$15 + 15 + 20 + 20 + 26 + 14 = 1{:}50$

(Sept. 22)

$=$

Polynomial Interpolation:

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \dots a_n x^n$$

polynomial of n degree.

$a_0, a_1 \dots a_n$ are constant co-efficients

there are $n+1$ co-efficients and $n+1$ terms

Polynomials can be thougt of as vector space

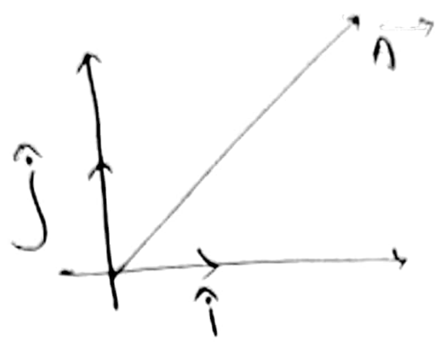
- you can add vectors
- multiply ~~vectors~~ scalars

these can be done in polynomials as well.

basis of a vector space:

basis is a set of vectors that spans the space.

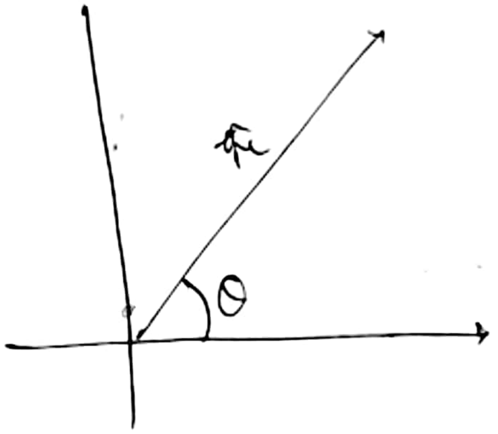
Okm. nushfique



$\vec{A} = A_x \hat{i} + A_y \hat{j}$

any vector can be expressed as a combination $\hat{i}\ \hat{j}$

So $\{\hat{i}, \hat{j}\} \longrightarrow$ basis



now, any vector can be expressed by $\{\hat{r}, \hat{\theta}\}$

Interesting thing here is no. of elements in basis corresponds to the dimention of the space.

$$\vec{A} = A_x \hat{i} + A_y \hat{j}$$
$$= A_r \hat{r} + A_\theta \hat{\theta}$$

3 points to remember: ① basis is a set vectors that span the space.

② They are not unique.

③ ...

$\hat{f} \rightarrow$ fourier transform of

$P_2(x) = a_0 1 + a_1 x + a_2 x^2$

$\{1, x, x^2\} \longrightarrow$ basis

$\rightarrow$ 3 dimensional space.

$\therefore P_n(x) = a_0 . 1 + a_1 x + a_2 x^2 + \ldots + a_n x^n$

$\{1, x, x^2, \ldots x^n\} \longrightarrow$ natural basis

$(n+1)$ dimensional space.

Function spaces:

they are also vector space. but with infinite dimension.

Fourier Series was

$$f(x) = \sum \left(\tilde{f} \sin(\ldots) + f' \cos(\ldots) \right)$$

infinite series (infinite no. of sins an cosines)

$\{\sin \ldots, \cos \ldots\} \rightarrow$ infinite basis natural basis

$f(x) = a_0 1 + a_1 x + a_2 x^2 +$

$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$

Polynomial of degreeen n, belongs a vector space of dimension $n+1$

$P_n(x) \in V^{n+1}$

function, $f(x) \in V^{\infty}$

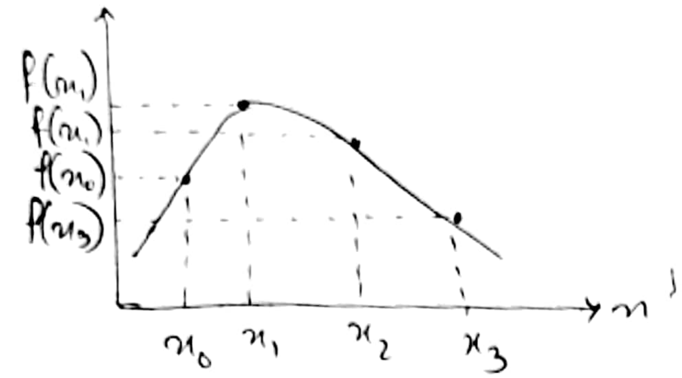
Weierstrass Approximation Theorem:

For a continuos function $f(x)$, on a bounded interval, this is always possible if you take a high enough degree polynomial

For any $f \in C([0,1])$ and any $\epsilon > 0$, there exists a polynomial such that

$$\max_{0 < x \leq 1} |f(x) - P(x)| \leq \epsilon$$

Polynomial Interpolation:

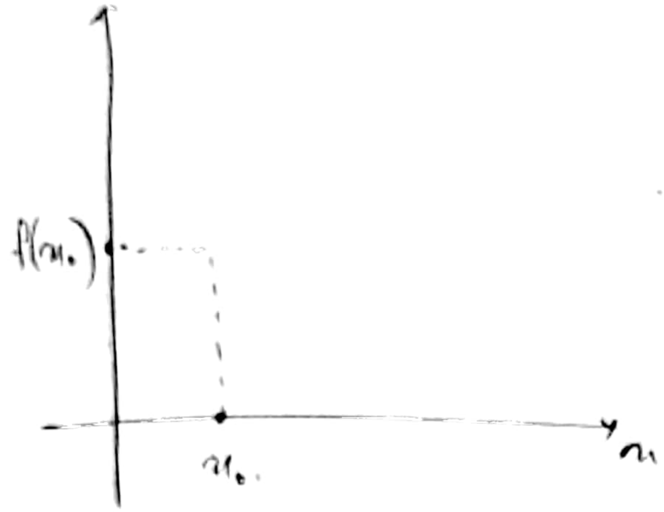


we know $\underbrace{x_0, x_1, x_2, x_3}_{\text{nodes}}$ and corresponding $f(x_0)$, $f(x_1)$, $f(x_2)$ and $f(x_3)$, we don't know the function.

the power of polynomial interpolation,
we can find the polynomial such that it goes through these points.

we can say we predict the value of the function upto some error ϵ.

