600
- 40x + 100y <= 24000
- 20x + 20y <= 6720
- x <= 180
- x >= 0, y>= 0

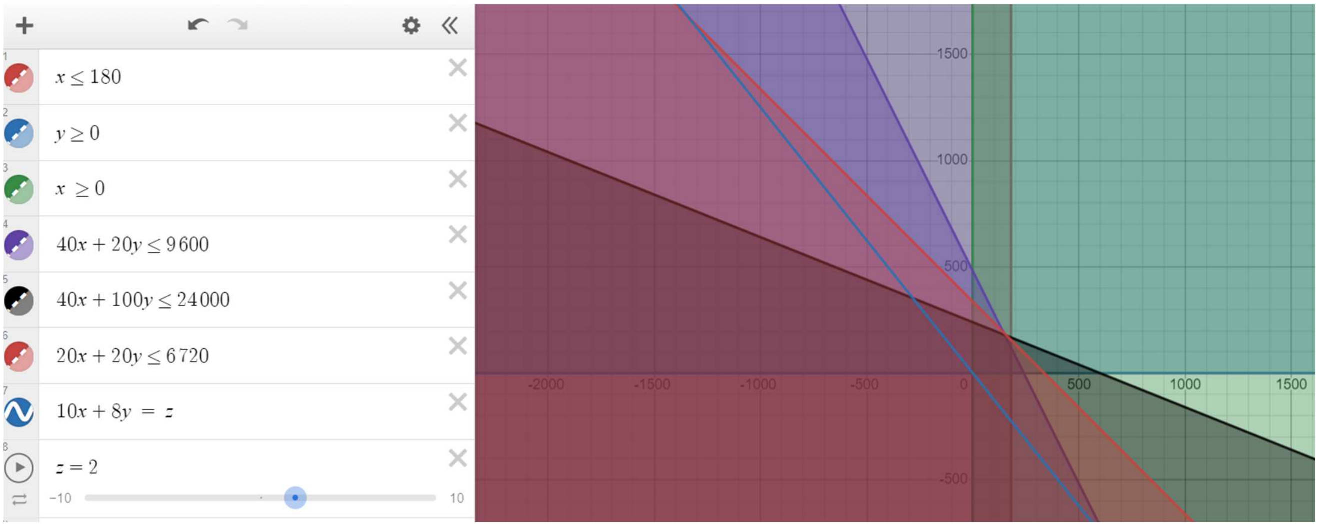
c) Use the graphical method to find the optimal solution. Show the feasible region and the optimal solution on the graph. Annotate all lines on your graph. What is the optimal daily profit for the factory? [10 marks]

Note: you can use graphical solvers available online but make sure that your graph is clear, all variables involved are clearly represented and annotated, and each line is clearly marked and related to the corresponding equation.

Our LP model combines all the equations together, giving us:

