GRAPHS

Definition of Graph : A graph G = (V, E) consists of V, a nonempty set of vertices (or nodes) and E, a set of edges. Each edge has either one or two vertices associated with it, called its endpoints. An edge is said to connect its endpoint.

Finite Graph : A graph with a finite vertex set is called a finite graph.

Infinite Graph : A graph with infinite vertex is called infinite Graph.

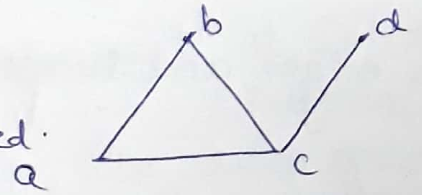
Directed and Undirected Graph :

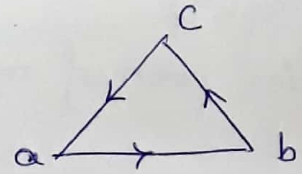
Directed Graph : A directed graph G consists of a set V of vertices and a set E of edges such that e∈E is associated with an ordered pair of vertices.

Each edge of a graph has direction.

Undirected Graph: It consists of set of vertices V and a set E of edges such that edge e∈E is associated with an unordered pair of vertices.

Undirected.





Directed

Utilities Problem

— travelling Salesman
— Round Robin
— Networking Problem

} Read from Book.

Types of Graph

→ Simple Graph

A graph which has neither loops nor multiple edges i.e where each edge connects two distinct vertices and no two edges connect same pair of vertices is called Simple Graph.

multigraph A graph which contain pair of Node by more than one edges, such edges called multiple or paralled edges.

psuedograph A graph in which loops and multiple edges are allowed, is called psuedograph.

GRAPH TERMINOLOGY

Degree of vertex : The degree of vertex in an Undirected graph is the number of edges incident withit except that a loop at a vertex contributes twice to the degree of that vertex. The degree of a vertex is denoted by deg( ).

* A vertex of degree zero, is called isolated node.
* A vertex of degree one is called pendant.

HANDSHAKING THEOREM imp

Let $G = (V, E)$ be an undirected graph with e edges. Then

$$2e = \sum_{u \in V} deg(u)$$

Note: This applies even if multiple edges and loops are present.

Proof: Since the degree of a vertex is the number of edges incident with that vertex, the sum of the degree counts the total number of times an edge is incident with a vertex. Since every edges is incident with exactly two vertices, each edge gets counted twice, once at each end. Thus the sum of the degree is equal twice the number of edges.

$G = (V, E)$, undirected Graph, e edges, Then

$$2e = \sum deg(u)$$
$$u \in V$$

mean each edge incident on two vertex exactly. So, some of ~~edge~~ deg count = total no. of edges.

Sum of deg = $2e$ (twice no. of edges).

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* An Undirected graph has an even number of vertices of odd degree.

Proof: Let $G=(V,E)$ a non directed graph. Let U denote the set of even degree vertices in G and W denote the set of odd degree vertices.

Then $\sum_{v_i \in V} deg_G(v_i) = \sum_{v_i \in U} deg_G(v_i) + \sum_{v_i \in W} deg_G(v_i)$

$\Rightarrow \quad 2e - \sum_{v_i \in U} deg_G(v_i) = \sum_{v_i \in W} deg_G(v_i)$ ---- (1)

Now $\sum_{v_i \in U} deg_G(v_i)$ is even as the sum of degrees of even degree vertices is always even.

Therefore, from (1)

$\sum_{v_i \in W} deg_G(v_i)$ is even

$\therefore$ Since for each $v_i \in W$, $deg_G(v_i)$ is odd, the number of odd vertices in G must be even.

<u>In degree</u> and <u>Out degree</u>

<u>Outdegree</u> : It is the number of edges beginning at v and. it is denoted by $indeg_G(v)$.

<u>Indegree</u> : It is the number of edges ending at v. $outdeg_G(v)$.

The sum of the indegree and outdegree of a vertex is called the total degree of the vertex.

* Show that the maximum number of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$.

Proof: By handshaking theorem.

$$\sum_{i=1}^{n} d(v_i) = 2e$$

where e is the number of edges with n vertices in the graph G.

$\Rightarrow \quad d(v_1) + d(v_2) + \ldots d(v_n) = 2e$. ———— (1)

Since the maximum deg of each vertex in the graph G can be $(n-1)$. Therefore, equation (1) reduces to.

$(n-1)+(n-1)+\ldots$ to n terms $= 2e$.

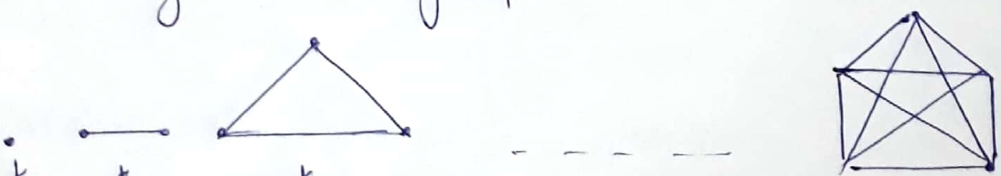
$n(n-1) = 2e$

$e = \frac{n(n-1)}{2}$.

Hence maximum number of edges in any simple graph with n vertices is $\frac{n(n-1)}{2}$.

Complete Graph: The Complete graph on n vertices, denoted by K_n, is the simple graph that Contains exactly one edge betn each pair of distinct vertices.

