SOS-Report

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Contents

1 Algorithm and Matroid Polytopes with preliminaries on polyhedra

1.1 Convex Function

A subset C of $\mathbb{R}^n$ is said to be convex if $\lambda x + (1-\lambda)y \in C \ \forall \ x, y \in C$ and $\lambda \in [0,1]$.

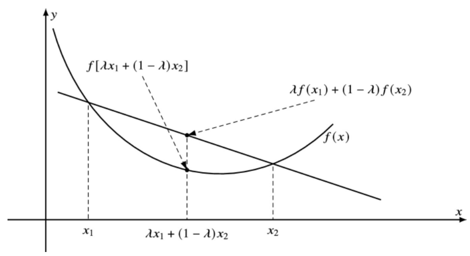

Figure 1: Example of a Convex Set

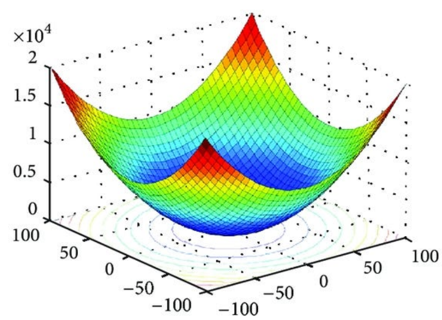

Figure 2: Example of Convex and Non-Convex Sets

- f is concave if $-f$ is convex.
- Geometrically, the line segment connecting $(x, f(x))$ to $(y, f(y))$ must sit above the graph of f.

Some examples are:

- e^{ax}
- $-\log(x)$
- x^a (defined on $\mathbb{R}_+$), $a \geq 1$ or $a \leq 0$
- $|x|^a$, $a \geq 1$
- $x \log(x)$ (defined on $\mathbb{R}_+$)

1.2 Convex Hull

The convex hull of a set $X \subseteq \mathbb{R}^n$ is the smallest convex set containing X. Formally, it is defined as the set of all convex combinations of points in X.

$$\text{conv.hull}(X) = \left\{ \sum_{i=1}^{k} \lambda_i x_i \mid x_i \in X, \lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = 1 \right\}$$

Theorem: For any $X \subseteq \mathbb{R}^n$ and $x \in \text{Conv.hull}(X)$, there exist affinely independent vectors $\vec{x_1}, \vec{x_2}, \ldots, \vec{x_k}$ in X with $x \in \text{Conv.hull}\{\vec{x_1}, \ldots, \vec{x_k}\}$.

This theorem is also known as Carathéodory's theorem.

Affine Halfspace: A subset H of $\mathbb{R}^n$ is called an affine halfspace if $H = \{x \mid A^T x \leq \delta\}$ for some $A \in \mathbb{R}^n$ with $A \neq 0$ and some $\delta \in \mathbb{R}$.

Linear Halfspace: A subset H of $\mathbb{R}^n$ is called an affine halfspace if $H = \{x \mid A^T x \leq \delta\}$ for some $A \in \mathbb{R}^n$ with $A \neq 0$ and some $\delta = 0$.

Uphull of X: X $\subseteq \mathbb{R}^n$. The set of Conv.hull X + $\mathbb{R}^n_+$
downhull of X: X $\subseteq \mathbb{R}^n$. The set of Conv.hull X - $\mathbb{R}^n_+$

1.3 Cone:

$C \subseteq \mathbb{R}^n$ is called Cone if $C \neq \emptyset$ and $\lambda X + \nu Y \in C$ and $\lambda, \nu \in \mathbb{R}_+$. Cone generated by

$$\text{Cone}(X) = \left\{ \sum_{i=1}^{k} \lambda_i x_i \mid x_i \in X, k \geq 0, \lambda_i \geq 0 \right\}$$

Theorem: for any $X \in \mathbb{R}^n$ and $X \in \text{Cone}X$. $\exists$ linearly independent vectors $\{\vec{x_1}, \vec{x}_2, ..., \vec{x}_k\} \in X$ with $x \in \text{Cone}\{\vec{x_1}, \vec{x}_2, ..., \vec{x}_k\}$

1.4 Polyhedral:

A cone C is polyhedral if there is a matrix A such that:

$$\text{C} = \{X | AX \leq 0\}$$

A polyhedral is intersection of finitely many linear halfspace.

1.5 Polyhedon:

$P \subseteq \mathbb{R}^n$ is called polyhedron, $\exists$ mXn matrix A and a vector $b \in \mathbb{R}^m$ $m \geq 0)$ such that:

$$\text{P} = \{X | AX \leq b\}$$

A polyhedron if intersection of finitely many affine halfspace.

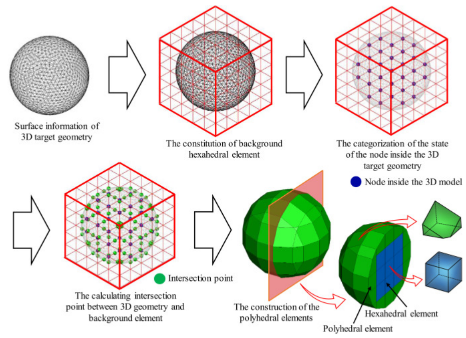

Figure 3: POlyhedral Visualisation

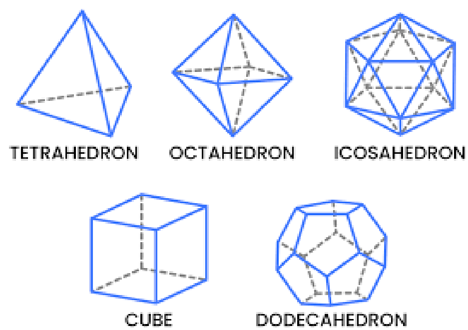

Figure 4: Example of polyhedron

1.6 Polytopes:

$P \subseteq \mathbb{R}^n$ is polytopes if it's convex hull finitely many vectors in $\mathbb{R}^n$

$$P = Q + C$$

where; P is Polyhedron, Q is some Polytopes and C is some Cone.