Taylor series :

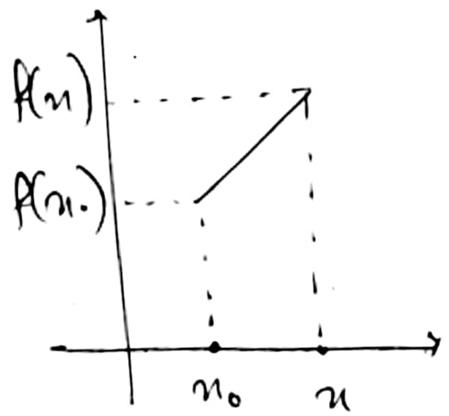


Let say I know $f(x_0)$, $f'(x_0)$, $f''(x_0)$ ...

if I know theses, can we predict the value of the function at some other point x_n.

Answer is yes.

Let say my function is a ~~strai~~ straight line.



$$\frac{f(x) - f(x_0)}{x - x_0} = \tan\theta$$

$$= f'(x_0)$$

or, $f(x) = f'(x_0)(x - x_0) + f(x_0)$

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2$$

$$+ \frac{f'''(x_0)}{3!}(x-x_0)^3 + \underbrace{\ldots\ldots}_{\text{error}}$$

if we don't go to ∞ terms but trunked till any power of x ~~when~~, then we get a polynomial

This is polynomial interpolation, with error we did not compute for.

Proof of Taylor series:

A function is a vector in a infinite vector space.

$$\{1, x, x^2 \ldots\}$$

$$f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \ldots$$

$$f'(x) = a_1 + 2a_2(x-x_0) + 3a_3(x-x_0)^2 + \ldots$$

$$f''(x) = 2a_2 + 3\times 2\, a_3(x-x_0) + \ldots$$

$f'''(x) = 6a_3 + \ldots$

$\therefore f(x_0) = a_0 \quad [\because \text{all } (x - x_0) = 0]$

$\therefore f'(x_0) = a_1 \quad [\because \text{all } (x - x_0) = 0]$

$\therefore f''(x_0) = 2a_2 \quad [\because \text{all } (x - x_0) = 0]$

$\therefore f'''(x_0) = 3!\, a_3$

$\therefore a_0 = f(x_0)$

$a_1 = f'(x_0)$

$a_2 = \dfrac{f''(x_0)}{2!}$

$a_3 = \dfrac{f'''(x_0)}{3!}$

$\vdots$

$a_n = \dfrac{f^n(x_0)}{n!}$

$\therefore f(x) = f(x_0) + f'(x_0)(x - x_0) + \dfrac{f''(x_0)}{2!}(x - x_0)^2 + \dfrac{f'''(x_0)}{3!}(x - x_0)^3 + \ldots$

Now $f(x) = \sin(x)$ and $x = 0$,

then $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

1st term : $f(0.1) \approx 0.1$

2nd term : $f(0.1) \approx x - \frac{x^3}{3!} \approx 0.1 - \frac{(0.1)^3}{3!} \approx 0.099833$

3rd term : $f(0.1) \approx 0.1 - \frac{(0.1)^3}{3!} + \frac{(0.1)^5}{5!} \approx 0.099833$

So, more the term, more accurate we get.

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \dots$$

Lets formalise the ~~#~~ error

ξ (greek letter zhi)

Let f be $n+1$ times differentiable on the interval a to b and let $f^{(n)}$ be continuous on $[a, b]$. If $x, x_0 \in [a, b]$ then there exists $\xi \in (a, b)$ such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{f^{n+1}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

$\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \longrightarrow$ taylor polynomial of degree n.

$\frac{f^{n+1}(\xi)}{(n+1)!}(x-x_0)^{n+1} \longrightarrow$ the lagrange form of the remainder

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$$f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

So, error is $\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$

So, error cannot be calculated, but it can be predicted at a bound as $\xi \in (a, b)$

Example: $f(x) = \sin x = \underbrace{x - \frac{x^3}{3!} + \frac{x^5}{5!}}_{P_6(x)} - \frac{x^7}{7!} + \dots$

↳ as next term of x^7

$\therefore f(x) = P_6(x) + \frac{f^{(7)}(\xi)}{7!}(x - x_0)^7$

$$|f(x) - P_6(x)| = \left| \frac{f^{7}(\xi)}{7!}(x - x_0)^7 \right|$$

Now, let $x = 0.1$, $\therefore f^{(7)}(x) = -\sin x$

$x_0 = 0$

$$|f(0.1) - P_6(0.1)| = \frac{1}{5040}(0.1)^7 \, |-\sin \xi|$$

$$\sin(\xi) \leq 1$$

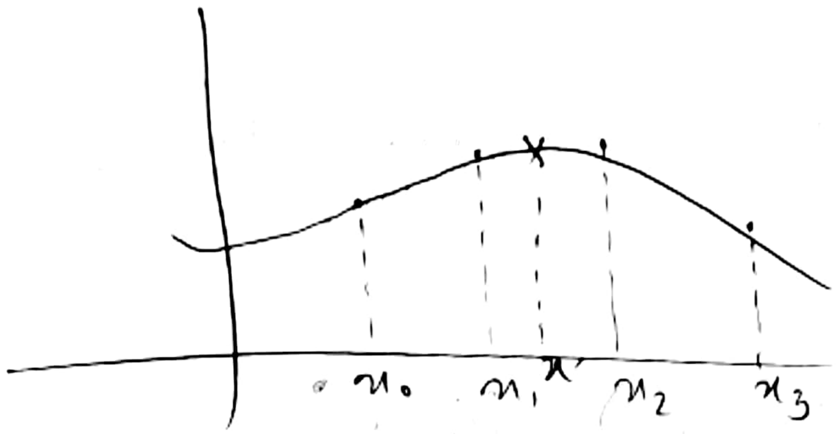
$$\therefore |f(0.1) - P_6(0.1)| \leq \frac{1}{5040}(0.1)^7$$

$$|f(0.1) - P_6(0.1)| \leq 1.984 \times 10^{-11}$$

1.983×10^{-11} exact

~~truct~~ Truncation of an infinite series gives us truncation error.

Polynomial Interpolation:



For interpolation, we will have multiple nodes (x_0, x_1, x_2, x_3) and we want to find the polynomial that go through the points.

and find $f(x')$ for any x'.