It should be noted that K_n has exactly $\frac{n(n-1)}{2}$ edges. The graph K_n for $n = 1, 2, 3 \ldots$



K_1 K_2 K_3 K_5

Null graph A graph which Contain isolated Node is called a null graph. Null graph is denoted by n vertices. by N.

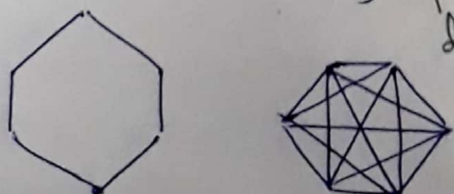
ex. N_4.

Regular Graph A graph in which all vertices are of Equal degree is Called regular graph.

If the degree of Each vertex is r, the the graph is called a regular graph of degree r.

Note, if G has 'n' vertices and regular of deg 'r' then G has $\left(\frac{1}{2}\right) \cdot r \cdot n$ edges.

r — deg, n — vertices



Size = no. of Edges in Graph

* What is the Size of an r-regular graph (p, q).

<u>Proof:</u> Since G is a 'r'-regular graph, by the definition of regularity of G, we have, $deg_G(v_i) = r$, for all $v_i \in V(G)$.

By handshaking theorem, $2q = \sum deg_G(v_i)$

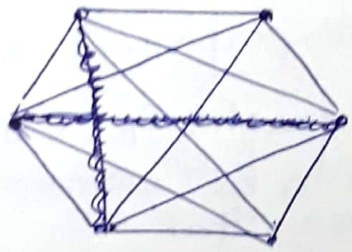
$$2q = \sum r = p \times r$$

$$\boxed{q = \frac{p \times r}{2}}$$

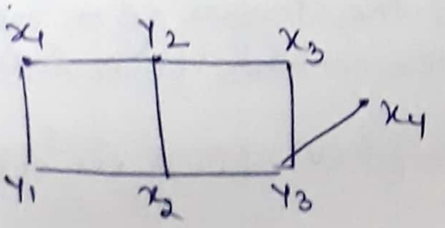
* Does there exist a 4-regular graph on 6 vertices? If so Construct

$$q = \frac{p \times r}{2} = \frac{6 \times 4}{2} = 12.$$

Yes it is possible



<u>Bipartite Graph</u> A graph $G = (V, E)$ is bipartite if the vertex set V can be partitioned into two subsets V_1 and V_2 such that every edge in E connects a vertex in V_1 and a vertex V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2). (V_1, V_2) is called a bipartition of G: obviously, a bipartite graph can have no loop.



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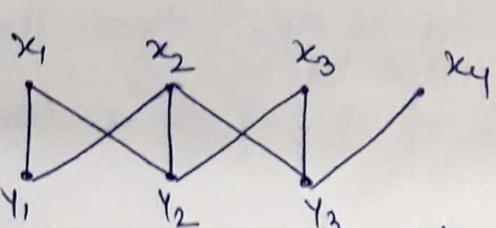
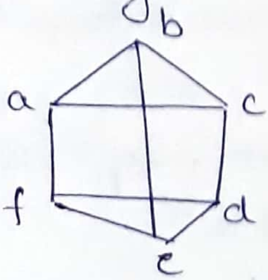
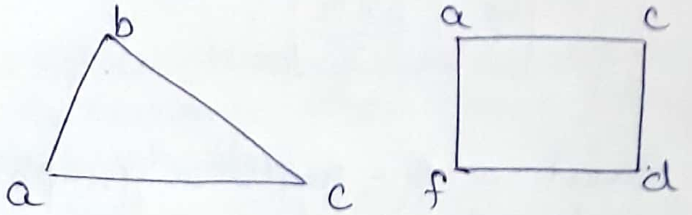


Complete Bipartite Graph: whose vertex set is partitioned into sets V_1 with m vertices and

Subgraph A subset of a graph is known as subgraph. A Subgraph is obtained by removing some vertex and its incident edges.

Ex.

G = 

Sub graph.  etc.

for a given graph G, there can be many subgraphs. Let $|V| = m$ and $|E| = n$. The total non-empty subsets of V is $2^m - 1$. and total no. of subsets of E is 2^n.

The number of subgraph is equal to $(2^m - 1) \times 2^n$.

Isomorphic Graph : Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be isomorphic if there exists a functions $f : V_1 \to V_2$ such that.

(i) f is one to one onto i.e, f is bijective.

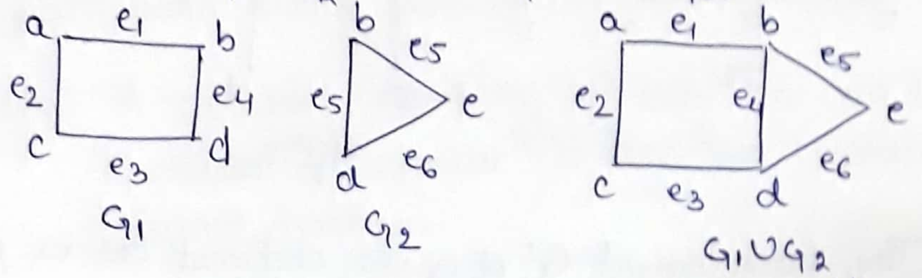
(ii) $\{a, b\}$ is an edge in E_1, if and only if $\{f(a), f(b)\}$ is an edge in E_2, for any two elements $a, b \in V_1$

Isomorphic Graph have properties:-

1. Same no. of vertices, i.e, $|V_1| = |V_2|$
2. Same number of edges, i.e $|E_1| = |E_2|$
3. Same degree sequences i.e, if the degree of a vertex v_i in G_1 is m, then the degree of the vertex $f(v_i)$ in G_2 must also be m.
4. If $\{v, v\}$ is a loop in G_1 then $\{f(v), f(v)\}$ is also a loop in G_2.

