Class Nos. 18, 19,20,21,22,23,24,25.
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MA 2302: Introduction to Probability and Statistics

# Bivariate Distributions

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# Two dimensional distributions

Consider the following table which shows the mark distribution of 100 students in two mid-semester tests. Observe that there are **2** students securing marks in the range 20-25 in the first test and 05-10 in the second test.

| Test I Marks | Test II Marks | | | | | | $f_X$ |
|---|---|---|---|---|---|---|---|
| | 00-05 | 05-10 | 10-15 | 15-20 | 20-25 | 25-30 | |
| 00-05 | 1 | 1 | 3 | 2 | 1 | 1 | 9 |
| 05-10 | 2 | 3 | 4 | 4 | 3 | 1 | 17 |
| 10-15 | 2 | 3 | **5** | 5 | 4 | 2 | 21 |
| 15-20 | 3 | 4 | 6 | 5 | 5 | 3 | 26 |
| 20-25 | 2 | **2** | 4 | 2 | 3 | 2 | 15 |
| 25-30 | 1 | 2 | 3 | 2 | 2 | 2 | 12 |
| $f_Y$ | 11 | 15 | 25 | 20 | 18 | 11 | 100 |

# Two dimensional distributions

The individual frequency distributions of $X$ and $Y$ are given by

| Test I Marks | No. of students |
|---|---|
| 00-05 | 9 |
| 05-10 | 17 |
| 10-15 | 21 |
| 15-20 | 26 |
| 20-25 | 15 |
| 25-30 | 12 |

| Test I I Marks | No. of students |
|---|---|
| 00-05 | 11 |
| 05-10 | 15 |
| 10-15 | 25 |
| 15-20 | 20 |
| 20-25 | 18 |
| 25-30 | 11 |

# Two dimensional distributions

We can write the two dimensional frequency distribution as a two dimensional probability distribution as follows:

| Test I Marks | Test II Marks | | | | | | $f_2(y)$ |
|---|---|---|---|---|---|---|---|
| | 00-05 | 05-10 | 10-15 | 15-20 | 20-25 | 25-30 | |
| 00-05 | 0.01 | 0.01 | 0.03 | 0.02 | 0.01 | 0.01 | 0.09 |
| 05-10 | 0.02 | 0.03 | 0.04 | 0.04 | 0.03 | 0.01 | 0.17 |
| 10-15 | 0.02 | 0.03 | 0.05 | 0.05 | 0.04 | 0.02 | 0.21 |
| 15-20 | 0.03 | 0.04 | 0.06 | 0.05 | 0.05 | 0.03 | 0.26 |
| 20-25 | 0.02 | **0.02** | 0.04 | 0.02 | 0.03 | 0.02 | 0.15 |
| 25-30 | 0.01 | 0.02 | 0.03 | 0.02 | 0.02 | 0.02 | 0.12 |
| $f_1(x)$ | 0.11 | 0.15 | 0.25 | 0.20 | 0.18 | 0.11 | 1.00 |

Here, $\mathbf{0.02} = \Pr\{20 \leq X < 25, 5 \leq Y < 10\}$.

# Two dimensional distributions

The individual probability distributions of $X$ and $Y$ are given by

| Test I Marks | No. of students |
|---|---|
| 00-05 | 0.11 |
| 05-10 | 0.15 |
| 10-15 | 0.25 |
| 15-20 | 0.20 |
| 20-25 | 0.18 |
| 25-30 | 0.11 |

| Test I I Marks | No. of students |
|---|---|
| 00-05 | 0.09 |
| 05-10 | 0.17 |
| 10-15 | 0.21 |
| 15-20 | 0.26 |
| 20-25 | 0.15 |
| 25-30 | 0.12 |

## Two dimensional distributions

*Discrete two-dimensional distributions*

If we replace the class intervals by the mid-values, we will get a bivariate prob. distribution.

| Test I Marks | Test II Marks | | | | | | $f_2(y)$ |
|---|---|---|---|---|---|---|---|
| | 2.5 | 7.5 | 12.5 | 17.5 | 22.5 | 27.5 | |
| 2.5 | 0.01 | 0.01 | 0.03 | 0.02 | 0.01 | 0.01 | 0.09 |
| 7.5 | 0.02 | 0.03 | 0.04 | 0.04 | 0.03 | 0.01 | 0.17 |
| 12.5 | 0.02 | 0.03 | 0.05 | 0.05 | 0.04 | 0.02 | 0.21 |
| 17.5 | 0.03 | 0.04 | 0.06 | 0.05 | 0.05 | 0.03 | 0.26 |
| 22.5 | 0.02 | **0.02** | 0.04 | 0.02 | 0.03 | 0.02 | 0.15 |
| 27.5 | 0.01 | 0.02 | 0.03 | 0.02 | 0.02 | 0.02 | 0.12 |
| $f_1(x)$ | 0.11 | 0.15 | 0.25 | 0.20 | 0.18 | 0.11 | 1.00 |

Here, $\mathbf{0.02} = \Pr\{X = 22.5, Y7.5\} = f(22.5,7.5)$.

## Two dimensional distributions

The individual (marginal) discrete probability distributions of $X$ and $Y$ are given by

| $x$ | $f_1(x)$ |
|---|---|
| 2.5 | 0.11 |
| 7.5 | 0.15 |
| 12.5 | 0.25 |
| 17.5 | 0.20 |
| 22.5 | 0.18 |
| 27.5 | 0.11 |

| $y$ | $f_2(y)$ |
|---|---|
| 2.5 | 0.09 |
| 7.5 | 0.17 |
| 12.5 | 0.21 |
| 17.5 | 0.26 |
| 22.5 | 0.15 |
| 27.5 | 0.12 |

If the distribution of one random variable is obtained from a joint distribution of two random variables, it is called a marginal distribution.

## Two dimensional distributions

The joint probability mass function $of$ $X$ and $Y$ has the property

$$f(x, y) = \Pr\{X = x, Y = y\} \geq 0, \sum_x \sum_y f(x, y) = 1.$$

The marginal pmf of $X$ is given by

$$f_1(x) = \Pr\{X = x\} = \sum_y f(x, y).$$

The marginal pmf of $Y$ is given by

$$f_2(y) = \Pr\{Y = y\} = \sum_x f(x, y).$$

$X$ and $Y$ are independent if

$$f(x, y) = f_1(x) f_2(y)$$

for all $x$ and $y$.

## Two dimensional distribution

Consider the following joint distribution and the marginal distributions:

| $x \downarrow y \rightarrow$ | 1 | 2 |
|---|---|---|
| 0 | **1/4** | **1/4** |
| 1 | **1/4** | **1/4** |

| $x$ | $f_1(x)$ |
|---|---|
| 0 | 1/2 |
| 1 | 1/2 |

| $y$ | $f_2(y)$ |
|---|---|
| 1 | 1/2 |
| 2 | 1/2 |

$$\boldsymbol{f(0,1)} = \frac{1}{4}, f_1(0) = f_2(1) = \frac{1}{2} \Rightarrow f(0,1) = f_1(0) \times f_2(1)$$

$$\boldsymbol{f(1,1)} = \frac{1}{4}, f_1(1) = f_2(1) = \frac{1}{2} \Rightarrow f(1,1) = f_1(1) \times f_2(1)$$

$$\boldsymbol{f(0,2)} = \frac{1}{4}, f_1(0) = f_2(2) = \frac{1}{2} \Rightarrow f(0,2) = f_1(0) \times f_2(2)$$

$$\boldsymbol{f(1,2)} = \frac{1}{4}, f_1(1) = f_2(2) = \frac{1}{2} \Rightarrow f(1,2) = f_1(1) \times f_2(2)$$

**Hence, in this case $X$ and $Y$ are independent.**

## Two dimensional distribution

Now consider the following bivariate discrete probability distribution:

| $y \rightarrow$ / $x \downarrow$ | 1 | 3 | 5 |
|---|---|---|---|
| 0 | 0.02 | 0.06 | 0.12 |
| 2 | 0.03 | 0.09 | 0.18 |
| 6 | 0.05 | 0.15 | 0.30 |

The marginal distributions are

| $x$ | $f_1(x)$ |
|---|---|
| 0 | 0.2 |
| 2 | 0.3 |
| 6 | 0.5 |

| $y$ | $f_2(y)$ |
|---|---|
| 1 | 0.1 |
| 3 | 0.3 |
| 5 | 0.6 |

Observe that

$$f(0,1) = 0.02 = 0.2 \times 0.1 = f_1(0) \times f_2(1).$$

$$f(0,3) = 0.06 = 0.2 \times 0.3 = f_1(0) \times f_2(3).$$

$$f(0,5) = 0.12 = 0.2 \times 0.6 = f_1(0) \times f_2(5).$$

$$f(2,1) = 0.03 = 0.3 \times 0.1 = f_1(2) \times f_2(1).$$

$$f(2,3) = 0.09 = 0.3 \times 0.3 = f_1(2) \times f_2(3).$$

$$f(2,5) = 0.18 = 0.3 \times 0.6 = f_1(2) \times f_2(5).$$

$$f(6,1) = 0.05 = 0.5 \times 0.1 = f_1(6) \times f_2(1).$$

$$f(6,3) = 0.15 = 0.5 \times 0.3 = f_1(6) \times f_2(3).$$

$$f(6,5) = 0.30 = 0.5 \times 0.6 = f_1(6) \times f_2(5).$$

**Since $f(x,y) = f_1(x) f_2(y)$ for all $x, y$, the random variables $X$ and $Y$ are independent.**

## Two dimensional distribution

For the following bivariate discrete probability distribution, the marginal distributions are same as the previous case.

| $y \rightarrow$ / $x \downarrow$ | 1 | 3 | 5 |
|---|---|---|---|
| 0 | 0.02 | 0.06 | 0.12 |
| 2 | 0.03 | 0.08 | 0.19 |
| 6 | 0.05 | 0.16 | **0.29** |

| $x$ | $f_1(x)$ |
|---|---|
| 0 | 0.2 |
| 2 | 0.3 |
| 6 | **0.5** |

| $y$ | $f_2(y)$ |
|---|---|
| 1 | 0.1 |
| 3 | 0.3 |
| 5 | **0.6** |

However, notice that

$$\boldsymbol{f(6,5)} = 0.29 \neq 0.5 \times 0.6 = \boldsymbol{f_1(6)} \times \boldsymbol{f_2(5)}.$$

Since $f(x,y) \neq f_1(x) f_2(y)$ for all $x, y$ the random variables $X$ and $Y$ are not independent.

