Class No. 1,2,3,4
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MA 2302: Introduction to Probability and Statistics

PROBABILITY

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Probability: Basic concepts

- An experiment which results whose outcome can not be predicted before it is performed, but the possible outcomes are known in advance, is known as a *random experiment*, which is commonly known as a *trial*.

- The set of all possible outcomes of a random experiment is known as a *sample space*.

- The elements of a sample space are known as *elementary outcomes*. A subset of the sample space (whose probability is computable) is known as an event.

- The probability of an event is the *degree of certainty* of occurrence of occurrence of the event.

Probability: Basic concepts

- If all the elementary outcomes are equally likely to occur, then $P(A) = \frac{\#(A)}{\#(S)}$, where # is the counting measure.
- Sometimes, the sample space contains infinite or uncountable number of points. In such a case, the above definition of probability fails.
- Thus, $P(A) = \frac{m(A)}{m(S)}$, where m may the counting measure or it measures length in 1D, area in 2D and volume in 3D.
- The empirical definition of probability is as follows: If a random experiment is repeated n tomes and if m is the number of times A occurs, then $P(A) = \lim_{n\to\infty} \frac{m}{n}$. Using this definition only, one can find that the probability of choosing an even number (event A) from the set of natural numbers (sample space S) is $1/2$.

Probability: Basic concepts

- The sample space S acts as an universal set and the events are subsets of S. The occurrence of the event A means in performing the random experiment (leading to the sample space S), a point e is obtained which belongs to A.

- $A \cup B$ occurs means $e \in A \cup B$, i.e. either A or B occurs. $A \cap B$ occurs means both A and B occur. Similarly, $A \cup B \cup C \cup D \cup, \ldots$ occurs means at least one of $A, B, C, D, \ldots$ occurs. $A - B = A \cap B^c$ occurs means, A occurs but (means and) B does not occur. $A \Delta B$ occurs means, either A occurs or B occurs but both A and B do not occur.

- Two events A and B are equally likely (to occur) if $P(A) = P(B)$. A and B are mutually exclusive if they are set theoretically disjoint leading to $P(A \cap B) = 0$. However, $P(A \cap B) = 0$ need not imply that A and B are mutually exclusive.

- If $E = \{1,3,5, \ldots\}, F = \{2,4,6, \ldots\}, G = \{1^2, 2^2, 3^2, \ldots\}$ and $A = E \cup G, B = F \cup G$, then $P(E) = P(F) = 1/2, P(G) = 0$. Hence $P(A) = P(B) = 1/2, P(A \cap B) = 0$, but $A \cap B \neq \phi$.

Addition rules

- If A and B are mutually exclusive then $P(A \cup B) = P(A) + P(B)$.

- In general $P(A \cup B) = P(A) + P(B) - A(A \cap \mathrm{B})$.

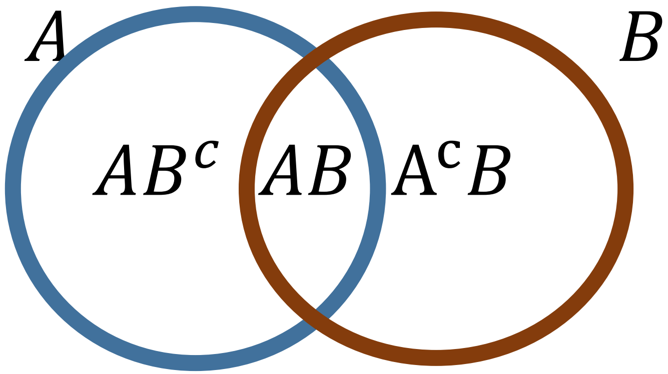
Proof: $A \cup B = A \cup (A^c \cap B)$, A and $A^c \cap B$ are mutually exclusive. Hence

$$P(A \cup B) = P(A) + P(A^c \cap B). \cdots\cdots \quad (1)$$

Furthermore, $B = (A \cap B) \cup (A^c \cap B)$, $A \cap B$ and $A^c \cap B$ are mutually exclusive. Hence

$$P(B) = P(A \cap B) + P(A^c \cap B). \cdots\cdots (2)$$

From (2), $P(A^c \cap B) = P(B) - P(A \cap B)$ and substituting in (1) we get the result.



