Production Function

Theory of Production

- Production involves transformation of inputs such as capital, equipment, labor, and land into output - goods and services
- In this production process, the manager is concerned with ***efficiency*** in the use of the inputs

  - technical vs. economical efficiency

Two Concepts of Efficiency

- **Economic efficiency**:
  - occurs when the cost of producing a given output is as low as possible
- **Technological efficiency**:
  - occurs when it is not possible to increase output without increasing inputs

You will see that basic production theory is simply an application of constrained optimization:

the firm attempts either to **minimize the cost of producing** a given level of output

or

to **maximize the output** attainable with a given level of cost.

Both optimization problems lead to same rule for the allocation of inputs and choice of technology

Production Function

- A *production function* is purely technical relation which **connects factor inputs & outputs**. It describes the transformation of factor inputs into outputs at any particular time period.

$$Q = f(L,K,R,L_d,T,t)$$

where

Q = output　　R= Raw Material

L= Labour　　L_d = Land

K= Capital　　T = Technology

t = time

For our current analysis, let's reduce the inputs to two, capital (K) and labor (L):

$$Q = f(L, K)$$

Production Table

Units of K Employed	Output Quantity (Q)							
8	37	60	83	96	107	117	127	128
7	42	64	78	90	101	110	119	120
6	37	52	64	73	82	90	97	104
5	31	47	58	67	75	82	89	95
4	24	39	52	60	67	73	79	85
3	17	29	41	52	58	64	69	73
2	8	18	29	39	47	52	56	52
1	4	8	14	20	27	24	21	17
	1	2	3	4	5	6	7	8
	Units of L Employed							

Same Q can be produced with different combinations of inputs, e.g. inputs are substitutable in some degree

Short-Run and Long-Run Production

- In the short run some inputs are fixed and some variable
  - e.g. the firm may be able to vary the amount of labor, but cannot change the amount of capital
  - in the short run we can talk about

    factor productivity / law of variable proportion/law of diminishing returns
- In the long run all inputs become variable
  - e.g. the long run is the period in which a firm **can adjust *all* inputs** to changed conditions
  - in the long run we can talk about ***returns to scale***

Short-Run Changes in Production
Factor Productivity

Units of K Employed	Output Quantity (Q)							
8	37	60	83	96	107	117	127	128
7	42	64	78	90	101	110	119	120
6	37	52	64	73	82	90	97	104
5	31	47	58	67	75	82	89	95
4	24	39	52	60	67	73	79	85
3	17	29	41	52	58	64	69	73
2	8	18	29	39	47	52	56	52
1	4	8	14	20	27	24	21	17
	1	2	3	4	5	6	7	8
	Units of L Employed							

How much does the quantity of Q change, when the quantity of L is increased?

Long-Run Changes in Production
Returns to Scale

Units of K Employed	Output Quantity (Q)							
8	37	60	83	96	107	117	127	128
7	42	64	78	90	101	110	119	120
6	37	52	64	73	82	90	97	104
5	31	47	58	67	75	82	89	95
4	24	39	52	60	67	73	79	85
3	17	29	41	52	58	64	69	73
2	8	18	29	39	47	52	56	52
1	4	8	14	20	27	24	21	17
	1	2	3	4	5	6	7	8
	Units of L Employed							

How much does the quantity of Q change, when the quantity of both L and K is increased?

Relationship Between Total, Average, and Marginal Product: Short-Run Analysis

- **Total Product** (TP) = total quantity of output
- **Average Product** (AP) = total product **per total input**
- **Marginal Product** (MP) = change in quantity when **one additional unit of input** used

The Marginal Product of Labor

- The marginal product of labor is the increase in output obtained by adding 1 unit of labor but holding constant the inputs of all other factors

Marginal Product of L:

$\mathbf{MP_L = \Delta Q/\Delta L}$ (holding K constant)

$= \delta Q/\delta L$

Average Product of L:

$\mathbf{AP_L = Q/L}$ (holding K constant)

Law of Diminishing Returns (Diminishing Marginal Product)

The law of diminishing returns states that when more and more units of a variable input are applied to a given quantity of fixed inputs, the total output may initially increase at an increasing rate and then at a constant rate but it will eventually increases at diminishing rates.