1.7 Charecteristics Cone:

If polyhedron $P \neq \emptyset$, then C is unique is called Char.Cone(P).

$$\text{Char.Cone}(P) = \{y \in \mathbb{R}^n | \forall x \in P, \forall \lambda \geq 0 : x + \lambda y \in P\}$$

If $P = \emptyset$, then $\text{Char.Cone}(P) = \{0\}$.
A set P is polytope iff P is polyhedron.
If polyhedron P is rational if it is determined by a rational system of linear inequalities.
Rational polytopes is convex hull of finite number of rational vector.

Farka's lemma: A system of $AX \leq b$ is called feasible if it has a solution X.

theorem: $AX \leq b$ is feasible $\iff$ $Y^T b \geq 0$ with $Y^T A = 0^T$.

Corollary: $AX = b$ has a solution $x \geq 0$ $\iff$ $y^T b \geq 0$ for each $y \geq 0$ with $y^T A \geq 0^T$.

Corollary: Let $AX \leq b$ be a feasible system of inequalities and let $C^T X \leq \delta$ be an inequality satisfied by each x with $AX \leq b$. Then for some $\delta' \leq \delta$, the inequality $C^T X \leq \delta'$ is a non-negative linear combination of inequalities in $AX \leq b$.

2 Linear and Integer programming

2.1 Linear programming

Linear programming (L.P.) means to concern the problem of maximizing or minimizing a linear function over a polyhedron.

$$\max \left\{ C^T X | AX \leq b \right\}$$

$$\min \left\{ C^T X | X \geq 0, AX \geq b \right\}$$

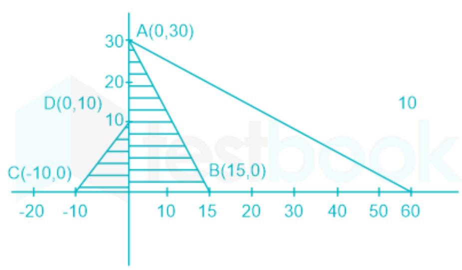

Figure 5: basic example of linear programming

- If a supremum of a linear function over a polyhedron is finite, then it is attained as a maximum (and for infimum and minimum)
- maximum is finite if the value set is non-empty and has an upperbound.

Duality theorem of linear programming: Let A be a matrix and b and c be vectors. Then:

$$\max\left\{C^T x | Ax \leq b\right\} = \min\left\{y^T b | y \geq 0, y^T A = C^T\right\}$$

atleast one of two optima is finite.

Corollary: If onr of two optima is finite means both of them are finite.
Weak duality $\rightarrow C^T X = y^T Ax \leq y^T b$

2.1.1 Equivalent Statements:

$$\max\left\{C^T x | x \geq 0, Ax \leq b\right\} = \min\left\{y^T b | y \geq 0, y^T A \geq C^T\right\}$$
$$\max\left\{C^T x | x \geq 0, Ax = b\right\} = \min\left\{y^T b | y \geq 0, y^T A \geq C^T\right\}$$
$$\min\left\{C^T x | x \geq 0, Ax \geq b\right\} = \max\left\{y^T b | y \geq 0, y^T A \leq C^T\right\}$$
$$\min\left\{C^T x | Ax \geq b\right\} = \max\left\{y^T b | y \geq 0, y^T A = C^T\right\}$$

One-to-one relation between constraint in a problem and variable in it's dual problem.

Maximize	**Minimize**
$\leq$ Constraint	Variable ≥ 0
$\geq$ Constraint	Variable ≤ 0
$=$ Constraint	Variable $=$
Variable ≥ 0	$\leq$ Constraint
Variable ≤ 0	$\geq$ Constraint
Unconstrained Variable	$=$ Constraint
Right Hand Side	Objective Function
Objective Function	Right Hand Side

Table 1: Some stuff

2.2 L.P. terminology:

- Polyhedron is Called feasible region.
- Any vector in polyhedron is called Feasible solution.
- Feasible region empty called Non-Feasible.
- Feasible region non-empty called Feasible.
- Function: $x \rightarrow c^T x$ called Objective function.
- Objective function is also known as Cost function.

