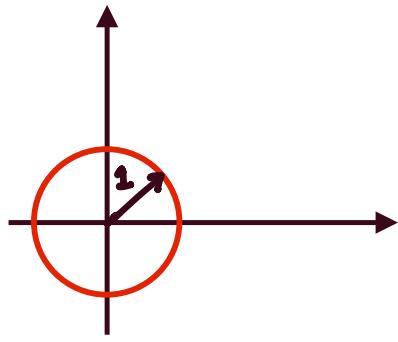


## Chapter 4: Complex Line Integral

### Parametrization

Consider,  $x^v + y^v = 1$



Question: What is the range of values of  $x$ ?  $\rightarrow [-1, 1]$

Question: Is there a way to generate the values of  $x$  in a functional manner?

$$x = \text{const} \quad \text{where } 0 \leq t \leq 2\pi$$

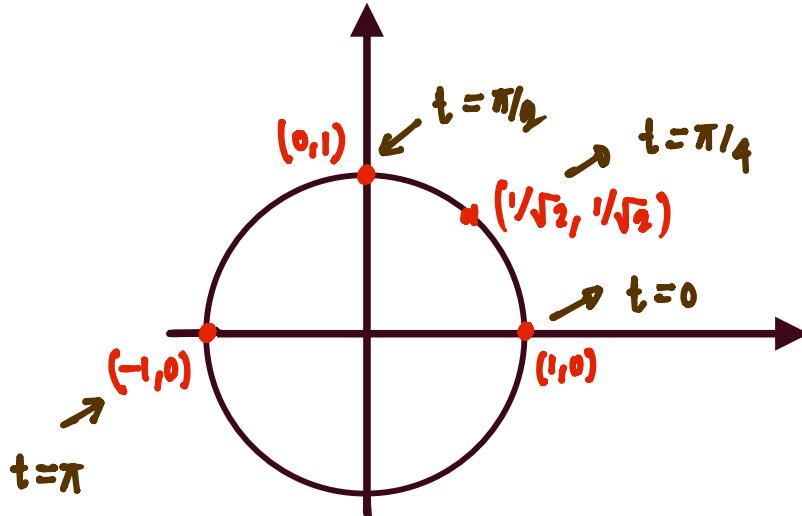
Question: What are the values of  $y$  then?

$$\begin{aligned} \cos^v t + y^v &= 1 \\ \Rightarrow y^v &= \sin^v t \\ \Rightarrow y &= \sin t \text{ (or } -\sin t) \end{aligned}$$

Now summarize,

$$\left. \begin{array}{l} x = \text{const} \\ y = \sin t \end{array} \right\} \text{ where } 0 \leq t \leq 2\pi \quad \text{--- (i)}$$

$t$	0	$\pi/4$	.....	$\pi/2$	.....	$\pi$	.....	$2\pi$
$(x, y)$	$(1, 0)$	$(1/\sqrt{2}, 1/\sqrt{2})$	.....	$(0, 1)$	.....	$(-1, 0)$	.....	$(0, 0)$



(i) Describes the curve  $C$ . These are called parametric equations for  $C$  &  $t$  is the parameter.

**Parametric curve (Defn):** A curve in 2-D plane can be described by,

$$\begin{aligned} x &= f(t) \\ y &= g(t) \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{ where } a \leq t \leq b$$

These are called parametric equations for that curve &  $t$  is the parameter.

Equivalently,

Complex form of parametric equations

$$z = f(t) + ig(t) \text{ where } a \leq t \leq b \rightarrow \text{Describes the same curve as } C$$

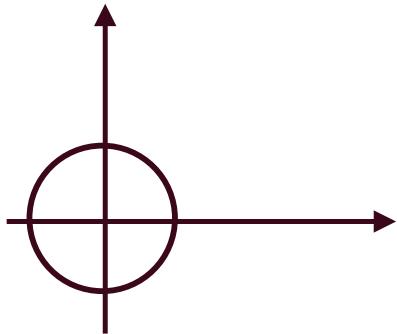
**Remarks:**

(i)  $(f(a), g(a)) \rightarrow$  initial point of the curve

(ii)  $(f(b), g(b)) \rightarrow$  terminal point of the curve

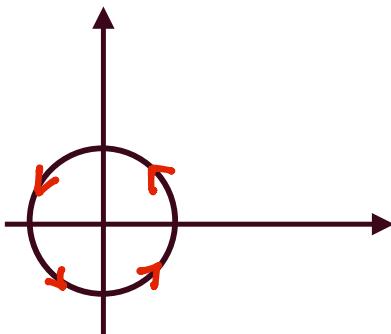
Difference between general and parametric curve:

$$x^2 + y^2 = 1$$



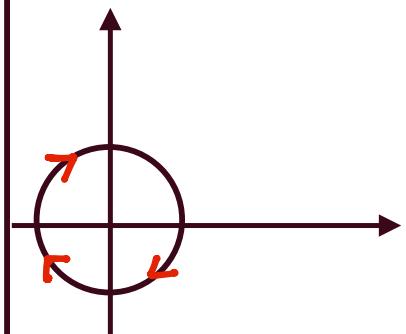
(No Direction)

$$\begin{aligned} x &= \cos t \\ y &= \sin t \\ 0 \leq t &\leq 2\pi \end{aligned}$$



Direction of increasing parameter

$$\begin{aligned} x &= \cos t \\ y &= -\sin t \\ 0 \leq t &\leq 2\pi \end{aligned}$$



Orientation

Orientation

## Problem

Plot the parametric curves defined by,

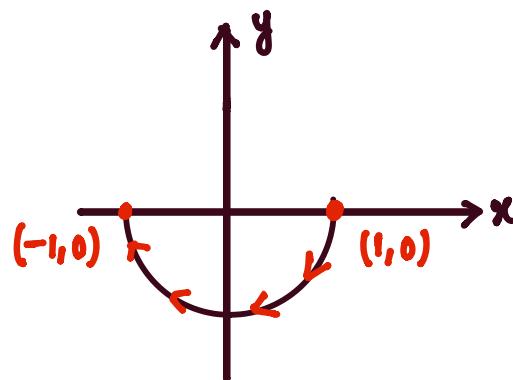
(a)  $x = \cos t, y = -\sin t ; 0 \leq t \leq \pi$

(b)  $x = t, y = 1 ; 0 \leq t \leq 4$

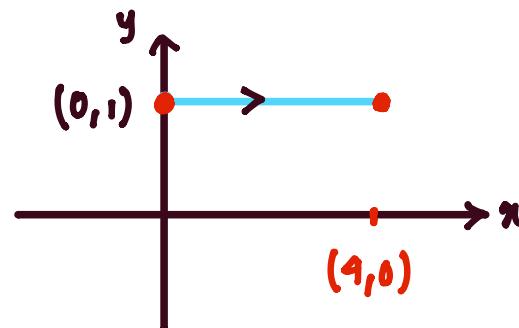
(c)  $z = 1 + it ; 0 \leq t \leq 1$

**Solution:**

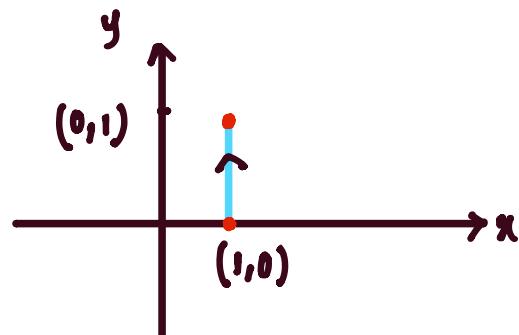
(a)  $\begin{cases} x = \cos t \\ y = -\sin t \end{cases} \quad 0 \leq t \leq \pi$



(b)  $\begin{cases} x = t \\ y = 1 \end{cases} \quad 0 \leq t \leq 4$

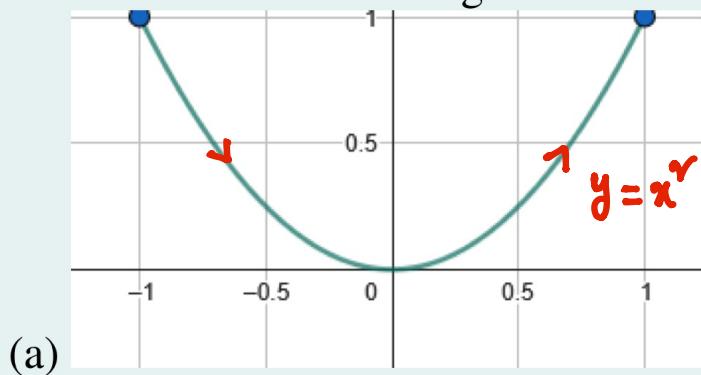


(c)  $\begin{cases} z = 1 \\ y = t \end{cases} \quad 0 \leq t \leq 1$



## Problem

Parametrize the following curves:

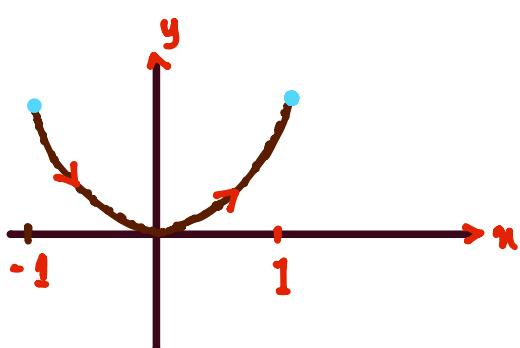


(a)

- (b) Straight line from  $(0, 3)$  to  $(2, 3)$ .  
(c) Straight line from  $(2, 3)$  to  $(2, 4)$ .  
(d) Straight line from  $(0, 3)$  to  $(2, 4)$ .

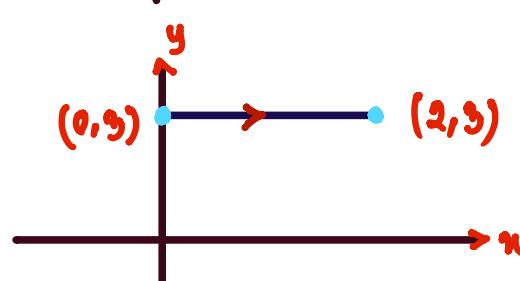
## Solution:

(a)



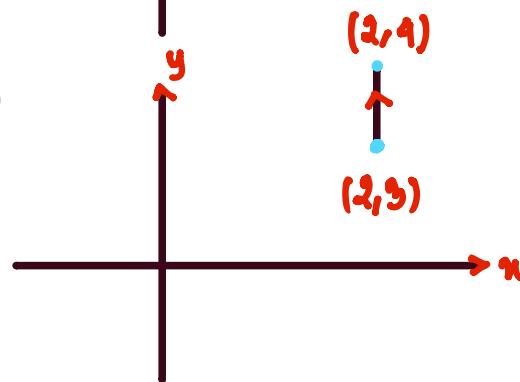
$$\begin{cases} x = t \\ y = t^2 \end{cases} \quad -1 \leq t \leq 1$$

(b)

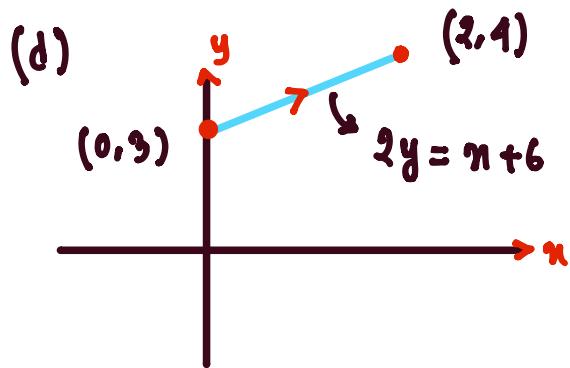


$$\begin{cases} x = t \\ y = 3 \end{cases} \quad 0 \leq t \leq 2$$

(c)



$$\begin{cases} x = 2 \\ y = t \end{cases} \quad 3 \leq t \leq 4$$



$$x = t$$

$$y = \frac{t}{2} + 3 \quad \text{where } 0 \leq t \leq 2$$

One parametrization

Straight line joining  $(x_1, y_1)$  &  $(x_2, y_2)$

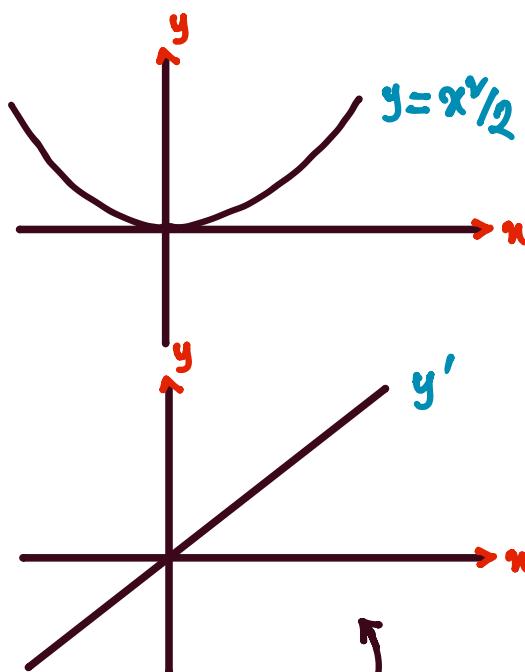
$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1)$$