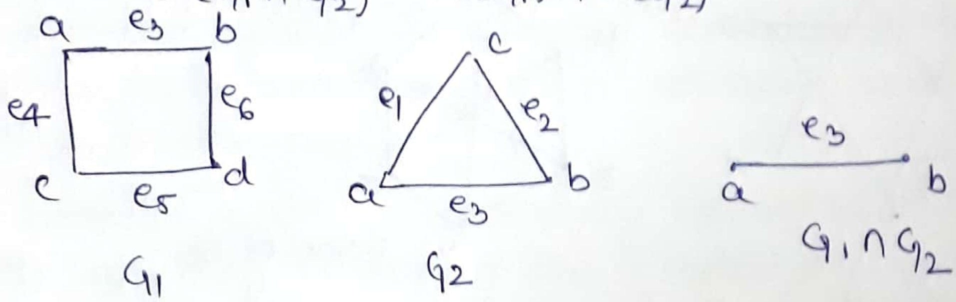
Operations of Graphs

* Union : $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$
  and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$

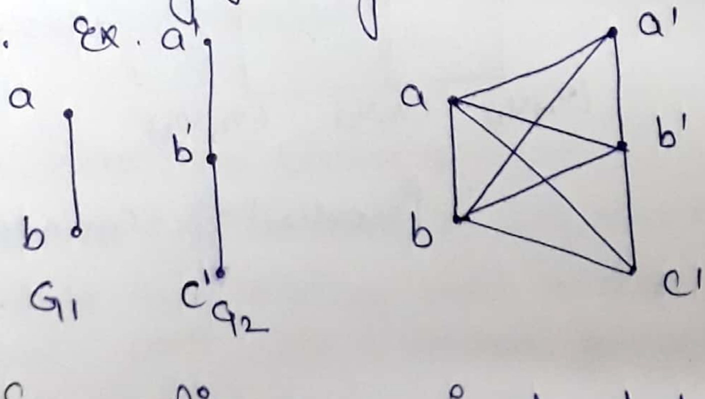


* Intersection: Given two G_1 and G_2 with atleast one vertex in common then their intersection will be a graph such that

$V(G_1 \cap G_2) = V(G_1) \cap V(G_2)$
$E(G_1 \cap G_2) = E(G_1) \cap E(G_2)$



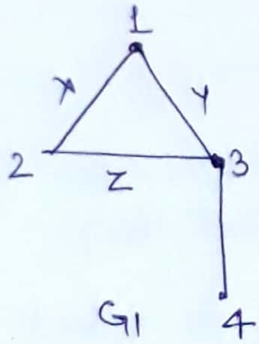
Sum of two Graphs If the graphs G_1 and G_2 such that $V(G_1) \cap V(G_2) = \phi$; then the sum of $G_1 + G_2$ is defined as the graph whose vertex set $V(G_1) + V(G_2)$ and the edge set is consisting of those edges, which are obtained by joining each vertex of G_1 to each vertex of G_2. Ex.



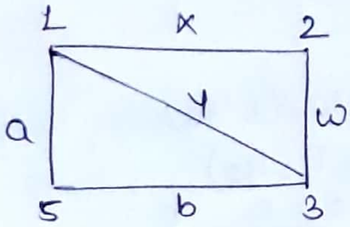
Ring Sum Ring sum is denoted by $G \oplus G_2$

(1) $V(G) = V(G_1) \cup V(G_2)$

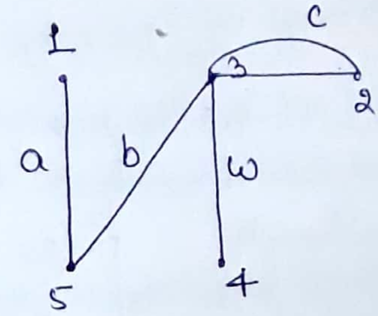
(2) $E(G) = E(G_1) \cup E(G_2) - E(G_1) \cap E(G_2)$



G_1



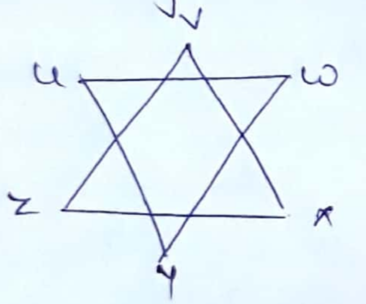
G_2

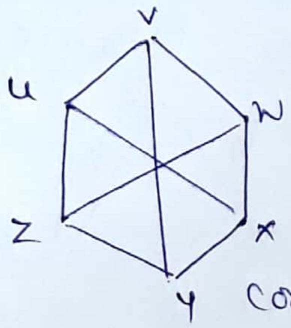


$G_1 \oplus G_2$

<u>Complement</u> The Complement G' of G is defined as a simple graph with the same vertex set as G and where two vertices u and v are adjacent only when they are not adjacent in G.

Ex –



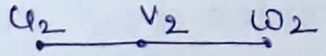


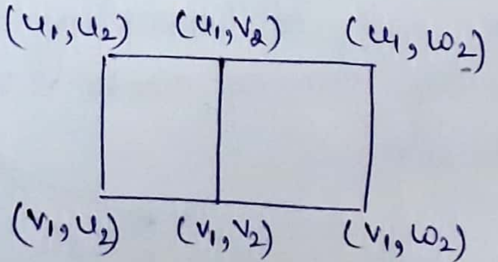
comp of G.

