Reliability Engineering Concepts

1. Reliability $R(t)$

Reliability is a measure of the ability of a system or component to perform its required functions under stated conditions for a specified period of time. Mathematically, reliability is defined as the probability that a system or component does not fail within a given time t.

The reliability function is expressed as:

$$R(t) = P(T > t)$$

where T is the random variable representing the time to failure.

2. Cumulative Failure Function $F(t)$

The cumulative failure function, also known as the cumulative distribution function (CDF) of the time to failure, represents the probability that the system or component will fail by a certain time t. It is the complement of the reliability function.

The cumulative failure function is expressed as:

$$F(t) = P(T \leq t) = 1 - R(t)$$

3. Failure Density Function $f(t)$

The failure density function, or probability density function (PDF) of the time to failure, represents the instantaneous rate of failure per unit time. It is the derivative of the cumulative failure function with respect to time.

The failure density function is expressed as:

$$f(t) = \frac{dF(t)}{dt} = \frac{d}{dt}(1 - R(t)) = -\frac{dR(t)}{dt}$$

4. Hazard Function $h(t)$

The hazard function, also known as the failure rate or force of mortality, represents the instantaneous rate of failure at a given time t, conditional on survival up to that time. It quantifies the likelihood of failure in the next instant given that the system has survived up to time t.

The hazard function is defined as:

$$h(t) = \lim_{\Delta t \to 0} \frac{P(t \leq T < t + \Delta t \mid T \geq t)}{\Delta t} = \frac{f(t)}{R(t)}$$

Derivation of Failure Rate

The failure rate, often denoted by $\lambda(t)$, represents the rate at which failures occur in a system or component at a specific time t, given that the system or component has survived up to that time. It is closely related to the reliability function $R(t)$ and the probability density function (PDF) of the time to failure.

Derivation of Failure Rate $\lambda(t)$

To derive the failure rate $\lambda(t)$, we start by expressing the conditional probability:

$$P(t \leq T < t + \Delta t \mid T > t) = \frac{P(t \leq T < t + \Delta t \text{ and } T > t)}{P(T > t)}$$

Since $T > t$ automatically implies $T \geq t$, we can simplify:

$$P(t \leq T < t + \Delta t \mid T > t) = \frac{P(t \leq T < t + \Delta t)}{P(T > t)}$$

The probability that $t \leq T < t + \Delta t$ is approximately:

$$P(t \leq T < t + \Delta t) \approx f(t)\Delta t$$

where $f(t)$ is the PDF of the time to failure. So, we can rewrite the conditional probability as:

$$P(t \leq T < t + \Delta t \mid T > t) \approx \frac{f(t)\Delta t}{R(t)}$$

Now, substituting this into the definition of $\lambda(t)$:

$$\lambda(t) = \lim_{\Delta t \to 0} \frac{\frac{f(t)\Delta t}{R(t)}}{\Delta t} = \frac{f(t)}{R(t)}$$

4. Final Expression for Failure Rate

The failure rate $\lambda(t)$ is given by the ratio of the PDF $f(t)$ to the reliability function $R(t)$:

$$\lambda(t) = \frac{f(t)}{R(t)}$$

This equation indicates that the failure rate is the instantaneous rate of failure at time t, normalized by the probability that the system has survived up to time t.

5. Relationship between the Functions R(t), F(t), f(t) and $\lambda(t)$

We have

$$\begin{aligned}\lambda(t) &= \frac{f(t)}{R(t)} \\ &= -\frac{R'(t)}{R(t)}\end{aligned}$$

integrating $\lambda(t)$ with respect to t on both sides within the limits $t = 0$ to $t = t$, we get

$$\begin{aligned}\int_0^t \lambda(t)\ dt &= -\log_e R(t)|_0^t \\ &= -\log_e R(t)\end{aligned}$$

$$\implies R(t) = e^{-\int_0^t \lambda(t)\ dt}$$

This shows that the reliability function is the exponential of the negative integral of the failure rate over time.

6. Example: Constant Failure Rate

For a system with a constant failure rate $\lambda(t) = \lambda$, the reliability function simplifies to:

$$R(t) = e^{-\lambda t}$$

The corresponding PDF of the time to failure is:

$$f(t) = \lambda e^{-\lambda t}$$

This is characteristic of the exponential distribution, where the time to failure is memoryless, and the failure rate is constant over time.

7. Mean Time to Failure (MTTF) and Median of the Random Variable T

MTTF is given by

$$\begin{aligned} MTTF = E(T) &= \int_0^\infty t f(t)\ dt \\ &= \int_0^\infty t\left\{-R'(t)\right\} dt \end{aligned}$$

Integrating by parts, we get

$$\begin{aligned} MTTF &= t\{-R(t\}|_0^\infty - \int_0^\infty (1)(-R(t))dt \\ &= \int_0^\infty R(t)dt \end{aligned}$$

NOTE : If failure rate is constant, i.e., $\lambda(t) = \lambda$, we have

$$R(t) = e^{-\int_0^t \lambda(t)\ dt} = e^{-\lambda t}$$

$$MTTF = \int_0^\infty R(t)dt = \int_0^\infty e^{-\lambda t} = \left.\frac{e^{-\lambda t}}{-\lambda}\right|_0^\infty = \frac{1}{\lambda}$$

8. Problems

1. Show that if the hazard rate of a component is constant, say λ, then the failure distribution of the component follows exponential distribution. Also show that $MTTF = \frac{1}{\lambda}$.