$$x = 2t - 6$$

$$y = t \quad \text{where } 3 \leq t \leq 4$$

Another parametrization

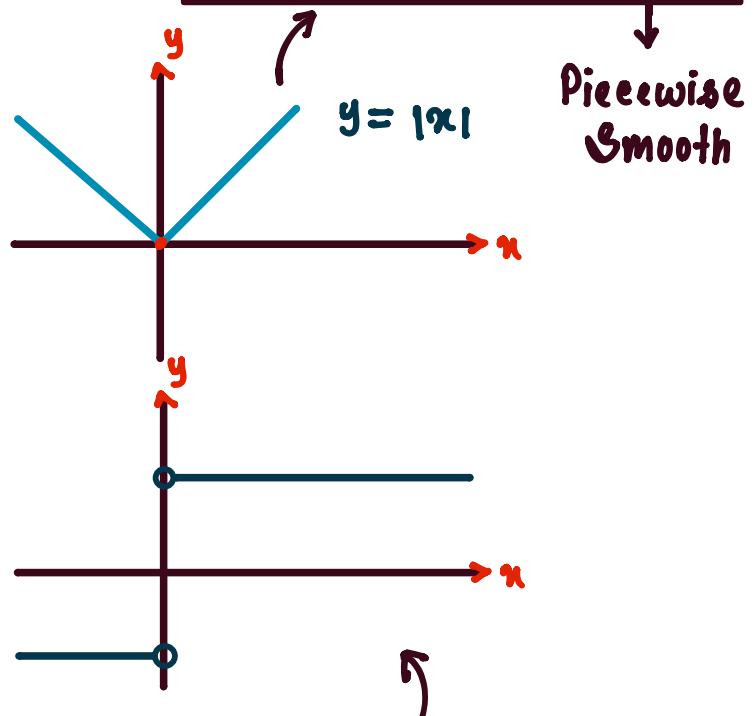
### Smooth Curve:



