

MAT215 COMPLETE PRACTICE SHEET SOLVE

Practice Sheet # 1

Part A

① Perform each of the indicated operations:

$$(i) (i-2) \{ 2(1+i) - 3(i-1) \}$$

$$(ii) \frac{8-i}{1-i}$$

$$= (i-2) \{ (2+2i) - 3i + 3 \}$$

$$= \frac{(8-i)(1+i)}{1-i^2}$$

$$= (i-2)(5-i)$$

$$= \frac{8+8i-i-i^2}{1+i}$$

$$= 5i - i^2 - 10 + 2i$$

$$= \frac{8+7i+1}{2}$$

$$= 1+7i-10$$

$$= \frac{9+7i}{2}$$

$$= 7i-9$$

$$(ii) \frac{(2+i)(3-2i)(1-i)}{(1-i)^2}$$

$$= \frac{(6-4i+3i-2i^2)(1-i)}{(1-i)^2}$$

$$= \frac{-2i^2 - i + 6}{1-i}$$

$$= \frac{-2(-1) - i + 6}{1-i} \quad \nearrow$$

$$(iii) (2i-1)^2 \left\{ \frac{4}{1-i} + \frac{2-i}{1+i} \right\}$$

$$= \left\{ \frac{4+4i+2-2i-i+i^2}{1-i^2} \right\} (2i-1)^2$$

$$= \frac{5+i}{1+i} (4i^2 - 4i + 1)$$

$$= \frac{(5+i)(-4-4i+1)}{2}$$

$$= \frac{(5+i)(-3-4i)}{2}$$

$$= \frac{-15-20i-3i-4i^2}{2}$$

$$= \frac{-15+4-23i}{2}$$

$$= \frac{-11-23i}{2}$$

$$(iv) 3\left(\frac{1+i}{1-i}\right)^2 - 2\left(\frac{1-i}{1+i}\right)^3$$

$$= 3\left(\frac{1+2i+i^2}{1-2i+i^2}\right) - 2\frac{(1-i)^2 \cdot (1-i)}{(1+i)^2 \cdot (1+i)}$$

$$= 3\left(\frac{1+2i-1}{1-2i-1}\right) - 2\left(\frac{1-2i-1}{1+2i-1}\right)\left(\frac{1-i}{1+i}\right)$$

$$= 3\left(\frac{2i}{-2i}\right) - 2\frac{-2i(1-i)}{2i(1+i)}$$

$$= -3 + 2\frac{1-i}{1+i}$$

$$= \frac{2-2i-3-3i}{1+i} = \frac{-1-5i}{1+i}$$

$$= \frac{(-1-5i)(1-i)}{1-i^2}$$

$$= \frac{-1+i-5i+5i^2}{1+i}$$

$$= \frac{-1-4i-5}{2} = \frac{-6-4i}{2}$$

$$= -3-2i$$

Ans.

$$(v) \frac{3i^{16} - i^{19}}{2i - 1}$$

$$\begin{aligned} &= \frac{3(i^2)^5 - i^{10} \cdot i^9}{2i - 1} \\ &= \frac{-3 - (i^2)^5 (i^3)^3}{2i - 1} \\ &= \frac{-3 + (-i)^3}{2i - 1} \\ &= \frac{-3 - i^3}{2i - 1} \\ &= \frac{(-3 + i^3)(2i + 1)}{(2i - 1)(2i + 1)} \\ &= \frac{-6i - 3 + 2i^2 + i}{4i^2 - 1} \\ &= \frac{-5i - 3 - 2}{-4 - 1} \end{aligned}$$

$$\begin{aligned} &= \frac{-5i - 5}{-5} \\ &= i + 1 \end{aligned}$$

(vi)

$$\frac{i^4 - i^9 + i^{16}}{2 - i^5 + i^{10} - i^{15}}$$

$$\begin{aligned} &= \frac{(i^2)^2 - (i^3)^3 + (i^2)^{16}}{2 - i^2 \cdot i^3 + (i^2)^5 - (i^3)^5} \\ &= \frac{1 - (-i)^3 + 1}{2 - i - 1 - (-i)^5} \\ &= \frac{1 - i + 1}{1 - i + i^2 \cdot i^3} \\ &= \frac{2 - i}{1 - i + i} \\ &= 2 - i \end{aligned}$$

$$1. |z| = |\bar{z}| \quad i = \sqrt{-1}$$

$$2. \overline{\bar{z}} = z \quad i^2 = -1$$

$$3. \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2 \quad i^3 = i^2 \cdot i$$

$$4. \overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2 \quad = -1 \cdot i = -i$$

$$5. \overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2 \quad i^4 = i^2 \cdot i^2 = 1$$

$$6. z + \bar{z} = 2 \cdot \operatorname{Re}(z)$$

$$7. z \cdot \bar{z} = |z|^2 = x^2 + y^2$$

- ② Show that the followings illustrate the associative law of addition.

$$(i) (5+3i) + \{(-1+2i) + (7-5i)\}$$

$$= (5+3i) + (6-3i)$$

= 11

$$(ii) \{ (5+3i) + (-1+2i) \} + (7-5i)$$

$$= (4+5i) + (7-5i)$$

= 11

∴ (i) & (ii) illustrate the associative law of addition.

- ③ If $z_1 = 1-i$, $z_2 = -2+4i$, $z_3 = \sqrt{3}-2i$, evaluate each of the followings.

$$(i) |2z_2 - 3z_1|^2 = |2(-2+4i) - 3(1-i)|^2$$

$$= |-4+8i - 3+3i|^2$$

$$= |-7+11i|^2$$

$$= \left(\sqrt{(-7)^2 + (11)^2} \right)^2$$

$$= (\sqrt{170})^2$$

$$= 170$$

$$(ii) \left| \frac{z_1 + z_2 + i}{z_1 - z_2 + i} \right|$$

$$= \left| \frac{1-i-2+4i+i}{1-i+2-4i+i} \right|$$

$$= \left| \frac{3i}{3-4i} \right|$$

$$= \frac{\sqrt{(3)^2}}{\sqrt{3^2 + (-4)^2}}$$

$$= \frac{3}{\sqrt{9+16}}$$

$$= \frac{3}{\sqrt{25}}$$

$$= \frac{3}{5}$$

$$(iii) \left| \frac{(z_2 + z_3)(z_1 - z_3)}{(-2+4i+\sqrt{3}-2i)(1-i-\sqrt{3}+2i)} \right|$$

$$= \frac{(\sqrt{3}-2+2i)(1-\sqrt{3}+i)}{+2i^2}$$

$$= \frac{-5+3\sqrt{3}+i3\sqrt{3}-2}{3\sqrt{3}-7+3\sqrt{3}i}$$

$$(iv) \operatorname{Re} \{ 2z_1^3 + 3z_2^3 - 5z_3^2 \}$$

$$= 2(1-i)^3 + 3(-2+4i)^3 - 5(\sqrt{3}-2i)^2$$

$$= 2(1-3i+3i^2-4i^3) + 3(-8+48i+96)$$

$$-64i) - 5(3-4\sqrt{3}i-4)$$

$$= 2-6i-6-2i-24+144i+288-192i$$

$$-15-20\sqrt{3}i+20$$

$$= 265-56i-20\sqrt{3}i$$

$$(v) \quad \operatorname{Im} \left\{ \frac{z_1 z_2}{z_3} \right\}$$

$$= \frac{(1-i)(-2+4i)}{\sqrt{3}-2i}$$

$$= \frac{-2+4i+2i-4i^2}{\sqrt{3}-2i}$$

$$= \frac{(2+6i)(\sqrt{3}+2i)}{(\sqrt{3})^2 - (2i)^2}$$

$$= \frac{2\sqrt{3} + 4i + 6\sqrt{3}i + 12i^2}{3+4}$$

$$= \frac{2\sqrt{3} - 12 + 4i + 6\sqrt{3}i}{7}$$

$$= \frac{4+6\sqrt{3}}{7}$$

$$(vi) \quad z_1^2 + 2z_1 - 3$$

$$= (1-i)^2 + 2(1-i) - 3$$

$$= 1-2i-1+2-2i-3$$

$$= -1-4i$$

$$(vii) \quad |z_1 \bar{z}_2 + z_2 \bar{z}_1|$$

$$= |(-i)(-2-4i) + (-2+4i)(1+i)|$$

$$= |-2-4i+2i+4i^2 - 2-2i+4i+4i^2|$$

$$= |-4+8i^2|$$

$$= |-4-8|$$

$$= | -12 |$$

$$= 12$$

$$(ix) \quad (z_3 - \bar{z}_3)^5$$

$$= \left\{ \sqrt{3}-2i - (\sqrt{3}+2i) \right\}^5$$

$$= (-4i)^5$$

$$= -1024i^5$$

$$= -1024i^2 \cdot i^3$$

$$= -1024i$$

$$[\text{If } z = x+iy]$$

$$\text{then } \bar{z} = x-iy]$$

(4) Express each of the following complex numbers in polar form and show them graphically.

$$(i) 2 + 2\sqrt{3}i$$

$$r = \sqrt{x^2 + y^2}$$

$$= \sqrt{2^2 + (2\sqrt{3})^2}$$

$$= \sqrt{4+12}$$

$$= \sqrt{16} = 4$$

$$\tan \theta = \frac{y}{x}$$

$$= \frac{2\sqrt{3}}{2}$$

$$= \sqrt{3}$$

$$= \tan 60^\circ = \tan \frac{\pi}{3}$$

$$\therefore \theta = \frac{\pi}{3}$$

$$\therefore \operatorname{Arg}(2 + 2\sqrt{3}i) = \frac{\pi}{3}$$

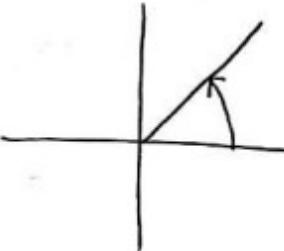
$$[r e^{i\theta} = r \operatorname{cis}(\theta + 2n\pi)]$$

$$\therefore \arg(2 + 2\sqrt{3}i) = \frac{\pi}{3} + 2n\pi$$

$$\therefore 2 + 2\sqrt{3}i = r \operatorname{cis}(\theta + 2n\pi)$$

$$= 4 \operatorname{cis}\left(\frac{\pi}{3} + 2n\pi\right); n = 0, \pm 1, \pm 2, \dots$$

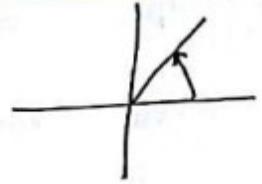
$$or, = 4 e^{i\pi/3}$$



(ii) $2\sqrt{2} + 2\sqrt{2}i$

$$\begin{aligned}\therefore r &= \sqrt{(2\sqrt{2})^2 + (2\sqrt{2})^2} \\ &= \sqrt{8+8} \\ &= \sqrt{16} \\ &= 4\end{aligned}$$

$$\begin{aligned}\tan \theta &= \frac{2\sqrt{2}}{2\sqrt{2}} \\ &= 1 \\ &= \tan 45^\circ = \tan \frac{\pi}{4} \\ \therefore \theta &= \frac{\pi}{4}\end{aligned}$$

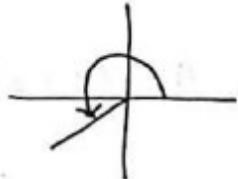


$$2\sqrt{2} + 2\sqrt{2}i = 4 \operatorname{cis}\left(\frac{\pi}{4} + 2n\pi\right); n=0, \pm 1, \pm 2, \dots$$

(iii) $-2\sqrt{3} - 2i$

$$\begin{aligned}r &= \sqrt{(-2\sqrt{3})^2 + (-2)^2} \\ &= \sqrt{12+4} \\ &= \sqrt{16} \\ &= 4\end{aligned}$$

$$\begin{aligned}\tan \theta &= \frac{-2}{-2\sqrt{3}} \\ &= \frac{1}{\sqrt{3}} \\ &= \tan 30^\circ = \tan \frac{\pi}{6} \\ \therefore \theta &= \pi + \frac{\pi}{6} = \frac{7\pi}{6}\end{aligned}$$



$$-2\sqrt{3} - 2i = 4 \operatorname{cis}\left(\frac{7\pi}{6} + 2n\pi\right); n=0, \pm 1, \pm 2, \dots$$

$$(iv) z = -1 + \sqrt{3}i$$

$$r = \sqrt{(-1)^2 + (\sqrt{3})^2}$$

$$= \sqrt{4}$$

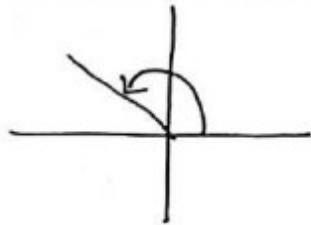
$$= 2$$

$$\tan \theta = \frac{\sqrt{3}}{-1}$$

$$= -\sqrt{3}$$

$$= -\tan 60^\circ = -\tan \frac{\pi}{3}$$

$$\therefore \theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$



$$\therefore -1 + \sqrt{3}i = 2 \operatorname{cis}\left(\frac{2\pi}{3} + 2n\pi\right); n=0, \pm 1, \pm 2, \dots$$

$$(v) -\sqrt{6} - \sqrt{2}i$$

$$r = \sqrt{(-\sqrt{6})^2 + (-\sqrt{2})^2}$$

$$= \sqrt{6+2}$$

$$= \sqrt{8}$$

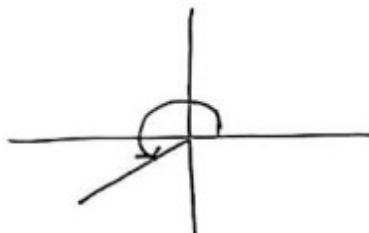
$$= 2\sqrt{2}$$

$$\tan \theta = \frac{-\sqrt{2}}{-\sqrt{6}}$$

$$= \frac{1}{\sqrt{3}}$$

$$= \tan 30^\circ = \tan \frac{\pi}{6}$$

$$\therefore \theta = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$$



$$\therefore -\sqrt{6} - \sqrt{2}i = 2\sqrt{2} \operatorname{cis}\left(\frac{7\pi}{6} + 2n\pi\right); n=0, \pm 1, \pm 2, \dots$$

$$\text{or } 2\sqrt{2} e^{i\frac{7\pi}{6}}$$

(5) Prove the followings:

$$\left[\begin{array}{l} \text{If } z = x + iy \\ \text{then, } \bar{z} = x - iy \end{array} \right]$$

(i) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

\Rightarrow Let, $z_1 = x_1 + iy_1$ & $z_2 = x_2 + iy_2$

$$L.H.S = \overline{z_1 + z_2} = \overline{x_1 + iy_1 + x_2 + iy_2}$$

$$= x_1 + x_2 - i(y_1 + y_2)$$

$$= (x_1 - iy_1) + (x_2 - iy_2)$$

$$= \overline{z}_1 + \overline{z}_2 = R.H.S.$$

(ii) $|z_1 z_2| = |\overline{z}_1| |\overline{z}_2|$

$$L.H.S. = |z_1 z_2| = |(x_1 + iy_1)(x_2 + iy_2)|$$

$$= |x_1 x_2 + ix_1 y_2 + ix_2 y_1 + i^2 y_1 y_2|$$

$$= |x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1)|$$

$$= \sqrt{(x_1 x_2 - y_1 y_2)^2 + (x_1 y_2 + x_2 y_1)^2}$$

=

$$= \sqrt{x_1^2 x^2 - 2x_1 x_2 y_1 y_2 + y_1^2 y_2^2 + x_1^2 y_2^2 + 2x_1 x_2 y_1 y_2 + x_2^2 y_1^2}$$

$$= \sqrt{x_1^2 (x_2^2 + y_2^2) + y_1^2 (x_2^2 + y_2^2)}$$

$$= \sqrt{(x_1^2 + y_1^2) (x_2^2 + y_2^2)}$$

$$= \sqrt{x_1^2 + y_1^2} \cdot \sqrt{x_2^2 + y_2^2}$$

$$= |z_1| \cdot |z_2|$$

$$= R.H.S.$$

$$\therefore L.H.S. = R.H.S.$$

[Proved]

$$(iii) |z_1 \pm z_2| \leq |z_1| + |z_2|$$

$$|z_1 + z_2|^2 = (z_1 + z_2) \cdot (\overline{z_1 + z_2}) \quad [\because |z|^2 = z \cdot \bar{z}]$$

$$= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$$

$$= z_1 \bar{z}_1 + z_1 \bar{z}_2 + \bar{z}_1 z_2 + z_2 \bar{z}_2$$

$$= |z_1|^2 + \overline{z_1 z_2} + \bar{z}_1 z_2 + |z_2|^2 \quad [\because \bar{\bar{z}} = z]$$

$$= |z_1|^2 + |z_2|^2 + 2 \operatorname{Re}(z, \bar{z}_2) \quad [\because z + \bar{z} = 2 \operatorname{Re}(z)]$$

$$\leq |z_1|^2 + |z_2|^2 + 2 |z_1| |z_2| \quad [\because \operatorname{Re}(z) \leq |z|]$$

$$\leq |z_1|^2 + 2 |z_1| |z_2| + |z_2|^2 \quad [\because |z| = |\bar{z}|]$$

$$\leq (|z_1| + |z_2|)^2$$

$$\therefore |z_1 + z_2| \leq |z_1| + |z_2| \quad [\text{Proved}]$$

Replacing z_2 by $-z_2$ we get,

$$|z_1 - z_2| \leq |z_1| + |-z_2|$$

$$\leq |z_1| + |z_2| \quad [\because |z| = |-z|]$$

[Proved]

$$(ii) |z_1 \pm z_2| \geq |z_1| - |z_2|$$

$$\begin{aligned}
 |z_1 - z_2|^2 &= (z_1 - z_2)(\overline{z_1 - z_2}) & [\because |z|^2 = z \cdot \bar{z}] \\
 &= (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) \\
 &= z_1 \bar{z}_1 - z_1 \bar{z}_2 - \bar{z}_1 z_2 + z_2 \bar{z}_2 \\
 &= |z_1|^2 - (\bar{z}_1 z_2 + \bar{z}_2 z_1) + |z_2|^2 & [\because \bar{\bar{z}} = z] \\
 &= |z_1|^2 - 2 \operatorname{Re}[\bar{z}_1 z_2] + |z_2|^2 & [\because z + \bar{z} = 2 \operatorname{R}(z)] \\
 &\geq |z_1|^2 - 2 |z_1| |z_2| + |z_2|^2 \\
 &\geq |z_1|^2 - 2 |z_1| |z_2| + |z_2|^2 \\
 &\geq (|z_1| - |z_2|)^2
 \end{aligned}$$

$$\therefore |z_1| - |z_2| \leq |z_1 - z_2| \quad \text{or, } |z_1 - z_2| \geq |z_1| - |z_2| \quad [\text{Proved}]$$

Replacing z_2 by $-z_2$ we get

$$\begin{aligned}
 |z_1 + z_2| &\geq |z_1| - |z_2| \\
 &\geq |z_1| - |z_2| & [\because |z| = |-z|]
 \end{aligned}$$

[Proved]

(6) State and prove the De Moivre's Theorem

→ Statement: For $n \in \mathbb{Z}$,

$$(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta)$$

Proof: The statement is true for $n=1$

Suppose the statement is true for $n=k$

i.e; $(\cos \theta + i \sin \theta)^k = \cos(k\theta) + i \sin(k\theta) \dots \dots \text{(i)}$

We will show that the statement is true for $n=k+1$

$$\begin{aligned} (\cos \theta + i \sin \theta)^{k+1} &= (\cos \theta + i \sin \theta)^k (\cos \theta + i \sin \theta) \\ &= (\cos(k\theta) + i \sin(k\theta)) \cdot (\cos \theta + i \sin \theta) \\ &= \cos(k\theta + \theta) + i \sin(k\theta + \theta) \\ &= \cos((k+1)\theta) + i \sin((k+1)\theta) \end{aligned}$$

Therefore, by the principle of mathematical induction the statement is true for $n \in \mathbb{N}$.

[Proved]

(7) Evaluate each of the following by De Moivre's Theorem:

$$(i) \frac{(8 \operatorname{cis} 40^\circ)^3}{(2 \operatorname{cis} 60^\circ)^4} = \frac{8^3 (\cos 3(40^\circ) + i \sin 3(40^\circ))}{2^4 (\cos 4(60^\circ) + i \sin 4(60^\circ))}$$

$$= \frac{2^5 (\cos 120^\circ + i \sin 120^\circ)}{(\cos 240^\circ + i \sin 240^\circ)}$$

$$= \frac{2^5 \left(-\frac{1}{2} + i \frac{\sqrt{3}}{2}\right)}{\left(-\frac{1}{2} - i \frac{\sqrt{3}}{2}\right)}$$

$$\frac{(8 \operatorname{cis} 40^\circ)^3}{(2 \operatorname{cis} 60^\circ)^4} = \frac{512 \operatorname{cis} 3 \times 40^\circ}{16 \operatorname{cis} 4 \times 60^\circ}$$

$$= \frac{2^5 \left(-\frac{1}{2} + i \frac{\sqrt{3}}{2}\right)^2}{\left(-\frac{1}{2}\right)^2 - \left(i \frac{\sqrt{3}}{2}\right)^2}$$

$$= 32 \operatorname{cis}(120 - 240)$$

$$= \frac{2^5 \left(-\frac{1}{2} + i \frac{\sqrt{3}}{2}\right)^2}{1/4 + 3/4}$$

$$= 32 \operatorname{cis}(-120)$$

$$= 2^5 \left(-\frac{1}{2} + i \frac{\sqrt{3}}{2}\right)^2$$

$$= 32 \left\{ \cos(-120) + i \sin(-120) \right\}$$

$$= 32 (\cos 120 - i \sin 120)$$

$$= 32 \left(-\frac{1}{2} - i \frac{\sqrt{3}}{2}\right)$$

$$= (-16 - 16\sqrt{3}i)$$

(ii)

$$\frac{(3e^{\pi i/6})(2e^{-5\pi i/4})(6e^{5\pi i/3})}{(4e^{2\pi i/3})^2}$$

$$= \frac{36 e^{\frac{\pi i}{6} - \frac{5\pi i}{4} + \frac{5\pi i}{3}}}{16 e^{\frac{4\pi i}{3}}}$$

$$= \frac{9}{4} e^{\frac{\pi i}{6} - \frac{5\pi i}{4} + \frac{5\pi i}{3} - \frac{4\pi i}{3}}$$

$$= \frac{9}{4} e^{\frac{2\pi i - 15\pi i + 20\pi i - 16\pi i}{12}}$$

$$= \frac{9}{4} e^{-\frac{9}{12}\pi i} = \frac{9}{4} e^{-\frac{3}{4}\pi i}$$

$$= \frac{9}{4} \left(\cos\left(-\frac{3\pi}{4}\right) + i \sin\left(-\frac{3\pi}{4}\right) \right) \quad [\because e^{i\theta} = \cos\theta + i \sin\theta]$$

$$= \frac{9}{4} \left(\cos \frac{3\pi}{4} - i \sin \frac{3\pi}{4} \right)$$

$$= \frac{9}{4} \left(-\frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}} \right)$$

$$= \frac{9}{4} \left(-\frac{1}{\sqrt{2}} \right) (1+i) = \frac{-9}{4\sqrt{2}} (1+i) = \frac{-9\sqrt{2}}{8} (1+i)$$

$$(iii) (5 \operatorname{cis} 20^\circ)(3 \operatorname{cis} 40^\circ)$$

$$= 5 (\cos 20^\circ + i \sin 20^\circ) 3 (\cos 40^\circ + i \sin 40^\circ)$$

$$= 15 (\cos 20^\circ \cos 40^\circ + \cos 20^\circ i \sin 40^\circ + i \sin 20^\circ \cos 40^\circ + i^2 \sin 20^\circ \sin 40^\circ)$$

$$= 15 \left\{ (\cos 20^\circ \cos 40^\circ - \sin 20^\circ \sin 40^\circ) + i (\sin 20^\circ \cos 40^\circ + \cos 20^\circ \sin 40^\circ) \right\}$$

$$= 15 \left\{ \cos(20^\circ + 40^\circ) + i \sin(20^\circ + 40^\circ) \right\}$$

$$\begin{aligned}\sin(A+B) &= \sin A \cos B + \cos A \sin B \\ \cos(A+B) &= \cos A \cos B - \sin A \sin B\end{aligned}$$

$$= 15 (\cos 60^\circ + i \sin 60^\circ)$$

$$= 15 \left(\frac{1}{2} + i \frac{\sqrt{3}}{2} \right)$$

$$(iv) (2 \operatorname{cis} 50^\circ)^6$$

$$= 2^6 \left\{ \cos 6(50^\circ) + i \sin 6(50^\circ) \right\}$$

$$= 2^6 (\cos 300^\circ + i \sin 300^\circ)$$

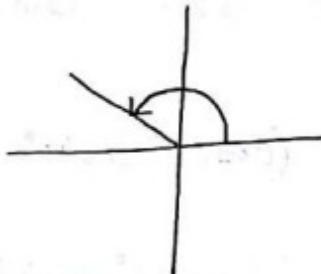
$$= 2^6 \left(\frac{1}{2} + i \left(-\frac{\sqrt{3}}{2} \right) \right)$$

$$= 2^6 \left(\frac{1}{2} - i \frac{\sqrt{3}}{2} \right)$$

(8) Find all the roots of the following equations

$$(i) (-1+i)^{1/3} = z$$

$$\text{Sol: } r = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$$



$$\theta = \tan^{-1} \left(\frac{1}{-1} \right) = \tan^{-1} (-1)$$

$$= \tan^{-1} (\tan(\pi/4))$$

$$= \tan^{-1} \tan(\pi - \pi/4)$$

$$= \frac{3\pi}{4}$$

$$(-1+i)^{1/3} = (\sqrt{2})^{1/3} \left\{ \cos\left(\frac{3\pi}{4} + 2\pi n\right) + i \sin\left(\frac{3\pi}{4} + 2\pi n\right) \right\}^{1/3}; n = 0, 1, 2$$

$$= 2^{1/6} \left\{ \cos\left(\frac{3\pi}{4} + 2\pi n\right)\left(\frac{1}{3}\right) + i \sin\left(\frac{3\pi}{4} + 2\pi n\right)\left(\frac{1}{3}\right) \right\}$$

$$= 2^{1/6} \left\{ \text{cis}\left(\frac{\pi}{4} + \frac{2}{3}\pi n\right) = 2^{1/6} \text{cis}\left(\frac{3\pi + 8\pi n}{12}\right) \right\}$$

$$\text{If } n=0, z_1 = 2^{1/6} \text{cis}\left(\frac{\pi}{4}\right)$$

$$\text{If } n=1, z_2 = 2^{1/6} \text{cis}\left(\frac{11\pi}{12}\right)$$

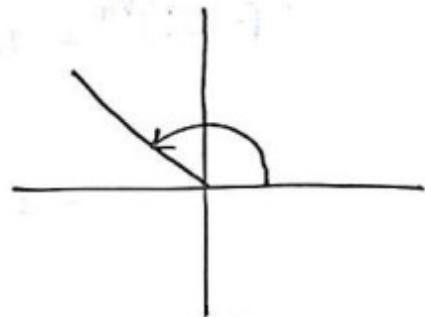
$$\text{If } n=2, z_3 = 2^{1/6} \text{cis}\left(\frac{19\pi}{12}\right)$$

