SIG787 - MATHEMATICS OF ARTIFICIAL INTELLIGENCE

ASSIGNMENT - 2

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① Question 1:

Consider these vectors $u = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $v = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix}$ and

$w = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 3 \end{bmatrix}$

(i) Determine which two vectors most similar to each other based on these norms:

(a) l_2 norm:

$$dist(x, y) = \|x - y\|_2 = \|y - x\|_2 \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$$

(b) l_2 norm:

$$dist(x, y) = \|x - y\|_1 = \|y - x\|_1 = \sum_{i=1}^{n} |x_i - y_i|$$

$$dist(x, y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$$

$$dist(\vec{u}, \vec{v}) = \sqrt{(1-0)^2 + (0-2)^2 + (1-1)^2 + (1-1)^2}$$

$$= \sqrt{5}$$

$$dist(\vec{v}, \vec{w}) = \sqrt{(0-1)^2 + (2-1)^2 + (1-0)^2 + (1-3}$$

$$= \sqrt{1+1+1+4} = \sqrt{7}$$

$$\text{dist}(\vec{u}, \vec{w}) = \sqrt{(1-1)^2 + (0-1)^2 + (1-0)^2 + (1-3)^2}$$
$$= \sqrt{6}$$

The vector $\vec{u}$ and $\vec{v}$ is the least so they are similar.

$$\sqrt{5} \leq \sqrt{6} \leq \sqrt{7}$$

$$\text{dist}(\vec{u}, \vec{v}) \leq \text{dist}(\vec{v}, \vec{w}) \leq \text{dist}(\vec{u}, \vec{w})$$

(b) l_1 norm:

$$\text{dist}(x, y) = \sum_{i=1}^{n} |x_i - y_i|$$

$$\text{dist}(\vec{u}, \vec{v}) = |1-0| + |0-2| + |1-1| + |1-1|$$
$$= |1| + |2|$$
$$= |3| = 3$$

$$\text{dist}(\vec{v}, \vec{w}) = |0-1| + |2-1| + |1-0| + |1-3|$$
$$= |-1| + |1| + |1| + |-2|$$
$$= 1 + 1 + 1 + 2 = 5$$

$$\text{dist}(\vec{u}, \vec{w}) = |1-1| + |0-1| + |1-0| + |1-3|$$
$$= 0 + |-1| + |1| + |2|$$
$$= 4$$

Here also, The distance is the "least" between the vectors $\vec{u}$ and $\vec{v}$. They are similar to each other.

$$3 \leq 4 \leq 5$$

$$dist(\vec{u}, \vec{v}) \leq dist(\vec{v}, \vec{w}) \leq dist(\vec{u}, \vec{w})$$

(ii) Determine which 2 vectors are most similar to each other based on cosine similarity measure:

$$\cos(\theta)_1 = \frac{x \cdot y}{||x|| \, ||y||}$$

since $\vec{u}$ and $\vec{v}$ are the 2 vectors similar to eachother

$$\therefore \cos(\theta u, v) = \frac{\vec{u} \cdot \vec{v}}{||\vec{u}|| \, ||\vec{v}||}$$

$$||\vec{u}|| = \sqrt{1^2 + 0 + 1^2 + 1^2} = \sqrt{3}$$

$$||\vec{v}|| = \sqrt{2^2 + 1^2 + 1^2 + 0} = \sqrt{6}$$

$$||\vec{w}|| = \sqrt{1^2 + 1^2 + 0 + 3^2} = \sqrt{11}$$

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$$\vec{u}\cdot\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

$$= (1)(0) + (0)(2) + (1)(1) + (1)(1)$$

$$\vec{u}\cdot\vec{v} = 0 + 0 + 1 + 1 = 2$$

$$\therefore \boxed{\vec{u}\cdot\vec{v} = 2}$$

$$\cos(\theta_{u,v}) = \frac{2}{\sqrt{3}\cdot\sqrt{6}}$$

$$\boxed{\therefore \sqrt{18} = 4.2426}$$

$$= \frac{2}{\sqrt{18}} \Rightarrow \sqrt{}$$

$$\cos(\theta_{u,v}) = \frac{2}{4.2426}$$

$$\boxed{\cos(\theta_{u,v}) = 0.4714} \Rightarrow 61.87^\circ$$

$$\cos(\theta_{u,w}) = \frac{\vec{u}\cdot\vec{w}}{\|\vec{u}\|\,\|\vec{w}\|}$$

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$$\vec{u} \cdot \vec{w} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 3 \end{bmatrix}$$