Assumptions. The law of diminishing returns is based on the following assumptions:

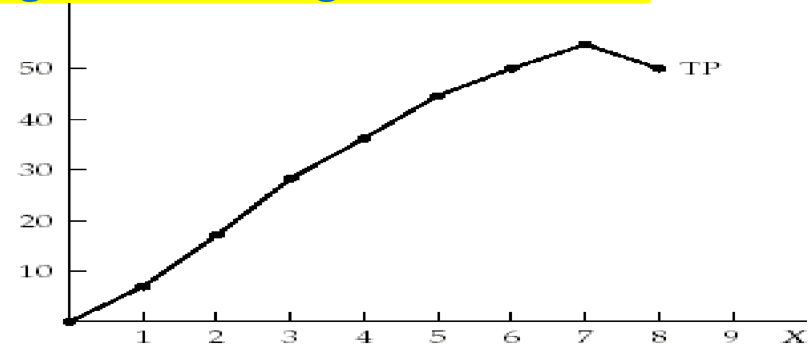
(i) the state of technology is given

(ii) labour is homogenous and

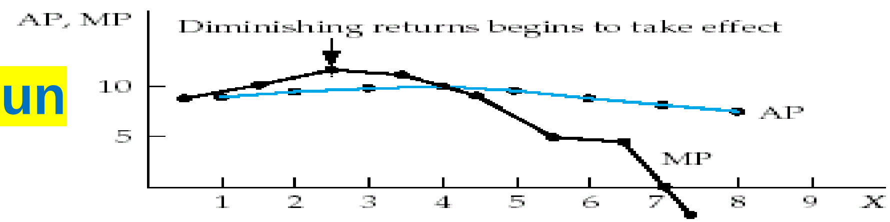
(iii) input prices are given.

Short-Run Analysis of Total, Average, and Marginal Product

- If MP > AP then AP is rising
- If MP < AP then AP is falling
- MP = AP when AP is maximized
- TP maximized when MP = 0

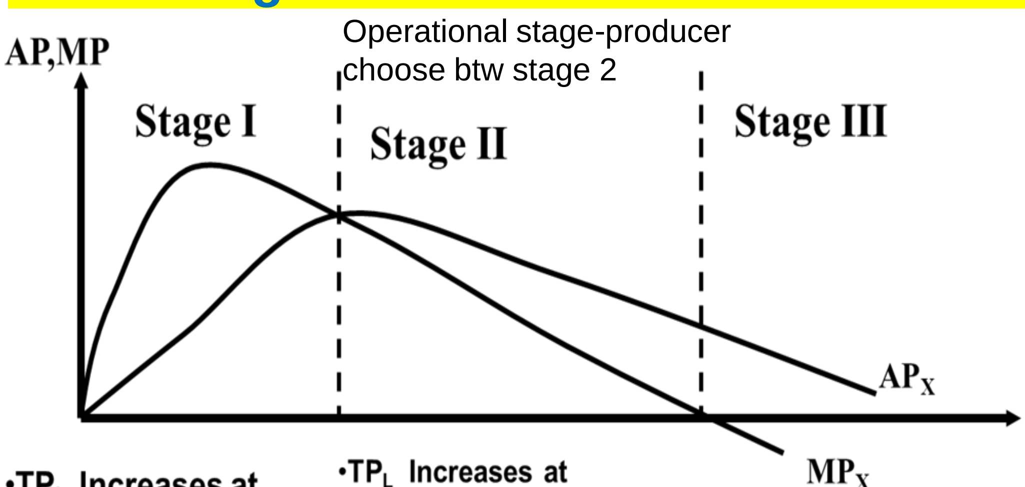


(a) Total product



(b) Average and marginal products

Three Stages of Production in Short Run



Stage I

- TP_L Increases at increasing rate.
- MP Increases at decreasing rate.
- AP is increasing and reaches its maximum at the end of stage I

