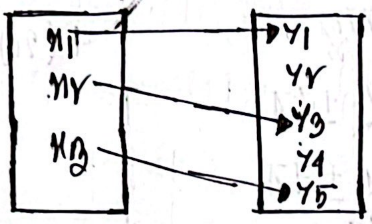
Lect-03: function of Complex Variables:

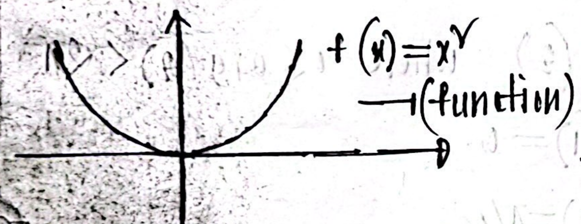
Function : Rule $\longrightarrow$ Definition of function

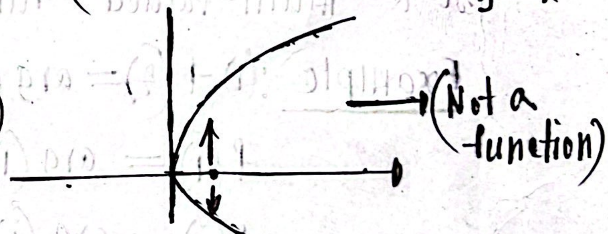


Domain co-domain

Defn (function) : A rule that assigns each element of a domain to exactly one element of the co-domain.

Example : $f: \mathbb{R} \longrightarrow \mathbb{R}$ by $f(x) = x^2$





Real valued function : A function $f: A \rightarrow \mathbb{R}$ where $A \subseteq \mathbb{R}$

Notation : $f(x) = y \longrightarrow$ Dependent variable or $f(x) = \square$
$x \longrightarrow$ Independent variable.

Complex valued function : A function of the form $f: S \rightarrow \mathbb{C}$ where $S \subseteq \mathbb{C}$

Notation : $f(z) = \square$ or $w = f(z)$
$z \rightarrow$ Independent variable
$w \longrightarrow$ Dependent variable.

Example : $f(z) = z^2, \ln z, e^z$

Single valued & Multiple valued function :

$f(z) = z^2$	$f(z) = Arg(z)$	$f(z) = z^{1/3}$
$f(i) = -1$	$f(i) = \pi/2$	$f(z_0) \to w_1, w_2, w_3$
$f(1) = 1$	$= \pi/2 \pm 2\pi$	
↓	$= \pi/2 \pm 4\pi$	
(Single-valued function)		

← Multiple Valued function →

Branch (of multiple valued function) : A choice of range st a multi-valued function becomes single valued.

Example : (i) $f(z) = \arg(z)$ where $0 \leq \arg(z) < 2\pi$

$f(1) = \arg(1) = 0$

$f(i) = \arg(i) = \pi/2$

$f(-i) = \arg(-i) = 3\pi/2$

(ii) $f(z) = \arg(z)$ where $0 < \arg(z) \leq 2\pi$ → $f(1) = 2\pi$

(iii) $f(z) = \arg(z)$ where $-\pi < \arg(z) \leq \pi$ → $f(-i) = -\pi/2$

Observation : Range selected in such a way to exclude all but one possible value for each element of the Domain.

Principal Branch : One particular choice of range for a multi-valued function.

Principal value : value of the function in principal branch.

Brain-teaser : $f(z) = \ln z \rightarrow$ single valued / multi-valued?

Soln : $z = x+iy$ or $z = re^{i\theta}$

$$f(z) = \ln(re^{i\theta})$$
$$= \ln(r) + \ln(e^{i\theta})$$
$$= \ln(r) + i\theta$$
$$f(z) = \ln(|z|) + i.\arg(z)$$
$$\Rightarrow f(z) \longrightarrow \text{multi-valued}$$

Problem : Principal value of $f(i) = ?$ if $f(z) = \ln z$ & principal branch is given by $2\pi \leq \arg(z) < 4\pi$

Soln :
$$f(z) = \ln z = \ln|z| + i\arg(z)$$
$$f(i) = \ln|i| + i\arg(i) = i\left(2\pi + \pi/2\right)$$

$\Rightarrow$ Principal value of $f(i) \rightarrow i\left(2\pi + \pi/2\right)$

Representation of Complex function :

$$W = f(z)$$

input $\rightarrow$ x-y plane
output $\rightarrow$ u-v plane

$$u + iv = f(x+iy)$$
$$u = u(x,y)$$
$$v = v(x,y)$$

input $\rightarrow$ r-θ-plane
output $\rightarrow$ u-v plane.

$$\therefore f(z) = u(x,y) + iv(x,y)$$

in polar co-ordinates

$$W = f(re^{i\theta})$$
$$u + iv = f(re^{i\theta})$$

$\Rightarrow u = u(r,\theta)$ & $v = v(r,\theta)$

$$f(z) = u(r,\theta) + iv(r,\theta)$$

example-01 : $f(z) = z^2$

$z = x+iy$

$f(z) = f(x+iy)$

$= (x+iy)^2$

$= x^2-y^2+i2xy$

$u(x,y) = x^2-y^2$

$v(x,y) = 2xy$

example-02 : $f(z) = z^2$

$z = re^{i\theta}$

$f(z) = f(re^{i\theta})$

$= (re^{i\theta})^2$

$= r^2\{\cos(2\theta) + i\sin(2\theta)\}$

$= r^2\cos 2\theta + i r^2\sin 2\theta$

$u(r,\theta) = r^2\cos 2\theta$

$v(r,\theta) = r^2\sin 2\theta$

Some examples of complex functions :

Question → How to plot complex function?