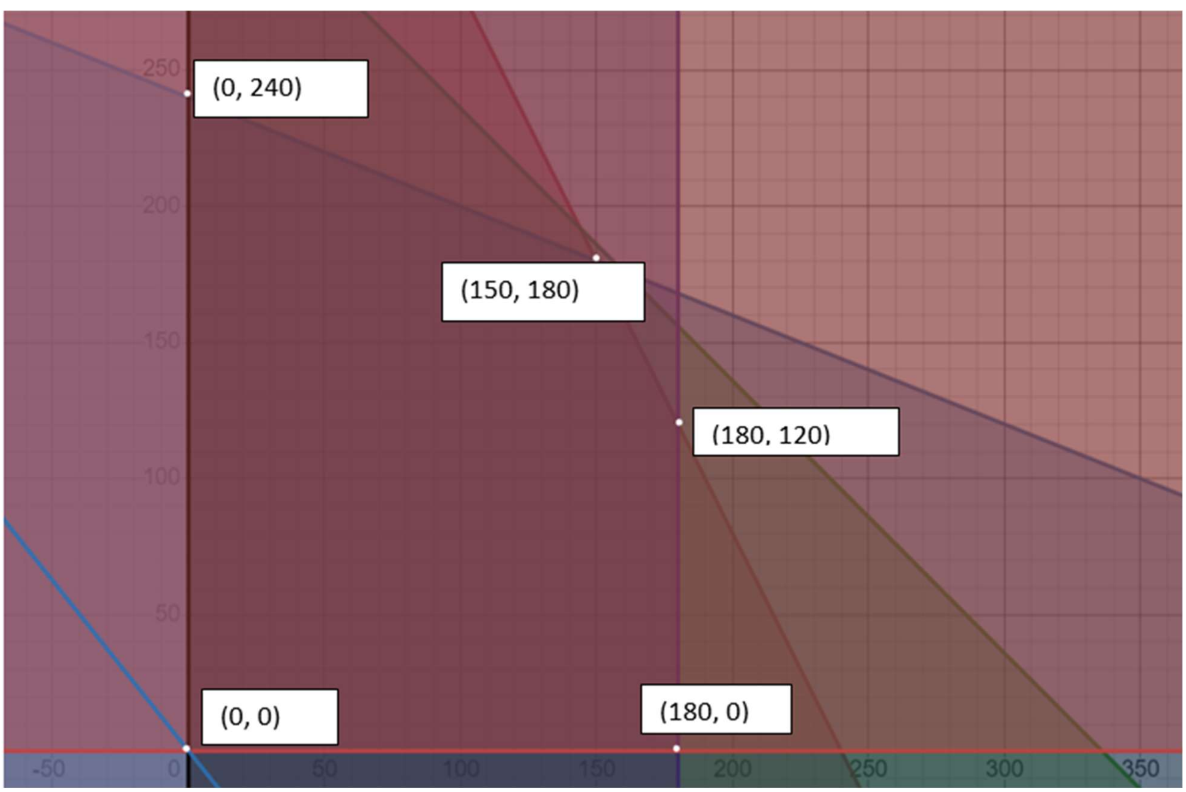
- 40x + 20y <= 9600
- 40x + 100y <= 24000
- 20x + 20y <= 6720
- x <= 180
- x >= 0, y>= 0
- 10x + 8y = z, This is the objective equation.

Demos is used here in order to get the clear graph,

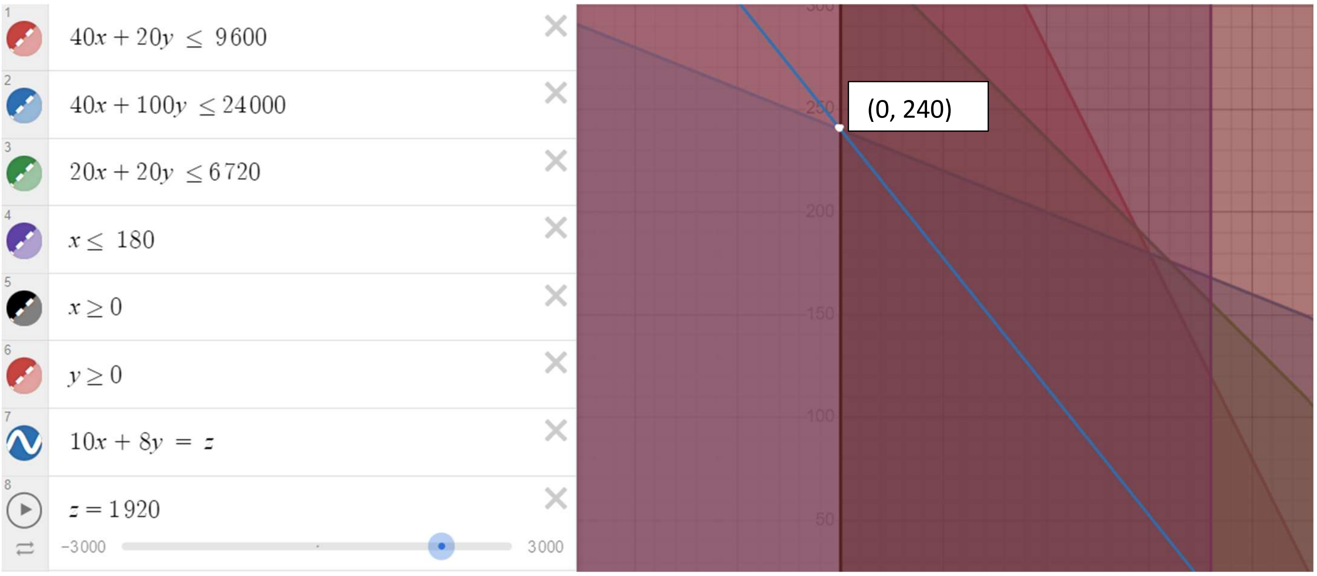


The feasible region is under the co-ordinates:

