Class No. 1,2,3,4
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MA 2302: Introduction to Probability and Statistics

PROBABILITY

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Probability

- An experiment which results whose outcome can not be predicted before it is performed, but the possible outcomes are known in advance, is known as a *random experiment*, which is commonly known as a *trial*.
- The set of all possible outcomes of a random experiment is known as a *sample space*.
- The elements of a sample space are known as *elementary outcomes*. A subset of the sample space (whose probability is computable) is known as an event.
- The probability of an event is the *degree of certainty* of occurrence of occurrence of the event.

- If all the elementary outcomes are equally likely to occur, then $P(A) = \frac{\#(A)}{\#(S)}$, where # is the counting measure.

- Sometimes, the sample space contains infinite or uncountable number of points. In such a case, the above definition of probability fails.

- Thus, $P(A) = \frac{m(A)}{m(S)}$, where m may the counting measure or it measures length in 1D, area in 2D and volume in 3D.

- The empirical definition of probability is as follows: If a random experiment is repeated n tomes and if m is the number of times A occurs, then $P(A) = \lim_{n\to\infty} \frac{m}{n}$. Using this definition only, one can find that the probability of choosing an even number (event A) from the set of natural numbers (sample space S) is $1/2$.

- The sample space S acts as an universal set and the events are subsets of S. The occurrence of the event A means in performing the random experiment (leading to the sample space S), a point e is obtained which belongs to A.

- $A \cup B$ occurs means $e \in A \cup B$, i.e. either A or B occurs. $A \cap B$ occurs means both A and B occur. Similarly, $A \cup B \cup C \cup D \cup, \ldots$ occurs means at least one of $A, B, C, D, \ldots$ occurs. $A - B = A \cap B^c$ occurs means, A occurs but (means and) B does not occur. $A \Delta B$ occurs means, either A occurs or B occurs but both A and B do not occur.

- Two events A and B are equally likely (to occur) if $P(A) = P(B)$. A and B are mutually exclusive if they are set theoretically disjoint leading to $P(A \cap B) = 0$. However, $P(A \cap B) = 0$ need not imply that A and B are mutually exclusive.

- If $E = \{1,3,5, \ldots\}, F = \{2,4,6, \ldots\}, G = \{1^2, 2^2, 3^2, \ldots\}$ and $A = E \cup G, B = F \cup G$, then $P(E) = P(F) = 1/2, P(G) = 0$. Hence $P(A) = P(B) = 1/2, P(A \cap B) = 0$, but $A \cap B \neq \phi$.

- If A and B are mutually exclusive then $P(A \cup B) = P(A) + P(B)$.
- In general $P(A \cup B) = P(A) + P(B) - P(A \cap \mathrm{B})$.

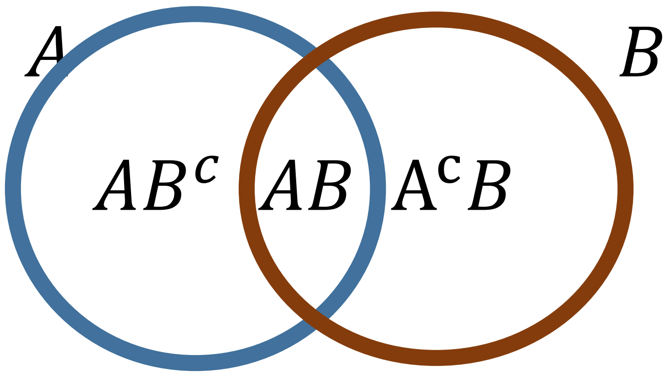
Proof: $A \cup B = A \cup (A^c \cap B)$, A and $A^c \cap B$ are mutually exclusive. Hence

$$P(A \cup B) = P(A) + P(A^c \cap B). \cdots\cdots \quad (1)$$

Furthermore, $B = (A \cap B) \cup (A^c \cap B)$, $A \cap B$ and $A^c \cap B$ are mutually exclusive. Hence

$$P(B) = P(A \cap B) + P(A^c \cap B). \cdots\cdots (2)$$

From (2), $P(A^c \cap B) = P(B) - P(A \cap B)$ and substituting in (1) we get the result.



