Chapter 6

Least square approximation.

A well defined system has equal number of variables & equations

Example:

$$x_1 + 2x_2 + x_3 = 0$$
$$x_1 - 9x_2 + 7x_3 = 2$$
$$2x_1 + 3x_2 + 5x_3 = 5$$

$$\underset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 1 & -9 & 7 \\ 2 & 3 & 5 \end{bmatrix}} \underset{x}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{b}{\begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}}$$

$n \times n \quad n \times 1 \rightarrow n \times 1$

If we have a system where no. of equations > no. of variables $\longrightarrow$ overdetermined system

Example:

$$x_1 + 2x_2 + x_3 = 0$$
$$x_1 - 9x_2 + 7x_3 = 2$$
$$x_1 + 3x_2 + 5x_3 = 4$$
$$2x_1 + 11x_2 - 9x_3 = 5$$
$$9x_1 + x_2 - x_3 = 7$$

$$\underset{m \times n}{\begin{bmatrix} 1 & 2 & 1 \\ 1 & -9 & 7 \\ 1 & 3 & 5 \\ 2 & 11 & -9 \\ 9 & 1 & -1 \end{bmatrix}} \underset{n \times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{m \times 1}{\begin{bmatrix} 0 \\ 2 \\ 4 \\ 5 \\ 7 \end{bmatrix}}$$

Least square approximation ~~is a~~ method is a way to find an approximate solⁿ of a over-determined system.

$$A \cdot x = b$$

$$m\times n \quad n\times 1 = m\times 1$$

How to solve?

$$A^T \cdot A \cdot x = A^T \cdot b$$ [A^T multiplied on both sides]

$$\underbrace{n\times m \cdot m\times n}_{\text{square matrix so can be inversed}} \quad n\times 1 \quad \underbrace{n\times m \; m\times 1}$$

$$n\times n \qquad n\times 1 \qquad n\times 1$$

$a_0 + a_1 x$ [no. of co-efficient = 2]

Example:

We want to fit a st. line through the following nodes

$x_0 = -3$ $\quad$ $x_1 = 0$ $\quad$ $x_2 = 6$

$f(x_0) = 0$ $\quad$ $f(x_1) = 0$ $\quad$ $f(x_2) = 2$.

no of nodes = 3

$P_1(x_0) = a_0 + a_1(x_0) = f(x_0) \Rightarrow a_0 - 3a_1 = 0$

$P_2(x_1) = a_0 + a_1 \times x_1 = f(x_1) \Rightarrow a_0 = 0$

$P_3(x_2) = a_0 + a_1 \times x_2 = f(x_2) \Rightarrow a_0 + 6a_1 = 2$

$$\underset{A}{\begin{bmatrix} 1 & -3 \\ 1 & 0 \\ 1 & 6 \end{bmatrix}} \underset{x}{\begin{bmatrix} a_0 \\ a_1 \end{bmatrix}} = \underset{b}{\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}}$$

Multiplying A^T on both sides

$$\begin{bmatrix} 1 & 1 & 1 \\ -3 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 1 & 0 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -3 & 0 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 \\ 3 & 45 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 12 \end{bmatrix}$$

Now you can apply gaussian elimination / LU / inverse method to find values of a_0 and a_1

inverse method:

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 45 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 12 \end{bmatrix}$$

$$= \begin{bmatrix} 3/7 \\ 5/21 \end{bmatrix}$$

$$P_1(x) = 3/7 + 5/21 \, x$$

Vector dot product:

matrix notation:

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad y = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$x^T . y = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$= 1 \times 4 + 2 \times 5 + 3 \times 6$$

$\downarrow$

= returns a scalar/number

vector notation:

$$\vec{x} . \vec{y}$$

$$= (1 \times 4) + (2 \times 5) + (3 \times 6)$$

$\downarrow$

= returns a scalar/number

Inner product or l_2-norm :

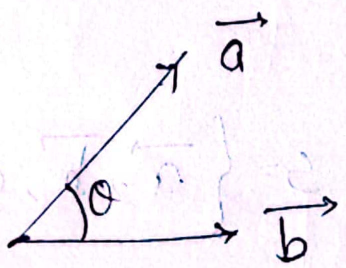
$\vec{x} \cdot \vec{x}$ or $x^T x$.

Magnitude of a vector :

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$|a| = \sqrt{(a_1)^2 + (a_2)^2 + (a_3)^2}$$

Dot product (second approach):



$a.b = |a||b| \cos\theta.$

or, $\cos\theta = \dfrac{a.b}{|a||b|}$

Orthogonality:

If $\angle$ between 2 vector $= 90°$,

or if 2 vectors are perpendicular to each other they are orthogonal.

if they are orthogonal then their dot product will be 0.