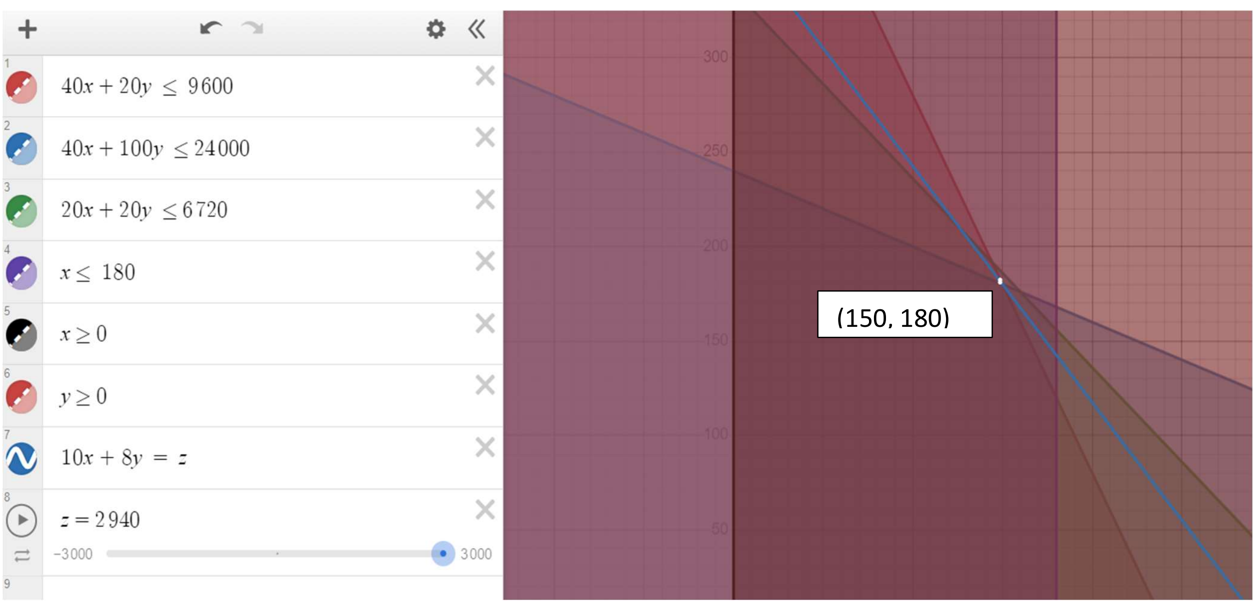
(0, 240), (150, 180), (180, 120), (180, 0), (0, 0)



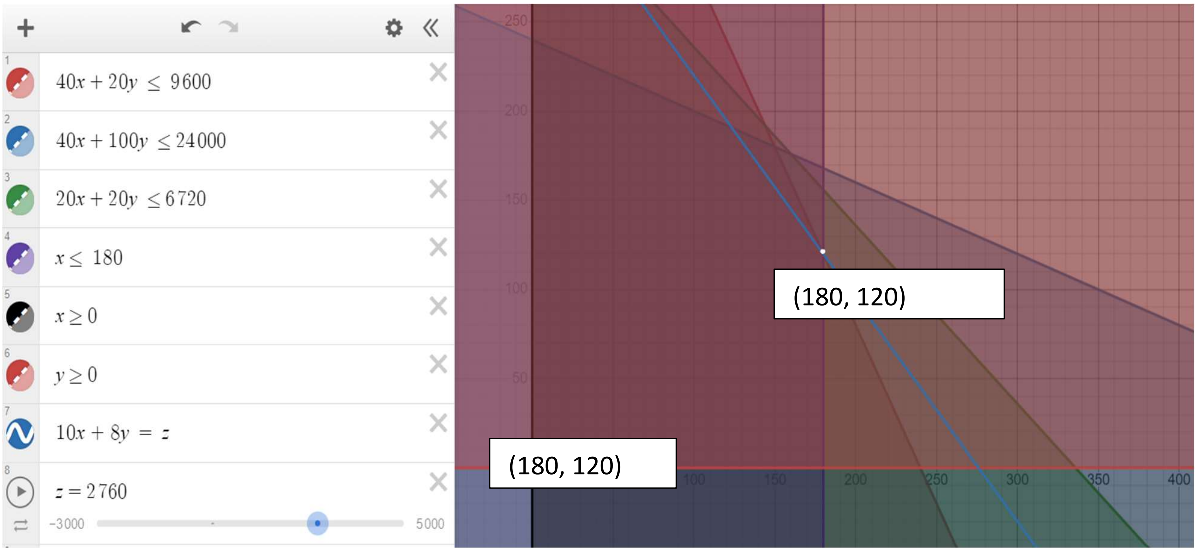
➢ For the co-ordinate value of (0, 240), The Max_z value is 1920.



