Practice Problems : Chapter 3

1. A function is given by $f(x) = x\,e^{-3x} + x^2$. Now answer the following up to five significant figures.
   a. Approximate the derivative of $f(x)$ at $x_0 = 2$ with step size $h = 0.1$ using the central difference method.
   b. Calculate the upper bound of the truncation error of $f(x)$ at $x_0 = 2$ using $h = 0.1$ using the central difference method.
   c. Compute $D^{(1)}_{0.1}$ at $x_0 = 2$ using Richardson extrapolation method and calculate the relative error.
2. During the derivation process, we explored the first-order Richardson extrapolated derivative using $(h \to \frac{h}{2})$,

$$D_h^{(1)} \equiv f'(x_0) - \frac{h^4}{480} + O(h^6)$$

   a. Using $h \to \frac{h}{2}$, derive the expression for $D^{(2)}_{\frac{h}{2}}$, which represents the second-order Richardson extrapolation.
   b. Now, starting from the definition of $D_h^{(1)}$ and applying $h \to \frac{h}{3}$, derive the expression for $D_h^{(1)}$.

3. Consider the function $g(x) = \ln(x)$,
   a. Approximate the derivative of $g(x)$ at $x_0 = 1.5$ with step size $h = 0.2$ using the central difference method up to 6 significant figures.
   b. Approximate the derivative of $g(x)$ at $x_0 = 1.5$ with step size $h = 0.2$ using the forward difference method up to 6 significant figures.
   c. Calculate the truncation error of $g(x)$ at $x_0 = 1.5$ using $h = 0.1, 0.01, 0.001$ in both the central difference and forward difference methods.
   d. Compute Compute $D^{(1)}_{0.2}$ at $x_0 = 1.5$ using Richardson extrapolation method up to 6 significant figures and calculate the truncation error. $D^{(1)}_{0.2}$ at $x_0 = 1.5$ using Richardson extrapolation method up to 6 significant figures and calculate the truncation error.
4. Given the function $f(x) = e^{-2x}$:
   a. Approximate the derivative of $f(x)$ at $x_0 = 0.5$ with step size $h = 0.2$ using the forward difference method up to 6 significant figures.
   b. Approximate the derivative of $f(x)$ at $x_0 = 0.5$ with step size $h = 0.2$ using the central difference method up to 6 significant figures.
   c. Calculate the truncation error of the derivative approximation at $x_0 = 0.5$ using $h = 0.1, 0.01, 0.001$ for both the forward and central difference methods.
   d. Compute $D^{(1)}_{0.2}$ at $x_0 = 0.5$ using Richardson extrapolation up to 6 significant figures and calculate the truncation/relative error.

5. Consider the following data table::

x	1.1	1.2	1.3
f(x)	-0.57941	-0.90730	-1.2807

a. Using the above data, compute f'(1.2) using the central difference method.
b. For the interval [1.1, 1.3], compute the error bound (truncation error) if the above data is generated by the function, $f(x) = x\,cos(x) - x^2 sin(x)$.
c. Also compute the actual error.