The roots are $\sqrt[6]{2} \text{ cis } 45^\circ$, $\sqrt[6]{2} \text{ cis } 165^\circ$ & $\sqrt[6]{2} \text{ cis } 285^\circ$

(ii)

$$z^5 = -4 + 4i$$

Sol: $r = \sqrt{(-4)^2 + 4^2} = \sqrt{32} = 4\sqrt{2}$



$$\theta = \tan^{-1}\left(\frac{4}{-4}\right) = \tan^{-1}(-1) = \tan^{-1} \tan\left(-\frac{\pi}{4}\right)$$

$$= \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

$$\therefore (-4+4i)^{1/5} = (4\sqrt{2})^{1/5} \left(\text{cis } \frac{3\pi}{4} + 2\pi n \right)^{1/5} \quad \text{where, } n=0,1,2,3,4$$

$$= (\sqrt{32})^{1/5} \left\{ \text{cis} \left(\frac{3\pi}{4} + 2\pi n \right) \frac{1}{5} \right\} = \sqrt[5]{32} \text{ cis} \left(\frac{3\pi}{20} + \frac{8\pi n}{20} \right)$$

The roots are.

$$\text{If } n=0, z_1 = \sqrt[5]{32} \text{ cis} \left(\frac{3\pi}{2} \right)$$

$$n=1, z_2 = \sqrt[5]{32} \text{ cis} \left(\frac{11\pi}{20} \right)$$

$$n=2, z_3 = \sqrt[5]{32} \text{ cis} \left(\frac{19\pi}{20} \right)$$

$$\text{If } n=3, z_4 = \sqrt[5]{32} \text{ cis} \left(\frac{27\pi}{20} \right)$$

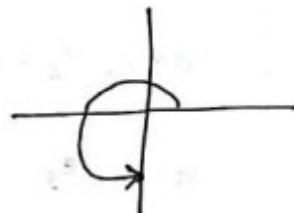
$$n=4, z_5 = \sqrt[5]{32} \text{ cis} \left(\frac{35\pi}{20} \right)$$

(iii)

$$z^4 = -16i$$

$$\Rightarrow r = \sqrt{0^2 + (-16)^2} = 16$$

$$\theta = \tan^{-1}\left(\frac{-16}{0}\right) = \tan^{-1} \infty = \tan^{-1} \tan\left(\frac{\pi}{2}\right)$$



$$(-16i)^{1/4} = 16^{1/4} \left\{ \cos\left(\frac{3\pi}{2} + 2\pi n\right) + i \sin\left(\frac{3\pi}{2} + 2\pi n\right) \right\}^{1/4}$$

$$= 2 \operatorname{cis}\left(\frac{3\pi + 4\pi n}{8}\right) \quad \text{where, } n=0, 1, 2, 3$$

$\text{If } n=0, z_1 = 2 \operatorname{cis}\left(\frac{3\pi}{8}\right)$ $n=1, z_2 = 2 \operatorname{cis}\left(\frac{7\pi}{8}\right)$	$\text{If } n=2, z_3 = 2 \operatorname{cis}\left(\frac{11\pi}{8}\right)$ $n=3, z_4 = 2 \operatorname{cis}\left(\frac{15\pi}{8}\right)$
---	---

$$(iv) z^6 = 64$$

$$\Rightarrow r = \sqrt{64^2 + 0^2} = 64$$



$$\theta = \tan^{-1}\left(\frac{0}{64}\right) = \tan^{-1} 0 = \tan^{-1} \tan 0 = 0$$

$$\therefore (64)^{1/6} = (64)^{1/6} \left\{ \cos(0+2\pi n) + i \sin(0+2\pi n) \right\}^{1/6}$$

$$= 2 \operatorname{cis}\left(\frac{\pi n}{3}\right) \quad \text{where, } n=0, 1, 2, 3, 4, 5$$

$$\text{If } n=0, z_1 = 2 \operatorname{cis} 0^\circ$$

$$n=1, z_2 = 2 \operatorname{cis}\left(\frac{\pi}{3}\right)$$

$$n=2, z_3 = 2 \operatorname{cis}\left(\frac{2\pi}{3}\right)$$

$$\text{If } n=3, z_4 = 2 \operatorname{cis}(\pi)$$

$$n=4, z_5 = 2 \operatorname{cis}\left(\frac{4\pi}{3}\right)$$

$$n=5, z_6 = 2 \operatorname{cis}(2\pi)$$

$$(iv) z^4 + z^2 + 1 = 0 \quad (\text{Polynomial eqn})$$

\Rightarrow Let, $x = z^2$

$$\therefore x^2 + x + 1 = 0$$

$$\therefore x = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1}$$

$$\therefore z^2 = \frac{-1 \pm \sqrt{-3}}{2}$$

$$\therefore z = \left(\frac{-1 \pm \sqrt{-3}}{2} \right)^{1/2}$$

$$z_1 = \left(\frac{-1}{2} + \frac{i\sqrt{3}}{2} \right)^{1/2}$$

$$r = \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$$

$$\theta = \tan^{-1} \left(\frac{\sqrt{3}/2}{-1/2} \right) = \tan^{-1} (-\sqrt{3})$$

$$= \tan^{-1} \tan \left(-\frac{\pi}{3} \right)$$

$$= \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

$$z_1 = 1^{1/2} \left\{ \text{cis} \left(\frac{2\pi}{3} + 2\pi n \right) \right\}^{1/2}$$

$$= \text{cis} \left(\frac{\pi + 3\pi n}{3} \right)$$

$$\text{When } n=0, z = \text{cis} \left(\frac{\pi}{3} \right)$$

$$n=1, z = \text{cis} \left(\frac{4\pi}{3} \right)$$

$$z_2 = \left(\frac{-1}{2} - \frac{i\sqrt{3}}{2} \right)^{1/2}$$

$$r = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2} = 1$$

$$\theta = \tan^{-1} \left(\frac{-\sqrt{3}/2}{-1/2} \right) = \tan^{-1} (\sqrt{3})$$

$$= \tan^{-1} \tan \left(\frac{\pi}{3} \right)$$

$$= \pi + \frac{\pi}{3} = \frac{4\pi}{3}$$

$$z_2 = 1^{1/2} \left\{ \text{cis} \left(\frac{4\pi}{3} + 2\pi n \right) \right\}^{1/2}$$

$$= \text{cis} \left(\frac{2\pi + 3\pi n}{3} \right)$$

$$\text{When } n=0, z = \text{cis} \left(\frac{2\pi}{3} \right)$$

$$n=1, z = \text{cis} \left(\frac{5\pi}{3} \right)$$

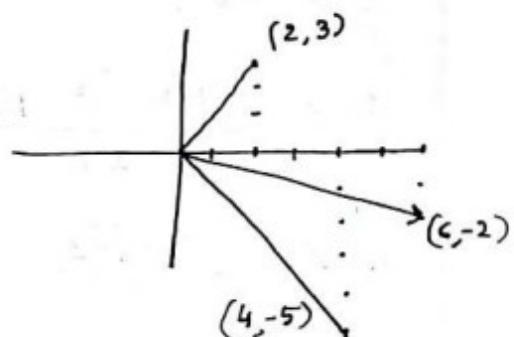
Part B

① Perform the indicated operations analytically and graphically.

(a) $(2 + 3i) + (4 - 5i)$

\Rightarrow Let, $z_1 = 2 + 3i$ & $z_2 = 4 - 5i$

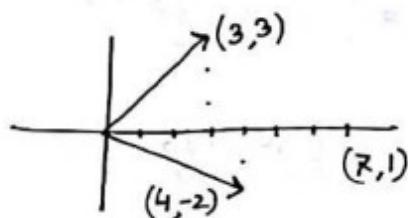
$\therefore z_1 + z_2 = 2 + 3i + 4 - 5i = 6 - 2i$



(b) $(7+i) - (4-2i)$

\Rightarrow Let, $z_1 = 7+i$ & $z_2 = 4-2i$

$\therefore z = z_1 - z_2 = 7+i - 4+2i = 3+3i$



② Describe geometrically the set of points z satisfying the following conditions:

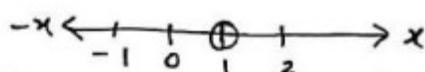
(a) $\operatorname{Re}(z) > 1$

\Rightarrow Now, $z = (x+iy)$

$\therefore \operatorname{Re}(z) = x$

$\therefore \operatorname{Re}(z) > 1$

$\Rightarrow x > 1$



$$(b) |2z + 3| > 4$$

Here,

$$z = x + iy$$

$$\therefore |2(x + iy) + 3| > 4$$

$$\Rightarrow |2x + 2iy + 3| > 4$$

$$\Rightarrow \sqrt{(2x+3)^2 + (2y)^2} > 4$$

$$\Rightarrow 4x^2 + 12x + 9 + 4y^2 > 16$$

$$\Rightarrow 4x^2 + 12x + 4y^2 > 7$$

$$\therefore x^2 + 3x + y^2 > \frac{7}{4}$$

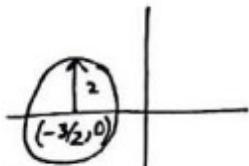
$$\Rightarrow x^2 + 2 \cdot x \cdot \frac{3}{2} + \left(\frac{3}{2}\right)^2 + y^2 > \frac{7}{4} + \left(\frac{3}{2}\right)^2$$

$$\Rightarrow \left(x + \frac{3}{2}\right)^2 + y^2 > \frac{7}{4} + \frac{9}{4}$$

$$\Rightarrow \left(x + \frac{3}{2}\right)^2 + y^2 > 4$$

$$\therefore \left(x + \frac{3}{2}\right)^2 + y^2 > 2^2$$

$$\text{Now, } \left(x + \frac{3}{2}\right)^2 + y^2 = 2^2$$



$$(c) \operatorname{Re}\left(\frac{1}{z}\right) > 1$$

$$\text{Here, } z = x + iy$$

$$\Rightarrow \frac{1}{z} = \frac{1}{x+iy}$$

$$= \frac{x - iy}{x^2 + y^2}$$

$$= \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2}$$

$$\therefore \operatorname{Re}\left(\frac{1}{z}\right) = \frac{x}{x^2 + y^2}$$

Now,

$$\operatorname{Re}\left(\frac{1}{z}\right) > 1$$

$$\Rightarrow \frac{x}{x^2 + y^2} > 1$$

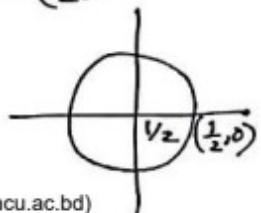
$$\Rightarrow x > x^2 + y^2$$

$$\Rightarrow 0 > x^2 + y^2 - x$$

$$\Rightarrow x^2 - 2 \cdot x \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^2 + y^2 < \left(\frac{1}{2}\right)^2$$

$$\Rightarrow \left(x + \frac{1}{2}\right)^2 + y^2 < \left(\frac{1}{2}\right)^2$$

$$\text{Now, } \left(x + \frac{1}{2}\right)^2 + y^2 = \left(\frac{1}{2}\right)^2$$



3. Using the properties of conjugate and modulus, show that:

(i) $\overline{z + 3i} = z - 3i$

\Rightarrow Here, $z = x + iy$

L.H.S. = $\overline{z + 3i}$

$$= \overline{x + iy + 3i}$$

$$= \overline{x - iy + 3i}$$

$$= \overline{x - i(y-3)}$$

$$= x + i(y-3)$$

$$= x + iy - 3i$$

$$= z - 3i$$

$$= R.H.S.$$

[Proved]

$$\begin{array}{ll}
 \text{(iii)} |(2\bar{z} + 5)(\sqrt{2} - i)| = \sqrt{3} |2z + 5| & \text{(iv)} |2z + 3\bar{z}| \leq 4 |\operatorname{Re}(z)| + |z|
 \end{array}$$

$$\begin{aligned}
 \text{L.H.S.} &= \left| \left\{ 2(\overline{x+iy}) + 5 \right\} (\sqrt{2} - i) \right| \\
 &= \left| \left\{ 2(x - iy) + 5 \right\} (\sqrt{2} - i) \right| \\
 &= \left| (2x - 2iy + 5) \right| \left| \sqrt{2} - i \right| \\
 &= \sqrt{(2x+5)^2 + (-2y)^2} \cdot \sqrt{(\sqrt{2})^2 + (-1)^2} \\
 &= \sqrt{(2x+5)^2 + (2y)^2} \cdot \sqrt{2+1} \\
 &= |2x+5 + 2yi| \cdot \sqrt{3} \\
 &= \sqrt{3} \cdot |2(x+iy) + 5| \\
 &= \sqrt{3} |2z + 5|
 \end{aligned}$$

$$\begin{aligned}
 \text{L.H.S.} &= |2z + 3\bar{z}| \\
 &= |2(x+iy) + 3(\overline{x+iy})| \\
 &= |2x + 2iy + 3(x-iy)| \\
 &= |2x + 2iy + 3x - 3iy| \\
 &= |5x - iy| \\
 &= |4x + (x - iy)| \\
 &\leq |4x| + |x - iy| \\
 &\leq 4|x| + |x - iy| \\
 &\leq 4|\operatorname{Re}(z)| + |\bar{z}| \\
 &\leq 4|\operatorname{Re}(z)| + |z| \\
 &\leq \text{R.H.S.}
 \end{aligned}$$

(4) Find the modulus and argument of the following complex number

(i) $\frac{2-i}{2+i}$

For Numerator

$$r = \sqrt{(2)^2 + (-1)^2} = \sqrt{5}$$

$$\theta = \tan^{-1}(-\frac{1}{2})$$

$$= 360^\circ - 26.57^\circ$$

$$= 333.43^\circ$$

$$\therefore (2-i) = \sqrt{5} \text{ cis } (333.43^\circ)$$

For denominator

$$r = \sqrt{(2)^2 + (1)^2} = \sqrt{5}$$

$$\theta = \tan^{-1}(\frac{1}{2})$$

$$= 26.57^\circ$$

$$\therefore (2+i) = \sqrt{5} \text{ cis } (333 - 26.57^\circ)$$

$$\therefore \frac{2-i}{2+i} = \frac{\sqrt{5} \text{ cis } (333.43^\circ)}{\sqrt{5} \text{ cis } (26.57^\circ)}$$

$$= \text{cis}(333.43^\circ - 26.57^\circ)$$

$$= 1 \cdot \text{cis}(307^\circ)$$

$$\therefore \text{Modulus} = 1 \quad \& \quad \text{Argument} = 307^\circ$$

4. (ii)

$$\frac{\sqrt{3}+i}{\sqrt{3}-i}$$

$$(iii) \quad \frac{(1+\sqrt{3}i)^2}{(1-\sqrt{3}i)^2}$$

$$= \frac{(\sqrt{3}+i)^2}{(\sqrt{3})^2 - i^2}$$

$$= \frac{3 + 2\sqrt{3}i - 1}{3 + 1}$$

$$= \frac{2 + 2\sqrt{3}i}{4}$$

$$= \frac{1+\sqrt{3}i}{2}$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2} i$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}/2}{1/2}\right)$$

$$= \frac{\pi}{3}$$

$$r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

$$\therefore \text{Arg} = \frac{\pi}{3}, \text{Mod} = 1$$

$$(iii) \left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i} \right)^2$$

For numerator

$$r = \sqrt{1^2 + (\sqrt{3})^2} = 2$$

$$\theta = \tan^{-1} \left(\frac{\sqrt{3}}{1} \right) = 60^\circ$$

$$= 60^\circ \text{ Q.E.D.} \quad \text{Ans} = 3$$

$$(1+\sqrt{3}i) = 2 \text{ cis } 60^\circ \text{ Q.E.D.}$$

For denominator

$$r = \sqrt{1^2 + (-\sqrt{3})^2} = 2$$

$$\theta = \tan^{-1} \left(-\frac{\sqrt{3}}{1} \right)$$

$$= 360^\circ - 30^\circ = 330^\circ$$

$$(-\sqrt{3}i) = 2 \text{ cis } 330^\circ$$

$$\left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i} \right)^2 = \left(\frac{2 \text{ cis } 60^\circ}{2 \text{ cis } 330^\circ} \right)^2$$

$$= \{ \text{cis}(60^\circ - 330^\circ) \}^2$$

$$= \text{cis}(-270^\circ) \times 2$$

$$= \text{cis}(-540^\circ)$$

$$\text{Modulus} = 1 \quad \& \quad \text{Argument} = -540^\circ$$

(5) Prove that $|z-i| = |z+i|$ represents a straight line.

$$\Rightarrow |x+iy-i| = |x+iy+i| \quad [\because z = x+iy]$$

$$\Rightarrow |x+i(y-1)| = |x+i(y+1)|$$

$$\Rightarrow \sqrt{x^2 + (y-1)^2} = \sqrt{x^2 + (y+1)^2}$$

$$\Rightarrow y^2 - 2y + 1 = y^2 + 2y + 1$$

$$\Rightarrow 4y = 0$$

$$y = 0$$

$\therefore y = 0 \quad \therefore x\text{-axis is a straight line.}$ [Proved]

(6) Prove that $|z+2i| + |z-2i| = 6$ represents an ellipse.

$$\Rightarrow |x+iy+2i| + |x+iy-2i| = 6$$

$$\Rightarrow |x+i(y+2)| + |x+i(y-2)| = 6$$

$$\Rightarrow \sqrt{x^2 + (y+2)^2} + \sqrt{x^2 + (y-2)^2} = 6$$

$$\Rightarrow \left(\sqrt{x^2 + (y+2)^2} \right)^2 = \left\{ 6 - \sqrt{x^2 + (y-2)^2} \right\}^2$$

$$\Rightarrow x^2 + (y+2)^2 = 36 - 12\sqrt{x^2 + (y-2)^2} + x^2 + (y-2)^2$$

$$\Rightarrow y^2 + 4y + 4 = 36 - 12\sqrt{x^2 + (y-2)^2} + y^2 - 4y + 4$$

$$\Rightarrow 8y = 36 - 12\sqrt{x^2 + (y-2)^2}$$

$$\Rightarrow 2y = 9 - 3\sqrt{x^2 + (y-2)^2}$$

$$\Rightarrow \left\{ 3\sqrt{x^2 + (y-2)^2} \right\}^2 = (9-2y)^2$$

$$\Rightarrow 9 \left\{ x^2 + (y-2)^2 \right\} = 81 - 36y + 4y^2$$

$$\Rightarrow 9x^2 + 9y^2 - 36y + 36 = 81 - 36y + 4y^2$$

$$\Rightarrow 9x^2 + 5y^2 = 45$$

$$\therefore \left(\frac{x^2}{5} + \frac{y^2}{9} \right) = 1$$

This an equation of ellipse.

[Proved]

(7) Find an equation of circle center at $(2, 3)$ with radius 3.

\Rightarrow We know,

$$|z - z_0| = R$$

Here,

$$R = 3$$

$$z_0 = 2 + 3i$$

$$\therefore |z - z_0| = 3$$

$$\Rightarrow |z - (2 - 3i)| = 3$$

(8) Sketch the region in x plane represented by the following set of points.

$$(1) \operatorname{Re}(\bar{z} - 1) = 2$$

Now,

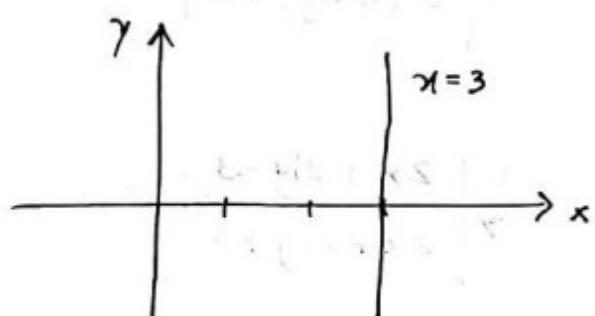
$$(\overline{x+iy} - 1) = (x - iy - 1)$$

$$= (x-1) - iy$$

$$\therefore \operatorname{Re}(\bar{z} - 1) = x - 1$$

$$\therefore x - 1 = 2$$

$$\therefore x = 3$$



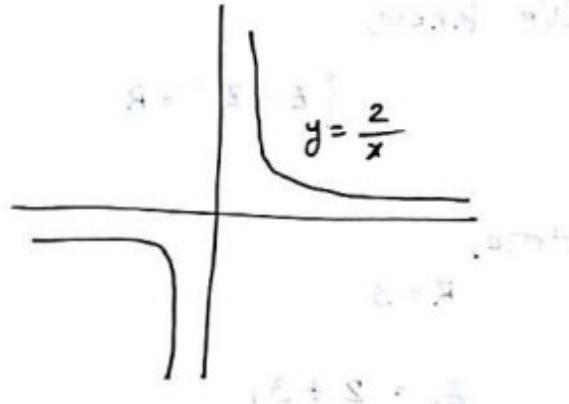
$$(ii) \operatorname{Im}(z^2) = 4$$

$$\begin{aligned} \Rightarrow z^2 &= (x+iy)^2 \\ &= x^2 + 2ixy + y^2 i^2 \\ &= (x^2 - y^2) + 2ixy \end{aligned}$$

$$\therefore \operatorname{Im}(z^2) = 2xy$$

$$\therefore 2xy = 4$$

$$\therefore y = \frac{2}{x}$$



(iii)

$$\left| \frac{2z-3}{2z+3} \right| = 1$$

$$\Rightarrow \left| \frac{2(x+iy)-3}{2(x+iy)+3} \right| = 1$$

$$\Rightarrow \left| \frac{2x+2iy-3}{2x+2iy+3} \right| = 1$$

$$\Rightarrow \left| \frac{(2x-3)+2iy}{(2x+3)+2iy} \right| = 1$$

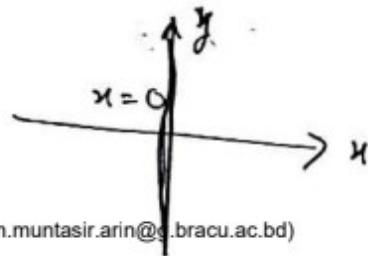
$$\Rightarrow \frac{\sqrt{(2x-3)^2 + (2y)^2}}{\sqrt{(2x+3)^2 + (2y)^2}} = 1$$

$$\Rightarrow (2x-3)^2 + 4y^2 = (2x+3)^2 + 4y^2$$

$$\Rightarrow 4x^2 - 12x + 9 = 4x^2 + 12x + 9$$

$$\Rightarrow 24x = 0$$

$$\therefore x = 0$$



$$(iv) \operatorname{Re}(z) + \operatorname{Im}(z) = 0$$

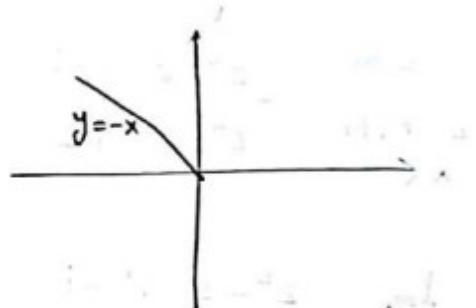
$$\Rightarrow \operatorname{Re}(z) = x$$

$$\operatorname{Im}(z) = y$$

$$\operatorname{Re}(z) + \operatorname{Im}(z) = 0$$

$$\therefore x + y = 0$$

$$\therefore y = -x$$



Practice Sheet # 2

① Evaluate the following limits:

$$(i) \lim_{z \rightarrow 1+i} \frac{z^2 - z + 1 - i}{z^2 - 2z + 2}$$

$$= \lim_{z \rightarrow 1+i} \frac{z^2 - z - i^2 - i}{z^2 - 2z + 1 + 1}$$

$$= \lim_{z \rightarrow 1+i} \frac{z^2 - i^2 - z - i}{(z-1)^2 + 1}$$

$$= \lim_{z \rightarrow 1+i} \frac{(z+i)(z-i) - (z+i)}{(z-1)^2 - i^2}$$

$$= \lim_{z \rightarrow 1+i} \frac{(z+i)(z-i-1)}{(z-1+i)(z-1-i)}$$

$$= \lim_{z \rightarrow 1+i} \frac{z+i}{z-1+i}$$

$$= \frac{1+i+i}{1+i-1+i} = \frac{1+2i}{2i}$$

$$= \frac{-i^2+2i}{-i^2 \cdot 2i} = \frac{-i(i-2)}{-i \cdot 2i^2}$$

$$(ii) \lim_{z \rightarrow 1+i} \left(\frac{z-1-i}{z^2-2z+2} \right)^2$$

$$= \lim_{z \rightarrow 1+i} \left(\frac{z-1-i}{z^2-2z+1+1} \right)^2$$

$$= \lim_{z \rightarrow 1+i} \left\{ \frac{z-1-i}{(z-1)^2 - i^2} \right\}^2$$

$$= \lim_{z \rightarrow 1+i} \left\{ \frac{(z-1-i)}{(z-1+i)(z-1-i)} \right\}^2$$

$$= \lim_{z \rightarrow 1+i} \left(\frac{1}{z-1+i} \right)^2$$

$$= \left(\frac{1}{1+i-1+i} \right)^2$$

$$= \left(\frac{1}{2i} \right)^2$$

$$= \frac{-1}{4} \quad \underline{\text{Ans}}$$

(iii)

$$\lim_{z \rightarrow i} \frac{z^2 + 1}{z^6 + 1}$$

$$= \lim_{z \rightarrow i} \frac{z^2 - i^2}{z^6 - i^6}$$

$$= \lim_{z \rightarrow i} \frac{z^2 - i^2}{(z^2)^3 - (i^2)^3}$$

$$= \lim_{z \rightarrow i} \frac{(z^2 - i^2)}{(z^2 - i^2)(z^4 + z^2 \cdot i^2 + i^4)}$$

$$= \lim_{z \rightarrow i} \frac{1}{z^4 - z^2 + 1}$$

$$= \frac{1}{i^4 - i^2 + 1}$$

$$= \frac{1}{1 + 1 + 1}$$

$$= \frac{1}{3} \quad \underline{\text{Ans.}}$$

(iv)

$$\lim_{z \rightarrow i/2} \frac{(2z-3)(4z+i)}{(iz-1)^2}$$

$$= \frac{(i-3)(2i+i)}{(i^2/2 - 1)^2}$$

$$= \frac{(i-3) \cdot 3i}{(-1/2 - 1)^2}$$

$$= \frac{3i^2 - 9i}{9/4}$$

$$= \frac{-3 - 9i}{9/4}$$

$$= \frac{-3(1+3i) \cdot 4}{9}$$

$$= -\frac{4}{3}(1+3i)$$

(2) If $f(z) = \frac{2z-1}{3z+2}$, prove that $\lim_{h \rightarrow 0} \frac{f(z_0+h)-f(z_0)}{h} = \frac{2}{(3z_0+2)^2}$

provided $z_0 \neq -\frac{2}{3}$.

$$\Rightarrow \because f(z) = \frac{2z-1}{3z+2}$$

$$\therefore f(z_0) = \frac{2z_0-1}{3z_0+2} \quad \& \quad f(z_0+h) = \frac{2(z_0+h)-1}{3(z_0+h)+2}$$

$$f(z_0+h) - f(z_0) = \frac{2(z_0+h)-1}{3(z_0+h)+2} - \frac{2z_0-1}{3z_0+2}$$

$$= \frac{(2z_0+2h-1)(3z_0+2) - (2z_0-1)(3z_0+3h+2)}{(3z_0+3h+2)(3z_0+2)}$$

$$= \frac{6z_0^2 + 4z_0 + 6hz_0 + 4h - 3z_0 - 2 - 6z_0^2 - 6hz_0 - 4z_0}{(3z_0+3h+2)(3z_0+2)}$$

$$= \frac{7h}{(3z_0+3h+2)(3z_0+2)}$$

$$\lim_{h \rightarrow 0} \frac{f(z_0+h)-f(z_0)}{h} = \lim_{h \rightarrow 0} \frac{7h}{(3z_0+3h+2)(3z_0+2)}$$

$$= \lim_{h \rightarrow 0} \frac{z}{(3z_0 + h + 2)(3z_0 + 2)}$$

$$= \frac{z}{(3z_0 + 2)(3z_0 + 2)}$$

$$= \frac{z}{(3z_0 + 2)^2} \quad [\text{Proved}]$$

Continuity

If the limit point and the function point of a function is equal then the function is continuous at that point.