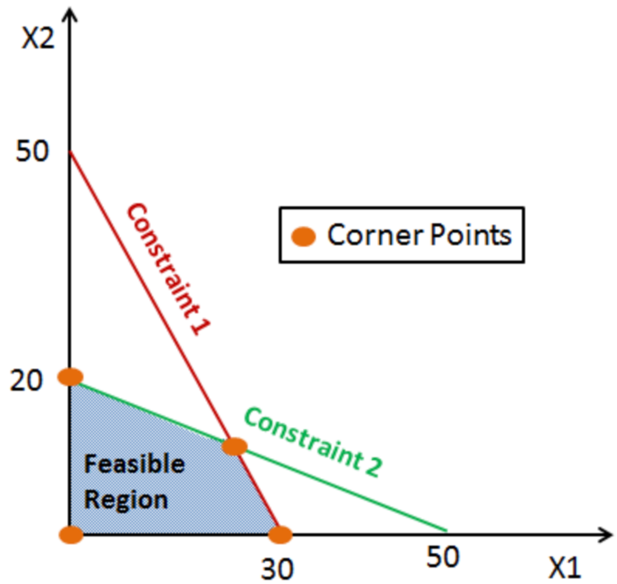

Figure 6: example of feasible region

- Any feasible solution attaining the optimum value is called Optimum solution.
- Any inequality $c^T x \leq \delta$ called Active or Tight for some x' if $cx' = \delta$.
- Minimization prorblem is called Dual problem of the maximization problem (then called Primal problem) Conversly.
- Feasible solution of dual problem is called Dual solution.

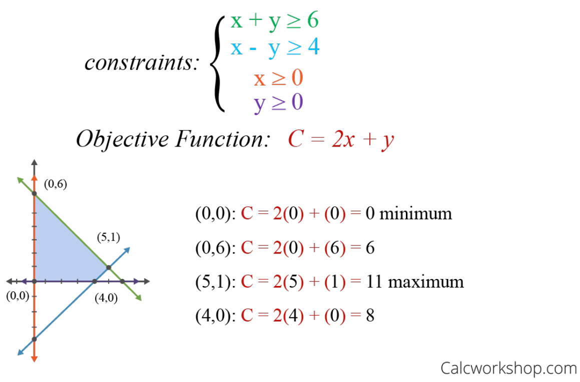

Figure 7: Example of objective function.

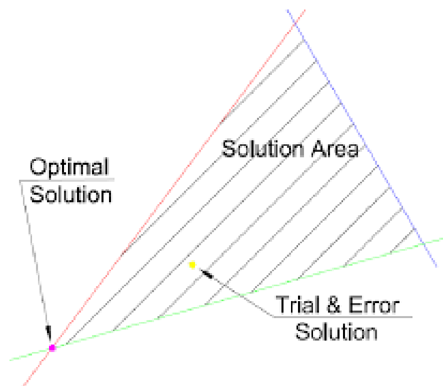

Figure 8: Example of optimum solution

Now let's move on another topic.

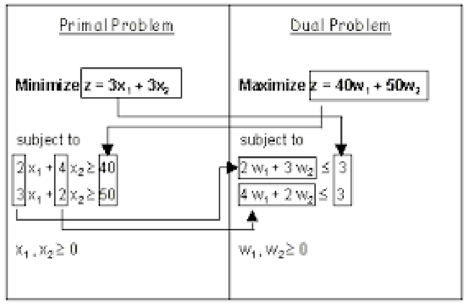

Figure 9: Example of dual problem

2.3 Complementary Slackness:

This condition of a pair of feasible solution x,y of the linear programs :- x and y are optimum solution iff $(Ax)_i = b_i$ for each i with $y_i > 0$.
It is similar for other dual linear programs.

It says that if a dual variable is greater than zero (slack) then the corresponding primal constraint must be an equality (tight).
It also says that if the primal constraint is slack then the corresponding dual variable is tight (or zero.)
Notice that if y0 were an extreme point in the dual, the complementary slackness condition relates a dual solution y0 to a point x0 in the set F in the primal. When we add to this, the fact that x0 is feasible, we may infer that both points should be optimal.

2.3.1 Theorem:

If the optimum value in L.P. problems is finite, then the minimum is attained by a vector $y \geq 0$ such that the rows of A corresponding to positive components of y are linearly independent.

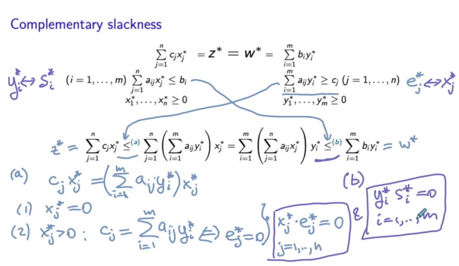

Figure 10: example complementary slackness

3 Face, Facets and Vertices:

Let,

$$P = \{x|Ax \leq b\}$$

be a polyhedron in $\mathbb{R}^n$

3.1 Supporting hyperplane:

H If C is non-zero vector and

$$\delta = max\left\{C^T x|Ax \leq b\right\},$$

the affine halfspace $\left\{x|C^T x = \delta\right\}$ is called supporting hyperplane of P.

3.2 Face:

SUbset F of P is called Face if

$$F = P$$

$$F = P \cap H$$

for some supporting hyperplane H of P.
F is a face of $P \iff F$ is the set of optimum solution of $\max\left\{C^T x|Ax \leq b\right\}$ for some $C \in \mathbb{R}^n$.

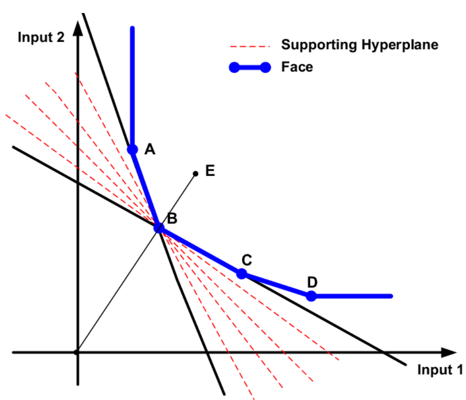

Figure 11: Example of supporting hyperplane and Face

3.3 Induce face / Determine face:

For any $C^T x \leq \delta$

$$F = \left\{x \in P \mid C^T x = \delta\right\}$$

$$F = \{x \in P|A' x = b'\}$$

for some subsystem $A'x \leq b'$ of $Ax \leq b$. So face non-empty polyhedron is non-empty.