2. The failure density function of the random variable T is given by $f(t) == \begin{cases} 0.012e^{-0.012t}, & t \geq 0, \\ 0, & \text{otherwise.} \end{cases}$
   Find

   (a) Reliability of the component

   (b) Reliability of the component for a 100 hours of mission time

   (c) Mean time to failure (MTTF)

   (d) What is the life of the component if a reliability of 0.96 is desired?

   Given: The failure density function of the random variable T is

   $$f(t) = \begin{cases} 0.012e^{-0.012t}, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

   This is an exponential distribution with rate parameter $\lambda = 0.012$.

(a) **Reliability of the component**

The reliability function R(t) is the probability that the component survives beyond time t, i.e.,

$$R(t) = P(T > t) = \int_t^\infty f(s)\,ds = e^{-\lambda t}$$

Hence,

$$R(t) = e^{-0.012t}$$

(b) **Reliability for 100 hours of mission time**

$$R(100) = e^{-0.012\times 100} = e^{-1.2} \approx 0.3012$$

(c) **Mean Time To Failure (MTTF)**

For the exponential distribution, the mean is given by:

$$\text{MTTF} = \frac{1}{\lambda} = \frac{1}{0.012} \approx 83.33 \text{ hours}$$

(d) **Life of the component if a reliability of 0.96 is desired**

We want:

$$R(t) = e^{-0.012t} = 0.96$$

Taking natural logarithm on both sides:

$$-0.012t = \ln(0.96) \Rightarrow t = \frac{-\ln(0.96)}{0.012}$$

$$t \approx \frac{-(-0.040821)}{0.012} \approx 3.40 \text{ hours}$$

3. The Hazard rate of a component is given by$H(t) = 0.6 \times 10^{-6}t$, where t is in years.

   (a) Calculate the reliability of the component for the first 2 years.

   (b) Calculate mean time to failure (MTTF).

Given: The hazard rate function of a component is

$$H(t) = 0.6 \times 10^{-6}t$$

(a) **Reliability of the component for the first 2 years**

The reliability function R(t) is related to the hazard function by:

$$R(t) = \exp\left(-\int_0^t H(s)\,ds\right)$$

Compute the integral:

$$\int_0^t H(s)\,ds = \int_0^t 0.6 \times 10^{-6} s\,ds = 0.6 \times 10^{-6}\int_0^t s\,ds = 0.6 \times 10^{-6} \cdot \frac{t^2}{2}$$

Therefore,

$$R(t) = \exp\left(-0.6 \times 10^{-6} \cdot \frac{t^2}{2}\right)$$

For t = 2 years:

$$R(2) = \exp\left(-0.6 \times 10^{-6} \cdot \frac{2^2}{2}\right) = \exp(-1.2 \times 10^{-6}) \approx 1 - 1.2 \times 10^{-6}$$

So,

$$R(2) \approx 0.9999988$$

(b) **Mean Time To Failure (MTTF)**

The MTTF is given by:

$$\text{MTTF} = \int_0^\infty R(t)\,dt = \int_0^\infty \exp\left(-0.6 \times 10^{-6} \cdot \frac{t^2}{2}\right) dt$$

Let:

$$\alpha = 0.6 \times 10^{-6} \cdot \frac{1}{2} = 0.3 \times 10^{-6}$$

Then,

$$\text{MTTF} = \int_0^\infty e^{-\alpha t^2} dt = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} = \frac{1}{2}\sqrt{\frac{\pi}{0.3 \times 10^{-6}}}$$

Compute:

$$\text{MTTF} = \frac{1}{2} \cdot \sqrt{\frac{3.1416}{0.3 \times 10^{-6}}} = \frac{1}{2} \cdot \sqrt{\frac{3.1416 \times 10^{6}}{0.3}} \approx \frac{1}{2} \cdot \sqrt{10.472 \times 10^{6}}$$

$$\text{MTTF} \approx \frac{1}{2} \cdot 3236.6 \approx 1618.3 \text{ years}$$

4. A company manufactures electrical components, and the lifetime (in hours) of a component is modelled as a continuous random variable X following an exponential distribution with a failure rate of =0.02 failures per hour.

   (a) Define the Reliability function R(t) and derive its expression for this system.

   (b) What is the probability that a randomly selected component survives beyond 50 hours?

   (c) Determine the Mean Time to Failure (MTTF) for this component.

   (d) Suppose a system consists of two identical components operating in parallel, meaning the system fails only when both components fail. Find the reliability function of the system and compute the probability that the system survives beyond 50 hours.