Introduction to Probability and Statistics Course ID:MA2203

Lecture-3

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Some Examples

1. An integer is chosen at random from two hundred digits $\{1, 2, \ldots, 200\}$. What is the probability that the integer is divisible by 6 or 8?
   Ans: The sample space $S = \{1, 2, \ldots, 200\}$. The event that the integer chosen is divisible by 6 is $A = \{6, 12, 18, \ldots, 198\}$, gives $|A| = 198/6 = 33$. Hence $P(A) = 33/200$. Similarly, the event that the integer is divisible by 8, $B = \{8, 16, 24, \ldots, 200\}$, $|B| = 200/8 = 25$, implies $P(B) = 25/200$. The LCM of 6 and 8 is 24. The number will be divisible by both 6 and 8 if it is divisible by 24. Hence $A \bigcap B = \{24, 48, \ldots, 192\}$. $|A \bigcap B| = 8$. This gives $P(A \bigcap B) = 8/200$. Hence, we have $P(A \bigcup B) = P(A) + P(B) - P(A \bigcap B) = 33/200 + 25/200 - 8/200 = 1/4$.

Ex. A and B alternatively cut a pack of cards and the pack is shuffled after each cut. If A starts and the game is continued until one cuts a diamond, what are the respective chances of A and B first cutting a diamond?

Ex. The sum of two non-negative quantities is equal to $2n$. Find the chance that their product is not less than $\frac{3}{4}$ times their greatest product.

2. The probability that a student passes a Physics test is $2/3$ and the probability that he passes both a Physics test and English test is $14/45$. The probability that he passes at least one test is $4/5$. What is the probability that he passes the English test?
   Sol'n: A= The student passes the Physics test, B=The student passes the English test. Given $P(A) = 2/3$, $P(B) =?$, $P(A \bigcap B) = 14/45$, $P(A \bigcup B) = 4/5$. We know $P(A \bigcup B) = P(A) + P(B) - P(A \bigcap B)$. This gives $P(B) = 4/9$.
3. A card is drawn from a pack of 52 cards. Find the probability of getting a king or heart or a red card?
   Ans: A=the card is a king, B=the card is a heart, C=the card is red. The events A, B and C are not mutually exclusive. We need to get $P(A \bigcup B \bigcup C) = P(A) + P(B) + P(C) - P(A \bigcap B) - P(B \bigcap C) - P(A \bigcap C) + P(A \bigcap B \bigcap C)$. Given $P(A) = 4/52$, $P(B) = 13/52$, $P(C) = 26/52$, $P(A \bigcap B) = 1/52$, $P(B \bigcap C) = 13/52$, $P(A \bigcap C) = 2/52$, $P(A \bigcap B \bigcap C) = 1/52$. Hence the required probability is 7/13.

- Conditional Probability: Sometimes it is essential to use the prior information regarding the happening of an event. Suppose, A and B are two events in a given sample space. The happening of A may be affected by the happening or non-happening of B. The probability of A, under the condition that another event, say B, has happened is called the conditional probability of A given B, this we denote by $P(A|B)$ and computed as,

$$P(A|B) = \frac{P(A \bigcap B)}{P(B)}, P(B) \neq 0.$$

  Moreover, we get $P(A \bigcap B) = P(A|B)P(B) = P(B|A)P(A)$, which is known as multiplication rule.
- Independent Events: Two events A and B are said to be independent if $P(A \bigcap B) = P(A)P(B)$. Hence, if the events A and B are independent we have the conditional probability as $P(A|B) = P(A)$, $P(B|A) = P(B)$ provided $P(A) \neq 0$, and $P(B) \neq 0$.
- The events $A_1, A_2, \ldots, A_n$ are said to be independent if

$$P(A_1 \bigcap A_2 \bigcap \cdots \bigcap A_n) = P(A_1)P(A_2)\ldots P(A_n)$$

  and for k different events $A_{j_1}, A_{j_2}, \ldots, A_{j_k}$,

$$P(A_{j_1} \bigcap A_{j_2} \bigcap \ldots A_{j_k}) = P(A_{j_1})P(A_{j_2})\ldots P(A_{j_k})$$

  where $k = 2, 3, \ldots, n-1$.

