Assignment - 1

Question 1

Derangement

P(at least one letter is in correct envelope)

= 1 - P(all letters in wrong envelope)

To find : P(all letters in wrong envelope)

Let A_i be the event that ith letter is in correct envelope.

$P(A_1 \cup A_2 \cup \dots \cup A_n)$ = P(at least one letter in correct envelope)

Using inclusion exclusion principle

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - S_2 + S_3 - \dots + (-1)^{n-1} S_n$$

$$= S_1 - S_2 + S_3 - \dots + (-1)^{n-1} S_n$$

$$= \frac{(n-1)!}{n!} n - \frac{(n-2)!}{n!} {}^nC_2 + \frac{(n-3)!}{n!} {}^nC_3 + \dots + (-1)^n \frac{1}{n!}$$

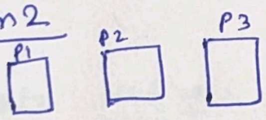
$$= \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} \dots + (-1)^n \frac{1}{n!}$$

for large n this converges to 1/e

$P_n = 1/e$

$$P\left(\bigcup_{i=1}^{n} A_i\right) = 1 - 1/e$$

Question 2



case of Monty Hall problem.

1 gift → 1000 dollars

2 empty

case 1 : when present 2 (P2) is not opened

Probability of 1000 dollars gift price in any present $= \frac{1}{3}$

case 2 : when P2 is opened

A) $P\left(\frac{\text{Host opens P2}}{\text{Money in P1}}\right) = \frac{1}{2}$

B) P(Money in P2) = 0

C) $P\left(\frac{\text{Host opens P2}}{\text{Money in P3}}\right) = 1$

$$P\left(\frac{A_3}{E}\right) = \frac{P(E/A_3)\,P(A_3)}{P(E/A_1)\,P(A_1) + P(E/A_3)\,P(A_3)}$$

$$= \frac{1 \cdot \frac{1}{3}}{\frac{1}{2} \cdot \frac{1}{3} + 1 \cdot \frac{1}{3}} = \frac{1/3}{1/3\left(\frac{1}{2}+1\right)} = \frac{2}{3}$$

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Expected winnings $= \frac{2}{3} \times 100 + \frac{0.1}{3}$

$= 666.67$ dollars.

Question 3

a) $P(B \cap C) > 0$ Given: A, B, C, D events

$P(A \cap B | C) = P(A | B \cap C)\, P(B | C)$

Using chain rule,

$$P(A \cap B \cap C) = P(A | B \cap C) \cdot P(B \cap C)$$

$$P(A \cap B | C) = \frac{P(A | B \cap C) \cdot P(B \cap C)}{P(C)}$$

$$= P(A | B \cap C) \cdot P(B | C)$$

TRUE

b) $P(A \cap B | C) = P(A|C) \cdot P(B|C)$

A, B are independent

$$P(A \cap B) = P(A) \cdot P(B)$$

$$P(A \cap B \cap C) = P(A \cap C) \cdot P(B \cap C)$$

$$P(A \cap B | C) = \frac{P(A \cap B \cap C)}{P(C)}$$

FALSE

$$= \frac{P(A \cap C)\, P(B \cap C)}{P(C)} = P(A|C)\, P(B \cap C)$$

c) $P(A | D \cap B^c) > P(A | D \cap B)$ Here D occurs

→ A is more likely when B ~~occurs~~ does not occur than when it occurs.