- If A, B and C are three events, then

$$P(A \cup B \cup C) = P\big((A \cup B) \cup C\big) = P(A \cup B) + P(C) - P\big((A \cup B) \cap C\big)$$

$$= P(A \cup B) + P(C) - P(AC \cup BC)$$

$$= P(A) + P(B) - P(AB) + P(C) - P(AC) - P(BC) + P(ABC)$$

$$= P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC).$$

Assignment 1: Derive the formula for $P(A \cup B \cup C \cup D)$.

Assignment 2: Guess the formula for $P(A \cup B \cup C \cup D \cup E)$.

Assignment 3: Observe that

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{1 \le i < j \le n} P\big(A_i A_j\big) + \sum_{1 \le i < j < k \le n} P\big(A_i A_j A_k\big) - \cdots + (-1)^{n-1} P(A_1 A_2 \cdots A_n).$$

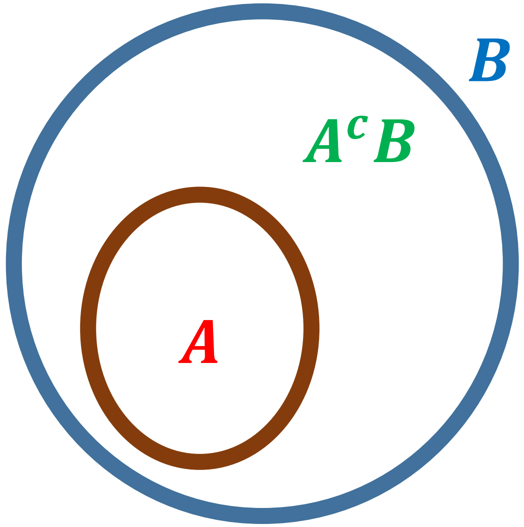
Note: If $p = P(A_i)$ for each i and $A_1, A_2, \ldots$ are independent i.e. $P(A_1 A_2 \ldots) = P(A_1)P(A_2) \ldots$, then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = np - \binom{n}{2} p^2 + \binom{n}{3} p^3 - \binom{n}{4} p^4 + \cdots + (-1)^{n-1} p^n$$

$$= 1 - (1 - p)^n.$$

- If A and B are two events and $A \subset B$, then $P(A) \leq P(B)$.

  Observe that $B = A \cup A^c B$, A and $A^c B$ are mutually exclusive. Hence, $P(B) = P(A) + P(A^c B)$. Since $P(A^c B) \geq 0$, the result follows.



- $P(A \cup B) \leq P(A) + P(B)$ follows from

  $P(A \cup B) = P(A) + P(B) - P(AB)$, where $P(AB) \geq 0$.

  In general, the Boole's inequality $P(\bigcup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} P(A_i)$

  follows from mathematical induction. (**Prove it.**)

- $P(AB) \geq P(A) + P(B) - 1$ follows from

  $P(A \cup B) = P(A) + P(B) - P(AB) \leq 1$. In general,

$$P\left(\bigcap_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} P(A_i) - (n-1).$$

  Prove it using mathematical induction.

Conditional Probability

If we perform any experiment, then the sample space always occurs. But a event A may or may not occur. Let A and B be two events and it is known that B has occurred. Then depending on B, the probability of A prior and posterior to the occurrence of B will certainly be affected.

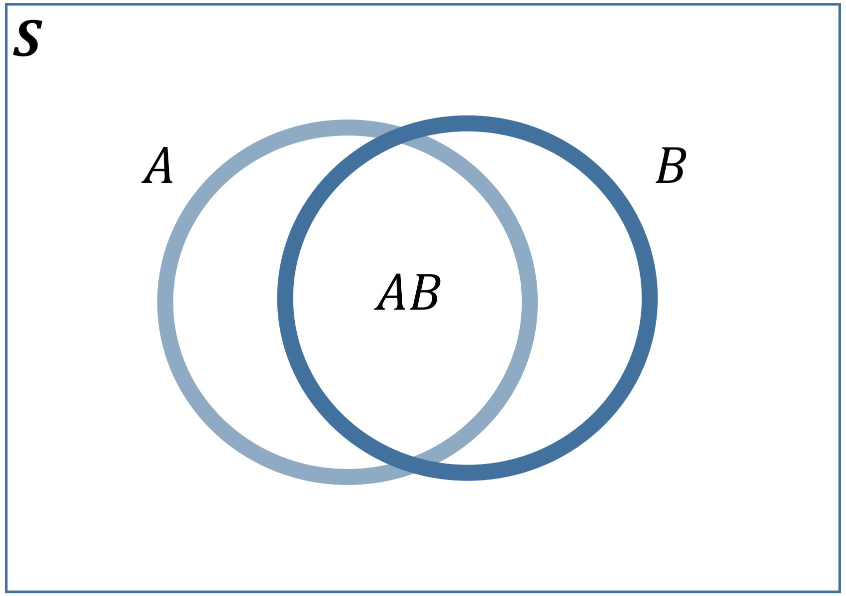
Example: Consider a box consisting of 50 resisters of 47 ohms each and40 resistors of 56 ohms each. Suppose that 2 resistors are drawn at random without replacement and we have to calculate the probability that both are of 56 ohms. Let A be the event that the first one drawn is of 56 ohms and B the event that the second one is of 56 ohms. Thus, we have to find $P(AB)$. Observe that the probability of second one being 56 ohms depends on what has been drawn in the first draw. And we use to calculate

