

Chapter 6

Least square approximation

A well defined system has equal number of variables

& equations

Example:

$$n_1 + 2n_2 + n_3 = 0$$

$$n_1 - 9n_2 + 7n_3 = 2$$

$$2n_1 + 3n_2 + 5n_3 = 5$$

$$\begin{bmatrix} A & | & b \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & | & 0 \\ 1 & -9 & 7 & | & 2 \\ 2 & 3 & 5 & | & 5 \end{bmatrix}$$

If we have a system where
no. of equations > no. of variables \rightarrow overdetermined system

Example:

$$n_1 + 2n_2 + n_3 = 0$$

$$n_1 - 9n_2 + 7n_3 = 2$$

$$n_1 + 3n_2 + 5n_3 = 4$$

$$2n_1 + 11n_2 - 9n_3 = 5$$

$$n_1 + n_2 - n_3 = 7$$

$$\begin{bmatrix} A & | & b \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & | & 0 \\ 1 & -9 & 7 & | & 2 \\ 1 & 3 & 5 & | & 4 \\ 2 & 11 & -9 & | & 5 \\ 1 & 1 & -1 & | & 7 \end{bmatrix}$$

~~Least square approximation is a method~~ is a way

~~to find an approximate solⁿ of a over-determined~~

system:

$$A \cdot x = b$$

$$0 = 0$$

$$C = 0$$

$$S = (s_{ij}) \quad 0 = (0) \quad 0 = (0)$$

$$A \cdot x = b$$

$\hat{e} = \text{column for } b$

$$0 = (0) \quad n \times 1 \quad (m \times 1) \quad (0) \cdot x + p = (0) \cdot q$$

How to solve?

$$S = (s_{ij}) \quad S^{T} \cdot S + D = (s_{ij})^T \quad A^T \cdot A \cdot x = A^T \cdot b$$

[A^T multiplied on both sides]

$$\begin{pmatrix} 0 \\ 0 \\ s \end{pmatrix}$$

$$\begin{pmatrix} n \times m & m \times n \end{pmatrix}$$

$$n \times 1$$

$$\begin{pmatrix} n \times m & m \times 1 \end{pmatrix}$$

square matrix

so can be inverted

$$\downarrow$$

$$\begin{matrix} & 1 \\ n \times n & \end{matrix}$$

$$\begin{matrix} & 1 \\ n \times 1 & \end{matrix}$$

$$\begin{matrix} & 1 \\ n \times 1 & \end{matrix}$$

$a_0 + a_1 n$ [no. of co-efficient = 2]

Example

We want to fit a st. line through the following nodes

$$n_0 = -3$$

$$n_1 = 0$$

$$n_2 = 6 \text{ m/s}$$

$$f(n_0) = 0$$

$$f(n_1) = 0$$

$$f(n_2) = 2$$

no. of nodes = 3

$$P_1(n) = a_0 + a_1(n) \xrightarrow{\text{at } n_0} a_0 + 3a_1 = 0$$

$$P_1(n_1) = a_0 + a_1 \times n_1 = f(n_1) \xrightarrow{\text{value of } a_1} a_1 = 0$$

$$P_1(n_2) = a_0 + a_1 \times n_2 \xrightarrow{\text{at } n_2} a_0 + 6a_1 = 2$$

[exp. method]

$$\begin{bmatrix} 1 & -3 \\ 1 & 0 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$A \quad \quad \quad b$

1 x 2 3 x 1 3 x 1

Multiplying A^T on both sides

$$\begin{bmatrix} 1 & 1 & 1 \\ -3 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -3 & 0 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 \\ 3 & 45 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 12 \end{bmatrix}$$

Now you can apply gaussian elimination / LU / inverse method to find values of a_0 and a_1 .

Inverse method:

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 45 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 12 \end{bmatrix}$$

$$= \begin{bmatrix} 3/1 \\ 5/21 \end{bmatrix}$$

$$P_1(x) = 3/7 + 5/21 x$$

Vector dot product:

(6)

matrix notation:

$$\mathbf{m} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$\mathbf{m}^T \cdot \mathbf{y} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$= 1 \times 4 + 2 \times 5 + 3 \times 6$$

= \downarrow
returns a scalar/number

vector notation:

$$\overrightarrow{\mathbf{m}} \cdot \overrightarrow{\mathbf{y}}$$

$$= (1 \times 4) + (2 \times 5) + (3 \times 6)$$

\downarrow

= returns a scalar/number

Inner product on \mathbb{R}^n -norm: $\| \cdot \|_{\text{Euclidean}}$

$\Rightarrow \vec{a} \cdot \vec{b} = \sum a_i b_i$ on \mathbb{R}^n .