Now,

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \dots a_n x^n$$

$$= \sum_{k=0}^{n} a_k x^k$$

So, for finding P_n, we need $a_0, a_1, \dots a_n$

Condition,

at the nodes $P_n(x_0) = f(x_0)$

$P_n(x_1) = f(x_1)$

$P_n(x_2) = f(x_2)$

$\vdots$

$\cancel{P_{n+1}(x_{n+1}) = f(x_{n+1})}$

$P_n(x_n) = f(x_n)$

example:

$n = 1$

we know $x_0 \rightarrow f(x_0)$
$x_1 \rightarrow f(x_1)$ } from this find a_0, a_1

$\therefore P_1(x) = a_0 + a_1 x$

we know,

$P_1(x_0) = a_0 + a_1 x_0 = f(x_0)$ ——— (i)

$P_1(x_1) = a_0 + a_1 x_1 = f(x_1)$ ——— (ii)

we need to get value of 2 values from 2 equations

subtracting we get

$a_1 (x_0 - x_1) = \underline{f(x_0) - f(x_1)}$

$$\therefore a_1 = \underbrace{\frac{f(x_0) - f(x_1)}{x_0 - x_1}}_{\text{all known.}}$$

putting a_0 in eq^n (i) or ii we can get a_1.

$$a_0 + \frac{f(x_0) - f(x_1)}{x_0 - x_1} x_0 = f(x_0)$$

$$a_0 = f(x_0) - x_0 \frac{f(x_0) - f(x_1)}{x_0 - x_1}$$

$$= \frac{f(x_0)(x_0 - x_1) - x_0 f(x_0) + x_0 f(x_1)}{x_0 - x_1}$$

$$= \frac{x_0 f(x_0) - x_1 f(x_0) - x_0 f(x_0) + x_0 f(x_1)}{x_0 - x_1}$$

$$= \frac{x_0 f(x_1) - x_1 f(x_0)}{x_0 - x_1}$$

$\therefore P_1(x) = a_0 + a_1 x$.

But doing this algebra is very hectic.

we can do this using matrix.

$$\begin{pmatrix} 1 & x_0 \\ 1 & x_1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} f(x_0) \\ f(x_1) \end{pmatrix}$$

$$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 1 & x_0 \\ 1 & x_1 \end{pmatrix}^{-1} \begin{pmatrix} f(x_0) \\ f(x_1) \end{pmatrix}$$

$$= \frac{1}{\det} \begin{pmatrix} x_1 & -x_0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} f(x_0) \\ f(x_1) \end{pmatrix}$$

HW $\longrightarrow$ finish the calculation.

Now for,

$x_0, x_1 \dots x_n \longrightarrow (n+1)$ nodes

So $(n+1)$ conditions.

So we need $(n+1)$ coeffs

$\therefore$ polynomial is of degree n.

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \dots a_n x^n$$

conditions:

$$P_n(x_i) = f(x_i)$$

Now, $f(x_0) = a_0 + a_1 x_0^1 + a_2 x_0^2 + \dots + a_n x_0^n$

$f(x_1) = a_0 + a_1 x_1^1 + a_2 x_1^2 + \dots a_n x_1^n$

$f(x_2) = a_0 + a_1 x_2^1 + a_2 x_2^2 + \dots a_n x_2^n$

$\vdots$

$f(x_n) = a_0 + a_1 x_n^1 + a_2 x_n^2 + \dots a_n x_n^n$

~~$a_0 \quad a_1 \quad a_2 \ \dots \ a_n$~~

~~a_1~~

~~a_0~~
~~a_1~~
~~a_2~~
~~a_3~~

$$\underbrace{\begin{pmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ 1 & x_2 & x_2^2 & \dots & x_2^n \\ \vdots & & & & \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{pmatrix}}_{\text{Vandermond Matrix}} \underbrace{\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}}_{\text{Need to find the co-effs}} = \begin{pmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{pmatrix}$$

$A \qquad V \qquad F$

$\therefore \; VA = F$

$A = V^{-1} F$

catch: V has to be invertible and inverse exist iff $\det(V) \neq 0$ and $\det(V) = \prod_{0 \le i < j \le n} (x_i - x_j)$

$$\det(V) = \underbrace{(x_0 - x_1)(x_1 - x_2) \dots (x_{n-1} - x_n)}$$

this is not zero iff $x_i \neq x_j$ if $i \neq j$

i.e. all the nodes has to be distinct.

Let say we have $x_0, x_1, \dots x_n \rightarrow (n+1)$ nodes

$$\begin{array}{cccc} x_0 & x_1 & \dots & x_n \\ \downarrow & \downarrow & & \downarrow \\ f(x_0) & f(x_1) & \dots & f(x_n) \end{array}$$

$\therefore (n+1)$ condition $\Rightarrow n+1$ coeffs

Theorem (Existence / Uniqueness):

Given $(n+1)$ distinct nodes, $x_0, x_1, \dots x_n$

there is a unique polynomial P_n that interpo-lates $f(x)$ at the nodes

~~Proof:~~

$\cancel{p_n \longleftrightarrow q_n}$

$\cancel{r_n := p_n - q_n}$

~~r_n should hav a degree $\leq n$~~

example:

$$f(x) = \cos(x)$$

$$x_0 = 0, \quad x_1 = \pi/2, \quad x_2 = \pi$$

$f(x_0) = 1$ ~~cos~~ $f(x_1) = 0$ $\quad f(x_2) = -1$

3 nodes, so $P_2(x)$

$$P_2(x_0) = a_0 + a_1 x_0^1 + a_2 x_0^2 = a_0 + 0 + 0$$

$$P_2(x_1) = a_0 + a_1 x_1^1 + a_2 x_1^2 = a_0 + a_1 \pi/2 + a_2 (\pi/2)^2$$

$$P_2(x_2) = a_0 + a_1 x_2^1 + a_2 x_2^2 = a_0 + a_1 \pi + a_2 \pi^2$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & \pi/2 & \pi^2/4 \\ 1 & \pi & \pi^2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & \pi/2 & \pi^2/4 \\ 1 & \pi & \pi^2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

CSE 72

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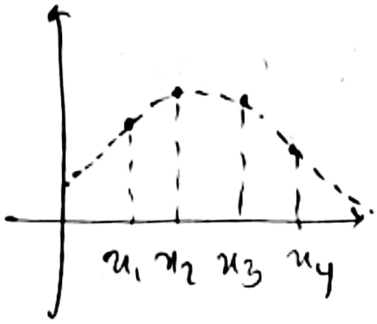
Langrang form



We didn't knew $f(x)$ but knew some points.