$$P(AB) = P(A)P(B : A\, has\, occured) = \frac{40}{90} \times \frac{39}{89}.$$

The probability $P(B : A\, has\, occured)$ is written as $P(B|A)$ and is called a conditional probability.

Conditional Probability

After the occurrence of B, if $P(B) > 0$, it acts as the sample space and the portion of the sample space outside B has zero probability. Thus, $P(A|B) = \frac{P(AB)}{P(B)}$, and consequently, $P(AB) = P(A|B)P(B) = P(A)P(B|A)$, which is known as the multiplication rule of probability.



If $P(B) = 0$, then $P(A|B) = P(A)$ since the nothing has occurred. However, if $P(B) = 1$, then also $P(A|B) = P(A)$ since occurrence of B is equivalent to the occurrence of the sample space which always occur. In particular $P(A|\phi) = P(A|S) = P(A)$.

Observe the following about conditional probability: (**To be proved by you**)

$$P(A|B) + P(A^c|B) = 1, P(A|B^c) + P(A^c|B^c) = 1$$

$$P(A \cup B|C) = P(A|C) + P(B|C) - P(A \cap B|C).$$

$$P(A|A \cup B) = \frac{P(A)}{P(A \cup B)}, P(A|A \cap B) = 1.$$

More generally, if $A \subset B, P(B) > 0$, then

$$P(A|B) = \frac{P(A)}{P(B)}.$$

If $A \subset B$, then

$$P(A|C) \leq P(B|C).$$

Independent events

Observe that, in the previous example if the resistors are chosen at random with replacement, then $P(AB) = P(A)P(B|A) = \frac{40}{90} \times \frac{40}{90} = P(A)P(B)$.

Definition: Two events A and B in a sample space S are said to be independent if $P(AB) = P(A)P(B)$. If there are more than two events, say $A_1, A_2, \dots, A_n$, then they are said to be independent if $P\left(A_{i_1} A_{i_2} \dots A_{i_r}\right) = P\left(A_{i_1}\right) P\left(A_{i_2}\right) \cdots P\left(A_{i_r}\right)$ for each nonempty subset $\{i_1, i_2, \dots, i_r\}$ of $\{1,2, \dots, n\}$. If $P\left(A_i A_j\right) = P(A_i)P(A_j)$ for all $i \neq j$, then $A_1, A_2, \dots, A_n$ are said to be pairwise independent.

Three events A, B and C are mutually independent if following are fulfilled:

$$P(AB) = P(A)P(B)$$

$$P(AC) = P(A)P(C)$$

$$P(BC) = P(B)P(C)$$

$$P(ABC) = P(A)P(B)\mathrm{P}(\mathrm{C})$$

Example 1. If A and B are independent, show that A^c and B are also independent.

Proof: Given $P(AB) = P(A)P(B)$. To prove $P(A^cB) = P(A^c)P(B)$.

