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MA 2302: Introduction to Probability and Statistics

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Bayes Theorem: Have you ever thought like this?

Let A and B be any two arbitrary events such that $P(B) > 0$. Then $B \subset (A \cup A^c)$ and hence $B = B \cap (A \cup A^c) = AB \cup A^c B$. Hence,

$$P(B) = P(AB) + P(A^c B) = P(A)P(B|A) + P(A^c)P(B|A^c)$$

which is the law of total probabilities and

$$P(A|B) = \frac{P(A)P(B|A)}{P(A)P(B|A) + P(A^c)P(B|A^c)}$$

which is the Bayes theorem. Also

$$P(A^c|B) = \frac{P(A^c)P(B|A^c)}{P(A)P(B|A) + P(A^c)P(B|A^c)}.$$

In addition,

$$P(A^c|B^c) = \frac{P(A^c)P(B^c|A^c)}{P(A^c)P(B^c|A^c) + P(A)P(B^c|A)}.$$

Ex 1. It is estimated that *2% of certain locality were infected by a virus. A certain plasma test is used to detect infection due the virus. About* 5% *of the test results are false positive and 10% are false negative. If a person was tested positive for the virus, find the probability that he was really infected. If a person was tested negative, what was the probability that he was actually not infected?*

Ans. Let A be the event that the person was infected by the virus and B, the event that that the person was tested positive for the virus. False positive means a person was tested positive given that he was not infected. False negative means a person was tested negative given that he was really infected. Thus,

$$P(A) = 0.02, P(A^c) = 0.98, P(B|A^c) = 0.05, P(B^c|A) = 0.10$$

Hence, $P(B^c|A^c) = 0.95, P(B|A) = 0.90$. (Observe that $P(A|B) + P(A^c|B) = 1$.) Hence,

$$\Pr\{A\ person\ was\ really\ infected\ given\ that\ he\ was\ tested\ positive\}$$

$$= P(A|B) = \frac{P(A)P(B|A)}{P(A)P(B|A) + P(A^c)P(B|A^c)} = \frac{0.02 \times 0.90}{0.02 \times 0.90 + 0.98 \times 0.05} = \frac{0.018}{0.018 + 0.049}$$
$$= 0.2686.$$

$$\Pr\{A\ person\ was\ really\ infected\ given\ that\ he\ was\ tested\ positive\}$$

$$= P(A^c|B^c) = \frac{P(A^c)P(B^c|A^c)}{P(A^c)P(B^c|A^c) + P(A)P(B^c|A)} = \frac{0.98 \times 0.95}{0.98 \times 0.95 + 0.02 \times 0.10} = \frac{0.931}{0.931 + 0.002}$$
$$= 0.9979.$$

Ex 2. A four digit number is formed using the numbers $0,1,2,3,4,5$ without using any number more than once. What is the probability that it is divisible by 4?

Ex 3. If the letters of the word 'REGULATIONS' are arranged at random, what is the chance that there will be exactly 4 letters between R and E ?

Ex 4. The sum of two non-negative quantities is equal to 2n. Find the chance that their product is not less than The sum of two non-negative quantities is equal to 2n. Find the chance that their product is not less than 3/4 times their greatest product.

Ex 5. Out of 11 chips numbered from 1 to 11, three are chosen at random (wor). What is the probability that they are in A.P.?

Ex 6. What is the chance that two positive integers chosen at random, will be relatively prime to each other ?

Hints. Let m and n be the two positive integers chosen at random and let p be a prime number. If any number is divisible by p, then the possible remainders are $0,1,2,\ldots,p-1$ and hence the probability that m is divisible by p is $\frac{1}{p}$. Similarly, the probability that n is divisible by p is also equal to $\frac{1}{p}$. Hence, the probability that both m and n are divisible by p is equal to $\frac{1}{p^2}$. Thus, the probability that at least one of m and n is not divisible by p is equal to $1-\frac{1}{p^2}$. Hence the probability that m and n are relatively prime is equal to

$$\prod_p \left(1-\frac{1}{p^2}\right)=\frac{6}{\pi^2}.$$

Ex 5. A piece of thread of length L is broken into three pieces. What is the probability that a triangle can be formed ?

Hints: $x, y, L - x - y$ lengths of

Three pieces. $x > 0, y > 0$

$x + y < L.$

Apply triangle property:

$x + y > z, x + z > y, y + z > x.$

$\Rightarrow x < \frac{L}{2}, y < \frac{L}{2}, x + y > \frac{L}{2}.$

Required probability

$= \frac{\frac{1}{2}\times\left(\frac{L}{2}\right)^2}{\frac{1}{2}\times L^2} = \frac{1}{4}.$