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MA 2302: Introduction to Probability and Statistics

Some Continuous Probability Distributions

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Continuous distributions

1. The Uniform or Rectangular Distribution

Let a and b be two real numbers such that $a < b$. Then the uniform or rectangular distribution in the interval $[a, b]$ has the density function

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise.} \end{cases}$$

If a random variable X has the above distribution, it is customary to write $X \sim U(a, b)$.

Example 1. The amount of time of a technician takes to fix the problem of gas leakage of a split AC is uniformly distributed between 30 minutes to 75 minutes. What percentage of the ACs need at least 40 minutes for the fixation of gas leakage problem? Out of 300 ACs, about how many can be repaired within 40 minutes?

Ans. Let X be the amount of time needed to fix the gas leakage problem of an AC. Then $X \sim U(30,75)$. Hence,

$$f(x) = \begin{cases} 1/45 & \text{if } 30 \le x \le 75 \\ 0 & \text{otherwise.} \end{cases}$$

Continuous distributions

The probability that a split AC needs at least 40 minutes for the fixation of gas leakage problem

$$= \Pr\{X \geq 40\} = \int_{40}^{\infty} f(x)\,dx = \int_{40}^{75} 1/45\,dx = \frac{7}{9} = 0.78.$$

Hence, about 78% ACs need at least 40 minutes for the fixation of gas leakage problem.

The probability that the gas leakage problem if an Ac can be fixed within 45 minutes

$$= \Pr\{X \leq 45\} = \int_{-\infty}^{45} f(x)\,dx = \int_{30}^{45} 1/45\,dx = \frac{1}{3}.$$

Thus, out of 300 ACs, about 100 can be repaired within 45 minutes.

Example 2: The amount of time the students of first year B. Tech. take to complete a one km walk is 7 minutes to 15 minutes. If a first year student is chosen at random, what is the probability that he will take any amount of time between 10 to 12 minutes? What percentage of students will take not more than 10 minutes to complete one km walk? What is the maximum time required by the speediest 10% students?

Continuous distributions

Ans. Let X be the amount of time needed by a randomly chosen first year student to walk 1 km. Then $X \sim U(7,15)$. Hence,

$$f(x) = \begin{cases} 1/8 \text{ if } 7 \le x \le 15 \\ \;0 \quad \text{otherwise.} \end{cases}$$

The probability that a randomly chosen 1st year student will take any amount of time between 10 to 12 minutes to walk 1 km

$$= \Pr\{10 \le X \le 12\} = \int_{10}^{12} f(x)\,dx = \int_{10}^{12} 1/8\,dx = \frac{1}{4}.$$

The probability that a student will take not more than 10 minutes to complete one km walk

$$= \Pr\{X \le 10\} = \int_{-\infty}^{10} f(x)\,dx = \int_{7}^{10} 1/8\,dx = \frac{3}{8} = 0.375.$$

Thus, 37.5% of students will not require more than 10 minutes to complete 1 km. Let t be the maximum time required by the speediest 10% students. Then

$$\Pr\{X \le t\} = 0.1 \Rightarrow t = 7.8$$

Continuous distributions

Moments of uniform distribution

Observe that if $X \sim U(a, b)$

$$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)\, dx = \int_a^b x^r \cdot \frac{1}{b-a} dx = \frac{b^{r+1} - a^{r+1}}{(r+1)(b-a)}, r = 1,2, \dots$$

Hence,

$$\mu = E(X) = \frac{b^2 - a^2}{2(b-a)} = \frac{b+a}{2}.$$

$$E(X^2) = \frac{b^3 - a^3}{3(b-a)} = \frac{b^2 + ba + a^2}{3}.$$

Hence, the variance of the distribution is given by

$$\sigma^2 = E(X^2) - \mu^2 = \frac{b^2 + ba + a^2}{3} - \left(\frac{b+a}{2}\right)^2 = \frac{(b-a)^2}{12}.$$

Exercise for you: Prove that $\boldsymbol{\mu_3 = 0}$. Find $\boldsymbol{\mu_4}$ using $\boldsymbol{\mu_r'}$ for $\boldsymbol{r = 1, 2, 3, 4}$.

Continuous distributions

Cumulative distribution function of uniform distribution:

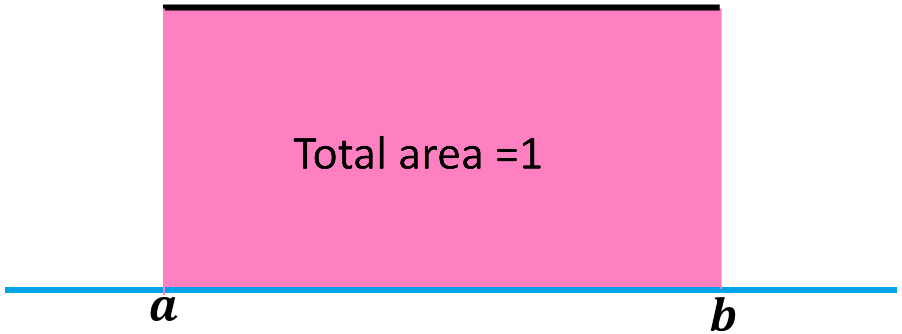
If $X \sim U(a,b)$, then $F(x) = 0$ if $x \le a$. For $a \le x \le b$,

$$F(x) = \int_{-\infty}^{x} f(u)\,du = \int_{a}^{x} \frac{1}{b-a}\,du = \frac{x-a}{b-a},$$

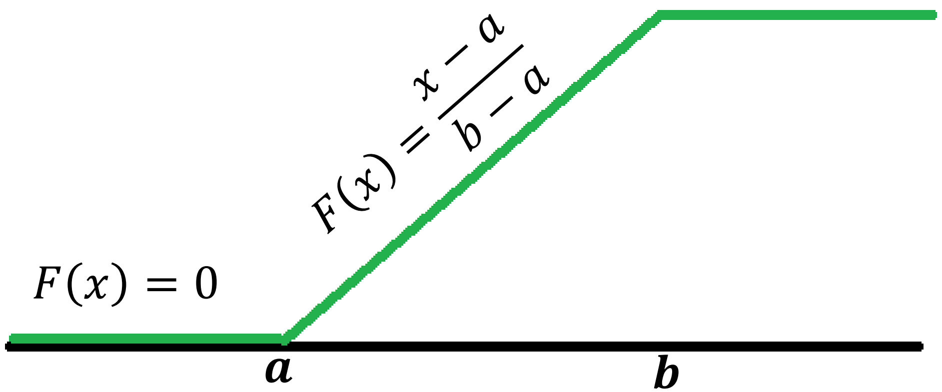
and for $x \ge b$, $F(x) = 1$. Hence

$$F(x) = \begin{cases} 0 & \text{if } x \le a \\ \dfrac{x-a}{b-a} & \text{if } a \le x \le b \\ 1 & \text{if } x \ge b. \end{cases}$$

$f(x) = \dfrac{1}{b-a}$



Sketch of $f(x)$



Sketch of $F(x)$

Continuous distributions

The Exponential distribution

The exponential distribution and the Poisson distribution are very much related in queuing theory. Under certain assumption, the arrival pattern in a queue (number of arrivals up to a given point of time) is Poisson while the inter arrival times are exponentially distributed.

A continuous random variable X is said to have an exponential distribution with parameter λ if its pdf is given by

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ \lambda e^{-\lambda x} & \text{if } x \geq 0. \end{cases}$$

Where $\lambda > 0$. Its cdf is given by $F(x) = 0$ if $x \leq 0$. For $x \geq 0$,

$$F(x) = \int_{-\infty}^{x} f(u)\, du = \int_{0}^{x} \lambda e^{-\lambda u}\, du = 1 - e^{-\lambda x}.$$

Hence

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 - e^{-\lambda x} & \text{if } x \geq 0. \end{cases}$$

Continuous distributions

Moments of exponential distribution:

The r-th moment about origin is given by

$$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)\, dx = \lambda \int_0^{\infty} x^r\, e^{-\lambda x} dx$$

For $r = 1{,}2$ one can easily evaluate the above integral using integration by parts once or twice and get

$$\mu = E(X) = \frac{1}{\lambda} \text{ and } E(X^2) = \frac{2}{\lambda^2}$$

from which the variance can be obtained as

$$\sigma^2 = E(X^2) - \mu^2 = \frac{1}{\lambda^2}$$

and consequently, the standard deviation $\sigma = 1/\lambda$. The exponential distribution has the unique property that its mean and standard deviation are equal.

To calculate the other moments, using repeated integration by parts creates problem. However, using gamma function, we can avoid complication.

