

Polynomial

$$P_n(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + \dots + a_n x^n$$

a_n Degree

$$P_3(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3$$

Degree = 3

coefficient = 4 $[a_0, a_1, a_2, a_3]$

Notice:

$$\text{Degree} = n$$

$$\text{Coefficient} = n+1$$

Example 1: If we have a polynomial with a degree of 27, how many co-efficient will it have?

$$\text{Degree, } n = 27$$

$$\begin{aligned}\therefore \text{Coefficient} &= n+1 \\ &= 27+1 \\ &= 28\end{aligned}$$

Vector Space

It is a region, where we can add vectors, multiply vectors with scalars.

Add vectors

$$\boxed{1+x+x^2} + \boxed{x^3}$$
$$= 1+x+x^2+x^3$$

multiply with scalars

$$(1+x+x^2) \times 5$$
$$= 5 + 5x + 5x^2$$

Basis

Basis is a set of vectors that spans the space.

$$P_3(x) = a_0 + a_1 x^1 + a_2 x^2 + a_3 x^3$$

Basis = $\{1, x^1, x^2, x^3\}$ [with this we can generate any sort of polynomial]

Dimensional space

The number of elements in a basis.

$$P_3(x) = a_0 + a_1 x^1 + a_2 x^2 + a_3 x^3$$

$$\text{Basis} = \{1, x^1, x^2, x^3\}$$

Dimensional Space = 4

Note: The dimensional space will be one more than the degree of polynomial.

Example: if we have a polynomial with a degree of 58, $P_{58}(x)$ then what will be the dimensional space & no of coefficient.

$$\text{dimensional space} = n+1 = 58+1 = 59$$

$$\text{coefficient} = 58+1 = 59 = \boxed{(n+1)}$$

Functional Space

$f(x)$ can go up to infinity

$$F(x) = 2 + 3x + 10x^2 + 14x^3 + 25x^4 + \dots$$

* There is no end to it

* It can go upto infinity

However, we can create a version of it using $P_n(x)$. Because $P_n(x)$ has a limit.

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$

→ It has a limit

Note: $f(x)$ belongs to the vector space of infinite dimension

$$f(x) \in \mathbb{V}^{\infty} \rightarrow \text{Vector space}$$

$$\text{On the other hand, } P_n(x) \in \mathbb{V}^{(n+1)} \rightarrow \text{dimension}$$

Now, suppose we have

$$f(x) = 2 + 3x + 4x^2 + 5x^3 + 8x^4$$

Error [Because]

But, we have $p_2(x) = a_0 + a_1x + a_2x^2$

we can't
duplicate
this portion

Now, The higher the degree, the lower the error

Because a higher degree polynomial can maintain similarity over a large portion.

Weierstrass Approximation Theorem

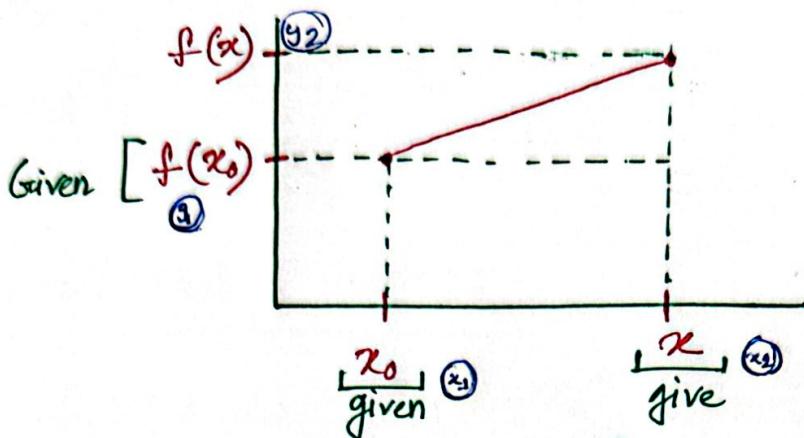
$$|f(x) - p_n(x)|$$

The larger the value of n , the more the error will decrease.

$$p_5(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$$

It creates less error.

Taylor Series



We have to determine $f(x)$. Additionally, $f'(x_0)$ is given also.

$$\begin{aligned}\text{Gradient, } f'(x_0) &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{f(x) - f(x_0)}{x - x_0}\end{aligned}$$

We have to find out $f(x)$

$$\text{So, } f'(x_0) = \frac{f(x) - f(x_0)}{x - x_0}$$

$$f'(x_0)(x - x_0) = f(x) - f(x_0)$$

$$\boxed{f(x) = f'(x_0)(x - x_0) + f(x_0)}$$

Taylor Series for straight line

Note: Actual Taylor series \rightarrow more complex & Infinitely large

Taylor Series (Not finite)

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)(x-x_0)^2}{2!} + \frac{f'''(x_0)(x-x_0)^3}{3!} + \dots$$