So we found some polynomials such that it goes throug the points (----)

(n+1) nodes $\longrightarrow$ P_n

Computation complexity is very high for vandermoude matrix.

So we found lagrange form.

basis:

~~We will chose a choose a basis such that~~

Previously,

	x	$f(x)$
x_0	2	30
x_1	5	45
x_2	7	60

$\therefore P_2(x) = a_0 x + a_1 x^1 + a_2 x^2$

but now $P_2(x) = l_0(x) f(x_0) + l_1(x) f(x_1) + l_2(x) f(x_2)$

l_0, l_1 and l_2 are lagrange basis.

$$l_0(x) = \frac{x - x_1}{x_0 - x_1} \times \frac{x - x_2}{x_0 - x_2}$$

$$l_1(x) = \frac{x - x_0}{x_1 - x_0} \times \frac{x - x_2}{x_1 - x_2}$$

$$l_2(x) = \frac{x - x_0}{x_2 - x_0} \times \frac{x - x_1}{x_2 - x_1}$$

Example.

	Time	Velocity
x_0	15	227.04
x_1	20	362.78
x_2	22.5	517.35

$P_2(17) = ?$

$$P_2(x) = l_0(x) f(x_0) + l_1(x) f(x_1) + l_2(x) f(x_2)$$

$$l_0(x) = \frac{x - x_1}{x_0 - x_1} \times \frac{x - x_2}{x_0 - x_2} = \frac{x-20}{15-20} \times \frac{x - 22.5}{15 - 22.5}$$

$$= \frac{2}{75}(x-20)(x-22.5)$$

$$l_1(x) = \frac{x - x_0}{x_1 - x_0} \times \frac{x - x_2}{x_1 - x_2} = \frac{x - 15}{20 - 15} \times \frac{x - 22.5}{20 - 22.5}$$

$$= -\frac{2}{25}(x-15)(x-22.5)$$

$$l_2(x) = \frac{x - x_0}{x_2 - x_0} \times \frac{x - x_1}{x_2 - x_1} = \frac{x-15}{22.5-15} \times \frac{x-20}{22.5-20}$$

$$= \frac{4}{75}(x-15)(x-20)$$

$$\therefore P_2(x) = \frac{2}{75}(x-20)(x-22.5) \times 227.04$$

$$- \frac{2}{25}(x-15)(x-22.5) \times 362.78$$

$$+ \frac{4}{75}(x-15)(x-20) \times 517.35$$

$$\therefore P_2(17) = \frac{2}{75}(17-20)(17-22.5) \times 227.04$$

$$- \frac{2}{25}(17-15)(17-22.5) \times 362.78$$

$$+ \frac{4}{75}(17-15)(17-20) \times 517.35$$

$f(x) = \cos(x)$

$x_0 = -\pi/4$

$x_1 = 0$

$x_2 = \pi/4$

$P_n(x) = ?$

first lets think what would be n?

$\therefore n = 2$ as three nodes are given.

$$P_2(x) = l_0(x)\, f(x_0) + l_1(x)\, f(x_2) + l_2(x)\, f(x_1)$$

$$l_0(x) = \frac{x - x_1}{x_0 - x_1} \times \frac{x - x_2}{x_0 - x_2} = \frac{x - 0}{-\pi/4} \times \frac{x - \pi/4}{-\pi/4 - \pi/4} = \frac{8x}{\pi^2}$$

$$l_1(x) = \frac{x - x_0}{x_1 - x_0} \times \frac{x - x_2}{x_1 - x_2} = \frac{x + \pi/4}{-\pi/4} \times \frac{x - \pi/4}{0 - \pi/4} = -\frac{16}{\pi^2} \; ($$

$$l_2(x) = \frac{x - x_0}{x_2 - x_0} \times \frac{x - x_1}{x_2 - x_1} = \frac{x + \pi/4}{\pi/4 + \pi/4} \times \frac{x - 0}{\pi/4 - 0} = \frac{8}{\pi^2}\left(x + \pi\right.$$

$f(x_0) = 1/\sqrt{2}$

$f(x_1) = 1$

$f(x_2) = 1/\sqrt{2}$

$$P_2(x) = \frac{8}{\pi^2} x \left(x - \pi/4\right) \frac{1}{\sqrt{2}} - \frac{16}{\pi^2} \left(x + \frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + \frac{8}{\pi^2} \left(x + \frac{\pi}{4}\right) \frac{1}{\sqrt{2}}$$

$\left(x - \frac{\pi}{4}\right)$

$+ \frac{\pi}{4}\Big)\left(x - \pi/4\right)$

$)$

Time	Velocity
0	0
10	227.4
15	362.8
20	517.35
22.5	602.97
30	901.67

= if mentiond $P_2(16)$, then have to consider only 3 nodes.

so we will take 3 nodes close to 16.

. . . . . .

Newton's divide and difference:

$$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)$$
$$(x-x_1)(x-x_2) + \dots \; a_n(x-x_0)(x-x_1)(x-x_2)\cdot(x-x$$

$$a_0 = f[x_0]$$
$$a_1 = f[x_0, x_1]$$
$$a_2 = f[x_0, x_1, x_2]$$
$$\vdots$$
$$a_n = f[x_0, x_1, x_2 \dots x_n]$$

$$\therefore P_4(x) = f[x_0]\,\cancel{(x-x_0)} + f[x_0, x_1](x-x_0) + f[x_0, x_1, x_2]$$
$$(x-x_0)(x-x_1)\,\cancel{(x-x_2)} + f[x_0, x_1, x_2, x_3]$$
$$(x-x_0)(x-x_1)(x-x_2) + f[x_0, x_1, x_2, x_3, x_4]$$
$$(x-x_0)(x-x_1)(x-x_2)(x-x_3)$$

Example:

	x	$f(x)$
x_0	-1	5
x_1	0	1
x_2	1	3
x_3	2	11
x_4	4	20

$$P_3(x) = f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1)$$
$$+ f[x_0,x_1,x_2,x_3](x-x_0)(x-x_1)(x-x_2)$$

$x_0 = -1 \quad f[x_0] = 5$

$f[x_0,x_1] = \dfrac{1-5}{0-(-1)} = -4$

$x_1 = 0 \quad f[x_1] = 1$

$f[x_0,x_1,x_2] = \dfrac{2-(-4)}{1-(-1)} =$

$f[x_1,x_2] = \dfrac{3-1}{1-0} = 2$

$x_2 = 1 \quad f[x_2] = 3$

$f[x_1,x_2,x_3] = \dfrac{8-2}{2-0} =$

$f[x_2,x_3] = \dfrac{11-3}{2-1} = 8$

$x_3 = 2 \quad f[x_3] = 11$

$f[x_2,x_3,x_4] = \dfrac{4.5-8}{3} =$

$f[x_3,x_4] = \dfrac{20-11}{4-2} = 9/2$

$x_4 = 4 \quad f[x_4] = 20$

$\therefore P_3(x) = 5 + (-4)(x - x_0) + 3(x - x_0)(x - x_1) + 0(x - x_0)(x - x_1)(x - $

Now, if x_4 comes, we need to incorporate to the previous polynomial.