$A^cB \cup AB = B$. A^cB and AB are mutually exclusive. Hence

$$P(B) = P(A^cB) + P(AB) = P(A^cB) + P(A)P(B)$$

Thus,

$$P(A^cB) = \mathrm{P}(B) - P(A)P(B) = \big(1 - P(A)\big)P(B) = P(A^c)P(B).$$

Q. If A and B are independent, show that A^c and B^c are also independent.

Proof: Given $P(AB) = P(A)P(B)$. To prove $P(A^cB^c) = P(A^c)P(B^c)$.

$$P(A^cB^c) = P((A \cup B)^c) = 1 - P(A \cup B) = 1 - P(A) - P(B) + P(AB)$$

$$= 1 - P(A) - P(B) + P(A)P(B) = \big(1 - P(A)\big)\big(1 - P(B)\big) = P(A^c)P(B^c)$$

Example 2. *If A, B and C are pairwise independent and A is independent of $B \cup C$ then prove that A, B and C are mutually independent.*

Ans. Given that $P(AB) = P(A)P(B), P(AC) = P(A)P(C)$ and $P(BC) = P(B)P(C)$ and $P\big(A \cap (B \cup C)\big) = P(A)P(B \cup C)$. To show that $P(ABC) = P(A)P(B)$P(C).

$P\big(A \cap (B \cup C)\big) = P(A)P(B \cup C)$implies that $P(AB \cup AC) = P(A)[P(B) + P(C) - P(BC)]$and hence,

$$P(AB)+P(AC) - P(ABC) = P(A)P(B)+ P(A)P(C) - P(A)P(BC).$$

Now using pairwise independence, one can get the desired results.

Example 3. *The probability of hitting a target in a single shot is p. What is the probability that out of ten shot fired, at least one shot will hit the target?*

Ans. Let A_i be the event that the i-th shot hits the target. Then $P(A_i) = p$ for each i. We are interested in

$$P(A_1 \cup A_2 \cup \cdots \cup A_{10}) = P[(A_1^c A_2^c \cdots A_{10}^c)^c] = 1 - P(A_1^c A_2^c \cdots A_{10}^c) = 1 - (1-p)^{10}.$$

A, B and C are pairwise independent but not mutually independent

Example 4. A box contains three chips bearing numbers $112,121,211$ *and* 222*. A chip is chosen at random. Let* A*,* B and C be the events that the first, second and third digit of the chip number is 1 *respectively. Prove that* A, B and C are pairwise independent but not mutually independent.

Ans: Observe that

$S = \{112,121,211,222\}$ $A = \{112,121\}$, $B = \{112,211\}$, $C = \{121,211\}$,

$AB = \{112\}$, $AC = \{121\}$, $BC = \{211\}$, $ABC = \phi$.

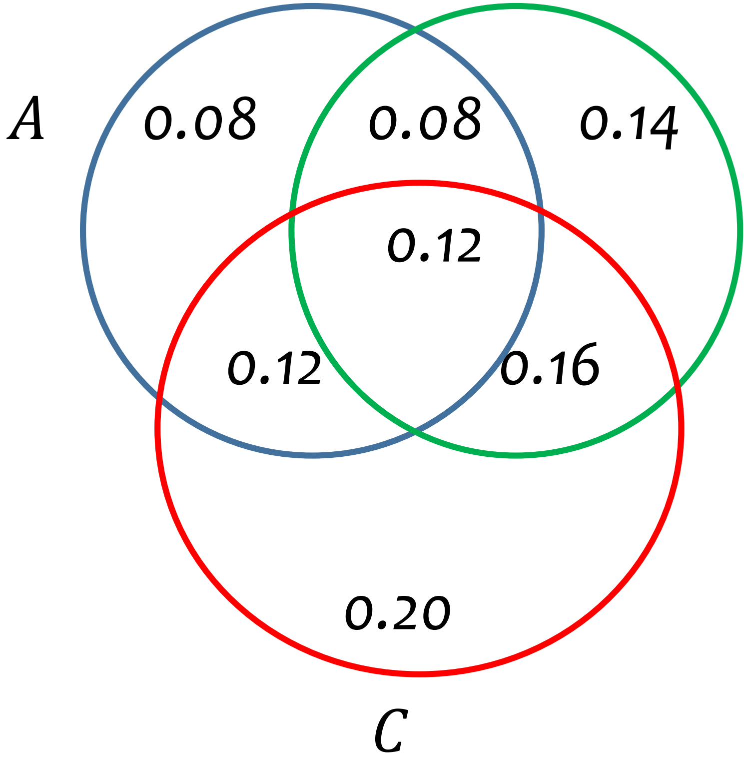
Hence, $P(A) = P(B) = P(C) = \frac{1}{2}$, $P(AB) = P(AC) = P(BC) = \frac{1}{4}$, $P(ABC) = 0$

Hence $P(AB) = P(A)P(B)$, $P(AC) = P(A)P(C)$ and $P(BC) = P(B)P(C)$ but

$P(ABC) = 0 \neq \frac{1}{8} = P(A)P(B)\mathrm{P(C)}$. Hence A, B and C are pairwise independent but not mutually independent.