$f(z_0)$ is continuous at z_0 if,

(i) $\lim_{z \rightarrow z_0} f(z)$ exist.

(ii) $f(z)$ is defined.

(iii) $\lim_{z \rightarrow z_0} f(z) = f(z_0)$

(3) Let $f(z) = \frac{z^2 + 4}{z - 2i}$ if $z \neq 2i$, while $f(2i) = 3 + 4i$, is $f(z)$ continuous at $z = 2i$.

$$\Rightarrow f(z) = \begin{cases} \frac{z^2 + 4}{z - 2i}, & z \neq 2i \\ 3 + 4i, & z = 2i \end{cases}$$

$$\begin{aligned} \lim_{z \rightarrow 2i} f(z) &= \lim_{z \rightarrow 2i} \frac{z^2 + 4}{z - 2i} \\ &= \lim_{z \rightarrow 2i} \frac{z^2 - 4i^2}{z - 2i} \\ &= \lim_{z \rightarrow 2i} \frac{(z + 2i)(z - 2i)}{(z - 2i)} \\ &= \lim_{z \rightarrow 2i} (z + 2i) \\ &= 4i \end{aligned}$$

But $f(2i) = 3 + 4i$

$$\therefore \lim_{z \rightarrow 2i} f(z) \neq f(2i)$$

$f(z)$ is not continuous at $z = 2i$

(4.) Find all points of discontinuity for the function $f(z) = \frac{2z-3}{z^2+2z+2}$

\Rightarrow The function will be discontinuous at those points where

$$(z^2 + 2z + 2) = 0$$

$$\Rightarrow z = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1}$$

$$= \frac{-2 \pm \sqrt{4-8}}{2}$$

$$= \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm \sqrt{2 \cdot 2 \cdot (-1)}}{2} = \frac{2(-1 \pm i)}{2} = (-1 \pm i)$$

Function will be discontinuous at $(-1+i)$ and $(-1-i)$ points.

Differentiability

The derivative of $f(z)$ is denoted by $f'(z)$ or $\frac{d}{dz} f(z)$ or $\frac{dw}{dz}$

and defined as,

$$f'(z) = \lim_{\Delta z \rightarrow 0} \left\{ \frac{f(z + \Delta z) - f(z)}{\Delta z} \right\}$$

If $f(z)$ is differentiable at $z = z_0$

$$f'(z) = \lim_{\Delta z \rightarrow 0} \left\{ \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \right\}$$

- ⑤ Using the definition, find the derivative of each function at the indicated points.

$$(i) f(z) = \frac{2z - i}{z + 2i} \quad \text{at } z = -i$$

\Rightarrow Here, $z_0 = -i$

$$f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$\therefore f'(-i) = \lim_{\Delta z \rightarrow 0} \frac{f(-i + \Delta z) - f(-i)}{\Delta z} \dots \dots (i)$$

Now,

$$f(-i + \Delta z) = \frac{2(-i + \Delta z) - i}{-i + \Delta z + 2i}$$

$$= \frac{-3i + 2\Delta z}{i + \Delta z} \dots \dots (ii)$$

Again,

$$f(-i) = \frac{2(-i) - i}{-i + 2i}$$

$$= \frac{-3i}{i} = -3 \dots \dots (iii)$$

Substituting the value from (ii) & (iii) in equation (i) we get,

$$f'(-i) = \lim_{\Delta z \rightarrow 0} \frac{\frac{-3i + 2\Delta z}{i + \Delta z} - (-3)}{\Delta z}$$

$$= \lim_{\Delta z \rightarrow 0} \frac{-3i + 2\Delta z + 3i + 3\Delta z}{i + \Delta z} \times \frac{1}{\Delta z}$$

$$= \lim_{\Delta z \rightarrow 0} \frac{5 \Delta z}{(i + \Delta z) \Delta z}$$

$$= \lim_{\Delta z \rightarrow 0} \frac{5}{i + \Delta z}$$

$$= \frac{5}{i} = \frac{5i}{i^2} = -5i$$

(ii) $f(z) = 3z^{-2}$ at $z = 1+i$

∴ Here,
 $z_0 = 1+i$

Now,
 $f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$

$$\therefore f'(1+i) = \lim_{\Delta z \rightarrow 0} \frac{f(1+i + \Delta z) - f(1+i)}{\Delta z} \dots \dots \text{(i)}$$

Now,

$$f(1+i + \Delta z) = 3(1+i + \Delta z)^{-2}$$

$$= \frac{3}{(1+i + \Delta z)^2} = \frac{3}{1+2i+i^2 + 2(1+i)\Delta z + \Delta z^2}$$

$$= \frac{3}{2i + 2\Delta z + 2\Delta z'i + \Delta z^2} \quad \dots \text{--- (ii)}$$

Again,

$$f(1+i) = 3(1+i)^{-2}$$

$$= \frac{3}{(1+i)^2} = \frac{3}{1+2i+i^2} = \frac{3}{2i} \quad \dots \text{--- (iii)}$$

Substituting the values from (ii) & (iii) in equation (i) we get,

~~$f'(1+i) = \text{before that } (ii) - (iii) \Rightarrow$~~

$$f(1+i + \Delta z) - f(1+i) = \frac{3}{2i + 2\Delta z + 2\Delta z'i + \Delta z^2} - \frac{3}{2i}$$

$$= \frac{6i - 6i - 6\Delta z - 6\Delta z'i - 3\Delta z^2}{2i(2i + 2\Delta z + 2\Delta z'i + \Delta z^2)}$$

$$= \frac{-3\Delta z(2 + 2i + \Delta z)}{2i(2i + 2\Delta z + 2\Delta z'i + \Delta z^2)}$$

From eqⁿ(i) we get,

$$f'(1+i) = \lim_{\Delta z \rightarrow 0} \frac{-3\Delta z(2 + 2i + \Delta z)}{2i(2i + 2\Delta z + 2\Delta z'i + \Delta z^2)} \times \frac{1}{\Delta z}$$

$$= \lim_{\Delta z \rightarrow 0} \frac{-3(2+2i+\Delta z)}{2i(2i+2\Delta z+2\Delta zi+\Delta z^2)}$$

$$= \frac{-3(2+2i)}{2i \cdot 2i}$$

$$= \frac{-6-6i}{4i^2}$$

$$= \frac{-2(3+3i)}{-4}$$

$$= \frac{3}{2}(1+i) \quad \underline{\text{Ans}}$$

⑥ Evaluate the following Limits using L' Hospital's rule.

$$(i) \lim_{z \rightarrow 2i} \frac{z^2 + 4}{2z^2 + (3-4i)z - 6i}$$

$$= \lim_{z \rightarrow 2i} \frac{2z}{4z + 3 - 4i} \quad [L' \text{ Hospital}]$$

$$= \frac{2 \cdot 2i}{4 \cdot 2i + 3 - 4i}$$

$$= \frac{4i}{8i + 3 - 4i}$$

$$= \frac{4i}{4i + 3}$$

$$= \frac{4i(4i - 3)}{(4i)^2 - 3^2}$$

$$= \frac{16i^2 - 12i}{-16 - 9}$$

$$= \frac{-16 - 12i}{-25}$$

$$= \frac{16}{25} + \frac{12}{25}i$$

$$(ii) \lim_{z \rightarrow 0} \frac{z - \sin z}{z^3}$$

$$= \lim_{z \rightarrow 0} \frac{1 - \cos z}{3z^2} \quad [L' \text{ Hospital}]$$

$$= \lim_{z \rightarrow 0} \frac{\sin z}{6z} \quad [L' \text{ Hospital}]$$

$$= \lim_{z \rightarrow 0} \frac{\cos z}{6}$$

$$= \frac{\cos 0^\circ}{6}$$

$$= \frac{1}{6} \quad \underline{\text{Ans.}}$$

Analytic Function: If $f'(z)$ exists at all points z of a region R , then $f(z)$ is said to be analytic in R .

$f'(z)$ exists
 z
 R

[$f'(z)$ is differentiable]

$f(z) = u(x, y) + i v(x, y)$ and $f'(z)$ exists at $z_0 = (x_0 + y_0)$ then 1st order partial derivatives of u and v must exist at (x_0, y_0) and they must satisfy.

$$\left. \begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array} \right\} \text{known as Cauchy - Reimann equation}$$

This is known as necessary condition.

Also $f'(z)$ can be written as at z_0 ,

$$f'(z_0) = u_x(x_0, y_0) + i v_x(x_0, y_0)$$

$\therefore f'(z_0) = (v_y - i u_y)$ which is known as sufficient condition.

Harmonic function: A real-valued function u of two real variables x and y is said to be harmonic in a given domain in the xy -plane if throughout that domain it has continuous partial derivatives of 1st & 2nd order and satisfies the partial differential equation.

$$U_{xx}(x,y) + U_{yy}(x,y) = 0 \quad \rightarrow \text{Laplace equation.}$$

- ⑦ Determine u is harmonic function. Find the conjugate harmonic function v and express $u+iv$ as an analytic function of z .

(i) $\underline{u = 3x^2y + 2x^2 - y^3 - 2y^2}$

\Rightarrow Now,

$$u_x = 6xy + 4x$$

$$u_{xx} = 6y + 4$$

$$\therefore U_{xx} + U_{yy} = 6y + 4 - 6y - 4 = 0$$

$\therefore u$ is a harmonic function.

Again,

$$u_y = 3x^2 - 3y^2 - 4y$$

$$u_{yy} = -6y - 4$$

Now,

$$v_x = (6xy + 4x) = v_y \dots \text{(i)}$$

$$v_y = (3x^2 - 3y^2 - 4y) = -v_x \dots \text{(ii)}$$

Keeping x constant & integrating both sides of eqn(i) with respect to y we get,

$$\int v_y dy = \int (6xy + 4x) dy$$

$$\Rightarrow v = 6x \frac{y^2}{2} + 4xy + \phi(x)$$

$$\therefore v = 3xy^2 + 4xy + \phi(x) \dots \text{(iii)}$$

Differentiating eqn (iii) with respect to x we get,

$$v_x = 3y^2 + 4y + \phi'(x)$$

$$\Rightarrow -(3x^2 - 3y^2 - 4y) = 3y^2 + 4y + \phi'(x) \quad [\text{From eqn(ii)}]$$

$$\Rightarrow -3x^2 + 3y^2 + 4y = 3y^2 + 4y + \phi'(x)$$

$$\Rightarrow \phi'(x) = -3x^2$$

$$\Rightarrow \int \phi'(x) dx = -3 \int x^2 dx$$

$$\therefore \phi(x) = \frac{x^3}{3}$$

Substituting the value of $\phi(x)$ in eqⁿ (iii) we get,

$$V = 3xy^2 + 4xy - x^3 + c$$

This is the conjugate harmonic function.

And analytic function of z is,

$$f(z) = (u + iv)$$

$$= (3x^2y + 2x^2 - y^3 - 2y^2) + i(3xy^2 + 4xy - x^3 + c)$$

(ii) $u = xe^x \cos y - ye^x \sin y$

Now,

$$u_x = e^x \cos y + xe^x \cos y - ye^x \sin y$$

$$u_{xx} = e^x \cos y + e^x \cos y + xe^x \cos y - ye^x \sin y$$

Again,

$$u_y = -xe^x \sin y - e^x \sin y - ye^x \cos y$$

$$u_{yy} = -xe^x \cos y - e^x \cos y - e^x \cos y + ye^x \sin y$$

$$U_{xx} + V_{yy} = 2e^x \cos y + xe^x \cos y - ye^x \sin y - xe^x \cos y \\ - 2e^x \cos y + ye^x \sin y$$

$$\therefore U_{xx} + V_{yy} = 0$$

$\therefore u$ is harmonic function.

Now,

$$u_x = e^x \cos y + xe^x \cos y - ye^x \sin y = V_y \dots (i)$$

$$u_y = -xe^x \sin y - e^x \sin y - ye^x \cos y = -V_x \dots (ii)$$

By integrating both sides of equation (i) with respect to y ,

$$\int V_y dy = \int (e^x \cos y + xe^x \cos y - ye^x \sin y) dy$$

$$\Rightarrow V = e^x \sin y + xe^x \sin y - e^x \left[y \int \sin y dy - \int \left(\frac{d}{dy}(y) \right) \sin y dy \right] dy$$

$$= e^x \sin y + xe^x \sin y - e^x [-y \cos y - \int -\cos y dy]$$

$$= e^x \sin y + xe^x \sin y - e^x (-y \cos y + \sin y) + \phi(x)$$

$$= e^x \sin y + xe^x \sin y + e^x y \cos y - \cancel{e^x \sin y} + \phi(x) \dots (iii)$$

$$= xe^x \sin y + e^x \cos y + \phi(x) \dots$$

Differentiating equation (iii) with respect to x we get,

$$V_x = xe^x \sin y + e^x \sin y + e^x y \cos y + \phi'(x)$$

$$\Rightarrow xe^x \sin y + e^x \sin y + ye^x \cos y = xe^x \sin y + e^x \sin y + e^x y \cos y + \phi'(x)$$

[From eqⁿ (ii)]

$$\Rightarrow \phi'(x) = 0$$

$$\Rightarrow \int \phi'(x) = \int 0 \cdot dx$$

$$\Rightarrow \phi(x) = C$$

Substituting the value of $\phi(x)$ in eqⁿ (iii) we get,

$V = xe^x \sin y + y \cdot e^x \cos y + C$; which is the conjugate harmonic

Analytic function, $f(z) = (u+iv)$

$$= xe^x \cos y - ye^x \sin y + i(xe^x \sin y + ye^x \cos y + C)$$

(iii) $u = e^{-x} (x \sin y - y \cos y)$

$$\Rightarrow u = e^{-x} x \sin y - e^{-x} y \cos y$$

$$u_x = -e^{-x} x \sin y + e^{-x} \sin y + e^{-x} y \cos y$$

$$u_{xx} = e^{-x} x \sin y - e^{-x} \sin y - e^{-x} \sin y - e^{-x} y \cos y$$

$$u_y = e^{-x} x \cos y - e^{-x} \cos y + e^{-x} y \sin y$$

$$u_{yy} = -e^{-x} x \sin y + e^{-x} \sin y + e^{-x} \sin y + e^{-x} y \cos y$$

$$\begin{aligned} \therefore u_{xx} + u_{yy} &= x e^{-x} \sin y - y \cdot e^{-x} \cos y - 2 e^{-x} \sin y - e^{-x} x \sin y \\ &\quad + 2 e^{-x} \sin y + e^{-x} y \cos y \end{aligned}$$

$$\therefore u_{xx} + u_{yy} = 0$$

$\therefore u$ is harmonic.

Now,

$$u_x = -e^{-x} x \sin y + e^{-x} \sin y + e^{-x} y \cos y = -V_y \dots \dots \text{(i)}$$

$$u_y = e^{-x} x \cos y - e^{-x} \cos y + e^{-x} y \sin y = V_x \dots \dots \text{(ii)}$$

Integrating equation (i) with respect to y , we get,

$$\int V_y \, dy = \int (-e^{-x} x \sin y + e^{-x} \sin y + e^{-x} y \cos y) \, dy$$

$$\Rightarrow V = x e^{-x} \cos y - e^{-x} \cos y + e^{-x} \left[y \int \cos y \, dy - \int \left\{ \frac{d}{dy}(y) \right\} \cos y \, dy \right] \, dy$$

$$= x e^{-x} \cos y - e^{-x} \cos y + e^{-x} (y \sin y - \int \sin y \, dy)$$

$$= xe^{-x} \cos y - e^{-x} \cos y + e^{-x} y \sin y + e^{-x} \cos y + \phi(x)$$

$$\therefore V = xe^{-x} \cos y + e^{-x} y \sin y + \phi(x) \quad \text{--- (iii)}$$

Differentiating (iii) with respect to x we get,

$$V_x = e^{-x} \cos y - xe^{-x} \cos y - e^{-x} y \sin y + \phi'(x)$$

$$\Rightarrow -xe^{-x} \cos y + e^{-x} \cos y - ye^{-x} \sin y = e^{-x} \cos y - xe^{-x} \cos y$$

$$[\text{From eqn (ii)}] \quad -e^{-x} y \sin y + \phi'(x)$$

$$\Rightarrow \phi'(x) = 0$$

$$\Rightarrow \int \phi'(x) dx = \int 0 dx$$

$$\Rightarrow \phi(x) = C$$

Substituting the value of $\phi(x)$ in equation (iii) we get,

$V = xe^{-x} \cos y + e^{-x} y \sin y + C$; which is the conjugate harmonic function

Analytic function, $f(z) = (u + iv)$

$$= (e^{-x} y \sin y - e^{-x} y \cos y) + i(xe^{-x} \cos y$$

- * Singular point: A point at which $f(z)$ fails to be analytic is called singular point or singularity of $f(z)$.
- * Isolated singularities: The function $f(z)$ has an isolated singularity at z_0 , if we can find $\delta > 0$, such that the circle $|z - z_0| = \delta$ enclosed no singular point of $f(z)$ other than z_0 .
- * Poles: If z_0 is an isolated singularity then we can find a positive integer n such that,

$$\lim_{z \rightarrow z_0} (z - z_0)^n f(z) = A \neq 0$$

then z_0 is called a pole of order n . If $n=1$, z_0 is called a simple pole.

⑧ For the following function locate and name the singularities in the finite z-plane and determine whether they are isolated singularities or not.

$$(a) f(z) = \frac{z}{(z^2 + 4)^2}$$

Sol: Here, $(z^2 + 4)^2 = 0$

$$\Rightarrow z^2 - 4i^2 = 0$$

$$\Rightarrow z^2 - (2i)^2 = 0$$

$$\Rightarrow (z+2i)(z-2i) = 0$$

$$\therefore z = 2i, -2i$$

\therefore Isolate singular points are $2i, -2i$

Here, $n = 2$

$$\lim_{z \rightarrow z_0} (z - z_0)^n f(z)$$

When $z_0 = 2i$

$$\lim_{z \rightarrow 2i} (z - 2i)^2 \frac{z}{(z^2 + 4)^2}$$

$$= \lim_{z \rightarrow 2i} (z - 2i)^2 \frac{z}{(z^2 - 4i^2)^2}$$

$$= \lim_{z \rightarrow 2i} (z - 2i)^2 \frac{z}{(z + 2i)^2(z - 2i)^2}$$

$$= \lim_{z \rightarrow 2i} \frac{z}{(z + 2i)^2}$$

$$= \frac{2i}{(2i + 2i)^2}$$

$$= \frac{2i}{(4i)^2}$$

$$= \frac{2i}{-16}$$

$$= \frac{1}{8i} \neq 0$$

$\therefore z = 2i$ is a pole of order 2.

When $z_0 = -2i$

$$\lim_{z \rightarrow -2i} (z + 2i)^2 \frac{z}{(z^2 + 4)^2}$$

$$= \lim_{z \rightarrow -2i} (z + 2i)^2 \frac{z}{(z^2 - 4i^2)^2}$$

$$= \lim_{z \rightarrow -2i} (z + 2i)^2 \frac{z}{\{(z + 2i)(z - 2i)\}^2}$$

$$= \lim_{z \rightarrow -2i} \frac{z}{(z - 2i)^2}$$

$$= \frac{-2i}{(-2i - 2i)^2}$$

$$= \frac{-2i}{(-4i)^2}$$

$$= \frac{-2i}{16i}$$

$$= \frac{-1}{8i} \neq 0$$

$\therefore z = -2i$ is a pole of order 2.

$$(b) f(z) = \frac{\ln(z-2)}{(z^2+2z+4)^4}$$

Now,

$$(z^2+2z+4)^4=0$$

$$\Rightarrow z^2 + 2z + 4 = 0$$

$$\Rightarrow z = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 4}}{2 \cdot 1}$$

$$= \frac{-2 \pm \sqrt{4-16}}{2}$$

$$= \frac{-2 \pm \sqrt{-12}}{2}$$

$$= \frac{-2 \pm 2\sqrt{-3}}{2}$$

$$= -1 \pm \sqrt{3}i;$$

Again,

$$\ln(0) = \text{undefined}$$

$$\therefore \ln(z-2) = 0$$

$$\Rightarrow z-2=0$$

$$\therefore z=2$$

Isolated singular points are $(-1+\sqrt{3}i)$, $(-1-\sqrt{3}i)$, $z=2$.

Hence, $n=4$

$$\lim_{z \rightarrow z_0} (z-z_0)^n f(z) = \lim_{z \rightarrow (-1+\sqrt{3}i)} (z+1-\sqrt{3}i)^4 \frac{\ln(z-2)}{(z^2+2z+4)^4}$$

$$= \lim_{z \rightarrow (-1+\sqrt{3}i)} \left\{ z + 1 - \sqrt{3}i \right\}^4 \frac{\ln(z-2)}{\left[\left\{ z - (-1+\sqrt{3}i) \right\} \left\{ z - (-1-\sqrt{3}i) \right\} \right]^4}$$

$$= \lim_{z \rightarrow (-1+\sqrt{3}i)} (z + 1 - \sqrt{3}i)^4 \frac{\ln(z-2)}{(z+1-\sqrt{3}i)^4 (z+1+\sqrt{3}i)^4}$$

$$= \lim_{z \rightarrow (-1+\sqrt{3}i)} \frac{\ln(z-2)}{(z+1+\sqrt{3}i)^4}$$

$$= \frac{\ln(-1+\sqrt{3}i-2)}{(-1+\sqrt{3}i+1+\sqrt{3}i)^4}$$

$$= \frac{\ln(\sqrt{3}i-3)}{(2\sqrt{3}i)^4} \neq 0$$

$\therefore z = (-1+\sqrt{3}i)$ is a pole of order 4.

Again, here, $n = 4$

$$= \lim_{z \rightarrow (-1 - \sqrt{3}i)} \frac{\{z - (-1 - \sqrt{3}i)\}^4}{\frac{\ln(z-2)}{\{z - (-1 + \sqrt{3}i)\} \{z - (-1 - \sqrt{3}i)\}^4}}$$

$$= \lim_{z \rightarrow (-1 - \sqrt{3}i)} \frac{(z+1+\sqrt{3}i)^4}{(z+1-\sqrt{3}i)^4} \frac{\ln(z-2)}{(z+1+\sqrt{3}i)^4}$$

$$= \lim_{z \rightarrow (-1 - \sqrt{3}i)} \frac{\ln(z-2)}{(z+1-\sqrt{3}i)^4}$$

$$= \frac{\ln(-1 - \sqrt{3}i - 2)}{(-1 - \sqrt{3}i + 1 - \sqrt{3}i)^4}$$

$$= \frac{\ln(-3 - \sqrt{3}i)}{(-2\sqrt{3}i)^4} \neq 0$$

$\therefore z = (-1 - \sqrt{3}i)$ is a pole of order 4.

Ans.

