Chapter 4

Non-linear equation:

the polynomials with degree > 1.

it can be a series, polynomial, rational function, etc.

the function must be continuous on some interval

$I = [a, b]$

We want to solve $f(x) = 0$

solution of $f(x) = 0$ is called root of the function.
and denoted by x_*. It is also called zero of the function.

$f(x_*) = 0 \Rightarrow$ exact solution.

$(x - x_*)$ is a faction of $f(x)$

How does the graph look like?

i) at $x = x_*$, the function will cross x-axis so the function changes sign.

ii) or it ~~midth~~ might just touch x-axis. Which implies first derivative changes sign at $x = x_*$

Here we will assume the graph will cross x-axis only once for $x \in I = [a,b]$

in other words factor $(x - x_*)$ has multiplicity 1.

We are looking for approximate solutions:

$f(x) \approx 0$, this also implies that $x - x_* \approx 0$.

What to do for the functions that

- With higher multiplicity

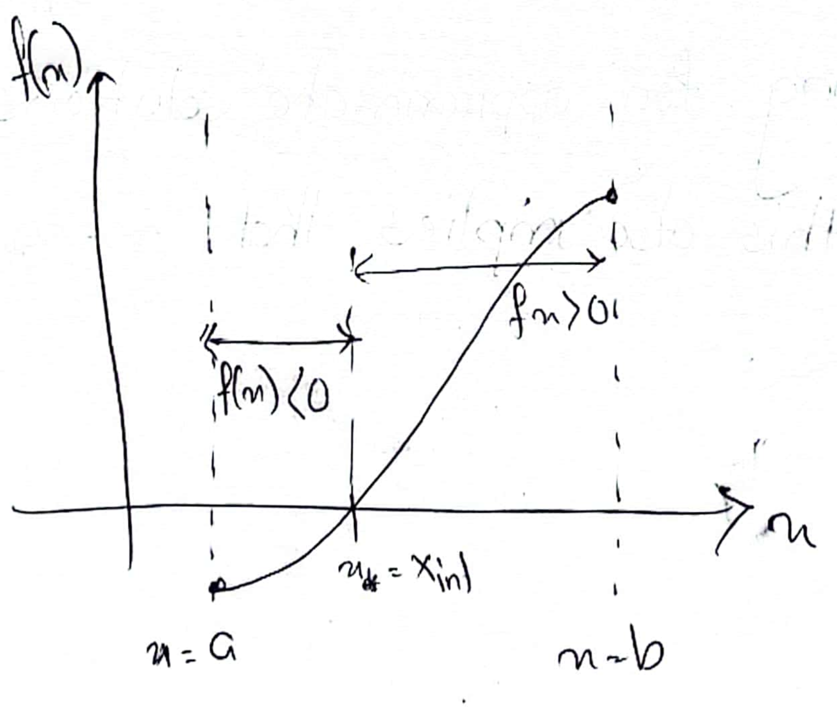
  solution: We divide the domain of the function:

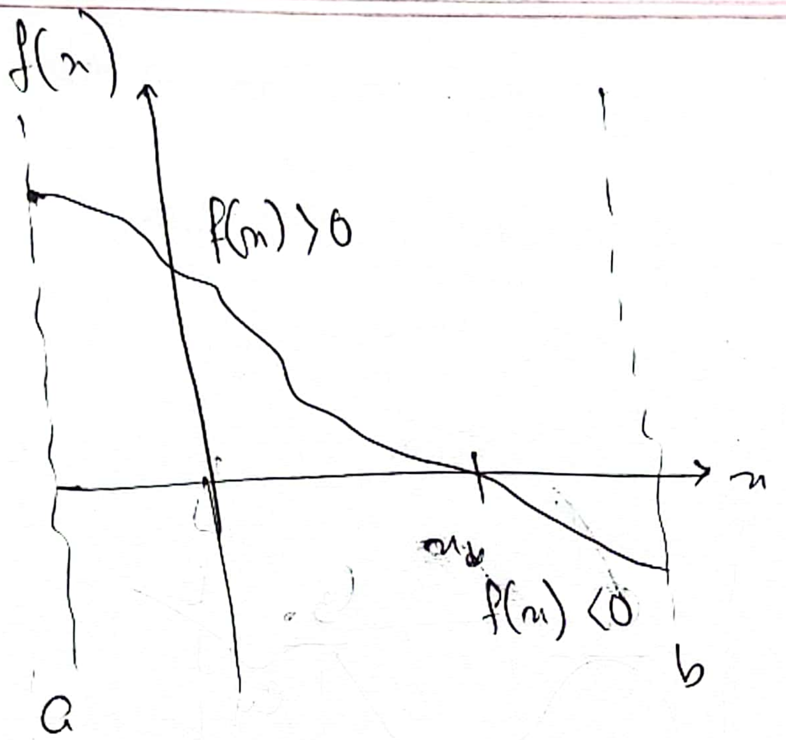
  $[a,b] = [a,c_1] \cup \dots \cup [c_n,b] = I_1 \cup \dots \cup I_n$

  $c_1, c_2, \dots$ are turning points.

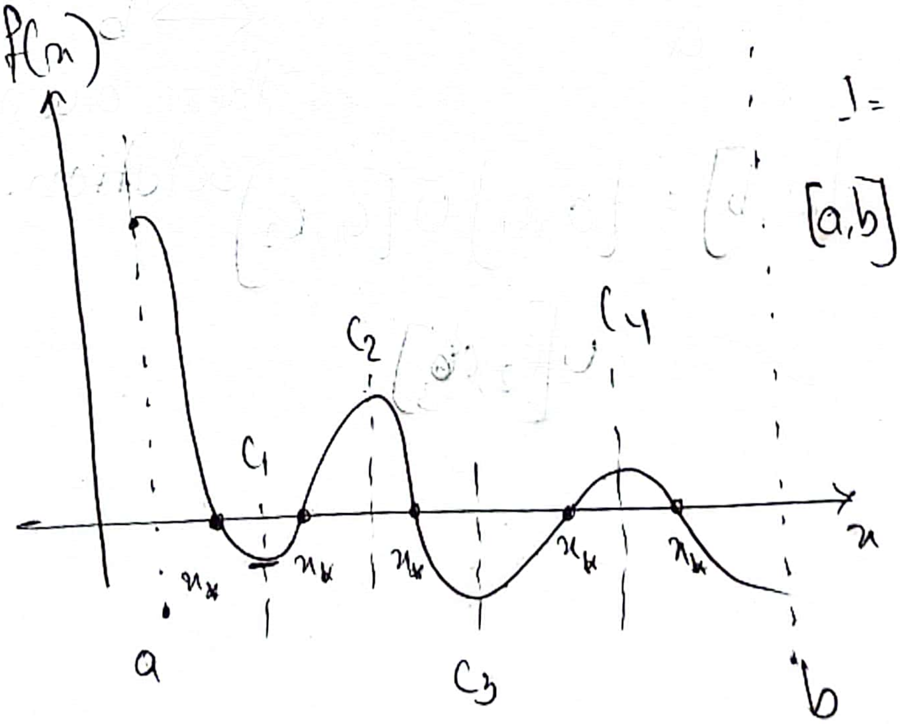
- only touch x-axis

  solution: we do the same as previous case.