Continuous distributions

The gamma function:

For $\alpha > 0$ is any real number, then we can define an integral

$$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1}\, dx.$$

Using integration by parts once, one can see that

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha).$$

By direct calculation, one can check that $\Gamma(1) = 1$. As an assignment, you verify that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\,.$$

Observe that, $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$, $\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3\sqrt{\pi}}{4}$,…. . Moreover, when n is a positive integer,

$$\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2) = \cdots = (n-1)(n-2)\cdots 2\cdot 1 = (n-1)!$$

Continuous distributions

Now, we are in a position to find all moments of exponential distribution. Observe that with a change of variable $y = \lambda x$,

$$\mu_r' = E(X^r) = \lambda \int_0^\infty x^r \, e^{-\lambda x} dx = \lambda \int_0^\infty \left(\frac{y}{\lambda}\right)^r e^{-y} \frac{dy}{\lambda}$$

$$= \frac{1}{\lambda^r} \int_0^\infty y^{r+1-1} \, e^{-y} dy = \frac{\Gamma(r+1)}{\lambda^r} = \frac{r!}{\lambda^r}, r = 1,2, \dots .$$

Hence,

$$\mu_1' = \frac{1}{\lambda}, \mu_2' = \frac{2}{\lambda^2}, \mu_3' = \frac{6}{\lambda^3}, \mu_4' = \frac{24}{\lambda^4}.$$

Thus,

$$\mu = \mu_1' = \frac{1}{\lambda}, \sigma^2 = \mu_2' - {\mu_1'}^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2{\mu_1'}^3 = \frac{2}{\lambda^3}, \mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'{\mu_1'}^2 - 3{\mu_1'}^4 = \frac{9}{\lambda^4}$$

Hence,

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = 4, \qquad \beta_2 = \frac{\mu_4}{\mu_2^2} = 9.$$

Continuous distributions

Moment generating function:

$$G(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx = \lambda \int_{0}^{\infty} e^{tx}\, e^{-\lambda x} dx$$

$$= \lambda \int_{0}^{\infty} e^{-(\lambda - t)x}\, dx = \frac{\lambda}{\lambda - t} = \left(1 - \frac{t}{\lambda}\right)^{-1}$$

Observe that the MGF exists for $t < \lambda$. Moreover,

$$\log G(t) = -\log\left(1 - \frac{t}{\lambda}\right) = \frac{t}{\lambda} + \frac{t^2}{\lambda^2} + \frac{t^3}{\lambda^3} + \frac{t^4}{\lambda^4} + \cdots$$

Hence,

$$\frac{G'(t)}{G(t)} = \frac{1}{\lambda} + \frac{2t}{\lambda^2} + \frac{3t^2}{\lambda^3} + \frac{4t^3}{\lambda^3} + \cdots.$$

implying that

$$G'(t) = \left(\frac{1}{\lambda} + \frac{2t}{\lambda^2} + \frac{3t^2}{\lambda^3} + \frac{4t^3}{\lambda^3} + \cdots\right) G(t)$$

Continuous distributions

Thus, $\mu = G'(0) = \frac{1}{\lambda}$. Now,

$$G''(t) = \left(\frac{1}{\lambda} + \frac{2t}{\lambda^2} + \frac{3t^2}{\lambda^3} + \cdots\right) G'(t) + \left(\frac{2}{\lambda^2} + \frac{6t}{\lambda^3} + \frac{12t^2}{\lambda^3} + \cdots\right) G(t)$$

Hence

$$E(X^2) = G''(0) = \frac{1}{\lambda} \times \frac{1}{\lambda} + \frac{2}{\lambda^2} = \frac{3}{\lambda^2}$$

and so on. Rest you calculate. Indeed,

$$G(t) = \left(1 - \frac{t}{\lambda}\right)^{-1} = 1 + \frac{t}{\lambda} + \frac{t^2}{\lambda^2} + \cdots = 1 + \sum_{r=1}^{\infty} \frac{t^r}{\lambda^r} = 1 + \sum_{r=1}^{\infty} \frac{r!}{\lambda^r} \times \frac{t^r}{r!}.$$

Hence,

μ_r' =Coefficient of $\frac{t^r}{r!}$ in the power series expansion of $G(t) = \frac{r!}{\lambda^r}$, $r = 1,2, \ldots$. Now substituting $r = 1,2,3,4$ one can find the first four moments of exponential distribution.

Continuous distributions

Example 3: The time between successive births in a city hospital is a random variable which is exponentially distributed with a mean of 6 hours. If a child birth takes place now, what is the probability that the next birth will occur not before the completion of next 8 hours?

Ans. Let X be the gap between successive births. It is given that the distribution of X is exponential with mean $\frac{1}{\lambda} = 6$. hence $\lambda = \frac{1}{6}$ and the pdf of X is given by

$$f(x) = \begin{cases} \frac{1}{6} e^{-\frac{1}{6}x} \text{ if } x > 0 \\ 0 \text{ otherwise.} \end{cases}$$

Hence, $\Pr\{\text{the next birth will occur not before the completion of next 8 hours}\}$

$$= \Pr\{X > 8\} = 1 - \Pr\{X \leq 8\} = 1 - F(8) = 1 - \left(1 - e^{-\frac{8}{6}}\right) = e^{-4/3} = 0.2636.$$

Continuous distributions

Memoryless property of exponential distribution:

If the distribution of X is exponential with parameter λ, then

$$\Pr\{X > x + t | X > t\} = \Pr\{X > x\}.$$

Proof. Observe that for $x > 0$, the CDF of exponential distribution is $F(x) = 1 - e^{-\lambda x}$. Hence

$$\Pr\{X > x\} = 1 - F(x) = e^{-\lambda x}.$$

Hence,

$$\Pr\{X > x + t | X > t\} = \frac{\Pr\{X > x + t, X > t\}}{\Pr\{X > t\}} = \frac{\Pr\{X > x + t\}}{\Pr\{X > t\}}$$

$$= \frac{1 - F(x + t)}{1 - F(x)} = \frac{e^{-\lambda(x+t)}}{e^{-\lambda t}} = e^{-\lambda x} = \Pr\{X > x\}.$$

Continuous distributions

The gamma distribution

We have already studied the integral

$$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1}\, dx, \alpha > 0.$$

Observe that with a change of variable $y = \lambda x$, we get

$$\int_0^\infty e^{-\lambda x} x^{\alpha-1}\, dx = \int_0^\infty e^{-y} \left(\frac{y}{\lambda}\right)^{\alpha-1} \frac{dy}{\lambda} = \frac{1}{\lambda^\alpha} \int_0^\infty e^{-y}\, y^{\alpha-1} dy = \frac{\Gamma(\alpha)}{\lambda^\alpha}.$$

Hence, if we substitute

$$f(x) = \begin{cases} \dfrac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda x} x^{\alpha-1} & \text{if } x > 0 \\ 0 & \text{otherwise,} \end{cases}$$

then, one can check that

$$\int_{-\infty}^\infty f(x)\, dx = \int_0^\infty f(x)\, dx$$

$f(x)$ so defined is known as the pdf of a gamma distribution. Observe that if $\alpha = 1$, then the gamma distribution reduces to the exponential distribution.

The Gamma Distribution

Moments of gamma distribution:

$$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)\,dx = \frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^\infty x^r e^{-\lambda x} x^{\alpha-1}\,dx$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^\infty e^{-\lambda x} x^{(\alpha+r)-1}\,dx = \frac{\lambda^\alpha}{\Gamma(\alpha)}\cdot\frac{\Gamma(\alpha+r)}{\lambda^{\alpha+r}} = \frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+r-1)}{\lambda^r}.$$

In particular,

$$\mu_1' = \frac{\alpha}{\lambda},\quad \mu_2' = \frac{\alpha(\alpha+1)}{\lambda^2},\quad \mu_3' = \frac{\alpha(\alpha+1)(\alpha+2)}{\lambda^3}, \mu_4' = \frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}{\lambda^4}.$$

Thus,

$$\mu = \mu_1' = \frac{\alpha}{\lambda}, \mu_2 = \sigma^2 = E(X^2) - \mu^2 = \frac{\alpha(\alpha+1)}{\lambda^2} - \left(\frac{\alpha}{\lambda}\right)^2 = \frac{\alpha}{\lambda^2}$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2{\mu_1'}^3 = \frac{2\alpha}{\lambda^3},\ \mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'{\mu_1'}^2 - 3{\mu_1'}^4 = \frac{3\alpha^2+6\alpha}{\lambda^4}.$$

Hence,

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{4}{\alpha},\qquad \beta_2 = \frac{\mu_4}{\mu_2^2} = 3 + \frac{6}{\alpha}.$$

Continuous distributions

Example 4: Find the harmonic mean of gamma distribution.