length of vectors and angles between them

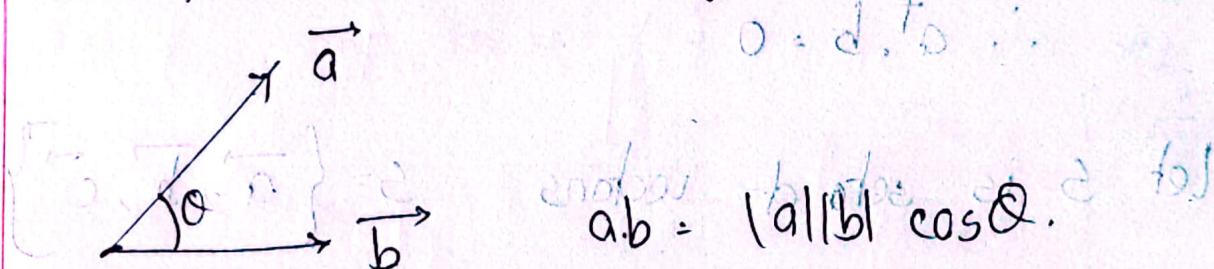
Magnitude of a vector: $\| \cdot \|_{\text{Euclidean}}$ and part

outgoing top $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ with length and part

$$\| \vec{a} \| = \sqrt{(a_1)^2 + (a_2)^2 + (a_3)^2}$$

$$[0 \text{ side } 20^\circ] \quad [2 \text{ side } 14^\circ] = \sqrt{d^2 + d^2} = d\sqrt{2}$$

Dot product (second approach):



$$a \cdot b = \|a\| \|b\| \cos \theta$$

If θ be lengthwise $\cos \theta = \frac{a \cdot b}{\|a\| \|b\|}$

$$\theta = \sqrt{d^2 + d^2} = \sqrt{2d^2} = d\sqrt{2}$$

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length of vectors in norm also θ

Orthogonality: when angle between 2 vectors is 90°

If angle between 2 vectors = 90° , then

or if 2 vectors are perpendicular to each other
they are orthogonal.

If they are orthogonal then their dot product will be 0.

$$\vec{a} \cdot \vec{b} = |a||b| \cos \theta \quad [\cos 90^\circ = 0]$$

$\therefore \vec{a} \cdot \vec{b} = 0$ (because $\cos 90^\circ = 0$)

$$\therefore \vec{a} \cdot \vec{b} = 0$$

Let S is set of vectors $S = \{\vec{a}, \vec{b}, \vec{c}\}$

Now S is orthogonal set iff

$$\vec{a} \cdot \vec{b} = 0$$

$$\vec{b} \cdot \vec{c} = 0$$

$$\vec{c} \cdot \vec{a} = 0$$

so each vector is perpendicular to each other

Orthonormality:

for being orthonormal they have to:

- 1) be orthogonal (dot product = 0)
- 2) length = 1 (unit vectors)

$$a = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}, b = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$$

length of a & b

length of a & b

$$\vec{a} \cdot \vec{b} = \cancel{4 \times 1} + 2 \times -3 + 1 \times 2 = 0$$

length of a & b

\therefore orthogonal

$$|a| \neq 1 \text{ and } |b| \neq 1$$

$$\text{So normalizing: } \hat{a} = \frac{\vec{a}}{|a|} = \frac{1}{\sqrt{21}} \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/\sqrt{21} \\ 2/\sqrt{21} \\ 1/\sqrt{21} \end{bmatrix}$$

$$|\hat{a}| = \sqrt{(4/\sqrt{21})^2 + (2/\sqrt{21})^2 + (1/\sqrt{21})^2} = 1$$

$$\hat{b} = \frac{\vec{b}}{|\vec{b}|} = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{14}} \\ \frac{-3}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \end{bmatrix}$$

(1) To find magnitude of vector \vec{b}

$$|\vec{b}| = \sqrt{\left(\frac{1}{\sqrt{14}}\right)^2 + \left(\frac{-3}{\sqrt{14}}\right)^2 + \left(\frac{2}{\sqrt{14}}\right)^2} = 1$$

the process of converting vectors to unit vectors is called normalization.

So normalization change magnitude but not direction.