**Observe that even if**

$$\boldsymbol{f(x,y) \neq f_1(x) f_2(y)}$$

**just for one pair of $(x,y)$, then $X$ and $Y$ cannot be independent.**

**Two dimensional distribution**

*Continuous two-dimensional distributions*

The joint probability density function of $X$ and $Y$ has the property

$$f(x,y) \geq 0, \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dxdy = 1.$$

Here, the total probability 1 is the volume of a three dimensional object.

The marginal pdf of $X$ is given by

$$f_1(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

The marginal pmf of $Y$ is given by

$$f_2(y) = \int_{-\infty}^{\infty} f(x,y)\,dx.$$

$X$ and $Y$ are independent if

$$f(x,y) = f_1(x)f_2(y)$$

for all $x$ and $y$.

**Two dimensional distribution**

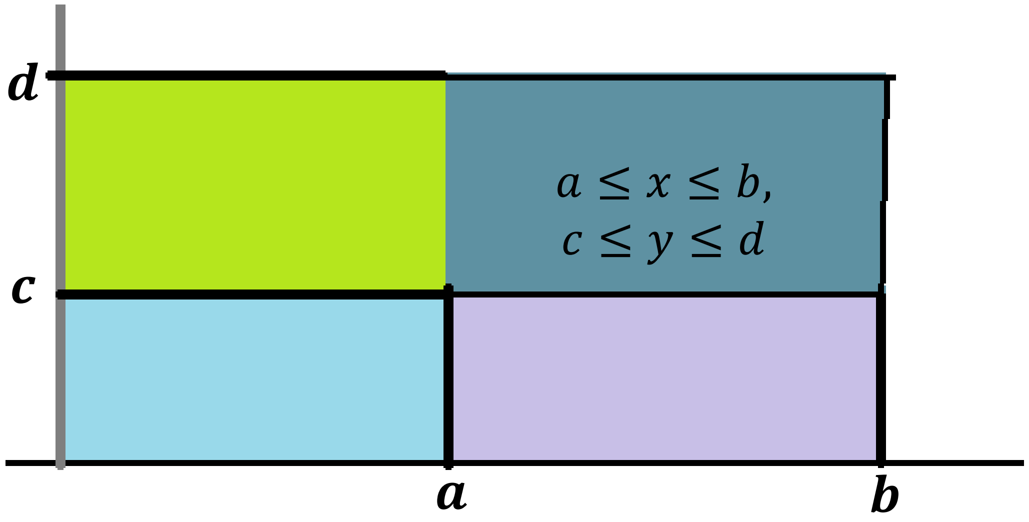
The distribution function of $(X, Y)$ is given by

$$F(x, y) = \Pr\{X \le x, Y \le y\} = \int_{-\infty}^{y}\int_{-\infty}^{x} f(u, v)\, du\, dv$$

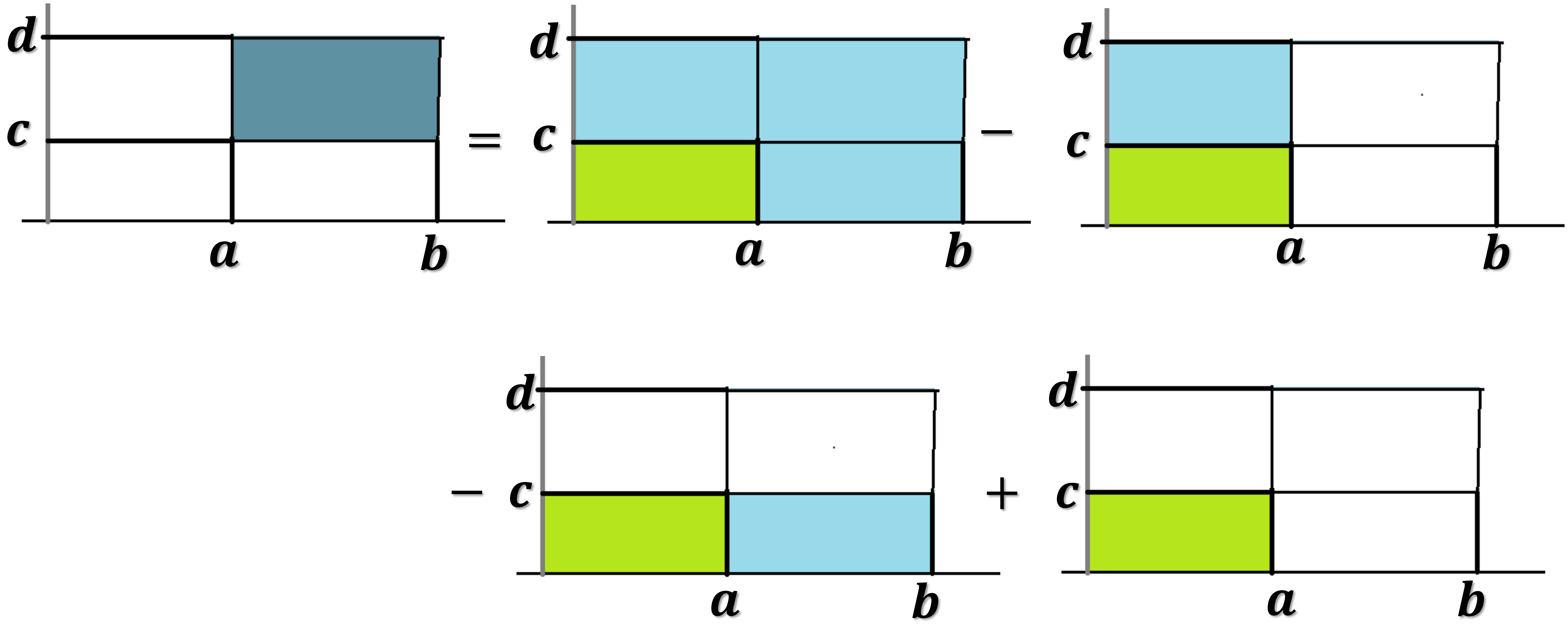
and

$$\Pr\{a \le X \le b, c \le Y \le d\} = \int_{c}^{d}\int_{a}^{b} f(x, y)\, dxdy.$$

Observe that the double integration is performed over the rectangle



## Two dimensional distribution



**Two dimensional distribution**

$$\int_{c}^{d}\int_{a}^{b} f(x,y)\,dxdy = \int_{-\infty}^{d}\int_{-\infty}^{b} f(x,y)\,dxdy$$

$$-\int_{-\infty}^{d}\int_{-\infty}^{a} f(x,y)\,dxdy - \int_{-\infty}^{c}\int_{-\infty}^{b} f(x,y)\,dxdy$$

$$+\int_{-\infty}^{c}\int_{-\infty}^{a} f(x,y)\,dxdy.$$

Hence,

$$\boldsymbol{Pr\{a \le X \le b, c \le Y \le d\} = F(b,d) - F(a,d) - F(b,c) + F(a,c)}.$$

**Two dimensional distribution**

**Example 1:** Let $(X, Y)$ has the pdf $f(x, y) = k$ if $0 < x < 5, 4 < y < 7$ and $f(x, y) = 0$ otherwise. Find $k$, $\Pr\{1 < X \leq 4, 3 \leq Y < 5\}$, $\Pr\{X > 3, Y \leq 5\}$ and $\Pr\{X \geq 5, Y \leq 7\}$.