Ans. The harmonic mean H of a distribution is given by

$$\frac{1}{H} = E\left(\frac{1}{X}\right) = \int_{-\infty}^{\infty} \frac{1}{x} \cdot f(x)\, dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty \frac{1}{x} \cdot e^{-\lambda x} x^{\alpha-1} dx$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{-\lambda x}\, x^{\alpha-2} dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \times \frac{\Gamma(\alpha-1)}{\lambda^{\alpha-1}} = \frac{\lambda}{\alpha-1}$$

which exists for $\alpha > 1$. Hence, $H = \frac{\alpha-1}{\lambda}$.

The moment generating function:

$$G(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{tx}\, e^{-\lambda x} x^{\alpha-1}\, dx$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{-(\lambda-t)x}\, x^{\alpha-1} dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \times \frac{\Gamma(\alpha)}{(\lambda-t)^\alpha} = \left(\frac{\lambda}{\lambda-t}\right)^\alpha = \left(1 - \frac{t}{\lambda}\right)^{-\alpha}.$$

Thus,

$$\log G(t) = -\alpha \log\left(1 - \frac{t}{\lambda}\right) = \alpha\left[\sum_{r=1}^{\infty} \frac{t^r}{\lambda^r}\right] = \alpha\left[\sum_{r=1}^{\infty} \frac{t^r}{r!} \times \frac{r!}{\lambda^r}\right]$$

Observe that the MGF exists for $t < \lambda$. Now, μ_r' =Coefficient of $\frac{t^r}{r!}$ in the power series expansion of $G(t) = \frac{\alpha r!}{\lambda^r}, r = 1,2, \ldots$ and the central moments can be calculated as usual.

Example 5: Find the mean of $\sqrt{X}$, where the distribution of X is gamma with parameter $\alpha = 4$ and $\lambda = 3$.

Ans. Here $f(x) = \frac{3^4}{(4-1)!} e^{-3x} x^{4-1} = \frac{27}{2} \times e^{-3x} x^3$ if $x > 0$ and $f(x) = 0$ otherwise. Hence

$$E(\sqrt{X}\,) = \int_{-\infty}^{\infty} \sqrt{x} f(x)\, dx = \frac{27}{2} \int_0^{\infty} \sqrt{x} e^{-3x} x^3\, dx = \frac{27}{2} \int_0^{\infty} e^{-3x} x^{3.5}\, dx$$

$$= \frac{27}{2} \times \frac{\Gamma(4.5)}{3^{4.5}} = \frac{3.5 \times 2.5 \times 1.5 \times 0.5 \times \Gamma(0.5)}{2 \times 3^{1.5}} = 0.63\sqrt{\pi}.$$

Continuous distributions

The normal distribution:

- The normal distribution is a continuous distribution which is widely used in practice. Many distributions approaches to normal distribution under certain limiting condition. The pdf of normal distribution is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, -\infty < x < \infty.$$

If X is a random variable with the above pdf, it is customary to write $X \sim N(\mu, \sigma^2)$.

- Here, the parameter μ is positive, negative or zero, while $\sigma > 0$. Later on, we will see that μ is the mean of the distribution, while σ is the standard deviation.

- If $\mu = 0$ and $\sigma = 1$, then the corresponding normal distribution is known as the standard normal distribution. If the distribution of X is standard normal, then it pdf is

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, -\infty < x < \infty.$$

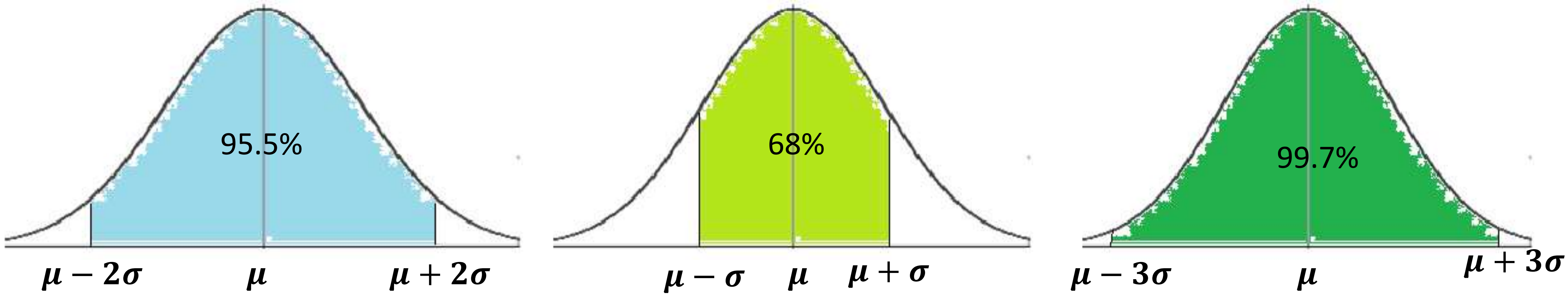
Thus, in this case $X \sim N(0,1)$. The standard normal distribution plays a very crucial role while dealing with any normal distribution.

Continuous distributions

The normal distribution:

- The curve of a normal distribution is perfectly bell shaped and is symmetric about $x = \mu$.

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$



- This shows that, though, for a normal distribution, the total area 1 is covered between $-\infty$ to ∞, one can see that there is hardly any are beyond $\mu - 3\sigma$ to $\mu + 3\sigma$.
- Mode is the value of x for which $f(x)$ is maximum. Using the pdf, one can check that the mode of the distribution is equal to μ where, maximum of $f(x)$ is $\frac{1}{\sqrt{2\pi}\,\sigma}$. Verify this using calculus.

Continuous distributions

The normal distribution:

- Because of symmetry, the area towards left of $x = \mu$ is equal to the area towards its right. That is

$$\int_{-\infty}^{\mu} f(x)\, dx = \int_{\mu}^{\infty} f(x)\, dx.$$

For example, with $u = \mu - x$,

$$\int_{-\infty}^{\mu} f(x)\, dx = -\frac{1}{\sqrt{2\pi}\,\sigma}\int_{\infty}^{0} e^{-\frac{1}{2u^2}}\, du = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{0}^{\infty} e^{-\frac{1}{2u^2}}\, du$$

and with $u = x - \mu$, one can check that

$$\int_{\mu}^{\infty} f(x)\, dx = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{0}^{\infty} e^{-\frac{1}{2u^2}}\, du.$$

Hence,

$$\int_{-\infty}^{\mu} f(x)\, dx = \frac{1}{2}$$

so that μ is also median of the distribution.

Continuous distributions

We will show that the mean of the distribution is also equal to μ. Thus, for a normal distribution, the mean, the median and the mode coincide.

Distribution function: The CDF of normal distribution is

$$F(x) = \Pr\{X \leq x\} = \int_{-\infty}^{x} f(u)\, du = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{x} e^{-\frac{1}{2}\left(\frac{u-\mu}{\sigma}\right)^2}\, du,$$

While that for the standard normal distribution is denoted by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}u^2}\, du.$$

This function is very widely used in statistics. For example, if $X \sim N(\mu, \sigma^2)$, then with a change of variable $z = \frac{u-\mu}{\sigma}$, i.e. $u = \mu + \sigma z$ and $du = \sigma dz$, then

$$\boldsymbol{F(x)} = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{x} e^{-\frac{1}{2}\left(\frac{u-\mu}{\sigma}\right)^2}\, du = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\frac{x-\mu}{\sigma}} e^{-\frac{1}{2}z^2}\, \sigma dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x-\mu}{\sigma}} e^{-\frac{1}{2}z^2}\, dz = \boldsymbol{\Phi\left(\frac{x-\mu}{\sigma}\right)}.$$

Continuous distributions

The relation

$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

is very important in finding probability for normal distribution. In particular, if $X \sim N(\mu, \sigma^2)$, then

$$\Pr\{a < X \leq b\} = F(b) - F(a) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

which helps in finding the area under the normal curve from $x = a$ to $x = b$.