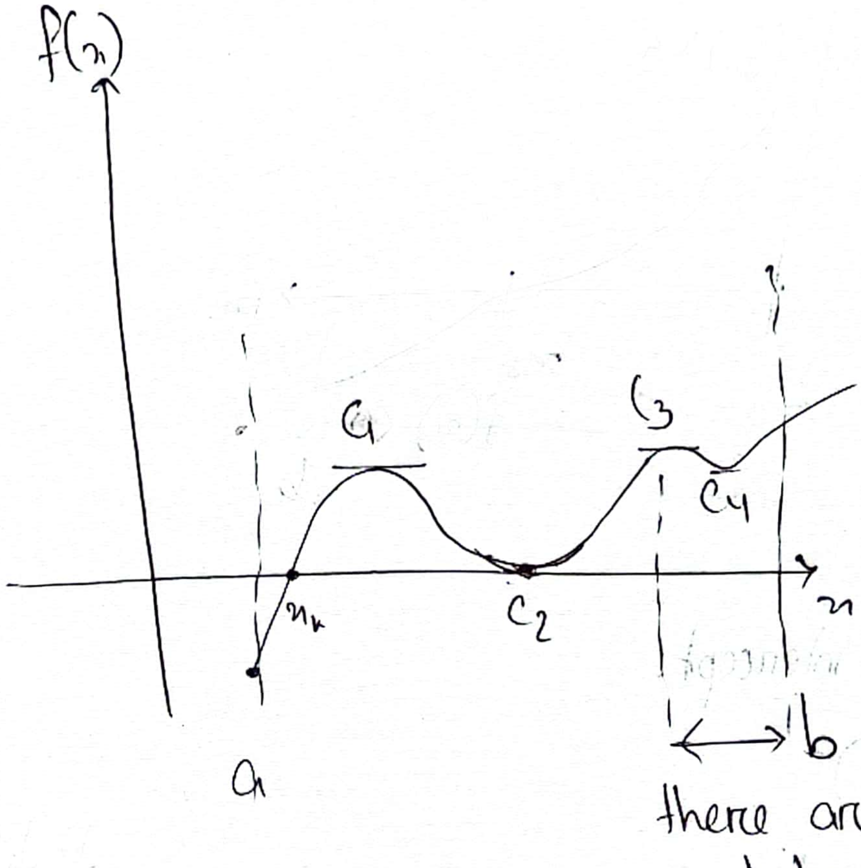
multiple intercept:



$I = I_1 \cup I_2 \cup I_3 \cup I$

$[a,b] = [a,c_1] \cup [c_1,c_2] \cup [c_2,c_3] \cup [c_3,c_4]$
$[c_4,b]$

just touch:



$[a,b] = [a,c_1] \cup [c_1,c_2] \cup [c_2,c_3]$

Interval bisection method:

Intermediate value theorem:

If a function $f(x)$ is continuous on an Interval $I = [a,b]$ then for each $x \in I$, there exists a real number c such that $f(x) = c$.

Now if $c = 0$, $f(x) = 0$ which is the root.

then we must have,

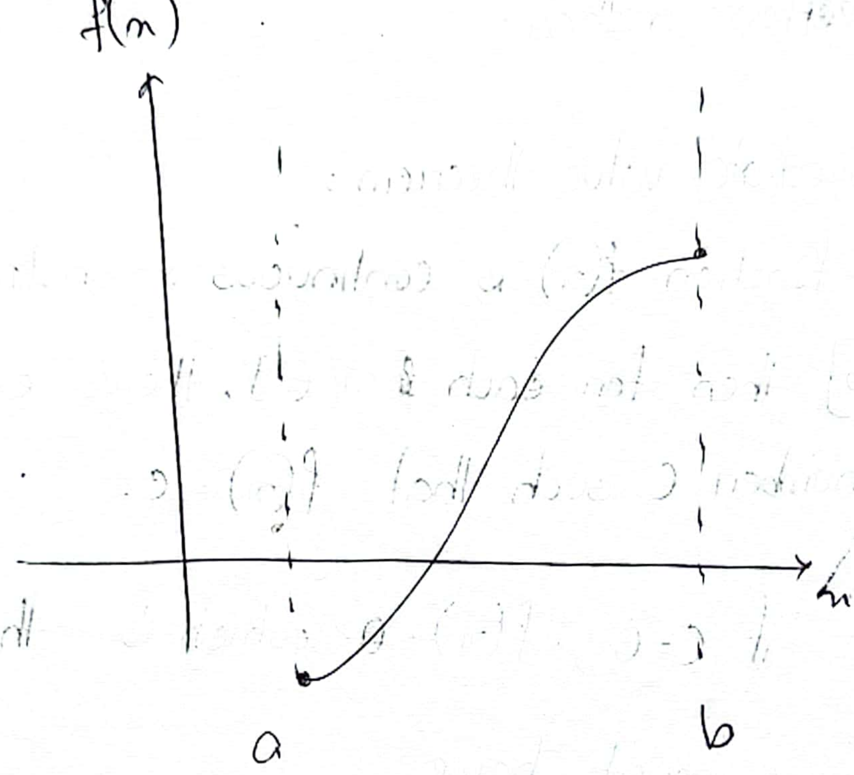
$f(a) > 0$ and $f(b) < 0$

or $f(a) < 0$ and $f(b) > 0$.

$\therefore f(a) f(b) < 0$

$\therefore$ if $f(a) f(b) > 0$, then there is not solution in that interval.

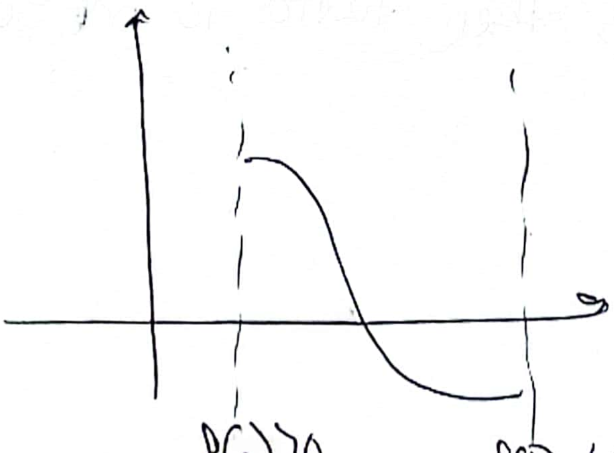
converse is also true



$f(a) < 0$ $\quad$ $f(b) > 0$.