$f: \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$

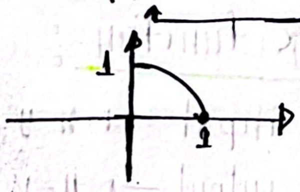
Dom → 1D
Co-Dom → 1D } 2D

$f: \mathbb{C} \to \mathbb{C}$ $f(z) = z^2$

Dom → 2D
Co-Dom → 2D } 4D

Problem-01 : $w = z+1$. Find image under w of

(Translation function)



Soln :

$z = x+iy$ → input x-y plane

$w = u+iv$ → output u-v plane

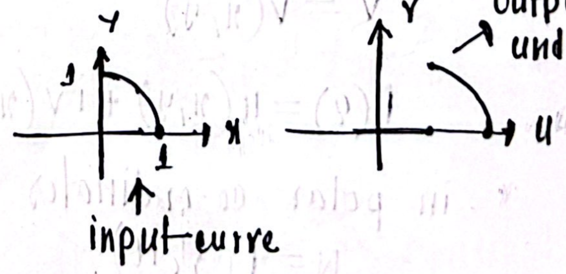
$u+iv = x+iy+1$

$u+iv = (x+1)+iy$

$u = x+1$

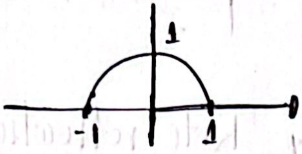
$v = y$

$(x,y) \longrightarrow (x+1, y)$



→ Reflection w.r.t x Axis. 5

Problem-02 : $f(z)=\bar{z}$. Find image of the following curve under f.



Soln : $z=x+iy \longrightarrow$ input x-y plane

$w=f(z)$

$u+iv=f(x+iy) \longrightarrow$ output u-v plane.

$u+iv=\overline{x+iy}$

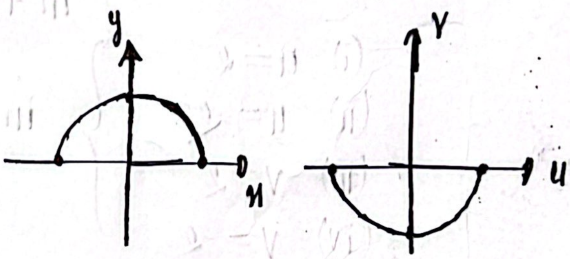
$\Rightarrow u+iv=x-iy$

$\Rightarrow u=x$

$v=-y$

input $\longrightarrow (x,y)$

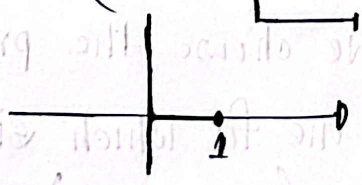
output $\longrightarrow (u,v)/(x,-y)$



Brain-teaser : Construct a complex function which reflects any point w.r.t y Axis.

Problem-03 : $f(z)=iz$. Find image under f of

$\longrightarrow$ (90° rotation)



Soln : $z=re^{i\theta} \longrightarrow$ input r-θ plane

$f(z)=f(re^{i\theta})=ire^{i\theta}=e^{i\pi/2}re^{i\theta}=re^{i(\theta+\pi/2)}$

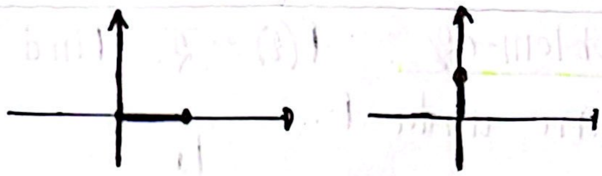
$\rho e^{i\psi} = f(z)=re^{i(\theta+\pi/2)} \longrightarrow$ output ρ-ψ plane

$= \rho e^{i\psi}=re^{i(\theta+\pi/2)}$

$\Rightarrow \rho = r$

$\psi = \theta + \pi/2$

$(r, \theta) \longrightarrow (r, \theta + \pi/2)$



$f(z) = i\bar{z} \longrightarrow$ Roto-reflection.

Brain-teaser:

(i) $w = e^z$. Image under w of $y = \pm 2, x = \pm 2$?

(ii) $w = z^2$. Find pre-image under w of in the z plane

(i) $u = 2$
(ii) $u = -2$
(iii) $v = 2$
(iv) $v = -2$

} in the w-plane

Hyperbolic function:

$\sinh z = \dfrac{e^z - e^{-z}}{2}$

$\cosh z = \dfrac{e^z + e^{-z}}{2}$

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

$\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$.