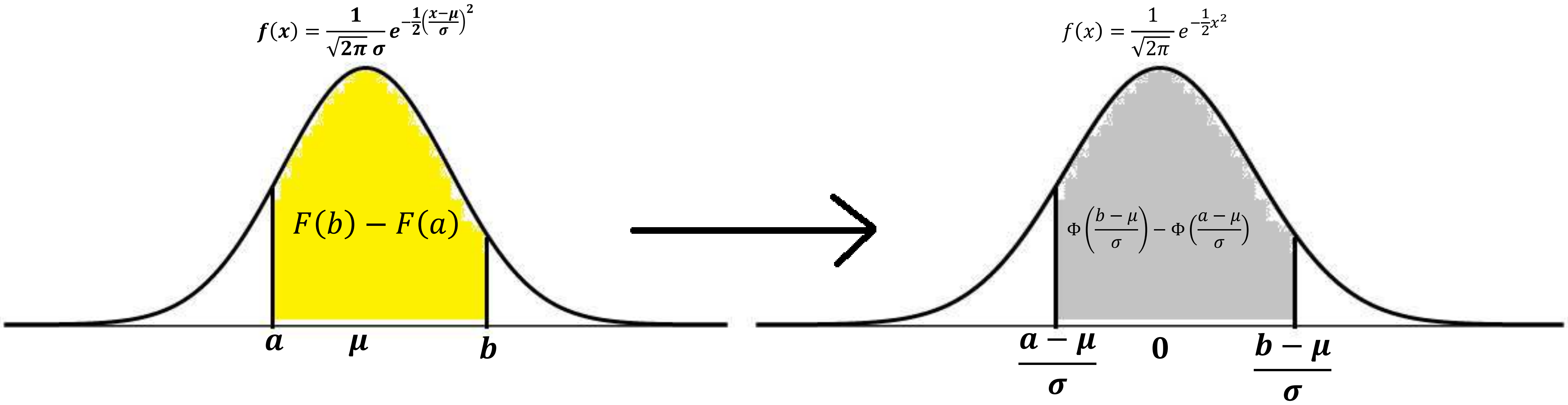


Table A7 Normal Distribution

Values of the distribution function $\Phi(z)$ [see (3), Sec. 24.8]. $\Phi(-z) = 1 - \Phi(z)$

z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$
	0.		**0.**		**0.**		**0.**		**0.**		**0.**
0.01	5040	0.51	6950	1.01	8438	1.51	9345	2.01	9778	2.51	9940
0.02	5080	0.52	6985	1.02	8461	1.52	9357	2.02	9783	2.52	9941
0.03	5120	0.53	7019	1.03	8485	1.53	9370	2.03	9788	2.53	9943
0.04	5160	0.54	7054	1.04	8508	1.54	9382	2.04	9793	2.54	9945
0.05	5199	0.55	7088	1.05	8531	1.55	9394	2.05	9798	2.55	9946
0.06	5239	0.56	7123	1.06	8554	1.56	9406	2.06	9803	2.56	9948
0.07	5279	0.57	7157	1.07	8577	1.57	9418	2.07	9808	2.57	9949
0.08	5319	0.58	7190	1.08	8599	1.58	9429	2.08	9812	2.58	9951
0.09	5359	0.59	7224	1.09	8621	1.59	9441	2.09	9817	2.59	9952
0.10	5398	0.60	7257	1.10	8643	1.60	9452	2.10	9821	2.60	9953
0.11	5438	0.61	7291	1.11	8665	1.61	9463	2.11	9826	2.61	9955
0.12	5478	0.62	7324	1.12	8686	1.62	9474	2.12	9830	2.62	9956
0.13	5517	0.63	7357	1.13	8708	1.63	9484	2.13	9834	2.63	9957
0.14	5557	0.64	7389	1.14	8729	1.64	9495	2.14	9838	2.64	9959
0.15	5596	0.65	7422	1.15	8749	1.65	9505	2.15	9842	2.65	9960
0.16	5636	0.66	7454	1.16	8770	1.66	9515	2.16	9846	2.66	9961
0.17	5675	0.67	7486	1.17	8790	1.67	9525	2.17	9850	2.67	9962
0.18	5714	0.68	7517	1.18	8810	1.68	9535	2.18	9854	2.68	9963
0.19	5753	0.69	7549	1.19	8830	1.69	9545	2.19	9857	2.69	9964
0.20	5793	0.70	7580	1.20	8849	1.70	9554	2.20	9861	2.70	9965
0.21	5832	0.71	7611	1.21	8869	1.71	9564	2.21	9864	2.71	9966
0.22	5871	0.72	7642	1.22	8888	1.72	9573	2.22	9868	2.72	9967
0.23	5910	0.73	7673	1.23	8907	1.73	9582	2.23	9871	2.73	9968
0.24	5948	0.74	7704	1.24	8925	1.74	9591	2.24	9875	2.74	9969
0.25	5987	0.75	7734	1.25	8944	1.75	9599	2.25	9878	2.75	9970
0.26	6026	0.76	7764	1.26	8962	1.76	9608	2.26	9881	2.76	9971
0.27	6064	0.77	7794	1.27	8980	1.77	9616	2.27	9884	2.77	9972
0.28	6103	0.78	7823	1.28	8997	1.78	9625	2.28	9887	2.78	9973
0.29	6141	0.79	7852	1.29	9015	1.79	9633	2.29	9890	2.79	9974
0.30	6179	0.80	7881	1.30	9032	1.80	9641	2.30	9893	2.80	9974
0.31	6217	0.81	7910	1.31	9049	1.81	9649	2.31	9896	2.81	9975
0.32	6255	0.82	7939	1.32	9066	1.82	9656	2.32	9898	2.82	9976
0.33	6293	0.83	7967	1.33	9082	1.83	9664	2.33	9901	2.83	9977
0.34	6331	0.84	7995	1.34	9099	1.84	9671	2.34	9904	2.84	9977
0.35	6368	0.85	8023	1.35	9115	1.85	9678	2.35	9906	2.85	9978
0.36	6406	0.86	8051	1.36	9131	1.86	9686	2.36	9909	2.86	9979
0.37	6443	0.87	8078	1.37	9147	1.87	9693	2.37	9911	2.87	9979
0.38	6480	0.88	8106	1.38	9162	1.88	9699	2.38	9913	2.88	9980
0.39	6517	0.89	8133	1.39	9177	1.89	9706	2.39	9916	2.89	9981
0.40	6554	0.90	8159	1.40	9192	1.90	9713	2.40	9918	2.90	9981
0.41	6591	0.91	8186	1.41	9207	1.91	9719	2.41	9920	2.91	9982
0.42	6628	0.92	8212	1.42	9222	1.92	9726	2.42	9922	2.92	9982
0.43	6664	0.93	8238	1.43	9236	1.93	9732	2.43	9925	2.93	9983
0.44	6700	0.94	8264	1.44	9251	1.94	9738	2.44	9927	2.94	9984
0.45	6736	0.95	8289	1.45	9265	1.95	9744	2.45	9929	2.95	9984
0.46	6772	0.96	8315	1.46	9279	1.96	9750	2.46	9931	2.96	9985
0.47	6808	0.97	8340	1.47	9292	1.97	9756	2.47	9932	2.97	9985
0.48	6844	0.98	8365	1.48	9306	1.98	9761	2.48	9934	2.98	9986
0.49	6879	0.99	8389	1.49	9319	1.99	9767	2.49	9936	2.99	9986
0.50	6915	1.00	8413	1.50	9332	2.00	9772	2.50	9938	3.00	9987

Table A8 **Normal Distribution**

Values of z for given values of $\Phi(z)$ [see (3), Sec. 24.8] and $D(z) = \Phi(z) - \Phi(-z)$
Example: $z = 0.279$ if $\Phi(z) = 61\%$; $z = 0.860$ if $D(z) = 61\%$.