Exponential function of Complex variable

$$f(z) = e^z = e^{(x+iy)}$$

$$= e^x \cdot e^{iy}$$

$$= e^x (\cos y + i \sin y) \quad [\because (\cos \theta + i \sin \theta) = e^{i\theta}]$$

$$\therefore |f(z)| = |e^z| = |e^x (\cos y + i \sin y)|$$

$$= |e^x| \cdot |\cos y + i \sin y|$$

$$= e^x \sqrt{\cos^2 y + \sin^2 y}$$

$$= e^x \cdot 1$$

$$\therefore |f(z)| = e^x > 0 \quad \text{for all } x \in \mathbb{R}$$

$$e^{(z+2\pi i)} = e^z \cdot e^{2\pi i}$$

$$= e^z (\cos 2\pi + i \sin 2\pi)$$

$$= e^z (1+0)$$

$$= e^z$$

$$z = r \cdot e^{i\theta}$$

$$\therefore z = r \cdot e^{i(\theta + 2\pi n)}$$

Practice Sheet # 3

(1) Show that :

$$(i) \exp(z \pm 3\pi i) = -e^z$$

Now,

$$\begin{aligned} e^{z+3\pi i} &= e^z \cdot e^{3\pi i} \\ &= e^z (\cos 3\pi + i \sin 3\pi) \\ &= e^z (-1+0) \\ &= -e^z \end{aligned}$$

Again,

$$\begin{aligned} e^{z-3\pi i} &= e^z \cdot e^{-3\pi i} \\ &= -e^z \end{aligned}$$

[Showed]

$$(ii) \exp\left(\frac{z+\pi i}{4}\right) = \sqrt{\frac{e}{2}}(1+i)$$

$$(iii) \exp(z + \pi i) = -e^z$$

Now,

$$\begin{aligned} e^{\frac{z+\pi i}{4}} &= e^{1/2} \cdot e^{\pi i/4} \\ &= e^{1/2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right) \\ &= e^{1/2} \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}}\right) \\ &= \frac{\sqrt{e}}{\sqrt{2}} (1+i) \end{aligned}$$

Now,

$$\begin{aligned} e^{z+\pi i} &= e^z \cdot e^{\pi i} \\ &= e^z (\cos \pi + i \sin \pi) \\ &= e^z (-1+0) \\ &= -e^z \end{aligned}$$

[Showed]

(2) Find all the values of z such that:

$$(i) e^z = -2$$

$$\text{Here, } x = -2 \quad \& \quad y = 0$$

$$r = \sqrt{(-2)^2 + 0} = 2$$

$$\theta = \tan^{-1}(0/-2)$$

$$= \tan^{-1}(0)$$

$$= \tan^{-1} \tan(\pi)$$

$$= \pi$$

$$\therefore e^z = r \cdot e^{i(\theta + 2\pi n)}$$

$$= 2 e^{i(2\pi n + \pi)}$$

$$\Rightarrow \ln e^z = \ln \{2 e^{i(2\pi n + \pi)}\}$$

$$\Rightarrow z = \ln 2 + \ln e^{(2\pi n + \pi)i}$$

$$= \ln 2 + (2\pi n + \pi)i$$

$$= \ln 2 + (2n+1)\pi i$$

Ans.

$$(ii) e^z = 1 + \sqrt{3}i$$

$$\text{Here, } x = 1 \quad \& \quad y = \sqrt{3}$$

$$r = \sqrt{1^2 + (\sqrt{3})^2} = 2$$

$$\theta = \tan^{-1}(\sqrt{3}/1)$$

$$= \tan^{-1} \tan\left(\frac{\pi}{3}\right)$$

$$= \frac{\pi}{3}$$

$$\therefore (1 + \sqrt{3}i) = r \cdot e^{i(\theta + 2\pi n)}$$

$$\therefore e^z = 2 e^{i\left(\frac{\pi}{3} + 2\pi n\right)}$$

$$\Rightarrow \ln e^z = \ln \left\{ 2 e^{i\left(\frac{\pi}{3} + 2\pi n\right)} \right\}$$

$$\Rightarrow z = \ln 2 + \ln e^{i\left(\frac{\pi}{3} + 2\pi n\right)}$$

$$= \ln 2 + i\left(\frac{\pi}{3} + 2\pi n\right); n=0,\pm 1,\pm 2\dots$$

Ans.

$$(iii) e^{2z-1} = 1$$

Here,

$$x=1 \quad \& \quad y=0$$

$$r = \sqrt{1^2 + 0^2} = 1$$

$$\theta = \tan^{-1}(0/1)$$

$$= \tan^{-1}(0)$$

$$= 0$$

$$\therefore 1 = r \cdot e^{i(\theta + 2\pi n)}$$

$$\therefore e^{2z-1} = 1 \cdot e^{i(0+2\pi n)}$$

$$\Rightarrow \ln e^{2z-1} = \ln(e^{2\pi ni})$$

$$\Rightarrow 2z-1 = 2\pi ni$$

$$\Rightarrow 2z = 2\pi ni + 1$$

$$\therefore z = \frac{1}{2} + i\pi n$$

Ans.

$$(iv) e^z = -1$$

Here, $x = -1 \quad \& \quad y = 0$

$$r = \sqrt{(-1)^2 + 0^2} = 1$$

$$\theta = \tan^{-1}(0/-1)$$

$$= \tan^{-1} \tan(\pi)$$

$$= \pi$$

$$\therefore -1 = r \cdot e^{i(\theta + 2\pi n)}$$

$$\therefore e^z = 1 \cdot e^{i(\pi + 2\pi n)}$$

$$\Rightarrow \ln e^z = \ln \left\{ e^{i(\pi + 2\pi n)} \right\}$$

$$\Rightarrow z = i(\pi + 2\pi n)$$

$$\therefore z = (2n+1)\pi i$$

Ans.

Trigonometric Functions

$$e^{\pm 2\pi i} = 1$$

Here,

$$\begin{aligned} e^{iz} &= \cos z + i \sin z & ; & e^{-iz} = \cos z - i \sin z \\ \therefore e^{iz} + e^{-iz} &= 2 \cos z & \therefore e^{iz} - e^{-iz} = 2i \sin z \\ \therefore \cos z &= \frac{1}{2}(e^{iz} + e^{-iz}) & \therefore \sin z &= \frac{1}{2i}(e^{iz} - e^{-iz}) \end{aligned}$$

③ Prove that:

$$(i) \sin(z+2\pi) = \sin z$$

$$\text{L.H.S.} = \sin(z+2\pi)$$

$$= \frac{1}{2i} \left\{ e^{i(z+2\pi)} - e^{-i(z+2\pi)} \right\}$$

$$= \frac{1}{2i} \left\{ e^{iz} \cdot e^{2\pi i} - e^{-iz} \cdot e^{-2\pi i} \right\}$$

$$= \frac{1}{2i} (e^{iz} \cdot 1 - e^{-iz} \cdot 1)$$

$$= \sin z$$

$$= \text{R.H.S.}$$

$$(ii) \cos(z+2\pi) = \cos z$$

$$\text{L.H.S.} = \cos(z+2\pi)$$

$$= \frac{1}{2} \left\{ e^{i(z+2\pi)} + e^{-i(z+2\pi)} \right\}$$

$$= \frac{1}{2} \left\{ e^{iz} \cdot e^{2\pi i} + e^{-iz} \cdot e^{-2\pi i} \right\}$$

$$= \frac{1}{2} (e^{iz} + e^{-iz})$$

$$= \cos z$$

$$= \text{R.H.S.}$$

Hyperbolic Function

$$1. \cosh z = \frac{1}{2}(e^z + e^{-z})$$

$$4. \frac{d}{dz}(\sinh z) = \cosh z$$

$$2. \sinh z = \frac{1}{2}(e^z - e^{-z})$$

$$5. \frac{d}{dz}(\cosh z) = \sinh z$$

$$3. \cosh^2 z - \sinh^2 z = 1$$

* Prove that:

$$(a) \sin z = (\sin x \cdot \cosh y + i \cos x \cdot \sinh y)$$

$$(b) \cos z = (\cos x \cdot \cosh y - i \sin x \cdot \sinh y)$$

Sol: We know, $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$

$$\therefore \cos z = \frac{1}{2}(e^{iz} + e^{-iz})$$

$$\therefore \sin iz = \frac{1}{2i}(e^{i^2 z} - e^{-i^2 z})$$

$$\therefore \cos iz = \frac{1}{2}(e^{i^2 z} + e^{-i^2 z})$$

$$= \frac{1}{2i}(e^{-z} - e^z)$$

$$= \frac{1}{2}(e^{-z} + e^z)$$

$$= \frac{-1}{2i}(e^z - e^{-z})$$

$$= \frac{1}{2}(e^z + e^{-z})$$

$$= \frac{-i}{2}(e^z - e^{-z})$$

$$= \cosh z \dots \dots \text{(ii)}$$

$$= \frac{-i}{2}(e^z - e^{-z})$$

$$= \cosh z \dots \dots \text{(ii)}$$

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 $= i \frac{1}{2}(e^z - e^{-z}) = i \sinh z \quad \text{(i)}$

$$(a) L.H.S. = \sin z$$

$$= \sin(x+iy)$$

$$= \sin x \cos iy + \cos x \sin iy$$

$$= \sin x \cdot \cosh y + \cos x \cdot i \sinh y$$

$$= \sin x \cdot \cosh y + i \cos x \cdot \sinh y$$

$$= R.H.S$$

$$(b) L.H.S. = \cos z$$

$$= \cos(x+iy)$$

$$= \cos x \cdot \cos iy - \sin x \cdot \sin iy$$

$$= \cos x \cdot \cosh y - \sin x \cdot i \sinh y$$

$$= \cos x \cdot \cosh y - i \sin x \cdot \sinh y$$

$$= R.H.S.$$

[Proved]

$$4(i) L.H.S. = \sin hz$$

$$= -i \sin iz \quad [*]$$

$$= -i \sin i(x+iy)$$

$$= -i \sin(ix-y)$$

$$= -i(\sin ix \cdot \cos y - \cos ix \cdot \sin y)$$

$$= -i \sin ix \cdot \cos y + i \cos ix \cdot \sin y$$

$$= -i^2 \sinh x \cdot \cos y + i \cosh x \cdot \sin y$$

$$= \sinh x \cdot \cos y + i \cosh x \cdot \sin y$$

$$= R.H.S.$$

$$4(ii) L.H.S. = \cos hz$$

$$= \cos iz$$

$$= \cos i(x+iy)$$

$$= \cos(ix-y)$$

$$= \cos ix \cdot \cos y + \sin ix \cdot \sin y$$

$$= \cos hx \cdot \cos y + i \sinhx \cdot \sin y$$

$$= R.H.S.$$

$$[*] \because i \sinh hx = \sin iz$$

$$\therefore i^2 \sinh hx = i \sin iz$$

$$\Rightarrow -\sinh hx = i \sin iz$$

$$\therefore \sinh hx = -i \sin iz$$

= -i

5.

Show that:

$$(a) \sin^{-1} z = -i \ln [iz \pm (1-z^2)^{1/2}]$$

$$(b) \cos^{-1} z = -i \ln [z \pm i(1-z^2)^{1/2}]$$

$$\Rightarrow \text{Let, } u = \sin^{-1} z$$

$$\Rightarrow z = \sin u$$

$$\Rightarrow z = \frac{1}{2i} (e^{iu} - e^{-iu})$$

$$\Rightarrow 2iz = e^{iu} - e^{-iu}$$

$$\Rightarrow 2iz \cdot e^{iu} = e^{2iu} - 1$$

$$\Rightarrow e^{2iu} - 2ze^{iu} - 1 = 0$$

$$\Rightarrow e^{iu} = \frac{-(-2iz) \pm \sqrt{(-2iz)^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1}$$

$$= \frac{2iz \pm \sqrt{4i^2 z^2 + 4}}{2}$$

$$= \frac{2iz \pm 2\sqrt{-z^2 + 1}}{2}$$

$$= iz \pm \sqrt{1-z^2}$$

$$\Rightarrow \ln e^{iu} = \ln (iz \pm \sqrt{1-z^2})$$

$$\Rightarrow iu = \ln (iz \pm \sqrt{1-z^2}) \quad [*]$$

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$$\text{Let, } u = \cos^{-1} z$$

$$\Rightarrow z = \cos u$$

$$\Rightarrow z = \frac{1}{2}(e^{iu} + e^{-iu})$$

$$\Rightarrow 2z = e^{iu} + e^{-iu}$$

$$\Rightarrow 2z \cdot e^{iu} = e^{2iu} + 1$$

$$\Rightarrow (e^{iu})^2 - 2ze^{iu} + 1 = 0$$

$$\Rightarrow e^{iu} = \frac{-(-2z) \pm \sqrt{(-2z)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1}$$

$$= \frac{2z \pm \sqrt{4z^2 - 4}}{2}$$

$$= \frac{2z \pm 2\sqrt{z^2 - 1}}{2}$$

$$= \frac{z \pm \sqrt{-i^2 z^2 + i^2}}{2}$$

$$= z \pm i\sqrt{1-z^2}$$

$$\Rightarrow \ln e^{iu} = \ln (z \pm i\sqrt{1-z^2})$$

$$\Rightarrow iu = \ln (z \pm i\sqrt{1-z^2})$$

(6) Solve the equation:

$$(i) \cosh z = \frac{1}{2}$$

$$\Rightarrow \frac{1}{2}(e^z + e^{-z}) = \frac{1}{2}$$

$$\Rightarrow e^z + e^{-z} = 1$$

$$\Rightarrow e^z \cdot e^z + e^{-z} \cdot e^z = e^z$$

$$\Rightarrow e^{2z} + 1 = e^z$$

$$\Rightarrow (e^z)^2 - e^z + 1 = 0$$

Now,

$$e^z = \frac{1}{2} + \frac{\sqrt{3}}{2} i$$

$$|z| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}/2}{1/2}\right)$$

$$= \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$$

$$\therefore \frac{1}{2} + \frac{\sqrt{3}}{2} i = r e^{i(\theta + 2\pi n)}$$

$$\Rightarrow e^z = 1 \cdot e^{i(\frac{\pi}{3} + 2\pi n)}$$

$$\therefore z = i\left(\frac{\pi}{3} + 2\pi n\right)$$

$$\therefore e^z = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1}$$

$$= \frac{1 \pm \sqrt{1-4}}{2}$$

$$= \frac{1 \pm \sqrt{3}i}{2}$$

$$= \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$\text{Again, } e^z = \frac{1}{2} - \frac{\sqrt{3}}{2} i$$

$$|z| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

$$\theta = \tan^{-1}\left(\frac{-\sqrt{3}/2}{1/2}\right)$$

$$= \tan^{-1}(-\sqrt{3}) = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$$

$$\therefore \frac{1}{2} - \frac{\sqrt{3}}{2} i = r e^{i(\theta + 2\pi n)}$$

$$\therefore e^z = 1 \cdot e^{i\left(\frac{5\pi}{3} + 2\pi n\right)}$$

$$\therefore z = i\left(\frac{5\pi}{3} + 2\pi n\right); n = 0, \pm 1, \pm 2, \dots$$

Ans.

$$(6) \text{ (ii)} \quad \sin hz = i$$

$$\Rightarrow \frac{1}{2}(e^z - e^{-z}) = i$$

$$\Rightarrow e^z - e^{-z} = 2i$$

$$\Rightarrow (e^z)^2 - 1 = 2ie^z$$

$$\Rightarrow (e^z)^2 - 2ie^z - 1 = 0$$

$$\therefore r = \sqrt{0^2 + 1^2} = 1$$

$$\therefore \theta = \tan^{-1}(1/0)$$

$$= \tan^{-1} \alpha$$

$$= \tan^{-1} \tan(\pi/2)$$

$$= \frac{\pi}{2}$$

$$\therefore z = r \cdot e^{i(\theta + 2\pi n)}$$

$$\therefore e^z = 1 \cdot e^{i(\frac{\pi}{2} + 2\pi n)}$$

$$\therefore z = i\left(\frac{\pi}{2} + 2\pi n\right) ; \quad n = 0, \pm 1, \pm 2, \pm 3, \dots$$

Now,

$$e^z = \frac{-(-2i) \pm \sqrt{(-2i)^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1}$$

$$= \frac{2i \pm \sqrt{4i^2 + 4}}{2}$$

$$= \frac{2i \pm 2\sqrt{-1+1}}{2} = i$$

(7)

Show that:

Logarithmic Function

$$(i) \ln(-1 + \sqrt{3}i) = \ln 2 + 2\left(n + \frac{1}{3}\right)\pi i \quad (ii) \ln(1-i) = \frac{1}{2}\ln 2 + \left(2n + \frac{7}{4}\right)\pi i$$

Sol: From $(-1 + \sqrt{3}i)$

$$x = -1 \quad \& \quad y = \sqrt{3}$$

$$\therefore r = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right)$$

$$= \tan^{-1}(-\sqrt{3})$$

$$= \tan^{-1} \tan\left(\pi - \frac{\pi}{3}\right) = \frac{2\pi}{3}$$

$$\therefore (-1 + \sqrt{3}i) = r e^{i(\theta + 2\pi n)}$$

$$\Rightarrow (-1 + \sqrt{3}i) = 2 e^{i\left(\frac{2\pi}{3} + 2\pi n\right)}$$

$$\Rightarrow \ln(-1 + \sqrt{3}i) = \ln 2 + \ln e^{i\left(\frac{2\pi}{3} + 2\pi n\right)}$$

$$= \ln 2 + 2\left(n + \frac{1}{3}\right)\pi i$$

Sol: From $(1-i)$

$$x = 1 \quad \& \quad y = -1$$

$$\therefore r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$\theta = \tan^{-1}(-1)$$

$$= \tan^{-1}(-1)$$

$$= \tan^{-1} \tan\left(2\pi - \frac{\pi}{4}\right) = \frac{7\pi}{4}$$

$$\therefore (1-i) = r e^{i(\theta + 2\pi n)}$$

$$\Rightarrow (1-i) = \sqrt{2} e^{i\left(\frac{7\pi}{4} + 2\pi n\right)}$$

$$\Rightarrow \ln(1-i) = \ln 2^{1/2} + \ln e^{\left(\frac{7\pi}{4} + 2\pi n\right)}\pi i$$

$$= \frac{1}{2}\ln 2 + \left(2n + \frac{7}{4}\right)\pi i$$

[Showed]

[Showed]

$$(iii) \ln(i^{1/2}) = \left(n + \frac{1}{4}\right)\pi i$$

Sol: From $i^{1/2}$, $x=0$ & $y=1$

$$\therefore r = \sqrt{0^2 + 1^2} = 1$$

$$\begin{aligned}\therefore \theta &= \tan^{-1}(1/0) = \tan^{-1} \infty \\ &= \tan^{-1}(\tan \pi/2) = \frac{\pi}{2}\end{aligned}$$

$$\therefore i = r \cdot e^{i(\theta + 2\pi n)}$$

$$\therefore i = 1 \cdot e^{i(\frac{\pi}{2} + 2\pi n)}$$

$$\Rightarrow \ln i = \ln e^{(2n + \frac{1}{2})\pi i}$$

$$\Rightarrow \ln i = (2n + \frac{1}{2})\pi i$$

$$\Rightarrow \frac{1}{2} \ln i = \frac{1}{2} (2n + \frac{1}{2})\pi i$$

$$\therefore \ln(i^{1/2}) = \left(n + \frac{1}{4}\right)\pi i$$

[shown]

Practice Sheet # 4

(1) Evaluate $\int_{(0,1)}^{(2,5)} (3x+y)dx + (2y-x)dy$ along

(a) The curve $y = x^2 + 1$

$$\text{Sol: } \Rightarrow dy = 2x dx$$

$$\begin{aligned} & \int_0^2 [(3x + x^2 + 1)dx + 2x(2x^2 + 2 - x)dx] \\ &= \left[\frac{3x^2}{2} + \frac{x^3}{3} + x + \frac{4x^4}{4} + \frac{4x^2}{2} - \frac{2x^3}{3} \right]_0^2 \\ &= \frac{3 \cdot 4}{2} + \frac{8}{3} + 2 + 16 + 8 - \frac{2 \cdot 8}{3} = \frac{88}{3} \quad \underline{\text{Ans.}} \end{aligned}$$

(b) The straight line joining $(0,1)$ and $(2,5)$

$$\Rightarrow \text{Equation of the straight line } \frac{x-0}{0-2} = \frac{y-1}{1-5}$$

$$\Rightarrow \frac{x}{-2} = \frac{y-1}{-4}$$

$$\Rightarrow 4x = 2y - 2$$

$$\Rightarrow y = 2x + 1$$

$$\therefore dy = 2dx$$

$$= \int_0^2 [(3x + 2x + 1) dx + (4x + 2 - x) 2 dx]$$

$$= \left[\frac{3x^2}{2} + x^2 + x + \frac{2 \cdot 3x^2}{2} + 2x \right]_0^2$$

$$= \frac{3 \times 4}{2} + 4 + 2 + \frac{3 \times 4}{2} + 4 \times 2 = 32$$

(c) The straight line from $(0, 1)$ to $(0, 5)$ and then from $(0, 5)$ to $(2, 5)$

\Rightarrow From $(0, 1)$ to $(0, 5)$

$$x = 0$$

$$\therefore dx = 0$$

$$\int_1^5 0 + (2y - 0) dy$$

$$= \left[\frac{2y^2}{2} \right]_1^5$$

$$= (25 - 1) = 24$$

Along $(0, 5)$ to $(2, 5)$

$$y = 5$$

$$\therefore dy = 0$$

$$\int_0^2 (3x + 5) dx + 0$$

$$= \left[\frac{3x^2}{2} + 5x \right]_0^2$$

$$= \frac{3}{2} \cdot 4 + 10 = 16$$

The required value is $= 24 + 16 = 40$

(d) The straight lines from $(0,1)$ to $(2,1)$ and then from $(2,1)$ to $(2,5)$

⇒ Along $(0,1)$ to $(2,1)$

$$y = 1$$

$$\therefore dy = 0$$

Along $(2,1)$ to $(2,5)$

$$x = 2$$

$$\therefore dx = 0$$

$$\int_0^2 (3x+1) dx + 0$$

$$= \left[\frac{3x^2}{2} + x \right]_0^2$$

$$= \frac{3}{2} \cdot 4 \times 2 = 8$$

$$\int_1^5 0 + (2y - 2) dy$$

$$= [y^2 - 2y]_1^5$$

$$= (25 - 10) - (1 - 2) = 16$$

The required value is $= 8 + 16 = 24$ Ans.

- ② Evaluate $\oint_C (x+2y)dx + (y-2x)dy$ around the ellipse C defined by $x = 4\cos\theta$, $y = 3\sin\theta$, $0 \leq \theta \leq 2\pi$ if C is described in a clockwise direction.

Sol ⇒ Given,

$$x = 4\cos\theta$$

$$dx = -4\sin\theta d\theta$$

$$\& \quad y = 3\sin\theta$$

$$\therefore dy = 3\cos\theta d\theta$$

$$\begin{aligned}
 & \oint_C (x+2y)dx + (y-2x)dy \\
 &= \int_0^{2\pi} \left[(4\cos\theta + 6\sin\theta)(-4\sin\theta d\theta) + (3\sin\theta - 8\cos\theta)(3\cos\theta d\theta) \right] \\
 &= \int_0^{2\pi} (-16\sin\theta\cos\theta d\theta - 24\sin^2\theta d\theta + 9\sin\theta\cos\theta d\theta - 24\cos^2\theta d\theta) \\
 &= \int_0^{2\pi} (-7\sin\theta\cos\theta d\theta - 24 d\theta) \\
 &= -7\sin 2\pi \cos 0 - 24 \int_0^{2\pi} d\theta - \frac{7}{2} \int_0^{2\pi} 2\sin\theta\cos\theta d\theta \\
 &= -24[\theta]_0^{2\pi} - \frac{7}{2} \int_0^{2\pi} (\sin 2\theta) d\theta \\
 &= -48\pi - \frac{7}{2} \left[\frac{\sin 4\theta}{2} \right]_0^{2\pi} \\
 &= -48\pi - \frac{7}{2} (\cos 4\pi - \cos 0) \\
 &= -48\pi - \frac{7}{2} (1-1) \\
 &= -48\pi
 \end{aligned}$$

Ans.

(3) Evaluate $\int_C (x^2 - iy^2) dz$ along

(a) the parabola $y = 2x^2$ from $(1,1)$ to $(2,8)$

$$\Rightarrow \text{Here, } y = 2x^2$$

$$\therefore dy = 4x dx$$

$$\text{And, } z = x + iy$$

$$dz = dx + i dy = dx + 4ix dx$$

$$\int_1^2 (x^2 - i4x^4)(dx + 4ix dx)$$

$$= \int_1^2 (x^2 dx + 4ix^3 dx - 4ix^4 dx - 16i^2 x^5 dx)$$

$$= \left[\frac{x^3}{3} + ix^4 - \frac{4ix^5}{5} + \frac{16x^6}{6} \right]_1^2$$

$$= \left(\frac{8}{3} + 16i - \frac{128i}{5} + \frac{416}{3} \right) - \left(\frac{1}{3} + i - \frac{4i}{5} + \frac{8}{3} \right)$$

$$= \underline{\frac{511}{3}} - \underline{\frac{49i}{5}} \quad \text{Ans}$$

(b) The straight line $(1,1)$ to $(1,8)$ then $(1,8)$ to $(2,8)$

\Rightarrow Along $(1,1)$ to $(1,8)$

$$x = 1$$

$$\therefore dx = 0$$

And,

$$z = x + iy$$

$$dz = dx + idy = idy$$

$$\int_1^8 (1 - iy^2) idy$$

$$= \int_1^8 (i - i^2 y^2) dy$$

$$= \left[iy + \frac{y^3}{3} \right]_1^8$$

$$= 8i + \frac{512}{3} - i - \frac{1}{3}$$

$$= 7i + \frac{511}{3}$$

Along $(1,8)$ to $(2,8)$

$$y = 8$$

$$dy = 0$$

$$\therefore dz = dx$$

$$\int_2^1 (x^2 - i64) dx$$

$$= \left[\frac{x^3}{3} - i64x \right]_2^1$$

$$= \left(\frac{1}{3} - 64i \right) - \left(\frac{8}{3} - 128i \right) = \frac{7}{3} - 64i$$

The required line value is $7i + \frac{511}{3} + \frac{7}{3} - 64i = \frac{518}{3} - 57i$

Ans.