$\exists\, x_* \in [a, b]$

$f(a)\, f(b) < 0$.



$f(a) > 0$ $\quad$ $f(b) < 0$

$x_* \in [a, b]$

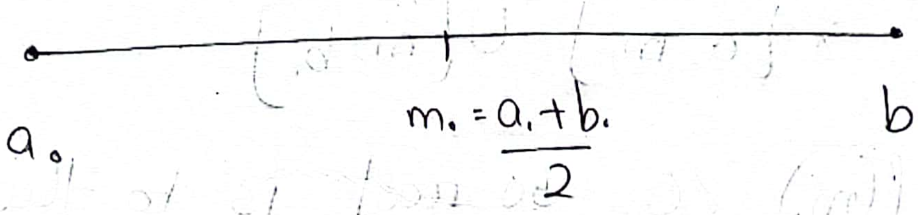
$f(a)\, f(b) < 0$.

How does interval bisection method works:

let $I = [a_0, b_0]$, we also assume $f(a_0) > 0$ and $f(b_0) < 0$.

Now midpoint $m_0 = (a_0 + b_0)/2$

$$[a_0, b_0] \to [a_0, m_0] \cup [m_0, b_0]$$



now, the root will fall in any of the sub-interval $[a_0, m_0]$ or $[m_0, b_0]$

$f(m_0) = 0$, the $x_* = m_0$

$f(m_0) > 0$ root is on the right.

$$|x_* - m_0| < \frac{1}{2}|a_0 - a_0| \qquad k = 0.$$

~~$J_1 = [a_1 m_1] \cup$~~

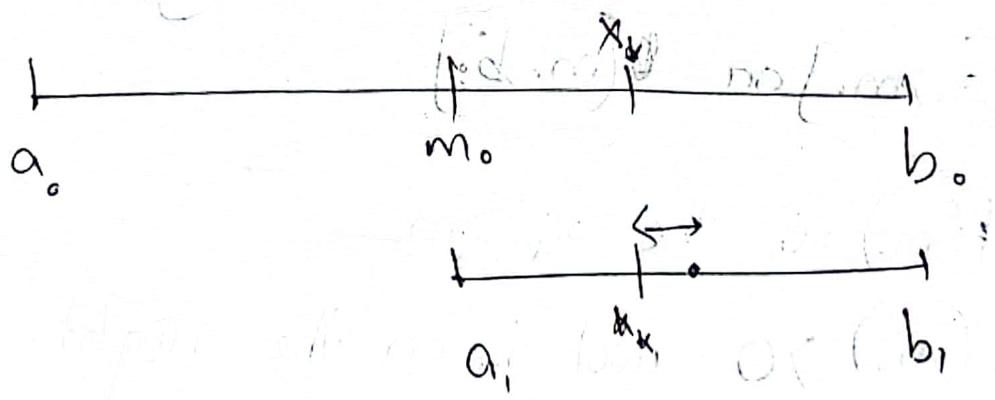
Now if error is within band, the m_0 is the root, else we continue.

$$J_1 = [m_0, b_0]$$
$$= [a_1, b_1]$$

$$m_1 = \frac{1}{2}(b_1 + a_1)$$

$$J \longrightarrow [a, m_1] \cup [m, b_1]$$

let $f(m_1) < 0$, so root is to the left



error now: $|x_* - m_1| \leq \frac{1}{2}|b_1 - a_1| = \frac{1}{4}(b_0 - a_0)$

after n iteration,

$$\left|x - m_n\right| \leq \frac{|b_0 - a_0|}{2^{n+1}}$$

$$\frac{|b_0 - a_0|}{2^{n+1}} \leq \delta \Rightarrow n \geq \frac{\log(|b - a_0|) - \log(\delta)}{\log(2)} - 1$$

if $a_0 = 1.5$, $b_0 = 3$ and $\delta = \epsilon_M = 1.1 \times 10^{-16}$

then, $$n \geq \frac{\log(3-1.5) - \log(1.1 \times 10^{-16})}{\log 2} - 1$$

$\therefore n \geq 53$ iteration.

Example:

Use interval bisection method to find solution accurate to within 10^{-3} for $f(x) = x^3 - 7x^2 + 14x - 6 = 0$, on interval $[1, 3.2]$

Soln: $a_0 = 1$ $b_0 = 3.2$

$f(1) = 1^3 - 7(1^2) + 14(1) - 6 = 2 > 0.$

and $f(3.2) = -0.11 < 0.$

Since $f(a)\, f(b) < 0$,

so $x_\alpha \in I_0 = [a_0, b_0]$

$$m_0 = \frac{a_0 + b_0}{2} =$$

k	a_k	b_k	m_k	$f(a_k)$	$f(b_k)$	$f(m_k)$	x_*
0	2.1	3.2	2.1	2	-0.11	1.79	
1	2.1	3.2	2.65	~~0.11~~ 1.79	-0.11	0.55	
2	2.65	3.2	2.925	0.55	-0.11	0.086	
3	2.925	3.2	3.0625	0.86	-0.11	-0.054	
4	2.925	3.0625					
⋮						⋮	
9	2.998	3.002	3.000195	1.96×10^{-3}	-2.3×10	$-1.95\times10^{-3} < 10^{-3}$	

stopping condition.

$x_* = 3.\underbrace{000}$

three places as error bound 10^{-3}.

Fixed point iteration:

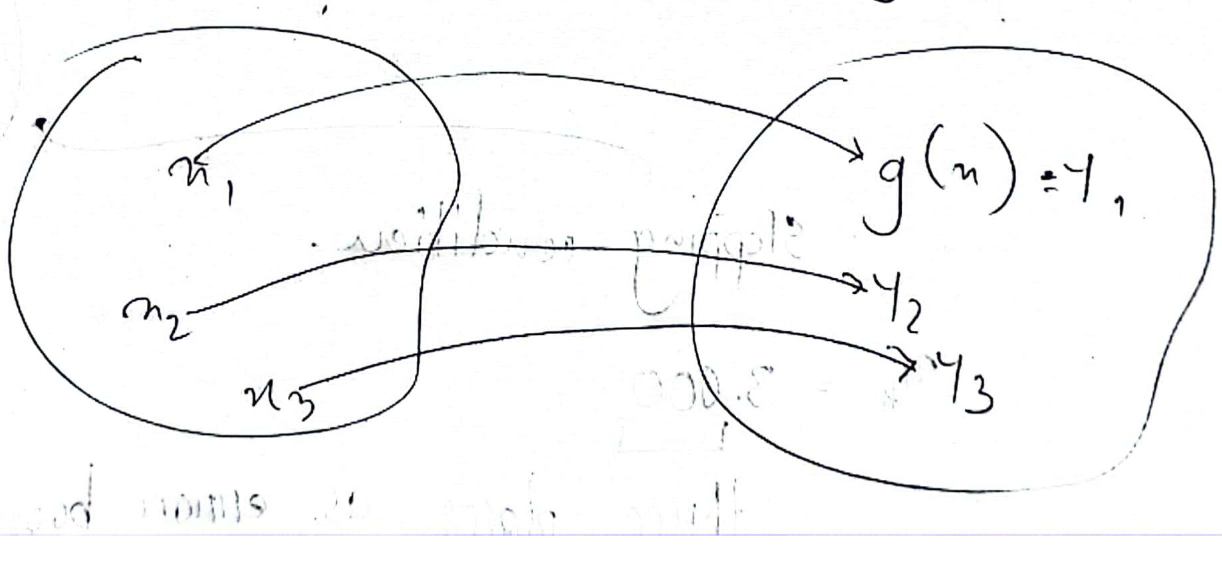
$$f(x) = 0 \longrightarrow g(x) - x = 0.$$