Tight/Active: $Ax \leq b \implies a^T x = \beta$. Constraint $a^T x \leq \beta$ from F face.

Implicitly equality: If $Ax \leq b \implies a^T x \leq \beta$.

3.3.1 Theorem:

Let$P = \{x|Ax \leq b\}$ be a polyhedron in $\mathbb{R}^n$. Let $A'x \leq b'$ be the subsystem of implicit inequalities in $Ax \leq b$. Then:

$$dim P = n - rank(A)'$$

.

3.3.2 Theorem:

Let $P = \{x|Ax \leq b\}$be a polyhedorn in $\mathbb{R}^n$ and let $z \in P$. Then z is a vertex og P iff $rank(A_Z) = n$

4 Polarity:

4.1 Polar:

$$\text{For } C \subseteq \mathbb{R}^n$$

$$\text{Polar of } C(C^*) = \left\{z \in \mathbb{R}^n | x^T z \leq 1, \forall x \in C\right\}$$

4.2 Polar Cone:

$$\text{For } C \subseteq \mathbb{R}^n$$

$$\text{Polar Cone of } C(C^*) = \left\{z \in \mathbb{R}^n | x^T z \leq 0, \forall z \in C\right\}$$

The polar C^* is equal to the cone generated by the temperature of the rows of A. So,
$C^{**} = C$ for each polyhedral cone$C$.

- Polar cone is always convex even if C is not convex.
- If C is empty set, $C^* = \mathbb{R}^n$.
- Polarity may be seen as a generalisation of orthogonality.

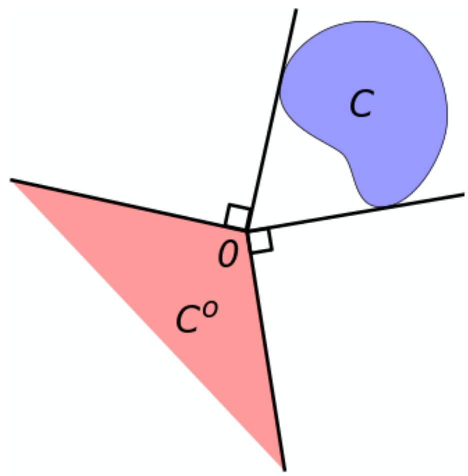

Figure 12: A set C and its polar cone C_o

Let C, C_1 and C_2 be non-empty sets in $\mathbb{R}^n$ then the following statements are true:-

- C^* is a closed convex cone.
- $C \subseteq C^{**}$ where C^{**} is a polar cone of C^*.
- $C_1 \subseteq C_2 \implies C_2^* \subseteq C_1^*$.

5 Blocking Polyhedra:

Blocking pairs of polyhedra are intimately related to maximum packing problems.
Anti-blocking pairs to minimum covering problems.

Subsets P of $\mathbb{R}^n$ of up-monotone if $x \in P$ and $y \geq x \implies y \in P$.

$$P^{\uparrow} = \{y \in \mathbb{R}^n | \exists x \in P : y \geq x\} = P + \mathbb{R}^n_+$$

Subsets P of $\mathbb{R}^n$ of down-monotone if $x \in P$ and $y \leq x \implies y \in P$.

$$P^{\downarrow} = \{u \in \mathbb{R}^n | \exists x \in P : y \leq x\}$$

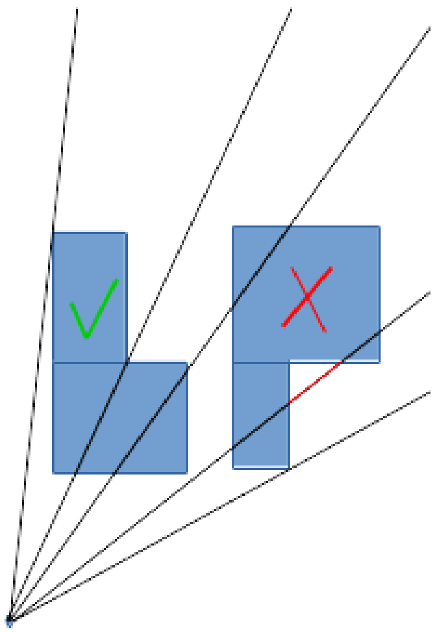

Figure 13: Example of monotone polyhedra

5.1 Theorem:

Let $P \subseteq \mathbb{R}^n$ be a polyhedron of blocking type. Then $B(P)$ is again a polyhedron of blocking type and $B(B(P)) = P$. Moreover, for any $x_1, x_2, ..., x_k \in \mathbb{R}^n_+$ holds iff

$$B(P) = \left\{ z \in \mathbb{R}^n_+ | x_i^+ Z \geq 1 for i = 1, 2, ..., k \right\}$$

6 Anti-blocking Polyhedra:

Set $P \subseteq \mathbb{R}^n$ is of antiblocking type if P is a non-empty closed convex subsets of $\mathbb{R}^n$ that is down-monotone in $\mathbb{R}^n_+$.
P is polyhedron of anti-blocking type iff:-

$$P = \left\{ x \in \mathbb{R}^n_+ | Ax \leq b \right\}$$

For any subsets P of $\mathbb{R}^n$, the anti-blocking set $A(P)$ of P is defined by :

$$A(P) = \left\{ z \in \mathbb{R}^n_+ | x^T z \leq 1 \text{ for each } x \in P \right\}$$

6.1 Theorem:

Let $P \subseteq \mathbb{R}^n_+$ be antiblocking type. Then $A(P)$ is again of antiblocking type and $A(A(P)) = P$

$P = conv.hull\,\{x_1, x_2, ..., x_k\}^{\downarrow} \cap \mathbb{R}^n_+$ iff $A(P) = \left\{z \in \mathbb{R}^n_+ | x_i^T z \leq 1 \text{ for } i = 1, 2, ..., k\right\}$

7 Method for linear programming:

7.1 Simplex method:

It consist of both finding a path in the 1-seleton of the feasible region, ending at an optimum vertex.