$$= (1)(1) + (0)(1) + (1)(0) + (1)(3)$$

$$\vec{u} \cdot \vec{w} = 1 + 0 + 0 + 3$$

$$\boxed{\vec{u} \cdot \vec{w} = 4}$$

$$\cos(\theta_{u,w}) = \frac{4}{\sqrt{3} \cdot \sqrt{11}} = \frac{4}{\sqrt{33}}$$

$$\cos(\theta_{u,w}) = \frac{4}{\sqrt{33}} \qquad \boxed{\because \sqrt{33} = 5.7445}$$

$$\cos(\theta_{u,w}) = \frac{4}{5.7445}$$

$$\boxed{\cos(\theta_{u,w}) = 0.6963} \Rightarrow 45.86^\circ$$

$$\cos(\theta_{v,w}) = \frac{\vec{v} \cdot \vec{w}}{||\vec{v}|| \, ||\vec{w}||}$$

$$\vec{v} \cdot \vec{w} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 3 \end{bmatrix} = (0)(1) + (2)(1) + (1)(0) + (1)(3)$$

$$\vec{v} \cdot \vec{w} = 0 + 2 + 0 + 3$$

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$$\boxed{\vec{v} \cdot \vec{w} = 5}$$

$$\cos(\theta v, w) = \frac{5}{\sqrt{6}\sqrt{11}} \qquad \boxed{\because \sqrt{66} = 8.1240}$$

$$\cos(\theta v, w) = \frac{5}{\sqrt{66}}$$

$$\cos(\theta v, w) = \frac{5}{8.1240} = 0.6154 \qquad \because \cos^{-1}(0.61\ldots) = 51^{\circ}3\ldots$$

$\vec{u}$ & $\vec{w}$ are closer to & smaller angle. They are similar to eachother

$$\boxed{\cos(\theta v, w) = 0.6154} \Rightarrow 52.03^{\circ}$$

(iii) Explain the reason behind the difference in results between (i) & (ii) if you observe any.

As per l_1 and l_2 norms, $\vec{u}$ and $\vec{v}$ are actually similar to eachother.

As per cosine similarity, $\vec{u}$ and $\vec{v}$ vectors are similar to eachother.

In l_2 norm, Magnitude of the vectors is considered in Euclidean space. In l_1 norm, the absolute differences between the components is considered.

But cosine similarity on the other hand, the direction of vectors is considered.

In layman terms,

(i) l_2 norm is Euclidean Distance between the vectors
↳ sensible to magnitude + directions.

(ii) l_1 norm is Manhattan Distance between the vectors.
↳ less sensible to outliers

(iii) Cosine Similarity gives the angle between the two vectors.

- Ranges from -1 to 1.
- Only the direction of vectors is considered.

(ii) Can the differences be resolved? Give the details of your suggestion, if you have any, and explain the outcome if suggestions are applied.

→ To resolve the differences, we can consider standardizing the vectors by dividing each component by its magnitude.

→ Magnitude of each vector will be 1. and the differences between norms + cosine similarity will be minimised.

② Question 2:

You are tasked with uncovered information about an incomplete matrix, some of whose entries are unknown and denoted as, a, b, c, and d:

$$A = \begin{bmatrix} -1 & 0 & a \\ b & 4 & c \\ d & 0 & 0 \end{bmatrix}$$

(i) Find the rank of A based on the values a, b, c and d.

(ii) Using the results you obtained in (i), how many distinct eigenvalue does A have considering a, b, c and d?

(i) We can try to use gaussian elimination to transform it into row echelon form.

Simplifying it as,

① $R_2 \rightarrow R_2 + bR_1$

Add b times the 1st Row to the 2nd Row

$$\begin{bmatrix} -1 & 0 & a \\ 0 & 4 & ab+c \\ d & 0 & 0 \end{bmatrix}$$

(2) $R_3 \rightarrow R_3 + dR_1$

Adding d times 1^{st} Row to 3^{rd} Row.

$$\begin{bmatrix} -1 & 0 & a \\ 0 & 4 & ab+c \\ 0 & 0 & ad \end{bmatrix}$$

$\Rightarrow$ 3^{rd} Row is 0 only if ad 0.
So if $a \neq 0$, $d \neq 0$ then 3^{rd} Row is actually Non-Zero. This actually contributes to Rank.

