Introduction to Probability and Statistics Course ID:MA2203

Lecture-2

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- Statistical or Empirical Probability (VON MISES): If an experiment is performed repeatedly under essentially homogeneous and identical conditions, then the limiting value of the ratio of the number of times the event occurs to the number of trials, as the number of trials becomes indefinitely large, is called the probability of happening of the event, it is being assumed that the limit is finite and unique. Mathematically, if for n trials an event A happens m times, then

$$P(A) = \lim_{n \to \infty} \frac{m}{n}.$$

- J. E. Kerrich conducted a coin tossing experiment with 10 sets of 1000 tosses ech during his confinement in World war-II. The number of heads found were: 502, 511, 529, 504, 476, 507, 520, 504, 529. The probability of getting a head in tossing a coin once is computed using the above definition, $5,079/10,000 = 0.5079 \approx 0.5$.

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- We also call the probability as the counterpart of relative frequency. If we denote $f(A)$ as the absolute frequency of an event A. Then Relative frequency of A is denoted by
$$f_{rel}(A) = \frac{f(A)}{n}.$$
- We can observe that, $0 \leq f_{rel}(A) \leq 1$, $f_{rel}(S) = 1$. For two mutually exclusive events, say A and B we have, $f_{rel}(A \bigcup B) = f_{rel}(A) + f_{rel}(B)$. So, having all these prior information regarding the probability, we will try to generalize the definition in such a way that, it should include all the previous definitions, and might be the best and most practical. This we call as Axiomatic definition of Probability, which we will discuss in the next.

Axiomatic Definition

- Let S be a given sample space, and $\mathbb{S}$ be a σ-field on it. Then probability P is defined as a set function $P : \mathbb{S} \to [0, 1]$, which satisfies the following axioms.

1. For every event $A \in \mathbb{S}$, $0 \leq P(A) \leq 1$.
2. The entire sample space has the probability, $P(S) = 1$.
3. For mutually exclusive events A and B,

$$P(A \bigcup B) = P(A) + P(B).$$

- The above axiom (3) can be extended to countable number of mutually exclusive events, say $A_1, A_2, \ldots$ that is,

$$P(A_1 \bigcup A_2 \bigcup \ldots) = P(A_1) + P(A_2) + \ldots.$$

Basic Results on Probability

1. Probability of the impossible event is zero, that is $P(\emptyset) = 0$. Observe that, the $\emptyset$ does not contain any elements, hence we have $S \bigcup \emptyset = S$. These two sets are disjoint. Using the third axiom, we have $P(S \bigcup \emptyset) = P(S) = P(S) + P(\emptyset)$, which implies that $P(\emptyset) = 0$.
   Note: $P(A) = 0$, does not imply that A is necessarily an empty set. In practice, probability '0' is assigned to the events which are so rare that they happen only once in a life time. For example, in a random tossing of a coin, the event that the coin will stand erect on its edge, is assigned the probability '0'.
2. Probability of the complementary event A^c of A is obtained as $P(A^c) = 1 - P(A)$. To prove this, observe that, A and A^c are disjoint events and $A \bigcup A^c = S$. Hence using axiom (3), we have $P(A \bigcup A^c) = P(S) = 1$, this implies $P(A) + P(A^c) = 1$, and the result.

Some More Results

3 For any two events A and B, we have (i) $P(A^c \bigcap B) = P(B) - P(A \bigcap B)$. (ii) If $B \subset A$, then (a) $P(A \bigcap B^c) = P(A) - P(B)$, (b) $P(B) \leq P(A)$. To prove (ii), the events B and $A \bigcap B^c$ are mutually exclusive, hence $P(A) = P(B \bigcup (A \bigcap B^c)) = P(B) + P(A \bigcap B^c)$, this implies that $P(A \bigcap B^c) = P(A) - P(B)$. Also $P(A \bigcap B^c) \geq 0$ which implies that $P(A) \geq P(B)$.

4 Addition Theorem of Probability: For events A and B in a sample space S, $P(A \bigcup B) = P(A) + P(B) - P(A \bigcap B)$.
Proof: The events $A - (A \bigcap B)$, $A \bigcap B$, and $B - (A \bigcap B)$ are mutually exclusive and its union is $A \bigcup B$. Hence applying axiom (3), we have $P(A \bigcup B) = P(A - (A \bigcap B)) + P(A \bigcap B) + P(B - (A \bigcap B)) = P(A) + P(B) - P(A \bigcup B)$. This proves the theorem.

Ex. Prove this theorem without using axioms of probability. Hint: Use set theory approach.

