Lect-05: Analyticity & harmonic conjugate :

Problem-01 : Show that, $\sin z$ is analytic.

Soln : Q → how to show a function is analytic ?

$f(z) = u + iv \; (u(x,y) + iv(x,y))$

u, v satisfy C-R eqns on R
& partial derivatives occuring in C-R eqns are continuous in R $\Rightarrow f(z)$ is analytic in R

Step-01 : convert $\sin z$ to cartesian representation :

$\sin z = \sin(x+iy) = \sin x \cos(iy) + \sin(iy)\cos x$

lemma : $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$, $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

$\cos(iy) = \dfrac{e^{iiy} + e^{-iiy}}{2}$, $\sin(iy) = \dfrac{e^{iiy} - e^{-iiy}}{2i}$

$= \dfrac{e^{-y} + e^{y}}{2}$ $\qquad = \dfrac{e^{-y} - e^{y}}{2i}$

$= \cosh(y)$ $\qquad = -\dfrac{1}{i}\sinh(y)$

$= i\sinh(y)$

$\Rightarrow f(z) = \sin z = \sin x \cosh y + i \cos x \sinh y$

Step-02 : $u = \sin x \cosh y$
$v = \cos x \sinh y$

$u_x = \cosh y \cos x$ $\qquad u_y = \sin x \sinh y$

$v_y = \cos x \cosh y$ $\qquad v_x = -\sinh y \sin x$

$u_x = v_y$ $\qquad u_y = -v_x$

Lect-05: Analyticity & harmonic conjugate:

Problem-01: Show that, $\sin z$ is analytic.

Soln: Q → how to show a function is analytic?

$f(z) = u + iv \; (u(x,y) + iv(x,y))$

u, v satisfy C-R eqns on R
& partial derivatives occuring in C-R eqns are continuous in R $\Rightarrow f(z)$ is analytic in R.

Step-01: convert $\sin z$ to cartesian representation:

$\sin z = \sin(x+iy) = \sin x \cos(iy) + \sin(iy)\cos x$

lemma: $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$, $\sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i}$

$$\cos(iy) = \frac{e^{iiy}+e^{-iiy}}{2} = \frac{e^{-y}+e^{y}}{2} = \cosh(y)$$

$$\sin(iy) = \frac{e^{iiy}-e^{-iiy}}{2i} = \frac{e^{-y}-e^{y}}{2i} = -\frac{1}{i}\sinh(y) = i\sinh(y)$$

$\Rightarrow f(z) = \sin z = \sin x \cosh y + i \cos x \sinh y$

Step-02: $u = \sin x \cosh y$
$v = \cos x \sinh y$

$u_x = \cosh y \cos x$ $\quad u_y = \sin x \sinh y$

$v_y = \cos x \cosh y$ $\quad v_x = -\sinh y \sin x$

$u_x = v_y$ $\quad u_y = -v_x$

$\Rightarrow$ C-R equations hold $\forall (x,y) \in \mathbb{R}^2$

Also, u_x, u_y, v_x, v_y are all continuous functions on $\mathbb{R}^2$ (being product of continuous functions)

$\therefore$ $f(z)$ is analytic on $\mathbb{R}^2 / \mathbb{C}$

Problems : Show that following functions are analytic on $\mathbb{C}$

(i) $\cos z$ (ii) $\sinh z$ (iii) $\cosh z$

Hint : (i) $\cos(x+iy) = \cos x \cos(iy) - \sin x \sin(iy)$

(ii) $\sinh z = \sinh(x+iy)$

$$= \frac{e^{x+iy} - e^{-x-iy}}{2}$$

$$= \frac{e^{x}(\cos y + i\sin y) - e^{-x}(\cos y - i\sin y)}{2}$$

$$= \cos y\left(\frac{e^{x}-e^{-x}}{2}\right) + i\sin y\left(\frac{e^{x}+e^{-x}}{2}\right)$$

$$= \sinh x \cos y + i\cosh x \sin y$$

Problem-02 : $\lim_{z \to m\pi i} (z - m\pi i)\frac{e^z}{\sin z}$, $m \neq 0$.

Sol^n : $\sin(m\pi) = 0$

$\sin(m\pi i) = \sin(0)\cosh(m\pi) + i\cos(0)\sinh(m\pi)$

$= i\sinh(m\pi) \neq 0$

$$\lim_{z \to m\pi i} (z - m\pi i)\frac{e^z}{\sin z} = 0 \quad \text{(Ans!)}$$

Problem-02 : $u(x,y) = e^{-x}(x\sin y - y\cos y)$.

(i) prove that u is harmonic

(ii) find v such that $f(z) = u + iv$ is analytic

(iii) Express $f(z)$ in terms of z.

Sol^n : (i) $u = e^{-x}(x\sin y - y\cos y)$

$$u_x = e^{-x}(x\sin y - y\cos y)' + (e^{-x})'(x\sin y - y\cos y)$$

$$= e^{-x}\{(x\sin y)' - (y\cos y)'\} - e^{-x}(x\sin y - y\cos y)$$

$$= e^{-x}\sin y - e^{-x}(x\sin y - y\cos y)$$

$$u_x = e^{-x}(\sin y - x\sin y + y\cos y)$$

$$u_{xx} = e^{-x}(\sin y - x\sin y + y\cos y)' - e^{-x}(\sin y - x\sin y + y\cos y)$$

$$\Rightarrow u_{xx} = -e^{-x}\sin y - e^{-x}(\sin y - x\sin y + y\cos y)$$

$$\Rightarrow u_{xx} = e^{-x}(-2\sin y + x\sin y \mp y\cos y)$$

Secondly, $u_y = e^{-x}(x\sin y - y\cos y)' + (e^{-x})'(x\sin y - y\cos y)$

$$= e^{-x}\{(x\sin y)' - (y\cos y)'\}$$

$$= e^{-x}[x\cos y - \{y(-\sin y) + \cos y\}]$$

$$u_y = e^{-x}(x\cos y + y\sin y - \cos y)$$

$$u_{yy} = e^{-x}(x\cos y + y\sin y - \cos y)'$$

$$= e^{-x}(-x\sin y + \sin y + y\cos y + \sin y)$$

$$u_{yy} = e^{x}(2\sin y - x\sin y + y\cos y)$$

$\therefore\ u_{xx} + u_{yy} = 0$

$\Rightarrow$ u is harmonic.