It's in practice on avg. quite efficient, but no polynomial-time worst-case running time bound has been proved.

Niki holds two part-time jobs, Job I and Job II. She never wants to work more than a total of 12 hours a week. She has determined that for every hour she works at Job I, she needs 2 hours of preparation time, and for every hour she works at Job II, she needs one hour of preparation time, and she cannot spend more than 16 hours for preparation. If she makes $40 an hour at Job I, and $30 an hour at Job II, how many hours should she work per week at each job to maximize her income?

Solution

In solving this problem, we will follow the algorithm listed above.

STEP 1. Set up the problem. Write the objective function and the constraints.

Since the simplex method is used for problems that consist of many variables, it is not practical to use the variables x, y, z etc. We use symbols x_1, x_2, x_3, and so on.

Let

- x_1 = The number of hours per week Niki will work at Job I and
- x_2 = The number of hours per week Niki will work at Job II.

It is customary to choose the variable that is to be maximized as Z.

The problem is formulated the same way as we did in the last chapter.

$$\begin{array}{ll} \textbf{Maximize} & Z = 40x_1 + 30x_2 \\ \textbf{Subject to:} & x_1 + x_2 \leq 12 \\ & 2x_1 + x_2 \leq 16 \\ & x_1 \geq 0; x_2 \geq 0 \end{array}$$

STEP 2. Convert the inequalities into equations. This is done by adding one slack variable for each inequality.

For example to convert the inequality $x_1 + x_2 \leq 12$ into an equation, we add a non-negative variable y_1, and we get

$$x_1 + x_2 + y_1 = 12$$

Figure 14:

Here the variable y_1 picks up the slack, and it represents the amount by which $x_1 + x_2$ falls short of 12. In this problem, if Niki works fewer than 12 hours, say 10, then y_1 is 2. Later when we read off the final solution from the simplex table, the values of the slack variables will identify the unused amounts.

We rewrite the objective function $Z = 40x_1 + 30x_2$ as $-40x_1 - 30x_2 + Z = 0$.

After adding the slack variables, our problem reads

$$\begin{array}{ll} \text{Objective function} & -40x_1 - 30x_2 + Z = 0 \\ \text{Subject to constraints:} & x_1 + x_2 + y_1 = 12 \\ & 2x_1 + x_2 + y_2 = 16 \\ & x1 \geq 0; x2 \geq 0 \end{array}$$

STEP 3. Construct the initial simplex tableau. Each inequality constraint appears in its own row. (The non-negativity constraints do *not* appear as rows in the simplex tableau.) Write the objective function as the bottom row.

Now that the inequalities are converted into equations, we can represent the problem into an augmented matrix called the initial simplex tableau as follows.

x1	x2	y1	y2	Z	C
1	1	1	0	0	12
2	1	0	1	0	16
–40	–30	0	0	1	0

Here the vertical line separates the left hand side of the equations from the right side. The horizontal line separates the constraints from the objective function. The right side of the equation is represented by the column C.

The reader needs to observe that the last four columns of this matrix look like the final matrix for the solution of a system of equations. If we arbitrarily choose $x_1 = 0$ and $x_2 = 0$, we get

$$\left[\begin{array}{ccccc} y_1 & y_2 & Z & | & C \\ 1 & 0 & 0 & | & 12 \\ 0 & 1 & 0 & | & 16 \\ 0 & 0 & 1 & | & 0 \end{array}\right]$$

which reads

Figure 15:

$y_1 = 12 \quad y_2 = 16 \quad Z = 0$

The solution obtained by arbitrarily assigning values to some variables and then solving for the remaining variables is called the **basic solution** associated with the tableau. So the above solution is the basic solution associated with the initial simplex tableau. We can label the basic solution variable in the right of the last column as shown in the table below.

x1	x2	y1	y2	Z		
1	1	1	0	0	12	y1
2	1	0	1	0	16	y2
−40	−30	0	0	1	0	Z

STEP 4. The most negative entry in the bottom row identifies the pivot column.

The most negative entry in the bottom row is -40; therefore the column 1 is identified.

x1	x2	y1	y2	Z		
1	1	1	0	0	12	y1
2	1	0	1	0	16	y2
−40	−30	0	0	1	0	Z
↑						

Question Why do we choose the most negative entry in the bottom row?

Answer The most negative entry in the bottom row represents the largest coefficient in the objective function - the coefficient whose entry will increase the value of the objective function the quickest.