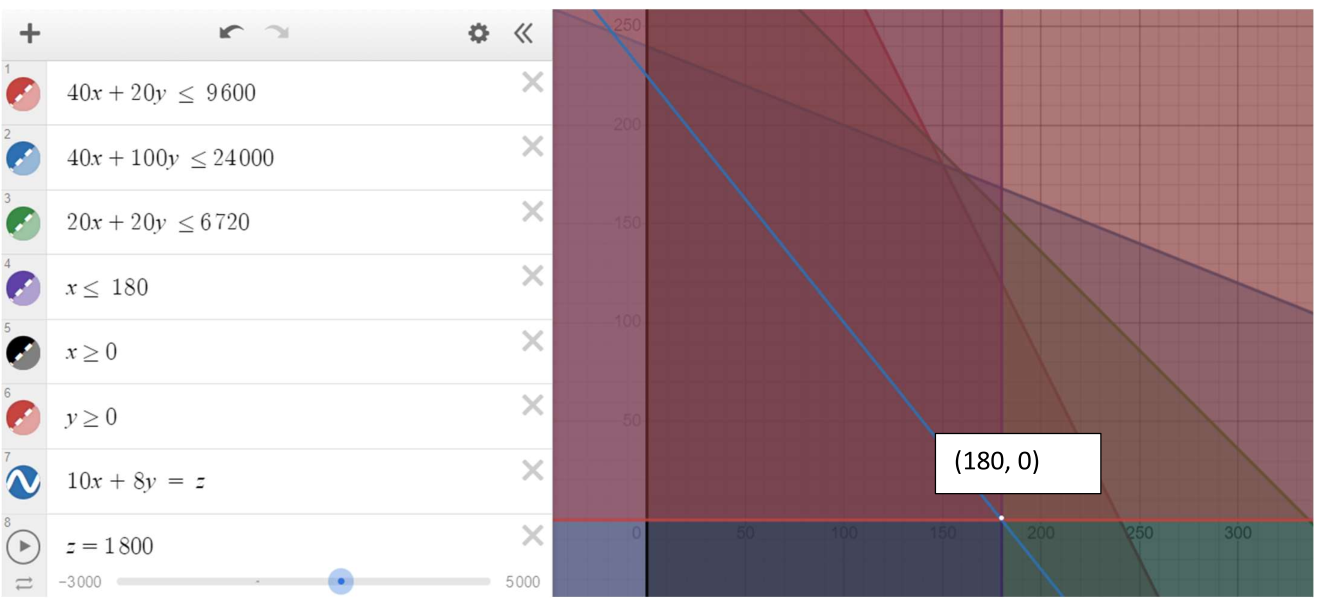
➢ For the co-ordinate value of (150, 180), The Max_z value is 2940.



➢ For the co-ordinate value of (180, 120), The Max_z value is 2760.



➢ For the co-ordinate value of (180, 0), The Max_z value is 1800.



The optimal solution on the graph is (150, 180) and the optimal daily profit for the factory is 2940.

d) Find the range for the profit ($) per shirt (if any) that can be obtained without affecting the optimal point of part (c). [5 marks]

Our LP model combines all the equations together, giving us:

- $40x + 20y <= 9600$
- $40x + 100y <= 24000$
- $20x + 20y <= 6720$
- $x <= 180$
- $x >= 0, y>= 0$
- $10x + 8y = \$ 2940$ as per above answer, This is the objective equation.

The way to approach this is by rewriting all the constraints in "slope-intercept" form, which is in the form of,

$$y = mx + c$$

$y < = 9600 – 40x/ 20$…. Eq 1

$y <= 24000 – 40x/ 100$….. Eq 2

$y <= 6720 – 20x/ 20$….. Eq 3

$8y <= 2940 – 10x$…. Eq 4

If we move from (150, 180) in a clockwise direction, makes the slope of objective function reach-2. The optimal solution changes from (150, 180) to (180, 0).

