Lecture 04–06

- Analysis of Algorithms

IT205: Data Structures (AY 2023/24 Sem II Sec B) — **Dr. Arpit Rana**

Algorithm

An algorithm is any ***well-defined computational*** procedure that –

- takes some value, or set of values, as ***input*** and
- produces some value, or set of values, as ***output***
- in a ***finite amount of time***.

Algorithm

Example: ***Sort*** a ***sequence of numbers*** into ***monotonically increasing order***.

- ***Input***: a sequence of n numbers $\langle a_1, a_2, \ldots, a_n \rangle$.
- ***Output***: a permutation (reordering) $\langle a'_1, a'_2, \ldots, a'_n \rangle$ of the input sequence such that $a'_1 \leq a'_2 \leq \ldots \leq a'_n$

Input sequence: $\langle 31, 41, 59, 26, 41, 58 \rangle$ is an ***instance*** of the sorting problem.

Algorithm

Example: House Rent Dataset (from Magicbricks, India): 4746 Records

	Posted On	BHK	Rent	Size	Floor	Area Type	Area Locality	City	Furnishing Status	Tenant Preferred	Bathroom	Point of Contact
311	2022-06-03	1	9000	450	Ground out of 3	Carpet Area	Salt Lake City Sector 5	Kolkata	Unfurnished	Bachelors/Family	1	Contact Agent
3869	2022-05-20	3	19500	1270	1 out of 2	Super Area	Madipakkam	Chennai	Semi-Furnished	Bachelors	2	Contact Owner
1368	2022-06-21	1	20000	310	Ground out of 7	Carpet Area	Malad West	Mumbai	Unfurnished	Bachelors	1	Contact Agent
1528	2022-06-13	2	16000	600	1 out of 2	Carpet Area	Girinagar	Bangalore	Unfurnished	Bachelors	2	Contact Owner
309	2022-06-25	3	13000	950	Ground out of 2	Carpet Area	Rabindrapally, Garia	Kolkata	Unfurnished	Bachelors/Family	2	Contact Owner

Keys

Satellite data

Records = Keys + Satellite data;

We focus on the keys to *sort* the associated satellite data.

Analysis of Algorithms

Analysis of algorithms means predicting the ***resources*** that the algorithm requires.

- Computational time — a.k.a. *time complexity*
- Memory — a.k.a. *space complexity*
- Communication bandwidth, or
- Energy consumption.

For this course, we focus more on ***time complexity*** in our analysis.

We do analyze the algorithms to find out most efficient algorithm out of several candidate algorithms.

Measuring Actual Running Time

- We can measure the actual running time of a program
  - Use ***wall clock time*** or ***insert timing code*** into program
- However, actual running time is not meaningful when comparing two algorithms
  - Coded in different languages
  - Using different data sets
  - Using different data structures
  - Running on different computers

Counting Operations

- Instead of measuring the actual timing, we count the ***number of operations***
    - Operations: ***arithmetic*** (e.g., add, subtract, multiply, divide, remainder, floor, ceiling), ***data movement*** (e.g., load, store, copy), ***comparison***, etc.
- Counting an algorithm's operations is a way to assess its efficiency
    - An algorithm's execution time is related to the number of operations it requires

Counting Operations

Example: How many operations are being performed?

```
for (int i = 1; i <= n; i++) {
    perform 100 operations;          // A
    for (int j = 1; j <= n; j++) {
        perform 2 operations;        // B
    }
}
```

- Total number of operations: $A + B = \sum_{i=1}^{n} 100 + \sum_{i=1}^{n} \left(\sum_{j=1}^{n} 2 \right)$

$$= 100n + 2n^2$$

Counting Operations

- Knowing the number of operations required by the algorithm, we can state that
  - the above algorithm takes $2n^2 + 100n$ operations to solve a problem of size n.
- If the time t needed for one operation is known, then we can state
  - the algorithm takes $(2n^2 + 100n)t$ time units.

Approximation of Analysis Results

- Suppose the time complexity of
  - Algorithm A is $3n^2 + 2n + \log n + 1/(4n)$
  - Algorithm B is $0.39n^3 + n$
- Intuitively, we know Algorithm A will outperform B
  - When solving a larger problem, i.e. for a larger n
- The dominating term $3n^2$ and $0.39n^3$ can tell us approximately how the algorithms perform
- The terms n^2 and n^3 are even simpler and preferred, and can be obtained through *asymptotic analysis*.