Addition rules

- If A, B and C are three events, then

$$P(A \cup B \cup C) = P\big((A \cup B) \cup C\big) = P(A \cup B) + P(C) - P\big((A \cup B) \cap C\big)$$

$$= P(A \cup B) + P(C) - P(AC \cup BC)$$

$$= P(A) + P(B) - P(AB) + P(C) - P(AC) - P(BC) + \ P(ABC)$$

$$= P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + \ P(ABC).$$

Assignment 1: Derive the formula for $P(A \cup B \cup C \cup D)$.

Assignment 2: Guess the formula for $P(A \cup B \cup C \cup D \cup E)$.

Assignment 3: Observe that

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{1 \le i < j \le n} P\big(A_i A_j\big) + \sum_{1 \le i < j < k \le n} P\big(A_i A_j A_k\big) - \cdots + (-1)^{n-1} P(A_1 A_2 \cdots A_n).$$

Note: If $p = P(A_i)$ for each i and $A_1, A_2, \ldots$ are independent i.e. $P(A_1 A_2 \ldots) = P(A_1) P(A_2) \ldots$, then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = np - \binom{n}{2} p^2 + \binom{n}{3} p^3 - \binom{n}{4} p^4 + \cdots + (-1)^{n-1} p^n$$

$$= 1 - (1-p)^n.$$

Conditional Probability

If we perform any experiment, then the sample space always occurs. But a event A may or may not occur. Let A and B be two events and it is known that B has occurred. Then depending on B, the probability of A prior and posterior to the occurrence of B will certainly be affected.

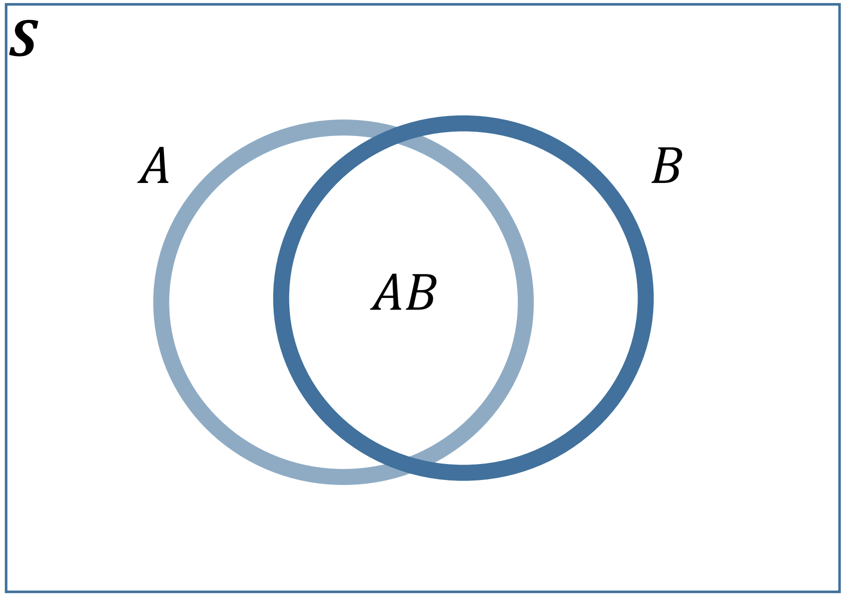
Example: Consider a box consisting of 50 resisters of 47 ohms each and40 resistors of 56 ohms each. Suppose that 2 resistors are drawn at random without replacement and we have to calculate the probability that both are of 56 ohms. Let A be the event that the first one drawn is of 56 ohms and B the event that the second one is of 56 ohms. Thus, we have to find $P(AB)$. Observe that the probability of second one being 56 ohms depends on what has been drawn in the first draw. And we calculate

$$P(AB) = P(A)P(B : A\, has\, occured) = \frac{40}{90} \times \frac{39}{89}.$$

The probability $P(B : A\, has\, occured)$ is written as $P(B|A)$ and is called a conditional probability.

