Polynomial

$$P_{\boxed{n}}(x) = a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3 + \dots + a_nx^n$$
$$= a_0 + a_1x^1 + a_2x^2 + a_3x^3 + \dots + a_nx^n$$

→ Degree

$$P_3(x) = a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3$$

Degree $= \boxed{3}$

coefficient $= 4$ $[a_0, a_1, a_2, a_3]$

Notice:

Degree $= n$

Coefficient $= n+1$

Example 1: If we have a polynomial with a degree of 27, how many co-efficient will it have?

Degree, $n = 27$

$\therefore$ Coefficient $= n+1$
$= 27+1$
$= 28$

Vector Space

It is a region, where we can add vectors, multiply vectors with scalers.

Add vectors

$\boxed{1+x+x^2} + \boxed{x^3}$

$= 1+x+x^2+x^3$

multiply with scalars

$(1+x+x^2) \times 5$

$= 5+5x+5x^2$

Basis

Basis is a set of vectors that spans the space.

$P_3(x) = a_0 + a_1 x^1 + a_2 x^2 + a_3 x^3$

Basis $= \{1, x^1, x^2, x^3\}$ [with this we can generate any sort of polynomial]

Dimensional space

The number of elements in a basis.

$P_3(x) = a_0 + a_1 x^1 + a_2 x^2 + a_3 x^3$

Basis $= \{1, x^1, x^2, x^3\}$

Dimensional Space $= 4$

Note: The dimensional space will be one more than the degree of polynomial, Example: if we have a polynomial with a degree of 58, $P_{58}(x)$ then what will be the dimensional space & no of coefficient.

$$\text{dimensional space} = n+1 = 58+1 = 59$$

$$\text{coefficient} = 58+1 = 59 = \boxed{(n+1)}$$

Functional Space

$f(x)$ can go up to infinity

$$F(x) = 2 + 3x + 10x^2 + 14x^3 + 25x^4 + \cdots\cdots$$

* There is no end to it
* It can go upto infinity

However, we can create a version of it using $P_n(x)$. Because $P_n(x)$ has a limit.

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

↳ It has a limit

$\boxed{\text{Note:}}$ $f(x)$ belongs to the vector space of infinite dimension

$$f(x) \in V^{\infty} \rightarrow \text{Vector space}$$

On the other hand, $P_n(x) \in V^{n+1} \rightarrow$ dimension

Now, Suppose we have

$$f(x) = 2 + 3x + 4x^2 + \underbrace{5x^3 + 8x^4}_{\text{Error}}$$

Error [Because we can't replicate this portion]

But, we have $P_2(x) = a_0 + a_1 x + a_2 x^2$

Now, The $\boxed{\text{higher the degree}}$, the lower the error

Because a higher degree polynomial can maintain similarity over a large portion.

Weierstrass Approximation Theorem

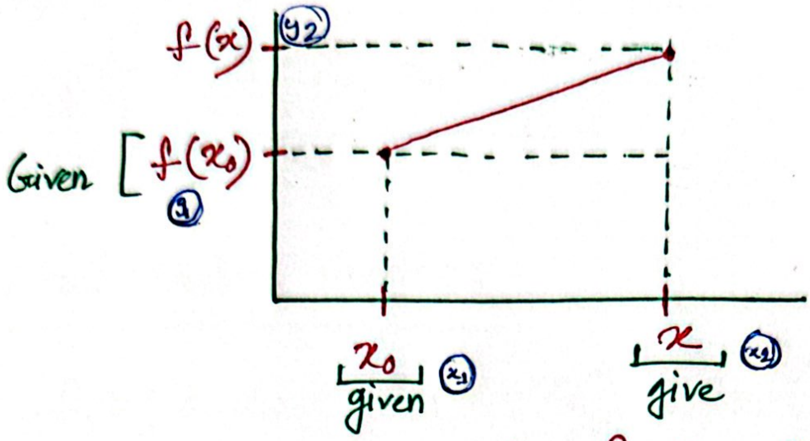
$$|f(x) - P_n(x)|$$

The larger the value of n, the more the error will decrease.

$$P_5(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$$

→ It creates less error.

Taylor Series



We have to determine $f(x)$. Additionally, $f'(x_0)$ is give also.

$$\text{Gradient, } f'(x_0) = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{f(x) - f(x_0)}{x - x_0}$$

We have to Find out $f(x)$

$$\text{So, } f'(x_0) = \frac{f(x) - f(x_0)}{x - x_0}$$

$$f'(x_0)(x - x_0) = f(x) - f(x_0)$$

$$\boxed{f(x) = f'(x_0)(x - x_0) + f(x_0)}$$

Taylor Series for straight line

Note: Actual taylor series → more comple & Infinitely large

Taylor Series (Not finite)

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)(x-x_0)^2}{2!}$$
$$+ \frac{f'''(x_0)(x-x_0)^3}{3!} + \cdots\cdots$$