So now since they are orthogonal and has magnitude = 1, so they are orthonormal.

$$(1) + (2) + (3) = 10$$

Example:

$$S = \left\{ \frac{1}{\sqrt{5}} [2, 1]^T, \frac{1}{\sqrt{5}} [1, 2]^T \right\} \quad T_B = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$S = \left\{ \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\} \quad U_1 \perp \text{ammonium}$$

$$\vec{U}_1 \cdot \vec{U}_2 = \frac{2}{\sqrt{5}} \times \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{5}} \times -\frac{2}{\sqrt{5}} = 0$$

[\because orthogonal]

$$|\vec{U}_1| = \sqrt{\left(\frac{2}{\sqrt{5}}\right)^2 + \left(\frac{1}{\sqrt{5}}\right)^2} = 1$$

$$|\vec{U}_2| = \sqrt{\left(\frac{1}{\sqrt{5}}\right)^2 + \left(-\frac{2}{\sqrt{5}}\right)^2} = 1$$

[\because Orthonormal]

$$Q^{-1} = Q^T$$

$$Q \cdot Q^T = I + \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} = I$$

$$Q^T \cdot Q = I$$

If $\{Q_i\}$ has column vectors that are from orthonormal sets.

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}$$

[Important]

$$1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}$$

$$1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}$$

[Important]

QR decomposition:

Gram-Schmidt Process:

lets have a basis (v_1, v_2, \dots, v_n) form a vector space. Gram-Schmidt process takes the basis (v_1, v_2, \dots, v_n) and forms new orthogonal basis (p_1, p_2, \dots, p_n) . We can later transform these orthogonal basis into orthonormal basis (q_1, q_2, \dots, q_n) .

original basis: v_1, v_2, v_3

↓ Gram Schmidt Process

orthogonal basis: p_1, p_2, p_3

↓ normalization

Orthonormal basis: q_1, q_2, q_3

$$\underline{1)} P_1 = U_1$$

mitzogen nach R2

$$\underline{2)} P_2 = U_2 - \frac{U_2 \cdot P_1}{P_1 \cdot P_1} P_1 \text{ Holmde - rezip.}$$

notwendig, dass (U_1, U_2, U_3) linear unabh. sind. P1 und P2

$$\underline{3)} P_3 = U_3 - \frac{U_3 \cdot P_1}{P_1 \cdot P_1} P_1 - \frac{U_3 \cdot P_2}{P_2 \cdot P_2} P_2 \text{ Holmde - rezip.}$$

linear unabh. aus (P_1, P_2, P_3) .

Example:

unabhängige Vektoren aus M. $(1, 1, 1)$ sind

$$U_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ linear unabh., } U_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, U_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ linear unabh.}$$

$$\underline{1)} P_1 = U_1 \Rightarrow U_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \text{ sind linear}$$

zusätzlich Holmde mindestens 1.

$$\underline{2)} P_2 = U_2 - \frac{U_2 \cdot P_1}{P_1 \cdot P_1} P_1 \text{ linear unabh.}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{(1 \times 1) + (0 \times -1) + (1 \times 1)}{(1 \times 1) + (-1 \times -1) + (1 \times 1)} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{aligned} & \left[\begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right] - \frac{2}{3} \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array} \right] = \left[\begin{array}{c} 1/3 \\ 2/3 \\ 1/3 \end{array} \right] = P_1 E \\ & = \left[\begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right] - \left[\begin{array}{c} 2/3 \\ -2/3 \\ 2/3 \end{array} \right] = \left[\begin{array}{c} 1/3 \\ 2/3 \\ 1/3 \end{array} \right] = P_1 \left[\begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right] \end{aligned}$$

$$\begin{aligned} 3) P_3 &= U_3 - \frac{U_3 \cdot P_1}{P_1 \cdot P_1} P_1 - \frac{U_3 \cdot P_2}{P_2 \cdot P_2} P_2 = P_3 E \\ &= \left[\begin{array}{c} 1 \\ 1 \\ 2 \end{array} \right] - \frac{(1 \times 1) + (1 \times -1) + (2 \times 1)}{(1 \times 1) + (-1 \times -1) + (1 \times 1)} \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array} \right] \\ &\quad - \frac{(1 \times 1/3) + (1 \times 2/3) + (2 \times 1/3)}{(1/3 \times 1/3) + (2/3 \times 2/3) + (1/3 \times 1/3)} \left[\begin{array}{c} 1/3 \\ 2/3 \\ 1/3 \end{array} \right] \end{aligned}$$

$$= \left[\begin{array}{c} 1 \\ 1 \\ 2 \end{array} \right] - \frac{2}{3} \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array} \right] - \frac{5}{2} \left[\begin{array}{c} 1/3 \\ 2/3 \\ 1/3 \end{array} \right]$$

$$= \left[\begin{array}{c} 1 \\ 1 \\ 2 \end{array} \right] - \left[\begin{array}{c} 2/3 \\ -2/3 \\ 2/3 \end{array} \right] - \left[\begin{array}{c} 5/6 \\ 5/3 \\ 5/6 \end{array} \right] = \left[\begin{array}{c} -1/2 \\ 0 \\ 1/2 \end{array} \right]$$

P_1, P_2, P_3 are orthogonal basis.