Ans. Observe that

$$k: \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dxdy = 1 \Rightarrow k\int_{4}^{7}\int_{0}^{5} dx\,dy\ = 1 \Rightarrow 15k = 1 \Rightarrow k = \frac{1}{15}.$$

$$\Pr\{1 < X \leq 4, 3 \leq Y < 5\} = \int_{3}^{5}\int_{1}^{4} f(x,y)\,dxdy = \int_{4}^{5}\int_{1}^{4} \frac{1}{15}\,dxdy = \frac{1}{5}.$$

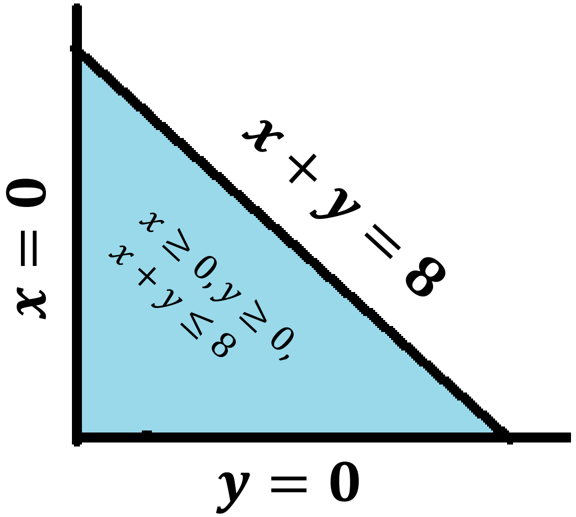
$$\Pr\{X > 3, Y \leq 5\} = \int_{-\infty}^{5}\int_{3}^{\infty} f(x,y)\,dxdy = \int_{4}^{5}\int_{3}^{5} \frac{1}{15}\,dxdy = \frac{2}{15}.$$

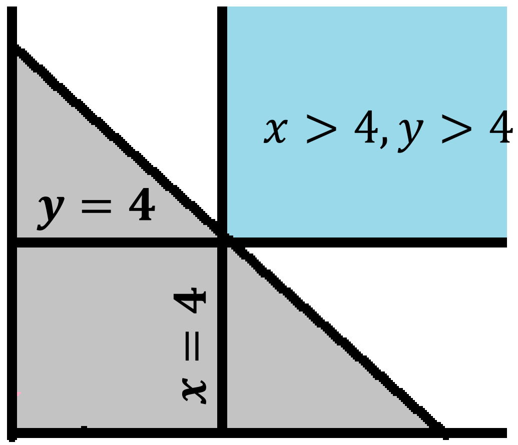
$$\Pr\{X \geq 5, Y \leq 7\} = \int_{5}^{\infty}\int_{-\infty}^{7} f(x,y)\,dxdy = \int_{5}^{\infty}\int_{-\infty}^{7} 0 \cdot dxdy = 0.$$

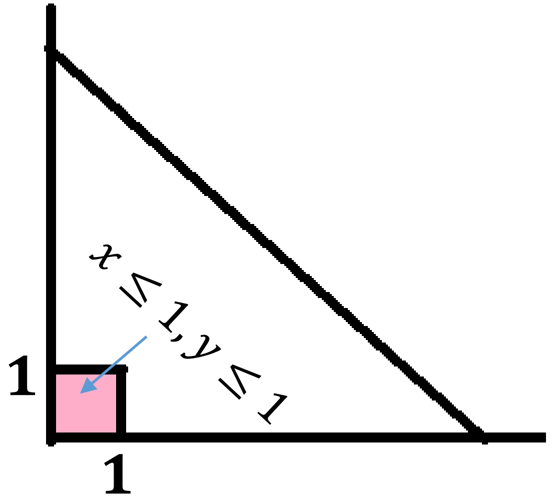
## Two dimensional distribution

**Example 2:** If $(X, Y)$ has the pdf $f(x, y) = \frac{1}{32}$ if $x \geq 0, y \geq 0, x + y \leq 8$ and $f(x, y) = 0$ otherwise. Find $\Pr\{X > 4, Y > 4\}$ and $\Pr\{X \leq 1, Y \leq 1\}$. Also find the marginal pdfs.

Ans. The range of $(X, Y)$ is given in the first figure.




To calculate $\Pr\{X > 4, Y > 4\}$, we have to integrate $f(x, y)$ over the blue region of the second figure where $f(x, y) = 0$. Hence $\Pr\{X > 4, Y > 4\} = 0$.

**Two dimensional distribution**

$$\Pr\{X \leq 1, Y \leq 1\} = \int_{-\infty}^{1}\int_{-\infty}^{1} f(x,y)\,dxdy = \int_{0}^{1}\int_{0}^{1} \frac{1}{32}\,dxdy = 1/32.$$

The marginal pdf of $x$ is given by

$$f_1(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{0}^{8-x} \frac{1}{32}\,dy = \frac{8-x}{32} \text{ if } 0 \leq x \leq 8 \text{ and } f_1(x) = 0 \text{ otherwise.}$$

Similarly,

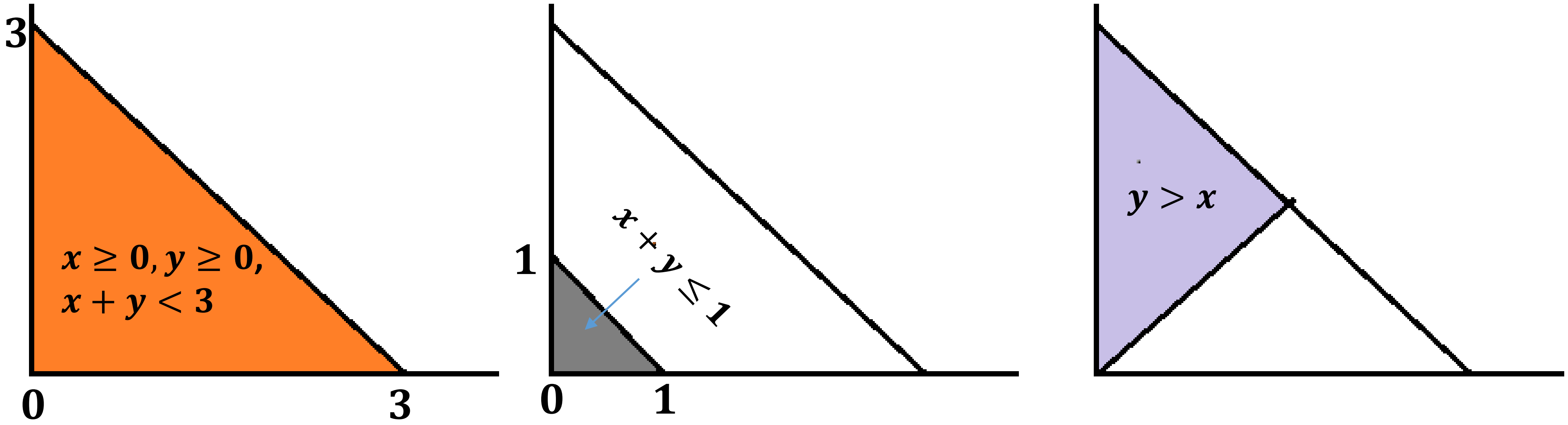
$$f_2(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{0}^{8-y} \frac{1}{32}\,dx = \frac{8-y}{32} \text{ if } 0 \leq y \leq 8 \text{ and } f_2(y) = 0 \text{ otherwise.}$$

**Remark**: Since $f(x,y) \neq f_1(x)f_2(y)$, $X$ and $Y$ are not independent.

**Two dimensional distribution**

**Example 3:** If $f(x,y) = k$ if $x > 0, y > 0, x + y < 3$ and $f(x,y) = 0$ otherwise, find $k$, $\Pr\{X + Y \leq 1\}$ and $\Pr\{Y > X\}$.

Ans. Observe that the nonvanishing region of $f(x,y)$ is a triangle as shown in the first figure:




The equation $f(x,y) = k$ defines a plane parallel to the $xy$-plane situated above at a distance $k$. Hence, the total probability 1 is a prism in three dimension whose base is the triangle marked in the first figure and whose height is $k$. Hence, the volume of the prism is

$$\frac{1}{2} \times 3^2 \times k = 1 \Rightarrow k = \frac{2}{9}.$$

**Two dimensional distribution**

Also, one can obtain $k$ as follows:

$$k: \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dxdy = 1 \Rightarrow \int_0^3\int_0^{3-y} k\, dxdy = 1 \Rightarrow k\int_0^3 (3-y)\, dy = 1 \Rightarrow k = \frac{2}{9}.$$

To find $\Pr\{X+Y \le 1\}$, one has to integrate $f(x,y)$ over the triangle shown in the second figure. This will result in the volume of a prism with base marked in the second figure and height $\frac{2}{9}$. Hence $\Pr\{X+Y \le 1\} = \frac{1}{2} \times 1^2 \times \frac{2}{9} = \frac{1}{9}$. Else, one has to find it as

$$\Pr\{X+Y \le 1\} = \int_0^1\int_0^{1-y} \frac{2}{9}\, dxdy = \frac{2}{9}\int_0^1 (1-y)\, dy.$$

Similarly, using the third figure, $\Pr\{Y > X\} = \frac{1}{2} \times \left(\frac{1}{2} \times 3^2\right) \times \frac{2}{9} = \frac{1}{2}$.

## Two dimensional distribution

**Example 4:** Show that the random variables with densities $f(x, y) = x + y$ and $g(x, y) = \left(x + \frac{1}{2}\right)\left(y + \frac{1}{2}\right)$ if $0 \leq x \leq 1, 0 \leq y \leq 1$ and $f(x, y) = g(x, y) = 0$ otherwise have the same marginal distributions. In the former case, $X$ and $Y$ are not independent whereas, in the later case $X$ and $Y$ are independent.