The simplex method begins at a corner point where all the main variables, the variables that have symbols such as x_1, x_2, x_3 etc., are zero. It then moves from a corner point to the adjacent corner point always increasing the value of the objective function. In the case of the objective function $Z = 40x_1 + 30x2$, it will make more sense to increase the value of x_1 rather than x_2. The variable x_1 represents the number of hours per week Niki works at Job I. Since Job I pays $40 per hour as opposed to Job II which pays only $30, the variable x_1 will increase the objective function by $40 for a unit of increase in the variable x_1.

STEP 5. Calculate the quotients. The smallest quotient identifies a row. The element in the intersection of the column identified in step 4 and the row identified in this step is identified as the pivot element.

Figure 16:

Following the algorithm, in order to calculate the quotient, we divide the entries in the far right column by the entries in column 1, excluding the entry in the bottom row.

x1	x2	y1	y2	Z			
1	1	1	0	0	12	y1	12 ÷ 1 = 12
$\boxed{2}$	1	0	1	0	16	y2	← 16 ÷ 2 = 8
−40	−30	0	0	1	0	Z	
↑							

The smallest of the two quotients, 12 and 8, is 8. Therefore row 2 is identified. The intersection of column 1 and row 2 is the entry 2, which has been highlighted. This is our pivot element.

Figure 17:

7.2 Solving linear programming problems: the Simplex Method

7.2.1 Standard form of the LP-problem

- All main constraints are equations.
- All variables are nonnegative.

In general, the standard form of an LP-problem can be written in the following form:

$$\begin{aligned}
\max \quad & z = \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1, \ldots, m \\
& x_j \geq 0, \quad j = 1, \ldots, n
\end{aligned}$$

This can be written in the following matrix form:

$$\begin{aligned}
\max \quad & \mathbf{c}^T \mathbf{x} \\
\text{s.t.} \quad & A\mathbf{x} = \mathbf{b} \\
& \mathbf{x} \geq 0
\end{aligned}$$

where $\mathbf{x} = (x_1, \ldots, x_n)^T$, $\mathbf{c} = (c_1, \ldots, c_n)^T$, $\mathbf{b} = (b_1, \ldots, b_m)^T$, and $A = [a_{ij}]$ is an $(m \times n)$ matrix.

7.2.2 How to convert an LP to standard form

1. Constraint $\sum a_{ij} x_j \leq b_i \quad \Rightarrow \quad \sum a_{ij} x_j + s_i = b_i, \quad s_i \geq 0$
2. Constraint $\sum a_{ij} x_j \geq b_i \quad \Rightarrow \quad \sum a_{ij} x_j - s_i = b_i, \quad s_i \geq 0$ where s_i is called a slack variable.
3. $x_j \leq 0 \quad \Rightarrow \quad x_j^{\text{new}} = -x_j \geq 0$
4. x_j is unrestricted in sign, i.e. x_j is a free variable $\quad \Rightarrow \quad x_j = x_j^+ - x_j^-$, where $x_j^+, x_j^- \geq 0$

7.3 Ellipsoid method:

In mathematical optimization, the ellipsoid method is an iterative method for minimizing convex functions over convex sets. The ellipsoid method generates a sequence of ellipsoids whose volume uniformly decreases at every step, thus enclosing a minimizer of a convex function.

When specialized to solving feasible linear optimization problems with rational data, the ellipsoid method is an algorithm which finds an optimal solution in a number of steps that is polynomial in the input size.

This method is very slow in practice.

It's decide polyhedron is empty or not.

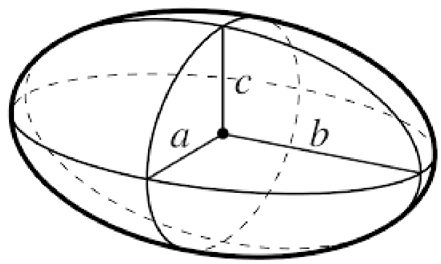

Figure 18: ellipsoid method

7.4 Numerical Example:

Unconstrained example: We consider the problem of minimizing a piecewise affine function:

$$\text{minimize } f(x) = \max_{i=1,2,\ldots,m}(a_i^T x + b_i)$$

with variable $x \in \mathbb{R}^n$. We use $n = 20$ variables and $m = 100$ terms , with problem data a_i and b_i generated from a unit normal distribution.

Basic ellipsoid algorithm: We use the basic ellipsoid algorithm described in $\S_1$ 1, starting with $P^{(0)} = I$, and $x^{(0)} = 0$.

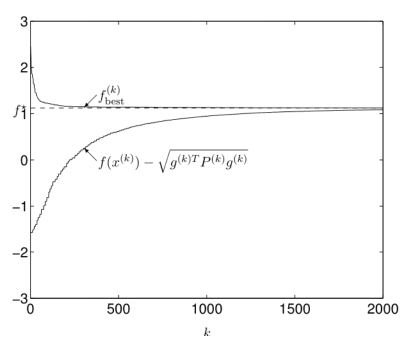

Figure 19: Convergence of $f_{\text{best}}^{(k)}$ and $f(x^{(k)}) - \sqrt{g^{(k)T}P^{(k)}g^{(k)}}$ (a lower bound on f^*) to f^* with the iteration number K.