Conditional Probability

After the occurrence of B, if $P(B) > 0$, it acts as the sample space and the portion of the sample space outside B has zero probability. Thus, $P(A|B) = \frac{P(AB)}{P(B)}$, and consequently, $P(AB) = P(A|B)P(B) = P(A)P(B|A)$, which is known as the multiplication rule of probability.



If $P(B) = 0$, then $P(A|B) = P(A)$ since the nothing has occurred. However, if $P(B) = 1$, then also $P(A|B) = P(A)$ since occurrence of B is equivalent to the occurrence of the sample space which always occur. In particular $P(A|\phi) = P(A|S) = P(A)$.

Independent events

Observe that, in the previous example if the resistors are chosen at random with replacement, then $P(AB) = P(A)P(B|A) = \frac{40}{90} \times \frac{40}{90} = P(A)P(B)$.

Definition: Two events A and B in a sample space S are said to be independent if $P(AB) = P(A)P(B)$. If there are more than two events, say $A_1, A_2, \ldots, A_n$, then they are said to be independent if $P\left(A_{i_1}A_{i_2} \ldots A_{i_r}\right) = P\left(A_{i_1}\right)P\left(A_{i_2}\right) \cdots P\left(A_{i_r}\right)$ for each nonempty subset $\{i_1, i_2, \ldots, i_r\}$ of $\{1,2, \ldots, n\}$. If $P\left(A_iA_j\right) = P(A_i)P(A_j)$ for all $i \neq j$, then $A_1, A_2, \ldots, A_n$ are said to be pairwise independent.

Three events A, B and C are mutually independent if following are fulfilled:

$$P(AB) = P(A)P(B)$$

$$P(AC) = P(A)P(C)$$

$$P(BC) = P(B)P(C)$$

$$P(ABC) = P(A)P(B)\mathrm{P}(\mathrm{C})$$

Independent events

Q. If A and B are independent, show that A^c and B are also independent.

Proof: Given $P(AB) = P(A)P(B)$. To prove $P(A^cB) = P(A^c)P(B)$.

$A^cB \cup AB = B$. A^cB and AB are mutually exclusive. Hence

$$P(B) = P(A^cB) + P(AB) = P(A^cB) + P(A)P(B)$$

Thus,

$$P(A^cB) = \mathrm{P}(B) - P(A)P(B) = \big(1 - P(A)\big)P(B) = P(A^c)P(B).$$

Q. If A and B are independent, show that A^c and B^c are also independent.

Proof: Given $P(AB) = P(A)P(B)$. To prove $P(A^cB^c) = P(A^c)P(B^c)$.

$$P(A^cB^c) = P((A \cup B)^c) = 1 - P(A \cup B) = 1 - P(A) - P(B) + P(AB)$$

$$= 1 - P(A) - P(B) + P(A)P(B) = \big(1 - P(A)\big)\big(1 - P(B)\big) = P(A^c)P(B^c)$$

Independent events

Q. *If A, B and C are pairwise independent and A is independent of $B \cup C$ then prove that A, B and C are mutually independent.*

Ans. Given that $P(AB) = P(A)P(B), P(AC) = P(A)P(C)$ and $P(BC) = P(B)P(C)$and $P\big(A \cap (B \cup C)\big) = P(A)P(B \cup C)$. To show that $P(ABC) = P(A)P(B)$P(C).

$P\big(A \cap (B \cup C)\big) = P(A)P(B \cup C)$implies that $P(AB \cup AC) = P(A)[P(B) + P(C) - P(BC)]$and hence,

$$P(AB)+P(AC) - P(ABC) = P(A)P(B)+ P(A)P(C) - P(A)P(BC).$$

Now using pairwise independence, one can get the desired results.

