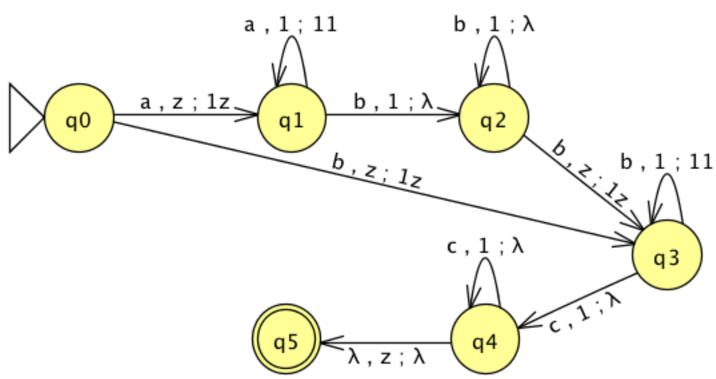
DCP3122 Introduction to Formal Languages, Spring 2015	5-May-2015

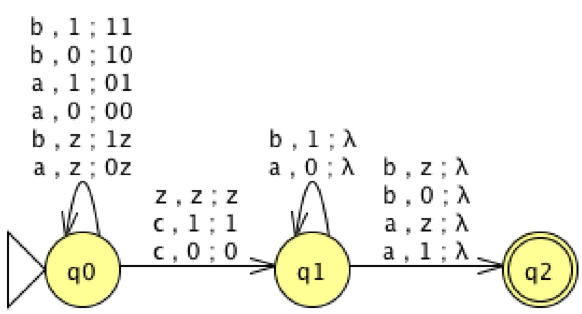
Homework 5 - Solution

Instructor: Prof. Wen-Guey Tzeng **Due: 18-May-2015**

1. Given $\Sigma = \{a, b, c\}$, find an NPDA that accepts $L = \{a^n b^{n+m} c^m : n \geq 0, m \geq 1\}$.
Ans. An NPDA that accepts L is $M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$, where $Q = \{q_0, q_1, \ldots, q_5\}$, $\Sigma = \{a, b, c\}$, $\Gamma = \{0, 1, z\}$, $F = \{q_5\}$, and the transition function δ is represented as the following graph



2. Given $\Sigma = \{a, b, c\}$, find an NPDA that accepts $L = \{w_1 c w_2 : w_1, w_2 \in \{a, b\}^*, w_1 \neq w_2^R\}$.
Ans. An NPDA that accepts L is $M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$, where $Q = \{q_0, q_1, q_2\}$, $\Sigma = \{a, b, c\}$, $\Gamma = \{0, 1, z\}$, $F = \{q_2\}$, and the transition function δ is represented as the following graph



3. What language is accepted by the PDA

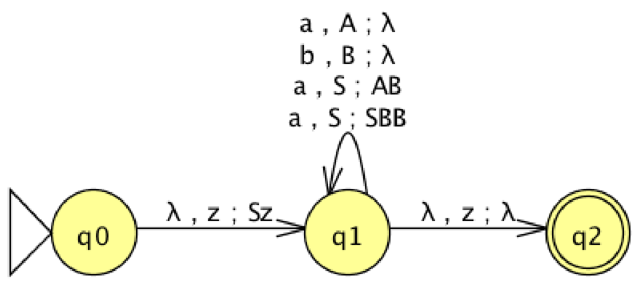
$$M = (\{q_0, q_1, \ldots, q_5\}, \{a, b\}, \{0, 1, a, z\}, \delta, z, q_0, \{q_5\}),$$

with

$$\begin{aligned}
\delta(q_0, b, z) &= \{(q_1, 1z)\},\\
\delta(q_1, b, 1) &= \{(q_1, 11)\},\\
\delta(q_2, a, 1) &= \{(q_3, \lambda)\},\\
\delta(q_3, a, 1) &= \{(q_4, \lambda)\},\\
\delta(q_4, a, z) &= \{(q_4, z), (q_5, z)\}?
\end{aligned}$$

Ans. $L = \emptyset$, because the transition function would never reach the final state.