Stage II

- TP_L Increases at Diminshing rate.
- MP_L Begins to decline.
- TP reaches maximum level at the end of stage II, MP = 0.
- AP_L declines

Stage III

- TPL begins to decline
- MP becomes negative
- AP continues to decline

Three Stages of Production				
Labor Unit (X)	Total Product (Q or TP)	Average Product (AP)	Marginal Product (MP)	Stages of Production
1	24	24	24	
2	72	36	48	I
3	138	46	66	Increasing
4	216	54	78	Returns
5	300	60	84	
6	384	64	84	
7	462	66	78	
8	528	66	66	II
9	576	64	48	Diminishing
10	600	60	24	Returns
11	594	54	-6	III
12	552	46	-42	Negative Returns

Application of Law of Diminishing Returns:

- It helps in identifying the rational and irrational stages of operations.
- It gives answers to question –

  How much to produce?

  What number of workers to apply to a given fixed inputs so that the output is maximum?

Production in the Long-Run

- All inputs are now considered to be variable (both L and K in our case)
- How to determine the optimal combination of inputs?

To illustrate this case we will use *production isoquants*.

An *isoquant* is a locus of all technically efficient methods or all possible combinations of inputs for producing a **given level of output.**

Production Table

Units of K Employed	Output Quantity (Q)							
8	37	60	83	96	107	117	[illegible]	128
7	42	64	78	90	101	110	119	120
6	37	52	64	73	82	90	97	104
5	31	47	58	67	75	82	89	95
4	24	39	52	60	67	73	79	85
3	17	29	41	52	58	64	69	73
2	8	18	29	39	47	52	56	52
1	4	8	14	20	27	24	21	17
	1	2	3	4	5	6	7	8

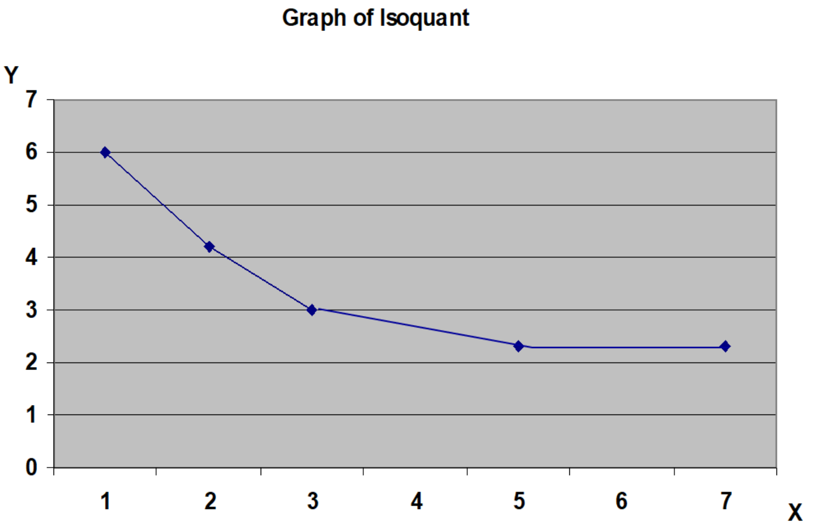
Units of L

Isoquant

Isoquant

Properties of Isoquants

- Isoquants have a negative slope.
- Isoquants are convex to the origin.
- Isoquants cannot intersect or be tangent to each other.
- Upper Isoquants represents higher level of output