%	$z(\Phi)$	$z(D)$	%	$z(\Phi)$	$z(D)$	%	$z(\Phi)$	$z(D)$
1	−2.326	0.013	41	−0.228	0.539	81	0.878	1.311
2	−2.054	0.025	42	−0.202	0.553	82	0.915	1.341
3	−1.881	0.038	43	−0.176	0.568	83	0.954	1.372
4	−1.751	0.050	44	−0.151	0.583	84	0.994	1.405
5	−1.645	0.063	45	−0.126	0.598	85	1.036	1.440
6	−1.555	0.075	46	−0.100	0.613	86	1.080	1.476
7	−1.476	0.088	47	−0.075	0.628	87	1.126	1.514
8	−1.405	0.100	48	−0.050	0.643	88	1.175	1.555
9	−1.341	0.113	49	−0.025	0.659	89	1.227	1.598
10	−1.282	0.126	50	0.000	0.674	90	1.282	1.645
11	−1.227	0.138	51	0.025	0.690	91	1.341	1.695
12	−1.175	0.151	52	0.050	0.706	92	1.405	1.751
13	−1.126	0.164	53	0.075	0.722	93	1.476	1.812
14	−1.080	0.176	54	0.100	0.739	94	1.555	1.881
15	−1.036	0.189	55	0.126	0.755	95	1.645	1.960
16	−0.994	0.202	56	0.151	0.772	96	1.751	2.054
17	−0.954	0.215	57	0.176	0.789	97	1.881	2.170
18	−0.915	0.228	58	0.202	0.806	97.5	1.960	2.241
19	−0.878	0.240	59	0.228	0.824	98	2.054	2.326
20	−0.842	0.253	60	0.253	0.842	99	2.326	2.576
21	−0.806	0.266	61	0.279	0.860	99.1	2.366	2.612
22	−0.772	0.279	62	0.305	0.878	99.2	2.409	2.652
23	−0.739	0.292	63	0.332	0.896	99.3	2.457	2.697
24	−0.706	0.305	64	0.358	0.915	99.4	2.512	2.748
25	−0.674	0.319	65	0.385	0.935	99.5	2.576	2.807
26	−0.643	0.332	66	0.412	0.954	99.6	2.652	2.878
27	−0.613	0.345	67	0.440	0.974	99.7	2.748	2.968
28	−0.583	0.358	68	0.468	0.994	99.8	2.878	3.090
29	−0.553	0.372	69	0.496	1.015	99.9	3.090	3.291
30	−0.524	0.385	70	0.524	1.036			
31	−0.496	0.399	71	0.553	1.058	99.91	3.121	3.320
32	−0.468	0.412	72	0.583	1.080	99.92	3.156	3.353
33	−0.440	0.426	73	0.613	1.103	99.93	3.195	3.390
34	−0.412	0.440	74	0.643	1.126	99.94	3.239	3.432
35	−0.385	0.454	75	0.674	1.150	99.95	3.291	3.481
36	−0.358	0.468	76	0.706	1.175	99.96	3.353	3.540
37	−0.332	0.482	77	0.739	1.200	99.97	3.432	3.615
38	−0.305	0.496	78	0.772	1.227	99.98	3.540	3.719
39	−0.279	0.510	79	0.806	1.254	99.99	3.719	3.891
40	−0.253	0.524	80	0.842	1.282			

Continuous distributions

In table $A7$, $\Phi(z)$ is given for only positive values z. Observe that $\Phi(0) = 0.5$ and using

$$\Phi(-z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-z} e^{-\frac{1}{2}x^2}\, dx$$

and with a change of variable $y = -x$, one can see that

$$\Phi(-z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-z} e^{-\frac{1}{2}x^2}\, dx = \frac{1}{\sqrt{2\pi}} \int_{z}^{\infty} e^{-\frac{1}{2}y^2}\, dy$$

Using the fact that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}\, dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}y^2}\, dy + \frac{1}{\sqrt{2\pi}} \int_{z}^{\infty} e^{-\frac{1}{2}y^2}\, dy = 1$$

one can see that

$$\Phi(-z) = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}y^2}\, dy = 1 - \Phi(z).$$

Continuous distributions

Example 5: If $X \sim N(50, 225)$ find (a) $\Pr\{40 \le X \le 70\}$ (b) $\Pr\{ X > 45\}$ (c) $\Pr\{ X \le 55\}$. Also (d) find c such that $\Pr\{ X > c\} = 0.35$ and (e) find d such that $\Pr\{ 41 < X < d\} = 0.40$.

Ans. (a) $X \sim N(50, 225) \Rightarrow \mu = 50, \sigma^2 = 225 \Rightarrow \sigma = 15$. Hence, (using Table A7)

$$\Pr\{40 \le X \le 70\} = \Phi\left(\frac{70-50}{15}\right) - \Phi\left(\frac{40-50}{15}\right) = \Phi(1.33) - \Phi(-0.67)$$

$$= \Phi(1.33) - \{1 - \Phi(0.67)\} = 0.9082 - 1 + 0.7486 = 0.6568.$$

(b) Again using Table A7,

$$\Pr\{ X > 45\} = 1 - \Pr\{X \le 45\} = 1 - \Phi\left(\frac{45-50}{15}\right)$$

$$= 1 - \Phi(-0.33) = \Phi(.33) = 0.6293.$$

(c) Again using Table A7,

$$\Pr\{ X \le 55\} = \Phi\left(\frac{55-50}{15}\right) = \Phi(0.33) = 0.6293.$$

Continuous distributions

(d) We have to find c such that $\Pr\{X > c\} = 0.35$. Hence,

$$1 - \Pr\{X \le c\} = 0.35 \Rightarrow \Pr\{X \le c\} = 0.65 \Rightarrow \Phi\left(\frac{c-50}{15}\right) = 0.65$$

From Table A8,

$$\frac{c-50}{15} = 0.385. \Rightarrow c = 50 + 15 \times 0.385 = 55.775$$

(e) We have to find d such that $\Pr\{41 < X < d\} = 0.40$.

$$\Rightarrow \Phi\left(\frac{d-50}{15}\right) - \Phi\left(\frac{41-50}{15}\right) = \Phi\left(\frac{d-50}{15}\right) - \Phi(-0.6) = \Phi\left(\frac{d-50}{15}\right) - 1 + \Phi(0.6)$$

$$= \Phi\left(\frac{d-50}{15}\right) - 1 + 0.7257 = 0.40.$$

$$\Rightarrow \Phi\left(\frac{d-50}{15}\right) = 0.6743 \Rightarrow \frac{d-50}{15} = 0.45$$

by virtue table A8. Hence, $d = 56.75$.

Continuous distributions

Example 6: The local authorities in a certain city decide to install 2,000 LED street lights in a street of the city. If the life time of a light is normally distributed with an average life of 10,000 hours and standard deviation of 2,000 hours, (a) about how many lights are expected to fail in the-first 7,000 hours of operation? (b) After what time 10% of the lights will fail? (c) After what time 90% lights would have been replaced?

Ans. Let X be the lifetime of a random LED street light. Then $X \sim N(10000, 2000^2)$.
Probability that an LED light will fail in the first 7,000 hours of operation

$$\Pr\{X \le 7000\} = \Phi\left(\frac{7000 - 10000}{2000}\right) = \Phi(-1.5) = 1 - \Phi(1.5) = 1 - 0.9332$$

$$= 0.0668.$$

Hence, expected number of failure in the first 7,000 hours of operation is
$2000 \times 0.0668 \approx 134.$

(b) $\Pr\{X < t\} = 0.10$, find t. (c) (b) $\Pr\{X < t'\} = 0.90$, find t'.

Continuous distributions

Example 7: The monthly incomes of a group of 10,000 persons were found to be normally distributed with mean ₹ 7500 and standard deviation Rs. 500. Show that, of this group, about 95% had income exceeding ₹ 6680 and only 5% had income exceeding ₹ 8320. What was the lowest income among the richest 1000?

Ans. Let X be the income of a random person. Then $X \sim N(7500, 500^2)$. The probability that the income of a random person exceeds ₹ 6680 is

$$\Pr\{X > 6680\} = 1 - \Pr\{X \leq 6680\} = 1 - \Phi\left(\frac{6680 - 7500}{500}\right)$$

$$= 1 - \Phi(-1.64) = \Phi(1.64) = 0.9495$$

Thus, about 95% had income exceeding ₹ 6680. Similarly, the probability that the income of a random person exceeds ₹ 8320 is

$$\Pr\{X > 8320\} = 1 - \Pr\{X \leq 8320\} = 1 - \Phi\left(\frac{8320 - 7500}{500}\right)$$

$$= 1 - \Phi(1.64) = 1 - 0.9495 = 0.0505$$

Hence about 5% had income exceeding ₹ 8320.

Continuous distributions

Example 7 (continuing): What was the lowest income among the richest 1000?

Let L be the lowest income of richest 1000. Then,

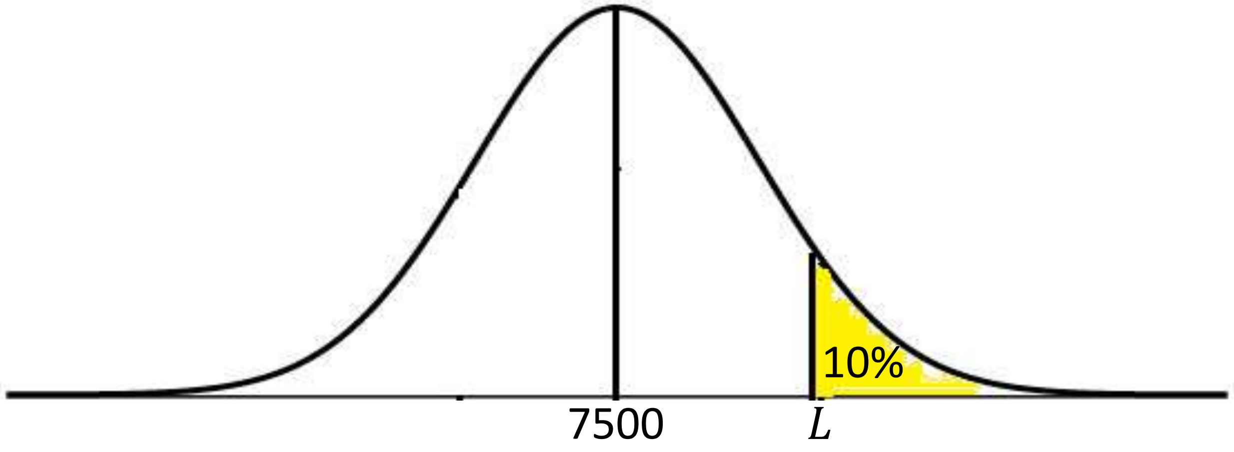
$$\Pr\{X \geq L\} = \frac{1000}{10000} = 0.1$$

$$\Rightarrow 1 - \Phi\left(\frac{L - 7500}{500}\right) = 0.1$$

$$\Rightarrow \Phi\left(\frac{L - 7500}{500}\right) = 0.9$$

$$\Rightarrow \frac{L - 7500}{500} = 1.282 \quad \text{(from Table A8)}$$

$$\Rightarrow L = 7500 + 500 \times 1.282 = 8141$$



Hence, the lowest income of richest 1000 persons is ₹ 8141.