$\vec{a} \cdot \vec{b} = |a||b| \cancel{\cos\theta}$ [$\cos 90° = 0$]

$\therefore \vec{a} \cdot \vec{b} = 0$

$\therefore a^T \cdot b = 0$

Let S is set of vectors $S = \{\vec{a}, \vec{b}, \vec{c}\}$

Now S is orthogonal set iff

$\vec{a} \cdot \vec{b} = 0$

$\vec{b} \cdot \vec{c} = 0$

$\vec{c} \cdot \vec{a} = 0$

So each vector is perpendicular to each other

Orthonormality:

for being orthonormal they have to:

1) be orthogonal (dot product = 0)

2) length = 1 (Unit vectors)

$a = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$

$\vec{a} \cdot \vec{b} = \cancel{4 \times 1}\ \vec{a} \cdot b = 4 \times 1 + 2 \times -3 + 1 \times 2 = 0$

$\therefore$ orthogonal

$|a| \neq 1$ and $|b| \neq 1$

So normalizing: $\hat{a} = \frac{\vec{a}}{|a|} = \frac{1}{\sqrt{21}} \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/\sqrt{21} \\ 2/\sqrt{21} \\ 1/\sqrt{21} \end{bmatrix}$

$|\hat{a}| = \sqrt{\left(4/\sqrt{21}\right)^2 + \left(2/\sqrt{21}\right)^2 + \left(1/\sqrt{21}\right)^2}$

$= 1$

$$\hat{b} = \frac{\vec{b}}{|b|} = \frac{1}{\sqrt{14}}\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{14} \\ -3/\sqrt{14} \\ 2/\sqrt{14} \end{bmatrix}$$

$$|\hat{b}| = \sqrt{\left(1/\sqrt{14}\right)^2 + \left(-3/\sqrt{14}\right)^2 + \left(2/\sqrt{14}\right)^2} = 1$$

the process of converting vectors to unit vectors is called normalization.

so normalization change magnitude but not direction.

so now since they are orthogonal and has magnitude = 1, so they are orthonormal

Example:

$$S = \left\{ \frac{1}{\sqrt{5}}(2,1)^T , \frac{1}{\sqrt{5}}(1,-2)^T \right\}$$

$$S = \left\{ \underbrace{\frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{u_1} , \underbrace{\frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -2 \end{bmatrix}}_{u_2} \right\}$$

$$\vec{u_1} \cdot \vec{u_2} = \frac{2}{\sqrt{5}} \times \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{5}} \times -\frac{2}{\sqrt{5}} = 0$$

[∵ othogonal]

$$|\vec{u_1}| = \sqrt{\left(\frac{2}{\sqrt{5}}\right)^2 + \left(\frac{1}{\sqrt{5}}\right)^2} = 1$$

$$|\vec{u_2}| = \sqrt{\left(\frac{1}{\sqrt{5}}\right)^2 + \left(-\frac{2}{\sqrt{5}}\right)^2} = 1$$

[∵ Orithonormal].

$$Q^{-1} = Q^T$$

$$Q \cdot Q^T = I$$

$$Q^T \cdot Q = I$$

if Q has ~~cottom~~ columns that are from orthonormal sets.

QR decomposition:

Gram - Schmidt Process:

lets have a basis $(v_1, v_2 \ldots v_n)$ form a vector space. Gram - Schmidt process takes the basis $(v_1, v_2 \ldots v_n)$ and forms new orthogonal basis $(P_1, P_2, \ldots P_n)$. We can later transform these ~~otho~~ orthogonal basis into orthonormal basis $(q_1, q_2 \ldots q_n)$

original basis : v_1, v_2, v_3

$\downarrow$ Gram schmidt Process

orthogonal basis : P_1, P_2, P_3

$\downarrow$ normalization

Orthonormal basis : q_1, q_2, q_3

1) $P_1 = U_1$

2) $P_2 = U_2 - \frac{U_2 \cdot P_1}{P_1 \cdot P_1} P_1$

3) $P_3 = U_3 - \frac{U_3 \cdot P_1}{P_1 \cdot P_1} P_1 - \frac{U_3 \cdot P_2}{P_2 \cdot P_2} P_2$

Example:

$$U_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, U_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, U_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

1) $P_1 = U_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

2) $P_2 = U_2 - \frac{U_2 \cdot P_1}{P_1 \cdot P_1} P_1$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{(1\times1)+(0\times-1)+(1\times1)}{(1\times1)+(-1\times-1)+(1\times1)} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 2/3 \\ -2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