Marginal Rate of Technical Substitution MRTS

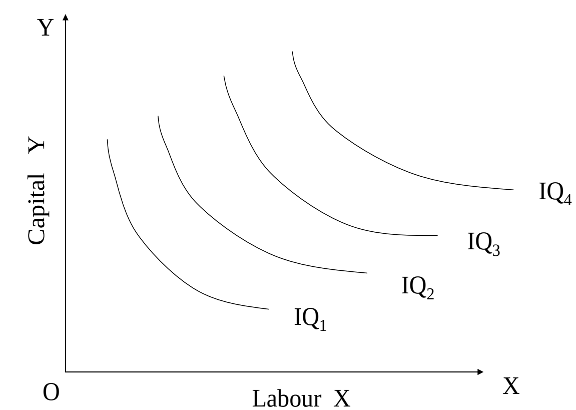
- The degree of imperfection in substitutability is measured with marginal rate of technical substitution (MRTS- Slope of Isoquant): **MRTS = ΔL/ΔK**

  (in this MRTS some of L is removed from the production and substituted by K to maintain the same level of output)

Isoquant Map

- Isoquant map is a **set of isoquants** presented on a two dimensional plain. Each isoquant shows various combinations of two inputs that can be used to produce a given level of output.

Figure : Isoquant Map



Laws of Returns to Scale

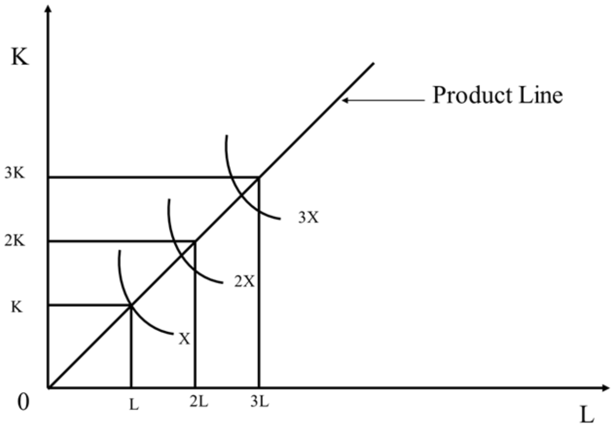
- It explains the behavior of output in response to a proportional and simultaneous change in input.
- When a firm increases both the inputs, there are three technical possibilities –

(i) TP may increase more than proportionately – **Increasing RTS** (output inc at higher proportion as inc in input)

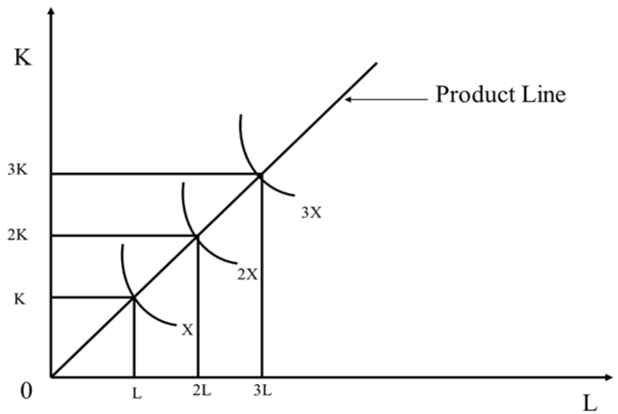
(ii) TP may increase proportionately **– constant RTS** (output inc at same proportion as inc in input)

(iii) TP may increase less than proportionately – **diminishing RTS** (output inc at lower proportion as inc in input)

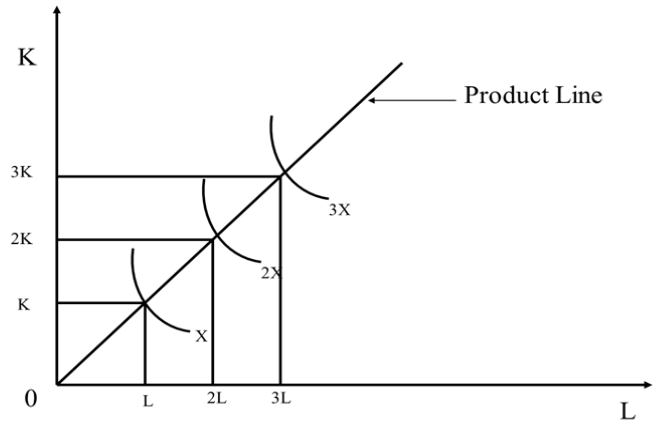
Increasing RTS



Constant RTS



Decreasing RTS



Elasticity of Factor Substitution

- ( ) is formally defined as the percentage change in the capital labour ratios (K/L) divided by the percentage change in marginal rate of technical substitution (MRTS), i.e