4. Construct an NPDA that accepts the language generated by the grammar $S \to aSbb|aab$.
Ans. The Greibach normal form of the grammar is $S \to aSBB|aAB$, $B \to b$, $A \to a$. An NPDA that accepts L is $M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$, where $Q = \{q_0, q_1, q_2\}$, $\Sigma = \{a, b\}$, $\Gamma = \{S, A, B, z\}$, $F = \{q_2\}$, and the transition function δ is represented as the following graph



5. Find a context-free grammar that generates the language accepted by the NPDA $M = (\{q_0, q_1\}, \{a, b\}, \{A, z\}, \delta, q_0, z, \{q_1\})$, with transitions

$$\begin{aligned}
\delta(q_0, a, z) &= \{(q_0, Az)\},\\
\delta(q_0, b, A) &= \{(q_0, AA)\},\\
\delta(q_0, a, A) &= \{(q_1, \lambda)\}.
\end{aligned}$$

Ans. We first transform the NPDA into a new one that satisfies the following two requirements:

(1) It has a single final state q_f that is entered if and only if the stack is empty;
(2) While $a \in \Sigma \cup \{\lambda\}$, all transitions must have the form $\delta(q_i, a, A) = \{c_1, c_2, \ldots, c_n\}$, where $c_i = (q_j, \lambda)$ or $c_i = (q_j, BC)$.

The NPDA satisfies requirement (2), but not (1). To satisfy the latter, we introduce a new final state q_2 so that $Q_{new} = \{q_0, q_1, q_2\}$, $F_{new} = \{q_2\}$, and δ_{new} is

$$\begin{aligned}
\delta(q_0, a, z) &= \{(q_0, Az)\},\\
\delta(q_0, b, A) &= \{(q_0, AA)\},\\
\delta(q_0, a, A) &= \{(q_1, \lambda)\},\\
\delta(q_1, \lambda, A) &= \{(q_1, \lambda)\},\\
\delta(q_1, \lambda, z) &= \{(q_2, \lambda)\}.
\end{aligned}$$

The last three transitions yield the corresponding productions

$$
\begin{aligned}
(q_0 A q_1) &\to a, \\
(q_1 A q_1) &\to \lambda, \\
(q_1 z q_2) &\to \lambda.
\end{aligned}
$$

From the first two transitions we get the set of productions

$$
\begin{aligned}
(q_0 z q_0) &\to a(q_0 A q_0)(q_0 z q_0)|a(q_0 A q_1)(q_1 z q_0)|a(q_0 A q_2)(q_2 z q_0), \\
(q_0 z q_1) &\to a(q_0 A q_0)(q_0 z q_1)|a(q_0 A q_1)(q_1 z q_1)|a(q_0 A q_2)(q_2 z q_1), \\
(q_0 z q_2) &\to a(q_0 A q_0)(q_0 z q_2)|a(q_0 A q_1)(q_1 z q_2)|a(q_0 A q_2)(q_2 z q_2), \\
(q_0 A q_0) &\to a(q_0 A q_0)(q_0 A q_0)|a(q_0 A q_1)(q_1 A q_0)|a(q_0 A q_2)(q_2 A q_0), \\
(q_0 A q_1) &\to a(q_0 A q_0)(q_0 A q_1)|a(q_0 A q_1)(q_1 A q_1)|a(q_0 A q_2)(q_2 A q_1), \\
(q_0 A q_2) &\to a(q_0 A q_0)(q_0 A q_2)|a(q_0 A q_1)(q_1 A q_2)|a(q_0 A q_2)(q_2 A q_2).
\end{aligned}
$$

To simplify the transitions, we remove the useless variables:

- A variable that does not occur on the left side of any production: $(q_1 z q_0)$, $(q_1 z q_1)$, $(q_1 z q_2)$, $(q_2 z q_0)$, $(q_2 z q_1)$, $(q_2 z q_2)$, $(q_1 A q_0)$, $(q_1 A q_1)$, $(q_1 A q_2)$, $(q_2 A q_0)$, $(q_2 A q_1)$, and $(q_2 A q_2)$.
- Non-reachable path: None.

The result grammar is

$$
\begin{aligned}
(q_0 A q_1) &\to a, \\
(q_1 A q_1) &\to \lambda, \\
(q_1 z q_2) &\to \lambda, \\
(q_0 z q_0) &\to a(q_0 A q_0)(q_0 z q_0), \\
(q_0 z q_1) &\to a(q_0 A q_0)(q_0 z q_1), \\
(q_0 z q_2) &\to a(q_0 A q_0)(q_0 z q_2), \\
(q_0 A q_0) &\to a(q_0 A q_0)(q_0 A q_0), \\
(q_0 A q_1) &\to a(q_0 A q_0)(q_0 A q_1), \\
(q_0 A q_2) &\to a(q_0 A q_0)(q_0 A q_2),
\end{aligned}
$$

with the start variable is $(q_0 z q_2)$.