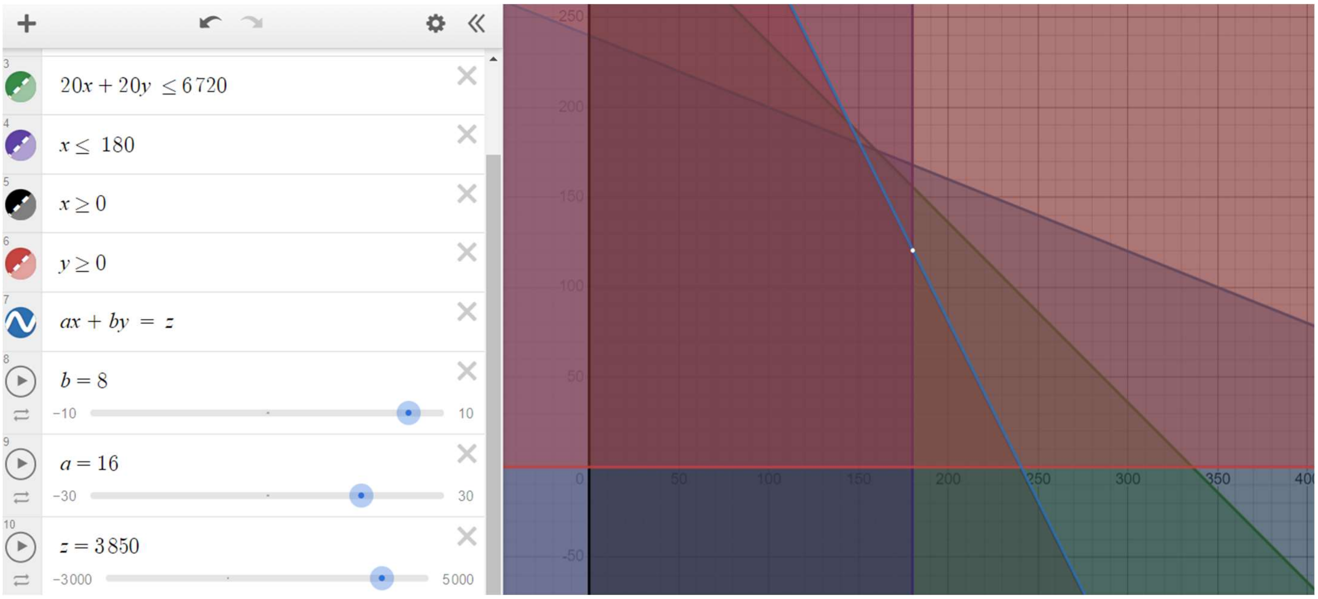
The optimal range for the profit ($) per shirt is found by replacing the co-efficient of interest in Eq 4 with a dummy variable c,