<u>Product of Graph.</u> It is $G_1 \times G_2$. Such that $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $V = V_1 \times V_2$. Then u and v are adjacent in $G_1 \times G_2$ whenever $[u_1 = v_1$ and u_2 and u_2 adj $v_2]$ or $[u_2 = v_1$ and u_1 adj $v_1]$

Ex.

G_1





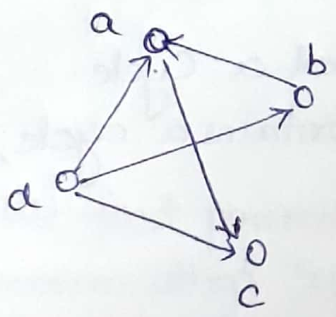
Product of G_1 and $G_2 = G_1 \times G_2$

Connectivity

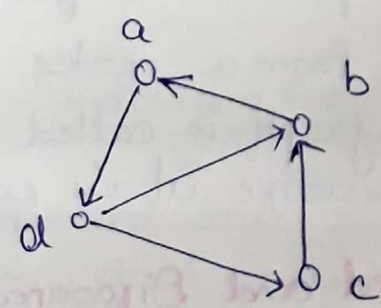
— An Undirected graph is called Connected if there is a path betn every pair of distinct Vertices in the graph.

Note: A graph Consisting of only one vertex is always Connected, because it does not Contain any pair of distinct vertices.

* A graph that is not connected is the union of two or more Connected Sub-graphs, each pair of which has no vertex in Common. These disjoint connected subgraphs are called the Connected Components of the graph.
* A directed graph is strongly connected if there is a path from a to b and from b to a whenever a and b are vertices in the graph.
* A directed graph is weakely connected if there is a path betn any two vertices in the underlying undirected Graph.



Weekely Connected



strongly Connected

Paths, Walk and Circuits:

→ Walk is finite alternating sequence of vertices and edges, beginning an ending with vertices such that edge is incident with the vertices preceding and following it.

→ No edge appear to traverse more than once in walk.

→ A vertex however may appear more than once.

→ Vertices with which a walk begins and ends are called terminal vertices.

→ If a walk start and end at same vertices it is called closed walk.

→

Path : An open walk in which no vertex appears more than once is called path.

- The no. of edges in path is called the length of Path.
- The terminal vertices of path are degree one and rest are of degree two.

Circuit : A closed walk in which no vertex other than starting and ending have a degree more than once is called Circuit.

Path: A path is a sequence of vertices such that there is an edge from each vertex to its successor.

A path is simple if each vertex is distinct.

A Path from a vertex to itself is called a Cycle.

A graph is called Cyclic if it contains a cycle; otherwise it is called acyclic.

Connected and Disconnected Components:-

Connected Graph : If there is atleast one path betw each pair of vertices in G.

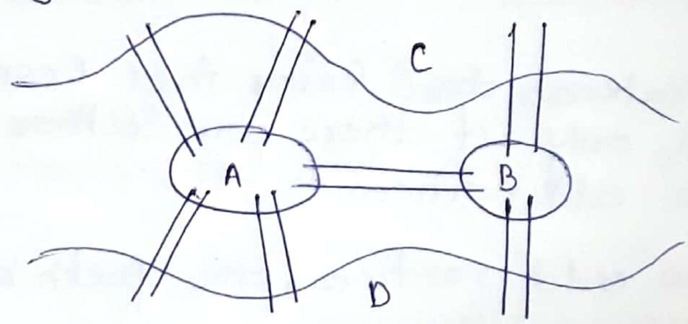
Disconnected Graph: If no path exist betw each Pair of vertices in G.

Connected Component : In a maximal Connected Subgraph of G. Each vertex belongs to exactly one connected component, as does each edge.

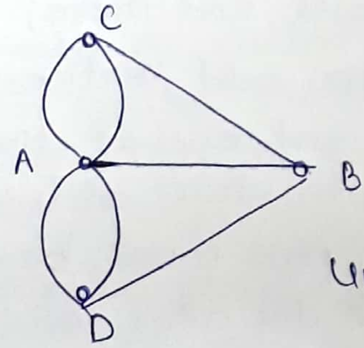
EULER AND HAMILTONIAN GRAPHS

The problem, often referred to as the bridges of Konigsberg problem, was first solved by Euler in Eighteen Century.

The Problem was rather Simple – the town of konigsberg Consists of two islands and seven bridges. It is possible, by beginning anywhere and ending anywhere, to walk through the town by crossing all seven bridges but not Crossing any bridge twice?



Konigsberg Bridges.



Underlying Graph

We first present some definitions and then present a theorem that Euler used to show that it is in fact impossible to walk through the town and traverse all the bridges only once.

Eulerian trial : An Euleriantrial is a trial that visits every edge of the graph once and only once. It can end on a vertex different from the one on which it began. A graph of this kind is said to be traversable.

Eulerian Circuit : An Eulerian circuit is an Eulerian trial that is a Circuit. That is, it begins and ends on the same Vertex.

Eulerian Graph : A graph is called Eulerian when it Contains a Eulerian Circuit.

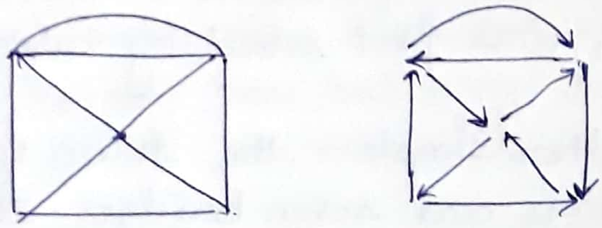


figure: An example of Eulerian trial.