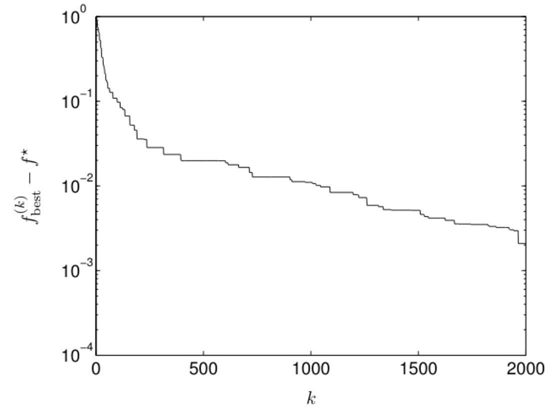

Figure 20: The suboptimality gap $f_{\rm best}^{(k)}$ - f^* versus iteration k for the basic ellipsoid algorithm and the deep-cut ellipsoid algorithm

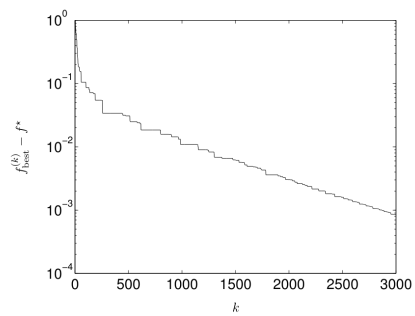

Figure 21: The suboptimality gap $f_{\rm best}^{(k)} - f^*$ versus iteration k for the constrained ellipsoid algorithm.

Constrained Example: Here we address a box-constrained version of:

$$\text{minimize } f(x) = \max_{i=1,2,\ldots,m}(a_i^T x + b_i),$$

$$\text{Subject to } |x_i| \leq 0.1,$$

with problem data generated as in the first example. We use the constrained ellipsoid method the solve the problem. Figure 21 shows the convergence of $f_{\rm best}^{(k)} - f^*$

8 Matroid:

In this chapter we assume that:

$$X + Y = X \cup Y$$

and

$$X - Y = X \backslash Y$$

8.1 Matroid definition:

A pair of $(S, \mathcal{I})$ is called *matroid*.

Where S is a finite set.
And $\mathcal{I}$ is a non-empty collection of subsets of S satisfying:-

1. if $I \in \mathcal{I}$ and $J \subseteq I,$ then $J \in \mathcal{I}.$ (Hereditary Property)
2. if $I, J \in \mathcal{I}$ and $|I| < |J|, then I + z \in \mathcal{I}$ for some $z \in J \backslash I,$ (Exchange Property)

Note: Matroid M = $(S, \mathcal{I})$, a subset $\mathcal{I}$ of S is called independent if $I \in \mathcal{I}$ and dependent otherwise.

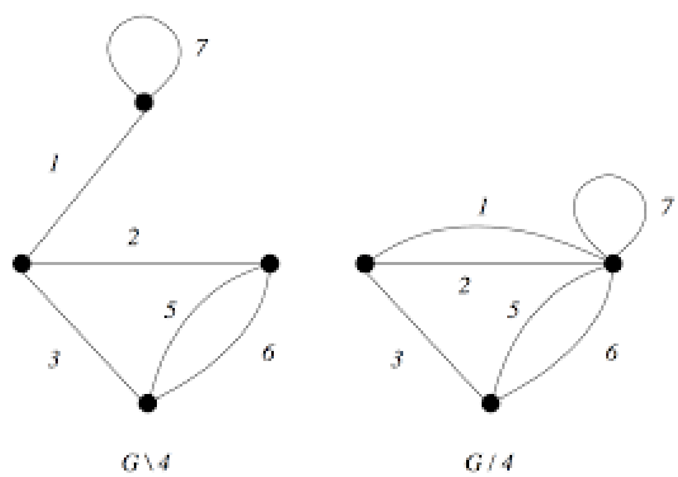

Figure 22: Example of Matroid

8.2 Base:

For the given $U \subseteq S$, a $B \subseteq U$ is called base of U if B is an inclusionwise maximal independent subsets of U.

$B \in \mathcal{I}$ and there is no $\mathcal{Z} \in \mathcal{I}$ with $B \subset \mathcal{Z} \subseteq U$

8.3 Ranks:

The common size of the bases of a subset U of S is called rank of U.

It is represented by $r_m(U)$ or $r(U)$.
All bases of U have same-size, called rank of U.
Common size of all bases is called rank of matroid.

8.3.1 Spanning:

It contains a base as a subsets.

8.4 Circuit:

Matroid is an inclusionwise minimal dependent set.

8.4.1 Loop:

It is an element S such that $\{S\}$ in a circuit.

8.4.2 Parallel:

Two element s, t of S are called parallel if $\{s, t\}$ is a circuit.

8.4.3 Theorem:

Let S be a finite set and let $\mathcal{I}$ be a non-empty collection of subsets satisfying if $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$.

Then if $I, J \in \mathcal{I}$ and $|I| < |J|$, then $I + z \in \mathcal{I}$ for some $z \in J \backslash I$ is equivalent to:-
if $I, J \in \mathcal{I}$ and $|I \backslash J| = 1, |J \backslash I| = 2,$ then $I + z \in \mathcal{I}$ for some $z \in J \backslash I$

8.5 Dual matroid:

Each matroid M, a dual matroid M^* can be associated, in such a way that $(M^*)^* = M$. Let $M = (S, \mathcal{I})$ be a matroid, and define:-

$$\mathcal{I}^* = \{I \subseteq S | S \backslash I \text{ is a spanning set of } M\}$$

8.5.1 Theorem:

$M^* = (S, \mathcal{I}^*)$ is a matroid.