$\Rightarrow$ 2^{nd} row is 0 only if the value of $ab+c = 0$.
If $a \neq 0$ Then 2^{nd} row is actually non-zero and which inturn contributes to Rank.

$\Rightarrow$ 1^{st} row is always Non-Zero here. Since it contains the non-zero elements.

Let us consider 2 cases:

Case 1:

Let us consider that $a \neq 0$ and $d \neq 0$.

All rows are Non-Zeroes. thus Rank is 3

Case 2:

Let us consider that $a = 0$, $d = 0$. No. of Non-Zero rows will be 2. Rank is 2 here.

(ii) The eigenvalues solving the characteristics equation is obtained by solving:

$$\det(A - \lambda I) = 0$$

I = Identity Matrix, λ = Eigen Value.

$$\det(A-\lambda I) = \det\begin{bmatrix} -1-\lambda & 0 & a \\ b & 4-\lambda & c \\ d & 0 & -\lambda \end{bmatrix}$$

To find the determinant cofactor: Row 2

$$b\begin{vmatrix} 0 & a \\ 0 & -\lambda \end{vmatrix} - (4-\lambda)\begin{vmatrix} -1-\lambda & a \\ d & -\lambda \end{vmatrix} + c\begin{vmatrix} -1-\lambda & 0 \\ d & 0 \end{vmatrix}$$

Here $\begin{vmatrix} 0 & a \\ 0 & -\lambda \end{vmatrix} = 0$.

$$\begin{bmatrix} -1-\lambda & a \\ d & -\lambda \end{bmatrix} = -\lambda(-1-\lambda) - ad$$

$$= \lambda + \lambda^2 - ad$$

$$= \lambda^2 + \lambda - ad \rightarrow (1)$$

$$\begin{vmatrix} -1-\lambda & 0 \\ d & 0 \end{vmatrix} = 0$$

$$\det(A - \lambda I) = (4-\lambda)(\lambda^2 + \lambda - ad)$$

$\Rightarrow$ Equating it 0

$$(4-\lambda)(\lambda^2+\lambda - ad) = 0 \rightarrow (2)$$

Let us consider our case 2 where $a=0$, $d=0$ $\rightarrow (3)$

Substitute (3) in (2)

$$(4-\lambda)(\lambda^2+\lambda) = 0$$

$$4\lambda + 4\lambda^2 - \lambda^2 - \lambda^3 = 0$$

$$-\lambda^3 + 3\lambda^2 + 4\lambda = 0$$

$$-\lambda(\lambda^2 - 3\lambda^2 - 4) = 0$$

$$\lambda(\lambda(\lambda-4) + (\lambda-4)) = 0$$

$$\boxed{\lambda(\lambda+1)(\lambda-4) = 0} \rightarrow (4)$$

Here From Eq. (4) We get:

$\lambda = 0$, $\lambda = -1$; $\lambda = 4$.

There are 3 distinct eigen values.

Let us consider our Case 1: where $a \neq 0$, $d \neq 0$ $\rightarrow (5)$

Using this Eq. (5) on Eq. (2)

$$(4-\lambda)(\lambda^2 + \lambda - ad) = 0$$

$$4\lambda^2 + 4\lambda - 4ad - \lambda^3 - \lambda^2 + \lambda ad = 0$$

$$-\lambda^3 + 3\lambda^2 + \lambda(4+ad) - 4ad = 0$$

$$\boxed{\lambda^3 - 3\lambda^2 - \lambda(4+ad) + 4ad = 0} \rightarrow (6)$$

In Equation (6), $\lambda = 1$ (Take):

$$1^3 - 3 - 1(4+ad) + 4ad = 0$$

$$1 - 3 - 4 - ad + 4ad = 0$$

$$-6 + 3ad = 0$$

$$\Rightarrow \boxed{ad = 2}$$

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In order to factor out $(\lambda-1)$, Synthetic division is used.