A vertex of odd if its degree is odd and even if its degree is even.

Theorem:- An Eulerian trial Exists in a Connected graph if and only if there are either no odd vertices or two odd vertices.

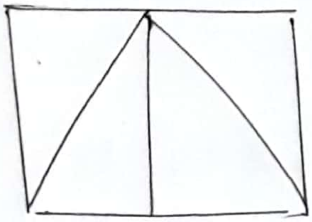
for the case of no odd vertices, the path begin at any vertex and will end there;
for the Case of two odd vertices, the path must begin at one odd vertex and end at the other. Any finite connected graph with two odd vertices is traversable. A traversable trail must begin at either odd vertex and will end at the other odd vertex.

Hamiltonian Graph

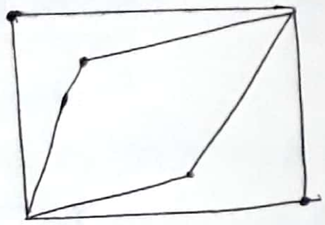
Hamiltonian Graph: If a graph has hamiltonian Circuit then graph is called hamiltonian Graph.

Hamiltonian Circuit: A hamiltonian Circuit in a graph is a closed Path that visits every vertex in graph Exactly once. It ends up with vertex where it started. Hamiltonian graphs are named after the Nineteenth Century Irish mathematician Sir William Rowan Hamilton (1805–1865)
This type of Problem is often referred to as the travelling Salesman or Postman Problem.

<u>Note:</u> An Euler Circuit traverses every edge in a Graph Exactly once, but may repeat vertices, while a hamiltonian Circuit visits each vertex in a Graph exactly once but may repeat Edges.



(a)
Hamiltonian But Not Euler

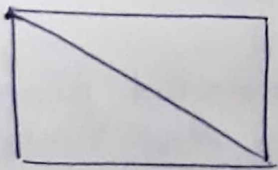


(b)
Euler but Not hamiltonian.

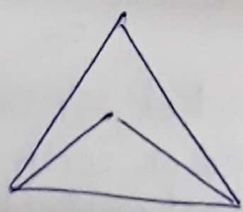
<u>Planar Graph</u> A graph is called Planar if it Can be drawn in the Plane without any edges Crossing. Such a drawing is Called a planar representation of Graph.

A Graph may be planar Even if it is usually drawn with crossings, because it may be possible to draw it in a different way without Crossings.

<u>Note:</u> A drawing of a geometric representation of a graph on any surface such that no edges intersect is Called Embedding.

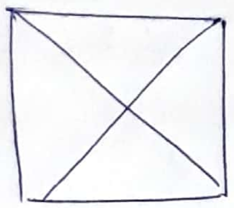


(a)

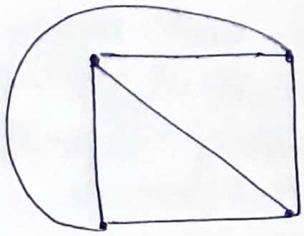


(b) Planar Graphs.

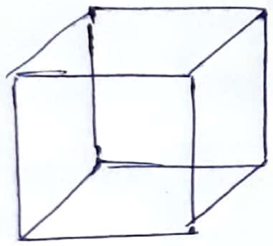
Ex.

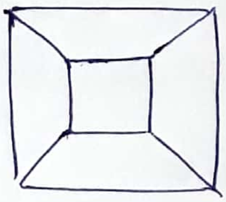


Graph K_4 with 2 edge Crossing



No Crossings.

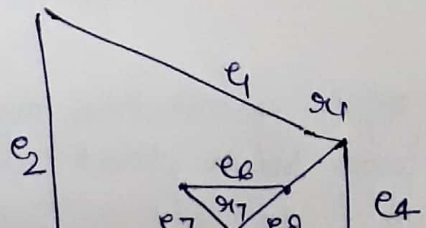




Planar representation.

Region of a Graph.

A region of a planar graph is defined to be an area of the plane that is bounded by edges and is not further divided into subareas.
The set of edges which bound a region of a planar graph is called its boundaries.



Euler's formula Let G be a Connected planar simple graph with e edges and vertices. Let r be the number of regions in a planar representation of G. Then $\boxed{r = e - v + 2}$ or

$$\boxed{n - e + r = 2}$$

$n/v \rightarrow$ nodes
$e \rightarrow$ edges.

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Proof: we prove theorem by induction on e, number of edges of G.

Basis of Induction. If $e=0$, then G must have just one vertex i.e, $n=1$ and one infinite region i.e, $r=1$. Then $n-e+r=1$

$$\boxed{1-0+1=2}$$

If $e=1$, then no of vertices of G is either 2 or 1, the first possibility of occuring when the edge is a loop. These two give rise to two regions.

$n=1$ 2

In the case of loop.

$\rightarrow n-e+r=1$

$\rightarrow 1-1+2=2$

In case of non loop

$\rightarrow n-e+r=2$

$\rightarrow 2-1+1=2$ Proof by Induction. Book Page 527

GRAPH COLORING

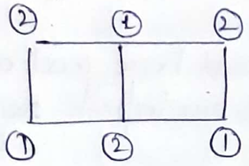
- Vertex Coloring
- Edge Coloring
- face coloring

There are many different ways.

No of colours required to colour a graph.

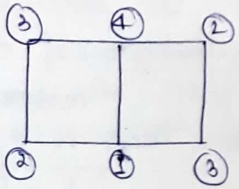
The assign of colours to the vertices of G, one color to each vertex, so that adjacent vertices are assigned different color is called proper coloring of G or simply vertex coloring.