Prove of Taylor Series:

$$f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3$$
$$+ \cdots\cdots$$

$$f'(x) = 0 + a_1 + 2a_2(x-x_0) + 3a_3(x-x_0)^2 + \cdots$$

$$f''(x) = 0 + 0 + 2a_2 + 3\times 2\, a_3(x-x_0)$$

$$f'''(x) = 0 + 0 + 0 + 3\times 2\, a_3$$

Now, Let $x = x_0$

$$f(x_0) = a_0 + a_1(x_0 - x_0) + a_2(x_0 - x_0)^2 + a_3(x_0-x_0)^3 + \cdots\cdots$$

$$= a_0 \quad \cdots\cdots (i)$$

$$f'(x_0) = a_1 \quad \cdots\cdots (ii)$$

$$f''(x_0) = 2a_2 \quad \cdots\cdots (iii)$$

$$f'''(x_0) = 3\times 2 a_3 \quad \cdots\cdots (iv)$$

From equation (iii)

$$f''(x_0) = 2a_2$$
$$a_2 = \frac{f''(x_0)}{2!}$$

From (iii)

$$f'''(x_0) = 3 \times 2\, a_3$$

$$a_3 = \frac{f'''(x_0)}{3!}$$

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)(x-x_0)^2}{2!} + \frac{f'''(x_0)(x-x_0)^3}{3!} + \cdots\cdots$$

Example:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\cdots$$

$$F(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f^4(x_0)(x-x_0)^2}{2!} + \frac{f'''(x_0)(x-x_0)^3}{3!} + \frac{f^4(x_0)(x-x_0)2}{4!} + \frac{f^5(x_0)(x-x_0)^2}{5!} + \cdots\cdots$$

We will always consider $x_0 = 0$ if nothing is given

$$f(x) = \sin(0) + \cos(0)(x) + \frac{(-\sin(0))(x)^2}{2!} + \frac{(-\cos(0))x^3}{3!} + \frac{\sin(0)x^4}{4!} + \frac{\cos(0)x^5}{5!} + \cdots$$

$$= 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\cdots$$

if $x = 0.1$

1st term $\Rightarrow f(0.1) \approx 0.1$,

2nd " $\Rightarrow f(0.1) \approx 0.1 - \frac{(0.1)^3}{3!} = 0.099833\ldots$

3rd " $\Rightarrow f(0.1) \approx 0.1 - \frac{(0.1)^3}{3!} + \frac{(0.1)^5}{5!}$

$= 0.0998334\ldots$

exact answer $\sin(0.1) = 0.0\underbrace{998334}_{6sf}$

Vandermonde Matrix

House Size (m^2)	Rent
20	200
30	250
40	275

nodal points/nodes/ Given values

$f(x)$ [Actual Values]

what will be the rent if the house size is 25?

In this case we have to generate a polynomial.

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots + a_n x^n$$

Here we have to follow some law

$$f(x) = P_n(x)$$

$$\boxed{P_n(30) = 250}$$

It means in a polynomial if we give 30 it needs to show 250

*

Age	Salary
25	25000
30	30,000
45	70,000

We can generate a polynomial

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

$$P_n(\text{nodes}) = f(\text{nodes})$$
$$P_n(30) = 30{,}000$$

The way to setup the degree:

$$P_2(x) = a_0 + a_1(x) + a_2 x^2$$

Here we need to know the coefficient. As we have three unknown coefficient or variable, we need three equations.

$$P_2(25) = a_0 + 25a_1 + 25^2 a_2$$
$$= a_0 + 25a_1 + 625a_2 = 25{,}000$$
$$P_2(30) = a_0 + 30a_1 + 900a_2 = 30{,}000$$
$$P_2(45) = a_0 + 45a_1 + 2025a_2 = 70{,}000$$

The degree will be one less than the number of node point given. $(n+1)$ nodes then degree will be n.

We usually use Vandermonde Matrix for degree=1.
We don't even use it for 2, 3, - - - - .

nodal points = 2
So, degree = 1

x_0	$f(x_0)$
$x_1 = 2$	5
$x_2 = 3$	6

$$P_1(x) = a_0 + a_1 x = f(x)$$
$$P_1(x_1) = a_0 + a_1 x_1 = f(x_1)$$
$$= a_0 + x_1 a_1 = f(x_1)$$
$$P_1(x_2) = a_0 + x_2 a_1 = f(x_2)$$

Coefficient Matrix:

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} f(x_1) \\ f(x_2) \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix}^{-1} \begin{bmatrix} f(x_1) \\ f(x_2) \end{bmatrix}$$

Proper Example

	Time	Velocity
nodes/nodal point = 2	15	362.8
Degree = 1	20	517.3

$$P_1(15) = a_0 + 15a_1 = 362.8$$

$$P_1(20) = a_0 + 20a_1 = 517.3$$

$$\begin{bmatrix} 1 & 15 \\ 1 & 20 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 362.8 \\ 517.3 \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 & 15 \\ 1 & 20 \end{bmatrix}^{-1} \begin{bmatrix} 362.8 \\ 517.3 \end{bmatrix}$$

$$= \frac{1}{20-15} \begin{bmatrix} 20 & -15 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 362.8 \\ 517.3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -3 \\ -1/5 & 1/5 \end{bmatrix} \begin{bmatrix} 362.8 \\ 517.3 \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 4 \times 362.8 - 3 \times 517.35 \\ -1/5 \times 362.8 - 1/5 \times 517.35 \end{bmatrix}$$

$$= \begin{bmatrix} -100.85 \\ 30.91 \end{bmatrix}$$

$a_0 = -100.85 \quad a_1 = 30.91$

In the case of higher degree polynomial matrix very large & that's why the time & space complexity will be increase. Also, it become too difficult to process.