Ans. If $X$ and $Y$ have the density $f(x, y) = x + y$, if $0 \leq x \leq 1, 0 \leq y \leq 1$ and $f(x, y) = 0$ otherwise, then for $0 \leq x \leq 1$, the marginal pdf of $X$ is

$$f_1(x) = \int_{-\infty}^{\infty} f(x, y)\, dy = \int_0^1 (x + y)\, dy = x + \frac{1}{2}$$

and $f_1(x) = 0$ otherwise. Similarly, for $0 \leq y \leq 1$, the marginal pdf of $Y$ is

$$f_2(y) = \int_{-\infty}^{\infty} f(x, y)\, dx = \int_0^1 (x + y)\, dx = y + \frac{1}{2}$$

and $f_2(y) = 0$ otherwise.

**Two dimensional distribution**

If $X$ and $Y$ have the density $g(x,y) = \left(x+\frac{1}{2}\right)\left(y+\frac{1}{2}\right)$ if $0 \le x \le 1, 0 \le y \le 1$ and $g(x,y) = 0$ otherwise, then for $0 \le x \le 1$, the marginal pdf of $X$ is

$$g_1(x) = \int_{-\infty}^{\infty} g(x,y)\,dy = \int_0^1 \left(x+\frac{1}{2}\right)\left(y+\frac{1}{2}\right)dy = \left(x+\frac{1}{2}\right)\int_0^1 \left(y+\frac{1}{2}\right)dy = x+\frac{1}{2}$$

and $g_1(x) = 0$ otherwise. Similarly, for $0 \le y \le 1$, the marginal pdf of $Y$ is

$$g_2(y) = \int_{-\infty}^{\infty} g(x,y)\,dx = \int_0^1 \left(x+\frac{1}{2}\right)\left(y+\frac{1}{2}\right)dx = \left(y+\frac{1}{2}\right)\int_0^1 \left(x+\frac{1}{2}\right)dx = y+\frac{1}{2}$$

and $g_2(y) = 0$ otherwise.

Hence, the marginal distributions corresponding to $f(x,y)$ and $g(x,y)$ are same. Since, $f(x,y) \neq f_1(x)f_2(y)$ for $0 \le x \le 1, 0 \le y \le 1$ , $X$ and $Y$ are not independent. In the later case, $g(x,y) = g_1(x)g_2(y)$ for all $x, y$, $X$ and $Y$ are independent.

## Two dimensional distribution

**Example 4:** If $X$ and $Y$ have the density $f(x,y) = k$ if $x^2 + y^2 < r^2$ and $f(x,y) = 0$ otherwise then find $k$. Also find $\Pr\{X^2 + Y^2 < r^2/4\}$ and the marginal pdfs of $X$ and $Y$.

Ans. $f(x,y) = k$ if $x^2 + y^2 < r^2$ and $f(x,y) = 0$ otherwise. Here,

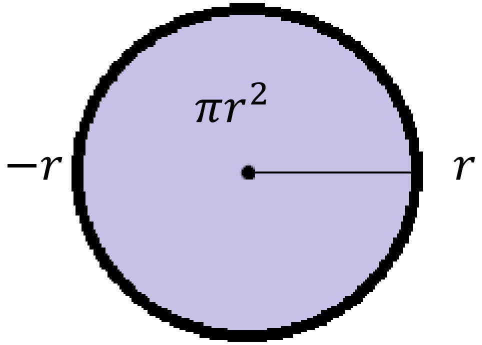
$$k: \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dxdy = 1 \Rightarrow \int_{-r}^{r}\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} k\,dxdy = 1$$

$$\Rightarrow 2k\int_{-r}^{r}\sqrt{r^2-y^2}\,dy = 1$$

$$\Rightarrow 4k\int_{0}^{r}\sqrt{r^2-y^2}\,dy = 1$$

$$\Rightarrow 4kr^2\int_{0}^{\pi/2}\cos^2\theta\,d\theta = 1 \Rightarrow k = \frac{1}{\pi r^2}.$$

$$y = r\sin\theta$$
$$\Rightarrow dy = r\cos\theta$$
$$\Rightarrow \sqrt{r^2-y^2} = r\cos\theta$$
$$y = 0 \Rightarrow \theta = 0$$
$$y = r \Rightarrow \theta = \pi/2$$




Range of $(X,Y)$

**Two dimensional distribution**

For $-r < x < r$,

$$f_1(x) = \int_{-\infty}^{\infty} f(x, y)\, dy = \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \frac{1}{\pi r^2}\, dy = \frac{2\sqrt{r^2 - x^2}}{\pi r^2}$$

and $f_1(x) = 0$ otherwise. Similarly, for $-r < y < r$, the marginal pdf of $Y$ is

$$f_2(y) = \int_{-\infty}^{\infty} f(x, y)\, dx = \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \frac{1}{\pi r^2}\, dx = \frac{2\sqrt{r^2 - y^2}}{\pi r^2}$$

and $f_2(y) = 0$ otherwise.

Since, $f(x, y) \neq f_1(x) f_2(y)$ for $-r < x < r, -r < y < r$, $X$ and $Y$ are not independent.

**Two dimensional distribution**

*Expectation in a joint distribution*

If $g(X,Y)$ is a function of the continuous random variables $X$ and $Y$, then the expectation (arithmetic mean) of $g(X,Y)$ is given by

$$E\{g(X,Y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f(x,y)\,dxdy$$

provided the double integral exists, i.e.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |g(x,y)|\,f(x,y)dxdy < \infty.$$

In particular, subject to the existence condition,

$$E(X+Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y)f(x,y)\,dxdy,$$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)\,dxdy.$$

**Two dimensional distribution**

If the random variables $X$ and $Y$ are discrete, then the expectation (arithmetic mean) of $g(X,Y)$ is given by

$$E\{g(X,Y)\} = \sum_x \sum_y g(x,y)\, f(x,y)$$

provided the double summation exists, i.e.

$$E\{g(X,Y)\} = \sum_x \sum_y |g(x,y)|\, f(x,y) < \infty.$$

In this case

$$E(X+Y) = \sum_x \sum_y (x+y)\, f(x,y)\,,$$

$$E(XY) = \sum_x \sum_y xy\, f(x,y)\,.$$

## Two dimensional distribution

Because of the linearity of the integrals and summations, if $a$ and $b$ are any two constants or at least nonrandom, then

$$E\{ag(X,Y) + bh(X,Y)\} = aE\{g(X,Y)\} + bE\{h(X,Y)\}.$$

In particular, if $g(X,Y) = X$ and $h(X,Y) = Y$, $a = b = 1$, then we get

$$E(X+Y) = E(X) + E(Y).$$

This is known as the addition of expectations and writing it as $\mu_{X+Y} = \mu_X + \mu_Y$, it is known as addition of means. Except existence condition for expectation of $X$ and $Y$, this result is true without further restrictions on $X$ and $Y$. However, for the $E(XY) = E(X)E(Y)$ to hold, in addition to existence of all expectations, we need an extra condition that $X$ and $Y$ be independent.

**Theorem:** If $E(X), E(Y)$ and $E(XY)$ exist and $X$ and $Y$ be independent, then $E(XY) = E(X)E(Y)$.

Proof: We will prove the theorem when $X$ and $Y$ are continuous random variables. If $X$ and $Y$ are discrete random variables, then the double integrations can be replaced by double summations and in that case $dxdy$ is to be dropped.

**Two dimensional distribution**

Since $X$ and $Y$ be independent, $f(x,y) = f_1(x)f_2(y)$ for all $x, y$. Hence

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)\,dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_1(x)f_2(y)\,dxdy$$

$$= \int_{-\infty}^{\infty} yf_2(y)\left\{\int_{-\infty}^{\infty} x\, f_1(x)dx\right\} dy = \int_{-\infty}^{\infty} yf_2(y)\mu_X\,dy$$

$$= \mu_X \int_{-\infty}^{\infty} yf_2(y)\,dy = \mu_X\mu_Y = E(X)E(Y).$$

If $X, Y, Z, \ldots$ are random variables then by a simple mathematical induction, one can check that $E(X + Y + Z + \cdots) = E(X) + E(Y) + E(Z) + \cdots$ and if they are independent, then $E(XYZ\cdots) = E(X)E(Y)E(Z)\cdots$, however, subject to the existence of expectations.