Continuous distributions

Exercise 8: In an university examination of a particular year, 60% of the students failed when mean of the marks was 50% and standard deviation was 15%. What was the pass mark? University decided to relax the conditions of passing by lowering the pass marks, to show its pass result 70%. Find the minimum marks for a student to pass, supposing the marks to be normally distributed and no change in the performance of students takes place.

Ans. Let K be the actual pass mark and X be the mark of a random student. Then $X \sim N(50, 15^2)$. A student fails if his mark is $< K$. As per the data given with the question,

$$\Pr\{X < K\} = 0.60 \Rightarrow \Phi\left(\frac{K - 50}{15}\right) = 0.60 \Rightarrow \frac{K - 50}{15} = 0.202 \Rightarrow K = 53.03 \approx 53.$$

Let K' be the revised pass mark so that 70% students can pass. Then

$$\Pr\{X \geq K'\} = 0.70 \Rightarrow 1 - \Phi\left(\frac{K' - 50}{15}\right) = 0.70 \Rightarrow \Phi\left(\frac{K' - 50}{15}\right) = 0.30$$

$$\Rightarrow \frac{K' - 50}{15} = -0.524 \Rightarrow K' = 42.14 \approx 42.$$

So, with the revised pass mark of 42%, the pass percentage will be 70.

Continuous distributions

Exercise 9: In NIT Rourkela, a student passes in a subject if his total mark is 35 or more. He gets an Ex grade if he secures 90 or more, A grade if he secures 80 or more but less than 90 and so on. In the year 2019, 5% students got Ex grade in MA 2203 while 9% student failed. Assuming a normal distribution of marks, what was the mean mark of MA 2203 in the year 2019 and what was the standard deviation? What percentage of students secured B grade?

Ans. Let X be the mark secured in MA 2203 by a random student in the year 2019. Let μ be the mean mark and σ be the standard deviation of marks. Then $X \sim N(\mu, \sigma^2)$. A student fails if his mark is < 35. As per the data given with the question,

$$\Pr\{X < 35\} = 0.09 \Rightarrow \Phi\left(\frac{35-\mu}{\sigma}\right) = 0.09 \Rightarrow \frac{35-\mu}{\sigma} = -1.341 \cdots\cdots (1).$$

$$\Pr\{X \geq 90\} = 0.05 \Rightarrow 1 - \Phi\left(\frac{90-\mu}{\sigma}\right) = 0.05$$

$$\Rightarrow \Phi\left(\frac{90-\mu}{\sigma}\right) = 0.95 \Rightarrow \frac{90-\mu}{\sigma} = 1.645 \cdots\cdots (2).$$

Continuous distributions

Exercise 9: Continued… What percentage of students secured B grade?

Ans. continued….

From Equations (1) and (2),

$$\frac{90-\mu}{35-\mu} = -1.2267 \Rightarrow 90-\mu = 1.2267\mu - 42.9695$$

$$\Rightarrow 2.2267\mu = 132.9695 \Rightarrow \mu = 59.7159 \approx 60$$

and substituting the value of μ in Equation (2), we get

$$\sigma = \frac{90-\mu}{1.645} = 18.4098.$$

Probability that a random student gets a B grade is

$$\Pr\{70 \leq X < 80\} = \Phi\left(\frac{80-59.72}{18.41}\right) - \Phi\left(\frac{70-59.72}{18.41}\right)$$

$$= \Phi(1.10) - \Phi(0.56) = 0.8643 - 0.7123 = 0.1520.$$

Thus, about 15% students have got B grade.

Continuous distributions

Exercise 10: Steel rods are manufactured to be 3 inches in diameter but they are acceptable if they are inside the limits $2{\cdot}99$ inches and $3{\cdot}01$ inches. It is observed that 5% are rejected as oversize and 5% are rejected as undersize. Assuming that the diameters are normally distributed. find the mean and standard deviation of the distribution. Hence calculate the proportion of rejections if the permissible limits were widened to $2{\cdot}985$ and $3{\cdot}015$ inches.

You do it.

Continuous distributions

Moments of normal distribution:

Let $X \sim N(\mu, \sigma^2)$. Then the mean is given by

$$\int_{-\infty}^{\infty} x\, f(x) dx = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} x\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} (\mu + \sigma z)\, e^{-\frac{1}{2}z^2} \sigma dz$$

$$z = \frac{x-\mu}{\sigma}$$
$$\Rightarrow x = \mu + \sigma z$$
$$\Rightarrow dx = \sigma\, dz$$

$$= \mu \cdot \frac{\mathbf{1}}{\sqrt{\mathbf{2\pi}}} \int_{-\infty}^{\infty} \boldsymbol{e^{-\frac{1}{2}z^2} dz} + \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \boldsymbol{z e^{-\frac{1}{2}z^2} dz}$$

$$= \mu \times \mathbf{1} + \frac{\sigma}{\sqrt{2\pi}} \times \mathbf{0} = \mu,$$

since $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} dz$ is the total area under the standard normal distribution which is equal to 1 and in $\int_{-\infty}^{\infty} z e^{-\frac{1}{2}z^2} dz$, the integrand is an odd function of z, the value of the integral is 0.

Continuous distributions

Now, the odd ordered central moments are given by

$$\mu_{2r-1} = \int_{-\infty}^{\infty} (x-\mu)^{2r-1} f(x)dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} (x-\mu)^{2r-1} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z)^{2r-1} e^{-\frac{1}{2}z^2} dz = \frac{\sigma^{2r-1}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^{2r-1} e^{-\frac{1}{2}z^2} dz = 0.$$

Hence, $\mu_{2r-1} = 0$ for $r = 1,2,\ldots$.

Continuous distributions

The even ordered central moments are given by

$$\mu_{2r} = \int_{-\infty}^{\infty} (x-\mu)^{2r} f(x)dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} (x-\mu)^{2r} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (\sigma z)^{2r} e^{-\frac{1}{2}z^2} dz = \frac{\sigma^{2r}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} z^{2r} e^{-\frac{1}{2}z^2} dz$$

$$= \frac{2\sigma^{2r}}{\sqrt{2\pi}}\int_{0}^{\infty} z^{2r} e^{-\frac{1}{2}z^2} dz = \frac{2\sigma^{2r}}{\sqrt{2\pi}}\int_{0}^{\infty} (2u)^{r} e^{-u} \frac{du}{\sqrt{2u}}$$

$$u = \frac{1}{2}z^2$$
$$\Rightarrow du = zdz$$
$$\Rightarrow dz = \frac{du}{\sqrt{2u}}$$

$$= \frac{2^r\sigma^{2r}}{\sqrt{\pi}}\int_{0}^{\infty} u^{r-\frac{1}{2}} e^{-u} du = \frac{2^r\sigma^{2r}}{\sqrt{\pi}} \times \Gamma\left(r+\frac{1}{2}\right)$$

$r = 1,2, \ldots$.