Derivative exists at all points and is continuous

Smooth curve

Made up of finitely many smooth segments

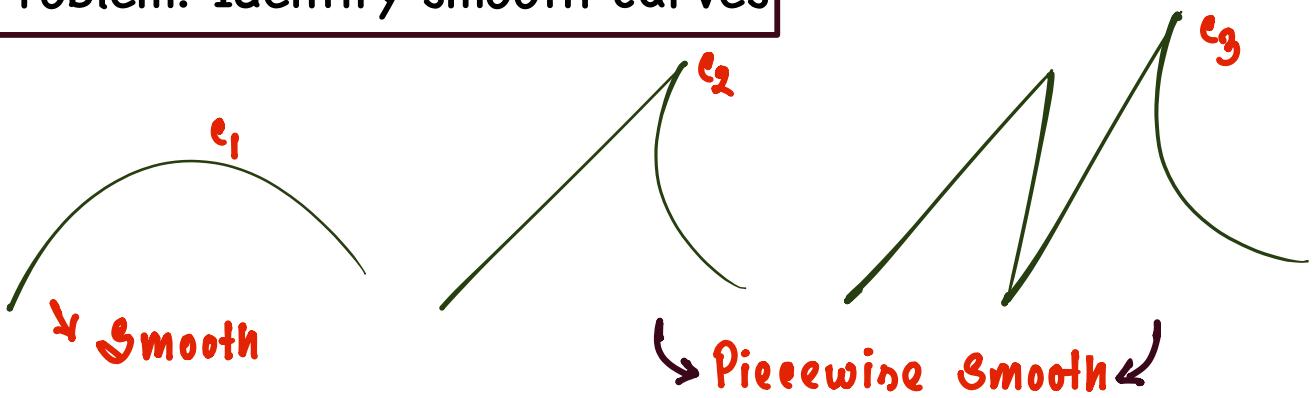


Piecewise Smooth

Derivative is not continuous

Not a smooth curve

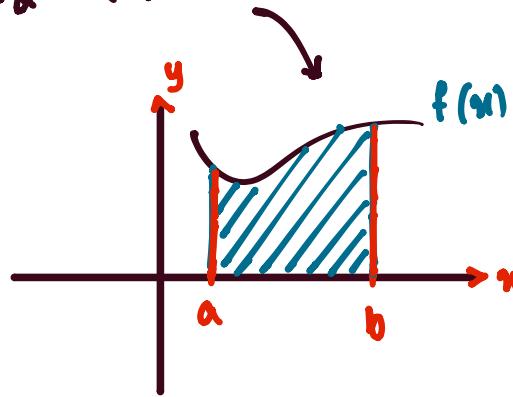
## Problem: Identify smooth curves



## Path in complex integration

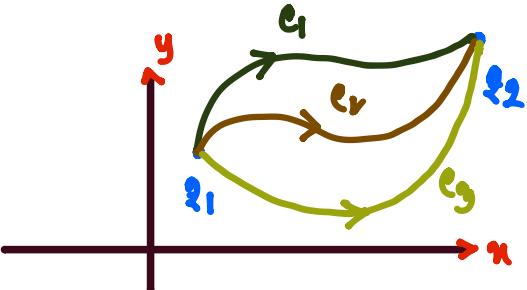
$f(x) \rightarrow$  real valued function

$$\int_a^b f(x) dx$$



$f(z) \rightarrow$  complex function

$$\int_{z_1}^{z_2} f(z) dz \rightarrow \text{meaningful? (x)}$$

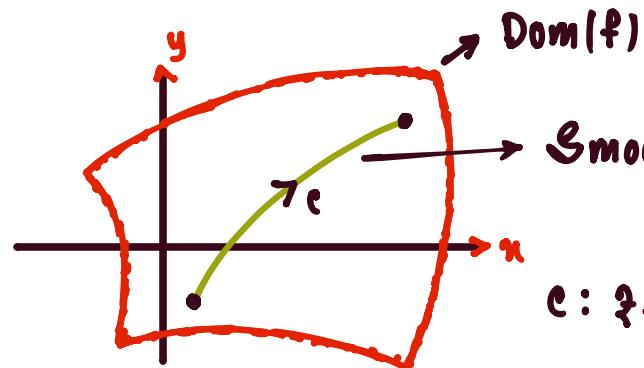


integration will be along a path  $e$

$$\int_e f(z) dz \rightarrow \text{integral of } f \text{ along } e$$

## Integral of a complex function along a smooth curve (Complex line integral)

$f(z) \rightarrow$  complex function



Smooth curve lying in the Domain of  $f$

$$c: z = h(t) + i g(t); \quad a \leq t \leq b \\ = \delta(t)$$

Then line integral (or complex line integral) of  $f$  along  $C$  is defined as

$$\int_C f(z) dz = \int_a^b f(\gamma(t)) \gamma'(t) dt$$

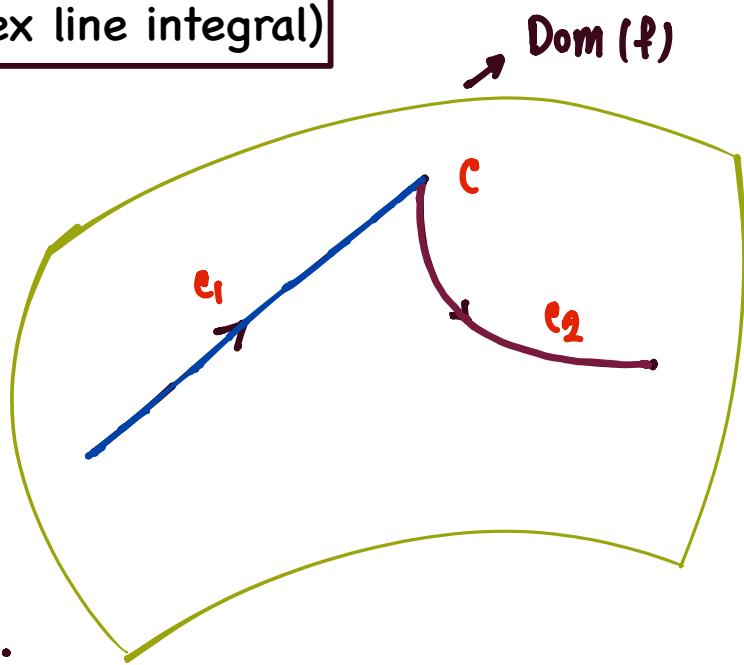
Integral of a complex function along a  
Piecewise smooth curve (Complex line integral)