Proof of Taylor Series:

$$f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \dots$$

$$f'(x) = 0 + a_1 + 2a_2(x-x_0) + 3a_3(x-x_0)^2 + \dots$$

$$f''(x) = 0 + 0 + 2a_2 + 3 \times 2 a_3(x-x_0)$$

$$f'''(x) = 0 + 0 + 0 + 3 \times 2 a_3$$

Now, Let $x = x_0$

$$\begin{aligned} f(x_0) &= a_0 + a_1(x_0 - x_0) + a_2(x_0 - x_0)^2 + \\ &\quad a_3(x_0 - x_0)^3 + \dots \\ &= a_0 \quad \dots \quad (i) \end{aligned}$$

$$f'(x_0) = a_1 \quad \dots \quad (ii)$$

$$f''(x_0) = 2a_2 \quad \dots \quad (iii)$$

$$f'''(x_0) = 3 \times 2 a_3 \quad \dots \quad (iv)$$

From equation (iii)

$$a_2 = \frac{f''(x_0)}{2!}$$

From (iii)

$$f'''(x_0) = 3 \times 2 a_3$$

$$a_3 = \frac{f'''(x_0)}{3!}$$

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)(x-x_0)^2}{2!} + \frac{f'''(x_0)(x-x_0)^3}{3!} + \dots$$

Example:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$F(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)(x-x_0)^2}{2!} + \frac{f'''(x_0)(x-x_0)^3}{3!} + \frac{f''''(x_0)(x-x_0)^4}{4!} + \frac{f^5(x_0)(x-x_0)^5}{5!} + \dots$$

We will always consider $x_0 = 0$ if nothing is given

$$\begin{aligned} f(x) &= \sin(0) + \cos(0)(x) + \frac{(-\sin(0))x^2}{2!} \\ &\quad + \frac{(-\cos(0))x^3}{3!} + \frac{\sin(0)x^4}{4!} + \frac{\cos(0)x^5}{5!} + \dots \\ &= 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + \dots \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \end{aligned}$$

if $x = 0.1$

1st term $\Rightarrow f(0.1) \approx 0.1$,

$$2\text{nd } " \Rightarrow f(0.1) \approx 0.1 - \frac{(0.1)^3}{3!} = 0.099833\ldots$$

$$3\text{rd } " \Rightarrow f(0.1) \approx 0.1 - \frac{(0.1)^3}{3!} + \frac{(0.1)^5}{5!} \\ = 0.0998334\ldots$$

exact answer $\sin(0.1) = \underbrace{0.0998334}_{6 \text{sf}}$

Vandermonde Matrix

House Size (m ²)	Rent
20	200
30	250
40	275

nodal points/nodes/
Given values

$f(x)$
[Actual
values]

what will be the rent if the house size
is 25?

In this case we have to generate a polynomial.

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots + a_n x^n$$

Hence we have to follow some law

$$f(x) = P_n(x)$$

$$P_n(30) = 250$$

It means in a polynomial if we give
30 it needs to show 250

*	Age	Salary
	25	25000
	30	30,000
	45	70,000

We can generate a polynomial

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$

$$P_n(\text{nodes}) = f(\text{nodes})$$

$$P_n(30) = 30,000$$

The way to setup the degree:

$$P_2(x) = a_0 + a_1(x) + a_2 x^2$$

Here we need to know the coefficient. As we have three unknown coefficient or variable, we need three equations.

$$P_2(25) = a_0 + 25a_1 + 25^2 a_2$$

$$= a_0 + 25a_1 + 625a_2 = 25,000$$

$$P_2(30) = a_0 + 30a_1 + 900a_2 = 30,000$$

$$P_2(45) = a_0 + 45a_1 + 2025a_2 = 70,000$$



The degree will be one less than the number of nodes point given. $(n+1)$ nodes then degree will be n .

We usually use Vandermonde Matrix for degree=1.
We don't even use it for 2, 3,

x_0	$f(x_0)$
$x_1=2$	5
$x_2=3$	6

nodal points = 2
So, degree = 1

$$P_1(x) = a_0 + a_1 x = f(x)$$

$$\begin{aligned} P_1(x_1) &= a_0 + a_1 x_1 = f(x_1) \\ &= a_0 + x_1 a_1 = f(x_1) \end{aligned}$$

$$P_1(x_2) = a_0 + x_2 a_1 = f(x_2)$$

Coefficient Matrix :

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} f(x_1) \\ f(x_2) \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix}^{-1} \begin{bmatrix} f(x_1) \\ f(x_2) \end{bmatrix}$$

Proper Example

	Time	Velocity
nodes/modal point = 2	15	362.8
Degree = 1	20	517.3

$$P_1(15) = a_0 + 15 a_1 = 362.8$$

$$P_1(20) = a_0 + 20 a_1 = 517.3$$

$$\begin{bmatrix} 1 & 15 \\ 1 & 20 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 362.8 \\ 517.3 \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 & 15 \\ 1 & 20 \end{bmatrix}^{-1} \begin{bmatrix} 362.8 \\ 517.3 \end{bmatrix}$$

$$= \frac{1}{20-15} \begin{bmatrix} 20 & -15 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 362.8 \\ 517.3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -3 \\ -1/5 & 1/5 \end{bmatrix} \begin{bmatrix} 362.8 \\ 517.3 \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 4 \times 362.8 - 3 \times 517.35 \\ -1/5 \times 362.8 - 1/5 \times 517.35 \end{bmatrix}$$

$$= \begin{bmatrix} -100.85 \\ 30.91 \end{bmatrix}$$

$$a_0 = -100.85 \quad a_1 = 30.91$$

In the case of higher degree polynomial matrix very large & that's why the time & space complexity will be increase. Also, it become too difficult to process.