3) $P_3 = U_3 - \frac{U_3 \cdot P_1}{P_1 \cdot P_1} P_1 - \frac{U_3 \cdot P_2}{P_2 \cdot P_2} P_2$

$$= \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \frac{(1\times 1) + (1\times -1) + (2\times 1)}{(1\times 1) + (-1\times -1) + (1\times 1)} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$- \frac{(1\times 1/3) + (1\times 2/3) + (2\times 1/3)}{(1/3\times 1/3) + (2/3\times 2/3) + (1/3\times 1/3)} \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - 2/3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - 5/2 \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2/3 \\ -2/3 \\ 2/3 \end{bmatrix} - \begin{bmatrix} 5/6 \\ 5/3 \\ 5/6 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix}$$

P_1, P_2, P_3 are orthogonal basis.

Now for normalization:

1) $q_1 = \frac{P_1}{|P_1|} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix} = \begin{bmatrix}1/\sqrt{3}\\-1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix}$

2) $q_2 = \frac{P_2}{|P_2|} = \frac{1}{\sqrt{6}/3}\begin{bmatrix}1/3\\2/3\\1/3\end{bmatrix} = \begin{bmatrix}\sqrt{6}/6\\\sqrt{6}/3\\\sqrt{6}/6\end{bmatrix}$

3) $q_3 = \frac{P_3}{|P_3|} = \frac{1}{2\sqrt{2}}\begin{bmatrix}-1/2\\0\\1/2\end{bmatrix} = \begin{bmatrix}-\sqrt{2}/2\\0\\\sqrt{2}/2\end{bmatrix}$:

QR decomposition:

Any $m \times n$ matrix A can be writhen in the form: $A = QR$

where Q is a $(m \times n)$ matrix with orthonormal co

R is a upper triangular matrix of shape $(n \times m)$

$$\underset{m \times n}{A} = \underset{m \times n}{Q} \times \underset{n \times m}{R}$$

$$Q^T A = Q^T Q R$$

$$Q^T A = I R \qquad [\because A A^T = I]$$

$$R = Q^T A$$

Example:

$x_0 = -3 \quad x_1 = 0 \quad x_2 = 6$

$f(x_0) = 0 \quad f(x_1) = 0 \quad f_1(x_2) = 2$

$$\begin{bmatrix} 1 & -3 \\ 1 & 0 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$\downarrow \quad \downarrow$

$U_1 \quad U_2$

$$P_1 = U_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$P_2 = U_2 - \frac{U_2 . P_1}{P_1 . P_1} P_1 = \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix} + \frac{(-3\times1)+(0\times1)+(6\times1)}{1\times1 + 1\times1 + 1\times1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ -1 \\ 5 \end{bmatrix}$$

$$P_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad P_2 = \begin{bmatrix} -4 \\ -1 \\ 5 \end{bmatrix}$$

$$q_1 = \frac{P_1}{|P_1|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$q_2 = \frac{P_2}{|P_2|} = \frac{1}{\sqrt{42}} \begin{bmatrix} -4 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} -4/\sqrt{42} \\ -1/\sqrt{42} \\ 5/\sqrt{42} \end{bmatrix}$$

$$Q = \begin{bmatrix} 1/\sqrt{3} & -4/\sqrt{42} \\ 1/\sqrt{3} & -1/\sqrt{42} \\ 1/\sqrt{3} & 5/\sqrt{42} \end{bmatrix}$$

$$R = Q^T A = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -4/\sqrt{42} & -1/\sqrt{42} & 5/\sqrt{42} \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 1 & 0 \\ 1 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{3} & \sqrt{3} \\ 0 & \sqrt{42} \end{bmatrix}$$

$$Ax = b$$

$$A^T A \cdot x = A^T \cdot b$$

$$(QR)^T \cdot (QR)x = (QR)^T b$$

$$R^T Q^T \cdot Q R x = R^T Q^T b$$

$$R^T \cdot I \cdot R x = R^T Q^T b$$

$$\boxed{Rx = Q^T b}$$

$$Rx = Q^T b$$

$$\begin{bmatrix} \sqrt{3} & \sqrt{3} \\ 0 & \sqrt{42} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -4/\sqrt{42} & -1/\sqrt{42} & 5/\sqrt{42} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{3} & \sqrt{3} \\ 0 & \sqrt{42} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{3} \\ 5\sqrt{2}/\sqrt{21} \end{bmatrix}$$

$$\begin{matrix} a_0 \\ a_1 \end{matrix} = \begin{bmatrix} 3/7 \\ 5/21 \end{bmatrix}$$