Asymptotic Analysis

- Asymptotic analysis is an analysis of algorithms that focuses on
  - Analyzing problems of ***large input size***
  - Consider only the ***leading term*** of the formula, why?
  - ***Ignore the coefficient*** of the leading term, why?

Asymptotic Analysis

- Asymptotic analysis is an analysis of algorithms that focuses on
  - Consider only the ***leading term*** of the formula

$f(n) = 2n^2 + 100n$

$f(1000) = 2(1000)^2 + 100(1000)$
$= 2{,}000{,}000 + 100{,}000$

$f(100000) = 2(100000)^2 + 100(100000)$
$= 20{,}000{,}000{,}000 + 10{,}000{,}000$

Lower order terms contribute lesser to the overall cost as the input grows larger. Hence, can be ignored.

Asymptotic Analysis: Examples of Leading Terms

- $a(n) = \frac{1}{2} n + 4$
  - Leading term: $\frac{1}{2} n$
- $b(n) = 240n + 0.001n^2$
  - Leading term: $0.001n^2$
- $c(n) = n \lg(n) + \lg(n) + n \lg(\lg(n))$
  - Leading term: $n \lg(n)$
  - Note that $\lg(n) = \log_2(n)$

Asymptotic Analysis

- Asymptotic analysis is an analysis of algorithms that focuses on
    - *Ignore the coefficient* of the leading term
        - Suppose two algorithms have $2n^2$ and $30n^2$ as the leading terms, respectively.
        - Although actual time will be different due to the different constants, the growth rates of the running time are the same.
        - Compare with another algorithm with leading term of n^3, the difference in growth rate is a much more dominating factor.

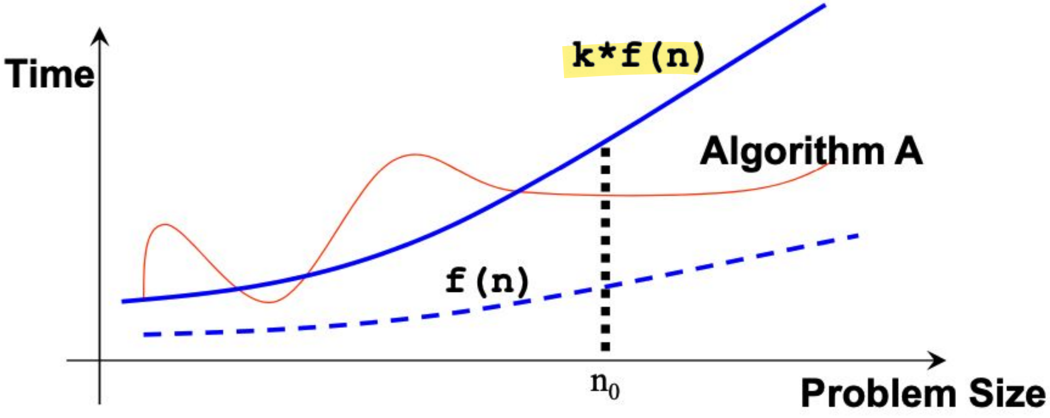
Upper Bound: The Big-O Notation

- If algorithm A requires time proportional to f(n)
    - Algorithm A is ***of the order of*** f(n)
    - Denoted as Algorithm **A is O(f(n))**
    - f(n) is the ***growth rate function*** for Algorithm A

The Big-O Notation: Formal Definition

Algorithm A is of $O(f(n))$ –

- if there exist a constant **k**, and a positive integer $\mathbf{n_0}$ such that
- Algorithm A requires no more than $k * f(n)$ time units to solve a problem of size $\mathbf{n \geq n_0}$.

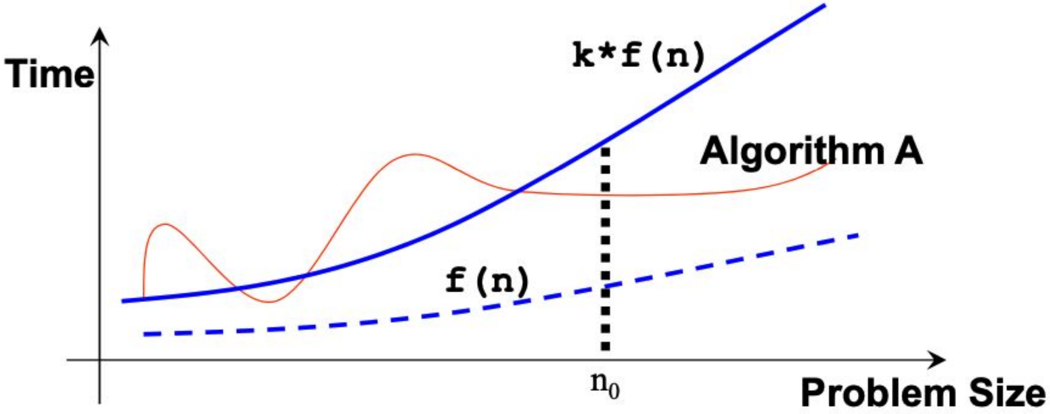


The Big-O Notation: Formal Definition

When problem size is larger than n_0, Algorithm A is bounded from above by $k * f(n)$