Problem-04: If we choose the principal branch of $\sin^{-1} z$ to be that one for which $\sin^{-1}(0) = 0$, prove that

$$\sin^{-1} z = \frac{1}{i} \ln\left(iz + \sqrt{1 - z^2}\right)$$

Soln: let, $\theta = \sin^{-1} z$

$\Rightarrow \sin\theta = z$

$\Rightarrow z = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

$\Rightarrow 2iz = e^{i\theta} - e^{-i\theta}$ $\quad$ let, $e^{i\theta} = u$

$\Rightarrow 2iz = u - \frac{1}{u}$

$\Rightarrow 2izu = u^2 - 1$

$\Rightarrow u^2 - 2izu - 1 = 0$

$\Rightarrow u = \frac{2iz \pm \sqrt{(2iz)^2 - 4(-1)}}{2}$

$\Rightarrow u = \frac{2iz \pm \sqrt{4 - 4z^2}}{2}$

$\Rightarrow u = \frac{2iz \pm 2\sqrt{1-z^2}}{2}$

$\Rightarrow u = iz \pm \sqrt{1-z^2}$

$\Rightarrow e^{i\theta} = iz \pm \sqrt{1-z^2}$

$-\sqrt{1-z^2}$
$= i^2\sqrt{1-z^2}$
$= \sqrt{i^4(1-z^2)}$
$= \sqrt{1-z^2}$

$\Rightarrow e^{i\theta} \cdot 1 = iz + \sqrt{1-z^2}$ (Since $\pm\sqrt{1-z^2}$ is implied by $\sqrt{1-z^2}$)

$\Rightarrow e^{i\theta} e^{2\pi k i} = iz + \sqrt{1-z^2}$, $k \in \mathbb{Z}$ $\qquad e^{2\pi k i} = \cos(2\pi k) + i\sin(2\pi k)$; $k \in \mathbb{Z}$ $= 1$

$\Rightarrow e^{i(\theta + 2\pi k)} = iz + \sqrt{1-z^2}$

$\Rightarrow i(\theta + 2\pi k) = \ln\left(iz + \sqrt{1-z^2}\right)$

$\Rightarrow \theta = -2\pi k + \frac{1}{i}\ln\left(iz + \sqrt{1-z^2}\right)$

$\Rightarrow \sin^{-1}(z) = -2\pi k + \frac{1}{i}\ln\left(iz + \sqrt{1-z^2}\right)$; $k \in \mathbb{Z}$

~~Ini~~ $\sin^{-1}(0) = -2\pi k + \frac{1}{i}\ln(1)$

$\Rightarrow 0 = -2\pi k$

$\Rightarrow k = 0$

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$$\Rightarrow \sin^{-1}(z) = \frac{1}{i}\ln\left(iz + \sqrt{1-z^2}\right)$$

Problem-05: If we choose the principal branch of $\tanh^{-1} z$ to be the one for which $\tanh^{-1}(0)=0$, prove that

$$\tanh^{-1}(z) = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)$$

Sol^n: let,

$$\theta = \tanh^{-1}(z)$$

$$\Rightarrow z = \tanh\theta$$

$$\Rightarrow z = \frac{\sinh\theta}{\cosh\theta}$$

$$\Rightarrow z = \frac{(e^{\theta}-e^{-\theta})/2}{(e^{\theta}+e^{-\theta})/2}$$

$$\Rightarrow z = \frac{e^{\theta}-e^{-\theta}}{e^{\theta}+e^{-\theta}} \qquad \text{let, } e^{\theta} = u$$

$$\Rightarrow z = \frac{u - 1/u}{u + 1/u}$$

$$\Rightarrow uz + \frac{1}{u}z = u - \frac{1}{u}$$

$$\Rightarrow (z-1)u = (-z-1)\frac{1}{u}$$

$$\Rightarrow (z-1)u^2 = -z-1$$

$$\Rightarrow u^2 = \frac{1+z}{1-z}$$

$$\Rightarrow e^{2\theta} = \frac{1+z}{1-z}$$

$$\Rightarrow e^{2\theta}e^{2\pi ki} = \frac{1+z}{1-z} \quad ; k \in \mathbb{Z}$$

$$\Rightarrow e^{2(\theta + \pi ki)} = \frac{1+z}{1-z}$$

$$\Rightarrow 2(\theta + \cancel{2}\pi k i) = \ln\left(\frac{1+z}{1-z}\right)$$

$$\Rightarrow \theta = -\pi k i + \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)$$

$$\Rightarrow \tanh^{-1}(z) = -\pi k i + \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)$$

$$\tanh^{-1}(0) = -\pi k i + \frac{1}{2}\ln(1) = 0$$

$$\Rightarrow -\pi k i = 0$$

$$\Rightarrow k = 0$$

$$\Rightarrow \tanh^{-1}(z) = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right).$$