**Two dimensional distribution**

*Variance of a sum*

Observe that the variance of a random variable $X$ is given by $\sigma_z^2 = E(Z-\mu_z)^2$. If $Z = X + Y$, then by the additivity of means, $\mu_Z = \mu_X + \mu_Y$ and hence,

$$\sigma_z^2 = E(Z-\mu_z)^2 = E\{(X+Y)-(\mu_X+\mu_Y)\}^2 = E\{(X-\mu_X)+(Y-\mu_Y)\}^2$$

$$= E\{(X-\mu_X)^2 + (Y-\mu_Y)^2 + 2(X-\mu_X)(Y-\mu_Y)\}$$

$$= E(X-\mu_X)^2 + E(Y-\mu_Y)^2 + 2E\{(X-\mu_X)(Y-\mu_Y)\}$$

The expectation $E\{(X-\mu_X)(Y-\mu_Y)\}$ is known as the covariance between $X$ and $Y$ and is denoted by $Cov(X,Y)$ or by $\sigma_{X,Y}$. Hence,

$$\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2 + 2\sigma_{X,Y}.$$

We can write it as

$$\boldsymbol{Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)}.$$

**Two dimensional distribution**

The covariance of $X$ and $Y$ is defined as

$$Cov(X,Y) = E\{(X-\mu_X)(Y-\mu_Y)\}$$

and reduces to the variance of $X$ if $X = Y$. Furthermore, using linearity of expectation, we get

$$E\{(X-\mu_X)(Y-\mu_Y)\} = E(XY - \mu_X Y - \mu_Y X + \mu_X\mu_Y)$$

$$= E(XY) - \mu_X E(Y) - \mu_Y E(X) + \mu_X\mu_Y E(1)$$

$$= E(XY) - \mu_X\mu_Y - \mu_Y\mu_X + \mu_X\mu_Y = E(XY) - \mu_X\mu_Y.$$

Thus,

$$Cov(X,Y) = E(XY) - \mu_X\mu_Y = E(XY) - E(X)\,E(Y).$$

When $X$ and $Y$ are independent, $E(XY) = E(X)\,E(Y)$ and hence $Cov(X,Y) = 0$ and

$$\boldsymbol{Var(X+Y) = Var(X) + Var(Y)} \text{ or } \boldsymbol{\sigma^2_{X+Y} = \sigma^2_X + \sigma^2_Y}.$$

Thus, in the presence of independence both mean and variance are additive and using induction one can see that $Var(X+Y+\cdots) = Var(X) + Var(Y) + \cdots$.

**Example 5:** Using addition of means and variances of independent random variables, find the mean and variance of the binomial distribution.

Ans. Consider $n$ Bernoulli trials with probability of success $p$ in each trial. For $i = 1,2, \dots, n$ let

$$X_i = \begin{cases} 0, & \text{if the } i^{\text{th}} \text{ trial is a failure} \\ 1, & \text{if the } i^{\text{th}} \text{ trial is a success.} \end{cases}$$

Then, $X_1, X_2, \dots, X_n$ are independent random variables and if $X = X_1 + X_2 + \cdots + X_n$, then $X \sim B(n, p)$. For each $i$, $\Pr\{X_i = 0\} = 1 - p = q$ (say) and $\Pr\{X_i = 1\} = p$. Hence,

$$E(X_i) = 0 \times q + 1 \times p = p, E\left(X_i^2\right) = 0^2 \times q + 1^2 \times p = p$$

for $i = 1,2, \dots, n$. Hence,

$$Var(X_i) = E\left(X_i^2\right) - E^2(X_i) = p - p^2 = pq.$$

Now, by additivity of means and variances for independent random variables,

$$E(X) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} p = np, Var(X) = \sum_{i=1}^{n} Var(X_i) = \sum_{i=1}^{n} pq = npq.$$

## Two dimensional distribution

**Example 6:** If certain sheets of wrapping paper have a mean weight of 10 g each, with a standard deviation of 0.05 g, what are the mean weight and standard deviation of a pack of 10,000 sheets?

Ans. Let $X_i$ be the weight of $i$-th paper sheet, $i = 1,2,\ldots,10{,}000$ and let $X = X_1 + X_2 + \cdots + X_{10000}$. Then $X_1, X_2, \ldots, X_{10000}$ are independent random variables each with mean 10 and standard deviation 0.05 (gram), i.e.

$$E(X_i) = 10, Var(X_i) = 0.05^2, i = 1,2,\ldots,10{,}000.$$

Type equation here.. By additivity of mean and variance,

$$E(X) = E(X_1) + E(X_2) + \cdots + E(X_{10000})$$

$$= 10{,}000 \times 10 = 1{,}00{,}000 \quad \text{(gm)}$$

$$Var(X) = Var(X_1) + Var(X_2) + \cdots + Var(X_{10000})$$

$$= 10{,}000 \times 0.05^2 = 25 \ \ (gm^2)$$

Hence, mean weight of 10,000 sheets is 100kg with standard deviation of 5 gm.

**Example 7:** If $X$ and Y are independent random variables such that $X \sim B\left(64, \frac{1}{4}\right)$ and $Y \sim \gamma(0.1, 7.5)$, find the mean and standard deviations of $10X - 4Y$.

Ans. Since $X \sim B\left(64, \frac{1}{4}\right)$, $n = 64, p = 1/4$ and

$$E(X) = np = 16, Var(X) = npq = 12.$$

Similarly, since $Y \sim \gamma(0.1, 7.5)$, $\lambda = 0.1$ and $\alpha = 7.5$. Hence,

$$E(X) = \frac{\alpha}{\lambda} = 75, Var(X) = \frac{\alpha}{\lambda^2} = 750.$$

Thus, by linearity of expectations,

$$E(10X - 4Y) = 10 \times E(X) + (-4) \times E(Y) = 10 \times 16 - 4 \times 75 = -140.$$

Since $X$ and $Y$ are independent, $10X$ and $-4Y$ are also independent. Hence,

$$Var(10X - 4Y) = Var(10X) + Var(-4Y) = 10^2 \times Var(X) + (-4)^2 \times Var(Y)$$

$$= 100 \times 12 + 16 \times 750 = 13{,}200 \Rightarrow S.D.(10X - 4Y) = \sqrt{13{,}200} = 114.89.$$

## Two dimensional distribution

**Example 8:** What are the mean thickness and the standard deviation of transformer cores each consisting of 50 layers of sheet metal and 49 insulating paper layers if the metal sheets have mean thickness 0.5 mm each with a standard deviation of 0.05 mm and the paper layers have mean 0.05 mm each with a standard deviation of 0.02 mm?

Ans. Let $X_i$ be the thickness of $i$-th metal sheet and $Y_j$, the thickness of $j$-th insulating paper layer $i = 1,2,\dots,50$ and $j = 1,2,\dots,49$. Then the thickness of the transformer core is equal to

$$Z = X_1 + X_2 + \cdots + X_{50} + Y_1 + Y_2 + \cdots + Y_{49}$$

Observe that $X_1, X_2, \dots, X_{50}, Y_1, Y_2, \dots, Y_{49}$ are independent random variables and for each $i, j$

$$E(X_i) = 0.5, Var(X_i) = 0.05^2, E(Y_j) = 0.05, Var(Y_j) = 0.02^2.$$

Hence,

$$E(Z) = E(X_1) + E(X_2) + \cdots + E(X_{50}) + E(Y_1) + E(Y_2) + \cdots + E(Y_{49})$$

$$= 50 \times 0.5 + 49 \times 0.05 = 27.45 \text{ (mm)}$$

$$Var(Z) = Var(X_1) + Var(X_2) + \cdots + Var(X_{50}) + Var(Y_1) + Var(Y_2) + \cdots + Var(Y_{49})$$

$$= 50 \times 0.05^2 + 49 \times 0.02^2 = 0.1446 \text{ (mm}^2\text{)} \Rightarrow S.D.(Z) = 0.38 \text{ mm}.$$

## Two dimensional distribution

**Ex 9:** A 5-gear assembly is put together with spacers between the gears. The mean thickness of the gears is 5.020 cm with a standard deviation of 0.003 cm. The mean thickness of the spacers is 0.040 cm with a standard deviation of 0.002 cm. Find the mean and standard deviation of the assembled units consisting of 5 randomly selected gears and 4 randomly selected spacers. (similar to Example 8, you do it.)

**Ex 10.** If the mean weight of certain (empty) containers is 5 lb the standard deviation is 0.2 lb, and if the filling of the containers has mean weight 100 lb and standard deviation 0.5 lb, what are the mean weight and the standard deviation of filled containers? (similar to Example 8, you do it.)

**Ex 11.** Let $X$ [cm] and $Y$ [cm] be the diameters of a pin and hole, respectively. Suppose that $(X, Y)$ has the density $f(x, y) = 625$ if $0.98 < x < 1.02, 1.00 < y < 1.02$ and $f(x, y)$ =0 otherwise. (a) Find the marginal distributions. (b) What is the probability that a pin chosen at random will fit the hole whose diameter is 1 cm.

**Ex 11.** Let $(X, Y)$ have the probability function $f(0,0) = f(1,1) = 1/8$ and $f(0,1) = f(1,0) = 3/8$. Are $X$ and $Y$ independent? (use a table to find the marginal distributions.)

**Two dimensional distribution**

*Conditional distributions*

Given the joint distribution of two discrete random variables $X$ and $Y$, it is sometimes necessary to find the distribution of one variable given that the other variable is fixed at some value in its range. If this value is arbitrary, the conditional distribution becomes a function of it.

Let $f(x, y) = \Pr\{X = x, Y = y\}$ joint probability mass function of $X$ and $Y$. Consider finding the probability $\Pr\{X = x|Y = y\}$ where $\Pr\{Y = y\} = f_2(y) > 0$. Using the formula for conditional probability, we can have

$$\Pr\{X = x|Y = y\} = \frac{\Pr\{X = x, Y = y\}}{\Pr\{Y = y\}} = \frac{f(x, y)}{f_2(y)}.$$

$\Pr\{X = x|Y = y\}$ is denoted by $f_{X|Y}(x|y)$. (In particular, $\Pr\{X = 5|Y = 2\}$ is denoted by $f_{X|Y}(5|2)$. ) Thus,

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_2(y)}, f_{Y|X}(y|x) = \frac{f(x, y)}{f_1(x)}.$$

**Two dimensional distribution**

*Conditional distributions*

If $f(x, y)$ is the joint probability density function of $X$ and $Y$, then the conditional pdf of $X$ given $Y = y$ with $f_2(y) > 0$ is defined as

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_2(y)}$$

and similarly, the conditional pdf of $Y$ given $X = x$ with $f_1(x) > 0$ is defined as

$$f_{Y|X}(y|x) = \frac{f(x, y)}{f_1(x)}.$$

The conditional distributions are univariate distributions. Unlike discrete distributions, none of $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$ is a probability when the joint distribution of $X$ and $Y$ is continuous.