Observations:

- n_0 and k are not unique
- there are many possible $f(n)$



Example: Finding n_0 and k

- Given complexity of Algorithm A is $2n^2 + 100n$
  - **Claim**: Algorithm A is of $O(n^2)$
- Solution:
  - $2n^2 + 100n < 2n^2 + n^2 = 3n^2$ whenever $n > 100$
  - Set the constants to be $k = 3$ and $n_0 = 100$
  - By definition, we say Algorithm A is $O(n^2)$
- Questions
  - Can we say A is $O(\cancel{2n}^2)$ or $O(\cancel{3n}^2)$?
  - Can we say A is $O(n^3)$?

Growth Terms

- In asymptotic analysis, a formula can be simplified to a single term with coefficient 1 (how?)
- Such a term is called a ***growth term*** (rate of growth, order of growth, order of magnitude)
- The most common growth terms can be ordered as follows (note that many others are not shown)...

O(1) < O(log n) < O(n) < O(n log n) < O(n2) < O(n3) < O(2n) < …

Here "log" means $\log_2$

- In big-O, log functions of different bases are all the same (why?)

Because base are consider as coefficient of that particular term.

Common Growth Rates

- $O(1)$ — constant time
  - Independent of n
- $O(n)$ — linear time
  - Grows as the same rate of n
  - E.g. double input size, double execution time
- $O(n^2)$ — quadratic time
  - Increases rapidly w.r.t. n
  - E.g. double input size, quadruple execution time
- $O(n^3)$ — cubic time
  - Increases even more rapidly w.r.t. n
  - E.g. double input size, 8 * execution time
- $O(2^n)$ — exponential time
  - Increases very very rapidly w.r.t. n

Example: Exponential-Time Algorithm

- Suppose we have a problem that, for an input consisting of **n** items, can be solved by going through **2^n** cases

- We use a ***supercomputer***, that analyses ***200 million cases per second***

  - Input with 15 items — 163 microseconds
  - Input with 30 items — 5.36 seconds
  - Input with 50 items — more than two months
  - Input with 80 items — 191 million years