$$( ) = \frac{\text{Percentage change in K/L}}{\text{Percentage change in MRTS}}$$

$$( ) = \frac{д(K/L) / (K/L)}{д(MRTS) / (MRTS)}$$

Cobb – Dougles Production function: -

$X = b_0 L^{b1} K^{b2}$

X= Out put

L = qty of Labour

K = qty of Capital

bo , b_1 , b_2 Coefficient

b1 - Labour

b2 - Capital

Characteristics of Cobb – Dougles Prodn function: -

1. The Marginal Product of Factor:

(a) $MP_L = dx/dl$

$X = b_0 L^{b1} K^{b2}$

$dx/dl = b_0 b_1 L^{b1-1} K^{b2}$

$= b_1 (boL^{b1} K^{b2}) L^{-1}$

$= b_1 X/L$

$= b_1 (AP_L)$

AP_L Average Product of Labour

Similarly

(b) $MP_K = dx/dk$

$= b_2 b_0 L^{b1} K^{b2-1}$

$= b_2 (b_0 L^{b1} K^{b2}) K^{-1}$

$= b_2 X/K$

$AP_k \longrightarrow$ Average Product of Capital

2. The Marginal rate of technical substitution

$$MRTS_{L.K} = \frac{MP_L}{MP_K}$$

$$= \frac{dx/dL}{dx/dk} = \frac{b_1(X/L)}{b_2(X/K)}$$

$$MRTS_{LK} = \frac{b_1}{b_2} \frac{K}{L}$$

3. The Elasticity of Substitution

$$\sigma = \frac{\Delta d\, k/L \;/\; k/l}{dMRTS \;/\; MRTS}$$

$$= \frac{dK/L/k/L}{\frac{b_1}{b_2}\,\frac{dk}{L} \Big/ \frac{b_1}{b_2}\,\frac{k}{L}}$$

EOS =1

This function is perfectly substitutable function.

4.Factor intensity: -

In cobb-Douglas function factor intensity is measured by ratio b1/b2.

The higher is the ratio (b1/b2), the more labour intensive is the technique.

Similarly, the lower the ratio (b1/b2) the more capital intensive is the technique.

OR

$\uparrow$ b1/b2 $\longrightarrow$ labour intensive

$\uparrow$ b1/b2 $\longrightarrow$ capital intensive

5. Returns to scale:-

In cobb – Dougles production function RTS is measured by the sum of the coefficients

$b_1+b_2 = V$

$x_0 = f(L,K)$

$X^* = f({}_kL, {}_KK)$

A homogenous function is a function such that if each of the inputs is multiplied by K i.e 'K' can the completely factored out. 'K' also has a power V which is called the degree of homogeneity and it measures RTS.

$X^* = K^v f(x_0)$

$X_0 = b_0 L^{b1} K^{b2}$

$X^* = b_0 (kL)^{b1} (kK)^{b2} = K^{b1+b2} (b_o L^{b1} K^{b2}) = K^v f(X_0)$

$(V=b_1 + b_2)$

$\therefore X^* = K^v f(X_o)$

In case when

V=1 we have constant RTS

V>1 we have increasing RTS

V<1 we have decreasing RTS

6.Efficiency of Production

The efficiency in the organization of the factor of production is measured by the coefficient b_0 :-

If two firms have the same K, L, b_1, b_2 and still produce different quantities of output, the difference can be due to superior organization and entrepreneurship of one of the firms, which results in different effectiveness.

The more efficient firm will have a larger b_0 than the less efficient one.