Example 5: $P(ABC) = P(A)P(B)P(C)$, but A, B and C are not pairwise independent and hence not mutually independent.

Q. Consider the following figure:



Observe that $P(A) = 0.4$, $P(B) = 0.5$, $P(C) = 0.6$, $\mathrm{P}(AB) = 0.2$, $P(AC) = 0.24$, $P(BC) = 0.28$, $P(ABC) = 0.12$. Hence $P(AB) = P(A)P(B)$, $P(AC) = P(A)P(C)$, $P(ABC) = P(A)P(B)P(C)$ but $P(BC) \neq P(B)P(C)$. Hence, A, B and C are not pairwise independent, though $P(ABC) = P(A)P(B)P(C)$. Hence they cannot be mutually independent because of failure of one condition $P(BC) = P(B)P(C)$. We can simply say that A and B are independent and A and C are independent.

We know that if A, B and C are independent events, then $P(ABC) = P(A)P(B)P(C)$. In the absence of independence, the following holds:

$$P(ABC) = P(A)P(B|A)P(C|AB).$$

Proof: $P(ABC) = P(AB)P(C|AB) = P(A)P(B|A)P(C|AB)$.

This can be further generalized.

Remark: Let A and B be two events with positive probabilities. Observe that if A and B are mutually exclusive, they cannot be independent. Moreover, if they are independent, then they cannot be mutually exclusive.

WHY???

Bayes theorem

Assume that in a sample space S, there are n events $A_1, A_2, \ldots, A_n$ which are pairwise mutually exclusive. Let B be another event such that $B \subset \bigcup_{i=1}^{n} A_i$. Then $B = B \cap (\bigcup_{i=1}^{n} A_i) = \bigcup_{i=1}^{n} A_i B$. Observe that the n events $A_1B, A_2B, \ldots, A_nB$ are also pairwise mutually exclusive. Hence, by addition rule of probability, $P(B) = P(\bigcup_{i=1}^{n} A_i B) = \sum_{i=1}^{n} P(A_iB)$

Hence,

$$\boldsymbol{P(B) = \sum_{i=1}^{n} P(A_i)P(B|A_i)}$$

This is known as the law of total probabilities. Now,

$$\boldsymbol{P(A_k|B)} = \frac{P(A_kB)}{P(B)} = \frac{\boldsymbol{P(A_k)P(B|A_k)}}{\boldsymbol{\sum_{i=1}^{n} P(A_i)P(B|A_i)}}$$

which is known as the Bayes theorem.

Example

Example 6. *A man is equally likely to choose one of the three Routs A, B and C from his house to the railway station. On a rainy day, the probability of missing the train is 0.2 if he starts before one hour of train time through Rout A. The same probabilities for Routs B and C are 0.1 and 0.15 respectively. (a) On a rainy day, what is the probability of his missing the train? (b) If on a rainy day, if he missed the train, what is the probability that his choice of rout was B*?

Ans. Let A_1, A_2 and A_3 be the event od choosing Routs A,B and C respectively and D, the event of missing the train. Then certainly $P(A_1) = P(A_2) = P(A_3) = 1/3$, $P(D|A_1) = 0.2$, $P(D|A_2) = 0.1$ and $P(D|A_3) = 0.15$. Hence,

Probability Of missing the train is (Law of total probabilities)

$$P(D) = \sum_{i=1}^{3} P(A_i)P(D|A_i) = \frac{1}{3} \times 0.2 + \frac{1}{3} \times 0.1 + \frac{1}{3} \times 0.15 = 0.15$$

Given that he missed the train, the probability that the Rout chosen by him was B is (by Bayes theorem)

$$P(A_2|D) = \frac{P(A_2)P(D|A_2)}{\sum_{i=1}^{3} P(A_i)P(D|A_i)} = \frac{\frac{1}{3} \times 0.1}{\frac{1}{3} \times 0.2 + \frac{1}{3} \times 0.1 + \frac{1}{3} \times 0.15} = \frac{2}{9}.$$

Bayes Theorem: Have you ever thought like this?