Continuous distributions

In particular,

Now,

$$\Gamma\left(r+\frac{1}{2}\right)=\left(r-\frac{1}{2}\right)\Gamma\left(r-\frac{1}{2}\right)=\left(r-\frac{1}{2}\right)\left(r-\frac{3}{2}\right)\Gamma\left(r-\frac{3}{2}\right)$$

$$=\cdots\cdots=\left(r-\frac{1}{2}\right)\left(r-\frac{3}{2}\right)\cdots\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$=\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2r-1)}{2^r}=\frac{(2r)!}{2^{2r}r!}\sqrt{\pi},$$

Hence,

$$\mu_{2r}=\frac{2^r\sigma^{2r}}{\sqrt{\pi}}\times\Gamma\left(r+\frac{1}{2}\right)=\frac{2^r\sigma^{2r}}{\sqrt{\pi}}\times\frac{(2r)!}{2^{2r}r!}\sqrt{\pi}$$

$$=\frac{(2r)!}{2^r r!}\cdot\sigma^{2r}, r=1,2,\ldots$$

Continuous distributions

Moments of normal distribution:

$$\mu_{2r-1} = 0, \qquad \mu_{2r} = \frac{(2r)!}{2^r r!} \cdot \sigma^{2r}, r = 1,2, \ldots$$

implies

$$\mu_2 = \frac{2!}{2^1 1!} \cdot \sigma^2 = \sigma^2,$$

$$\mu_3 = 0,$$

$$\mu_4 = \frac{4!}{2^2 2!} \cdot \sigma^4 = 3\sigma^4.$$

Now,

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = 0, \qquad \beta_2 = \frac{\mu_4}{\mu_2^2} = 3.$$

Continuous distributions

Moment generating function:

$$G(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{tx} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\mu+\sigma z)} e^{-\frac{1}{2}z^2} dz = e^{t\mu} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t\sigma z} e^{-\frac{1}{2}z^2} dz$$

$$z = \frac{x-\mu}{\sigma}$$
$$\Rightarrow x = \mu + \sigma z$$
$$\Rightarrow dx = \sigma\, dz$$

$$= e^{t\mu} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2\sigma t z)} dz$$

$$= e^{t\mu} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-\sigma t)^2 + \frac{1}{2}\sigma^2 t^2} dz$$

$$= e^{\mu t + \frac{1}{2}\sigma^2 t^2} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-\sigma t)^2} dz = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

since $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2-\sigma t)} dz$ represents the total area of $N(\sigma t, 1)$ distribution.

Continuous distributions

Moment generating function:

$$G(t) = e^{\mu t+\frac{1}{2}\sigma^2 t^2}$$

implies that

$$\log G(t) = \mu t + \frac{1}{2}\sigma^2 t^2$$

and hence,

$$\frac{G'(t)}{G(t)} = \mu + \sigma^2 t \Rightarrow G'(t) = (\mu + \sigma^2 t)G(t).$$

$$\Rightarrow G''(t) = (\mu + \sigma^2 t)G'(t) + \sigma^2 G(t)$$

Hence,

$$\mu_1' = G'(0) = \mu,\ \mu_2' = G''(0) = \mu^2 + \sigma^2,$$

from which one can easily obtain $\mu_2 = \mu_2' - {\mu_1'}^2 = \sigma^2$. By successive differentiation of $G(t)$, other moments can be obtained.

Continuous distributions

Exercise 11: Prove that if $X \sim N(\mu, \sigma^2)$, then the quartile deviation, the mean deviation and the standard deviation are approximately in the ratio $10 : 12 : 15$.

Ans. There are three quartiles Q_1, Q_2 and Q_3 that divide a distribution into four equal parts. For continuous distributions Q_i has the property that the area up to Q_i under $f(x)$ is equal to $i/4, i = 1,2,3$. Thus, $F(Q_i) = i/4$. The *quartile deviation* (also known as *semi inter quartile range*) is a measure of dispersion defined by

$$Q.D. = \frac{Q_3 - Q_1}{2}.$$

The *mean deviation* (from mean) is defined as the arithmetic mean or expectation of the absolute deviations from mean. For frequency distribution, it is defined as

$$M.D. = \frac{1}{N}\sum_{i=1}^{n} f_i |x_i - \bar{x}|.$$

For continuous distribution

$$M.D. = E|X - \mu|.$$

Continuous distributions

For a normal distribution

$$F(Q_1) = 0.25, F(Q_3) = 0.75 \Rightarrow \Phi\left(\frac{Q_1 - \mu}{\sigma}\right) = 0.25, \Phi\left(\frac{Q_3 - \mu}{\sigma}\right) = 0.75.$$

In view of Table A8,

$$\frac{Q_1 - \mu}{\sigma} = -0.674, \qquad \frac{Q_3 - \mu}{\sigma} = 0.674 \qquad \Rightarrow Q.D. = \frac{Q_3 - Q_1}{2} = 0.674\sigma \approx \frac{2}{3}\sigma.$$

Now,

$$M.D. = E|X - \mu| = \int_{-\infty}^{\infty} |x - \mu|\, f(x)\, dx = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} |x - \mu|\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\, dx$$

$$z = \frac{x - \mu}{\sigma} \Rightarrow x = \mu + \sigma z \Rightarrow dx = \sigma\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |\sigma z| e^{-\frac{1}{2}z^2}\, dz = \frac{2\sigma}{\sqrt{2\pi}} \int_0^{\infty} z\, e^{-\frac{1}{2}z^2} dz = \frac{2\sigma}{\sqrt{2\pi}} \int_0^{\infty} \sqrt{2u}\, e^{-u} \frac{du}{\sqrt{2u}}$$

$$= \sqrt{2/\pi}\sigma \approx \frac{4}{5}\sigma.$$

$$u = \frac{1}{2}z^2 \Rightarrow du = z dz \Rightarrow dz = \frac{du}{\sqrt{2u}}$$

Hence,

$$Q.D.: M.D.: S.D. = \frac{2}{3}\sigma : \frac{4}{5}\sigma : \sigma = 10 : 12 : 15$$

Continuous distributions

Normal approximation to binomial distribution:

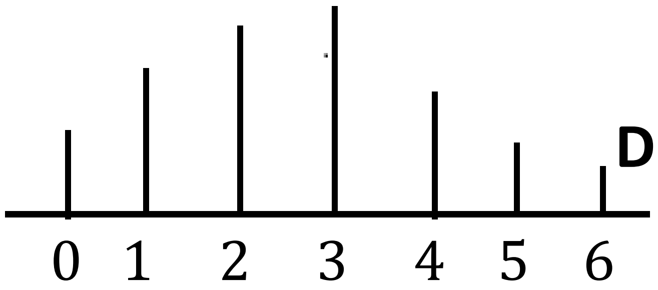
Assume that $X \sim B(n, p)$ and n is large (say >30). Let a and b be integers, $0 \le a \le b \le n$. Then by De Moivre-Laplace theorem,

$$\Pr\{a \le X \le b\} = \Phi(b^*) - \Phi(a^*)$$

where,

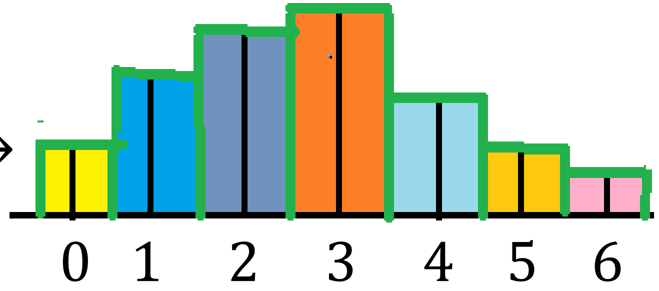
$$a^* = \frac{a - np - 0.5}{\sqrt{npq}}, b^* = \frac{b - np + 0.5}{\sqrt{npq}}.$$

Somebody may ask the role of ± 0.5. The binomial distribution is a discrete distribution, while the normal distribution is a continuous distribution.



Discrete to Continuous →

$$\boldsymbol{Hight = Area}$$



The probability $\Pr\{X = k\}$ has been replaced by $\Pr\{k - 1/2 \le X \le k + 1/2\}$.

Continuous distributions

Example 12: The germination success rate for papaya seeds is 75%. In a package of 400 seeds, what is the probability that more than 80% of the seeds germinate? What is the probability that the number of germinations is anywhere between 290 to 308 seeds?

Ans. Let X be the number of seeds germinate in a pack of 400. Since, 75% of the seeds germinate, p =probability that any seed germinate$=0.75$. Hence $X \sim B(400, 0.75)$. Hence,

$$np = 300, npq = 75.$$

$$\Pr\{\text{More than 80\% germination}\} = \Pr\{X > 320\} = 1 - \Pr\{X \leq 320\}$$

$$= 1 - \Phi\left(\frac{320 - 300 + 0.5}{\sqrt{75}}\right) = 1 - \Phi(2.37) = 1 - 0.9911 = 0.0089.$$

$$\Pr\{290 \leq X \leq 308\} = \Phi\left(\frac{308 - 300 + 0.5}{\sqrt{75}}\right) - \Phi\left(\frac{290 - 300 - 0.5}{\sqrt{75}}\right)$$

$$= \Phi(0.98) - \Phi(-1.21) = \Phi(0.98) - 1 + \Phi(1.21)$$

$$= 0.8365 - 1 + 0.8869 = 0.7234.$$

Continuous distributions

Example 13: How many times a fair coin must be tossed so as to obtain at least 500 heads with a probability of 0.4? (Use normal approximation of binomial distribution.)