$P(A | D^c \cap B^c) > P(A | D^c \cap B)$ Here D does not occurs

→ A is more likely when B does not occur ~~than~~ than when it occurs

$\therefore P(A|B) < P(A|B^c)$

so, $P(A|B) > P(A|B^c)$ → incorrect FALSE

~~2)~~
4)
a) Intuition $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges $\sum_{n=1}^{\infty} \frac{1}{n^2}$ does not :)

$\Rightarrow E(X^2) = \sum_{n=1}^{\infty} \frac{1}{n}$

$\Rightarrow$ define the distribution as

$X : \Omega \rightarrow \mathbb{R}$

$$P(n) = \frac{c}{n^3}$$

$$\Rightarrow E(X^2) = \sum_{n=1}^{\infty} \frac{1}{n}$$

$E(X) = \sum_{n=1}^{\infty} \frac{1}{n^2}$ which converges

$\Rightarrow E(X)$ finite

(c) $E(e^{-X}) < 1/3$ Jensen

$E[f(X)] \geq f(E(X))$ for convex fun.

$e^{-E(X)} = \frac{1}{e}$

So. $E(e^{-X}) \geq 1/e > 1/3$

So not possible

$E(e^{-X}) < 1/3$ will not be possible for any X

(b) $E(X)$ finite - $E(X^2)$ not

Similar as (a) not taking $x < 1$ because c/x^3 will diverge

$$\int_1^{\infty} x \, p(x) = 1 \quad \left(p(x) = c/x^3 \right)$$

$$\left[-\frac{c}{2x^2} \right]_1^{\infty} = 1$$

$\Rightarrow c = 2$

$$\Rightarrow p(x) = \begin{cases} 2/x^3 & x \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

5) Maximum money obtained

$= \max(X_1, X_2, \ldots, X_n) = \text{say } m$

$$E[M] = \sum_{m=1}^{N} \left[m \cdot \left(\frac{m}{N}\right)^n - m \left(\frac{m-1}{N}\right)^n \right]$$

Basically $\left(\frac{m}{N}\right)^n$ is prob all no. $\leq m$ - the case where all nos. $< m$.

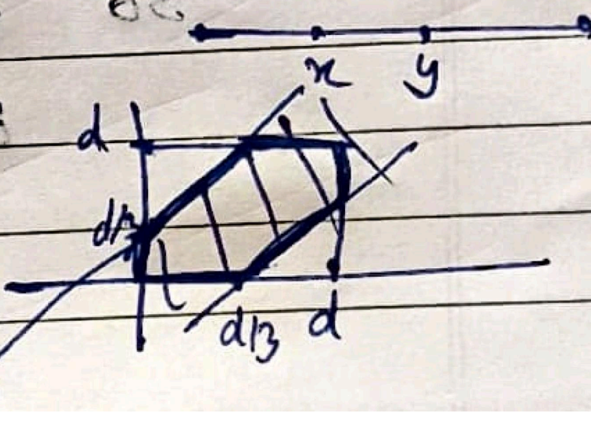
$$E[M] = \sum_{k=1}^{N} k \left(\left(\frac{k}{N}\right)^n - \left(\frac{k-1}{N}\right)^n \right)$$

6) $d^2 - \frac{4d^2}{9} \Rightarrow \frac{5d^2}{9d^2}$

Two eq^n

$y = x + d/3$

& $y = x - d/3$



Prob. $= \frac{\text{shaded area}}{d^2} \Rightarrow 5/9$

(2) $n+1$ people
not the originator Prob $= \frac{n}{n+1}$

probability that in all r times not the originator

(interdependence)

P no returning to originator $= \left(\frac{n}{n+1}\right)^r$

No repeats

$P_{\text{no repeat}}$ $\frac{n}{n+1} \cdot \frac{n-1}{n+1} \cdot \frac{n-r+1}{n+1}$

$$\prod_{k=0}^{r-1} \frac{n-k}{n+1}$$

$$P = \frac{n(n-1)\cdots(n-r+1)}{(n+1)^r}$$

$$P = \frac{n!}{(n-r)!\,(n+1)^r}$$

$r \leq n$

P_0 no in grp :-

$$P_0 = \frac{{}^{n}C_N}{{}^{n+1}C_N}$$

for r series $\left(\frac{{}^{n}C_N}{{}^{n+1}C_N}\right)^r$

No repeats

$$P = \frac{{}^{n}C_N \; {}^{n-N}C_N \; {}^{n-2N}C_N \cdots {}^{n-(r-1)N}C_N}{({}^{n+1}C_N)^r}$$

for $rN \leq n$

$$P \text{ no repeats in grp} = \frac{\prod_{k=0}^{r-1} \left({}^{n-kN}C_N\right)}{\left(({}^{n+1}C_N)\right)^r}$$