→ No, two adjacent vertices can have same number of colours.



(a)

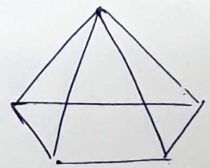
Examples.



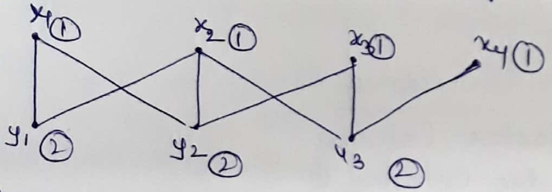
The chromatic no of a graph is the minimum no of colors required to color the vertices of the graph G, and is denoted by $\chi(G)$. Thus, a graph G is k-chromatic if $\boxed{\chi(G) = k}$

The chromatic number of some familiar graph can be easily determined.

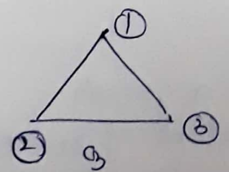
① $\chi(K_n) = n$, where K_n is the complete graph of n vertices.



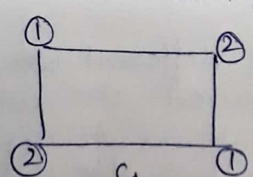
② $\chi(K_{m,n}) = 2$, i.e the chromatic number of every bipartite graph is 2.



③ $\chi(C_n) = 2$, if n is even and $\chi(C_n) = 3$, if n is odd.



C_3



C_4

Chromatic Polynomial

Let G be an undirected graph and λ be the number of colors that are available for proper coloring the vertices of G. The polynomial $P(\lambda)$ gives the number of different ways the graph can be properly colored using of fewer colors is called chromatic polynomial of G.

Que Write down chromatic polynomial of a given graph on n vertices.

Soln Let G be a graph on n vertices. Let C_i denote the different ways of properly coloring of G using exactly i distinct colors. These i colors can be chosen out of λ colors in ${}^{\lambda}C_i$ distinct ways.

Thus, total no. of distinct ways a proper coloring to a graph with i colors out of colors is possible in $C_i\,{}^{\lambda}C_i$ ways. Chromatic Polynomial is:

$$P_n(G,x) = \sum_{i=1}^{n} C_i\,{}^{\lambda}C_i$$

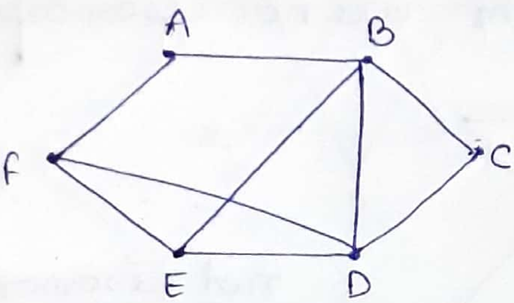
$$\Rightarrow C_1\,{}^{\lambda}C_1 + C_2\,{}^{\lambda}C_2 + \ldots C_n\,{}^{\lambda}C_n$$

$$= C_1\lambda + C_2\frac{\lambda(\lambda-1)}{2!} + \ldots C_n\frac{\lambda(\lambda-1)(\lambda-2)\ldots(\lambda-n+1)}{n!}$$

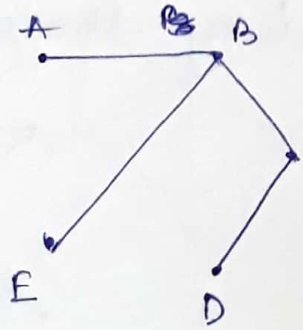
Each C_i can be evaluated individually.

Dijkshtra Algoro — from Book

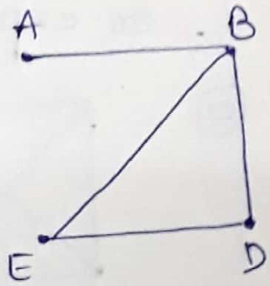
Oue Subgraphs.



(a)



(G_1)

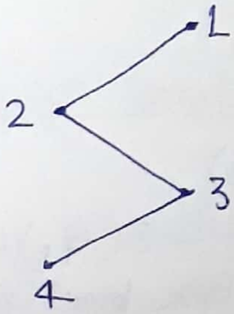


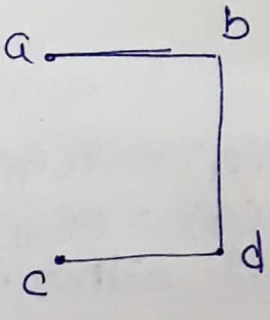
(G_2)

G_1 is subgraph of G but not an induced subgraph whereas G_2 is an induced subgraph.

Note: If G is a graph with vertex set V and U is a subset of V then the subgraph G(U) of G whose vertex set is U and whose edge set comprises exactly the edges of E which join vertices in U is termed an induced subgraph of G.

Isomorphic Graph.





$V(G_1) = \{1, 2, 3\ 4\}$
$V(G_2) = \{a, b, c, d\}$
$E(G_1) = \{\{1,2\}, \{2,3\} \{3,4\}\}$
$E(G_2) = \{\{a,b\}, \{b,d\} \{d,c\}\}$

define a function $f : V(G_1) \rightarrow V(G_2)$ as

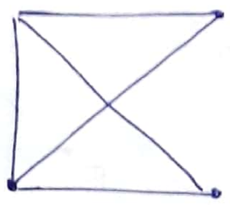
$f(1) = a \quad f(2) = b$
$f(3) = d \quad f(4) = c.$

So, f is clearly one-to-one and onto, hence an isomorphism.