$f(x)$ and $g(x)$ must be continuous fun^c in some interval $I = [a,b]$

advantage: multiple root can be taken care of.

$f(x) = 0$, if solution exists, this implies that $g(x) = x$.

this is fixed equation or mapping.

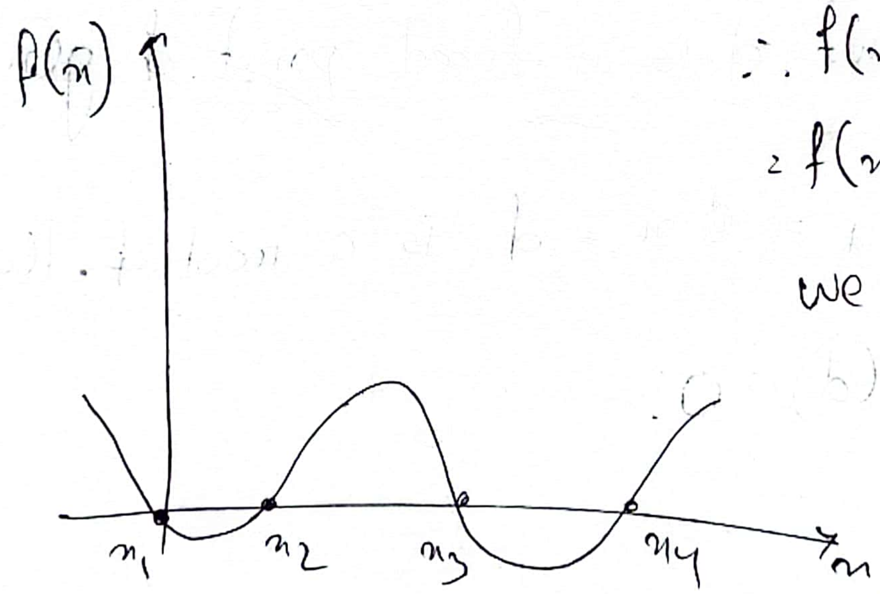


Now for non-trivial mapping $y = g(x) \neq x$.

$\Rightarrow$ in trivial it could be.

So, Any point x that remains same under nontrivial mapping g is called the fixed point under the mapping.

if $x_0 \in I \Rightarrow g(x_0) = x_0$, then we call x_0 a fixed point.



$\therefore f(x_1) = f(x_2) = f(x_3) = f(x_4) = 0$.

we want $g(x)$

i.e. $g(x_1) = x_1$

$g(x_2) = x_2$

$g(x_3) = x_3$

$g(x_4) = x_4$

fixed points

graph of $g(x) = x$

- It is clear that a line through the origin (0,0) and (c,c) must have slope 1 (one).

Therefore any graph of $g(x)$ v/s x that intersect the line with slope one must have a fixed point.

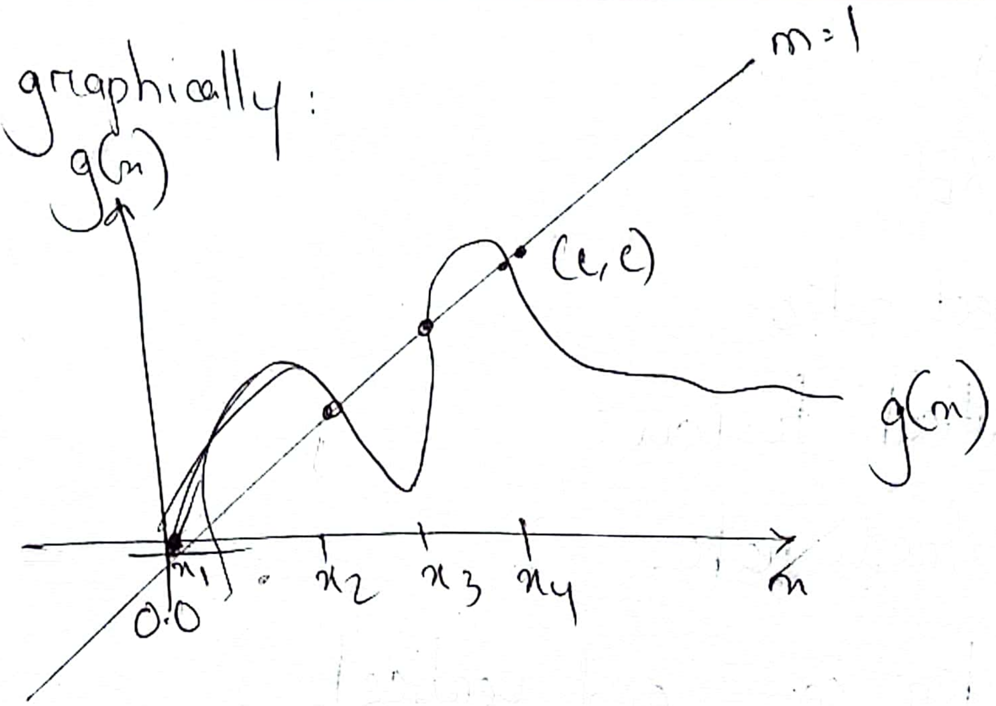
if $g(x)$ intersect $x = d$, then $g(d) = d$