If we did the same for lagrange l_0, l_1 ~~and~~ l_2 and l_3 would have to be calculated again.

But in newtons method it is very easy.

$$f[x_0, x_1, x_2, x_3] = \frac{3-3}{2+1} = 0.$$

$$f[x_1, x_2, x_3, x_4] = \frac{-\frac{7}{6} - 3}{4-0} = \frac{-15}{24}$$

$$f[x_0, x_1, x_2, x_3, x_4] = \frac{\frac{-15}{24} - 0}{4+1} = -\frac{5}{24}$$

$$\therefore P_4(x) = 5 + (-4)(x - x_0) + 3(x - x_0)(x - x_1) + 0(x - x_0)(x - x_1)(x - x_2) + -\frac{5}{24}(x - x_0)(x - x_1)(x - x_2)(x - x_3).$$

Time	Velocity
10	227.4
15	362.8
20	517.35
30	901.67

a) $P_n(x)$ using Newton's method

b) $P_n(17) = ?$

c) $P_n(x)$ for $x_3 = 30$, $f(x_3) = 901.67$

a) $P_2(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$

$\cancel{+ f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2)}$

$x_0 = 10 \quad f[x_0] = 227.4$

$f[x_0, x_1] = \dfrac{362.8 - 227.4}{15 - 10} = 27.08$

$x_1 = 15 \quad f[x_1] = 362.8$

$f[x_0, x_1, x_2] = \dfrac{13.91 - 27.08}{20 - 10} = 0.383$

$f[x_1, x_2] = \dfrac{517.35 - 362.8}{20 - 15} = 13.91$

$f[x_0, x_1, x_2, x_3]$

$x_2 = 20 \quad f[x_2] = 517.35$

$f[x_1, x_2, x_3] =$

$f[x_2, x_3] = \dfrac{901.67 - 517.35}{30 - 20} =$

$x_3 = 30 \quad f[x_3] = 901.67$

$\therefore P_2(x) = 227.4 + 27.08$

b) Putting $x = 17$,

c)

d

$\therefore P_2(x) = 227.4 + 27.08(x-x_0) + 0.383(x-x_0)(x-x_1)$.

b) Putting $x = 17$,

1.67

c)

$= 0.383$.

$f[x_0, x_1, x_2, x_3]$.

Cauchy's theorem

$$\left| f(x) - P_n(x) \right| = \left| \frac{f^{n+1}(\xi)}{(n+1)!} (x-x_0)(x-x_1)\dots(x-x_n) \right|$$

Example:

Given, $f(x) = \cos(x)$

x values

$$\left\{ \underset{x_0}{-\pi/4},\ \underset{x_1}{0},\ \underset{x_2}{\pi/4} \right\}$$

Now, calculate the upper bound of error using Cauchy's theorem.

[Interval of ξ may or may not be given]

$$\left| f(x) - P_2(x) \right| = \frac{f^3(\xi)}{3!} (x+\pi/4)(x-0)(x-\pi/4)$$

$$= \left| \frac{\sin(\xi)}{3!} \right| \left| (x+\pi/4)(x-0)(x-\pi/4) \right|$$

normally the highest

$$= \left| \frac{\sin(1)}{3!} \right| \left| (x+\pi/4)(x-0)(x-\pi/4) \right|$$

NB: for any function, if get 0 after differentiating, then point give the maximum value

$$\omega(x) = (x+\pi/4)\, x\, (x-\pi/4)$$

$$= x\left(x^2 - \frac{\pi^2}{16}\right)$$

$$= x^3 - x\frac{\pi^2}{16} \quad \text{—(i)}$$

$$\omega'(x) = 3x^2 - \frac{\pi^2}{16} = 0.$$

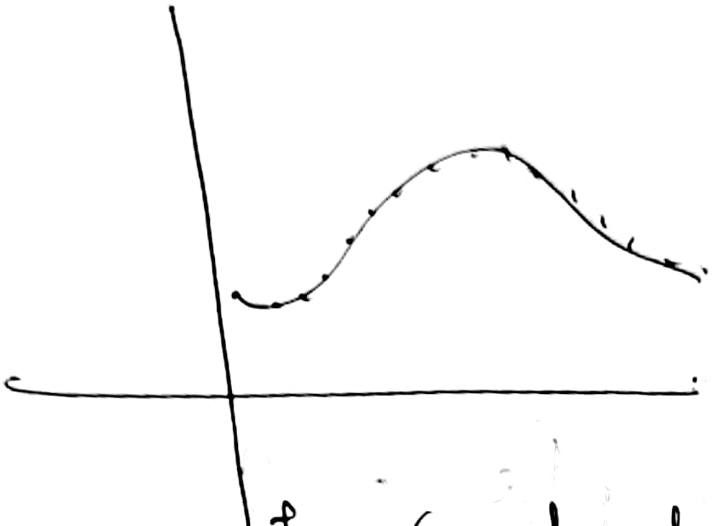
$$\therefore x = \sqrt{\frac{\pi^2}{3\times 16}}$$

$$= \pm \pi/4\sqrt{3}$$

Now, putting $x = \pm \pi/4\sqrt{3}$ in (i)

x	$\lvert\omega(x)\rvert$
$-\pi/4\sqrt{3}$	0.186
$+\pi/4\sqrt{3}$	0.186
	0.383
1	0.383

Chebychev Nodes



if n(no. of nodes) $\longrightarrow \alpha$,

then error should become 0.