$Z = cx + 8y$

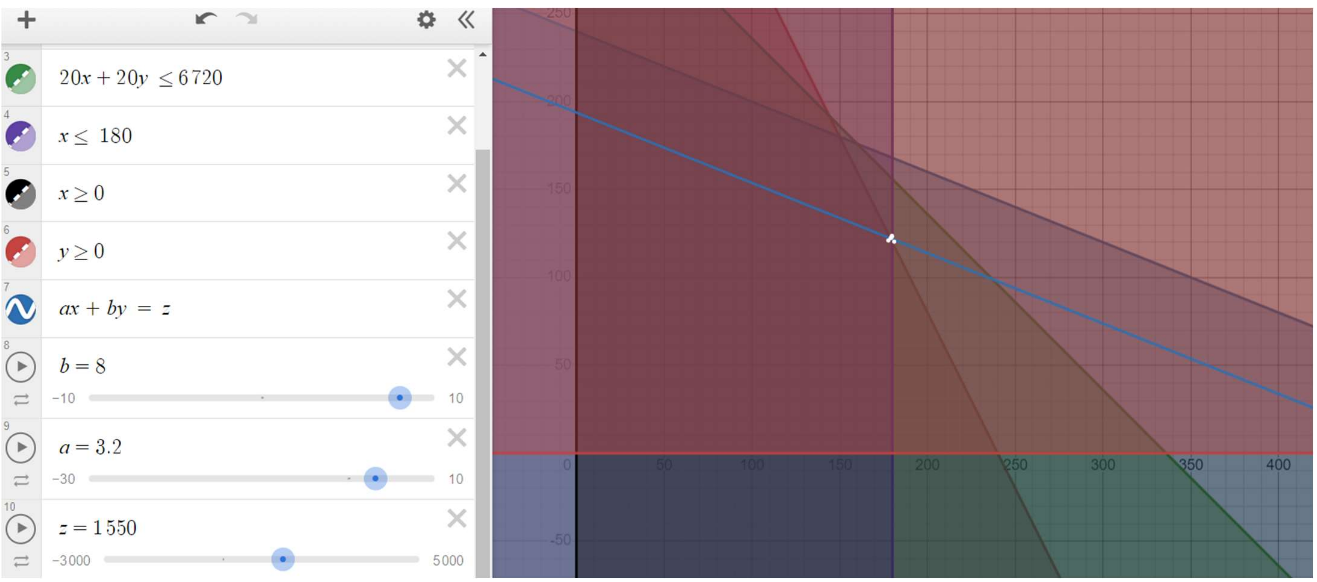
And the c ranges between,

$$3.2 <= a <= 16$$

For a = 16, Max profit for this is $ 3850.



For a = 3.2, Max profit for this is $ 1550:



d) The range for the profit ($) per shirt,

3.2 <= a <= 16

that can be obtained without affecting the optimal point of part.

2. A factory makes three products called Bloom, Amber, and Leaf, from three materials containing Cotton, Wool and Nylon. The following table provides details on the sales price, production cost and purchase cost per ton of products and materials respectively.

	Sales price	Production cost		Purchase price
Bloom	\$60	\$5	Cotton	\$40
Amber	\$55	\$4	Wool	\$45
Leaf	\$60	\$5	Nylon	\$30

TABLE 1

The maximal demand (in tons) for each product, the minimum cotton and wool proportion in each product are as follows,

	Demand	min Cotton proportion	min Wool proportion
Bloom	4200	50%	40%
Amber	3200	60%	40%
Leaf	3500	50%	30%

a) Formulate an LP model for the factory that maximises the profit, while satisfying the demand and the cotton and wool proportion constraints. [20 Marks]

On using the hint,

1. Let $x_{ij} \geq 0$ be a decision variable that denotes the number of tons of products j for j ∈ {1 = Bloom, 2 = Amber, 3 = Leaf} to be produced from Materials i ∈ {C=Cotton, W=Wool, N=Nylon}.

2. The proportion of a particular type of Material in a particular type of Product can be calculated as: e.g., the proportion of Cotton in product Bloom is given by: $x_{C1} / (x_{C1} + x_{W1} + x_{N1})$

The objective function is to maximise the profits which is the difference of sales price, production cost and purchase price.

Max. Profit = Sales Price- Production Cost- Purchase Price

On performing the above function on the table 1,

We get,

Max. Profit = [$(60-5-40)x_{C1} + (55-4-40)x_{C2} + (60-5-40)x_{C3} + (60-5-45)x_{W1} +$

$(55-4-45)x_{W2} + (60-5-45)x_{W3} + (60-5-30)x_{N1} + (55-4-30)x_{N2} +$

$(60-5-30)x_{N3}$]

On solving it we get,

Max. Profit = [$15x_{C1} + 11x_{C2} + 15x_{C3} + 10x_{W1} + 6x_{W2} + 10x_{W3} + 25x_{N1} + 21x_{N2} + 25x_{N3}$]

The next step is to form the constraint equations:

- Demand Constraints:

$$x_{C1} + x_{W1} + x_{N1} <= 4200$$

$$x_{C2} + x_{W2} + x_{N2} <= 3200$$

$$x_{C3} + x_{W3} + x_{N3} <= 3500$$

- Cotton proportion constraints:

  X_{C1} >= 50% ($x_{C1} + x_{W1} + x_{N1}$), This is re-written as:
  $X_{C1} >= 0.5(x_{C1} + x_{W1} + x_{N1})$
  On solving we get,

$$0.5x_{C1} - 0.5x_{W1} – 0.5x_{N1} >= 0$$

  Solving in similar way we get the following equations:

$$0.4x_{C2} - 0.6x_{W2} – 0.6x_{N2} >= 0$$
$$0.5x_{C3} - 0.5x_{W3} – 0.5x_{N3} >= 0$$