Scanned by CamScanner

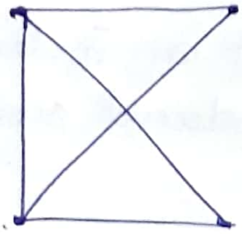
Que for each pair of graphs shown, either label the graphs so as to exibit an isomorphism or explain why the graphs are not isomorphic.

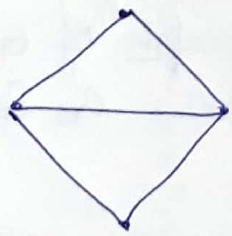
(a)



not isomophic.

(b)

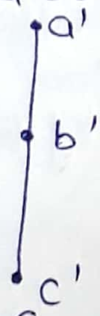




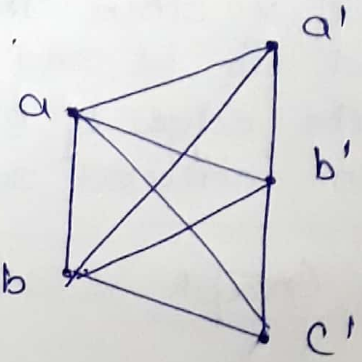
are isomophic.

Que sum of two graphs.

G_1



G_2



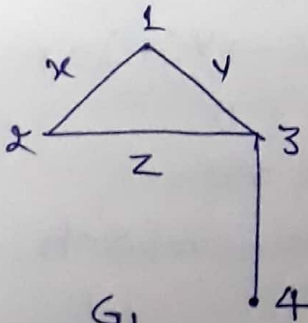
$G_1 + G_2$

Que Ring Sum.

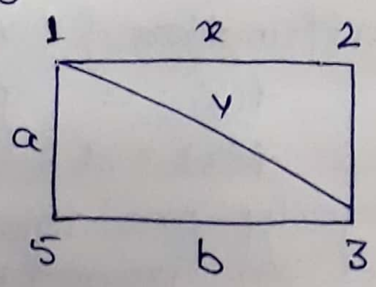
$$V(G) = V(G_1) \cup V(G_2)$$

$$E(G) = E(G_1) \cup E(G_2) - E(G_1) \cap E(G_2)$$

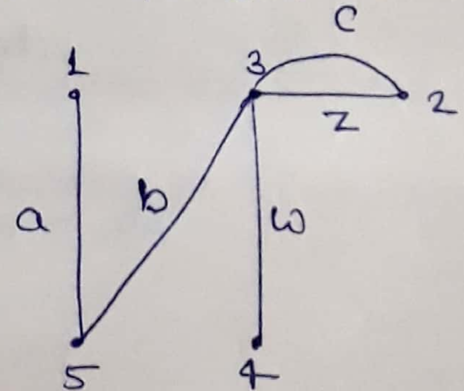
i.e the edges that either in G_1 or G_2 but not in both