(c) the straight line from $(1,1)$ to $(2,8)$

$$\Rightarrow \text{Equation of the line } \frac{x-1}{1-2} = \frac{y-1}{1-8}$$

$$\Rightarrow \frac{x-1}{-1} = \frac{y-1}{-7}$$

$$\Rightarrow 7x - 7 = y - 1$$

$$\Rightarrow y = 7x - 6$$

$$dy = 7 dx$$

$$\int_1^2 \left\{ x^2 - i(7x-6)^2 \right\} (1+7i) dx$$

$$= (1+7i) \int_1^2 (x^2 - 49ix^2 + 84x - 36i) dx$$

$$= (1+7i) \left[\frac{x^3}{3} - \frac{49x^3i}{3} + \frac{42x^2i}{2} - 36ix \right]_1^2$$

$$= (1+7i) \left[\left(\frac{8}{3} - \frac{392i}{3} + \frac{424i}{2} - 72i \right) - \left(\frac{1}{3} - \frac{49i}{3} + \frac{42i}{2} - 36i \right) \right]$$

$$= (1+7i) \left(\frac{7}{3} - \frac{73i}{3} \right)$$

$$= \frac{7}{3} - \frac{73i}{3} + \frac{49i}{3} - \frac{511i}{3} = \frac{518}{3} - 8i$$

Ans.

Again,

$$z = x + iy$$

$$dz = dx + i dy$$

$$= dx + i 7dx$$

$$= (1+7i) dx$$

(4) Evaluate $\oint_C |z|^2 dz$ around the square with vertices at

$(0,0), (1,0), (1,1), (0,1)$

$$z = \sqrt{x^2 + y^2} \quad |z|^2 = x^2 + y^2$$

$$dz = dx + i dy$$

Sol \Rightarrow Along $(0,0)$ to $(1,0)$

$$y=0$$

$$\therefore dy=0$$

$$\int_0^1 (x^2 + 0) dx \\ = \left[\frac{x^3}{3} \right]_0^1 = \frac{1}{3}$$

Along $(1,0)$ to $(1,1)$

$$x=1$$

$$\therefore dx=0$$

$$\int_0^1 (1+y^2) i dy \\ = \left[iy - \frac{iy^3}{3} \right]_0^1 = i + \frac{i}{3} = \frac{4i}{3}$$

Along $(1,1)$ to $(0,1)$

$$y=1$$

$$dy=0$$

$$\int_0^1 (x^2 + 1) dx \\ = \left[\frac{x^3}{3} + x \right]_0^1 = -\left(\frac{1}{3} + 1\right) = -\frac{4}{3}$$

Along $(0,1)$ to $(0,0)$

$$x=0$$

$$\therefore dx=0$$

$$\int_1^0 (0+y^2) i dy \\ = i \left[y^3 / 3 \right]_1^0 = -i \left[\frac{1}{3} \right] = -\frac{i}{3}$$

$$\text{The required value} = \frac{1}{3} + \frac{4i}{3} - \frac{4}{3} - \frac{i}{3} = (i-1) \underline{\text{Ans}}$$

(5) (a) Evaluate $\int_C (z^2 + 3z) dz$ along the circle $|z|=2$ from $(2,0)$ to $(0,2)$ in a counterclockwise direction.

$$\text{Sol} \Rightarrow \text{Equation of the line } \frac{x-2}{2-0} = \frac{y-0}{0-2}$$

Again,
 $z = x + iy$

$$\Rightarrow -x + 2 = y$$

$$\Rightarrow dy = -dx$$

$$dz = dx + i dy$$

$$= dx - i dx$$

$$= (1-i) dx$$

$$\begin{aligned} & \int_2^0 \{(x+iy)^2 + 3(x+iy)\} (1-i) dx \\ &= (1-i) \int_2^0 (x^2 + 2ixy - y^2 + 3x + 3iy) dx \\ &= (1-i) \int_2^0 \{x^2 + 3x + 2ix(-x+2) - (-x+2)^2 + 3i(-x+2)\} dx \\ &= -(1-i) \int_0^2 (x^2 + 3x - 2ix^2 + 4ix - 4 + 4x - x^2 - 3ix + 6i) dx \\ &= -(1-i) \left[\frac{x^3}{3} + \frac{3x^2}{2} - \frac{2ix^3}{3} + 2ix^2 - 4x + 2x^2 - \frac{x^3}{3} - \frac{3ix^2}{2} + 6ix \right]_0^2 \\ &= -(1-i) \left(\frac{8}{3} + 6 - \frac{16i}{3} + 8i - 8 + 8 - \frac{8}{3} - 6i + 12i \right) \\ &= -(1-i) (6 + \frac{26i}{3}) = -\frac{4}{3} (11 + 2i) \quad \underline{\text{Ans.}} \end{aligned}$$

(5)(b) Evaluate $\int_C (z^2 + 3z) dz$ along the straight line from $(2,0)$ to $(2,2)$
 and then from $(2,2)$ to $(0,2)$. $dz = dx + i dy$

Sol \Rightarrow Along $(2,0)$ to $(2,2)$

$$x = 2$$

$$dx = 0$$

$$\int_0^2 \{(x+iy)^2 + 3(x+iy)\} i dy$$

$$= \int_0^2 (4 + 4iy + iy^2 + 6 + 3iy) i dy$$

$$= [4y + 2y^2 - \frac{iy^3}{3} + 6y + \frac{3iy^2}{2}]_0^2$$

$$= 8i + 8i - \frac{8i}{3} + 12i + 6i - 6i$$

$$= i \int_0^2 (4 + 4iy + iy^2 + 6 + 3iy) dy$$

$$= i \int_0^2 (10y + 7iy - y^2) dy$$

$$= i \left[10y + \frac{7iy^2}{2} - \frac{y^3}{3} \right]_0^2$$

$$= 20i + 14i^2 - \frac{8i}{3}$$

$$= \frac{52i}{3}$$

Along $(2,2)$ to $(0,2)$

$$y = 2$$

$$dy = 0$$

$$\int_2^0 \{(x+iy)^2 + 3(x+iy)\} dx$$

$$= - \int_0^2 (x^2 + 4ix + 4i^2 + 3x + 6i) dx$$

$$= - \left[\frac{x^3}{3} + 2ix^2 - 4x + \frac{3x^2}{2} + 6ix \right]_0^2$$

$$= - \left[\frac{8}{3} + 8i - 8 + 6 + 12i \right]$$

$$= - \left[\frac{2}{3} + 20i \right]$$

\therefore The required value =

$$= \frac{52i}{3} - 14 - \frac{2}{3} - 20i = -\frac{44}{3} - \frac{8i}{3}$$

Ans.

⑥ (a) Evaluate $\int\limits_i^{2-i} (3xy + iy^2) dz$ along the straight line joining

$$z = i \quad \text{and} \quad z = 2-i$$

$$\text{Sol} \Rightarrow \text{Here, } z_1 = i = 0 + 1 \cdot i \Rightarrow (0, 1)$$

$$z_2 = 2-i = 2 + i(-1); (2, -1)$$

And,

$$z = x + iy$$

$$dz = dx + idy$$

$$= -dy + idy = (i-1)dy$$

Along $(0, 1)$ to $(2, -1)$

$$\frac{x-0}{0-2} = \frac{y-1}{1+1} \Rightarrow 2x = -2(y-1)$$

$$\Rightarrow x = -y + 1$$

$$\Rightarrow y = 1-x$$

$$\Rightarrow dy = -dx \Rightarrow dx = -dy$$

$$\int\limits_i^{-1} \{3y(-y+1) + i(y^2)\} (i-1) dy$$

$$= \int\limits_1^{-1} (3y^2 - 3y + iy^2) (i-1) dy$$

$$= (i-1) \left[-y^3 + \frac{3y^2}{2} + \frac{iy^3}{3} \right]_1^{-1}$$

$$= (i-1) \left[\left\{ -(-1) + \frac{3}{2} - \frac{i}{3} \right\} - \left(-1 + \frac{3}{2} + \frac{i}{3} \right) \right]$$

$$= (i-1) \left(2 - \frac{2i}{3} \right) = \frac{8i}{3} - \frac{4}{3}$$

⑦(а) Evaluate $\oint (\bar{z})^2 dz$ around the circles $|z|=1$

$$\text{Sol} \Rightarrow |z|=1$$

$$\Rightarrow z = e^{i\theta}$$

$$\Rightarrow dz = ie^{i\theta} d\theta$$

$$\int_0^{2\pi} (e^{-2i\theta}) ie^{i\theta} d\theta$$

$$= i \int_0^{2\pi} e^{i\theta - 2i\theta} d\theta$$

$$= i \int_0^{2\pi} e^{-i\theta} d\theta$$

$$= i \left[\frac{e^{-i\theta}}{-i} \right]_0^{2\pi}$$

$$= \frac{i}{-i} [e^{-i2\pi} - e^0]$$

$$= -1 (1 - 1)$$

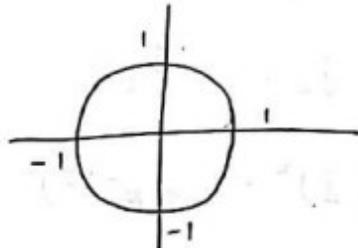
$$= 0$$

Ans.

$$\therefore z = e^{i\theta}$$

$$\therefore \bar{z} = e^{-i\theta}$$

$$\therefore (\bar{z})^2 = (e^{-i\theta})^2 = e^{-2i\theta}$$



[Limit of circle is always
0 to 2π]

(7)(b) Evaluate $\oint_C (\bar{z})^2 dz$ around the circles $|z-1|=1$.

$$\Rightarrow \text{Here, } |z-1|=1$$

$$\Rightarrow z-1 = e^{i\theta}$$

$$\Rightarrow z = 1 + e^{i\theta}$$

$$\bar{z} = 1 + e^{-i\theta}$$

$$dz = ie^{i\theta} d\theta$$

$$\begin{aligned}(\bar{z})^2 &= (1 + e^{-i\theta})^2 \\&= 1 + 2e^{-i\theta} + e^{-2i\theta}\end{aligned}$$

$$\begin{aligned}&\int_0^{2\pi} (1 + 2e^{-i\theta} + e^{-2i\theta}) ie^{i\theta} d\theta \\&= i \int_0^{2\pi} (e^{i\theta} + 2e^{i\theta-i\theta} + e^{i\theta-2i\theta}) d\theta \\&= i \int_0^{2\pi} (e^{i\theta} + 2 + e^{-i\theta}) d\theta \\&= i \left[\frac{e^{i\theta}}{i} + 2\theta + \frac{e^{-i\theta}}{-i} \right]_0^{2\pi} \\&= \left(e^{2\pi i} + 2\pi i + -e^{-2\pi i} \right) - \left(e^0 + 0 - e^0 \right) = (1 + 4\pi i - 1) - (1 - 1)\end{aligned}$$

⑧ Evaluate $\oint_C \frac{dz}{z-2}$ around (a) the circle $|z-2|=4$, (b) $|z-1|=9$

$$\text{Sol: } (a) |z-2|=4$$

$$\Rightarrow z-2 = 4e^{i\theta}$$

$$\Rightarrow z = 4e^{i\theta} + 2$$

$$\therefore dz = 4ie^{i\theta} d\theta$$

$$\int_0^{2\pi} \frac{4ie^{i\theta} d\theta}{4e^{i\theta}}$$

$$= i \int_0^{2\pi} e^{i\theta-i\theta} d\theta$$

$$= i \int_0^{2\pi} d\theta$$

$$= i [\theta]_0^{2\pi}$$

$$= 2\pi i$$

Ans.

$$(b) |z-1|=9$$

$$\Rightarrow z-1 = 9e^{i\theta}$$

$$\Rightarrow z = 9e^{i\theta} + 1$$

$$\Rightarrow dz = 9ie^{i\theta} d\theta$$

$$\int_0^{2\pi} \frac{9ie^{i\theta} d\theta}{9e^{i\theta} + 1 - 2}$$

$$= \int_0^{2\pi} \frac{9ie^{i\theta} d\theta}{9e^{i\theta} - 1}$$

$$= \int_0^{2\pi} \frac{du}{u}$$

$$= [\ln u]_0^8$$

$$= \ln 8 - \ln 0 = 0$$

$$\text{Let, } 9e^{i\theta} - 1 = u$$

$$\therefore du = 9ie^{i\theta} d\theta$$

$$\text{If } \theta=0, \text{ then, } u=9-1=8$$

$$\text{If } \theta=2\pi \text{ then, } u=9-1=8$$

Ans.

(9) Evaluate $\oint_C (5z^4 - z^3 + 2) dz$ around the circle $|z|=1$.

\Rightarrow Here, $|z|=1$

$$z = e^{i\theta}$$

$$dz = ie^{i\theta} d\theta$$

$$\int_0^{2\pi} (5e^{4i\theta} - e^{3i\theta} + 2) ie^{i\theta} d\theta$$

$$= i \int_0^{2\pi} (5e^{5i\theta} - e^{4i\theta} + 2e^{i\theta}) d\theta$$

$$= i \left[\frac{5e^{5i\theta}}{5i} - \frac{e^{4i\theta}}{4i} + \frac{2e^{i\theta}}{i} \right]_0^{2\pi}$$

$$= \left[e^{5i\theta} - \frac{e^{4i\theta}}{4} + 2e^{i\theta} \right]_0^{2\pi}$$

$$= e^{10i} \left[\cos 5\theta + i \sin 5\theta \right]_0^{2\pi} - \frac{1}{4} \left[\cos 4\theta + i \sin 4\theta \right]_0^{2\pi} + 2 \left[\cos \theta + i \sin \theta \right]_0^{2\pi}$$

$$= (1+0-1+0) - \frac{1}{4} (1+0-1+0) + 2 (1+0-1+0)$$

$$= 0 - 0 + 0$$

$$= 0$$

Ans.

Cauchy's Integral Formula

① Evaluate (a) $\oint \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$ where C is the circle $|z|=3$

Sol. Now,

$$\frac{1}{(z-1)(z-2)} = \left(\frac{1}{z-2} - \frac{1}{z-1} \right)$$

$$\begin{aligned} \therefore \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz &= \oint_C \left[\frac{1}{z-2} - \frac{1}{z-1} \right] (\sin \pi z^2 + \cos \pi z^2) dz \\ &= \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2} dz - \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1} dz \end{aligned}$$

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = 2\pi i f(2) - 2\pi i f(1) \dots \dots (i)$$

Now,
 $f(z) = \sin(\pi z^2) + \cos(\pi z^2)$

$$\begin{aligned} \therefore f(2) &= \sin 4\pi + \cos 4\pi & \& f(1) &= \sin \pi + \cos \pi \\ &= 0 + 1 = 1 & \& &= 0 - 1 = -1 \end{aligned}$$

From eqⁿ (i) \Rightarrow

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = 2\pi i - 2\pi i(-1) = 4\pi i$$

Ans.

Formula: $f^n(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz$

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(1) (b) Evaluate $\oint_C \frac{e^{2z}}{(z+1)^4} dz$ where C is the circle $|z|=3$

Sol: Here, $(n+1)=4$ and $f(z)=e^{2z}$ $\therefore z_0 = -1$

We know, Cauchy's integral formula,

$$f^n(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz$$

$$\Rightarrow f^3(z_0) = \frac{3!}{2\pi i} \oint_C \frac{e^{2z}}{(z+1)^4} dz$$

$$\Rightarrow \oint_C \frac{e^{2z}}{(z+1)^4} dz = \frac{2\pi i}{3!} f^3(-1)$$

$$= \frac{2\pi i}{3!} \cdot 8e^{-2}$$

$$= \frac{8\pi i e^{-2}}{3} \quad \underline{\text{Ans.}}$$

(3) Evaluate $\oint_C \frac{e^{3z}}{z-\pi i} dz$ where C is the circle $|z-1|=4$.

Sol: Here, $z_0 = \pi i$

$$f(z) = e^{3z} \quad \& \quad f(z_0) = e^{3\pi i} = \cos(3\pi) + i \sin(3\pi) \\ = -1 + 0$$

$$= -1$$

We know, Cauchy's Integral Formula,

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)} dz$$

$$\Rightarrow (-1) = \frac{1}{2\pi i} \oint_C \frac{e^{3z}}{(z - \pi i)} dz$$

$$\therefore \oint_C \frac{e^{3z}}{z - \pi i} dz = -2\pi i$$

Ams.

(4) Evaluate $\frac{1}{2\pi i} \oint_C \frac{e^{zt}}{(z^2 + 1)^2} dz$ if $t > 0$ and C is the circle $|z| = 3$

Sol: Here, $(n+1) = 2 \quad \therefore n = 1$

$$\frac{1}{z^2 + 1} = \frac{1}{z^2 - i^2} = \frac{1}{(z-i)(z+i)} = \frac{1}{2i} \left[\frac{1}{z-i} - \frac{1}{z+i} \right]$$

$$\frac{1}{2\pi i} \oint_C \frac{e^{zt}}{z^2 + 1} dz = \frac{1}{2\pi i} \oint_C \frac{1}{2i} \left[\frac{1}{z-i} - \frac{1}{z+i} \right] e^{zt} dz$$

$$= \frac{1}{4\pi i^2} \left[\oint_C \frac{e^{zt}}{z-i} dz - \oint_C \frac{e^{zt}}{z+i} dz \right]$$

$$= \frac{1}{4\pi i^2} \left[2\pi i f(i) - 2\pi i f(-i) \right] = \frac{1}{2i} [f(i) - f(-i)]$$

Now,

$$\frac{1}{2\pi i} \oint \frac{e^{zt}}{(z^2+1)^{1+1}} dz = \frac{1}{2i} [f'(i) - f'(-i)] \dots \text{(i)}$$

$$\therefore f(z) = e^{zt}$$

$$\therefore f'(z) = t e^{zt}$$

$$\therefore f'(i) = t e^{it} \quad \& \quad f'(-i) = t e^{-it}$$

From equation (i) we get,

$$\begin{aligned}\frac{1}{2\pi i} \oint \frac{e^{zt}}{(z^2+1)^2} dz &= \frac{1}{2i} [t e^{it} - t e^{-it}] \\ &= \frac{t}{2i} [(cost + isint) - (cost - isint)] \\ &= \frac{t}{2i} \cdot 2isint \\ &= t \cdot sint\end{aligned}$$

Ans.

Practice Sheet # 5

- ① Expand each of the following functions in a Taylor series about the indicated points.

(i) e^{-z} at $z = 0$

$$\Rightarrow \begin{array}{ll} f(z) = e^{-z} & f(0) = e^{-0} = 1 \\ f'(z) = -e^{-z} & f'(0) = -e^{-0} = -1 \\ f''(z) = e^{-z} & f''(0) = e^{-0} = 1 \\ f'''(z) = -e^{-z} & f'''(0) = -e^{-0} = -1 \\ f^{iv}(z) = e^{-z} & f^{iv}(0) = e^{-0} = 1 \end{array}$$

According to Taylor's theorem,

$$f(z) = f(0) + \frac{f'(0)}{1!}(z-0) + \frac{f''(0)}{2!}(z-0)^2 + \dots + \frac{f^{iv}(0)}{n!}(z-0)^n + \dots$$

$$\therefore f(z) = f(0) + f'(0)(z-0) + \frac{f''(0)}{2!}(z-0)^2 + \frac{f'''(0)}{3!}(z-0)^3 + \frac{f^{iv}(0)}{4!}(z-0)^4$$

$$= 1 + (-1)z + \frac{1}{2!} \cdot z^2 + \frac{-1}{3!} \cdot z^3 + \frac{1}{4!} z^4$$

$$= 1 + (-1) \frac{z}{1!} + \frac{z^2}{2!} + (-1) \frac{z^3}{3!} + \frac{z^4}{4!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} (-1)^n$$

$$(ii) \cos z \quad \text{at} \quad z = \frac{\pi}{2}$$

Sol:

$$f(z) = \cos z$$

$$f\left(\frac{\pi}{2}\right) = \cos \frac{\pi}{2} = 0$$

$$f'(z) = -\sin z$$

$$f'\left(\frac{\pi}{2}\right) = -\sin \frac{\pi}{2} = -1$$

$$f''(z) = -\cos z$$

$$f''\left(\frac{\pi}{2}\right) = -\cos \frac{\pi}{2} = 0$$

$$f'''(z) = \sin z$$

$$f'''\left(\frac{\pi}{2}\right) = \sin \frac{\pi}{2} = 1$$

$$f^{iv}(z) = \cos z$$

$$f^{iv}\left(\frac{\pi}{2}\right) = \cos \frac{\pi}{2} = 0$$

$$f^v(z) = -\sin z$$

$$f^v\left(\frac{\pi}{2}\right) = -\sin \frac{\pi}{2} = -1$$

According to Taylor series,

$$f(z) = f\left(\frac{\pi}{2}\right) + \frac{f'\left(\frac{\pi}{2}\right)}{1!} \left(z - \frac{\pi}{2}\right) + \frac{f''\left(\frac{\pi}{2}\right)}{2!} \left(z - \frac{\pi}{2}\right)^2 + \frac{f'''\left(\frac{\pi}{2}\right)}{3!} \left(z - \frac{\pi}{2}\right)^3$$

$$+ \frac{f^{iv}\left(\frac{\pi}{2}\right)}{4!} \left(z - \frac{\pi}{2}\right)^4 + \frac{f^v\left(\frac{\pi}{2}\right)}{5!} \left(z - \frac{\pi}{2}\right)^5$$

$$= 0 + (-1) \left(z - \frac{\pi}{2}\right) + 0 + \frac{1}{3!} \left(z - \frac{\pi}{2}\right)^3 + 0 + \frac{(-1)}{5!} \left(z - \frac{\pi}{2}\right)^5$$

$$= (-1) \frac{1}{1!} \left(z - \frac{\pi}{2}\right)^1 + (-1)^2 \cdot \frac{1}{3!} \left(z - \frac{\pi}{2}\right)^3 + (-1)^3 \cdot \frac{1}{5!} \left(z - \frac{\pi}{2}\right)^5$$

$$= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} (-1)^{n+1} \left(z - \frac{\pi}{2}\right)^{2n+1}$$

$$(iii) z^3 - z^2 + 4z + 2 \quad \text{at } z=2$$

$$\underline{\text{Sol:}} \quad f(z) = z^3 - z^2 + 4z + 2$$

$$f'(z) = 3z^2 - 2z + 4$$

$$f''(z) = 6z - 2$$

$$f'''(z) = 6$$

$$f^{iv}(z) = 0$$

$$f(2) = 14$$

$$f'(2) = 12$$

$$f''(2) = 10$$

$$f'''(2) = 6$$

$$f^{iv}(2) = 0$$

From Taylor Series,

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \dots + \frac{f^n(a)}{n!}(z-a)^n + \dots$$

$$\therefore f(z) = f(2) + f'(2)(z-2) + \frac{f''(2)}{2!}(z-2)^2 + \frac{f'''(2)}{3!}(z-2)^3 + \frac{f^{iv}(2)}{4!}(z-2)^4$$

$$= 14 + 12(z-2) + \frac{10}{2!}(z-2)^2 + \frac{6}{3!}(z-2)^3 + 0$$

$$= 14 + 12(z-2) + 5(z-2)^2 + (z-2)^3$$

Ans.

(iv.) ze^{2z} at $z = -1$

Sol: $f(z) = ze^{2z}$

$$f'(z) = z \cdot 2e^{2z} + e^{2z} \cdot 1$$

$$\begin{aligned} f''(z) &= 4ze^{2z} + 2e^{2z} + 2e^{2z} \\ &= 4(ze^{2z} + e^{2z}) \end{aligned}$$

$$\begin{aligned} f'''(z) &= 4(2ze^{2z} + 2e^{2z} + 2e^{2z}) \\ &= 8ze^{2z} + 12e^{2z} \end{aligned}$$

$$f(-1) = -e^{-2}$$

$$f'(-1) = -2e^{-2} + e^{-2} = -e^{-2}$$

$$f''(-1) = 0$$

$$\begin{aligned} f'''(-1) &= -8e^{-2} + 12e^{-2} \\ &= 4e^{-2} \end{aligned}$$

From Taylor's Series,

$$f(z) = f(-1) + f'(-1)(z+1) + \frac{f''(-1)}{2!}(z+1)^2 + \frac{f'''(-1)}{3!}(z+1)^3$$

$$= -e^{-2} - e^{-2}(z+1) + 0 + \frac{4e^{-2}}{3!}(z+1)^3$$

$$= e^{-2}(-1 - (z+1) + \frac{4}{3!}(z+1)^3)$$

Ans.