Q. *The probability of hitting a target in a single shot is p. What is the probability that out of ten shot fired, at least one shot will hit the target?*

Ans. Let A_i be the event that the i-th shot hits the target. Then $P(A_i) = p$ for each i. We are interested in

$$P(A_1 \cup A_2 \cup \cdots \cup A_{10}) = P[(A_1^c A_2^c \cdots A_{10}^c)^c] = 1 - P(A_1^c A_2^c \cdots A_{10}^c) = 1 - (1-p)^{10}.$$

Independent events

Example 1: A, B and C are pairwise independent but not mutually independent

Q. A box contains three chips bearing numbers $112,121,211$ *and* $222.$ *A chip is chosen at random. Let* A, B and C be the events that the first, second and third digit of the chip number is 1 *respectively. Prove that* A, B and C are pairwise independent but not mutually independent.

Solun: Observe that

$S = \{112,121,211,222\}$ $A = \{112,121\}$, $B = \{112,211\}$, $C = \{121,211\}$,

$AB = \{112\}$, $AC = \{121\}$, $BC = \{211\}$, $ABC = \phi$.

Hence, $P(A) = P(B) = P(C) = \frac{1}{2}$, $P(AB) = P(AC) = P(BC) = \frac{1}{4}$, $P(ABC) = 0$

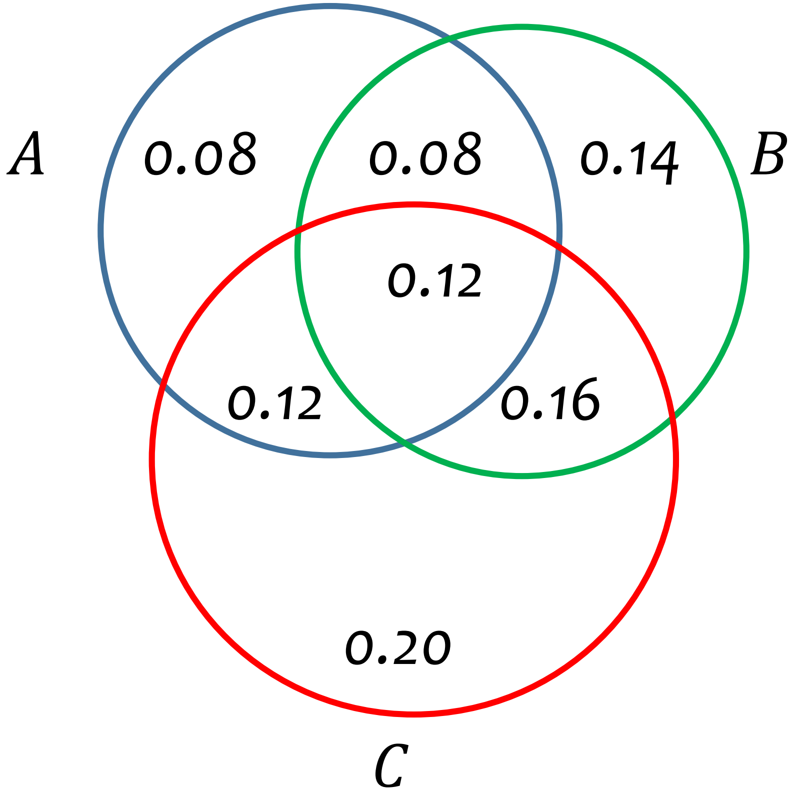
Hence $P(AB) = P(A)P(B)$, $P(AC) = P(A)P(C)$ and $P(BC) = P(B)P(C)$ but

$P(ABC) = 0 \neq \frac{1}{8} = P(A)P(B)$P(C). Hence A, B and C are pairwise independent but not mutually independent.