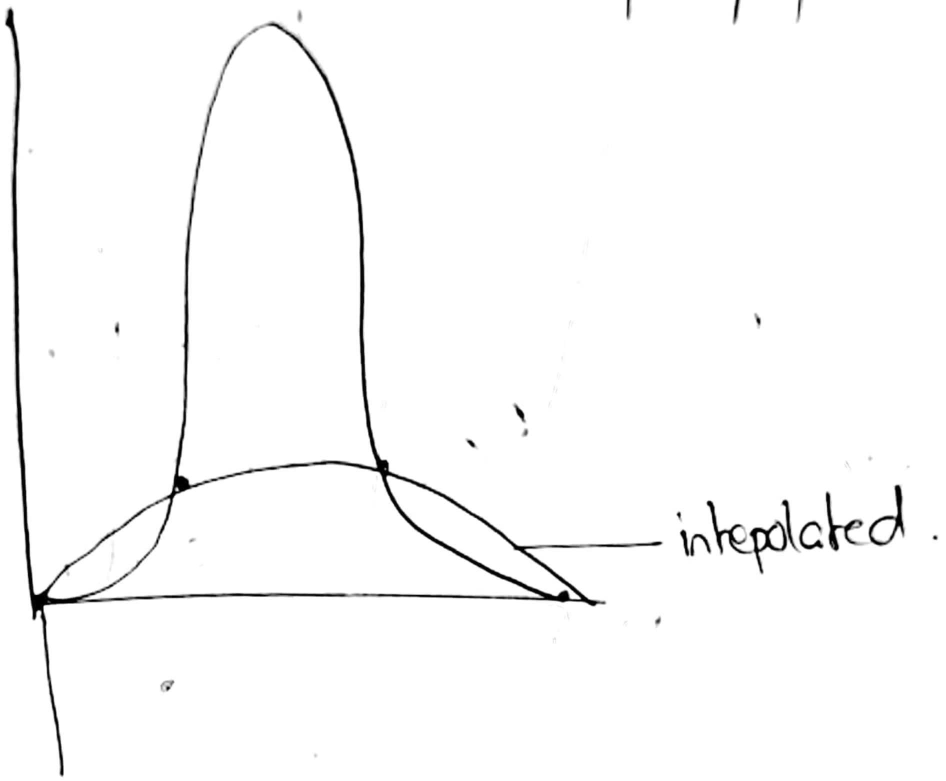
but for some function this is not true.

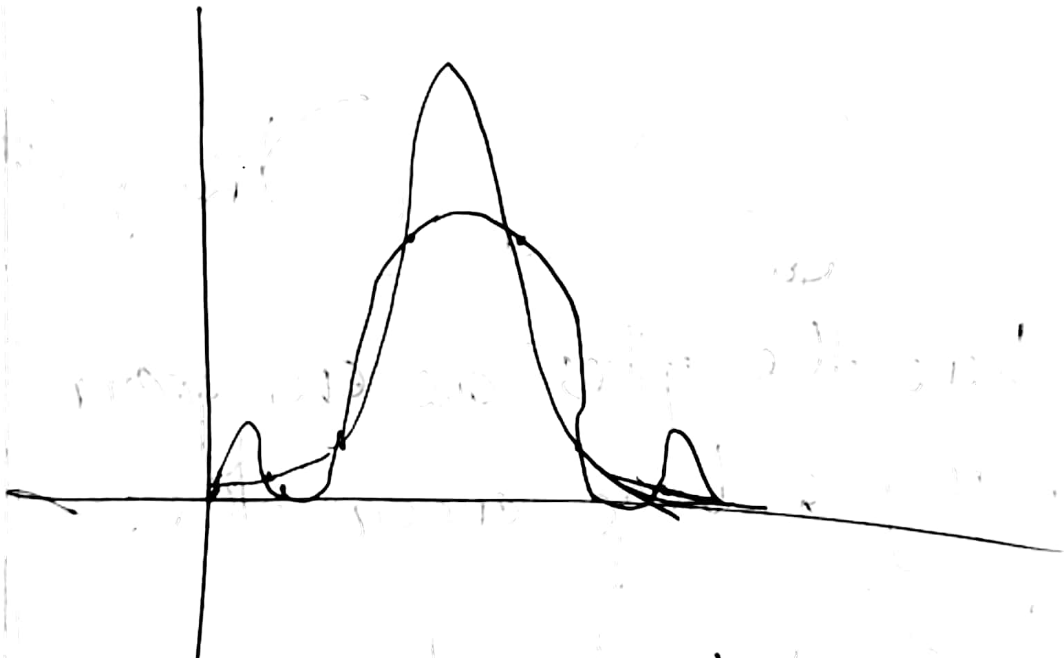
Runge 1901

$$f(x) = \frac{1}{1+25x^2} \text{ on } [-1, 1].$$

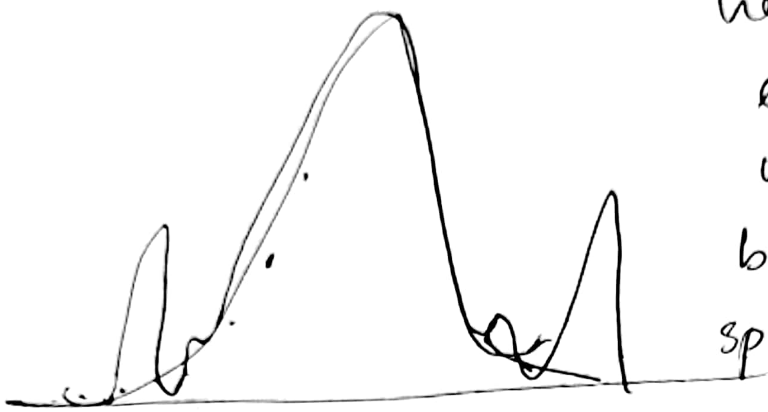
n=3

equally spaced nodes.