Let A and B be any two arbitrary events such that $P(B) > 0$. Then $B \subset (A \cup A^c)$ and hence $B = B \cap (A \cup A^c) = AB \cup A^cB$. Hence,

$$P(B) = P(AB) + P(A^cB) = P(A)P(B|A) + P(A^c)P(B|A^c)$$

which is the law of total probabilities and

$$P(A|B) = \frac{P(A)P(B|A)}{P(A)P(B|A) + P(A^c)P(B|A^c)}$$

which is the Bayes theorem. Also

$$P(A^c|B) = \frac{P(A^c)P(B|A^c)}{P(A)P(B|A) + P(A^c)P(B|A^c)}.$$

In addition,

$$P(A^c|B^c) = \frac{P(A^c)P(B^c|A^c)}{P(A^c)P(B^c|A^c) + P(A)P(B^c|A)}.$$

Example 7. It is estimated that *2% of certain locality were infected by a virus. A certain plasma test is used to detect infection due the virus. About* 5% *of the test results are false positive and 10% are false negative. If a person was tested positive for the virus, find the probability that he was really infected. If a person was tested negative, what was the probability that he was actually not infected?*

Ans. Let A be the event that the person was infected by the virus and B, the event that that the person was tested positive for the virus. False positive means a person was tested positive given that he was not infected. False negative means a person was tested negative given that he was really infected. Thus,

$$P(A) = 0.02, P(A^c) = 0.98, P(B|A^c) = 0.05, P(B^c|A) = 0.10$$

Hence, $P(B^c|A^c) = 0.95, P(B|A) = 0.90$. (Observe that $P(A|B) + P(A^c|B) = 1$.) Hence,

$$\Pr\{A\ person\ was\ really\ infected\ given\ that\ he\ was\ tested\ positive\}$$

$$= P(A|B) = \frac{P(A)P(B|A)}{P(A)P(B|A) + P(A^c)P(B|A^c)} = \frac{0.02 \times 0.90}{0.02 \times 0.90 + 0.98 \times 0.05} = \frac{0.018}{0.018 + 0.049}$$
$$= 0.2686.$$

$$\Pr\{A\ person\ was\ really\ infected\ given\ that\ he\ was\ tested\ positive\}$$

$$= P(A^c|B^c) = \frac{P(A^c)P(B^c|A^c)}{P(A^c)P(B^c|A^c) + P(A)P(B^c|A)} = \frac{0.98 \times 0.95}{0.98 \times 0.95 + 0.02 \times 0.10} = \frac{0.931}{0.931 + 0.002}$$
$$= 0.9979.$$

Example 8. *Balls are drawn one at random and with replacement from a box consisting of 40 white and 50 red balls. What is the probability of drawing the third white ball before the second red ball?*

Ans: We can realise the third white ball before the second red ball as follows

$$WWW, RWWW, WRWW, WWRW.$$

Since balls are drawn **with replacement, successive draws are independent** and hence

$$P(WWW) = \frac{40}{90} \times \frac{40}{90} \times \frac{40}{90}, P(RWWW) = \frac{50}{90} \times \frac{40}{90} \times \frac{40}{90} \times \frac{40}{90}$$

$$P(WRWW) = \frac{40}{90} \times \frac{50}{90} \times \frac{40}{90} \times \frac{40}{90}, P(WWRW) = \frac{40}{90} \times \frac{40}{90} \times \frac{50}{90} \times \frac{40}{90}.$$

Hence, the required probability is $\left(\frac{4}{9}\right)^3 \left(1 + 3 \times \frac{5}{9}\right)$.

If balls are drawn without replacement then ???