- Wool proportion constraints:

  X_{W1} >= 40% ($x_{C1} + x_{W1} + x_{N1}$), This is re-written as:
  $X_{W1} >= 0.4(x_{C1} + x_{W1} + x_{N1})$
  On solving we get,

$$-0.4x_{C1} + 0.6x_{W1} – 0.4x_{N1} >= 0$$

  Solving in similar way we get the following equations:

$$-0.4x_{C2} + 0.6x_{W2} – 0.4x_{N2} >= 0$$
$$-0.3x_{C3} + 0.7x_{W3} – 0.3x_{N3} >= 0$$

➢ <u>Non-Negativity Constraints:</u>

x_{ij} >= 0, for all values materials and products.

b) Solve the model using R/R Studio. Find the optimal profit and optimal values of the decision variables. [20 Marks]

The code is present in the file attached "BPriyankaa-code".pdf

<u>Inference:</u> The optimal profit is 141850 and optimal values of the decision variables are 2100, 1920, 1750, 1680, 1280, 1050, 420, 0, 700.

3. Two construction companies, Giant and Sky, bid for the right to build in a field. The possible bids are $ 10 Million, $ 20 Million, $ 30 Million, $ 35 Million and $ 40 Million. The winner is the company with the higher bid. The two companies decide that in the case of a tie (equal bids), Giant is the winner and will get the field. Giant has ordered a survey and, based on the report from the survey, concludes that getting the field for more than $ 35 Million is as bad as not getting it (assume loss), except in case of a tie (assume win). Sky is not aware of this survey.

(a) State reasons why/how this game can be described as a two-players-zero-sum game [5 Marks]

➢ Two-players-zero-sum game is a game consisting of two players and outcomes being either win or loss. When we add the win and loss value, it becomes zero.
➢ Here the Giant and Sky are the two construction companies (Two Players) and since the gain of one Player will equal the loss of the other Player. The value becomes zero.

This is why this game can be described as two-players-zero-sum game.

These companies must bid high to get the field. The possible bids are $ 10,20,30,35 & 40 million.

<u>Cases:</u>

- Gaint will be the winner if there is an equal bid.
- If Gaint gets the field for more than $ 35 Million, it will win. Bid >= $ 35 million

- In both the cases, Company Sky will lose.

Here lets, assume win as 1 and loss as-1.

Both the companies has equal number of strategies = 5.

(b) Considering all possible combinations of bids, formulate the payoff matrix for the game. [5 Marks]

The strategy used by the companies is the payoff matrix.

The notation for this problem is:

The choices of Player 1 are represented in the rows of a matrix and the choices of Player 2 are represented in the columns of a matrix

m, number of strategies for Player 1, Company Giant

n, number of strategies for Player 2, Company Sky

a_i = ith strategy for Player 1

b_j = jth strategy for Player 2

v_{ij} is the Payoff-matrix

Payoff of Player 1 = (- Payoff of Player 2)

This payoff matrix has 5 strategies chosen by the Player 1.

Based on the cases given, this matrix is formulated.

This following image has the Pay-off matrix,

		Player 2					**V**
Player 1		10 M	20 M	30 M	35 M	40 M	
	10 M	1	-1	-1	-1	-1	1
	20 M	1	1	-1	-1	-1	1
	30 M	1	1	1	-1	-1	1
	35 M	1	1	1	1	-1	1
	40 M	-1	-1	-1	-1	1	1

(c) Explain what is a saddle point. Verify: does the game have a saddle point? [5 Marks]

	Player 2						V	Min
Player 1		10 M	20 M	30 M	35 M	40 M		
	10 M	1	-1	-1	-1	-1	1	-1
	20 M	1	1	-1	-1	-1	1	-1
	30 M	1	1	1	-1	-1	1	-1
	35 M	1	1	1	1	-1	1	-1
	40 M	-1	-1	-1	-1	1	1	-1
	MAX	1	1	1	1	1		

A saddle point in the two-player zero-sum game is a point where it is the minimum of row values and maximum of the column values.

Then Min column represents the minimum value in each row and MAX represents the maximum value of each column. A saddle point occurs when there is atleast one value that is same in Min and Max column.