(2) Expand $f(z) = \frac{z}{(z-1)(2-z)}$ in a Laurent series valid for

$$(i) |z| < 1$$

Sol:

$$\frac{z}{(z-1)(2-z)} = \frac{A}{z-1} + \frac{B}{2-z}$$

$$\Rightarrow z = A(2-z) + B(z-1)$$

$$\text{When, } z = 1$$

$$1 = A(2-1) + B(1-1)$$

$$\therefore A = 1$$

$$\text{When, } z = 2$$

$$2 = A(2-2) + B(2-1)$$

$$\therefore B = 2$$

$$\therefore f(z) = \frac{1}{z-1} + \frac{2}{2-z} = \frac{1}{-(z+1)} + \frac{2}{z(1-\frac{z}{2})}$$

$$= -(1-z)^{-1} + (1-\frac{z}{2})^{-1}$$

$$= -[1 + z + z^2 + z^3 + \dots] + \left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \dots\right]$$

$$= -[z + z^2 + z^3 + \dots] + \left[\frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \dots\right]$$

$$= \sum_{n=0}^{\infty} (-1) z^n + \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n = \sum_{n=0}^{\infty} \left(-1 + \frac{1}{2^n}\right) z^n$$

$$(ii) |z| < 1 & |z| < 2$$

Sol: Here, $|z| > 1$

$$\Rightarrow \frac{1}{|z|} < 1$$

$$\Rightarrow \frac{|z|}{2} < 1$$

$$f(z) = \frac{1}{z-1} + \frac{2}{2-z} = \frac{1}{z(1-1/z)} + \frac{2}{2(1-z/2)}$$

$$= \frac{1}{z} \left(1 - \frac{1}{z}\right)^{-1} + \left(1 - \frac{z}{2}\right)^{-1}$$

$$= \frac{1}{z} \left[1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \dots \right] + \left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \dots \right]$$

$$= \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n + \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n$$

Ans.

$$(iii) |z| > 2$$

Sol: $f(z) = \frac{1}{z-1} + \frac{2}{2-z} = \frac{1}{z(1-1/z)} + \frac{2}{z(1-z/2)(2/z-1)}$

$$= \frac{1}{z} \left(1 - \frac{1}{z}\right)^{-1} + \frac{2}{z} \left(1 - \frac{2}{z}\right)^{-1}$$

Here,

$$\frac{1}{|z|} < \frac{1}{2}$$

$$\therefore \frac{2}{|z|} < 1$$

$$= \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n - \frac{2}{z} \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n$$

$$(iv) |z-1| > 1$$

Sol: Let, $z-1 = w$
 $\Rightarrow w+1 = z$

$$f(z) = \frac{1}{z-1} + \frac{2}{2-z}$$

$$\begin{aligned} f(w) &= \frac{1}{w+1-1} + \frac{2}{2-w-1} \\ &= \frac{1}{w} + \frac{2}{1-w} \end{aligned}$$

$$= \frac{1}{w} + \frac{-2}{w(1-1/w)}$$

$$= \frac{1}{w} - \frac{2}{w} \sum_{n=0}^{\infty} \left(\frac{1}{w}\right)^n$$

$$\therefore f(z) = \frac{1}{z-1} - \frac{2}{z-1} \sum_{n=0}^{\infty} \left(\frac{1}{z-1}\right)^n$$

Ans.

(3) Expand $f(z) = \frac{1}{z(z-2)}$ in a Laurent series valid for

(i) $0 < |z| < 2$

& (ii) $|z| > 2$

$$(v) 0 < |z-2| < 1$$

Sol: Let, $z-2 = w$
 $\therefore z = w+2$

$$f(z) = \frac{1}{z-1} + \frac{2}{2-z}$$

$$f(w) = \frac{1}{w+2-1} + \frac{2}{2-w-2}$$

$$= \frac{1}{w+1} + \frac{2}{-w}$$

$$= \sum_{n=0}^{\infty} (-w)^n - \frac{2}{w}$$

$$\therefore f(z) = \sum_{n=0}^{\infty} (-z+2)^n - \frac{2}{z-2}$$

Ans.

$$\text{Sol: } \frac{1}{z(z-2)} = \frac{A}{z} + \frac{B}{z-2}$$

$$\Rightarrow 1 = A(z-2) + Bz$$

When, $z=0$

$$1 = -2A$$

$$\therefore A = -\frac{1}{2}$$

$$(i) f(z) = \frac{1}{z(z-2)} = \frac{-1/2}{z} + \frac{1}{z-2}$$

$$= \frac{1}{2z-4} - \frac{1}{2z}$$

$$= \frac{1}{-4(1-z/2)} - \frac{1}{2z}$$

$$= \frac{-1}{4} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n - \frac{1}{2z}$$

Ans.

When, $z=2$

$$1 = 0 + 2B$$

$$\therefore B = \frac{1}{2}$$

$$(ii) f(z) = \frac{-1}{2z} + \frac{1}{2z-4}$$

$$= \frac{1}{2z(1-2/z)} - \frac{1}{2z}$$

$$= \frac{1}{2z} \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n - \frac{1}{2z}$$

Ans.

④ Evaluate $\oint_C \frac{z^2}{2z^2 + 5z + 2} dz$ using the residue at the poles where C is the unit circle $|z|=1$.

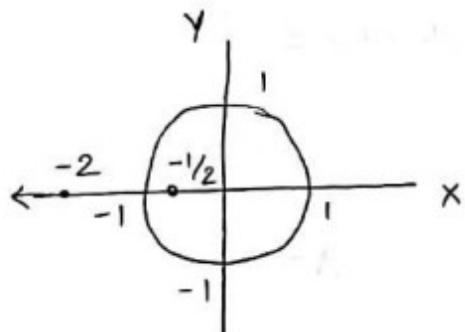
Sol: Here,

$$2z^2 + 5z + 2 = 0$$

$$\Rightarrow 2z^2 + 4z + z + 2 = 0$$

$$\Rightarrow 2z(z+2) + 1(z+2) = 0$$

$$\Rightarrow (z+2)(2z+1) = 0$$



$$\therefore |z| = 1$$

$$\therefore z+2=0 \quad \text{on, } 2z+1=0$$

$$\therefore z = -2 \quad \therefore z = -\frac{1}{2}$$

$\therefore z = -\frac{1}{2}$ is valid for $|z|=1$.

\therefore Residue at $z = -\frac{1}{2}$

Here, $a = -\frac{1}{2}$ & $m = 1$

According to Cauchy's - Residue Theorem,

$$\frac{1}{2\pi i} \oint_C f(z) dz = \lim_{m \rightarrow \infty} \frac{1}{(m-1)!} \cdot \frac{d^{m-1}}{dz^{m-1}} \left\{ (z-a)^m f(z) \right\}$$

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$$\therefore R = \lim_{z \rightarrow -1/2} \frac{1}{0!} \cdot 1 \cdot \left\{ \left(z + \frac{1}{2}\right)^1 \frac{z^2}{(z+2)(z+\frac{1}{2})} \right\}$$

$$= \lim_{z \rightarrow -1/2} \frac{z^2}{z+2}$$

$$= \frac{1/4}{3/2}$$

$$= \frac{1}{4} \times \frac{2}{3} = \frac{1}{6}$$

$$\therefore \oint_C \frac{z^2}{2z^2 + 5z + 2} dz = 2\pi i \cdot R$$

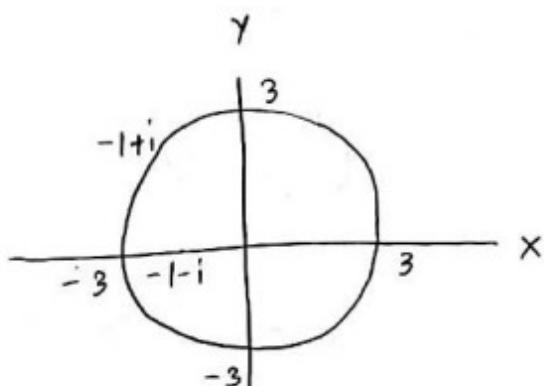
$$= 2\pi i \times \frac{1}{6} = \frac{\pi i}{3} \quad \underline{\text{Ans.}}$$

(5) Evaluate $\oint_C \frac{z^2 + 4}{z^3 + 2z^2 + 2z} dz$ using the residue at the poles, around the circle $|z| = 3$.

Sol: Here,

$$z^3 + 2z^2 + 2z = 0$$

$$\Rightarrow z(z^2 + 2z + 2) = 0$$



$$\Rightarrow z(z^2 + 2z + 1 + 1) = 0$$

$$\Rightarrow z \{ (z+1)^2 + 1 \} = 0$$

$$\Rightarrow z \{ (z+1)^2 - i^2 \} = 0$$

$$\Rightarrow z(z+1+i)(z+1-i) = 0$$

$$\therefore z = 0, (-1-i), (-1+i)$$

Since all poles are inside the circle, so all are valid at $|z|=3$

Residue at $z=0$

$$\text{Here, } a = 0 \quad \& \quad m = 1$$

$$\therefore R_1 = \lim_{z \rightarrow 0} \frac{1}{0!} \frac{d^{m+1}}{dz^{m+1}} \{ (z-a)^m f(z) \}$$

$$= \lim_{z \rightarrow 0} 1 \cdot 1 \cdot (z-0) \frac{z^2 + 4}{z(z+1+i)(z+1-i)}$$

$$= \frac{0+4}{(1+i)(1-i)}$$

$$= \frac{4}{1-i^2} = \frac{4}{2} = 2$$

$$R_2 = \lim_{z \rightarrow -1-i} \frac{(z+1+i)(z^2+4)}{z(z+1+i)(z+1-i)}$$

$$= \lim_{z \rightarrow -1-i} \frac{z^2+4}{z(z+1-i)}$$

$$= \frac{(-1-i)^2 + 4}{z(-1-i)(-1-i+1-i)}$$

$$= \frac{1+2i+i^2+4}{(-1-i)(-2i)}$$

$$= \frac{2i+4}{2i+2i^2}$$

$$= \frac{(2i+4)(2i+2)}{(2i+2)(2i+2)}$$

$$= \frac{4i^2+4i+8i+8}{(2i)^2-2^2}$$

$$= \frac{12i+4}{-8} = \frac{4(3i+1)}{-8}$$

$$= -\frac{1}{2} - \frac{3i}{2}$$

$$R_3 = \lim_{z \rightarrow -1+i} \frac{(z+1-i)(z^2+4)}{z(z+1+i)(z+1-i)}$$

$$= \lim_{z \rightarrow -1+i} \frac{z^2+4}{z(z+1+i)}$$

$$= \frac{(-1+i)^2 + 4}{(-1+i)(-1+i+1+i)}$$

$$= \frac{1-2i+i^2+4}{(-1+i)2i}$$

$$= \frac{4-2i}{2i^2-2i}$$

$$= \frac{(4-2i)(-2+2i)}{(-2-2i)(-2+2i)}$$

$$= \frac{-8+8i+4i-4i^2}{(-2)^2-(2i)^2}$$

$$= \frac{12i-4}{8}$$

$$= -\frac{1}{2} + \frac{3i}{2}$$

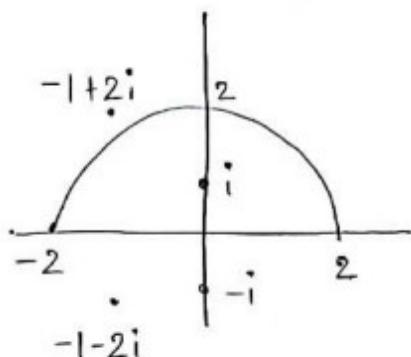
$$\begin{aligned}
 \oint_C \frac{z^2 + 4}{z^3 + 2z^2 + 2z} dz &= 2\pi i (R_1 + R_2 + R_3) \\
 &= 2\pi i \left(2 - \frac{1}{2} - \frac{3i}{2} - \frac{1}{2} + \frac{3i}{2} \right) \\
 &= 2\pi i (2 - 1) \\
 &= 2\pi i
 \end{aligned}$$

Ans.

- ⑥ Evaluate $\oint_C \frac{ze^{i\pi z}}{(z^2 + 2z + 5)(z^2 + 1)^2} dz$ using the residue at the poles where C is the upper half circle of equation $|z|=2$.

Sol: Considering the denominator,

$$\begin{aligned}
 &(z^2 + 2z + 5)(z^2 + 1)^2 \\
 &= (z^2 + 2z + 1 + 4)(z^2 - i^2)^2 \\
 &= \{(z+1)^2 + 2^2\} \{(z+i)(z-i)\}^2 \\
 &= \{(z+1)^2 - (2i)^2\} (z+i)^2 (z-i)^2 \\
 &= (z+1+2i)(z+1-2i)(z+i)^2(z-i)^2
 \end{aligned}$$



$$\therefore z = (-1-2i), (-1+2i), -i, +i$$

Here,

$f(z)$ has simple poles at $z = (-1+2i), (-1-2i)$

& $f(z)$ has pole of order 2 at $z = -i, i$

Since $|z|=2$, so only $z = +i, -i$ is valid for $|z|=2$

$\therefore z = +i$ is valid for ~~upper~~ upper half circle of $|z|=2$

Now, Residue at $z = i$.

Here, $a = i$ & $m = 2$

$$R = \lim_{z \rightarrow a} \frac{1}{(m-1)!} \cdot \frac{d^{m-1}}{dz^{m-1}} \left\{ (z-a)^m f(z) \right\}$$

$$= \lim_{z \rightarrow i} \frac{1}{1!} \left(\frac{\partial}{\partial z} \right) (z-i)^2 \left(\frac{ze^{iz}}{(z+1+2i)(z+1-2i)(z+i)^2(z-i)} \right)$$

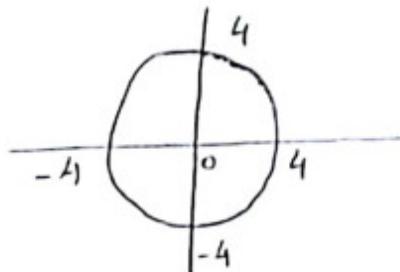
(7) Evaluate $\frac{1}{2\pi i} \oint_C \frac{z^2 - z + 2}{z^4 + 10z^2 + 9} dz$ using the residue at the poles around the circle C with the equation $|z| = 4$

Sol: Here, the denominator,

$$\begin{aligned} & z^4 + 10z^2 + 9 \\ &= z^4 + 9z^2 + z^2 + 9 \\ &= z^2(z^2 + 9) + 1(z^2 + 9) \\ &= (z^2 + 9)(z^2 + 1) \end{aligned}$$

$$\begin{aligned} &= \{z^2 - (3i)^2\} \{z^2 - i^2\} \\ &= (z + 3i)(z - 3i)(z + i)(z - i) \end{aligned}$$

$$\therefore z = +3i, -3i, +i, -i$$



Since all the poles are inside the circle

So all poles are valid for $|z| = 4$

We know, Residue Theorem,

$$\frac{1}{2\pi i} \oint f(z) = \lim_{z \rightarrow a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left\{ (z-a)^m f(z) \right\}$$

$$R_1 = \lim_{z \rightarrow -i} \frac{(z+i)(z^2 - z + 2)}{(z+i)(z-i)(z+3i)(z-3i)}$$

$$= \lim_{z \rightarrow -i} \frac{z^2 - z + 2}{(z-i)(z+3i)(z-3i)}$$

$$= \frac{-1+i+2}{(-2i) \cdot 2i \cdot (-4i)}$$

$$= \frac{1+i}{-16i} = \frac{-1}{16i} - \frac{i}{16i}$$

$$= \frac{i}{16} - \frac{1}{16}$$

$$R_3 = \lim_{z \rightarrow -3i} \frac{(z+3i)(z^2 - z + 2)}{(z+i)(z-i)(z+3i)(z-3i)}$$

$$= \lim_{z \rightarrow -3i} \frac{z^2 - z + 2}{(z+i)(z-i)(z-3i)}$$

$$= \frac{-9+3i+2}{(-2i)(-4i)(-6i)}$$

$$= \frac{3i-7i}{+48i} = \frac{-7}{48i} + \frac{3}{48}$$

$$= \frac{+7i}{48} + \frac{1}{16}$$

$$R_2 = \lim_{z \rightarrow i} \frac{(z-i)(z^2 - z + 2)}{(z+i)(z-i)(z+3i)(z-3i)}$$

$$= \lim_{z \rightarrow i} \frac{z^2 - z + 2}{(z+i)(z+3i)(z-3i)}$$

$$= \frac{-1-i+2}{2i \cdot 4i \cdot (-2i)}$$

$$= \frac{1-i}{16i} = \frac{1}{16i} - \frac{i}{16i}$$

$$= \frac{-i}{16} - \frac{1}{16}$$

$$R_4 = \lim_{z \rightarrow 3i} \frac{(z-3i)(z^2 - z + 2)}{(z+i)(z-i)(z+3i)(z-3i)}$$

$$= \lim_{z \rightarrow 3i} \frac{z^2 - z + 2}{(z+i)(z-i)(z+3i)}$$

$$= \frac{-9-3i+2}{4i \cdot 2i \cdot 6i}$$

$$= \frac{-7-3i}{-48i} = \frac{3}{48} + \frac{7i}{48i}$$

$$= \frac{1}{16} - \frac{7i}{48}$$

$$\begin{aligned}
 \frac{1}{2\pi i} \oint_C \frac{z^2 - z + 2}{z^4 + 10z^2 + 9} dz &= R_1 + R_2 + R_3 + R_4 \\
 &= \frac{i}{16} - \frac{1}{16} - \frac{i}{16} - \frac{1}{16} + \frac{1}{16} + \frac{7i}{48} \\
 &\quad + \frac{1}{16} - \frac{7i}{48} \\
 &= 0
 \end{aligned}$$

Ans.

Practice Sheet #6

① Show that $\int_0^\infty \frac{\ln(x^2+1)}{x^2+1} dx = \pi \ln 2$

Sol: Lets consider, $\int_C \frac{\ln(z+i)}{z^2+1} dz$

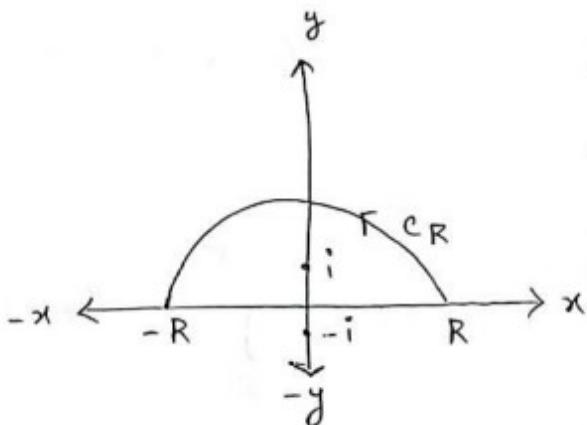
$$\therefore f(z) = \frac{\ln(z+i)}{z^2+1}$$

$$z^2 + 1 = 0$$

$$\Rightarrow z^2 = -1$$

$$\Rightarrow z^2 = i^2$$

$$\therefore z = \pm i$$



Only $z = i$ is valid for above region.

\therefore Residue at $z = i$

$$R = \lim_{z \rightarrow i} (z-i) \frac{\ln(z+i)}{(z+i)(z-i)} = \frac{\ln(2i)}{2i}$$

$$\int_C f(z) dz = 2\pi i \cdot R$$

Here,

$$x=0, y=1 \quad \& \quad r=1$$

c

$$= 2\pi i \frac{\ln(2i)}{2i}$$

$$\therefore \theta = \tan^{-1}\left(\frac{1}{0}\right)$$

$$= \pi \ln(2i)$$

$$= \tan^{-1}(\infty) = \tan^{-1} \tan\left(\frac{\pi}{2}\right)$$

$$= \pi \ln 2 + \pi \ln i$$

$$= \frac{\pi}{2}$$

$$= \pi \ln 2 + \pi \ln e^{i\pi/2}$$

$$\therefore z = re^{i\theta}$$

$$\therefore i = e^{i\frac{\pi}{2}}$$

$$= \pi \ln 2 + \pi \left(\frac{i\pi}{2}\right)$$

$$= \pi \ln 2 + \frac{i\pi^2}{2}$$

We know,

$$\int_C f(z) dz = \int_{-R}^R f(z) dz + \int_{C_R} f(z) dz$$

$$\therefore \pi \ln 2 + \frac{i\pi^2}{2} = \int_{-R}^0 \frac{\ln(x+i)}{x^2+1} dx + \int_0^R \frac{\ln(x+i)}{x^2+1} dx + \int_{C_R} f(z) dz$$

$$= \int_0^R \frac{\ln(i-x)}{x^2+1} dx + \int_0^R \frac{\ln(x+i)}{x^2+1} dx + \int_{C_R} f(z) dz$$

Here,

$$\therefore x=1, y=0$$

$$\& \pi = 1$$

$$= \ln(i^2 - x^2)$$

$$\therefore \theta = \tan(\theta/1)$$

$$= \ln(-1-x^2)$$

$$= \tan^{-1} \tan(\pi)$$

$$= \ln\{(-1)(1+x^2)\}$$

$$= \pi$$

$$= \ln(-1) + \ln(x^2 + 1)$$

$$\therefore z = e^{i\pi}$$

$$= \ln e^{2\pi i} + \ln(x^2 + 1)$$

$$\therefore i^2 = e^{2\pi i}$$

$$= 2\pi i + \ln(x^2 + 1)$$

$$\therefore \int_C f(z) dz = \int_0^R \frac{1}{x^2+1} \left\{ 2\pi i + \ln(x^2+1) \right\} dx + \int_{C_R} f(z) dz$$

By equating the real part of the equation we get,

$$\pi \ln 2 = \int_0^R \frac{\ln(x^2+1)}{x^2+1} dx$$

$$\Rightarrow \lim_{R \rightarrow \infty} \int_0^R \frac{\ln(x^2+1)}{x^2+1} dx = \pi \ln 2$$

$$\therefore \int_0^\infty \frac{\ln(x^2+1)}{x^2+1} dx = \pi \ln 2$$

Ans.

(2) Show that $\int_0^\alpha \frac{1}{x^4 + x^2 + 1} dx = \frac{\pi\sqrt{3}}{6}$

Sol: Let's consider, $\oint_C \frac{1}{z^4 + z^2 + 1} dz$

Here,

$$f(z) = \frac{1}{z^4 + z^2 + 1}$$

For poles,

$$z^4 + z^2 + 1 = 0$$

$$\Rightarrow z^4 + 2z^2 + 1 - z^2 = 0$$

$$\Rightarrow (z^2 + 1)^2 - z^2 = 0$$

$$\Rightarrow (z^2 + 1 + z)(z^2 + 1 - z) = 0$$

$$\therefore z^2 + z + 1 = 0$$

$$\text{or, } z^2 - z + 1 = 0$$

$$\Rightarrow z = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1}$$

$$\Rightarrow z = \frac{1 \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1}$$

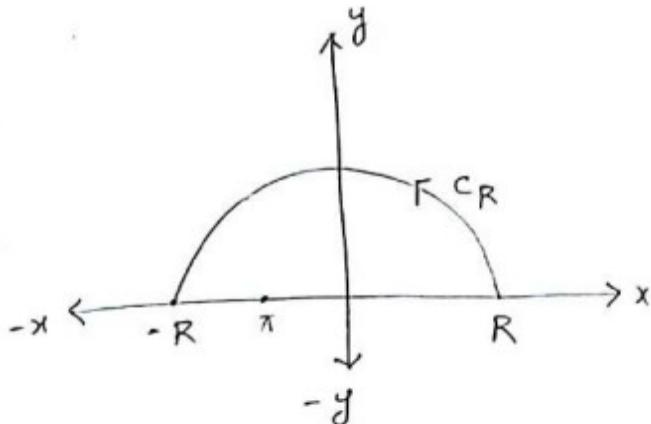
$$= \frac{-1 \pm \sqrt{3}i}{2}$$

$$= \frac{1 \pm \sqrt{-3}}{2}$$

$$= \frac{-1}{2} \pm \frac{i\sqrt{3}}{2}$$

$$= \frac{1}{2} \pm \frac{i\sqrt{3}}{2}$$

So, the poles are $z = \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right), \left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$ lie inside C .



But only $z = \frac{1}{2} + \frac{i\sqrt{3}}{2}$, $-\frac{1}{2} + \frac{i\sqrt{3}}{2}$ lies inside C.

$$R_1 = \lim_{z \rightarrow \frac{1}{2} + \frac{i\sqrt{3}}{2}} \frac{(z - \frac{1}{2} - \frac{i\sqrt{3}}{2})}{(z + \frac{1}{2} - \frac{i\sqrt{3}}{2})(z - \frac{1}{2} + \frac{i\sqrt{3}}{2})(z^2 + z + 1)}$$

$$= \lim_{z \rightarrow \frac{1}{2} + \frac{i\sqrt{3}}{2}} \frac{1}{(z - \frac{1}{2} + \frac{i\sqrt{3}}{2})(z^2 + z + 1)}$$

$$= \frac{1}{(\frac{1}{2} + \frac{i\sqrt{3}}{2} - \frac{1}{2} + \frac{i\sqrt{3}}{2}) \left\{ (\frac{1}{2} + \frac{i\sqrt{3}}{2})^2 + (\frac{1}{2} + \frac{i\sqrt{3}}{2}) + 1 \right\}}$$

$$= \frac{1}{i\sqrt{3} \left\{ \left(\frac{1}{4} + \frac{i\sqrt{3}}{2} - \frac{3}{4} \right) + \frac{1}{2} + \frac{i\sqrt{3}}{2} + 1 \right\}}$$

$$= \frac{1}{i\sqrt{3}(1+i\sqrt{3})(1-i\sqrt{3})} = \frac{1-i\sqrt{3}}{i\sqrt{3}(1+3)} = \frac{1-i\sqrt{3}}{4\sqrt{3}i}$$

$$= \frac{1}{i\sqrt{3}-3} \quad \checkmark$$

$$R_2 = \lim_{z \rightarrow -\frac{1}{2} + \frac{i\sqrt{3}}{2}} \frac{(z + \frac{1}{2} - \frac{i\sqrt{3}}{2})}{(z + \frac{1}{2} - \frac{i\sqrt{3}}{2})(z + \frac{1}{2} + \frac{i\sqrt{3}}{2})(z^2 - z + 1)}$$

$$= \lim_{z \rightarrow -\frac{1}{2} + \frac{i\sqrt{3}}{2}} \frac{1}{(z + \frac{1}{2} + \frac{i\sqrt{3}}{2})(z^2 - z + 1)}$$

$$= \frac{1}{(-\frac{1}{2} + \frac{i\sqrt{3}}{2} + \frac{1}{2} + \frac{i\sqrt{3}}{2}) \left\{ (-\frac{1}{2} + \frac{i\sqrt{3}}{2})^2 - (-\frac{1}{2} + \frac{i\sqrt{3}}{2}) + 1 \right\}}$$

$$= \frac{1}{i\sqrt{3} \left\{ (\frac{1}{4} - \frac{i\sqrt{3}}{2} - \frac{3}{4}) + \frac{1}{2} - \frac{i\sqrt{3}}{2} + 1 \right\}}$$

$$= \frac{1}{i\sqrt{3}(1-i\sqrt{3})(1+i\sqrt{3})} = \frac{1+i\sqrt{3}}{i\sqrt{3}(1+3)} = \frac{1+i\sqrt{3}}{4\sqrt{3}i}$$

$$= \frac{1}{i\sqrt{3} + 3} \checkmark$$

$$\begin{aligned} \oint_C f(z) dz &= 2\pi i (R_1 + R_2) = 2\pi i \left(\frac{1-i\sqrt{3}}{4\sqrt{3}i} + \frac{1+i\sqrt{3}}{4\sqrt{3}i} \right) \\ &= 2\pi i \left(\frac{2\cancel{\sqrt{3}}}{4\sqrt{3}i} \right) \end{aligned}$$

$$= \frac{\pi}{\sqrt{3}}$$

We know,

$$\int_C f(z) dz = \int_{-R}^R f(x) dx + \int_{C_R} f(z) dz$$

$$\Rightarrow \frac{\pi}{\sqrt{3}} = \int_{-\alpha}^{\alpha} \frac{1}{x^4+x^2+1} dx + 0$$

$$= \int_{-\alpha}^0 \frac{dx}{x^4+x^2+1} + \int_0^{\alpha} \frac{dx}{x^4+x^2+1}$$

$$= \int_{-\alpha}^0 \frac{(-) dx}{x^4+x^2+1} + \int_0^{\alpha} \frac{dx}{x^4+x^2+1} \quad [\text{Replacing } x \text{ by } (-x)]$$

$$= \int_0^{\alpha} \frac{dx}{x^4+x^2+1} + \int_0^{\alpha} \frac{dx}{x^4+x^2+1}$$

$$= 2 \int_0^{\alpha} \frac{dx}{x^4+x^2+1}$$

$$\Rightarrow \int_0^{\alpha} \frac{dx}{x^4+x^2+1} = \frac{\pi}{2\sqrt{3}} = \frac{\pi\sqrt{3}}{6} \quad [\text{Showed}]$$

$$③ \text{ Show that } \int_0^\alpha \frac{1}{x^4+1} dx = \frac{\pi}{2\sqrt{2}}$$

Sol: Let's consider, $\int_C \frac{dz}{z^4+1}$ where C is the contour.