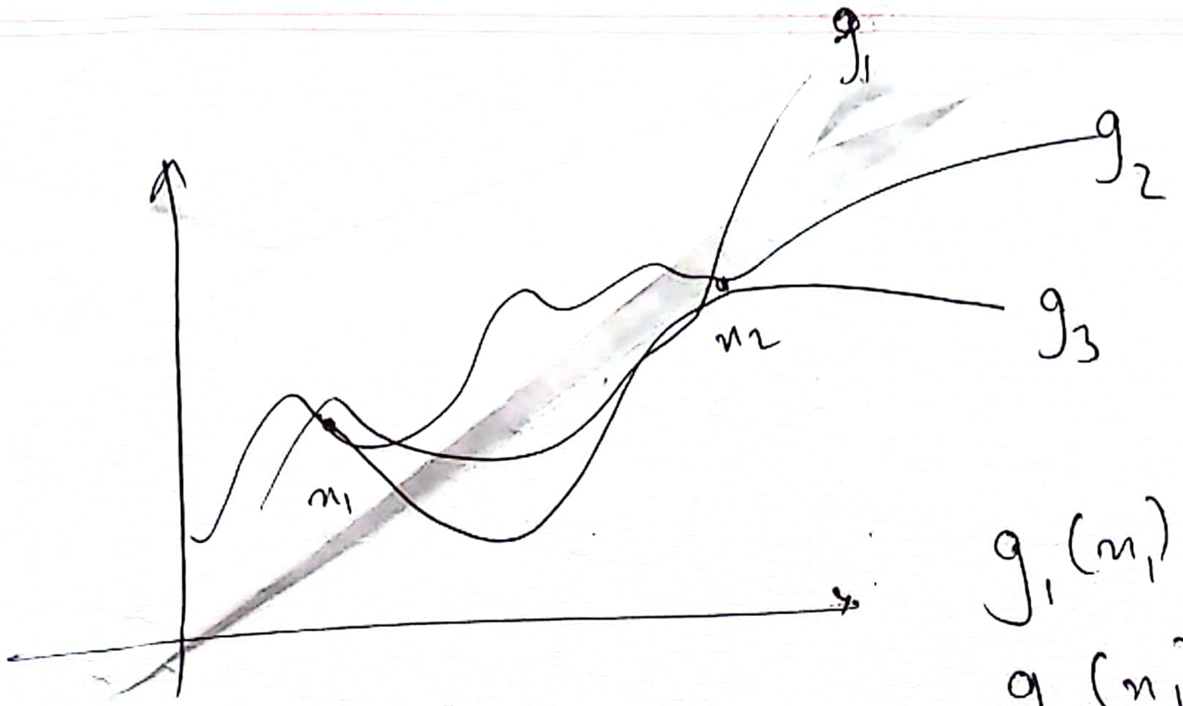
$\therefore$ $x = d$ is a fixed point of $g(x)$

$\therefore$ $x_* = x = d$ is a root of the func.

$f(d) = 0$.

graphically:





$g_1(x_1) = x_1$

$g_2(x_1) = x_1$

$g_3(x_1) = x_1$

$g_1(x_2) = x_2$

$g_2(x_2) = x_2$

$g_3(x_2) = x_2$

$g(x) = \dfrac{x+2}{2}$

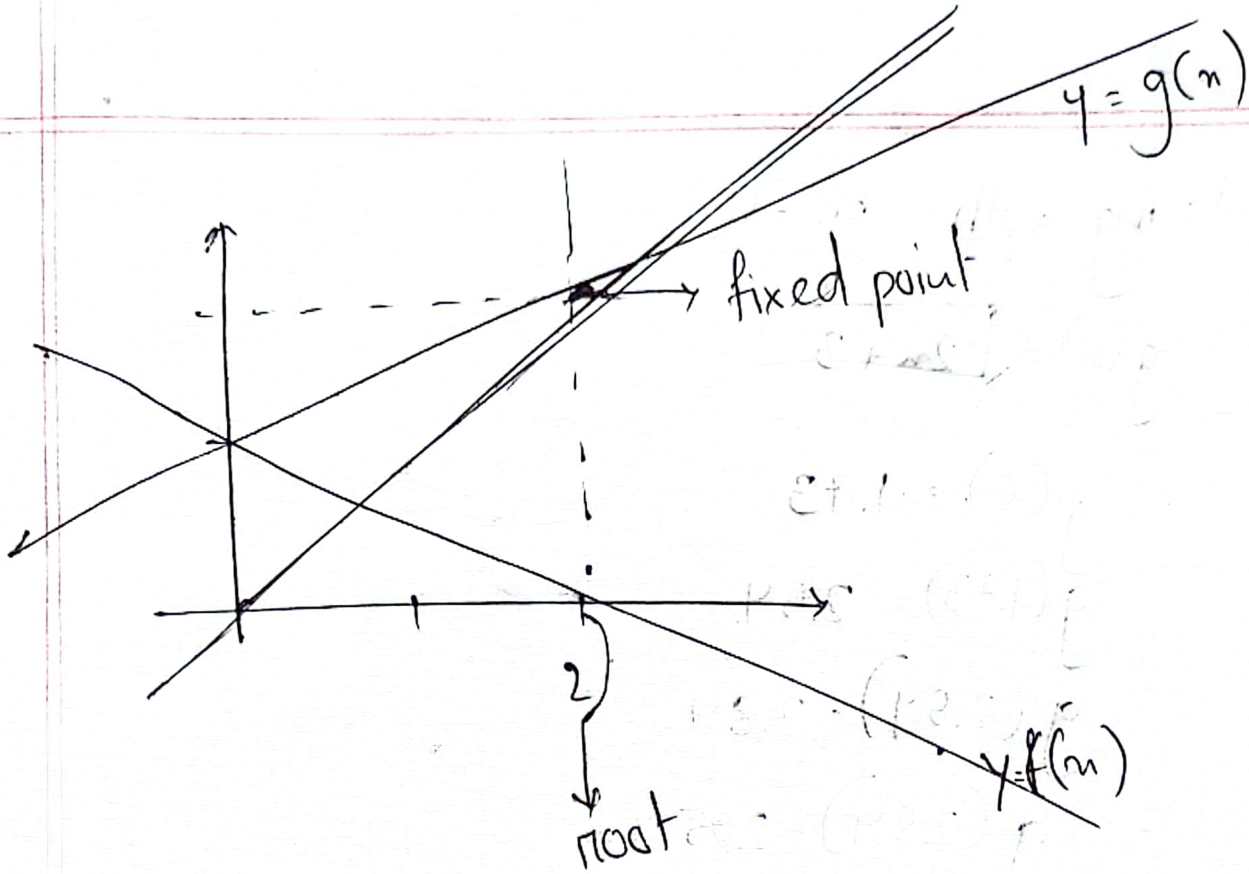
$g(0) = 1 \Rightarrow x=0$ is not fixed point

$g(1) = 1.5 \Rightarrow x = 1.5$ " " " "

$g(2) = 2 \Rightarrow x = 2$ a fixed point

$\therefore f(x) = g(x) - x = \dfrac{x+2}{2} - x = -\frac{1}{2}x + 1$

has a root at 2.