Put $ad=2$ in Eq. 6

$$\lambda^3 - 3\lambda^2 - (4+2)\lambda + 4(2) = (\lambda-1)\,Q(\lambda)$$

$$\boxed{Q(\lambda) = \lambda^2 - 2\lambda - 8} \rightarrow 7$$

Solving the Equation:

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

From Eq. 7:

$a=1$, $b=-2$, $c=-8$

$$\lambda = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-8)}}{2(1)}$$

$$\lambda = \frac{2 \pm \sqrt{36}}{2}, \quad \lambda = \frac{2 \pm 6}{2}$$

$$\lambda = \frac{8}{2} = 4; \quad \lambda = \frac{-4}{2} = -2$$

When $\lambda = 1$, $\boxed{ad=2}$

$\lambda = 4 \Rightarrow$ Got from solving Eq. 7.

$\lambda = -2 \Rightarrow$ Got from solving Eq. 7

$\therefore$ There are 3 distinct eigenvalues.

(3) Question (3): Consider the following matrix:

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

(i) Find the full solution for $Ax = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $x \in R^3$

$$Ax = b$$

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(1) $R_1 \to R_1(-1)$ [-1 times R_1] $\begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

(2) $R_2 \to R_2(-1)$ [1 times R_2]

$$(-1)R_2 \Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

(3) $R_1 \to R_1 + R_2$; Adding $R_1 + R_2$ and that becomes R_1

$R_1 \to R_1 + R_2 \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$

$$Ax = b$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$(-1)(x_1) + (1)(x_2) + x_3(0) = 1$

$\boxed{-x_1 + x_2 = 1} \to (1)$

$0(x_1) + (-1)(x_2) + (1)(x_3) = 1$

$\boxed{-x_2 + x_3 = 1} \to (2)$

Here $\boxed{x_2 = 1 - x_3} \to (3)$

Here $x_3 = 1$, as it is a free vector.

then: $x_2 = 1 - x_3 = 1 - 1 = 0$ $\to (4)$

Put (4) in Eq: (1)

$$-x_1 + 0 = 1$$

$$\boxed{x_1 = -1}$$

Let $x_3 = t$

$x_2 = t - 1$

$x_1 = x_2 - 1$

$x_1 = (t-1) - 1 = (t-2)$

Hence : $x = \begin{bmatrix} t-2 \\ t-1 \\ t \end{bmatrix}$

(ii) Find the Rank (A):

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

① $R_1 \to (-1) R_1$ [-1 times R_1 column]

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

② $R_2 \to R_2 + R_1$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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Rank (A) = 2

Which means there are 2 Non-zero rows.

→ Based on the column space,

∴ C(A) is spanned by pivot columns

$$C(A) = \text{span}\left\{\begin{bmatrix}-1\\0\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\}$$

→ Row Space Basis $C(A^T)$

$$C(A^T) = \text{span}\left\{\begin{bmatrix}1 & -1 & 0\end{bmatrix}, \begin{bmatrix}0, & 0, & 1\end{bmatrix}\right\}$$

→ Basis for Null space $N(A)$

$$A\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 0$$

The null space of A is the set of all solutions to the homogeneous equation $AX = 0$ represented above

$$A\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 0 \Rightarrow \begin{cases}-x_1 + x_2 = 0 \rightarrow ① \\ -x_2 + x_3 = 0 \rightarrow ②\end{cases}$$

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From Eq. (1): $x_1 = x_2$ So $\boxed{x_1 = x_2 = x_3}$

From Eq. (2) $x_3 = x_2$

Now Null Space Becomes,

$$N(A) = \text{Span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$$

→ To find the left Null space $N(A^T)$

Here, $A^T y = 0$

$$A^T = \begin{bmatrix}-1 & 0\\1 & -1\\0 & 1\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix} = 0 \Rightarrow \begin{cases}-y_1 = 0\\ y_1 - y_2 = 0\\ y_2 = 0\end{cases}$$

$$\boxed{y_1 = y_2 = 0}$$

$$N(A^T) = \text{span}\left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$$

This is an empty set and only contains 0 values.

