Introduction to Probability and Statistics Course ID:MA2203

Lecture-5

Course Teacher: Dr. Manas Ranjan Tripathy

Department of Mathematics
National Institute of Techonology, Rourkela

- Mean and Variance of a Distribution: Let X be a random variable which is defined on a sample space S. The mean of the distribution of X is denoted by μ and is defined as

$$\begin{aligned}\mu = E(X) &= \sum_j x_j P(X = x_j), \ \textit{if } X \textit{ is discrete rv} \\ &= \int_{-\infty}^{\infty} x f(x) dx, \ \textit{if } X \textit{ is continuous rv}.\end{aligned}$$

- The variance of the distribution of X is denoted by σ^2 and is defined as

$$\begin{aligned}\sigma^2 = Var(X) &= \sum_j (x_j - \mu)^2 P(X = x_j), \ \textit{if } X \textit{ is discrete rv} \\ &= \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx, \ \textit{if } X \textit{ is continuous rv}.\end{aligned}$$

- Note that $\sigma^2 > 0$. The positive square root of σ^2 is called the standard deviation and is denoted by σ.

- Sometimes it is difficult to calculate the variance σ^2 using the above formula. We have an alternative formula. Observe that

$$\sigma^2 = E(X - EX)^2 = EX^2 - (EX)^2.$$

- Examples: (i) In the coin (fair) tossing problem, X is the number of heads. $P(X = 0) = \frac{1}{2} = P(X = 1)$. Hence mean of the distribution is $E(X) = \mu = 0.\frac{1}{2} + 1.\frac{1}{2} = \frac{1}{2}$. Further variance $V(X) = \sigma^2 = (0 - \frac{1}{2})^2 \frac{1}{2} + (1 - \frac{1}{2})^2 \frac{1}{2} = \frac{1}{4}$.
- (ii) Let X be a continuous type random variable having probability density $f(x) = \frac{1}{b-a}$, if $a < x < b$, and zero otherwise. This is a uniform distribution in the interval (a, b). Now the mean of the distribution is $\mu = E(X)$ given by

$$E(X) = \mu = \int_a^b x \frac{1}{b-a} dx = \frac{a+b}{2}.$$

Further the variance σ^2 is given by

$$\sigma^2 = V(X) = \int_a^b (x - \frac{a+b}{2})^2 \frac{1}{b-a} dx$$

- Mode and Median of a Distribution:
  (a) The median of a discrete random variable is the "middle" value. It is the value of X for which $P(X < x)$ is greater than or equal to 0.5 and $P(X > x)$ is greater than or equal to 0.5.
  (b) If X is a discrete random variable, the mode is the value of X (i.e, $X = x$) at which the probability mass function takes its maximum value. In other words, it is the value that is most likely to be sampled.
  (c) If X is a continuous type random variable, let M be its median, then we must have

$$\int_{-\infty}^{M} f(x)dx = \int_{M}^{\infty} f(x)dx = \frac{1}{2}.$$

  Solving the equation for M will give the median.
  (d) The mode is that value which maximizes the probability density function. Hence the mode can be obtained by differentiating the density function and equating to zero, obtaining the critical points. Then check for the points at which the density function is maximum.

- Mean of a Symmetric Distribution: If a distribution is symmetric about a point, say $x = c$, that is $f(c-x) = f(c+x)$ then $\mu = c$. For example, in Example (i) it is symmetric about $c = 1/2$, in fact any point in the interval $(0, 1)$. In the case of (ii) also it is symmetric about the point $x = \frac{a+b}{2}$.
- Transformation of Mean and Variance: (i) If a random variable X has mean μ and variance σ^2, then the random variable $Y = aX + b$ has the mean $\mu* = a\mu + b$ and variance $\sigma^2* = a^2\sigma^2$.
  (ii) In particular, the standardized random variable $Z = \frac{X-\mu}{\sigma}$ has the mean 0 and variance 1.
  Proof (i): Let X be a continuous random variable.

$$
\begin{aligned}
E(Y) &= \int_{-\infty}^{\infty} (ax+b)f(x)dx, \\
&= a\int_{-\infty}^{\infty} xf(x)dx + b\int_{-\infty}^{\infty} f(x)dx = a\mu + b = \mu * .
\end{aligned}
$$

$$
\begin{aligned}
V(Y) &= \int_{-\infty}^{\infty} (ax+b-\mu*)^2 f(x)dx, \\
&= a\int_{-\infty}^{\infty} (ax-a\mu)^2 f(x)dx, \\
&= a^2\int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx = a^2 V(X) = a^2\sigma^2 = \sigma^2 * .
\end{aligned}
$$

- (ii) Taking $a = \frac{1}{\sigma}$ and $b = -\frac{\mu}{\sigma}$ and using the result (i), we can prove (ii).
- Expectation and Moments: Let X be a random variable defined on a sample space S. Let $g(x)$ be a non constant and continuous for all x, then $g(X)$ is a random variable. The mathematical expectation or its expectation $E(g(X))$ is defined as

$$\begin{aligned} E(g(X)) &= \sum_j g(x_j)P(X = x_j), \ \textit{if } X \textit{ is discrete} \\ &= \int_{-\infty}^{\infty} g(x)f(x)dx, \ \textit{if } X \textit{ is continuous.} \end{aligned}$$

- Moment of Order k or k^{th} moment of X:

$$\begin{aligned} E(X^k) &= \sum_j x_j^k P(X = x_j), \ \textit{if } X \textit{ is Discrete} \\ &= \int_{-\infty}^{\infty} x^k f(x)dx, \ \textit{if } X \textit{ is Continuous.} \end{aligned}$$

- If we choose $k = 1$, we get $E(X) = \mu$.

- Central Moments: The k^{th} order central moments of X is defined as

$$\begin{aligned} E(X-\mu)^k &= \sum_j (x_j-\mu)^k P(X=x_j), \text{ if } X \text{ is Discrete} \\ &= \int_{-\infty}^{\infty} (x-\mu)^k f(x)dx, \text{ if } X \text{ is Continuous.} \end{aligned}$$

- If we choose, $k=2$ we get the variance $\sigma^2 = E(X-\mu)^2$.
- Note that, (i) $E(k)=k$, (ii) $V(k)=0$.
- Denote $\mu_k = E(X-\mu)^k$, then Skewness (it represents the measure of symmetry) is given by

$$\alpha_3 = \frac{\mu_3}{(\mu_2)^{3/2}} = \frac{E(X-\mu)^3}{\sigma^3}.$$

- Kurtosis (it measures the peakedness or flatness of the top of a distribution) is given by

$$\alpha_4 = \frac{\mu_4}{(\mu_2)^2} = \frac{E(X-\mu)^4}{\sigma^4}.$$

- Ex. Show that for a symmetric distribution whose third central moment exists, the Skewness is zero.