(SHELLY SINGHAL) → 230963

Question 1

a) $P(1 \to 1) = 0.5$ $P(2 \to 1) = 0.25$ $P(3 \to 3) = 0.25$ $P(4 \to 3) = 0.75$

$P(1 \to 2) = 0.5$ $P(2 \to 2) = 0.75$ $P(3 \to 4) = 0.75$ $P(3 \to 4) = 0.25$

Transition Matrix $Q =$

$$\begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.25 & 0.75 & 0 & 0 \\ 0 & 0 & 0.25 & 0.75 \\ 0 & 0 & 0.75 & 0.25 \end{bmatrix}$$

b) There are two classes :- $\{1,2\}$, $\{3,4\}$

All states in both classes are recurrent → chain will return to eventually, with probability 1.

∴ No state is transient.

c) Stationary distribution on class $\{1,2\}$

Let $\pi = (\pi_1, \pi_2, 0, 0)$ where $\pi_1 + \pi_2 = 1$ — ①

$\pi Q = \pi$

$[\pi_1\ \pi_2, 0, 0]\, Q = \pi$

$\pi_1 = 0.5\pi_1 + 0.25\pi_2$

$\pi_2 = 0.5\pi_1 + 0.75\pi_2$

From eq ①

$\pi_1 = 0.5\pi_1 + 0.25\pi_2$

$= 0.5\pi_1 = 0.25\pi_2$

$\pi_2 = 2\pi_1$

$\pi_1 + 2\pi_1 = 1$

$3\pi_1 = 1$ $\pi_1 = \frac{1}{3}$ $\pi_2 = \frac{2}{3}$

$$\boxed{\pi^{(1)} = \left(\frac{1}{3}, \frac{2}{3}, 0, 0\right)}$$

Stationary distribution on class $\{3,4\}$

Let $\pi = (0, 0, \pi_3, \pi_4)$ $\pi_3 + \pi_4 = 1$

$\pi Q = \pi$

$$\begin{bmatrix} 0 & 0 & \pi_3 & \pi_4 \end{bmatrix}_{1\times 4} \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.25 & 0.75 & 0 & 0 \\ 0 & 0 & 0.25 & 0.75 \\ 0 & 0 & 0.75 & 0.25 \end{bmatrix}_{4\times 4} = [0, 0, \pi_3\ \pi_4]$$

$\pi_3 = 0.25\pi_3 + 0.75\pi_4$

$\pi_4 = 0.75\pi_3 + 0.25\pi_4$

$\pi_3 = 0.25\pi_3 + 0.75\pi_4$

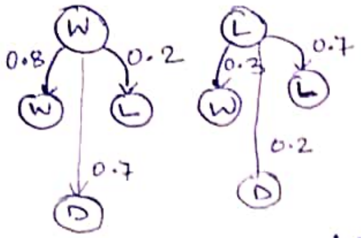
$0.\cancel{75}\pi_3 = 0.\cancel{75}\pi_4$

$\pi_3 = \pi_4$

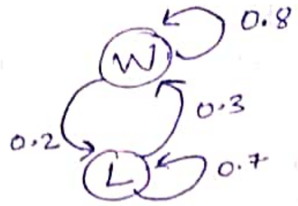
$2\pi_3 = 1$ $\pi_3 = \frac{1}{2}$

$$\boxed{\pi^{(2)} = \left(0, 0, \frac{1}{2}, \frac{1}{2}\right)}$$

Question 2 $\longrightarrow$ SHELLY Game



$\longrightarrow$



Markov chain

Two state Win & Lose

a) Transition Matrix $Q = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}$

$\pi = [\pi_W, \pi_L]$ $\qquad \pi_W + \pi_L = 1$

$\pi Q = \pi$

$[\pi_W \ \pi_L] \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} = [\pi_W \ \pi_L]$

$\pi_W = 0.8\pi_W + 0.3\pi_L$, $\pi_L = 0.2\pi_W + 0.7\pi_L$

$0.2\pi_W = 0.3\pi_L$

$\pi_W = \frac{3}{2}\pi_L$

$\frac{3}{2}\pi_L + \pi_L = 1 \qquad \frac{5\pi_L}{2} = 1 \qquad \pi_L = \frac{2}{5}$

$\pi_L = 0.4$

$\pi_W = 0.6$

In the long run, the team wins 60% of the games.

b) $\pi_W \cdot 0.7 + \pi_L \cdot 0.2 = 0.6 \cdot 0.7 + 0.4 \times 0.2$

$= 0.42 + 0.08 = 0.5$

50% of games result in a team dinner

c) $\frac{1}{0.5} = 2$ [Expected waiting time $= \frac{1}{P(\text{having dinner in any game})}$]

c) Expected no. of games until a dinner $= 2$.