## Two dimensional distribution

**Example 12**: Let $X$ and $Y$ be random variables such that $f(0,0) = 0.1, f(0,1) = 0.2$, $f(1,0) = 0.3$ and $f(1,1) = 0.4$. Find the conditional distribution of $X$ given $Y = 1$ and that of $Y$ given $X = 0$.

Ans. We can write the joint pmf in a tabular form as follows:

| $x \downarrow$ | $y \rightarrow$ | 0 | 1 | $f_1(x) \downarrow$ |
|---|---|---|---|---|
| 0 | | 0.1 | 0.2 | 0.3 |
| 1 | | 0.3 | 0.4 | 0.7 |
| $f_2(y) \rightarrow$ | | 0.4 | 0.6 | 1 |

Conditional distribution of $X$ given $Y = 1$.

$$f_{X|Y}(x|1) = \frac{f(x,1)}{f_2(1)}.$$

Hence,

$$f_{X|Y}(0|1) = \frac{f(0,1)}{f_2(1)} = \frac{0.2}{0.6} = \frac{1}{3},$$

$$f_{X|Y}(1|1) = \frac{f(1,1)}{f_2(1)} = \frac{0.4}{0.6} = \frac{2}{3}.$$

## Two dimensional distribution

### Conditional distribution of $Y$ given $X = 0$.

$$f_{Y|X}(y|0) = \frac{f(0,y)}{f_1(0)}.$$

Hence,

$$f_{Y|X}(0|0) = \frac{f(0,0)}{f_1(0)} = \frac{0.1}{0.3} = \frac{1}{3},$$

$$f_{Y|X}(1|0) = \frac{f(0,1)}{f_1(0)} = \frac{0.2}{0.3} = \frac{2}{3}.$$

### Conditional distribution of $X$ given $Y = 1$.

| $x$ | $f_{X\|Y}(x\|1)$ |
|---|---|
| 0 | 1/3 |
| 1 | 2/3 |

### Conditional distribution of $Y$ given $X = 0$.

| $y$ | $f_{Y\|X}(y\|0)$ |
|---|---|
| 0 | 1/3 |
| 1 | 2/3 |

**Two dimensional distribution**

**Example 13:** If $(X, Y)$ has the pdf $f(x, y) = \frac{1}{32}$ if $x \geq 0, y \geq 0, x + y \leq 8$ and $f(x, y) = 0$ otherwise, find the conditional density of $X$ given $Y = y$ for $0 < y < 8$ and that of $Y$ given $X = 2$.

Ans. The marginal densities of $X$ and $Y$ are given by

$$f_1(x) = \begin{cases} \frac{8-x}{32} & \text{if } 0 \leq x \leq 8 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f_2(y) = \begin{cases} \frac{8-y}{32} & \text{if } 0 \leq y \leq 8 \\ 0 & \text{otherwise.} \end{cases}$$

The conditional density of $X$ given $Y = y$ is given by

$f_{X|Y}(x|y) = \frac{f(x,y)}{f_2(y)} = \frac{1/32}{(8-y)/32} = \frac{1}{8-y}\, if\ 0 \leq x \leq 8 - y$ and $f_{X|Y}(x|y) = 0$ otherwise.

Similarly, the conditional pdf of $Y$ given $X = 2$ is given by

$f_{Y|X}(y|2) = \frac{f(2,y)}{f_1(2)} = \frac{1/32}{(8-2)/32} = \frac{1}{6}$ if $0 \leq y \leq 6$ and $f_{X|Y}(y|2) = 0$ otherwise.

## Two dimensional distribution

**Example 14:** If $(X, Y)$ has the pmf $f(x, y) = k(x^2 + y)$ if $x = 0,1,2,3, y = 0,1,2,3$, and $x + y \leq 4$ and $f(x, y) = 0$ otherwise, find the conditional pmf of $X$ given $Y = 1$ and that of $Y$ given $X = 3$.

Ans. The joint distribution of $X$ and $Y$ can be described in tabular form as follows:

| $x \downarrow y \rightarrow$ | **0** | **1** | **2** | **3** | |
|---|---|---|---|---|---|
| **0** | $\mathbf{0}$ | $\boldsymbol{k}$ | $\boldsymbol{2k}$ | $\boldsymbol{3k}$ | |
| **1** | $\boldsymbol{k}$ | $\boldsymbol{2k}$ | $\boldsymbol{3k}$ | $\boldsymbol{4k}$ | |
| **2** | $\boldsymbol{4k}$ | $\boldsymbol{5k}$ | $\boldsymbol{6k}$ | $\mathbf{0}$ | |
| **3** | $\boldsymbol{9k}$ | $\boldsymbol{10k}$ | $\mathbf{0}$ | $\mathbf{0}$ | |
| | | | | | $\boldsymbol{50k}$ |

Observe that $f(2,3) = f(3,2) = f(3,3) = 0$ since for $(x, y) = (2,3), (3,2), (3,3), x + y > 4$. $f(0,0) = 0$ since $k(0^2 + 0) = 0$. The total probability i.e.

$$\sum_x \sum_y f(x, y) = 50k = 1.$$

Hence, $\boldsymbol{k = \frac{1}{50}}$.

## Two dimensional distribution

On substituting $k = 1/50$, we have the following table for the joint pmf of $(X, Y)$ and the marginal pmfs of $X$ and $Y$.

| $x \downarrow y \rightarrow$ | **0** | **1** | **2** | **3** | $f_1(x)$ |
|---|---|---|---|---|---|
| **0** | 0 | 0.02 | 0.04 | 0.06 | **0.12** |
| **1** | 0.02 | 0.04 | 0.06 | 0.08 | **0.20** |
| **2** | 0.08 | 0.10 | 0.12 | 0 | **0.30** |
| **3** | 0.18 | 0.20 | 0 | 0 | **0.38** |
| $f_2(y)$ | **0.28** | **0.36** | **0.22** | **0.14** | $\mathbf{1.00}$ |

| $x$ | $f_1(x)$ |
|---|---|
| 0 | 0.12 |
| 1 | 0.20 |
| 2 | 0.30 |
| 3 | 0.38 |

| $y$ | $f_2(y)$ |
|---|---|
| 0 | 0.28 |
| 1 | 0.36 |
| 2 | 0.22 |
| 3 | 0.14 |

## Two dimensional distribution

| $x\downarrow y\rightarrow$ | **0** | **1** | **2** | **3** | $f_1(x)$ |
|---|---|---|---|---|---|
| **0** | 0 | 0.02 | 0.04 | 0.06 | **0.12** |
| **1** | 0.02 | 0.04 | 0.06 | 0.08 | **0.20** |
| **2** | 0.08 | 0.10 | 0.12 | 0 | **0.30** |
| **3** | 0.18 | 0.20 | 0 | 0 | **0.38** |
| $f_2(y)$ | **0.28** | **0.36** | **0.22** | **0.14** | **1.00** |

| $y$ | $f_2(y)$ |
|---|---|
| 0 | 0.28 |
| 1 | 0.36 |
| 2 | 0.22 |
| 3 | 0.14 |

| $x$ | $f_{X\|Y}(x\|1)$ |
|---|---|
| 0 | 1/18 |
| 1 | 2/18 |
| 2 | 5/18 |
| 3 | 10/18 |

The conditional pmf of $X$ given $Y = 1$ is given by

$$f_{X|Y}(x|1) = \frac{f(x,1)}{f_2(1)}.$$

Hence,

$$f_{X|Y}(0|1) = \frac{f(0,1)}{f_2(1)} = \frac{0.02}{0.36} = \frac{1}{18}, \qquad f_{X|Y}(1|1) = \frac{f(1,1)}{f_2(1)} = \frac{0.04}{0.36} = \frac{1}{9},$$

$$f_{X|Y}(2|1) = \frac{f(2,1)}{f_2(1)} = \frac{0.10}{0.36} = \frac{5}{18}, \qquad f_{X|Y}(3|1) = \frac{f(3,1)}{f_2(1)} = \frac{0.20}{0.36} = \frac{5}{9}.$$

## Two dimensional distribution

| $x \downarrow y \rightarrow$ | 0 | 1 | 2 | 3 | $f_1(x)$ |
|---|---|---|---|---|---|
| 0 | 0 | 0.02 | 0.04 | 0.06 | 0.12 |
| 1 | 0.02 | 0.04 | 0.06 | 0.08 | 0.20 |
| 2 | 0.08 | 0.10 | 0.12 | 0 | 0.30 |
| 3 | 0.18 | 0.20 | 0 | 0 | 0.38 |
| $f_2(y)$ | 0.28 | 0.36 | 0.22 | 0.14 | 1.00 |

| $x$ | $f_1(x)$ |
|---|---|
| 0 | 0.12 |
| 1 | 0.20 |
| 2 | 0.30 |
| 3 | 0.38 |

| $y$ | $f_{Y\|X}(y\|3)$ |
|---|---|
| 0 | 9/19 |
| 1 | 10/19 |

Observe that when $X$ is fixed at 3, the only allowed values of $Y$ are 0 and 1 since for $X = 2, 3,\ X + Y$ exceeds 4.