Boole's Inequality

- For n events $A_1, A_2, \dots, A_n$, we have

$$(i)\ P(\bigcap_{i=1}^{n} A_i) \geq \sum_{i=1}^{n} P(A_i) - (n-1).$$

$$(ii)\ P(\bigcup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} P(A_i).$$

Proof:(i) This can be proved by the method of mathematical induction. Verify the result for $n = 2$ that is, $P(A_1 \bigcup A_2) = P(A_1) + P(A_2) - P(A_1 \bigcap A_2) \leq 1$, this implies $P(A_1 \bigcap A_2) \geq P(A_1) + P(A_2) - 1$. The result is true for $n = 2$. Assume that the result holds true for $n = k$, that is

$$P(\bigcap_{i=1}^{k} A_i) \geq \sum_{i=1}^{k} P(A_i) - (k-1).$$

To prove that the result is true for $n = k + 1$.

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\begin{aligned}
L.H.S : P(\bigcap_{i=1}^{k+1} A_i) &= P((\bigcap_{i=1}^{k} A_i) \bigcap A_{k+1}) \\
&\geq P(\bigcap_{i=1}^{k} A_i) + P(A_{k+1}) - 1 \\
&\geq \sum_{i=1}^{k} P(A_i) - (k-1) + P(A_{k+1}) - 1 \\
&= \sum_{i=1}^{k+1} P(A_i) - k : R.H.S
\end{aligned}
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Proof (ii): Now applying the above inequality, with the events $A_1^c, A_2^c, \ldots, A_n^c$, we have

$$\begin{aligned} P(\bigcap A_i^c) &\geq \sum_{i=1}^{n} P(A_i^c) - (n-1) \\ &= 1 - P(A_1) + \cdots + 1 - P(A_n) - (n-1) \\ &= 1 - \sum_{i=1}^{n} P(A_i). \end{aligned}$$

Hence, we have

$$\begin{aligned} \sum_{i=1}^{n} P(A_i) &\geq 1 - P(\bigcap A_i^c) \\ &= 1 - P([\bigcup A_i]^c) \\ &= P(\bigcup A_i). \end{aligned}$$

Bonferroni's Inequality

- Given n events, $A_1, A_2, \ldots, A_n$, we have

$$\sum_{i=1}^{n} P(A_i) \geq P(\bigcup_{i=1}^{n} A_i) \geq \sum_{i=1}^{n} P(A_i) - \sum_{1 \leq i \leq j \leq n} P(A_i \bigcap A_j).$$

Proof: This can be proved by the method of mathematical induction. Check that, the result is true for $n = 3$.

$$\begin{aligned} P(\bigcup_{i=1}^{3}) &= \sum_{i=1}^{3} P(A_i) - \sum_{1 \leq i < j \leq 3} P(A_i \bigcap A_j) + P(\bigcap_{i=1}^{3} A_i) \\ &\geq \sum_{i=1}^{3} P(A_i) - \sum_{1 \leq i < j \leq 3} P(A_i \bigcap A_j). \end{aligned}$$

Let the result is true for $n = k$. To prove the result for $n = k + 1$.

Proof Continue...

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\begin{aligned}
P(\bigcup_{i=1}^{k+1} A_i) &= P(\bigcup_{i=1}^{k} A_i \bigcup A_{k+1}) \\
&= P(\bigcup_{i=1}^{k} A_i) + P(A_{k+1}) - P[(\bigcup_{i=1}^{k} A_i) \bigcap A_{k+1}] \\
&= P(\bigcup_{i=1}^{k} A_i) + P(A_{k+1}) - P[\bigcup_{i=1}^{k} (A_i \bigcap A_{k+1})] \\
&\geq \{\sum_{i=1}^{k} P(A_i) - \sum_{1\leq i<j\leq n} P(A_i \bigcap A_j)\} \\
&\quad + P(A_{k+1}) - P\{\bigcup_{i=1}^{k} (A_i \bigcap A_{k+1})\}.
\end{aligned}
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From Boole's inequality, we have

$$P\left(\bigcup_{i=1}^{k}(A_i \bigcap A_{k+1})\right) \leq \sum_{i=1}^{k} P(A_i \bigcap A_{k+1})$$

Using this, we get

$$\begin{aligned} P\left(\bigcup_{i=1}^{k+1} A_i\right) &\geq \sum_{i=1}^{k+1} P(A_i) - \sum_{1 \leq i < j \leq n} P(A_i \bigcap A_j) - \sum_{i=1}^{k} P(A_i \bigcap A_{k+1}) \\ &= \sum_{i=1}^{k+1} P(A_i) - \sum_{1 \leq i < j \leq k+1} P(A_i \bigcap A_j). \end{aligned}$$