$f(z) \rightarrow$  complex function

$C \rightarrow$  piecewise smooth curve  
lying in the  $\text{Dom}(f)$

$$C_1: z = h_1(t) + i g_1(t) = \gamma_1(t); \quad a \leq t \leq b$$

$$C_2: z = h_2(t) + i g_2(t) = \gamma_2(t); \quad c \leq t \leq d$$



Then line integral along  $C$  of  $f$  is defined as,

$$\begin{aligned} \int_C f(z) dz &= \int_{C_1} f(z) dz + \int_{C_2} f(z) dz \\ &= \int_a^b f(\gamma_1(t)) \gamma_1'(t) dt + \int_c^d f(\gamma_2(t)) \gamma_2'(t) dt \end{aligned}$$

## Solving complex integrals

Procedure for solving integral of the form  $\int_C f(z) dz$

(i) Find parametric equations for  $C$  in the complex form

$$C: z = \gamma(t); a \leq t \leq b$$

(ii) Apply the definition

$$\int_C f(z) dz = \int_a^b f(\gamma(t)) \gamma'(t) dt$$

### Problem

Evaluate  $\int_C \bar{z} dz$  from  $z = 0$  to  $z = 4 + 2i$  along the curve  $C$  given by  $z = t^2 + it$ .

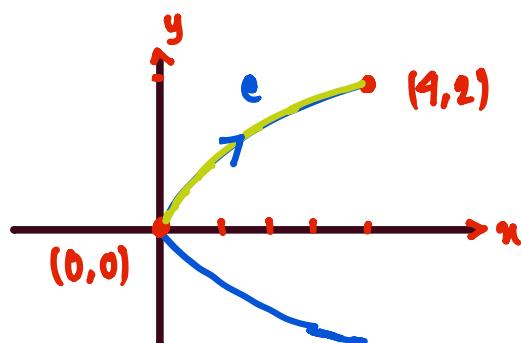
Solution: Step-01: finding parametric equations for  $C$  (in the complex form)

$$C: z = t^2 + it$$

$$\text{initial point: } z=0 \Rightarrow t^2+it=0 \Rightarrow t=0$$

$$\text{terminal point: } z=4+2i \Rightarrow t^2+it=4+2i \Rightarrow t=2$$

$$\therefore C: z = t^2 + it; 0 \leq t \leq 2$$



Step-02: Apply the definition

$$\int_C \bar{z} dz = \int_0^2 \overline{t^2 + it} (2t + i) dt$$

$$\begin{aligned}
 &= \int_0^2 (t^2 - it) (2t + i) dt \\
 &= \int_0^2 (2t^3 - it^2 + t) dt \\
 &= 10 - 8i \quad (\text{Ans:})
 \end{aligned}$$

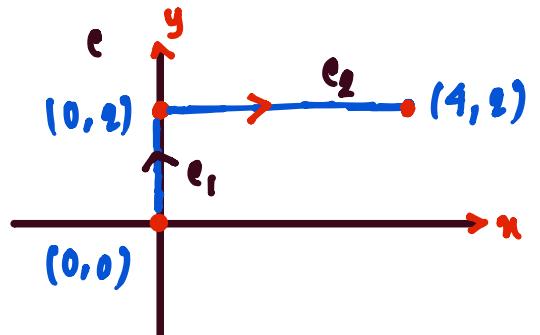
### Problem

Evaluate  $\int_C \bar{z} dz$  along the line from  $z = 0$  to  $z = 2i$  and then from  $z = 2i$  to  $z = 4 + 2i$ .

Sol<sup>n</sup>: Step-01: Find parametric equations for  $C$  in complex form

$$C_1: z = 0 + it; \quad 0 \leq t \leq 2$$

$$C_2: z = t + 2i; \quad 0 \leq t \leq 4$$



Step-02: Apply the definition

$$\begin{aligned}
 \int_C f(z) dz &= \int_{C_1} f(z) dz + \int_{C_2} f(z) dz \\
 &= \int_{C_1} \bar{z} dz + \int_{C_2} \bar{z} dz \\
 &= \int_0^2 (-it) i dt + \int_0^4 \overline{t+2i} \cdot 1 dt \\
 &= \int_0^2 t dt + \int_0^4 (t-2i) dt \\
 &= 10 - 8i \quad (\text{Ans:})
 \end{aligned}$$

Real line integral:

$$C: \begin{cases} x = f(t) \\ y = g(t) \end{cases} \text{ where } a \leq t \leq b \rightarrow \text{Smooth curve}$$