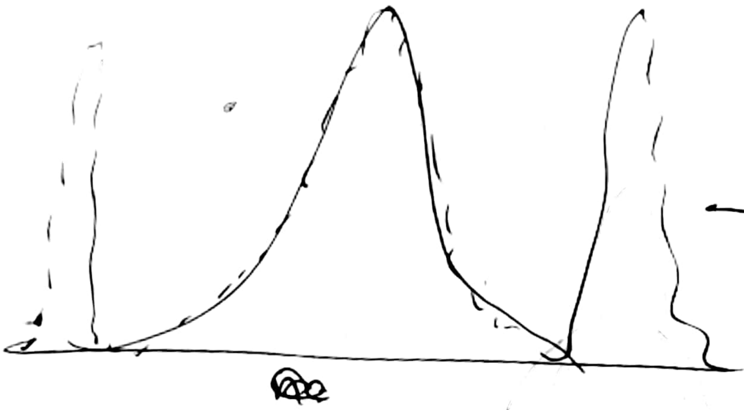


n-11



here the error is very close but ~~bot~~ spikes at the end.

n=15



here the spikes are even greater creating a huge error. This is called runge phenomena

For proof look at paper by runge.
beyond scope of course

if $n \to \infty$, error $\to \infty$

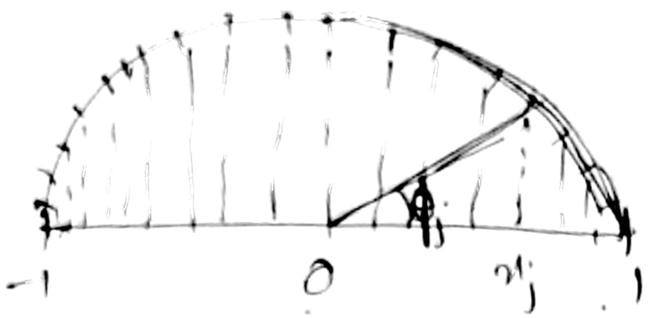
$$\omega(x) = (x-x_0)(x-x_1)\dots(x-x_n)$$

$\hookrightarrow$ we saw this in cauchy's theorem, that this was part of the error.

now, for $\omega_n \approx n!$ (for equally spaced nodes)

So, $\omega(x)$ diverges when there is equally spaced node.

So, one way to counter this is taking unequal spaced nodes. Now as errors are at the ~~edg~~ corners we take more nodes at the corners



so we take a semi circular on the number line with equally angled division. project them on the number line.

$$\phi_j = \frac{(2j+1)\pi}{2(n+1)} \qquad j: 0 \ldots n.$$

$$x_j : \text{radius} \times \cos[\phi_j]$$

$$= \cos\left[\frac{(2j+1)\pi}{2(n+1)}\right] \qquad j: 0 \ldots n$$

$$f(x) = \frac{1}{1+25x^2}, \quad n = 3$$

$$x_0 = \cos\frac{(2\times 0+1)}{2\times 4}\pi \Rightarrow x_0 = \cos \pi/8$$

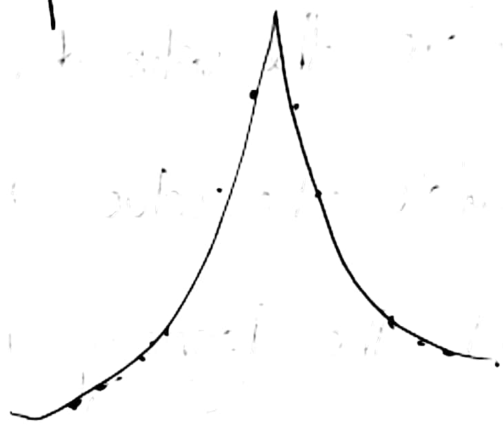
$$x_1 = \cos\frac{3\pi}{8}$$

equal

not equal

$$x_2 = \cos\frac{5\pi}{8}$$

$$x_3 = \cos\frac{7\pi}{8}$$



Given ~~the~~ Runge function,

$$f(x) = \frac{3}{4+9x^2} \quad [-4,4]$$

$n=3$

a) Calculate the ~~value of~~ equal angled points of ϕ's

b) Calculate the value of Chebyshev nodes (x value

c) Find the lagrange basis, $l_2(x)$

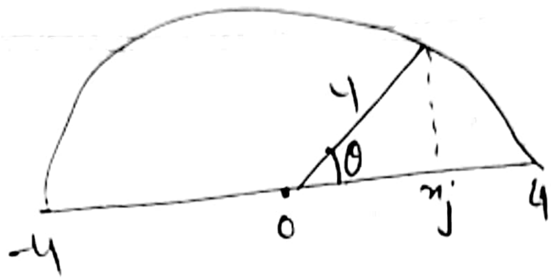
$$\Rightarrow \quad \phi_j = \frac{(2j+1)\pi}{2(n+1)} \qquad j = 0 \ldots 3$$

$$\phi_0 = \frac{\pi}{8}$$

$$\phi_1 = \frac{3\pi}{8}$$

$$\phi_2 = \frac{5\pi}{8}$$

$$\phi_3 = \frac{7\pi}{8}$$

$\Rightarrow$ 

$$\cos \theta_j = \frac{x_j}{4}$$

$$x_j = 4\cos \theta_j$$

$$x_0 = 4 \times \cos \frac{\pi}{8}$$

$$x_1 = 4 \times \cos \frac{3\pi}{8}$$

$$x_2 = 4 \times \cos \frac{5\pi}{8}$$

$$x_3 = 4 \times \cos \frac{7\pi}{8}$$

$$\Rightarrow \quad l_2(x) = \frac{x - x_0}{x_2 - x_0} \times \frac{x - x_1}{x_2 - x_1} \times \frac{x - x_3}{x_2 - x_3}$$

$$= \frac{x - 4\cos\frac{\pi}{8}}{4\cos\frac{5\pi}{8} - 4\cos\frac{\pi}{8}} \times \frac{x - 4\cos\frac{3\pi}{8}}{4\cos\frac{5\pi}{8} - 4\cos\frac{3\pi}{8})}$$

x	$f(x)$

$$\times \frac{x - 4\cos\frac{7\pi}{8}}{4\cos\frac{5\pi}{8} - 4\cos\frac{7\pi}{8}}$$

Hermite Interpolation:

so far we used to see,

x	$f(x)$	$f'(x)$
3	15	40
5	27	20
9	35	37

So the degree used to be 2.

For hermite interpolation the first derivative will also be given.