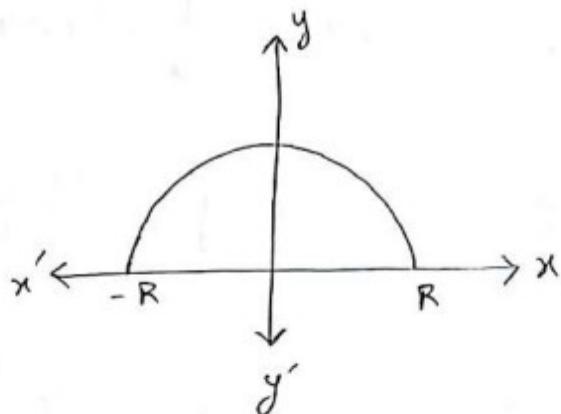
$$\text{For pole, } z^4 + 1 = 0$$

$$\Rightarrow z^4 = -1$$

$$\therefore z = (-1)^{1/4}$$

Here,

$$x = -1 \quad \& \quad y = 0$$



We know,

$$z = e^{i(\theta + 2\pi n)}$$

$$\therefore \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\therefore (-1) = e^{i(\pi + 2\pi n)}$$

$$= \tan^{-1}\left(\frac{0}{-1}\right)$$

$$\Rightarrow (-1)^{1/4} = e^{i(\pi + 2\pi n)\frac{1}{4}}$$

$$= \tan^{-1}(0)$$

$$= e^{i\left(\frac{\pi}{4} + \frac{n\pi}{2}\right)} ; n=0,1,2,3$$

$$= \pi$$

$$n=0 : z_1 = e^{i\pi/4} = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$

$$n=1 : z_2 = e^{i3\pi/4} = \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$

$$n=3 : z_3 = e^{i\left(\frac{\pi}{4} + \frac{3\pi}{2}\right)} = \cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right) = -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$$

$$n=2 : z_1 = e^{i\left(\frac{5\pi}{4}\right)} = \cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$$

$\therefore z = \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \right), \left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \right)$ poles are inside C.

$$R_1 = \lim_{z \rightarrow \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}} \frac{1}{(z + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}})(z - \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}})(z + \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}})}$$

$$= \frac{1}{\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} - \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)}$$

$$= \frac{1}{\sqrt{2} \left(\frac{2i}{\sqrt{2}}\right) \left(\sqrt{2} + \frac{2i}{\sqrt{2}}\right)}$$

$$= \frac{1}{2i \cdot \sqrt{2} \left(1 + \frac{2i}{2}\right)}$$

$$= \frac{1}{2\sqrt{2}i(1+i)}$$

$$= \frac{1}{2\sqrt{2}i - 2\sqrt{2}}$$

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$$\begin{aligned}
 R_2 &= \lim_{z \rightarrow -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}} \frac{1}{(z - \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}})(z - \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}})(z + \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}})} \\
 &= \frac{1}{(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}})(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} - \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}})(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}})} \\
 &= \frac{1}{(-\sqrt{2})(-\sqrt{2} + i\sqrt{2})(i\sqrt{2})} \\
 &= \frac{1}{-2i \cdot \sqrt{2}(i-1)} \\
 &= \frac{1}{-2\sqrt{2}i(i-1)} \\
 &= \frac{1}{+2\sqrt{2} + 2\sqrt{2}i} \\
 &= \frac{+1}{2\sqrt{2}(i+1)}
 \end{aligned}$$

$$\therefore \oint_C \frac{dz}{z^4 + 1} = 2\pi i [R_1 + R_2] = 2\pi i \left[\frac{1}{2\sqrt{2}(i-1)} + \frac{1}{2\sqrt{2}(i+1)} \right]$$

$$\frac{2\pi i}{2\sqrt{2}} \left[\frac{1}{i-1} + \frac{1}{i+1} \right]$$

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$$= \frac{\pi i}{\sqrt{2}} \left[\frac{i+1 + i-1}{(i+1)(i-1)} \right]$$

$$= \frac{2\pi i^2}{\sqrt{2}(i^2 - 1^2)}$$

$$= \frac{-2\pi i}{-2\sqrt{2}}$$

$$= -\frac{\pi i}{\sqrt{2}} \quad \frac{\pi}{2}$$

We know,

$$\oint_C f(z) dz = \int_{-R}^R f(x) dx + \int_{C_R} f(z) dz$$

$$\Rightarrow \frac{\pi}{2} = \lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx + \lim_{R \rightarrow \infty} \int_{C_R} f(z) dz$$

$$= \int_{-\infty}^{\infty} \frac{dx}{x^4 + 1} + 0$$

$$= \int_{-\infty}^0 \frac{dx}{x^4 + 1} + \int_0^{\infty} \frac{dx}{x^4 + 1} = \int_{-\infty}^0 \frac{-dx}{x^4 + 1} + \int_0^{\infty} \frac{dx}{x^4 + 1}$$

[Replacing x by $(-x)$]

$$= \int_0^\alpha \frac{dx}{x^4+1} + \int_0^\alpha \frac{dx}{x^4+1}$$

$$= 2 \int_0^\alpha \frac{dx}{x^4+1}$$

$$\therefore \int_0^\alpha \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}} \quad [\text{Showed}]$$

(4) Show that $\int_0^\alpha \frac{\cos 2\pi x}{x^4+x^2+1} dx = \frac{-\pi}{2\sqrt{3}} e^{-\pi\sqrt{3}}$

Sol: Lets consider $\int_C \frac{e^{i2\pi z}}{z^4+z^2+1} dz$

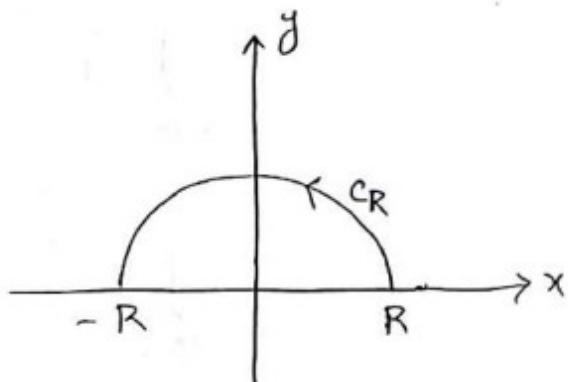
$$\therefore f(z) = \frac{e^{i2\pi z}}{z^4+z^2+1}$$

For poles,
 $(z^4+z^2+1) = 0$

$$\Rightarrow z^4 + 2z^2 + 1 - z^2 = 0$$

$$\Rightarrow (z^2+1)^2 - z^2 = 0$$

$$\Rightarrow (z^2+1+z)(z^2+1-z) = 0$$



$$\therefore z^2 + z + 1 = 0$$

$$\text{or, } z^2 - z + 1 = 0$$

$$\Rightarrow z = \frac{-1 \pm i\sqrt{3}}{2}$$

$$= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}$$

$$\Rightarrow z = \frac{-(-1) + \sqrt{(-1)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1}$$

$$\Rightarrow z = \frac{1 \pm i\sqrt{3}}{2} = \frac{1}{2} \pm \frac{i\sqrt{3}}{2}$$

$\therefore z = \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right), \left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$ poles are inside C.

$$\begin{aligned}
 R_1 &= \lim_{z \rightarrow -\frac{1}{2} + \frac{i\sqrt{3}}{2}} \frac{e^{i2\pi z}}{(z + \frac{1}{2} + \frac{i\sqrt{3}}{2})(z - \frac{1}{2} - \frac{i\sqrt{3}}{2})(z - \frac{1}{2} + \frac{i\sqrt{3}}{2})} \\
 &= \frac{e^{i2\pi(-\frac{1}{2} + \frac{i\sqrt{3}}{2})}}{(-\frac{1}{2} + \frac{i\sqrt{3}}{2} + \frac{1}{2} + \frac{i\sqrt{3}}{2})(-\frac{1}{2} + \frac{i\sqrt{3}}{2} - \frac{1}{2} - \frac{i\sqrt{3}}{2})(-\frac{1}{2} + \frac{i\sqrt{3}}{2} - \frac{1}{2} + \frac{i\sqrt{3}}{2})} \\
 &= \frac{e^{-\pi i} \cdot e^{-\sqrt{3}\pi}}{i\sqrt{3} \cdot (-1) (i\sqrt{3} - 1)} = \frac{(-1) e^{-\pi\sqrt{3}}}{-i^2 \cdot 3 + i\sqrt{3}} = \frac{-e^{-\pi\sqrt{3}}}{i\sqrt{3} + 3} \\
 &= \frac{-e^{-\pi\sqrt{3}}}{\sqrt{3}(i + \sqrt{3})}
 \end{aligned}$$

$$\begin{aligned}
 R_2 &= \lim_{z \rightarrow \frac{1}{2} + \frac{i\sqrt{3}}{2}} \frac{e^{i2\pi z}}{(z + \frac{1}{2} + \frac{i\sqrt{3}}{2})(z + \frac{1}{2} - \frac{i\sqrt{3}}{2})(z - \frac{1}{2} + \frac{i\sqrt{3}}{2})} \\
 &= \frac{e^{i2\pi(\frac{1}{2} + \frac{i\sqrt{3}}{2})}}{(\frac{1}{2} + \frac{i\sqrt{3}}{2} + \frac{1}{2} + \frac{i\sqrt{3}}{2})(\frac{1}{2} + \frac{i\sqrt{3}}{2} + \frac{1}{2} - \frac{i\sqrt{3}}{2})(\frac{1}{2} + \frac{i\sqrt{3}}{2} - \frac{1}{2} + \frac{i\sqrt{3}}{2})} \\
 &= \frac{e^{i\pi} \cdot e^{-\sqrt{3}\pi}}{(1+i\sqrt{3}) \cdot 1 \cdot i\sqrt{3}} = \frac{(-1) e^{-\sqrt{3}\pi}}{i\sqrt{3} - 3} = \frac{-e^{-\sqrt{3}\pi}}{\sqrt{3}(i - \sqrt{3})}
 \end{aligned}$$

$$\begin{aligned}
 \therefore \oint_C f(z) dz &= 2\pi i [R_1 + R_2] \\
 &= 2\pi i \left[\frac{-e^{-\pi\sqrt{3}}}{\sqrt{3}(i+\sqrt{3})} + \frac{-e^{-\pi\sqrt{3}}}{\sqrt{3}(i-\sqrt{3})} \right] \\
 &= 2\pi i \times \frac{-e^{-\pi\sqrt{3}}}{\sqrt{3}} \cdot \left[\frac{1}{i+\sqrt{3}} + \frac{1}{i-\sqrt{3}} \right] \\
 &= \frac{-2\pi i e^{-\pi\sqrt{3}}}{\sqrt{3}} \left[\frac{i-\sqrt{3} + i+\sqrt{3}}{i^2 - (\sqrt{3})^2} \right] \\
 &= \frac{-2\pi i e^{-\pi\sqrt{3}}}{\sqrt{3}} \left[\frac{2i}{-1-3} \right] = \frac{-4\pi i^2 e^{-\pi\sqrt{3}}}{-4\sqrt{3}} \\
 &= \frac{-\pi e^{-\pi\sqrt{3}}}{\sqrt{3}}
 \end{aligned}$$

Equating the real parts,

$$\oint_C \frac{\cos 2\pi z}{z^4 + z^2 + 1} dz = -\frac{\pi}{\sqrt{3}} e^{-\pi\sqrt{3}} = \oint_C f(z) dz$$

We know,

$$\oint_C f(z) dz = \int_{-R}^R f(x) dx + \int_{CR} f(z) dz$$

$$\Rightarrow -\frac{\pi}{\sqrt{3}} e^{-\pi\sqrt{3}} = \lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx + \lim_{R \rightarrow \infty} \int_{CR} f(z) dz$$

$$= \int_{-\alpha}^{\alpha} \frac{\cos 2\pi x}{x^4 + x^2 + 1} dx + 0$$

$$= 2 \int_0^{\alpha} \frac{\cos 2\pi x}{x^4 + x^2 + 1} dx$$

$$\therefore \int_0^{\alpha} \frac{\cos 2\pi x}{x^4 + x^2 + 1} dx = \frac{-\pi e^{-\pi\sqrt{3}}}{2\sqrt{3}} \quad [\text{Showed}]$$

Laplace Transformation of some elementary functions:

	<u>$F(t)$</u>	<u>$L\{F(t)\} = f(s)$</u>
1.	1	$\frac{1}{s}$
2.	t	$\frac{1}{s^2}$
3.	t^n	$\frac{n!}{s^{n+1}}$
4.	e^{at}	$\frac{1}{s-a}$
5.	$\sin at$	$\frac{a}{s^2+a^2}$
6.	$\cos at$	$\frac{s}{s^2+a^2}$
7.	$\sin ht$	$\frac{a}{s^2-a^2}$
8.	$\cos ht$	$\frac{s}{s^2-a^2}$
9.	$\frac{t^n}{(n+1)!}$	$\frac{1}{s^{n+1}} ; n > -1$

Laplace Transformation

Practice Sheet # 7

① Find the laplace transformation of each of following function:

$$(a) 3e^{-2t}$$

$$\text{Sol: } L\{3e^{-2t}\}$$

$$= 3 L\{e^{-2t}\}$$

$$= 3 \cdot \frac{1}{s - (-2)}$$

$$= \frac{3}{s+2}$$

$$(b) 4t^3 - e^{-t}$$

$$4L\{t^3\} - L\{e^{-t}\}$$

$$= 4 \cdot \frac{3!}{s^{3+1}} - \frac{1}{s - (-1)}$$

$$= \frac{24}{s^4} - \frac{1}{s+1}$$

$$(c) 7\sin 2t - 3\cos 2t$$

$$= 7 L\{\sin 2t\} - 3 L\{\cos 2t\}$$

$$= 7 \cdot \frac{2}{s^2 + 2^2} - 3 \cdot \frac{s}{s^2 + 2^2}$$

$$= \frac{14}{s^2 + 4} - \frac{3s}{s^2 + 4} = \frac{14 - 3s}{s^2 + 4}$$

$$(d) (t^2 + 1)^2$$

$$= t^4 + 2t^2 + 1$$

$$= L\{t^4\} + 2 L\{t^2\} + L\{1\}$$

$$= \frac{4!}{s^{4+1}} + 2 \cdot \frac{2!}{s^{2+1}} + \frac{1}{s}$$

$$= \frac{24}{s^5} + \frac{4}{s^3} + \frac{1}{s}$$

$$\begin{aligned}
 \text{(e)} \quad & (4e^{2t} - 2)^3 & (a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3 \\
 & = 64e^{6t} - 3 \cdot 16e^{4t} \cdot 2 + 3 \cdot 4e^{2t} \cdot 4 - 8 \\
 & = 64e^{6t} - 96e^{4t} + 48e^{2t} - 8 \\
 & = 64 L\{e^{6t}\} - 96 L\{e^{4t}\} + 48 L\{e^{2t}\} - 8 L\{1\} \\
 & = 64 \cdot \frac{1}{s-6} - 96 \cdot \frac{1}{s-4} + 48 \cdot \frac{1}{s-2} - 8 \cdot \frac{1}{s} \\
 & = \frac{64}{s-6} - \frac{96}{s-4} + \frac{48}{s-2} - \frac{8}{s} \quad \underline{\text{Ans.}}
 \end{aligned}$$

2. Evaluate each of the following:

$$\text{(i)} \quad L\{t^3 e^{-3t}\}$$

$$= f(s+3)$$

$$= \frac{6}{(s+3)^4}$$

Ans.

Here,

$$f(s) = L\{t^3\}$$

$$= \frac{3!}{s^{3+1}} = \frac{6}{s^4}$$

$$\therefore f(s-(-3)) = f(s+3) = \frac{6}{(s+3)^4}$$

$$(ii) L \{ 5e^{3t} \sin 4t \}$$

$$= 5 L \{ e^{3t} \sin 4t \}$$

$$= 5 f(s-3)$$

$$= 5 \cdot \frac{4}{(s-3)^2 + 16}$$

$$= \frac{20}{s^2 - 6s + 25} \quad \underline{\text{Ans.}}$$

$$(iii) L \{ (t+2)^2 e^t \}$$

$$= L \{ (t^2 + 4t + 4) e^t \}$$

$$= L \{ t^2 e^t \} + L \{ 4t e^t \} + 4 L \{ e^t \}$$

$$= \frac{2}{(s-1)^3} + \frac{4}{(s-1)^2} + \frac{4}{s-1}$$

Ans.

Here,

$$f(s) = L \{ \sin 4t \} = \frac{4}{s^2 + 16}$$

$$f(s-1) = \frac{4}{(s-1)^2 + 16}$$

Here,

$$(i) f(s) = L \{ t^2 \} = \frac{2!}{s^2 + 1} = \frac{2}{s^3}$$

$$\therefore f(s-1) = \frac{2}{(s-1)^3}$$

$$(ii) f(s) = L \{ t \} = \frac{1}{s^2}$$

$$\therefore f(s-1) = \frac{1}{(s-1)^2}$$

$$(iii) f(s) = L \{ 1 \} = \frac{1}{s}$$

$$\therefore f(s-1) = \frac{1}{s-1}$$

$$(iv) L \{ e^{-t} (3 \sinh 2t - 5 \cosh 2t) \}$$

$$= 3 L \{ e^{-t} \sinh 2t \} - 5 L \{ e^{-t} \cosh 2t \}$$

$$= 3 \cdot \frac{2}{(s+1)^2 - 4} - 5 \cdot \frac{s+1}{(s+1)^2 - 4}$$

$$= \frac{6}{(s+1)^2 - 4} - \frac{5s+5}{(s+1)^2 - 4}$$

$$= \frac{6 - 5s + 5}{(s+1)^2 - 4}$$

$$= \frac{1 - 5s}{s^2 + 2s - 3} \quad \underline{\text{Ans.}}$$

$$(v) L \{ e^{-4t} \cosh 2t \}$$

Here,

$$f(s) = L \{ \cosh 2t \} = \frac{s}{s^2 - 2^2}$$

$$\therefore f(s-(-4)) = f(s+4) = \frac{s+4}{(s+4)^2 - 4}$$

$$\therefore L \{ e^{-4t} \cosh 2t \} = \frac{s+4}{s^2 + 2^2} \quad \underline{\text{Ans.}}$$

Here,

$$(i) f(s) = L \{ \sinh 2t \}$$

$$= \frac{2}{s^2 - 2^2}$$

$$\therefore f(s-(-1)) = f(s+1)$$

$$= \frac{2}{(s+1)^2 - 4}$$

$$(ii) f(s) = L \{ \cosh 2t \}$$

$$= \frac{s}{s^2 - 2^2}$$

$$\therefore f(s+1) = \frac{s+1}{(s+1)^2 - 4}$$

$$(vi) L \left\{ e^{2t} (3 \sin 4t - 4 \cos 4t) \right\}$$

$$= 3 L \left\{ e^{2t} \sin 4t \right\} - 4 L \left\{ e^{2t} \cos 4t \right\}$$

Here,

$$(i) f(s) = L \left\{ \sin 4t \right\}$$

$$= \frac{4}{s^2 + 4^2}$$

$$\therefore f(s-2) = \frac{4}{(s-2)^2 + 16}$$

$$(ii) f(s) = L \left\{ \cos 4t \right\}$$

$$= \frac{s}{s^2 + 4^2}$$

$$\therefore f(s-2) = \frac{s-2}{(s-2)^2 + 16}$$

Now,

$$3 L \left\{ e^{2t} \sin 4t \right\} - 4 L \left\{ e^{2t} \cos 4t \right\} = \frac{12}{(s-2)^2 + 16} - \frac{4s-8}{(s-2)^2 + 16}$$

$$= \frac{12 - 4s + 8}{(s-2)^2 + 16}$$

$$= \frac{20 - 4s}{s^2 - 4s + 20} \quad \underline{\text{Ans.}}$$

3. Determine each of the following:

$$(i) L^{-1} \left\{ \frac{12}{4-3s} \right\}$$

$$= L^{-1} \left\{ \frac{12}{-3(s - 4/3)} \right\}$$

$$= L^{-1} \left\{ \frac{-4}{s - 4/3} \right\}$$

$$= -4 L^{-1} \left\{ \frac{1}{s - 4/3} \right\}$$

$$= -4 e^{4/3 t}$$

Ans.

$$(ii) L^{-1} \left\{ \frac{2s-5}{s^2-9} \right\}$$

$$= L^{-1} \left\{ \frac{2s}{s^2-3^2} \right\} - L^{-1} \left\{ \frac{5}{s^2-3^2} \right\}$$

$$= 2 L^{-1} \left\{ \frac{s}{s^2-3^2} \right\} - \frac{5}{3} L^{-1} \left\{ \frac{3}{s^2-3^2} \right\}$$

$$= 2 \cos 3t - \frac{5}{3} \sin 3t$$

Ans.

$$(iii) L^{-1} \left\{ \frac{23s-15}{s^2+8} \right\}$$

$$= L^{-1} \left\{ \frac{23s}{s^2+(2\sqrt{2})^2} \right\} - L^{-1} \left\{ \frac{15}{s^2+(2\sqrt{2})^2} \right\}$$

$$= 23 L^{-1} \left\{ \frac{s}{s^2+(2\sqrt{2})^2} \right\} - \frac{15}{2\sqrt{2}} L^{-1} \left\{ \frac{2\sqrt{2}}{s^2+(2\sqrt{2})^2} \right\}$$

$$= 23 \cos 2\sqrt{2}t - \frac{15}{2\sqrt{2}} \sin 2\sqrt{2}t$$

Ans.

$$(vi) L^{-1} \left\{ \frac{1}{s^4} \right\} = L^{-1} \left\{ \frac{1}{s^{3+1}} \right\} = \frac{t^3}{3!} = \frac{t^3}{6}$$

Ans.

$$(iv) L^{-1} \left\{ \frac{1}{s^{3/2}} \right\}$$

$$= L^{-1} \left\{ \frac{1}{s^{\frac{1}{2}} + 1} \right\}$$

$$= \frac{t^{1/2}}{\Gamma(\frac{1}{2} + 1)}$$

$$= \frac{t^{1/2}}{\frac{1}{2} \Gamma(\frac{1}{2})}$$

$$= \frac{t^{1/2}}{\frac{1}{2} \sqrt{\pi}}$$

$$= 2 \sqrt{\frac{t}{\pi}} \quad \underline{\text{Ans.}}$$

$$(v) L^{-1} \left\{ \frac{s+1}{s^{4/3}} \right\}$$

Here,

$$\frac{s+1}{s^{4/3}} = \frac{s}{s^{4/3}} + \frac{1}{s^{4/3}} = \frac{1}{s^{1/3}} + \frac{1}{s^{4/3}} = \frac{1}{s^{-2/3} + 1} + \frac{1}{s^{1/3} + 1}$$

$$L^{-1} \left\{ \frac{s+1}{s^{4/3}} \right\} = L^{-1} \left\{ \frac{1}{s^{-\frac{2}{3}} + 1} \right\} + L^{-1} \left\{ \frac{1}{s^{\frac{1}{3}} + 1} \right\}$$

$$= \frac{t^{-2/3}}{(-\frac{2}{3} + 1)!} + \frac{t^{1/3}}{(\frac{1}{3} + 1)!} = \frac{t^{-2/3}}{\Gamma(\frac{1}{3})} + \frac{t^{1/3}}{\Gamma(\frac{4}{3})}$$

Ans.

$$(vii) L^{-1} \left\{ \frac{1}{\sqrt{2s+3}} \right\}$$

$$= L^{-1} \left\{ \frac{1}{\sqrt{2(s + \frac{3}{2})}} \right\}$$

$$= \frac{1}{\sqrt{2}} L^{-1} \left\{ \frac{1}{(s + \frac{3}{2})^{\frac{1}{2}}} \right\}$$

$$= \frac{1}{\sqrt{2}} e^{-\frac{3}{2}t} L^{-1} \left\{ \frac{1}{s^{\frac{1}{2}}} \right\}$$

$$= \frac{e^{-\frac{3t}{2}}}{\sqrt{2}} L^{-1} \left\{ \frac{1}{s^{-\frac{1}{2}}+1} \right\}$$

$$= \frac{e^{-\frac{3t}{2}}}{\sqrt{2}} \frac{t^{-\frac{1}{2}}}{(\frac{1}{2})!}$$

$$= \frac{1}{\sqrt{2\pi}} t^{-\frac{1}{2}} e^{-\frac{3t}{2}}$$

Ans.