$\uparrow b_0 \longrightarrow$ More efficient is firm

Constant Elasticity Substitution (CES) Production Functions

- The CES production function is expressed as

$$Q = A \left[\alpha K^{-\beta} + (1-\alpha) L^{-\beta}\right]^{-\upsilon/\beta}$$

$$\text{Subject to } (A > 0,\ 0 < \alpha < 1,\ and\ \beta > -1)$$

$$\text{where L - labour, K = capital, and A, } \alpha \text{ and}$$

$$\beta \text{ are the three parameters}$$

1. ‘A’ is the **efficiency parameter** and shows the scale effect. It indicates state of technology and entrepreneurial organizational aspects of production.

   Higher Value of A ➝ higher output (given same inputs)

2. □ *is the capital intensity factor coefficient and (1-* □) is the labour intensity of coefficient . The value of □ indicates the relative contribution of capital input and labour input to total output.

3. Value of Elasticity of Substitution (□) depends upon the value of substitution parameter ‘β’

$$\beta = \left(1 - \frac{1}{\sigma}\right)$$

4. The parameter υ represents degree of returns to scale.
5. Marginal Products of labour and capital are always positive if we assume constant return to scale.

Equilibrium of the firm: Choice of optimal combination of factors of prod^n

- Assumptions:

  1. The goal of the firm is profit maximization i.e maximization of difference

  $\prod$ - Profit

  R- Revenue

  C-Cost

  2. The price of o/p is given, Px
  3. The price of factors are given w is the given wage rate r is given price capital

Single Decision of the firm

(a) **Maximize profit $\prod$, subject to cost constraint.** In this case total cost & prices are given and maximization of $\prod$ is if X is maximised since c & Px are given constant.

$$\prod = R - C$$
$$= P_x X - \overline{C}$$

(b) **Maximise Profit $\prod$ for a given level of o/p.** Maximisation of $\prod$ is achieved in this case if cost c is minimized , given that X & Px are given constants.

$$\prod = \underline{R} - \underline{C}$$
$$\prod = P_x X - C$$

We will use isoquant map (1) and isoquant line (2)

Figure : Isoquant Map (1)

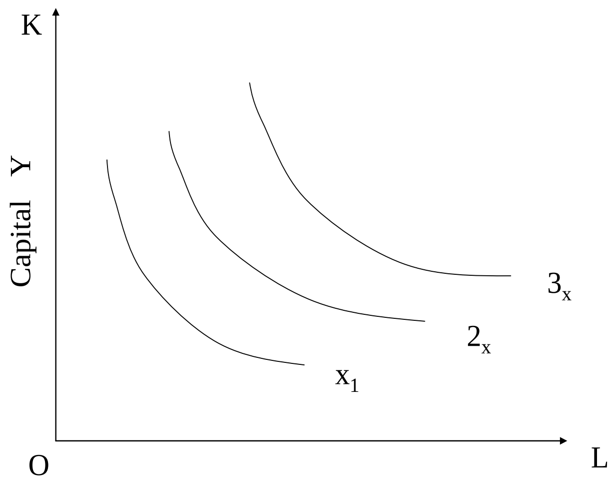
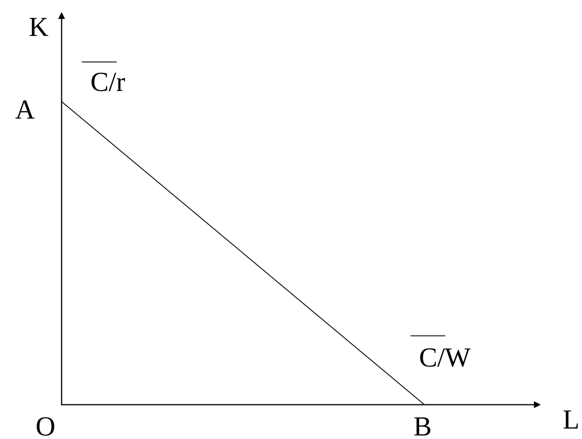


Figure : Isoquant Line (2)