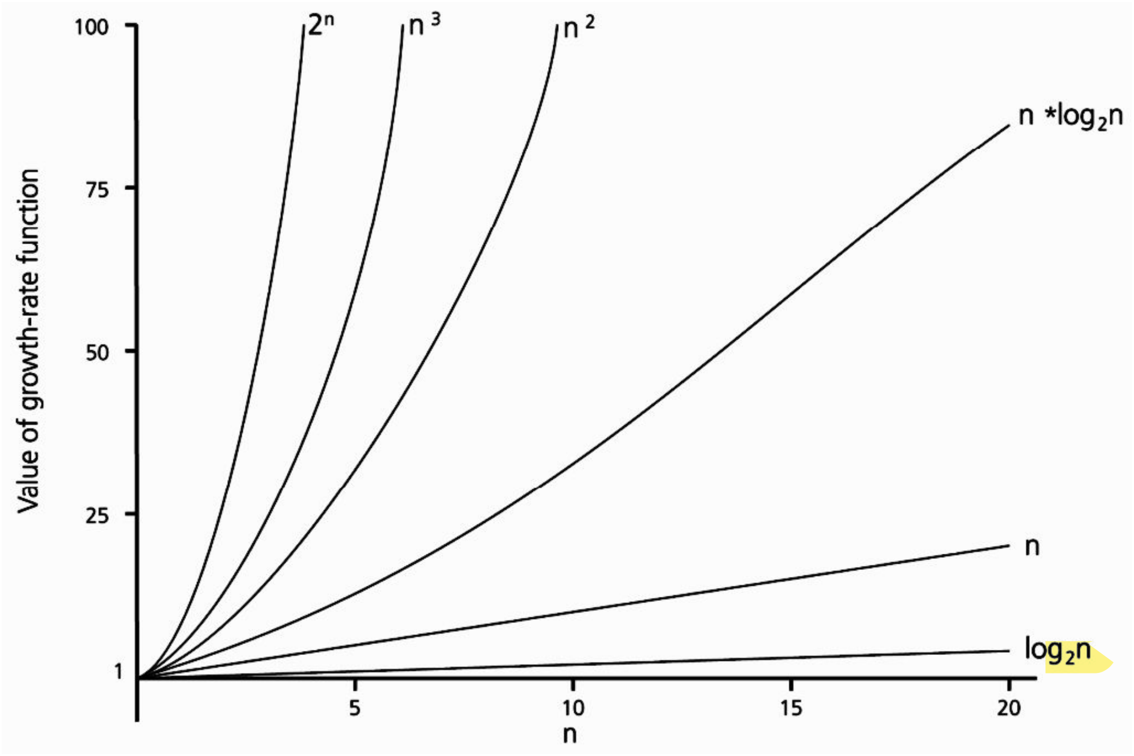
Example: Quadratic-Time Algorithm

- Suppose solving the same problem with another algorithm will use $300n^2$ *clock cycles* on a Handheld PC, running at *33 MHz*
    - Input with 15 items — 2 milliseconds
    - Input with 30 items — 8 milliseconds
    - Input with 50 items — 22 milliseconds
    - Input with 80 items — 58 milliseconds
- Therefore, to speed up program, don't simply rely on the raw power of a computer
    - *Very important to use an efficient algorithm*

Comparing Growth Rates

Function	10	100	1,000	10,000	100,000	1,000,000
1	1	1	1	1	1	1
$\log_2 n$	3	6	9	13	16	19
n	10	10^2	10^3	10^4	10^5	10^6
$n * \log_2 n$	30	664	9,965	10^5	10^6	10^7
n^2	10^2	10^4	10^6	10^8	10^{10}	10^{12}
n^3	10^3	10^6	10^9	10^{12}	10^{15}	10^{18}
2^n	10^3	10^{30}	10^{301}	$10^{3,010}$	$10^{30,103}$	$10^{301,030}$

Comparing Growth Rates



How to Find Complexity?

Some rules of thumb

- In general, just count the ***number of statements executed***.
- If there are only a small number of simple statements in a program – O(1)
- If there is a ‘for’ loop dictated by a loop index that goes up to **n** – O(n)
- If there is a nested ‘for’ loop with outer one controlled by **n** and the inner one controlled by **m** – O(n*m)
- For a loop with a range of values **n**, and each iteration reduces the range by a fixed constant fraction (e.g.: ½) – O(log n)
- For a ***recursive*** method, each call is usually O(1). So
  - If **n** calls are made – O(n)
  - If **n log n** calls are made – O(n log n)

Example: Finding Complexity

What is the complexity of the following code fragment?

```
int sum = 0;
for (int i = 1; i < n; i = i*2) {
    sum++;
}
```

It is clear that sum is incremented only when

$i = 1, 2, 4, 8, \ldots, 2^k$ where $k = \lfloor \log_2 n \rfloor$

There are only ***k+1*** iterations. So the complexity is $O(k)$ or $O(\log_2 n)$.

Example: Finding Complexity

What is the complexity of the following code fragment? For simplicity, let's assume that n is some power of 3.

```
int sum = 0;
for (int i = 1; i <= n; i = i*3) {
    for (int j = 1; j <= i; j++) {
        sum++;}}
```

$$
\begin{aligned}
f(n) &= 1 + 3 + 9 + 27 + \ldots + 3^{(\log_3 n)} \\
&= 1 + 3 + \ldots + n/9 + n/3 + n \\
&= n + n/3 + n/9 + \ldots + 3 + 1 \\
&= n * (1 + 1/3 + 1/9 + \ldots) \\
&\le n * (3/2) \\
&= 3n/2 \\
&= O(n)
\end{aligned}
$$

Analysis 1: Tower of Hanoi

For a given N number of disks, the way to accomplish the task in a minimum number of steps is:

- Move the top N−1 disks to an intermediate peg.
- Move the bottom disk to the destination peg.
- Finally, move the N−1 disks from the intermediate peg to the destination peg.

Analysis 1: Tower of Hanoi

- Number of moves made by the algorithm is $2^n - 1$. Prove it!
  - Hints: $f(1)=1$, $f(n) = f(n-1) + 1 + f(n-1)$, and prove by induction
- Assume each move takes *c* time, then

  $$f(n) = c(2^n - 1) = O(2^n)$$

- The Tower of Hanoi algorithm is an ***exponential*** time algorithm.