$$(viii) L^{-1} \left\{ \frac{1}{(s+4)^{\frac{5}{2}}} \right\}$$

$$= e^{-4t} L^{-1} \left\{ \frac{1}{s^{\frac{5}{2}}} \right\}$$

$$= e^{-4t} L^{-1} \left\{ \frac{1}{s^{\frac{3}{2}}+1} \right\}$$

$$= e^{-4t} \cdot \frac{t^{\frac{3}{2}}}{(\frac{3}{2}+1)!}$$

$$= e^{-4t} \frac{t^{\frac{3}{2}}}{\frac{3}{2} \sqrt{\frac{3}{2}}} \quad [m+1 = n \Gamma(n)]$$

$$= e^{-4t} \frac{t^{\frac{3}{2}}}{\frac{3}{2} \sqrt{\frac{1}{2}+1}}$$

$$= e^{-4t} \frac{t^{\frac{3}{2}}}{\frac{3}{2} \cdot \frac{1}{2} \sqrt{\frac{1}{2}}}$$

$$= \frac{4}{3} e^{-4t} \frac{t^{\frac{3}{2}}}{\pi} \quad \underline{\text{Ans}}$$

4. Evaluate each of the following using partial fraction:

$$(i) L^{-1} \left\{ \frac{6s-4}{s^2-4s+20} \right\}$$

$$= L^{-1} \left\{ \frac{6(s-2)+8}{(s-2)^2+4^2} \right\}$$

$$= 6L^{-1} \left\{ \frac{6(s-2)}{(s-2)^2+4^2} \right\} + 2L^{-1} \left\{ \frac{8}{(s-2)^2+4^2} \right\}$$

$$= 6e^{2t} \cos 4t + 2e^{2t} \sin 4t$$

$$= 2e^{2t} (3 \cos 4t + \sin 4t)$$

Ans.

$$(ii) L^{-1} \left\{ \frac{4s+12}{s^2+8s+16} \right\}$$

$$= L^{-1} \left\{ \frac{4(s+4)-4}{(s+4)^2} \right\}$$

$$= 4L^{-1} \left\{ \frac{s+4}{(s+4)^2} \right\} - 4L^{-1} \left\{ \frac{1}{(s+4)^2} \right\}$$

$$= 4L^{-1} \left\{ \frac{1}{s+4} \right\} - 4e^{-4t} \cdot t$$

$$= 4e^{-4t} - 4te^{-4t} = 4e^{-4t}(1-t)$$

$$(iii) L^{-1} \left\{ \frac{2s^2-4}{(s+1)(s-2)(s-3)} \right\}$$

$$\text{Sol: } \frac{2s^2-4}{(s+1)(s-2)(s-3)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{s-3}$$

$$\therefore 2s^2-4 = A(s-2)(s-3) + B(s-3)(s+1) + C(s+1)(s-2)$$

$$s=2 : 2 \cdot 2^2 - 4 = B \cdot 3 \cdot (-1) \Rightarrow 8 - 4 = -3B \Rightarrow B = -\frac{4}{3}$$

$$s=3 : 2 \cdot 3^2 - 4 = C \cdot 4 \cdot 1 \Rightarrow 18 - 4 = 4C \Rightarrow C = \frac{7}{2}$$

$$s=-1 : 2 \cdot 1 - 4 = A(-3)(-4) \Rightarrow 2 - 4 = 12A \Rightarrow A = -\frac{1}{6}$$

$$\therefore \frac{2s^2 - 4}{(s+1)(s-2)(s-3)} = -\frac{1}{6(s+1)} - \frac{4}{3(s-2)} + \frac{7}{2(s-3)}$$

$$L^{-1} \left\{ \frac{2s^2 - 4}{(s+1)(s-2)(s-3)} \right\} = -\frac{1}{6} L^{-1} \left\{ \frac{1}{s+1} \right\} - \frac{4}{3} L^{-1} \left\{ \frac{1}{s-2} \right\} + \frac{7}{2} L^{-1} \left\{ \frac{1}{s-3} \right\}$$

$$= -\frac{1}{6} e^{-t} - \frac{4}{3} e^{2t} + \frac{7}{2} e^{3t}$$

Ans.

$$(v) L^{-1} \left\{ \frac{3s+1}{(s^2+1)(s-1)} \right\}$$

$$\text{Sol: } \frac{3s+1}{(s^2+1)(s-1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1} = \frac{A}{s-1} + \frac{Bs}{s^2+1} + \frac{C}{s^2+1}$$

$$\therefore 3s+1 = A(s^2+1) + Bs(s-1) + C(s-1)$$

$$s=1 : 4 = 2A \Rightarrow A = 2$$

$$s=0 : 1 = A - C \Rightarrow C = 2 - 1 = 1$$

$$s=-1 : -3+1 = 2A + 2B - 2 \Rightarrow B = \frac{-2 - 4 + 2}{2} = -2$$

$$\frac{3s+1}{(s^2+1)(s-1)} = \frac{2}{s-1} - \frac{2s}{s^2+1} + \frac{1}{s^2+1}$$

$$\therefore L^{-1} \left\{ \frac{3s+1}{(s^2+1)(s-1)} \right\} = 2L^{-1} \left\{ \frac{1}{s-1} \right\} - 2L^{-1} \left\{ \frac{s}{s^2+1^2} \right\} + L^{-1} \left\{ \frac{1}{s^2+1^2} \right\}$$

$$= 2e^t - 2\cos t + \sin t$$

Ans.

$$(vii) L^{-1} \left\{ \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} \right\}$$

Sol: Here,

$$\begin{aligned} & \frac{s^2+2s+3}{\{(s^2+2 \cdot s \cdot 1 + 1^2) + 1\} \{(s^2+2 \cdot s \cdot 1 + 1^2) + 2^2\}} \\ &= \frac{s^2+2s+3}{\{(s+1)^2+1^2\} \{(s+1)^2+2^2\}} \end{aligned}$$

Now,

$$\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = \frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2+2s+5}$$

$$\therefore s^2+2s+3 = (As+B)(s^2+2s+5) + (Cs+D)(s^2+2s+2)$$

$$= As^3 + 2As^2 + 5As + Bs^2 + 2Bs + 5B + Cs^3 + 2Cs^2 + \\ 2Cs + Ds^2 + 2Ds + D$$

$$= s^3(A+c) + (2A+B+2C+D)s^2 + s(5A+2B+2C+2D) + (5B+2D)$$

By equating both sides we get,

$$A+c = 0 \quad \dots \dots \dots \quad (i)$$

$$2A+B+2C+D = 1 \quad \dots \dots \dots \quad (ii)$$

$$5A+2B+2C+2D = 2 \quad \dots \dots \dots \quad (iii)$$

$$5B+2D = 3 \quad \dots \dots \dots \quad (iv)$$

$$\text{From eqn (i)} \Rightarrow 2(A+c) + B + D = 1$$

$$\Rightarrow B+D = 1 \quad \dots \dots \dots \quad (v)$$

$$(iv) - 2 \times (v) \Rightarrow 5B+2D=3$$

$$\begin{array}{r} -2B-2D=-2 \\ \hline 3B=1 \end{array}$$

$$\therefore B = \frac{1}{3}$$

$$\begin{aligned} \text{From eqn (v)} \Rightarrow D &= 1-B \\ &= 1 - \frac{1}{3} \end{aligned}$$

$$= \frac{2}{3}$$

Substituting the value of B , D & $A=-c$ in eqn (iii) we get,

$$-5C + 2C + 2\left(\frac{1}{3}\right) + 2\left(\frac{2}{3}\right) = 2$$

$$\Rightarrow -3C + \frac{2}{3} + \frac{4}{3} - 2 = 0$$

$$\Rightarrow \frac{2+4-6}{3} = 3C$$

$$\Rightarrow 0 = 3C$$

$$\therefore C = 0$$

From eqⁿ (i) $\Rightarrow A + 0 = 0$

$$\therefore A = 0$$

$$\therefore L^{-1} \left\{ \frac{s^2 + 2s + 3}{(s^2 + 2s + 2)(s^2 + 2s + 5)} \right\} = L^{-1} \left\{ \frac{0 + \frac{1}{3}}{s^2 + 2s + 2} + \frac{0 + \frac{2}{3}}{s^2 + 2s + 5} \right\}$$

$$= \frac{1}{3} L^{-1} \left\{ \frac{1}{s^2 + 2s + 2} \right\} + \frac{2}{3} L^{-1} \left\{ \frac{1}{s^2 + 2s + 5} \right\}$$

$$= \frac{1}{3} L^{-1} \left\{ \frac{1}{(s+1)^2 + 1^2} \right\} + \frac{2}{3} L^{-1} \left\{ \frac{2}{(s+1)^2 + 2^2} \right\}$$

$$= \frac{1}{3} e^{-t} \sin t + \frac{2}{3} e^{-t} \sin 2t = \frac{1}{3} e^{-t} (\sin t + \sin 2t)$$

$$(iv) L^{-1} \left\{ \frac{5s^2 - 15s - 11}{(s+1)(s-2)^3} \right\}$$

Sol: Here,

$$\begin{aligned} \frac{5s^2 - 15s - 11}{(s+1)(s-2)^3} &= \frac{A}{s+1} + \frac{B}{(s-2)} + \frac{C}{(s-2)^2} + \frac{D}{(s-2)^3} \\ \therefore 5s^2 - 15s - 11 &= A(s-2)^3 + B(s-2)^2(s+1) + C(s+1)(s-2) + D(s+1) \\ &= A(s^3 - 3s^2 \cdot 2 + 3s \cdot 2^2 - 2^3) + B(s^3 - 2 \cdot s \cdot 2 + 2^2)(s+1) \\ &\quad + C(s^2 - 2s + s - 2) + D(s+1) \\ &= A(s^3 - 6s^2 + 12s - 8) + B(s^3 - 4s^2 + 4s + s^2 - 4s + 4) \\ &\quad + C(s^2 - s - 2) + D(s+1) \\ &= As^3 - 6As^2 + 12As - 8A + Bs^3 - 3Bs^2 + 4B + Cs^2 - Cs \\ &\quad - 2C + Ds + D \\ &= s^3(A+B) + s^2(-6A - 3B + C) + s(12A - C + D) + (-8A + 4B - 2C + D), \end{aligned}$$

By equating both sides we get,

$$A+B=0 \dots \dots \dots (i)$$

$$12A - C + D = -15 \dots \dots \dots (iii)$$

$$-6A - 3B + C = 5 \dots \dots \dots (ii)$$

$$-8A + 4B - 2C + D = -11 \dots \dots \dots (iv)$$

Now,

$$\text{From eqn (i)} \Rightarrow B = -A \dots (v)$$

$$\text{eqn (ii)} \Rightarrow -6A - 3(-A) + C = 5$$

$$\Rightarrow -6A + 3A + C = 5$$

$$\Rightarrow C = 5 + 3A \dots \dots \dots (vi)$$

$$\text{From (iii)} \Rightarrow 12A - 5 - 3A + D = -15$$

$$\Rightarrow D = -9A - 10 \dots \dots \dots (vii)$$

$$(iv) \Rightarrow -8A - 4A - 2(3A + 5) - 9A - 10 = -11$$

$$\Rightarrow -21A - 6A - 10 - 10 = -11$$

$$\Rightarrow -27A = 9$$

$$\therefore A = -1/3$$

$$(v) \Rightarrow B = \frac{1}{3}$$

$$(vi) \Rightarrow C = 5 + (-1) = 4$$

$$(vii) \Rightarrow D = 3 - 10 = -7$$

$$\begin{aligned}
 L^{-1} \left\{ \frac{5s^2 - 15s - 11}{(s+1)(s-2)^3} \right\} &= -\frac{1}{3} L^{-1} \left\{ \frac{1}{s+1} \right\} + \frac{1}{3} L^{-1} \left\{ \frac{1}{s-2} \right\} + 4 L^{-1} \left\{ \frac{1}{(s-2)^2} \right\} \\
 &\quad - 7 L^{-1} \left\{ \frac{1}{(s-2)^3} \right\} \\
 &= -\frac{1}{3} e^{-t} + \frac{1}{3} e^{2t} + 4t e^{2t} - 7e^{2t} L^{-1} \left\{ \frac{1}{s^2+1} \right\} \\
 &= -\frac{1}{3} e^{-t} + \frac{1}{3} e^{2t} + 4t e^{2t} - \frac{7}{2} t^2 e^{2t} \quad \underline{\text{Ans.}}
 \end{aligned}$$

5. Solve the given differential equation:

$$(i) \quad y'' - 3y' + 2y = 4e^{2t}, \quad y(0) = -3, \quad y'(0) = 5$$

$$\begin{aligned}
 \text{Sol: } L \{ y'' - 3y' + 2y \} &= L \{ 4e^{2t} \} \\
 \Rightarrow L \{ y'' \} - 3L \{ y' \} + 2L \{ y \} &= 4L \{ e^{2t} \} \\
 \Rightarrow [s^2 L \{ y \} - s^1 y(0) - s^0 y'(0)] - 3[s L \{ y \} - s^0 y(0)] + 2[L \{ y \}] &= 4 \left(\frac{1}{s-2} \right) \\
 \Rightarrow [s^2 y - sy(0) - y'(0)] - 3[sy - y(0)] + 2y &= \frac{4}{s-2} \quad [\text{Let, } L \{ y \} = y]
 \end{aligned}$$

$$\Rightarrow [s^2y - s(-3) - 5] - 3[sy - (-3)] + 2y = \frac{4}{s-2}$$

$$\Rightarrow s^2y + 3s - 5 - 3sy - 9 + 2y = \frac{4}{s-2}$$

$$\Rightarrow s^2y - 3sy + 2y = \frac{4}{s-2} - 3s + 14$$

$$\Rightarrow y(s^2 - 3s + 2) = \frac{4 - 3s^2 + 6s + 14s - 28}{s-2}$$

$$\Rightarrow y = \frac{20s - 3s^2 - 24}{(s^2 - 3s + 2)(s-2)} = \frac{20s - 3s^2 - 24}{(s^2 - 2s + 1 - s + 1)(s-2)}$$

$$= \frac{20s - 3s^2 - 24}{\{(s-1)^2 - (s-1)\}(s-2)}$$

$$= \frac{20s - 3s^2 - 24}{(s-1)(s-1-1)(s-2)}$$

$$= \frac{-3s^2 + 20s - 24}{(s-1)(s-2)^2} \quad \dots \text{(i)}$$

Now,

$$\frac{-3s^2 + 20s - 24}{(s-1)(s-2)^2} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{(s-2)^2} \quad \dots \text{(ii)}$$

$$\Rightarrow 20s - 3s^2 - 24 = A(s-2)^2 + B(s-2)(s-1) + C(s-1) \quad \dots \text{(iii)}$$

$$s=2 \therefore -12 + 40 - 24 = 0 + 0 + C(2-1) \Rightarrow C = 4$$

$$s=1 \therefore -3 + 20 - 24 = A \Rightarrow A = -7$$

$$s=0 \therefore -24 = 4A + 2B - C = -28 + 2B - 4 \Rightarrow B = \frac{8}{2} = 4$$

From equation (ii) we get,

$$\frac{-3s^2 + 20s - 24}{(s-1)(s-2)^2} = \frac{-7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$$

$$\text{From eqn(i)} \Rightarrow y = -\frac{7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$$

$$\Rightarrow L\{Y\} = -\frac{7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$$

$$\Rightarrow Y = L^{-1}\left\{-\frac{7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}\right\}$$

$$= -7L^{-1}\left\{\frac{1}{s-1}\right\} + 4L^{-1}\left\{\frac{1}{s-2}\right\} + 4L^{-1}\left\{\frac{1}{(s-2)^2}\right\}$$

$$= -7e^t + 4e^{2t} + 4te^{2t}$$

Ans.

$$(ii) \quad y'' + 9y' = \cos 2t, \quad y(0) = 1, \quad y(\pi/2) = -1$$

Sol: Here, $y'(0)$ is unknown, so assume $y'(0) = C$

$$\therefore L\{y'' + 9y'\} = L\{\cos 2t\}$$

$$\Rightarrow L\{y''\} + 9L\{y'\} = \frac{s}{s^2+2^2}$$

$$\Rightarrow [s^2 L\{y\} - sy(0) - y'(0)] + 9y = \frac{s}{s^2+4} \quad [\text{Let } L\{y\} = y]$$

$$\Rightarrow s^2y - s - C + 9y = \frac{s}{s^2+4}$$

$$\Rightarrow y(s^2 + 9) = \frac{s}{s^2+4} + s + C$$

$$\Rightarrow y = \frac{s \cancel{(s^2+9)}}{(s^2+9)(s^2+4)} + \frac{s}{(s^2+9)} + \frac{C}{s^2+9}$$

$$\Rightarrow L\{y\} = \frac{s}{s(s^2+4)} - \frac{s}{s(s^2+9)} + \frac{s}{s^2+9} + \frac{C}{s^2+9}$$

$$\Rightarrow y = L^{-1} \left\{ \frac{s}{s(s^2+4)} + \frac{4}{9} \frac{s}{(s^2+9)} + \frac{C}{s^2+9} \right\}$$

$$y(t) = \frac{1}{5} \cos 2t + \frac{4}{9} \cos 3t + \frac{C}{3} \sin 3t$$

$$\therefore Y\left(\frac{\pi}{2}\right) = \frac{4}{5}(0) + \frac{C}{3}(-1) + \frac{1}{5}(-1)$$

$$\Rightarrow -1 = -\frac{C}{3} - \frac{1}{5}$$

$$\Rightarrow C = \left(1 - \frac{1}{5}\right) \times \frac{1}{3} = \frac{12}{5}$$

$$\therefore Y = \frac{4}{5} \cos 3t + \frac{4}{5} \sin 3t + \frac{1}{5} \cos 2t \quad \underline{\text{Ans.}}$$

$$(iii) \quad y'' + 2y' + 5y = e^{-t} \sin t, \quad y(0) = 0, \quad y'(0) = 1$$

$$\text{Sol: } L\{y''\} + 2L\{y'\} + 5L\{y\} = L\{e^{-t} \sin t\}$$

$$\Rightarrow [s^2 L\{y\} - s y(0) - y'(0)] + 2[s L\{y\} - y(0)] + 5y = \frac{1}{(s+1)^2 + 1}$$

$$\Rightarrow [s^2 y - s \cdot 0 - 1] + 2[s y - 0] + 5y = \frac{1}{(s^2 + 2s + 1) + 1}$$

$$\Rightarrow s^2 y - 1 - 2sy + 5y = \frac{1}{s^2 + 2s + 2}$$

$$\Rightarrow y(s^2 - 2s + 5) = \frac{1}{s^2 + 2s + 2} + 1$$

$$\Rightarrow y = \frac{1 + s^2 + 2s + 2}{(s^2 + 2s + 2)(s^2 - 2s + 5)}$$

$$L\{Y\} = \frac{s^2 + 2s + 3}{(s^2 + 2s + 2)(s^2 + 2s + 5)} \dots \text{(i)}$$

[Same as 4(vii)]

Now,

$$\frac{s^2 + 2s + 3}{(s^2 + 2s + 2)(s^2 + 2s + 5)} = \frac{As + B}{s^2 + 2s + 2} + \frac{Cs + D}{s^2 + 2s + 5} \dots \text{(ii)}$$

$$\Rightarrow s^2 + 2s + 3 = (As + B)(s^2 + 2s + 5) + (Cs + D)(s^2 + 2s + 2)$$

$$\begin{aligned} &= As^3 + 2As^2 + 5As + Bs^2 + 2Bs + 5B \\ &\quad Cs^3 + 2Cs^2 + 2Cs + Ds^2 + 2Ds + 2D \end{aligned}$$

$$\begin{aligned} &= s^3(A + C) + s^2(2A + B + 2C + D) + s(5A + 2B + 2C + 2D) \\ &\quad + (5B + 2D) \end{aligned}$$

Equating both sides of the equation we get,

$$A + C = 0 \dots \text{(iii)}$$

$$2A + B + 2C + D = 1 \dots \text{(iv)}$$

$$5A + 2B + 2C + 2D = 2 \dots \text{(v)}$$

$$5B + 2D = 3 \dots \text{--- (vi)}$$

Now, (iii) $\Rightarrow A = 0$ *(vii)*
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$$(vi) \Rightarrow 5B + 2D = 3$$

$$(vii) \Rightarrow B = \frac{3-2D}{5} \dots \dots (viii)$$

$$(iv) \Rightarrow 2(-C) + \frac{3-2D}{5} + 2C + D = 1$$

$$\Rightarrow \frac{3-2D}{5} + D = 1$$

$$\Rightarrow 3 + 3D = 5$$

$$\therefore D = \frac{2}{3}$$

$$(viii) \Rightarrow B = \frac{3 - \frac{4}{3}}{5} = \frac{9-4}{3} \times \frac{1}{5} = \frac{1}{3}$$

$$(v) \Rightarrow 5(-C) + 2 \cdot \frac{1}{3} + 2C + 2 \cdot \frac{2}{3} = 2$$

$$\Rightarrow -3C + \frac{2}{3} + \frac{4}{3} = 2$$

$$\Rightarrow \frac{2+4-6}{3} = 3C$$

$$\Rightarrow C = \frac{0}{3}$$

$$\therefore C = 0$$

Substituting the value of A, B, C & D in eqⁿ (ii)

$$\frac{s^2 + 2s + 3}{(s^2 + 2s + 2)(s^2 + 2s + 5)} = \frac{0.5 + \frac{1}{3}}{s^2 + 2s + 2} + \frac{0.5 + \frac{2}{3}}{s^2 + 2s + 5}$$

From eqⁿ (i)

$$Y = L^{-1} \left\{ \frac{1}{3} \cdot \frac{1}{(s+1)^2 + 1^2} + \frac{2}{3} \cdot \frac{1}{(s+1)^2 + 2^2} \right\}$$

$$= \frac{1}{3} L^{-1} \left\{ \frac{1}{(s+1)^2 + 1^2} \right\} + \frac{1}{3} L^{-1} \left\{ \frac{2}{(s+1)^2 + 2^2} \right\}$$

$$= \frac{1}{3} e^{-t} \sin t + \frac{1}{3} e^{-t} \sin 2t$$

$$= \frac{1}{3} e^{-t} (\sin t + \sin 2t)$$

Ans.

$$(iv) Y''' - 3Y'' + 3Y' - Y = e^t t^2 \quad Y(0)=0, \quad Y'(0)=0, \quad Y''(0)=-2$$

Sol: Let, $L \{Y\} = y$

$$\therefore L \{Y'''\} - 3L \{Y''\} + 3L \{Y'\} - L \{Y\} = L \{e^t t^2\}$$

$$\Rightarrow [s^3 L \{Y\} - s^2 Y(0) - s Y'(0) - Y''(0)] - 3[s^2 L \{Y\} - s Y(0) - Y'(0)]$$

$$+ 3 [sL\{y\} - y(0)] - y = \frac{2!}{(s-1)^{2+1}}$$

$$\Rightarrow [s^3y + 2] - 3[s^2y - 0 - 0] + 3[sy - 0] - y = \frac{2}{(s-1)^3}$$

$$\Rightarrow s^3y + 2 - 3s^2y + 3sy - y = \frac{2}{(s-1)^3}$$

$$\Rightarrow y(s^3 - 3s^2 + 3s - 1) = \frac{2}{(s-1)^3} - 2$$

$$\Rightarrow y = \frac{2 - 2(s-1)^3}{(s^3 - 3s^2 + 3s - 1)(s-1)^3}$$

$$\Rightarrow y(s-1)^3 = \frac{2}{(s-1)^3} - 2$$

$$\Rightarrow y = \frac{2}{(s-1)^6} - \frac{2}{(s-1)^3}$$

$$\Rightarrow L\{y\} = \frac{2}{(s-1)^6} - \frac{2}{(s-1)^3}$$

$$\Rightarrow y = 2L^{-1}\left\{\frac{1}{(s-1)^6}\right\} - 2L^{-1}\left\{\frac{1}{(s-1)^3}\right\}$$

$$= 2e^t \frac{t^5}{5!} - 2e^t \frac{t^2}{2!}$$

$$= \frac{e^t t^5}{60} - e^t t^2$$

Ans.

$$(v) tY'' + 2Y' + tY = 0 \quad Y(0)=1, \quad Y(t)=0, \quad Y'(0)=c$$

$$\text{Sol: } L\{tY''\} + 2L\{Y'\} + L\{tY\} = L\{0\}$$

$$\Rightarrow -\frac{d}{ds} [s^2 L\{Y\} - sY(0) - Y'(0)] + 2[sL\{Y\} - Y(0)]$$

$$-\frac{d}{ds}(y) = 0$$

$$\Rightarrow -\frac{d}{ds} [s^2 y - s - c] + 2(sy - 1) - \frac{dy}{ds} = 0$$

$$\Rightarrow -[s^2 \frac{dy}{ds} + 2sy - 1] + 2sy - 2 - \frac{dy}{ds} = 0$$

$$\Rightarrow -s^2 \frac{dy}{ds} - \frac{dy}{ds} - 2sy + 1 + 2sy - 2 = 0$$

$$\Rightarrow -\frac{dy}{ds} [s^2 + 1] = 1$$

$$\Rightarrow \frac{dy}{ds} = \frac{-1}{s^2 + 1}$$

$$\therefore y = -\tan^{-1}s + c_1$$

$$= -\tan^{-1}s + \frac{\pi}{2}$$

$$= +\tan^{-1}\left(\frac{1}{s}\right)$$

$$\therefore Y = L^{-1}\left\{\tan^{-1}\left(\frac{1}{s}\right)\right\} = \frac{\sin t}{t}$$

$$-\tan^{-1} \alpha + c_1 = 0$$

$$\Rightarrow -\frac{\pi}{2} + c_1 = 0$$

$$\therefore c_1 = \frac{\pi}{2}$$