Now for normalization:

$$1) q_1 = \frac{P_1}{\|P_1\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$2) q_2 = \frac{P_2}{\|P_2\|} = \frac{1}{\sqrt{6/3}} \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} \sqrt{6}/6 \\ \sqrt{6}/3 \\ \sqrt{6}/6 \end{bmatrix}$$

$$3) q_3 = \frac{P_3}{\|P_3\|} = \frac{1}{2\sqrt{2}} \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix} = \begin{bmatrix} -\sqrt{2}/4 \\ 0 \\ \sqrt{2}/4 \end{bmatrix}$$

$$\begin{bmatrix} e^{ix} \\ e^{ix} \\ e^{ix} \end{bmatrix} = (e^{ix}) + (e^{ix}) + (e^{ix})$$

$$\begin{bmatrix} e^{ix} \\ e^{ix} \\ e^{ix} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{ix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} e^{ix} \\ e^{ix} \\ e^{ix} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{ix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

QR decomposition:

Any $m \times n$ matrix A can be written in the form: $A = Q R$

where Q is a $(m \times n)$ matrix with orthonormal columns
 R is a upper triangular matrix of shape $(n \times m)$

$$m \times n \quad m \times n \quad n \times m$$

$$Q^T A = Q^T Q R$$

$$Q^T A = I R \quad [\because A A^T = I]$$

$$R = Q^T A$$

Example:

$$u_1 = -3 \quad u_1 = 0 \quad u_2 = 6$$

$$f(u_1) = 0 \quad f(u_1) = 0 \quad f(u_2) = 2$$

$$\begin{bmatrix} 1 & -3 \\ 1 & 0 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$\downarrow \quad \downarrow$$

$$U_1 \quad U_2$$

$$P_1 = U_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$P_2 = U_2 - \frac{U_2 \cdot P_1}{P_1 \cdot P_1} P_1 = \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix} + \frac{(-3 \times 1) + (0 \times 1) + (6 \times 1)}{1 \times 1 + 1 \times 1 + 1 \times 1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ -1 \\ 5 \end{bmatrix}$$

$$P_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad P_2 = \begin{bmatrix} -4 \\ -1 \\ 5 \end{bmatrix}$$

$$d = xA$$

$$d^T A = x \cdot A^T A$$

$$q_1 = \frac{P_1}{|P_1|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$q_2 = \frac{P_2}{|P_2|} = \frac{1}{\sqrt{42}} \begin{bmatrix} -4 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} -4/\sqrt{42} \\ -1/\sqrt{42} \\ 5/\sqrt{42} \end{bmatrix}$$

$$Q = \begin{bmatrix} 1/\sqrt{3} & -4/\sqrt{42} \\ 1/\sqrt{3} & -1/\sqrt{42} \end{bmatrix}$$

$$d^T Q = x R$$

$$R = Q^T A = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -4/\sqrt{42} & -1/\sqrt{42} & 5/\sqrt{42} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1/\sqrt{3} \\ -4/\sqrt{42} \end{bmatrix} \begin{bmatrix} \sqrt{3} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$Ax = b$$

$$A^T A \cdot x = A^T b$$

$$(QR)^T \cdot (QR)x = (QR)^T b$$

$$R^T Q^T \cdot Q R x = R^T Q^T b$$

$$\boxed{R^T \cdot I \cdot R x = R^T Q^T b}$$

$$Rx = Q^T b$$

$$\begin{bmatrix} \sqrt{3} & \sqrt{3} \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \sqrt{1}/\sqrt{3} & \sqrt{1}/\sqrt{3} \\ -4/\sqrt{2} & \sqrt{1}/\sqrt{2} + 5/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{3} & \sqrt{3} \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 2\sqrt{3} \\ 5\sqrt{2}/\sqrt{2} \end{bmatrix}$$

$$\begin{aligned} a_0 &= \begin{bmatrix} 3/7 \\ 5/7 \end{bmatrix} \\ a_1 &= \begin{bmatrix} 5/7 \\ 2/7 \end{bmatrix} \end{aligned}$$