$f(x) = x^2 - 2x - 3 = 0.$

in reality $x^2 - 2x - 3 = (x-3)(x+1)$, $f(x)$ has two roots at $x = 3, -1$

lets try out $g(x)$

1) $x^2 - 2x - 3 = 0 \Rightarrow x = \sqrt{2x+3}$

2) $x^2 - 2x - 3 = x(x-2) - 3 \Rightarrow x = \dfrac{3}{x-2}$

3) $x^2 - 2x - 3 = 0 \Rightarrow x = x^2 - x - 3$

4) $x^2 - 2x - 3 = 0 \Rightarrow 2x^2 - 2x = x^2 + 3 \Rightarrow x = \dfrac{x^2+3}{2x-2}$

starting with $x_0 = 0$

$g(x) = \sqrt{2x+3}$

$g(0) = 1.73$

$g(1.73) = 2.54$

$g(2.54) = 2.84$

$g(2.84) = 2.95$

$g(2.95) = 2.98$

$g(2.98) = 2.99$

$g(2.99) = 3.00$

$g(x) = x^2 - x - 3$ → not convergent

$g(0) = -300$

$g(-3) = 9.00$

$g(9) = 69$

$g(69) = 4.69 \times 10^3$

$$g(x) = \frac{x^2+3}{2x-2}$$

$g(0) = -1.5$

$g(-1.5) = -1.05$

$g(-1.05) = -1.00$

question is what $g(x)$ to take, what x to take.

answer → "Contraction Mapping theorem"

$g(x) = \sqrt{2x+3}$

$g(42) = 9.33$	$g(0) = 1.73$
$g(9.33) = 4.65$	$g(1.73) = 2.54$
$g(4.65) = 3.51$	$g(2.54) = 2.84$
$g(3.51) = 3.17$	$g(2.84) = 2.95$
$g(3.17) = 3.06$	$g(2.95) = 2.98$
$g(3.06) = 3.02$	$g(2.98) = 2.99$
$g(3.02) = 3.01$	$g(2.99) = 3.00$
$g(3.01) = 3.00.$	$g(3.00) = 3$

Both $x_0 = 0$ & $x_0 = 42$ converges to root, $x_* = 3$

Even though $x_0 = 0$ is closer to $x_* = -1$, it converges to $x_* = 3$

$g(x) = x^2 - x - 3$

$g(42) = 1.72 \times 10^3$

$g(1.72 \times 10^3) = 2.95 \times 10^6$

$g(2.95 \times 10^6) = 8.72 \times 10^{12}$

$g(0) = -3$

$g(-3) = 9$

$g(9) = 69$

$\vdots$

$\therefore$ both diverges

$g(x) = \dfrac{x^2+3}{2x-2}$

$g(42) = 21.6$

$g(21.6) = 11.4$

$g(11.4) = 6.39$

$g(6.39) = 4.07$

$g(4.07) = 3.19$

$g(3.19) = 3.01$

$g(3.01) = 3.00$

$g(0) = -1.5$

$g(-1.5) = -1.05$

$g(-1.05) = -1$

$g(-1) = -1$

$\therefore$ $x_0 = 0$ converges to nearest root -1

$$\lambda = \left|\frac{\Delta g(x_k)}{\Delta x}\right| = \left|\frac{g(x_{k+1}) - g(x)}{x_{k+1} - x_k}\right| < 1$$

Contraction mapping theorem:

if g is a contraction mapping on $L = [a, b]$ then

1) There exists a unique fixed point $x_* \in L$ with $g(x_*) = x_*$

2) For any $x_0 \in L$, the iteration $x_{k+1} = g(x_k)$ will converge to x_* as $k \to \infty$

$\lambda = |g'(\text{root})|$

$g(x) = \sqrt{2x+3} = (2x+3)^{1/2}$

$g'(x) = \frac{1}{2}(2x+3)^{-1/2}$

$\lambda = |g'(1)| = .1$ (not < 1)

$\lambda = |g'(3)| = \frac{1}{3}$ (<1)

$\therefore$ both $x_0 = 0$, $x_0 = 42$, converge to $x_* = 3$

~~but $x_0 = 0$, $x_0 = 42$ converge to~~

$\lambda = 0 \rightarrow$ super linear convergance

$\hookrightarrow$ fastest convergence less iteration required

$0 < \lambda < 1 \longrightarrow$ linear convergence.

$\hookrightarrow$ slowly will converge.

$\lambda = 1$ ignore.

$\lambda > 1 \longrightarrow$ Diverge.

$g(x) = x^2 - x - 3$

$g'(x) = 2x - 1$

$\lambda = |g'(-1)| = |-3| = 3$ (not <1)

$\lambda = |g'(3)| = 5$ (not <1)

$\therefore$ diverge.

$g(x) = \dfrac{x^2+3}{2x-2}$ $\dfrac{dx}{dy} = \dfrac{x^2-2x-3}{2(x-1)^2}$

$\lambda = |g'(-1)| = 0.$

$\lambda = |g'(3)| = 0$

$\therefore x_0 = 0 \longrightarrow -1$

$x_0 = 42 \longrightarrow 3$

Example:

$$f(x) = x^3 - 2x^2 - x + 2.$$

a) state the actual root of $f(x)$

b) construct 3 differe ~~fg~~ $g(x)$ such that $f(x) = 0$.

c) find the converging rate $g(x)$ and will root it will conver to.

a) $f(x) = x^3 - 2x^2 - x + 2 = 0$.

$$x^2(x-2) - 1(x-2) = 0.$$

$$(x^2-1)(x-2) = 0.$$

$x = 1$

-1

2

(b) i) $x^3 - 2x^2 - x + 2 = 0.$

$$x = x^3 - 2x^2 + 2$$

ii) $x^3 - 2x^2 - x + 2 = 0$

$$x(x^2 - 2x - 1) = -2$$

$$x = \frac{-2}{x^2 - 2x - 1}$$

iii) $x^3 - 2x^2 - x + 2 = 0.$

$$2x^2 = x^3 - x + 2$$

$$x = \frac{1}{\sqrt{2}}\sqrt{x^3 - x + 2}.$$

c i) $g(x) = x^3 - 2x^2 + 2$

$g'(x) = 3x^2 - 4x$

$$\lambda = |g'(x_*)| = \begin{cases} 7 \longrightarrow x_* = -1 \\ 1 \longrightarrow x_* = 1 \\ 4 \longrightarrow x_* - 2. \end{cases}$$

all divergent

ii) $g(x) = \dfrac{-2}{x^2 - 2x - 1}$.