PRIYANKA ARORA
230799

3) Cat chain

$$P_R = \begin{bmatrix} P(1 \to 1) & P(1 \to 2) \\ P(2 \to 1) & P(2 \to 2) \end{bmatrix}$$

$$= \begin{bmatrix} 0.2 & 0.8 \\ 0.8 & 0.2 \end{bmatrix}$$

$$\pi_R P_R = \pi_R$$

$$0.2\pi_1 + 0.8\pi_2 = \pi_1$$

$$\Rightarrow \pi_1 = 0.5$$

distribution of cat chain $(0.5, 0.5)$

Mouse chain

$$P_M = \begin{bmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{bmatrix}$$

$$\mu_1(0.7) + \mu_2(0.6) = \mu_1$$

$$\mu_1 = 2\mu_2 \Rightarrow \mu_1 = 0.66$$

$$0.3\mu_1 + 0.4\mu_2 = \mu_2 \qquad \mu_2 = 0.33$$

Stationary distribution of mouse $\left(\frac{2}{3}, \frac{1}{3}\right)$

Joint states are

→ (C1, M1) → (C1, M2)
→ (C2, M1) → (C2, M2)

Yes Z_n is Markov chain because cat & mouse move independently & they are both Markov chains
⇒ So Joint process is also Markov chain on product space
(Next state depends only on current state)

Q5)

drift = (0.1) × (0.01) + (0.85 × 0)

= 0.05 = 0.005

tendency to ↑

irreducible & countable state Markov chain hence not all states are recurrent
It has a +ve drift so once it increases, it may never return.

classmate

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state space here is infinite
(prices can keep rising)
upward drift means transient.

No limiting distribution

Ans No, stationary distribution does not exist.

(c) Basically simulation for 2160 steps
(5 seconds per step in 3hrs)
If it reaches 130 for X no. of trials out of Y no. of total trials we can get

$$P(\text{payoff of ₹}5) = \frac{X}{Y}$$

Question 4 → SHELL

Markov chain where each square is a state.

Type	Number of such states (squares)	Legal Moves	~~Prob per square~~
Corner	4	3	
Edge (not a corner)	$6 \times 4 = 24$	5	
Any other square	$64 - 28 = 36$	8	

* In a Markov chain, if all transitions symmetric & equally likely

$\pi(s) \propto \deg(s)$ ↓ no. of legal moves from s

$$\pi(s) = \frac{d(s)}{\Sigma d(s)}$$

$$\Sigma d(s) = 4 \times 3 + 24 \times 5 + 36 \times 8$$
$$= 12 + 120 + 288$$
$$= 420$$

Square type	Total Contribution	Legal moves	Probability per square
Corner	12	3	$3/420 = 1/140$
Edge (Ex. corner)	120	5	$5/420 = 1/84$
Any other sq.	288	8	$8/420 = 2/105$
	420		

Question 6 a) → SHELLY

Current permutation $p \in S_{26}$
Next permutation $q \in S_{26}$

two distinct position elements swap for $p \to q$

Number of swaps possible $^{26}C_2 = \frac{\overset{13}{\cancel{26}}.25}{\cancel{2}} = 325$

$P(p \to q) = \frac{1}{325}$ otherwise 0.
[one swap]

$\therefore P(p \to q) = \begin{cases} \frac{1}{325} & \text{if swapping one time (two entries)} \\ 0 & \text{otherwise} \end{cases}$ ↳ we need in one step only.

Stationary distribution: $\pi(g)$

state space has $\boxed{26!}$ elements.

Symmetric chain → probability of going from $p \to q$ is same as $q \to p$ (only one swap difference)

∴ Uniform distribution is stationary

Every permutation is reachable

no permutation is more likely than any other in long time

$$\boxed{\pi(g) = \frac{1}{26!}}$$

Q6 (b)

$$q(g,h) = P(g \to h)$$

$P_{rop}(g \to h) = \frac{1}{325}$ if h differs from g by a swap

Acceptence = $\min\left(1, \frac{s(h)}{s(g)}\right)$

So final prob.

$$q(g,h) = \frac{1}{325} \min\left(1, \frac{s(h)}{s(g)}\right)$$

$$q(h,g) = \frac{1}{325} \min\left(1, \frac{s(g)}{s(h)}\right)$$

assume

$s(h) < s(g)$

$$q(g,h) = \frac{1}{325} \frac{s(h)}{s(g)}$$

$$q(h,g) = \frac{1}{325}$$

$$s(g)\, q(g,h) = s(h) \cdot q(h,g)$$

Hence chain is reversible wrt distribution

$\pi(g) \propto s(g)$

& this is stationary distribution