The conditional pmf of $Y$ given $X = 3$ is given by

$$f_{Y|X}(y|3) = \frac{f(3,y)}{f_1(3)}.$$

Hence,

$$f_{Y|X}(0|3) = \frac{f(3,0)}{f_1(3)} = \frac{0.18}{0.38} = \frac{9}{19}, \qquad f_{Y|X}(1|3) = \frac{f(3,1)}{f_1(3)} = \frac{0.20}{0.38} = \frac{`10}{19},$$

$$f_{Y|X}(2|3) = \frac{f(3,2)}{f_1(3)} = \frac{0}{0.38} = 0, \qquad f_{Y|X}(3|3) = \frac{f(3,0)}{f_1(3)} = \frac{0}{0.38} = 0.$$

**Two dimensional distribution**

**Example 15:** If $(X, Y)$ has the pdf $f(x, y) = Kxy$ if $x \geq 0, y \geq 0, x + y \leq 1$ and $f(x, y) = 0$ otherwise, find $K$, the conditional density of $X$ given $Y = 0.25$ and that of $Y$ given $X = 0.5$.

Ans. Observe that

$$K: \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dxdy = 1 \Rightarrow K \int_0^1 \int_0^{1-y} xy\, dxdy = 1$$

$$\Rightarrow K \int_0^1 y \left\{ \int_0^{1-y} x\, dx \right\} dy = 1 \Rightarrow \frac{K}{2} \int_0^1 y(1-y)^2\, dy = 1$$

$$\frac{K}{2} \int_0^1 (y - 2y^2 + y^3)\, dy = 1 \Rightarrow \frac{K}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = 1$$

$$\frac{K}{24} = 1 \Rightarrow K = 24$$

**Two dimensional distribution**

For $0 < x < 1$,

$$f_1(x) = \int_{-\infty}^{\infty} f(x, y)\, dy = \int_0^{1-x} 24xy\, dy$$

$$= 24x \int_0^{1-x} y\, dy = 12x(1 - x)^2$$

and $f_1(x) = 0$ otherwise. Similarly, for $0 < y < 1$, the marginal pdf of $Y$ is

$$f_2(y) = \int_{-\infty}^{\infty} f(x, y)\, dx = \int_0^{1-y} 24xy\, dx$$

$$= 24y \int_0^{1-y} x\, dx = 12y(1 - y)^2$$

and $f_2(y) = 0$ otherwise.

**Two dimensional distribution**

The conditional pdf of $X$ given $Y = y$ is

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_2(y)} = \frac{24xy}{12y(1-y)^2} = \frac{2x}{(1-y)^2}$$

if $0 < x < 1 - y$ and $f_{X|Y}(x|y) = 0$ otherwise. Hence, the conditional pdf of $X$ given $Y = 0.25$ is

$$f_{X|Y}(x|0.25) = \frac{2x}{(1-0.25)^2} = \frac{32}{9}x$$

if $0 < x < 0.75$ and $f_{X|Y}(x|0.25) = 0$ otherwise. In other words,

$$f_{X|Y}(x|0.25) = \begin{cases} \frac{32}{9}x & \text{if } 0 < x < \frac{3}{4} \\ 0 & \text{otherwise.} \end{cases}$$

**Two dimensional distribution**

The conditional pdf of $Y$ given $X = x$ is

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_1(x)} = \frac{24xy}{12x(1-x)^2} = \frac{2y}{(1-x)^2}$$

if $0 < y < 1 - x$ and $f_{Y|X}(y|x) = 0$ otherwise. Hence, the conditional pdf of $Y$ given $X = 0.5$ is

$$f_{Y|X}(y|0.5) = \frac{f(0.5,y)}{f_1(0.5)} = \frac{2y}{(1-0.5)^2} = 8y$$

if $0 < y < 0.5$ and $f_{Y|X}(y|0.5) = 0$ otherwise.

$$f_{Y|X}(y|0.5) = \begin{cases} 8y \text{ if } 0 < y < 0.5 \\ 0 \qquad \text{otherwise.} \end{cases}$$

## Two dimensional distribution

*Conditional expectation and conditional variance*

If $X$ and $Y$ are two **discrete random variables**, then the conditional expectation of $X$ given $Y = y$ is a function of $y$ and is defined as

$$\mu_{X|y} = E(X|Y = y) = \sum_{x} x\, f_{X|Y}(x|y)$$

and the conditional expectation of $Y$ given $X = x$ is a function of $x$ and is defined as

$$\mu_{Y|x} = E(Y|X = x) = \sum_{y} y\, f_{Y|X}(y|x).$$

If $X$ and $Y$ are **continuous**, then

$$\mu_{X|y} = E(X|Y = y) = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)dx$$

and

$$\mu_{Y|x} = E(Y|X = x) = \int_{-\infty}^{\infty} y\, f_{Y|X}(y|x)dy.$$

## Two dimensional distribution

In general, if $X$ and $Y$are discrete, then the conditional expectation of g($X$) given $Y = y$ is defined as

$$E(g(X)|Y = y) = \sum_x g(x)\, f_{X|Y}(x|y)$$

and the conditional expectation of $h(Y)$ given $X = x$ is is defined as

$$E(h(Y)|X = x) = \sum_y h(y)\, f_{Y|X}(y|x).$$

If $X$ and $Y$are continuous, then

$$E(g(X)|Y = y) = \int_{-\infty}^{\infty} g(x)\, f_{X|Y}(x|y)dx$$

and

$$E(h(Y)|X = x) = \int_{-\infty}^{\infty} h(y)\, f_{Y|X}(y|x)dy.$$

**Two dimensional distribution**

Similarly, if $X$ and $Y$ are discrete, then the conditional variance of $X$ given $Y = y$ is also a function of $y$ and is defined as

$$\sigma^2_{X|y} = Var(X|Y = y) = E\left\{\left(X - \mu_{X|y}\right)^2 |Y = y\right\} = \sum_x \left(x - \mu_{X|y}\right)^2 f_{X|Y}(x|y)$$

and the conditional variance of $Y$ given $X = x$ is also function of $x$ and is defined as

$$\sigma^2_{Y|x} = Var(Y|X = x) = E\left\{\left(Y - \mu_{Y|x}\right)^2 |X = x\right\} = \sum_y \left(y - \mu_{Y|x}\right)^2 f_{Y|X}(y|x).$$

If $X$ and $Y$ are continuous, then

$$\sigma^2_{X|y} = Var(X|Y = y) = E\left\{\left(X - \mu_{X|y}\right)^2 |Y = y\right\} = \int_{-\infty}^{\infty} \left(x - \mu_{X|y}\right)^2 f_{X|Y}(x|y)dx$$

and

$$\sigma^2_{Y|x} = Var(Y|X = x) = E\left\{\left(Y - \mu_{Y|x}\right)^2 |X = x\right\} = \int_{-\infty}^{\infty} \left(y - \mu_{Y|x}\right)^2 f_{Y|X}(y|x)dy.$$

**Two dimensional distribution**

*Conditional expectation and conditional variance*

Observe that

$$\sigma_{X|y}^2 = Var(X|Y=y) = \int_{-\infty}^{\infty} \left(x-\mu_{X|y}\right)^2 f_{X|Y}(x|y)dx$$

$$= \int_{-\infty}^{\infty} \left(x^2 - 2x\mu_{X|y} + \mu_{X|y}^2\right) f_{X|Y}(x|y)dx$$

$$= \int_{-\infty}^{\infty} x^2 f_{X|Y}(x|y)dx - 2\mu_{X|y}\int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)dx + \mu_{X|y}^2 \int_{-\infty}^{\infty} f_{X|Y}(x|y)\,dx$$

$$= \int_{-\infty}^{\infty} x^2 f_{X|Y}(x|y)dx - \mu_{X|y}^2 = E(X^2|Y=y) - E^2(X|Y=y).$$

which is analogous to

$$\sigma^2 = E(X^2) - \mu^2.$$

Similar thing happens if $X$ and $Y$ are discrete.

# Two dimensional distribution

**Example 16**: Let $X$ and $Y$ be random variables such that $f(0,0) = 0.1, f(0,1) = 0.2,$ $f(1,0) = 0.3$ and $f(1,1) = 0.4$. Find the conditional expectation and conditional variance of $X$ given $Y = 1$ and those of $Y$ given $X = 0$.