So we have additional 3 conditions, so now we will have degree 5. $(3+3=6$, so $n-1=5)$

P_{2n+1}
$\downarrow$
previous degree

$$
\begin{array}{lll}
x & f(x) & f'(x) \\
\hline
x_0 & \longrightarrow f(x_0) \longrightarrow & f'(x_0) \\
x_1 & \longrightarrow f(x_1) \longrightarrow & f'(x_1) \\
x_2 & \longrightarrow f(x_2) \longrightarrow & f'(x_2) \\
x_3 & \longrightarrow f(x_3) \longrightarrow & f'(x_3)
\end{array}
$$

$\therefore$ here degree $= 7$.

advantage with limited knowledge we can calculate ~~higher~~ better $P(x)$.

let, $f(x) = \sin(x)$ $\quad f'(x) = \cos(x)$

Now

	x	$f(x)$	$f'(x)$
x_0	0	0	1
x_1	$\pi/2$	1	0

$$P_3(x) = h_0(x) f(x_0) + h_1(x) f(x_1) + \hat{h}_0(x) f'(x_0) + \hat{h}_1(x) f'(x_1)$$

$\cancel{P_3(x)}$,

$$h_k(x) = \left(1-2(x-x_k)\, l_k'(x_k)\right)\left(l_k(x)\right)^2$$

$$\hat{h}_k(x) = (x-x_k)\left(l_k(x)\right)^2$$

$$P_3(x) = 0 + h_1(x) f(x_1) + \hat{h}_0(x_0)\, f'(x_0) + 0$$

$\cancel{h_1(x)} =$

$$l_1(x) = \frac{x-x_0}{x_1-x_0} = \frac{x-0}{\pi/2-0} = \frac{2x}{\pi}$$

$$l_1'(x) = \frac{2}{\pi}$$

$$\therefore h_1(x) = \left\{1-2\left(x-\pi/2\right)\times\frac{2}{\pi}\right\}\left(\frac{2x}{\pi}\right)^2$$

$$\hat{h}_0(x) = (x-x_0)\left(l_0(x)\right)^2$$

$$= (x-0)\left(\frac{x-x_1}{x_0-x_1}\right)^2$$

$$= x\left(\frac{x-\frac{\pi}{2}}{-\pi/2}\right)^2 = x\left(\frac{2x}{2}-\frac{\pi}{2} \times -\frac{2}{\pi}\right)^2$$

$$= x\left(-\frac{(2x-\pi)}{\pi}\right)^2$$

$$= x\left(1-\frac{2}{\pi}x\right)^2$$

$$P_3(x) = \left\{1-\frac{4}{\pi}\left(x-\frac{\pi}{2}\right)\right\}\left(\frac{2}{\pi}x\right)^2 \times 1$$

$$+\ x\left(1-\frac{2}{\pi}x\right)^2 \times 1$$

Example:

	x	$f(x)$	$f'(x)$
x_0	-1	1	2
x_1	0	0	2
x_2	1	1	0

degree = 5

$$P_5(x) = h_0(x)f(x_0) + h_1(x)\cancel{f(x_1)}^{\to 0} + h_2(x)f(x_2)$$

$$+\ \hat{h}_0(x)f'(x_0) + \hat{h}_1(x)f'(x_1) + \hat{h}(x)\cancel{f'(x_2)}$$

$$= h_0(x)f(x_0) + h_2(x)f(x_2)$$

$$+\ \hat{h}_0(x)f'(x_0) + \hat{h}_1(x)f'(x_1)$$

$$l_0(x) = \frac{x - x_1}{x_0 - x_1} \times \frac{x - x_2}{x_0 - x_2}$$

$$h_k(x) = \left\{1 - 2(x - x_k)\, l_k'(x_k)\right\} \left(l_k(x)\right)^2$$

$$\hat{h}_k(x) = (x - x_k)\left(l_k(x)\right)^2$$

$$h_0(x) = \left\{1 - 2(x - x_0)\, l_0'(x_0)\right\} \left(l_0(x)\right)^2$$

$$l_0(x) = \frac{x - 0}{-1 - 0} \times \frac{x - 1}{-1 - 1} = -x \times \frac{x-1}{-2} = \frac{x(x-1)}{2}$$

$$l_0'(x) = \frac{2x}{2} - \frac{1}{2} \Rightarrow l_0'(x_0) = -1 - \frac{1}{2}$$

$$\cancel{h_k(x) = \left\{1 - 2\,(x - (-1)(-1))\right\} \left(\frac{x^2}{2}\right)^2}$$

$$h_0(x) = \left\{1 - 2\left(x - (-1)\right)\left(-1 - \frac{1}{2}\right)\right\} \left(\frac{x^2 - x}{2}\right)^2$$

$$= \left\{1 - 2\,(x + 1) \times \left(-\frac{3}{2}\right)\right\} \left(\frac{x^2 - x}{2}\right)^2$$

$$= (1 + 3x + 3) \left(\frac{x^2 - x}{2}\right)^2$$

$$h_2(x) = \left\{1 - 2(x - x_2)\, l_2'(x_2)\right\} \left(l_2(x)\right)^2$$

$$l_2(x) = \frac{x - x_0}{x_2 - x_0} \times \frac{x - x_1}{x_2 - x_1}$$

$$= \frac{x+1}{1+1} \times \frac{x-0}{1-0} = \frac{x+1}{2} \times x = \frac{x^2+x}{2}$$

$$l_2'(x) = x + \frac{1}{2} \Rightarrow l_2'(x_2) = 1 + \frac{1}{2}$$

$$h_2(x) = \left\{1 - 2(x-1)\left(1 + \frac{1}{2}\right)\right\} \left(\frac{x^2+x}{2}\right)^2$$

$$= (1 - 3x + 3) \left(\frac{x^2+x}{2}\right)^2$$

$$\hat{h}_0(x) = (x - x_0)\left(l_0(x)\right)^2$$

$$= (x+1)\left\{\frac{x(x-1)}{2}\right\}^2$$

$$\hat{h}_1 = (x - x_1)\left(l_1(x)\right)^2$$

$$= x\left(\frac{x-x_0}{x_1-x_0} \times \frac{x-x_2}{x_1-x_2}\right)^2$$

$$= x\left(\frac{x+1}{1} \times \frac{x-1}{-1}\right)^2$$

$$= x\left(\frac{x^2-1}{-1}\right)^2$$

$$= x\left(1-x^2\right)^2$$

$$\cancel{P_5(x) = \{ 1 - 2\,(x - x_1)}$$

$$\cancel{P_5(x) = (4+3x)\left(\frac{x^2-x}{2}\right)^2 \times 1 + (4-3x)\left(\frac{x^2+x}{2}\right)^2}$$