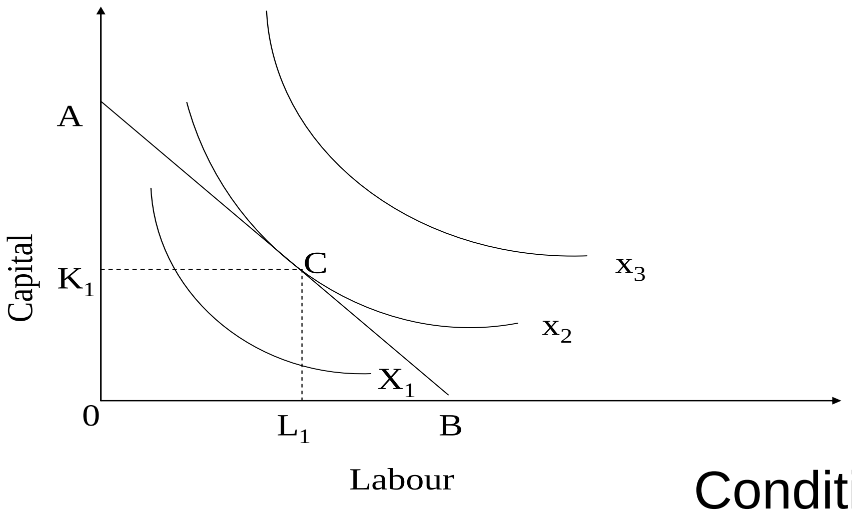
The cost line is defined by cost equation

C= (r) (k) + (w) (L)

W $\longrightarrow$ wage rate r= price of capital service

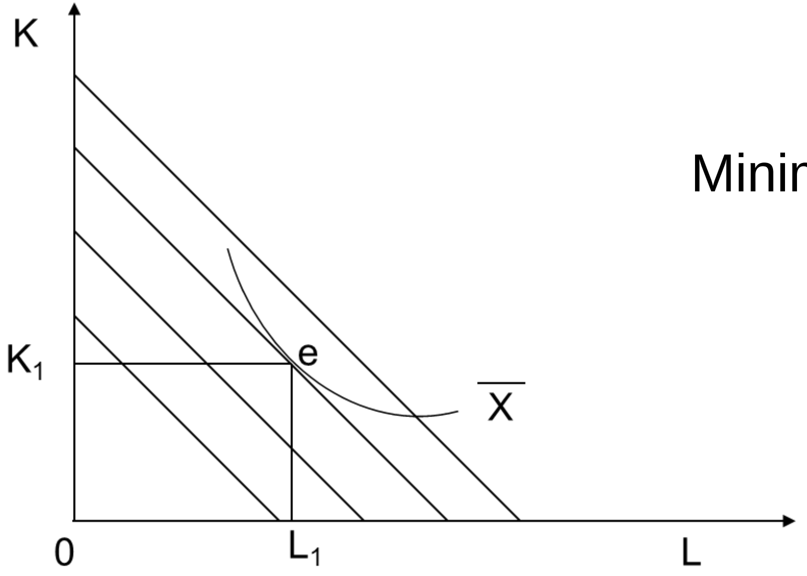
Case I

Maximization of output subject to cost constraint



Condition for Equilibrium

- At point of tendency slope of isocost line (w/r) = slope of isoquant. (MP_L/MP_K)
- The isoquants should be convex to origin



Case II

Minimization of cost for given level of output

Given the following production function and input prices,
estimate the the **optimum input combination of L and K**,
assuming that the **firm has only Rs. 6000/-** to spend.
Additionally, assume **profit maximization** as the objective function of the firm:

$$Q=LK-80L$$

P_L=60, Pk=30

Slope of isoquant curve = - mpl/mpk
Slope of isocost curve = -pl/pk
Mpl/mpk=pl/pk=60/30=2
Mpl=d(q)/d(l)= k-80
Mpk= d(q)/d(k)= l
K-80/l=2
=>K-80=2l
K*pk+l*pl=6000
=>K*30+l*60=6000
=>K+2l=200
=>2k-80=200
K=140
L=30