1. A Hopfield associative memory network has activities for individual units, s_i for $i = 1, 2, \ldots, N$, that take values of either $+1$ or -1, and are updated at every discrete time step of the network dynamics (assume that $\Delta t = 1$) by the rule

$$s_i(t+1) = \mathrm{sgn}\left(\sum_{j=1}^{N} M_{ij} s_j(t)\right), \tag{1}$$

where

$$\mathrm{sgn}(z) = \begin{cases} +1 & \text{if} \quad z \geq 0 \\ -1 & \text{if} \quad z < 0\,. \end{cases}$$

Here M_{ij} is the i, j element of a matrix constructed from P "memory" vectors u_i^a ($a = 1, 2, \ldots P$ and $i = 1, 2, \ldots, N$) through the outer product

$$M_{ij} = \sum_{a=1}^{P} u_i^a u_j^a \quad \text{if } i \neq j \quad \text{and} \quad M_{ii} = 0\,. \tag{2}$$

Consider a 100-element network ($N = 100$). Construct P (specific values of P will be given below) memory states by randomly assigning $+1$ and -1 values with equal probabilities to each of the N elements u_i^a of each of the P memory vectors. Using these memory vectors, set the matrix of synaptic weights according to equation 2. Then, study the behavior of the network by iterating equation 1.

To measure how close the state of the network at time t, given by the variables $s_i(t)$, is to a particular memory state, define the overlap function

$$q(t) = \frac{1}{N} \sum_{i=1}^{N} s_i(t) u_i^1\,.$$

This is equal to 1 if the $s_i = u_i^1$ for all i and is near zero if there is no similarity between the state of the network, given by the set of s_i values, and the memory vector.

In order to set the initial values $s_i(0)$ so that a particular value of $q(0)$ is obtained (at least approximately), use the following procedure. For each i, set $s_i(0) = u_i^1$ with probability $q(0)$ (i.e. if a random number you generate is less than $q(0)$). With probability $1 - q(0)$ (i.e. if the random number you generated is not less than $q(0)$), set $s_i(0)$ randomly to either $+1$ or -1 (with an equal probability of 50% for either case).

Plot $q(t)$ as the network evolves from this state according to equation 1. Final values of $q(t)$ near one indicate successful recovery of the memory. Provide sample plots of $q(t)$ from which you obtain the answers to the following questions:

a) What is the range of positive $q(0)$ values that assures successful memory recovery for $P = 1, 5,$ and 10? Also, determine how accurately the memory is recovered in these three cases by reporting the final value of q obtained from your simulations.

b) Increase P until memory recovery fails (q ends up considerably less than 1) even for $q(0) = 1$. At what P value does this occur?