8) $P(\cap A_i^c) = \prod_{i=1}^{n} P(A_i^c)$

Since independent events

$\cancel{\prod P(A_i)}$

$$\prod_{i=1}^{n} P(A_i^c) = \prod_{i=1}^{n} (1-P(A_i)) \geq 0$$

$\cancel{e^x > x \text{ for } x>0}$

$\cancel{\prod_{i=1}^{n} (1-P(A_i)) \leq}$

$$1 - x \leq e^{-x} \quad \text{where } x = P(A_i)$$

$$1 - P(A_i) \leq e^{-P(A_i)}$$

multiply for all $1 \leq i \leq n$

$\Rightarrow$ $\cancel{P(\bigcap_{i=1}^{n} A_i)}$

$$P\left(\bigcap_{i=1}^{n} A_i^c\right) \leq e^{-\sum_{i=1}^{n} P(A_i)}$$

9) $H(x) = (F * G)(x)$

$X \sim F$ & $Y \sim G$

$$H(x) = P(X + Y \leq x) = \int_{-\infty}^{\infty} F(x-y)\,dG(y)$$

$\rightarrow$ Non decreasing

$F(x_1 - y) \leq F(x_2 - y)$

$x_1 < x_2$ then $H(x_1) = \int_{-\infty}^{\infty} F(x_1 - y)\,dG(y)$

$\Rightarrow H(x_1) \leq H(x_2)$

⇒ Limit at $+\infty$ $\quad x \to \infty$

$F(x-y) \Rightarrow F(\infty)$

Since F is distribution func.

we get

$$\lim_{x\to\infty} H(x) = \int 1 \, dG(y) = 1$$

at $x \to -\infty$

$$\lim_{x\to -\infty} H(x) = \int F(-\infty) \, dG(y) = 0$$

⇒ Right cont.

$$\lim F(x_n^+) = F(x)$$

Since F is right continuous

$x_n \downarrow x$

$\Rightarrow F(x_n - y) \downarrow F(x-y)$

$$\text{so } \lim_{n\to\infty} H(x_n) = \lim_{n\to\infty} \int F(x_n - y) \, dG(y)$$

$$= \int F(x-y) \, dG(y)$$

$$= H(x)$$

10) $$\int_{\Omega}\int_{0}^{\infty} 1_{[0, X(\omega))}(x)\, dx$$

$$= \int_{\Omega} X(\omega)\, dP(\omega)$$

$$= E[X]$$

Using Fubinis theorem we switch integral order.

$$\int_{\Omega} 1_{[0, X(\omega))}(x)\, dP(\omega)$$

$$\boxed{x < X(\omega)}$$

$$\to P(X > x)$$

$$\Rightarrow \int_{0}^{\infty} (1 - F(x))\, dx$$

$$\Rightarrow E(X) = \int_{0}^{\infty} (1 - F(x))\, dx$$

11) $X \sim N(\mu, \sigma^2)$

$$E[e^{uX}] = \int_{-\infty}^{\infty} e^{ux} f(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{ux} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(ux - \frac{(x-\mu)^2}{2\sigma^2}\right) dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-(\mu+u\sigma^2))^2}{2\sigma^2}\right) dx$$

$$E[e^{uX}] = e^{u\mu + \frac{1}{2}u^2\sigma^2}$$

$\phi(x) = e^{ux}$ is convex

$$E[e^{uX}] \geq e^{u\,E[X]} = e^{\mu u}$$

$$E[e^{uX}] = e^{u\mu + 1/2\, u^2\sigma^2} \geq e^{u\mu}$$