(ii) $f(z) = u + iv$ is analytic

$\Rightarrow$ u, v satisfies Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{—(i)}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{—(ii)}$$

from (i),

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

$$\Rightarrow \frac{\partial v}{\partial y} = e^{-x}\left(\sin y - x\sin y + y\cos y\right)$$

$$\Rightarrow v(x,y) = -e^{-x}\cos y - e^{-x}x(-\cos y) + e^{-x}\int y\cos y\, dy + F(x)$$

$$\Rightarrow v(x,y) = e^{-x}\left(-\cos y + x\cos y\right) + e^{-x}\left\{y(\sin y) - \int 1(\sin y)\,dy\right\} + F(x)$$

$$= e^{-x}\cos y\,(x-1) + e^{-x}\left(y\sin y + \cos y\right) + F(x)$$

$$\Rightarrow v(x,y) = xe^{-x}\cos y + e^{-x}y\sin y + F(x)$$

$$\Rightarrow v(x,y) = e^{-x}\left(x\cos y + y\sin y\right) + F(x)$$

$$\Rightarrow \frac{\partial v}{\partial x} = e^{-x}\left(x\cos y + y\sin y\right)' + (e^{-x})'\left(x\cos y + y\sin y\right) + F'(x)$$

$$\Rightarrow -\frac{\partial u}{\partial y} = e^{-x}\cos y - e^{-x}\left(x\cos y + y\sin y\right) + F'(x)$$

$\Rightarrow -e^{-x}\left(x\cos y + y\sin y - \cos y\right) = e^{-x}\left(\cos y - x\cos y - y\sin y\right)$
$+ F'(x)$

$\Rightarrow \quad F'(x) = 0$

$\Rightarrow \quad F(x) = c$ where c is any constant

$\therefore v(x,y) = e^{-x}\left(x\cos y + y\sin y\right) + c$

$\Rightarrow v(x,y) = e^{-x}\left(x\cos y + y\sin y\right) \longrightarrow$ (one particular such v)

Soln to (iii):

$f(z) = u + iv$

$= e^{-x}\left(x\sin y - y\cos y\right) + ie^{-x}\left(x\cos y + y\sin y\right)$

$= e^{-x}\left\{x\left(\frac{e^{iy}-e^{-iy}}{2i}\right) - y\left(\frac{e^{iy}+e^{-iy}}{2}\right)\right\}$

$+ ie^{-x}\left\{x\left(\frac{e^{iy}+e^{-iy}}{2}\right) + y\left(\frac{e^{iy}-e^{-iy}}{2i}\right)\right\}$

$= e^{-x}e^{iy}\left(\frac{x}{2i} - \frac{y}{2} + i\frac{x}{2} + \frac{y}{2}\right)$

$e^{-x}e^{-iy}\left(-\frac{x}{2i} - \frac{y}{2} + i\frac{x}{2} - \frac{y}{2}\right)$

$= e^{-(x-iy)}.0 + e^{-x-iy}\left(\frac{x}{2}i + \frac{x}{2}i - y/2 - y/2\right)$

$= e^{-x-iy}\left(xi - y\right)$

$= e^{-(x+iy)}\left(xi + i^2 y\right)$

$= ie^{-(x+iy)}\left(x + iy\right)$

$= ie^{-z}.z$

$\therefore f(z) = ize^{-z}$ (Ans)

Lect-03 : Reference page :

(i) $\frac{d}{dx}(\sin x) = \cos x$, $\frac{d}{dx}(\cos x) = -\sin x$

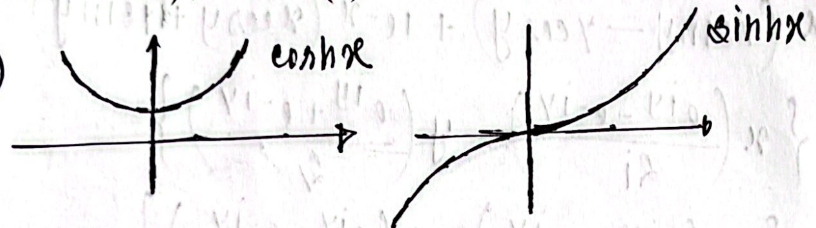
(ii) $\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x$

$\frac{d}{dx}(\cosh x) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{1}{2}(e^x - e^{-x}) = \sinh x$

(iii) Sum, Substraction or product of continuous functions is continuous

$\sinh(x)$, $\cosh(x) \longrightarrow$ continuous

(iv)



(v) $\sin(a+b) = \sin a \cos b + \sin b \cos a$

$\cos(a+b) = \cos a \cos b - \sin a \sin b$

(vi) $\frac{d}{dx}\left(f(x) \pm g(x)\right) = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$

or, $(f \pm g)' = f' \pm g'$ (Additivity of differentiation)

(vii) $\int uv\,dx = u\int v\,dx - \int\left(u'\int v\,dx\right)dx$

<u>choosing u, v</u> : LIATE formula