(iii) Construct $B = AA^T$

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \quad A^T = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$B = AA^T = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$B = AA^T = \begin{bmatrix} (-1)(-1) + (1)(1) + 0 & (-1)(0) + (1)(-1) + 0 \\ 0 + (-1)(1) + 0 & 0 + (-1)(-1) + 1 \end{bmatrix}$$

$$B = AA^T = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Finding eigen values of B by solving,

$$\det(B - \lambda I) = 0$$

$$B - \lambda I = \begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix} \rightarrow 5$$

$$\det(B-\lambda I) = (2-\lambda)^2 - (-1)^2$$
$$= 4 + \lambda^2 - 4\lambda - 1$$
$$= \lambda^2 - 4\lambda + 3$$
$$= (\lambda-3)(\lambda-1)$$

Now, $(\lambda-3)(\lambda-1)=0$

$\lambda_1 = 3;\ \lambda_2 = 1$

① Case 1: $\lambda_1 = 3$, substitute in Eq. ⑤

$$(B-3\lambda) = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$-x - y = 0 \Rightarrow \boxed{x = -y}$$

When $\lambda_1 = 3$, The eigenvalues is (eigen vector)

$$u_1 = \begin{bmatrix} -1 \\ +1 \end{bmatrix}$$

② Case 2: $\lambda_2 = 1$, substitute in Eq. ⑤

$$(B - I) = \begin{bmatrix} -1 & -1 \\ +1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$x - y = 0 \Rightarrow \boxed{x = y}$$

For $\lambda_2 = 1$, the eigen vectors is :

$$u_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Constructing D

$$D = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{bmatrix}$$

$$\sigma_i = \sqrt{\lambda_i}$$

$$\sigma_1 = \sqrt{3} \; ; \; \sigma_2 = \sqrt{1} = 1$$

$$D = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(iv) If the eigenvectors of B are orthonormal, orthonormal then

Making U matrix using orthonormal vector

$$u_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Normalize u_1 & u_2:

$$||u_1|| = \sqrt{(1)^2 + (-1)^2} = \sqrt{2}$$

$$||u_2|| = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$$

$$\boxed{u_1 = u_1 / ||u_1||} \rightarrow (6)$$

$$u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\boxed{u_2 = u_2 / ||u_2||} \rightarrow (7)$$

$$u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$U = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$$

$$\therefore \underline{U} = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \rightarrow (8)$$

Next step is to check the orthogonality of U

$$\Rightarrow U^T U = I$$

$$U^T = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\therefore U^T U = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$U^T U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \rightarrow (9)$$

$$\left(\frac{-1}{\sqrt{2}} \times \frac{-1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow \frac{1}{2} + \frac{1}{2}$$

$$\Rightarrow (1)$$

So $\boxed{U^T U = I}$ → This is proved Eq. (9)

Therefore U is orthogonal.

(v) Find eigen values & eigenvectors of matrix, $G = A^T A$ and orthonormalize them.

$G = A^T A$

$$A^T = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1+0 & -1+0 & 0+0 \\ -1+0 & 1+1 & 0-1 \\ 0+0 & 0-1 & 0+1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

To find the eigenvalue of G,

$$\det(G - \lambda I) = 0$$

$$(G - \lambda I) = \begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-1 \end{bmatrix}$$

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$$\det(G-\lambda I) = (1-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ 0 & 1-\lambda \end{vmatrix} + 0$$

$$\det(G-\lambda I) = (1-\lambda)\big((2-\lambda)(1-\lambda) - (-1)(-1)\big) - (-1)(-1)(1-\lambda)$$

$$\det(G-\lambda I) = (1-\lambda)(2-\lambda-2\lambda+\lambda^2-1) - (1-\lambda)$$

$$\det(G-\lambda I) = (1-\lambda)(\lambda^2-3\lambda+1) - (1-\lambda)$$

$$= -\lambda^3+4\lambda^2-3\lambda \rightarrow (10)$$

Eq. (10) to Zero.

$$\Rightarrow \lambda^3 - 4\lambda^2 + 3\lambda = 0$$

$$\lambda(\lambda-3)(\lambda-1) = 0$$

$$\boxed{\lambda_1 = 0, \quad \lambda_2 = 3, \quad \lambda_3 = 1}$$

(1) Eigenvectors of G for $\boxed{\lambda_2 = 3}$

$$(G-3I) = \begin{bmatrix} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \rightarrow (11)$$