Here there is no such saddle point.

In this model lower bound(L) is 0, and upper bound(U) is Inf.

Since L < U then players will make use of mixed strategies.

(d) Construct a linear programming model for Company Sky in this game. [5 Marks]

Using the linear programming model for Player 2,

a_i = ith strategy for Player 1

b_j = jth strategy for Player 2

v_{ij} is the Payoff-matrix

Here $a_i = x_i$ = ith strategy for Player 1; $b_j = y_j$ = jth strategy for Player 2

Player 2							V
		10 M	20 M	30 M	35 M	40 M	
Player 1	10 M	1	-1	-1	-1	-1	1
	20 M	1	1	-1	-1	-1	1
	30 M	1	1	1	-1	-1	1
	35 M	1	1	1	1	-1	1
	40 M	-1	-1	-1	-1	1	1

Suppose the Player 2, chooses the mixed strategy (y_1, y_2, y_3, y_4, y_5)

- If the Player 1 chooses the strategy 1, then expected answer is $y_1 - y_2 - y_3 - y_4 - y_5 >= 0$
- If the Player 1 chooses the strategy 2, then expected answer is $y_1 + y_2 - y_3 - y_4 - y_5 >= 0$
- If the Player 1 chooses the strategy 3, then expected answer is $y_1 + y_2 + y_3 - y_4 - y_5 >= 0$
- If the Player 1 chooses the strategy 4, then expected answer is $y_1 + y_2 + y_3 + y_4 - y_5 >= 0$
- If the Player 1 chooses the strategy 5, then expected answer is $-y_1 - y_2 - y_3 - y_4 + y_5 >= 0$

The objective function is min v where,

$$v >= y_1, \; v - y_1 >= 0$$

$$v >= y_2, \; v - y_2 >= 0$$

$$v >= y_3, \; v - y_3 >= 0$$

$$v >= y_4, \; v - y_4 >= 0$$

$$v >= y_5, \; v - y_5 >= 0$$

On adding all the constraints, it must sum upto 1, $y_1 + y_2 + y_3 + y_4 + y_5 = 1$

In this model lower bound(L) is 0, and upper bound(U) is Inf.

Since L < U then players will make use of mixed strategies.

(e) Produce an appropriate code to solve the linear programming model in part (d). [5 Marks]

The code is under the “QUESTION-3” section in the name named “BPriyankaa-code.R”

(f) Solve the game for Sky using the linear programming model and the code you constructed in parts (d) and (e). Interpret your solution. [5 Marks]

The linear programming model is coded for the game of Company Sky.

- This is the necessary library to create a linear programming model and for consistency – lpSolveAPI.
- Linear programming model for Company Sky with 6 constraints and 6 decision variables.
- This is the objective function for Company Sky, v- minimizing the returns.
- Let y_1, y_2, y_3, y_4, y_5- constraints.
- The payoff matrix is coded in r as rows.

The sum of the total probability must be 1 and it is also set into the row.

The equality conditions and RHS values are set.

The values are all set and solve the model. The optimal value and optimal values of the decision variables are obtained.

```
get.objective(BidModel) # Get the optimal value
[1] 0
> get.variables(BidModel) # optimal values of the decision variables
[1] 0.5 0.0 0.0 0.0 0.5 0.0
> get.constraints(BidModel)
[1] 0 0 0 0 0 1
```

Since the objective function is min v where,

$$v >= y_1,\ v - y_1 >= 0$$

$$v >= y_2,\ v - y_2 >= 0$$

$$v >= y_3,\ v - y_3 >= 0$$

$$v >= y_4,\ v - y_4 >= 0$$

$$v >= y_5,\ v - y_5 >= 0$$

On adding all the constraints, it must sum upto 1, $y_1 + y_2 + y_3 + y_4 + y_5 = 1$

The optimal value is 0 and the optimal values of decision variables are

0.5, 0, 0, 0.5, 0, 0

Of the linear programming model of the game of the Company Sky.

REFERENCE

Jose M Sallan, Orial Lordan, Vicenc Fernandez. Modeling and solving linear programming with R (2015)

lp_solve, Konis K, Schwendinger F, Hornik K (2023). _lpSolveAPI: R Interface to 'lp_solve' Version 5.5.2.0_. R package version 5.5.2.0-17.11, <https://CRAN.R-project.org/package=lpSolveAPI>.