6. Show that the language $L = \{w \in \{a,b,c\}^* : n_a(w) = n_b(w) \leq n_c(w)\}$ is not context-free.
Ans. We pick a string $w = a^m b^m c^m \in L$, $m \in \mathbb{N}$. There are many ways to decompose w as $w = uvxyz$ with $|vxy| \leq m$ and $|vy| \geq 1$. However, for all of them have a winning countermove such that $uv^i xy^i z \notin L$:

   - $v = a^k, y = a^k$: $uv^0xy^0z = a^{m-2k}b^m c^m \notin L$.
   - $v = a^k, y = a^k b^k$: $uv^0xy^0z = a^{m-2k}b^{m-k} c^m \notin L$.
   - $v = a^k, y = b^\ell$: $uv^0xy^0z = a^{m-k}b^{m-\ell} c^m \notin L$.
   - $v = a^k b^k, y = b^k$: $uv^0xy^0z = a^{m-k}b^{m-2k} c^m \notin L$.
   - $v = b^k, y = b^k$: $uv^0xy^0z = a^{m}b^{m-2k} c^m \notin L$.
   - $v = b^k, y = b^k c^k$: $uv^0xy^0z = a^{m}b^{m-2k} c^{m-k} \notin L$.

- $v = b^k, y = c^k$ $uv^0xy^0z = a^m b^{m-k} c^{m-k} \notin L$.
- $v = b^k c^k, y = c^k$: $uv^0xy^0z = a^m b^{m-2k} c^{m-k} \notin L$.
- $v = c^k, y = c^k$: $uv^0xy^0z = a^m b^m c^{m-2k} \notin L$.

Therefore, by the pumping lemma for context-free languages, L is not context-free.

7. Show that the language $L = \{a^n b^m : n \text{ and } m \text{ are both prime}\}$ is not context-free.
Ans. We pick a string $w = a^m b^m \in L$, m is an odd prime. There are many ways to decompose w as $w = uvxyz$ with $|vxy| \le m$ and $|vy| \ge 1$. However, for all of them have a winning countermove such that $uv^ixy^iz \notin L$:

   - $v = a^k, y = a^\ell$ with $k + \ell$ is odd: $uv^0xy^0z = a^{m-k-\ell} b^m \notin L$ since $m - k - \ell$ is even.
   - $v = a^{k_1}, y = a^{k_2} b^\ell$ with one of k_1 and k_2 is odd, or ℓ is odd: $uv^0xy^0z = a^{m-k_1-k_2} b^{m-\ell} \notin L$ since $m - k_1 - k_2$ is even or $m - \ell$ is even.
   - $v = a^k, y = b^\ell$ with one of k and ℓ is odd: $uv^0xy^0z = a^{m-k} b^{m-\ell} \notin L$ since $m - k$ is even or $m - \ell$ is even.
   - $v = a^\ell b^{k_1}, y = b^{k_2}$ with one of k_1 and k_2 is odd, or ℓ is odd: $uv^0xy^0z = a^{m-\ell} b^{m-k_1-k_2} \notin L$ since $m - \ell$ is even or $m - k_1 - k_2$ is even.
   - $v = b^k, y = b^\ell$ with $k + \ell$ is odd: $uv^0xy^0z = a^m b^{m-k-\ell} \notin L$ since $m - k - \ell$ is even.

   Therefore, by the pumping lemma for context-free languages, L is not context-free.

8. Determine whether or not $L = \{a^n b^j a^k b^l : n \le k, j \le l\}$ is context-free. You have to prove your answer.
Ans. We pick a string $w = a^m b^m a^m b^m \in L$, $m \in \mathbb{N}$. There are many ways to decompose w as $w = uvxyz$ with $|vxy| \le m$ and $|vy| \ge 1$. However, for all of them have a winning countermove such that $uv^ixy^iz \notin L$:

   - [1st a] $v = a^k, y = a^k$ with $k \in \mathbb{N}$: $uv^ixy^iz = a^{m+2(i-1)k} b^m a^m b^m \notin L$ for $i = 2, 3, \ldots$.
   - [1st a, 1st b] $v = a^k, y = a^k b^k$: $uv^ixy^iz = a^{m+2(i-1)k} b^{m+(i-1)k} a^m b^m \notin L$ for $i = 2, 3, \ldots$.
   - [1st a, 1st b] $v = a^k, y = b^k$: $uv^ixy^iz = a^{m+(i-1)k} b^{m+(i-1)k} a^m b^m \notin L$ for $i = 2, 3, \ldots$.
   - [1st a, 1st b] $v = a^k b^k, y = b^k$: $uv^ixy^iz = a^{m+(i-1)k} b^{m+2(i-1)k} a^m b^m \notin L$ for $i = 2, 3, \ldots$.
   - [1st b] $v = b^k, y = b^k$: $uv^ixy^iz = a^m b^{m+2(i-1)k} a^m b^m \notin L$ for $i = 2, 3, \ldots$.
   - [1st b, 2nd a] $v = b^k, y = b^k a^k$: $uv^ixy^iz = a^m b^{m+2(i-1)k} a^{m+(i-1)k} b^m \notin L$ for $i = 2, 3, \ldots$.
   - [1st b, 2nd a] $v = b^k, y = a^k$: $uv^ixy^iz = a^m b^{m+(i-1)k} a^{m+(i-1)k} b^m \notin L$ for $i = 2, 3, \ldots$.
   - [1st b, 2nd a] $v = b^k a^k, y = a^k$: $uv^ixy^iz = a^m b^{m+(i-1)k} a^{m+2(i-1)k} b^m \notin L$ for $i = 2, 3, \ldots$.
   - [2nd a] $v = a^k, y = a^k$: $uv^0xy^0z = a^m b^m a^{m-2k} b^m \notin L$.
   - [2nd a, 2nd b] $v = a^k, y = a^k b^k$: $uv^0xy^0z = a^m b^m a^{m-2k} b^{m-k} \notin L$.
   - [2nd a, 2nd b] $v = a^k, y = b^k$: $uv^0xy^0z = a^m b^m a^{m-k} b^{m-k} \notin L$.
   - [2nd a, 2nd b] $v = a^k b^k, y = b^k$: $uv^0xy^0z = a^m b^m a^{m-k} b^{m-2k} \notin L$.
   - [2nd b] $v = b^k, y = b^k$: $uv^0xy^0z = a^m b^m a^m b^{m-2k} \notin L$.

   Therefore, by the pumping lemma for context-free languages, L is not context-free.

9. Show that the family of context-free languages is not closed under difference in general, but is closed under regular difference, that is, if L_1 is context-free and L_2 is regular, then $L_1 - L_2$ is context-free.
Ans. The answer contains the following two parts:

- Let L_1 and L_2 be context-free languages. By theorem, we have that the family of context-free languages is not close under intersection and complement. Thus, $L_1 - L_2 = L_1 \cap \overline{L}_2$ is not close under difference in general.
- Let L_1 be a context-free language and L_2 a regular language. By the closure property of regular languages, we know that $\overline{L}_2$ is also regular. By theorem, we know that $L_1 \cap L_2$ is context-free by the closure property under regular intersection. Therefore, $L_1 - L_2 = L_1 \cap \overline{L}_2$ is also context-free by the closure property under regular intersection.

10. Show that $L = \{w \in \{a,b\}^* : n_a(w) = n_b(w); w \text{ does not contain a substring } aab\}$ is context-free.
Ans. Let $L_1 = \{w \in \{a,b\}^* : n_a(w) = n_b(w)\}$ and $L_2 = \{w \in \{a,b\}^* : w \text{ contains 'aab' as a string}\}$. The following shows that L_1 is context-free and L_2 is regular.

- An NPDA that accepts L_1 is $M_1 = (\{q_0, q_1\}, \{a, b\}, \{a, b, z\}, \delta, q_0, z, \{q_1\})$, where the transition function δ is defined as follows:

$$\begin{aligned}
\delta(q_0, \lambda, z) &= (q_1, z),\\
\delta(q_0, a, z) &= (q_0, az),\\
\delta(q_0, a, a) &= (q_0, aa),\\
\delta(q_0, a, b) &= (q_0, \lambda),\\
\delta(q_0, b, z) &= (q_0, bz),\\
\delta(q_0, b, a) &= (q_0, \lambda),\\
\delta(q_0, b, b) &= (q_0, bb).
\end{aligned}$$

- An NFA that accepts L_2 is $M_2 = (\{q_0, q_1, q_2, q_3\}, \{a, b\}, \delta, q_0, \{q_3\})$ accepts L'', where the transition function δ is defined as follows:

$$\begin{aligned}
\delta(q_0, a) &= q_1,\\
\delta(q_0, b) &= q_0,\\
\delta(q_1, a) &= q_2,\\
\delta(q_1, b) &= q_0,\\
\delta(q_2, a) &= q_0,\\
\delta(q_2, b) &= q_3,\\
\delta(q_3, a) &= q_3,\\
\delta(q_3, b) &= q_3.
\end{aligned}$$

Therefore, $L = L_1 \cap L_2$ is context-free by the closure property under regular intersection.