Analysis 2: Sequential Search

Check whether an item *x* is in an unsorted array ***a[ ]***

- If found, it returns position of ***x*** in array
- If not found, it returns *-1*

```
public int seqSearch(int a[], int len, int x) {
    for (int i = 0; i < len; i++) {
        if (a[i] == x)
            return i;
        }
    return -1;
}
```

Analysis 2: Sequential Search

- Time spent in each iteration through the loop is at most some constant $\mathbf{c_1}$
- Time spent outside the loop is at most some constant $\mathbf{c_2}$
- Maximum number of iterations is **n**
- Hence, the asymptotic upper bound is $c_1 n + c_2 = O(n)$
- Observation
  - In general, a loop of **n** iterations will lead to **O(n)** growth rate
  - This is an example of ***Worst Case Analysis***.

Analysis 3: Binary Search

- Important characteristics
  - Requires array to be sorted
  - Maintain sub-array where *x* might be located
  - Repeatedly compare *x* with ***m***, the middle of current sub-array
    - If *x* = ***m***, found it!
    - If *x* > ***m***, eliminate ***m*** and positions before ***m***
    - If *x* < ***m***, eliminate ***m*** and positions after ***m***
- Iterative and recursive implementations

Binary Search (Recursive)

Exercise part->Binary search

```
int binarySearch(int a[], int x, int low, int high) {
    if (low > high)     // Base Case 1: item not found
        return -1;

    int mid = (low+high) / 2;

    if (x > a[mid])
        return binarySearch(a, x, mid+1, high);
    else if (x < a[mid])
        return binarySearch(a, x, low, mid-1);
    else
        return mid;     // Base Case 2: item found
}
```

Binary Search (Iterative)

```
int binSearch(int a[], int len, int x) {
    int mid, low = 0;
    int high = len-1;

    while (low <= high) {
        mid = (low+high) / 2;
        if (x == a[mid])
            return mid;
        else if (x > a[mid])
            low = mid+1;
        else
            high = mid-1;
    }
    return -1; // item not found
}
```

Analysis 3: Binary Search (Iterative)

- Time spent outside the loop is at most c_1
- Time spent in each iteration of the loop is at most c_2
- For inputs of size **n**, if the program goes through at most **f(n)** iterations, then the complexity is $c_1 + c_2 f(n)$ or $O(f(n))$
  - i.e. the complexity is decided by the number of iterations (loops)

Analysis 3: Binary Search (Iterative) — Finding f(n)

- At any point during binary search, part of array is "alive" (might contain ***x***)
- Each iteration of loop eliminates at least half of previously "alive" elements
- At the beginning, all ***n*** elements are "alive", and after
  - One iteration, at most **n/2** are left, or alive
  - Two iterations, at most **(n/2)/2 = n/4 =** $n/2^2$ are left
  - Three iterations, at most **(n/4)/2 = n/8 =** $n/2^3$ are left
  - ...
  - ***k*** iterations, at most $n/2^k$ are left
  - At the final iteration, at most 1 element is left,

Analysis 3: Binary Search (Iterative) — Finding f(n)

- In the worst case, we have to search all the way up to the last iteration k with only one element left
- We have: $n/2^k = 1 \Rightarrow 2^k = n \Rightarrow k = \log_2(n) = \lg(n)$
- Hence, the binary search algorithm takes $O(f(n))$, or $O(\lg(n))$ time
- Observation
  - In general, when the domain of interest is reduced by a fraction for each iteration of a loop, then it will lead to **O(log n)** growth rate.

Analysis of Different Cases

- For an algorithm, three different cases of analysis
    - Worst-Case Analysis
        - Look at the worst possible scenario
    - Best-Case Analysis
        - Look at the ideal case | Usually not useful
    - Average-Case Analysis
        - Probability distribution should be known | Hardest/impossible to analyze

Analysis of Different Cases

- For example, in case of Sequential Search
    - **Worst-Case**: target item at the tail of array
    - **Best-Case**: target item at the head of array
    - **Average-Case**: target item can be anywhere

Summary of Lectures on Algorithms

- Algorithm Definition
- Algorithm Analysis
  - Counting operations
  - Asymptotic Analysis
  - Big-O notation (Upper-Bound)
- Three cases of analysis
  - Best-case
  - Worst-case
  - Average-case

Next Lecture

- Arrays: The Data Structure