- Pairwise Independent: The events $A_1, A_2, \ldots, A_n$ are said to be pairwise independent if $P(A_i \bigcap A_j) = P(A_i)P(A_j)$, for all $i, j = 1, 2, \ldots, n$ such that $i \neq j$.
- Note that, independent implies pairwise independent but the converse may not be true.
- Problem 1: Prove that if A and B are independent, then (i) A and B^c, (ii) A^c and B and (iii) A^c and B^c are independent.
  Proof (i): $P(A \bigcap B^c) = P(A) - P(A \bigcap B) = P(A)(1 - P(B)) = P(A)P(B^c)$. (iii) $P(A^c \bigcap B^c) = P((A \bigcup B)^c) = 1 - P(A \bigcup B) = 1 - P(A) - P(B) + P(A)P(B) = (1 - P(A))(1 - P(B)) = P(A^c)P(B^c)$.
- Problem 2: If the events A, B and C are independent then $A \bigcup B$ and C are also independent. (Home work)
- Problem 3: Let A and B be two events such that $P(A) = 3/4$, and $P(B) = 5/8$ then show that (i) $P(A \bigcup B) \geq 3/4$, (ii) $3/8 \leq P(A \bigcap B) \leq 5/8$.
  Proof(i): Since $A \subset (A \bigcup B)$, $P(A) \leq P(A \bigcup B)$ this implies $3/4 \leq P(A \bigcup B)$. (ii) Also $(A \bigcap B) \subset B$, so $P(A \bigcap B) \leq P(B) = 5/8$. Further $(A \bigcup B) = P(A) + P(B) - P(A \bigcap B) \leq 1$. This implies $3/4 + 5/8 - 1 \leq P(A \bigcap B)$. This gives $P(A \bigcap B) \geq 3/8$. Combining (i) and (ii), we have $3/8 \leq P(A \bigcap B) \leq 5/8$.

- Bayes' Theorem: If $A_1, A_2, \ldots, A_n$ are mutually disjoint events with $P(A_i) \neq 0, i = 1, 2, \ldots, n$ then for any arbitrary event B which is a subset of $\bigcup A_i$ such that $P(B) > 0$, we have

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_{i=1}^{n} P(A_i)P(B|A_i)} = \frac{P(A_i)P(B|A_i)}{P(B)}.$$

- Here we can see that $P(B) = \sum_{i=1}^{n} P(A_i)P(B|A_i)$. This is called the total probability.

- Problem (1): There are two bags A and B. A contains n white and 2 black balls and B contains 2 white and n black balls. One of the two bags is selected at random and two balls are drawn from it without replacement. If both the balls drawn are white and the probability that the bag A was used to draw the ball is $6/7$, find the value of n.
  Ans: Let E_1 be the event that the bag A was selected, E_2 be the event that the bag B is selected. Let E be the event that the two balls drawn are white. Hence we have $P(E_1) = P(E_2) = 1/2$, $P(E|E_1) = \frac{C(n,2)}{C(n+2,2)}$, $P(E|E_2) = \frac{C(2,2)}{C(n+2,2)}$. From Bayes theorem,

$$P(E_1|E) = \frac{P(E_1)P(E|E_1)}{P(E_1)P(E|E_1) + P(E_2)P(E|E_2)} = 6/7.$$

  Substituting all the values in the left hand side, we have after simplification $\frac{n(n-1)}{n(n-1)+2} = 6/7$. Which gives $n^2 - n - 12 = 0$ and consequently after solving we get $n = 4, -3$. Since n can not be negative we have, $n = 4$.