Ans. Let n be the number of times a fair coin is tossed and let X be the number of heads obtained. Then $X \sim B(n, 1/2)$. Hence,

$$np = n/2, npq = n/4.$$

$$\begin{aligned}
\Pr\{\text{at least 500 heads}\} &= \Pr\{X \geq 500\} \\
&= 1 - \Pr\{X \leq 499\} \\
&= 1 - \Phi\left(\frac{499 - \frac{n}{2} + 0.5}{\sqrt{\frac{n}{4}}}\right) = 0.4 \\
&\Rightarrow \Phi\left(\frac{999 - n}{\sqrt{n}}\right) = 0.6
\end{aligned}$$

Continuous distributions

$$\Rightarrow \frac{999-n}{\sqrt{n}} = 0.253 \Rightarrow 999 - n = 0.253\sqrt{n}$$

$$\Rightarrow 0.253\sqrt{n} = 999 - n \Rightarrow 0.064n = 998001 + n^2 - 1998n.$$

$$\Rightarrow n^2 - 1998.064n + 998001 = 0 \Rightarrow n \approx 1007 \text{ or } 991.$$

If $n = 1007$, then $\Phi\left(\frac{999-n}{\sqrt{n}}\right) < 0.5$ and then

$$\Pr\{X \geq 500\} = 1 - \Pr\{X \leq 499\} > 0.5$$

Hence, the coin should be tosses 991 times to ensure at least 500 heads with probability 0.4. Observe that if $n = 991$, then

$$\Phi\left(\frac{999-n}{\sqrt{n}}\right) = \Phi(0.25) = 0.60$$

and hence

$$\Pr\{X \geq 500\} = 1 - \Pr\{X \leq 499\} = 0.4.$$

Continuous distributions

Example 14: (Bernoulli law of large numbers) . In an experiment let an event A is an event with probability p such that $0 < p < 1$ and let X be the number of times A happens in n independent trials. Show that for any given $\epsilon > 0$

$$\lim_{n\to\infty} \Pr\left\{\left|\frac{X}{n} - p\right| \leq \epsilon\right\} = 1\,.$$

Proof. Observe that X$\sim B(n,p)$. Now,

$$\Pr\left\{\left|\frac{X}{n} - p\right| \leq \epsilon\right\} = \Pr\{-n\epsilon \leq X - np \leq n\epsilon\} = \Pr\left\{-\sqrt{\frac{n}{pq}}\epsilon \leq \frac{X-np}{\sqrt{npq}} \leq \sqrt{\frac{n}{pq}}\epsilon\right\}.$$

Since, by De Moivre Laplace theorem, the distribution of $\frac{X-np}{\sqrt{npq}}$ approaches to standard normal (that is $N(0,1)$) as $n \to \infty$, hence

$$\Pr\left\{-\sqrt{\frac{n}{pq}}\epsilon \leq \frac{X-np}{\sqrt{npq}} \leq \sqrt{\frac{n}{pq}}\epsilon\right\} \sim \Phi\left(\sqrt{\frac{n}{pq}}\epsilon\right) - \Phi\left(-\sqrt{\frac{n}{pq}}\epsilon\right) \to \Phi(\infty) - \Phi(-\infty) = 1 - 0$$
$$= 1$$

Continuous distributions

Ex13: If the lifetime X of a certain kind of automobile battery is normally distributed with a mean of 5 years and a standard deviation of 1 year, and the manufacturer wishes to guarantee the battery for 4 years, what percentage of the batteries will he have to replace under the guarantee?

Hints: $X \sim N(5, 1^2)$, Ans. $100 \times \Pr\{X < 4\}$.

Ex.14 A manufacturer knows from experience that the resistance of resistors he produces is normal with mean $150\ \Omega$ and standard deviation $5\ \Omega$. What percentage of the resistors will have resistance between and Between $148\ \Omega$ and $152\ \Omega$? Between $140\ \Omega$ and $160\ \Omega$?

Hints: $X \sim N(150, 5^2)$, Ans. $100 \times \Pr\{148 \leq X \leq 152\}$.

Ex 15. The breaking strength X [kg] of a certain type of plastic block is normally distributed with a mean of 1500 kg and a standard deviation of 50 kg. What is the maximum load such that we can expect no more than 5% of the blocks to break?

Hints: $X \sim N(1500, 50^2)$, Ans. Find L such that $\Pr\{L \geq X\} \leq 0.05 \Rightarrow F(L) \leq 0.05 \Rightarrow L$?.

Continuous distributions

Ex16: A producer sells electric bulbs in cartons of 1000 bulbs. Find the probability that any given carton contains not more than 1% defective bulbs, assuming the production process to be a Bernoulli experiment with $p = 1\%$ (probability that any given bulb will be defective).

Hints: X= def. bulbs in a cartoon. $X \sim B(1000, 0.01)$, Find $\Pr\{X \leq 10\} = \Phi\left(\frac{10-10+0.5}{\sqrt{1000\times0.01\times0.99}}\right)$

Ex.17 If sick-leave time X used by employees of a company in one month is (very roughly) normal with mean 1000 hours and standard deviation 100 hours, how much time t should be budgeted for sick leave during the next month if t is to be exceeded with probability of only 20%?

Hints: $X \sim N(1000, 100^2)$. Find t such that $\Pr\{X > t\} = 0.20$.

Ex 18. If the monthly machine repair and maintenance cost X in a certain factory is known to be normal with mean ₹ 120,000 and standard deviation ₹20,000, what is the probability that the repair cost for the next month will exceed the budgeted amount of ₹150,000?

Hints: $X \sim N(120000, 20000^2)$. Find $\Pr\{X > 150000\}$.

Continuous distributions

Ex19: If the resistance X of certain wires in an electrical network is normal with mean $0.01\ \Omega$ and standard deviation $0.001\ \Omega$, how many of 1000 wires will meet the specification that they have resistance between $0.009\ \Omega$ and $0.011\ \Omega$?

Hints: $X \sim N(0.01, 0.001^2)$, $\Pr\{0.009 \le X \le 0.011\} = \Phi\left(\frac{0.009-0.01}{0.001}\right) - \Phi\left(\frac{0.011-0.01}{0.001}\right)$ (×1000)

Example 20: Prove that

$$\frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = 1.$$

$$z = \frac{x-\mu}{\sigma} \Rightarrow x = \mu + \sigma z \Rightarrow dx = \sigma\, dz$$

Proof:

$$\frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} dz$$

Integrand is an even function of z.

$$= \frac{2}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-\frac{1}{2}z^2} dz = \frac{2}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-u} \frac{du}{\sqrt{2u}}$$

$$u = \frac{1}{2}z^2 \Rightarrow du = z dz \Rightarrow dz = \frac{du}{\sqrt{2u}}$$

$$= \frac{1}{\sqrt{\pi}}\int_{0}^{\infty} e^{-u} u^{\frac{1}{2}-1} du = \frac{1}{\sqrt{\pi}} \times \Gamma\left(\frac{1}{2}\right) = 1.$$

Continuous distributions

Example 20: If $X \sim N(\mu, \sigma^2)$ and $Y = \left(\frac{X-\mu}{\sigma}\right)^2$, then prove that the distribution of Y is gamma with $\alpha = \lambda = 1/2$.

Proof: Since If $X \sim N(\mu, \sigma^2)$,

$$\int_{-\infty}^{\infty} f(x)\, dx = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = 1.$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} dz = 1 \Rightarrow \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{1}{2}z^2} dz = 1$$

$$z = \frac{x-\mu}{\sigma} \Rightarrow x = \mu + \sigma z \Rightarrow dx = \sigma\, dz$$

$$\Rightarrow \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{1}{2}y} \frac{dy}{2\sqrt{y}} = 1 \Rightarrow \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{1}{2}y}\, y^{\frac{1}{2}-1} dy = 1$$

$$y = z^2 \Rightarrow dy = 2z dz \Rightarrow dz = \frac{dy}{2\sqrt{y}}$$

$$\Rightarrow \frac{\left(\frac{1}{2}\right)^{1/2}}{\Gamma\left(\frac{1}{2}\right)} \int_{0}^{\infty} e^{-\frac{1}{2}y}\, y^{\frac{1}{2}-1} dy = 1 \Rightarrow \int_{0}^{\infty} g(y)\, dy = 1,$$

where $g(y) = \frac{(1/2)^{1/2}}{\Gamma(1/2)} e^{-\frac{1}{2}y} y^{\frac{1}{2}-1}$ if $y > 0$ and $g(y) = 0$ otherwise, which is a gamma distribution.