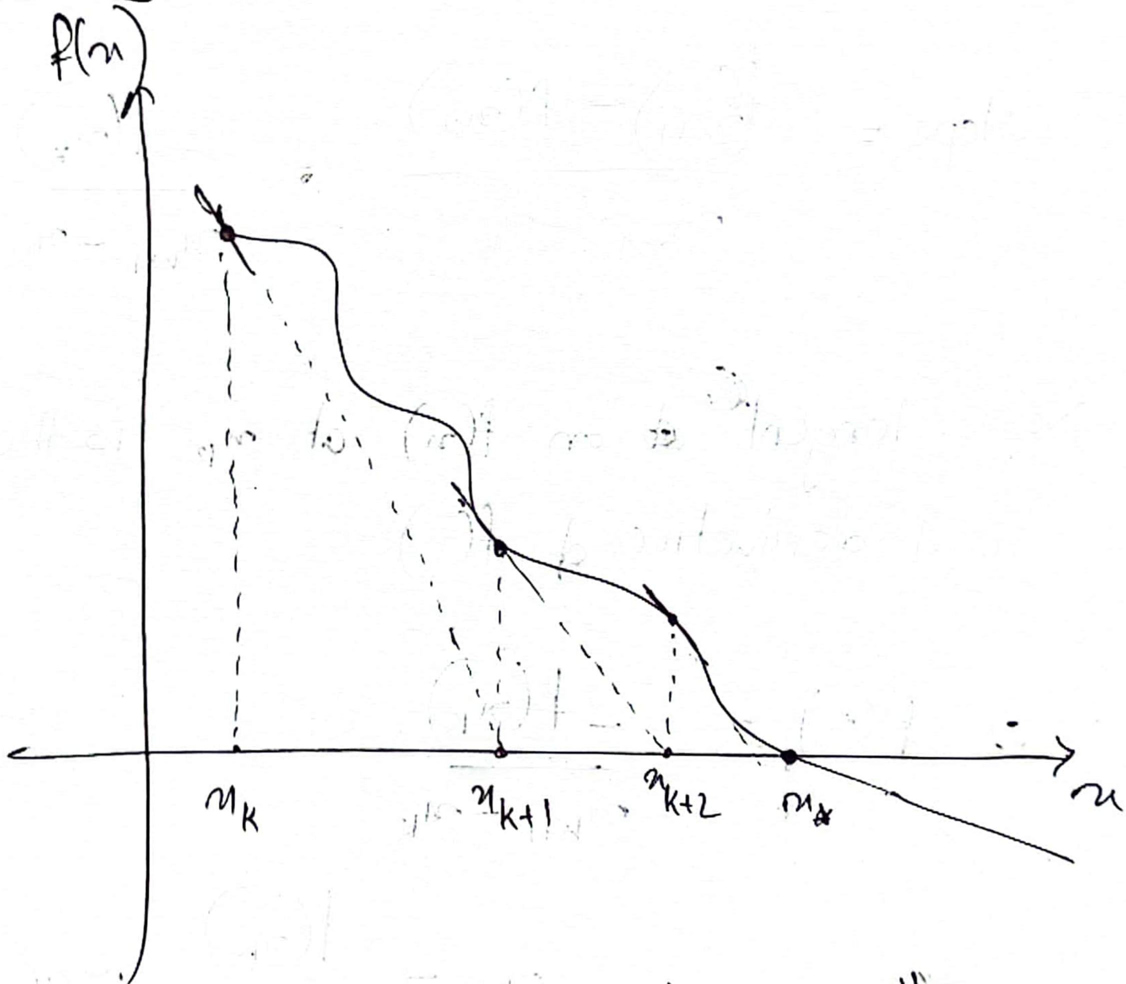
$g'(x) = \dfrac{4(x-1)}{(x^2 - 2x - 1)^2}$

$\lambda = g'(x_*) = \begin{matrix} 2 & x_* = -1 \\ 0 & x_* = 1 \end{matrix}$

$\therefore$ only converge for $x_* = 1$

Newton's method: Also called Newton Raphson method

method of finding $g(x)$ is different
choosing x_0 is also different



$x_* - x_k$ is getting smaller.

$x_* - x_k \approx 0$ as $k \to \infty$

According to the discuss on the previous case, it is clear that x_k and x_{k+1} are both on same tangent line.

$$\text{slope} = \frac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k} = \frac{-f(x_k)}{x_{k+1} - x_k}$$

Now, tangent on $f(x)$ at x_k is the first derivative of $f(x)$

$$\therefore f'(x)_0 = \frac{-f(x_k)}{x_{k+1} - x_k}$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \approx g(x_k)$$

$$g'(x) = \frac{d}{dx}\left(x - \frac{f(x)}{f'(x)}\right)$$

$$= 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{\{f'(x)\}^2}$$

$$= \frac{f(x)\,f''(x)}{(f'(x))^2}$$

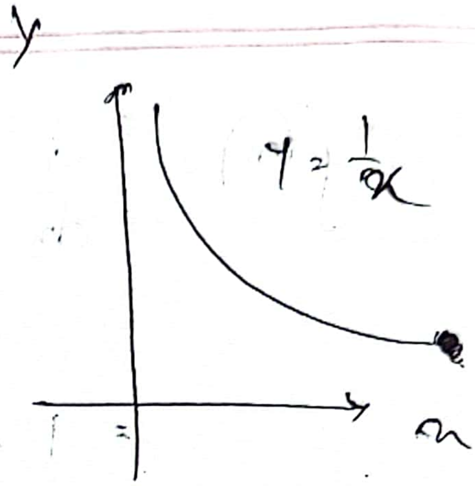
$$\lambda = g'(x_*) = \left.\frac{f(x)\,f''(x)}{(f'(x))^2}\right|_{x = x_*}$$

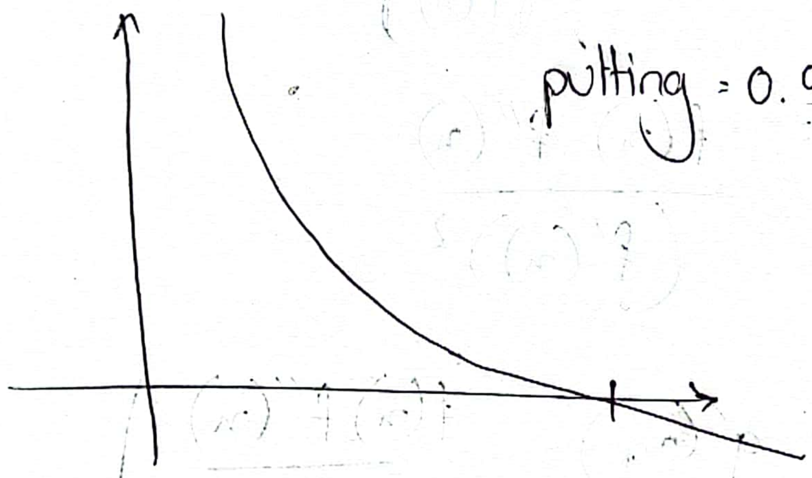
$$= \frac{\cancel{f(x_*)}^{\,0}\, f''(x_*)}{(f'(x_*))^2} = 0.$$

$\therefore$ Newton's method is super-linear.