Then the real line integral of  $P(x, y) dx + Q(x, y) dy$  along a curve  $C$  is defined as,

$$\begin{aligned} & \int_C P(x, y) dx + Q(x, y) dy \\ &= \int_a^b P(f(t), g(t)) f'(t) dt + Q(f(t), g(t)) g'(t) dt \\ &= \int_a^b \{P(f(t), g(t)) f'(t) + Q(f(t), g(t)) g'(t)\} dt \end{aligned}$$

Procedure for solving integral of the form

$$\int_C P(x, y) dx + Q(x, y) dy$$

(i) Find parametric equations for  $C$

$$C: \begin{cases} x = f(t) \\ y = g(t) \end{cases} \quad a \leq t \leq b$$

(ii) Apply the definition

$$\int_C P(x, y) dx + Q(x, y) dy = \int_a^b P(f(t), g(t)) f'(t) + Q(f(t), g(t)) g'(t) dt$$

### Problem

Evaluate  $\int_{(0,3)}^{(2,4)} (2y + x^2) dx + (3x - y) dy$  along

(a) the parabola  $x = 2t, y = t^2 + 3$ .

(b) the straight lines from  $(0, 3)$  to  $(2, 3)$  and then from  $(2, 3)$  to  $(2, 4)$ .

(c) the straight line from  $(0, 3)$  to  $(2, 4)$ .

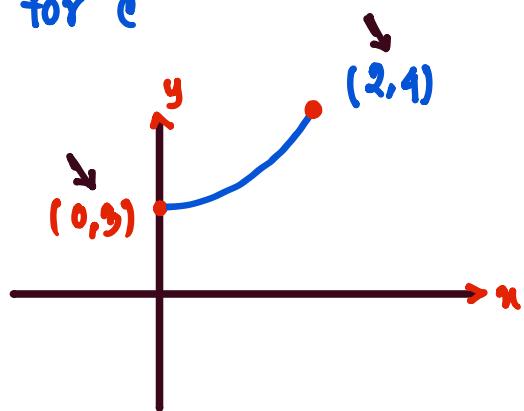
(a) Step-01: Parametric equations for  $\epsilon$

$$\epsilon: \begin{aligned} x &= 2t \\ y &= t^2 + 3 \end{aligned}$$

$$\text{initial point: } 0 = 2t \Rightarrow t = 0$$

$$\text{terminal point: } 2 = 2t \Rightarrow t = 1$$

$$\therefore \epsilon: x = 2t, y = t^2 + 3; 0 \leq t \leq 1$$

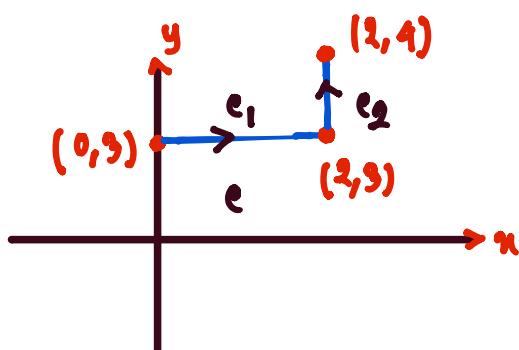


Step-02: Apply the definition

$$\begin{aligned} & \int_{\epsilon} (2y + x^2) dx + (3x - y) dy \\ &= \int_0^1 \left\{ (2t^2 + 6 + 4t^2) \cdot 2 + (6t - t^2 - 3) \cdot 2t \right\} dt \\ &= \frac{33}{2} \quad (\text{Ans:}) \end{aligned}$$

(b) Step-01: Parametric equations for  $\epsilon$

$$\epsilon_1: \begin{aligned} x &= t && \text{where } 0 \leq t \leq 2 \\ y &= 3 \end{aligned}$$



$$\epsilon_2: \begin{aligned} x &= 2 && \text{where } 0 \leq t \leq 1 \\ y &= t \end{aligned}$$

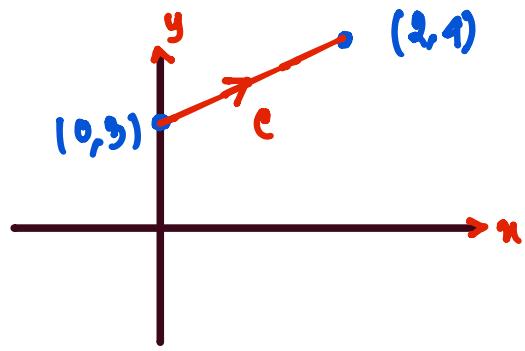
Step-02: Apply the definition

$$\begin{aligned} & \int_{\epsilon} (2y + x^2) dx + (3x - y) dy \\ &= \int_{\epsilon_1} (2y + x^2) dx + (3x - y) dy + \int_{\epsilon_2} (2y + x^2) dx + (3x - y) dy \\ &= \int_0^2 (6 + t^2) \cdot 1 dt + \int_0^1 (6 - t) \cdot 1 dt = \frac{40}{3} + \frac{5}{2} \quad (\text{Ans:}) \end{aligned}$$