G_1

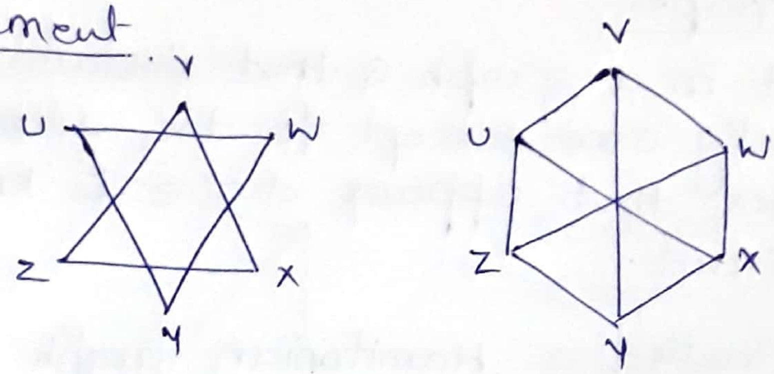


G_2

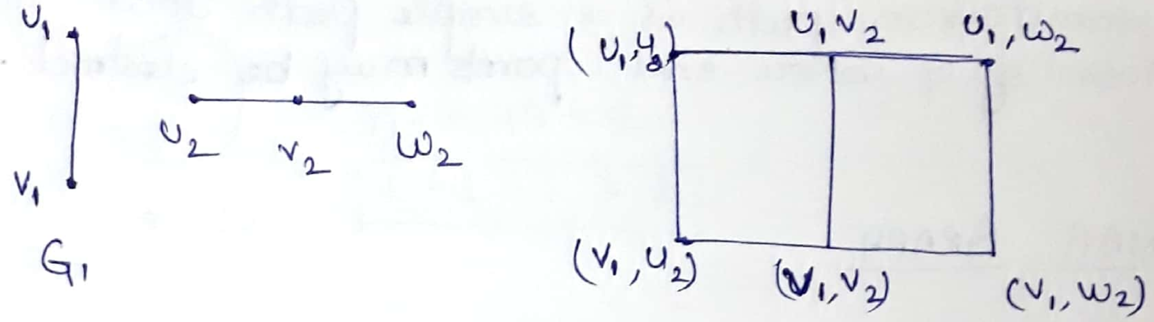


$G_1 \oplus G_2$

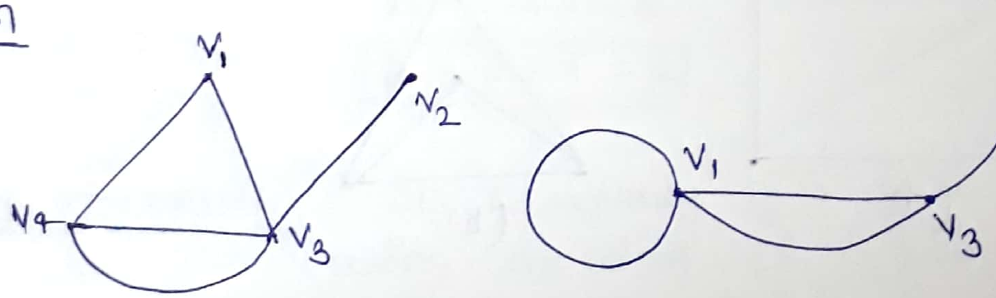
Complement



Product of Graphs



Fusion



Eulerian Graph:- A Path in a connected graph G is called Euler path if it includes every edge exactly once. Since the path contains every edge exactly once, it is also called Euler trail.

Euler Circuit: A Euler path that is circuit is called Euler Circuit. A connected graph which has a Eulerian Circuit is called an Eulerian Graph.

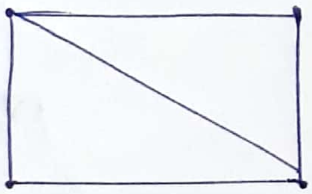
Hamiltonian Graphs.

A circuit in a graph G that contains each vertex in G exactly once, except for the starting and ending vertex that appears twice is known as Hamiltonian Circuit.

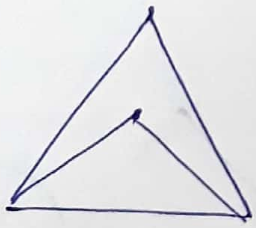
A Graph G is called a Hamitonian graph if it contains a Hamiltonian Circuit.

A Hamitonian path is a simple path that contain all vertices of G where end points may be distinct.

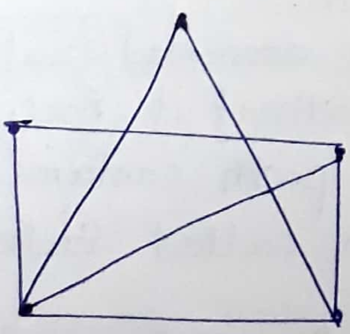
PLANAR GRAPH



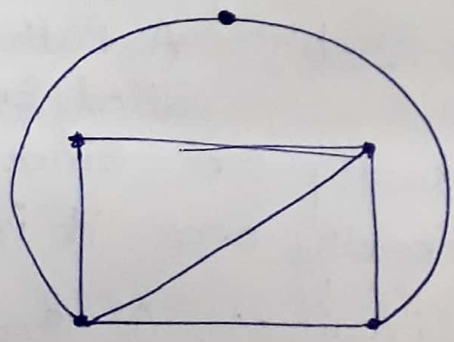
(a)

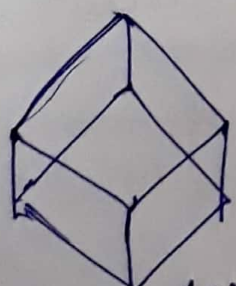


(b)

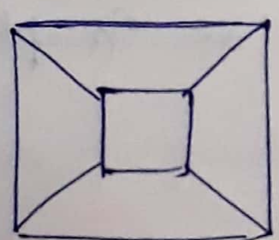


(a)



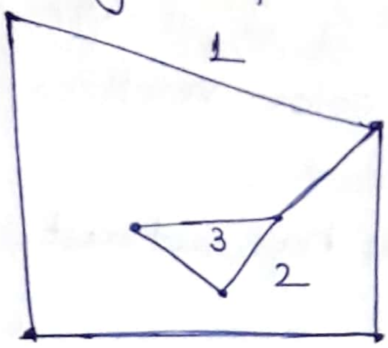


(a)



(b).

Region of Graph



$n - e + r = 2$

vertices.
edges.
region $\boxed{n - e + r = 2}$

$n = 1$

$n - e + r = 2$

$\boxed{1 - 1 + 2 = 2}$

2

$n - e + r = 2$

$\boxed{2 - 1 + 1 = 2}$

GRAPH COLORING The n coloring of G is a coloring of G using n-colors. If G has n coloring, then G is said to be n-colorable.

The chromatic number of a graph G is the minimum no. of colors to color the vertices of the graph G and is denoted $\chi(G)$. Thus a graph is n-colorable if

$\boxed{\chi(G) \leq n}$

There is no easy way of finding $\chi(G)$ of a graph G. The following rules may be helpful in finding $\chi(G)$.

1. A graph consisting of only isolated vertices is 1-chromatic.
2. $\chi(G) \leq |V|$, where $|V|$ is the number of vertices of G.

3. If a subgraph of G requires m colors then $\chi(G) \geq m$.

4. If the degree of a vertex G is d, then almost d colors are required to color vertices adjacent to it

5. Every k – chromatic graph has atleast k vertices v such that degree $(v) \geq k-1$.

6. for any graph G, $\chi(G) \leq 1+\Delta(G)$ where $\Delta(G)$ is the largest degree of any vertex of G.

7. The following statements are equivalent.
   1. A graph G is 2 colorable
   2. G is bipartite.
   3. Every cycle of G has even length.

8. If $\delta(G)$ is the minimum deg of any vertex of G, the $\chi(G) \geq \frac{|V|}{|V| - \delta(G)}$

9. If G is a tree with at least one edge then G is 2 – chromatic.