8.5.2 Theorem:

The rank function r_m^* of the dual matroid M^* satisfies, for $U \subseteq S$:-

$$r_m^*(U) = |U| + r_m(S \backslash U) - r_M(S)$$

Some different stuff:-

- Circuit M^* are called cocircuit of M.
- M^* loops co-loops or bridge of M. Parallel elements of M^* are co-parallel or series of M.
- M is connected if $r_m(U) + r_m(S \backslash U) > r_m(S)$.

8.6 Deletion:

For $A \subseteq S$, the deletion $M \backslash A$ is the matroid on $S \backslash A$ with independent sets being $I \cap (S \backslash A)$ for $I \in \mathcal{I}$.

8.7 Contraction:

For $A \subseteq S$, the contraction $M \backslash A$ is defined on $S \backslash A$ with independent sets.

$$\{I \subseteq S|A : I \cup J \in \mathcal{I} \text{ for some } J \in \mathcal{I} \text{ for some independent } J \subseteq A\}$$

8.8 Truncation:

The truncation $T_K(M)$ of a matroid M reduces the rank of every subsets to a maximum of K.

Note: Restriction of M to Y $M|Y$

$$Y = S \backslash Z \text{ with } \mathcal{Z} \subseteq S \text{ and,}$$

$$S \backslash Z = \{x \in S | x \notin Z\}$$

$$Y = S \backslash Z \text{ then we denote } M \cdot Y = M \backslash Z$$

$$r_{M/Z}(X) = r(X \cup Z) - r(Z)$$

$$r_{M \backslash X/Y}(Z) = r_{M \backslash X}(Z \cup \{Y\}) - r_m(\{Y\}) = r_M(Z \cup \{Y\}) - r_M(\{Y\}) = r_{M/Y}(Z) = r_{M/Y \backslash X}$$

$M = (S, \mathcal{I})$ be a matroid and let K be a natural number($\mathbb{N}$) define :
$\mathcal{I}' = \{I \in \mathcal{I} | \, |I| \leq k\}$. Then $(S, \mathcal{I}')$ is again a matroid called K-truncation of MAtroid.

9 Some types of Matroid:

9.1 Uniform Matroids:

Matroid in which the independent sets are exactly the sets containing at most r elements, for some fixed integer r.

Note: Determined by a set S and a number K : the independent sets are the subsets I of S with $|I| \leq k$.

Trivially a matroid called a $K-$Uniform matroid by U_n^K, where $n = |S|$.

9.2 Linear Matroid:

A matroid is linear over $\mathbb{F}$ if $M \cong M(E, \mathbb{F})$ for some $E \subseteq \mathbb{F}^K$

Note: $\mathbb{F}$ be a field and let $E \subseteq \mathbb{F}^K$ be a finite set of vectors.

$$\mathcal{I} = \{F \subseteq E \,|\, \text{f is a linearli independent over } \mathbb{F}\}$$

$$M(E, \mathbb{F}) = (E, \mathcal{I}) \text{ is a linear matroid.}$$

9.3 Binary Matroid:

A matroid representable over $GF(2)$- the field with two elements is called binary matroid.

A matroid M is binary iff for each choice of circuits

$$c_1, ..., c_t \text{ the set } c_1 \Delta ... \Delta c_t$$

can be partitioned into circuits.
Example: $U_4^2 \implies$ 2 uniform matroid on 4 elements

9.3.1 Theorem:

A matroid is binary iff it has no U_4^2 minor.

9.4 Regular matroid:

A regular matroid is representable over each field.

9.4.1 Theorem:

It is equivalent to requiring that it can be represented over $\mathbb{R}$ by columns of a totally unimodular matrix.

9.5 Algebraic matroid:

Let $\mathbb{K}$ be an extension field of $\mathbb{F}$. A set $\{x_1, x_2, ..., x_k\} \subseteq \mathbb{K}$ is algebraically dependent over $\mathbb{F}$ if there exists a polynomial p with coefficients in $\mathbb{F}$ such that $p(x_1, ..., x_k) = 0$.

Let $\mathbb{K}$ be an extension field of $\mathbb{F}$ and $E \subseteq \mathbb{K}$ be finite.

$$\mathcal{I} = \{f \subseteq E \,| f \text{ is algebraically independent over } \mathbb{F}\}$$

Then $M(E, \mathbb{F}) = (E, \mathcal{I})$ is an algebraic matroid.

9.6 Graphical matroid:

Let $G = (V, E)$ be an unidirected graph and let

$$\mathcal{I} = \{f \subseteq E \,| (V, F) \text{ is a forest}\}$$

then $M(G) \cong M(H) \implies G \cong H$ for any graph H.

9.6.1 Facts:

G be graph. If G is a 3-Vertex-Connected, then

$$M(G) \cong M(H) \implies G \cong H$$

for any graph H.

Note: Independent Sets are acyclic subgraphs.

9.6.2 Cographic matroids:

Derived from the dual of a graph's cyclic matroid.

9.7 Transversal matroid:

Let $A_1, ..., A_n$; $T \subseteq E$.
T is a partial transversal of $(A_1, ..., A_n)$ if $\exists$ an injective map $\phi : T \to \{1, 2, ..., n\}$ such that $t \in A_{\phi(t)}$ $\forall t \in T$.

Note:

Given $A = (A_1, ..., A_n) \in (2^E)^n$, let

$$I = \{f \subseteq E \,| f \text{ is a partial transversal of } A\}$$

Then, $M(A) = (E, \mathcal{I})$ is transversal matroids.

Note: Universal matroids are transversal matroids.

$$M(k_4) \text{ isn't transversal}$$

___________Mid-Term End___________

Thank you