Example 9. *A pressure control apparatus contains 3 electronic tubes. The apparatus will not work unless all tubes are operative. If the probability of failure of each tube during some interval of time is 0.04, what is the corresponding probability of failure of the apparatus?*

Ans: Let A_i be the event that the i-th tube is operative, $i = 1,2,3$. The apparatus will not work if al least one tube fails. We need to calculate $P(A_1^c \cup A_2^c \cup A_3^c)$. But,

$$P(A_1^c \cup A_2^c \cup A_3^c) = P\big((A_1 \cap A_2 \cap A_3)^c\big) = 1 - P(A_1 \cap A_2 \cap A_3) = 1 - P(A_1)P(A_2)P(A_3)$$
$$= 1 - (1 - 0.04)^3 = 1 - 0.96^3 = 0.1153$$

10. Suppose that in a production of spark plugs the fraction of defective plugs has been constant at 2% over a long time and that this process is controlled every half hour by drawing and inspecting two plugs just produced. Find the probabilities of getting ***(a)*** *no defectives,* ***(b)*** *1 defective,* ***(c)*** *2 defectives. What is the sum of these probabilities*?

Ans. The probability of a plug being defective is 0.02. Let A and B be the events that the first and the second plug are defective respectively. (a) Prob. of no defective plug $= (1 - 0.02)^2$, (b)Prob. of one defective plug $= 2 \times 0.02(1 - 0.02)$, (c) b)Prob. of two defective plugs $= 0.02 \times 0.02$. Sum of probabilities is equal to 1.

Example 11. *The numbers* $1,2,\ldots,10$ *are arranged at random. What is the probability that none of the numbers are in their original position?* ***If*** $\mathbf{1},\mathbf{2},\ldots,\boldsymbol{n}$ ***and*** $\boldsymbol{n}\to\infty$***, then ?????***

Ans. Let A_i be the event that the number i is in the i-th position, $\mathrm{i}=1,2,\ldots,10$. we need to find

$P(A_1^c \cap A_2^c \cap \cdots \cap A_{10}^c) = P\big((A_1 \cup A_2 \cup \cdots \cup A_{10})^c\big) = 1 - P(A_1 \cup A_2 \cup \cdots \cup A_{10})$,

$P(A_1 \cup A_2 \cup \cdots \cup A_{10}) = \sum_{i=1}^{10} P(A_i) - \sum_{1\le i<j\le 10} P\big(A_iA_j\big) + \sum_{1\le i<j<k\le 10} P\big(A_iA_jA_k\big) - \cdots + (-1)^{10-1}P(A_1A_2\cdots A_{10})$

Observe that $P(A_i) = \frac{9!}{10!}$ for each i, $P\big(A_iA_j\big) = \frac{8!}{10!}$ for $i \neq j$, $P\big(A_iA_jA_k\big) = \frac{7!}{10!}$ for

$i \neq j \neq k$, …….. and finally, $P(A_1A_2\cdots A_{10}) = \frac{1}{10!}$.

Thus,

$$P(A_1 \cup A_2 \cup \cdots \cup A_{10}) = \binom{10}{1}\cdot\frac{9!}{10!} - \binom{10}{2}\cdot\frac{8!}{10!} + \binom{10}{3}\cdot\frac{7!}{10!} - \cdots + (-1)^9\cdot\frac{1}{10!}$$

and hence the required probability is

$$1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + \frac{1}{10!}.$$

Example 12. *Three capacitors are chosen at random and with replacement from a lot consisting of 8 capacitors of capacity 500 μF and 16 capacitors of capacity 1000 μF. Find the probability that, out of the three chosen capacitors, (a) there is at least one capacitors of capacity 500 μF and (b) not more that two capacitors of capacity 1000 μF?*

Ans. Let A be the event of choosing a 500 μF capacitor and B, the event of choosing a 1000 μF capacitor from the lot if one is chosen at random. The certainly, $P(A) = \frac{1}{3}$ and $P(B) = \frac{2}{3}$. The three choices can result in any one of the following:

Outcomes	AAA	AAB	BAA	ABA	ABB	BAB	BBA	BBB
Probability	1/27	2/27	2/27	2/27	4/27	4/27	4/27	8/27
No. of 0.5F	3	2	2	2	1	1	1	0
No. of 1F	0	1	1	1	2	2	2	3

Thus, the probability of choosing 0,1,2,3, 500 μF capacitors are respectively 8/27, 12/27, 6/27, 1/27. The probability of choosing 0,1,2,3, 1000 μF capacitors are respectively 1/27, 6/27, 12/27, 8/27. The probability of choosing at least one capacitors of capacity 500 μF is equal to 12/27+6/27+1/27=19/27. The probability of choosing not more then two capacitors of capacity 1000 MFD is equal to 1/27+6/27+12/27=19/27.