Ans. The conditional distributions of $X$ given $Y = 1$ and of $Y$ given $X = 0$ are (from Ex. 12)

| $x$ | $f_{X\|Y}(x\|1)$ |
|---|---|
| **0** | **1/3** |
| **1** | **2/3** |

| $y$ | $f_{Y\|X}(y\|0)$ |
|---|---|
| **0** | **1/3** |
| **1** | **2/3** |

Hence,

$$E(X|Y=1) = \sum_x x\, f_{X|Y}(x|1) = 0 \times f_{X|Y}(0|1) + 1 \times f_{X|Y}(1|1) = \mathbf{0} \times \frac{\mathbf{1}}{\mathbf{3}} + \mathbf{1} \times \frac{\mathbf{2}}{\mathbf{3}} = \frac{\mathbf{2}}{\mathbf{3}}.$$

$$E(Y|X=0) = \sum_y y\, f_{Y|X}(y|0) = 0 \times f_{Y|X}(0|0) + 1 \times f_{Y|X}(1|0) = \mathbf{0} \times \frac{\mathbf{1}}{\mathbf{3}} + \mathbf{1} \times \frac{\mathbf{2}}{\mathbf{3}} = \frac{\mathbf{2}}{\mathbf{3}}.$$

Hence,

$$E(X^2|Y=1) = \sum_x x^2 f_{X|Y}(x|1) = 0^2 \times f_{X|Y}(0|1) + 1^2 \times f_{X|Y}(1|1) = 0 \times \frac{1}{3} + 1 \times \frac{2}{3} = \frac{2}{3}$$

and

$$E(Y^2|X=0) = \sum_y y^2 f_{Y|X}(y|0) = 0^2 \times f_{Y|X}(0|0) + 1^2 \times f_{Y|X}(1|0) = 0 \times \frac{1}{3} + 1 \times \frac{2}{3} = \frac{2}{3}.$$

Thus,

$$Var(X|Y=1) = E(X^2|Y=1) - E^2(X|Y=1) = \frac{2}{3} - \frac{4}{9} = \frac{2}{9}$$

and

$$Var(Y|X=0) = E(Y^2|X=1) - E^2(Y|X=1) = \frac{2}{3} - \frac{4}{9} = \frac{2}{9}.$$

Note: $E(X|Y=1) = E(Y|X=0)$ and $Var(X|Y=1) = Var(Y|X=0)$ are mere coincidences.

**Two dimensional distribution**

**Example 17:** If $(X,Y)$ has the pmf $f(x,y) = \frac{x^2+y}{50}$ if $x = 0,1,2,3, y = 0,1,2,3,$ and $x + y \le 4$ and $f(x,y) = 0$ otherwise, find the conditional expectations and variances of $X$ given $Y = 1$ and $Y$ given $X = 3$.

Ans. The conditional distributions of $X$ given $Y = 1$ and of $Y$ given $X = 0$ are (from Ex. 14)

| $\boldsymbol{x}$ | $\boldsymbol{f_{X\|Y}(x\|1)}$ |
|---|---|
| 0 | 1/18 |
| 1 | 2/18 |
| 2 | 5/18 |
| 3 | 10/18 |

| $\boldsymbol{y}$ | $\boldsymbol{f_{Y\|X}(y\|3)}$ |
|---|---|
| 0 | 9/19 |
| 1 | 10/19 |

Hence,

$$E(X|Y=1) = \sum_x x\, f_{X|Y}(x|1) = 0 \times f_{X|Y}(0|1) + 1 \times f_{X|Y}(1|1) + 2 \times f_{X|Y}(2|1)$$

$$+3 \times f_{X|Y}(3|1) = 0 \times \frac{1}{18} + 1 \times \frac{2}{18} + 2 \times \frac{5}{18} + 3 \times \frac{10}{18} = \frac{7}{3}.$$

## Two dimensional distribution

$$E(Y|X=3)=\sum_{y} y\, f_{Y|X}(y|3)=0\times f_{Y|X}(0|3)+1\times f_{Y|X}(1|3)$$

$$=0\times\frac{9}{19}+1\times\frac{10}{19}=\frac{10}{19}.$$

$$E(X^2|Y=1)=\sum_{x} x^2\, f_{X|Y}(x|1)=0^2\times f_{X|Y}(0|1)+1^2\times f_{X|Y}(1|1)+2^2\times f_{X|Y}(2|1)$$

$$+3^2\times f_{X|Y}(3|1)=0\times\frac{1}{18}+1\times\frac{2}{18}+4\times\frac{5}{18}+9\times\frac{10}{18}=\frac{56}{9}.$$

| 12 | $f_{Y\|X}(y\|3)$ |
|---|---|
| 0 | 9/19 |
| 1 | 10/19 |

| $x$ | $f_{X\|Y}(x\|1$ |
|---|---|
| 0 | 1/18 |
| 1 | 2/18 |
| 2 | 5/18 |
| 3 | 10/18 |

$$E(Y^2|X=3)=\sum_{y} y^2\, f_{Y|X}(y|3)=0^2\times f_{Y|X}(0|3)+1^2\times f_{Y|X}(1|3)$$

$$=0\times\frac{9}{19}+1\times\frac{10}{19}=\frac{10}{19}.$$

Hence,

$$Var(X|Y=1)=E(X^2|Y=1)-E^2(X|Y=1)=\frac{56}{9}-\frac{49}{9}=\frac{7}{9},$$

$$Var(Y|X=3)=E(Y^2|X=3)-E^2(Y|X=3)=\frac{10}{19}-\frac{100}{361}=\frac{90}{361}.$$

## Two dimensional distribution

**Example 18:** If $(X, Y)$ has the pdf $f(x, y) = \frac{1}{32}$ if $x \geq 0, y \geq 0, x + y \leq 8$ and $f(x, y) = 0$ otherwise, find the conditional expectations and variances of $X$ given $Y = y$ for $0 < y < 8$ and $Y$ given $X = 2$.

Ans. In view of Ex 13, the conditional distributions of $X$ given $Y = y$ and $Y$ given $X = 2$ are

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{1}{8-y} & \text{if } 0 \leq x \leq 8-y \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_{Y|X}(y|2) = \begin{cases} \dfrac{1}{6} & \text{if } 0 \leq y \leq 6 \\ 0 & \text{otherwise.} \end{cases}$$

**Two dimensional distribution**

Hence,

$$E(X|Y=y)=\int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)dx=\int_{0}^{8-y}\frac{x}{8-y}dx=\frac{8-y}{2},$$

$$E(Y|X=2)=\int_{-\infty}^{\infty} y\, f_{Y|X}(y|2)dy=\int_{0}^{6} y\cdot\frac{1}{6}dy=3,$$

$$E(X^2|Y=y)=\int_{-\infty}^{\infty} x^2\, f_{X|Y}(x|y)dx=\int_{0}^{8-y}\frac{x^2}{8-y}dx=\frac{(8-y)^2}{3}$$

and

$$E(Y^2|X=2)=\int_{-\infty}^{\infty} y^2\, f_{Y|X}(y|2)dy=\int_{0}^{6} y^2\cdot\frac{1}{6}dy=12.$$

Hence,

$$Var(X|Y=y)=E(X^2|Y=y)-E^2(X|Y=y)=\frac{(8-y)^2}{3}-\left(\frac{8-y}{2}\right)^2=\frac{(8-y)^2}{12}$$

$$Var(Y|X=2)=E(Y^2|X=2)-E^2(Y|X=2)=12-3^2=3.$$

**Two dimensional distribution**

**Example 19:** If $(X, Y)$ has the pdf $f(x,y) = 24xy$ if $x \geq 0, y \geq 0,\ x + y \leq 1$ and $f(x,y) = 0$ otherwise, find the conditional expectations and variances of $X$ given $Y = 0.25$ and $Y$ given $X = 0.5$.

Ans. In view of Ex. 15, the conditional distributions of $X$ given $Y = 0.25$ and $Y$ given $X = 0.5$ are

$$f_{X|Y}(x|0.25) = \begin{cases} \dfrac{32}{9}x & \text{if } 0 < x < \dfrac{3}{4} \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_{Y|X}(y|0.5) = \begin{cases} 8y & \text{if } 0 < x < 0.5 \\ 0 & \text{otherwise.} \end{cases}$$

## Two dimensional distribution

Hence,

$$\boldsymbol{E(X|Y=0.25)=\int_{-\infty}^{\infty}x\,f_{X|Y}(x|0.25)dx=\int_{0}^{3/4}x\cdot\frac{32}{9}x\,dx=\frac{32}{9}\times\frac{27}{192}=\frac{1}{2}},$$

$$\boldsymbol{E(Y|X=0.5)=\int_{-\infty}^{\infty}y\,f_{Y|X}(y|0.5)dy=\int_{0}^{1/2}y\cdot 8y\,dy=8\times\frac{1}{24}=\frac{1}{3}},$$

$$E(X^2|Y=0.25)=\int_{-\infty}^{\infty}x^2\,f_{X|Y}(x|0.25)dx=\int_{0}^{3/4}x^2\cdot\frac{32}{9}x\,dx=\frac{32}{9}\times\frac{81}{1024}=\frac{9}{32}$$

and

$$E(Y^2|X=0.5)=\int_{-\infty}^{\infty}y^2\,f_{Y|X}(y|0.5)dy=\int_{0}^{1/2}y^2\cdot 8y\,dy=8\times\frac{1}{64}=\frac{1}{8}.$$

Hence,

$$\boldsymbol{Var(X|Y=0.25)=E\left(X^2\middle|Y=0.25\right)-E^2(X|Y=0.25)=\frac{9}{32}-\left(\frac{1}{2}\right)^2=\frac{1}{32}},$$

$$\boldsymbol{Var(Y|X=0.5)=E\left(Y^2\middle|X=0.5\right)-E^2(Y|X=0.5)=\frac{1}{8}-\left(\frac{1}{3}\right)^2=\frac{1}{72}}.$$