From Eq. (11) We get:

$$\left.\begin{array}{ll} -2x - y = 0 & \Rightarrow x = y/2 \\ -x - y - z = 0 & \\ -y - 2z = 0 & \Rightarrow z = y/2 \end{array}\right\} \rightarrow \text{So } \boxed{x = z}$$

For $\lambda_2 = 3$, the eigenvectore: $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

(2) Eigenvector of G for $\lambda_3 = 1$

$$(G-I) = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \rightarrow (12)$$

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From Eq. (12), We get:

$$\left.\begin{array}{l} -y = 0 \\ -x + y - z = 0 \\ -y = 0 \end{array}\right] \rightarrow \therefore \boxed{y = 0}$$

$$-x - z = 0$$

$$\therefore \boxed{x = -z}$$

So the eigen vector for $\lambda_3 = 1$, is $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

Eigenvectors for $\lambda = 0$

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \rightarrow (13)$$

From Eq. (13), we get:

$$\Rightarrow \left[\begin{array}{l} +x - y = 0 \\ -x + 2y - z = 0 \\ -y + z = 0 \end{array}\right.$$

We get: $x = y$, $y = -z$, $x = -z$ $\Rightarrow x = y = z$

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The eigenvector of $\lambda_1 = 0$, is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$V_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad V_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad V_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

<u>Next Normalizing V_1, V_2, V_3:</u>

For $V_2 \Rightarrow V_2 / \|V_2\|$

$$\|V_2\| = \sqrt{(1)^2 + (-2)^2 + (1)^2} = \sqrt{6}$$

$$V_2 = \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \rightarrow 14$$

$V_3 \Rightarrow V_3 / \|V_3\|$

$$\|V_3\| = \sqrt{(1)^2 + 0 + (-1)^2} = \sqrt{2}$$

$$V_3 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \rightarrow 15$$

For $V_3 \Rightarrow V_3 / \|V_1\|$

$$\|V_1\| = \sqrt{(1)^2+(1)^2+(1)^2} = \sqrt{3}$$

$$V_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \rightarrow (16)$$

(vi) Orthogonalise $\{V_1, V_2, V_3\}$ and construct V

$$V = \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ -2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$

Calculating the orthogonality of V:

$$V^T V = I$$

$$V^T = \begin{bmatrix} 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

$$V^T V = \begin{bmatrix} 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ -2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$

R_1:

$$V^T V = \left(\frac{1}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{6}}\right) + \left(\frac{-2}{\sqrt{6}}\right)\left(\frac{-2}{\sqrt{6}}\right) + \left(\frac{1}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{6}}\right) = 1$$

$$\left(\frac{1}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{2}}\right) + 0 + \left(\frac{1}{\sqrt{6}}\right)\left(\frac{-1}{\sqrt{2}}\right) = 0$$

$$\left(\frac{1}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{3}}\right) + \left(\frac{-2}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{3}}\right) = 0$$

R_2:

$$\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{6}}\right) + 0 + \left(\frac{-1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{6}}\right) = 0$$

$$\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + 0 + \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right) = 1$$

$$\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{3}}\right) + 0 + \left(-\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{3}}\right) = 0$$

R_3:

$$\left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{6}}\right) + \left(\frac{1}{\sqrt{3}}\right)\left(\frac{-2}{\sqrt{6}}\right) + \left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{6}}\right) = 0$$

$$\left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{2}}\right) + 0 + \left(\frac{1}{\sqrt{3}}\right)\left(-1/\sqrt{2}\right) = 0$$

$$\left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right)+\left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right)+\left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right)=1$$

$$V^T \cdot V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \rightarrow (17)$$

$V^T \cdot V = I \rightarrow$ This is proved from Eq. (17).

So, we can confirm that, V is actually an orthogonal matrix.