Independent events

Example 2: $P(ABC) = P(A)P(B)P(C)$, but A, B and C are not pairwise independent and hence not mutually independent.

Q. Consider the following figure:



Observe that $P(A) = 0.4$, $P(B) = 0.5$, $P(C) = 0.6$, $\mathrm{P}(AB) = 0.2, P(AC) = 0.24, P(BC) = 0.28, P(ABC) = 0.12$. Hence $P(AB) = P(A)P(B)$, $P(AC) = P(A)P(C)$, $P(ABC) = P(A)P(B)P(C)$ but $P(BC) \neq P(B)P(C)$. Hence, A, B and C are not pairwise independent, though $P(ABC) = P(A)P(B)P(C)$. Hence they cannot be mutually independent because of failure of one condition $P(BC) = P(B)P(C)$. We can simply say that A and B are independent and A and C are independent.

Independent events

We know that if A, B and C are independent events, then $P(ABC) = P(A)P(B)P(C)$. In the absence of independence, the following holds:

$$P(ABC) = P(A)P(B|A)P(C|AB).$$

Proof: $P(ABC) = P(AB)P(C|AB) = P(A)P(B|A)P(C|AB)$.

This can be further generalized.

Remark: Let A and B be two events with positive probabilities. Observe that if A and B are mutually exclusive, they cannot be independent. Moreover, if they are independent, then they cannot be mutually exclusive.

WHY???

Bayes theorem

Assume that in a sample space S, there are n events $A_1, A_2, \dots, A_n$ which are pairwise mutually exclusive. Let B be another event such that $B \subset \bigcup_{i=1}^{n} A_i$. Then $B = B \cap (\bigcup_{i=1}^{n} A_i) = \bigcup_{i=1}^{n} A_i B$. Observe that the n events $A_1B, A_2B, \dots, A_nB$ are also pairwise mutually exclusive. Hence, by addition rule of probability, $P(B) = P(\bigcup_{i=1}^{n} A_i B) = \sum_{i=1}^{n} P(A_i B)$

Hence,

$$\boldsymbol{P(B) = \sum_{i=1}^{n} P(A_i)P(B|A_i)}$$

This is known as the law of total probabilities. Now,

$$\boldsymbol{P(A_k|B)} = \frac{P(A_k B)}{P(B)} = \frac{\boldsymbol{P(A_k)P(B|A_k)}}{\boldsymbol{\sum_{i=1}^{n} P(A_i)P(B|A_i)}}$$

which is known as the Bayes theorem.

Examples

1. *A man is equally likely to choose one of the three Routs A, B and C from his house to the railway station. On a rainy day, the probability of missing the train is 0.2 if he starts before one hour of train time through Rout A. The same probabilities for Routs B and C are 0.1 and 0.15 respectively. (a) On a rainy day, what is the probability of his missing the train? (b) If on a rainy day, if he missed the train, what is the probability that his choice of rout was B*?

Ans. Let A_1, A_2 and A_3 be the event od choosing Routs A,B and C respectively and D, the event of missing the train. Then certainly $P(A_1) = P(A_2) = P(A_3) = 1/3$, $P(D|A_1) = 0.2, P(D|A_2) = 0.1$ and $P(D|A_3) = 0.15$. Hence,

Probability Of missing the train is (Law of total probabilities)

$$P(D) = \sum_{i=1}^{3} P(A_i)P(D|A_i) = \frac{1}{3} \times 0.2 + \frac{1}{3} \times 0.1 + \frac{1}{3} \times 0.15 = 0.15$$

Given that he missed the train, the probability that the Rout chosen by him was B is (by Bayes theorem)

$$P(A_2|D) = \frac{P(A_2)P(D|A_2)}{\sum_{i=1}^{3} P(A_i)P(D|A_i)} = \frac{\frac{1}{3} \times 0.1}{\frac{1}{3} \times 0.2 + \frac{1}{3} \times 0.1 + \frac{1}{3} \times 0.15} = \frac{2}{9}\,.$$