Example: $f(x) = \frac{1}{x} - a$

$x_* = \frac{1}{a}$





putting $= 0.5$

$x_0 \in [0, 2]$

$x_k - \dfrac{\frac{1}{x_k} - 0.5}{-\frac{1}{x_k^2}}$

$x_k - \dfrac{1 - 0.5x_k}{\cancel{x_k}} \times -\dfrac{x_k^2}{1}$

$x_k + x_k - 0.5x_k^2$

$2x_k - 0.5x_k^2$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$= x_k - \frac{\frac{1}{x_k} - 0.5}{\frac{d}{dx}\left(\frac{1}{x_k} - 0.5\right)}$$

$$\therefore x_{k+1} = 2x_k - 0.5x_k^2$$

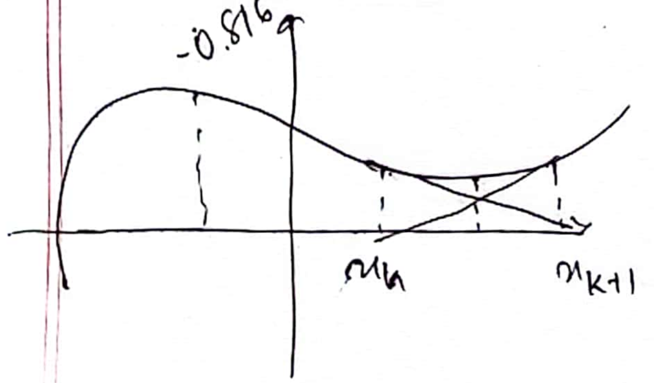
Lets start the iteration at $x_0 = 1$

k	x_k	x_{k+1}
0	1	1.5
1	1.5	1.875
2	1.875	
3	1.9921875	
4	1.999962482	
5		
6		

k	x	
0	1	
1	1.5	
2	1.875	
3	1.9921875	
4	1.999962482	
5	2	
6	2	

When Newton's method do not work:

when there is a turning point



if $f(x) = x^3 - 2x + 2$

finding turning point:

$$f'(x) = 3x^2 - 2 = 0$$

$$x = \pm\sqrt{2/3}$$

So make sure $x_0 < -0.816$.

Aitke Acceleration:

In this method we learn how to accelerate the convergance rate for a linear convergent function.

⇒ for λ = constant $\neq 0$.

$$\cancel{x^* = \frac{x_{k+1} - \lambda x_k}{1-\lambda}}$$

iteration formulat:

$$\hat{x}_{k+2} = x_k - \frac{(x_{k+1} - x_k)^2}{x_{k+2} - 2x_{k+1} + x_k}$$

accelerated value

for every 2 iteration acceleration occurs

$$x_0 \longrightarrow x_1 \longrightarrow x_2 \longrightarrow \hat{x}_2 \longrightarrow x_3 \longrightarrow x_4$$

$$\hat{x}_6 \longleftarrow x_6 \longleftarrow x_5 \longleftarrow \hat{x}_4$$

Example: $f(x) = \frac{1}{x} - 0.5$ [x_* is at 2]

construct $g(x)$ such that $x = 2$ is a fixed point of $g(x)$.

let $g(x) = x + \frac{1}{16} f(x) = x + \frac{1}{16}\left(\frac{1}{x} - 0.5\right)$

valid since $g(2) = 2$.

$$\lambda = g'(x_*) = 1 + \frac{1}{16}\left(-\frac{1}{x^2}\right)$$

$$= 0.984375 < 1$$

linear convergence but very slow.

- lets start with $x_0 = 1.5$.
- we consider upto 7 significant values.

$x_0 = 1.5$

$x_1 = g(x_0) = 1.510417$ $0.489583 \rightarrow (x_1 - x_*)$

$x_2 = g(x_1) = 1.520546$ $0.479454 \rightarrow (x_2 - x_*)$

$x_3 = g(x_2) = 1.530400$

$\vdots$

$x_{818} = g(x_{817}) = 1.999999$ 0.000000

$\hookrightarrow (x_{818} - x_*)$

Aitken acceleration:

$x_0 = 1.5$

$x_1 = g(x_0) = 1.510417$ $\qquad |x_1 - x_*| = 0.489583$

$x_2 = g(x_1) = 1.520546$ $\qquad |x_2 - x_*| = 0.479454$

$$\hat{x}_2 = x_0 - \frac{(x_1 - x_0)^2}{x_2 - 2x_1 + x_0} = 1.877604.$$

$|\hat{x}_2 - x_*| = 0.122396$

$x_3 = g(\hat{x}_2) = 1.879641$ $\qquad |x_3 - x_*| = 0.120359$

$x_4 = g(x_3) = 1.881642$ $\qquad |x_4 - x_*| = 0.007366$

$$\hat{x}_4 = \hat{x}_2 - \frac{(x_3 - \hat{x}_2)}{x_4 - 2x_3 + \hat{x}_2} = 1.992364$$

$|\hat{x}_4 - x_*| = 0.007366$

$\vdots$

$\hat{x}_8 = g(x_7) = 2{,}000000 \qquad |\hat{x}_8 - x_*| \leq 0{,}000000$

Secant method

Also called Quasi Newton Method

$$x_{k+1} = x_k - \frac{f(x_k)\left(x_k - x_{k-1}\right)}{f(x_k) - f(x_{k-1})}$$

Example: $f(x) = x^3 - 0.165x^2 + 3.993 \times 10^{-3}$

$$\left.\begin{aligned} X_0 &= 0.003 \\ X_{-1} &= 0.002 \end{aligned}\right\} \text{ will always be given.}$$

Show 3 iterations.

$$x_1 = x_0 - \frac{f(x_0)\left(x_0 - x_{-1}\right)}{f(x_0) - f(x_{-1})}$$

$$= 0.03 - \frac{f(0.03)\,(0.03 - 0.02)}{f(0.03) - f(0.02)}$$

H.W.

Example: $f(x) = \frac{1}{x} - 0.5 \qquad x_0 = 0.25, \; x_1 = 0.5$

$$x_{k+1} = x_k - \frac{\left(\frac{1}{x_k} - 0.5\right)(x_k - x_{k-1})}{\left(\frac{1}{x_k} - 0.5\right)\left(\frac{1}{x_{k-1}} - 0.5\right)}$$

k	x_k
0	0.25
1	0.5
2	0.6875
3	1.01562
4	1.3540
5	1.68205
6	1.8973
7	1.98367
⋮	
12	2.0000