(vii) Find vectors $\{w_1, w_2, w_3\}$

$$w_i = \frac{1}{\sigma_i} A^T \; ; \; w_i, i = 1, 2$$

$w_3 \perp w_1$; $w_3 \perp w_2$ and $\|w_3\| = 1$

Obtained σ_1, σ_2 from (iv)

$$u_1 = \begin{bmatrix} -1/\sqrt{2} \\ +1/\sqrt{2} \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\boxed{\sigma_1 = \sqrt{3}}$$

$$A^T = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$A^T u_1 = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$A^T u_1 = \begin{bmatrix} 1/\sqrt{2} \\ -\sqrt{2} \\ +1/\sqrt{2} \end{bmatrix}$$

$$\Rightarrow \sigma_1 = \sqrt{3}$$

$$w_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1/\sqrt{2} \\ -\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ -\sqrt{2}/\sqrt{3} \\ 1/\sqrt{6} \end{bmatrix}$$

$$A^T u_2 = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & +1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$A^T u_2 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$\Rightarrow \boxed{\sigma_1 = 1}$$

$$w_2 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

To find W_3 such that $W_3 \perp W_1$; $W_3 \perp W_2$, and $||W_3|| = 1$

Lets start with any vector V not in span of W_1 and W_2 and orthogonalize it against W_1 and W_2.

$$\text{Here} \quad W_3 = V - \frac{V \cdot W_1}{W_1 \cdot W_1} W_1 - \frac{V \cdot W_2}{W_2 \cdot W_2} W_2$$

$$\text{Let} \quad V = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$V \cdot W_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} \\ -\sqrt{2}/\sqrt{3} \\ 1/\sqrt{6} \end{bmatrix} = -\left(-\frac{1}{\sqrt{6}} + \frac{\sqrt{2}}{\sqrt{3}} - \frac{1}{\sqrt{6}}\right)$$

$$= -\left(-\frac{\cancel{2}^{\sqrt{2}}}{\cancel{\sqrt{2}}\sqrt{3}} + \frac{\sqrt{2}}{\sqrt{3}}\right)$$

$$= -\left(-\frac{\sqrt{2}}{\sqrt{3}} + \frac{\sqrt{2}}{\sqrt{3}}\right)$$

$$\boxed{V \cdot W_1 = 0}$$

$$V \cdot W_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = -1/\sqrt{2} + 0 + 1/\sqrt{2}$$

$$\boxed{V \cdot W_3 = 0}$$

This means that V is orthogonal to W_1 & W_2.

$\therefore V \cdot W_1 = 0 = V \cdot W_2$

W_3 can be normalized & given as:

$$W_3 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

Here:

$$W_1 = \begin{bmatrix} 1/\sqrt{6} \\ -\sqrt{2}/\sqrt{3} \\ 1/\sqrt{6} \end{bmatrix} \quad W_2 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \quad W_3 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$W_1 = V_1$; $W_2 = V_2$; $W_3 = V_3$

∙ Sets of $\{W_1, W_2, W_3\}$ & $\{V_1, V_2, V_3\}$ are similar to eachother except for the permutation of the order.

(viii) Compute UDV^T

$$U = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \qquad D = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$V^T = \begin{bmatrix} 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

$$UD = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -\sqrt{3}/\sqrt{2} & 1/\sqrt{2} & 0 \\ \sqrt{3}/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$$

$$UDV^T = \begin{bmatrix} -\sqrt{3}/\sqrt{2} & 1/\sqrt{2} & 0 \\ \sqrt{3}/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & -2/\sqrt{6} & \frac{1}{\sqrt{6}} \\ -1/\sqrt{2} & 0 & \frac{1}{\sqrt{2}} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

On simplifying the above:

$$UDV^T = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$\therefore \boxed{UDV^T = A}$. This is the singular value decomposition of Matrix A.

(ix) Compare the columns U and V

* Columns of U has u_1 and u_2 that form an orthonormal basis for $C(A^T)$ for the rowspace of A. This happens because

they are orthonormal eigenvectors AA^T that spans the row space of A.

* Columns of $V \to v_1\ v_2\ v_2$ forms orthogonal normal basis for C(A) [the column space of A].
This happens because they are orthonormal eigenvectors of $A^T \cdot A$, the column space of A.

→ For $N(A^T)$, Based on orthonormal basis; first 2 columns of $V \to v_1$ and v_2.

→ The last column of $v_1\ v_3$ form an orthonormal basis for N(A)., [The null space of A].

In summary, the columns of U, V are closely related to basis of A, Matrix.

Columns of U → Forms orthonormal basis for $C(A^T)$

Columns of V → Forms an orthonormal basis f[illegible] C(A), N(A) & $N(A^T)$.

The orthonormalization highlights the dimensional and orthogonality of the subspaces of Matrix