### (e) Step-01: Parametric equations

$$\begin{aligned} n &= 2t - 6 \\ y &= t \end{aligned}$$

where  $3 \leq t \leq 4$



### Step-02: Apply the definition

$$\int_C (2y + nx) dn + (3n - y) dy$$

$$= \int_3^4 \{2t + (2t-6)^2\} \cdot 2 + \{3(2t-6) - t\} \cdot 1 dt$$

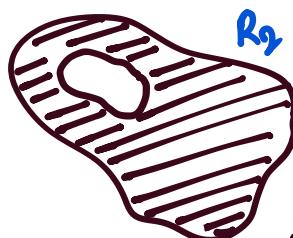
$$= \frac{9x}{6} \quad (\text{Ans.})$$

**Simply Connected region:** A region with no holes

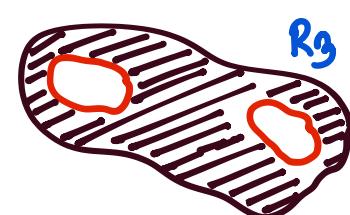
**Multiply connected region:** A region with one or more holes



Simply connected



Multiply connected

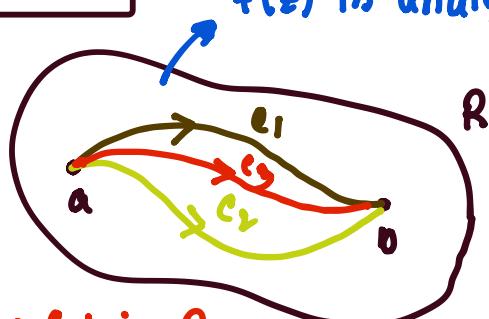


A Result for simply connected region:

$R$ : Simply connected region

$f(z) \rightarrow$  Analytic in  $R$

$\int_a^b f(t) dz \rightarrow$  independent of the path in  $R$  joining points  $a$  &  $b$  in  $R$



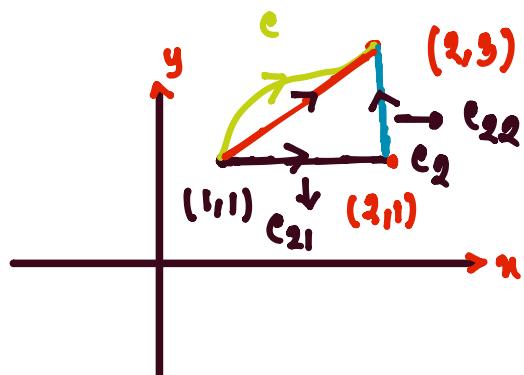
### Problem

If  $C$  is the curve defined by  $y = x^3 - 3x^2 + 4x - 1$  joining  $(1, 1)$  to  $(2, 3)$ , then find  $\int_C (12z^2 - 4iz) dz$

### Solution: Step-01: Reformulation

$f(z) = 12z^2 - 4iz$  is analytic everywhere in the complex plane.

& Entire complex plane is simply connected



The integral is independent of the path joining  $(1,1)$  to  $(2,3)$

Let,

$e_2$ : line segment from  $(1,1)$  to  $(2,1)$  & then from  $(2,1)$  to  $(2,3)$

$$\int_e (12z^2 - 4iz) dz = \int_{e_2} (12z^2 - 4iz) dz$$

Step-02:  $\int_{e_2} (12z^2 - 4iz) dz$  computation

(2.1)  $e_2$  parametric form:

$$C_{21}: z = t + i ; 1 \leq t \leq 2$$

$$C_{22}: z = 2 + it ; 1 \leq t \leq 3$$

(2.2) Apply the definition

$$\int_{e_2} (12z^2 - 4iz) dz = \int_{C_{21}} (12z^2 - 4iz) dz + \int_{C_{22}} (12z^2 - 4iz) dz$$

$$= \int_1^2 \{12(t+i)^2 - 4i(t+i)\} \cdot 1 dt$$

$$+ \int_1^3 \{12(2+it)^2 - 4i(2+it)\} \cdot i dt$$

$$= -156 + 38i$$

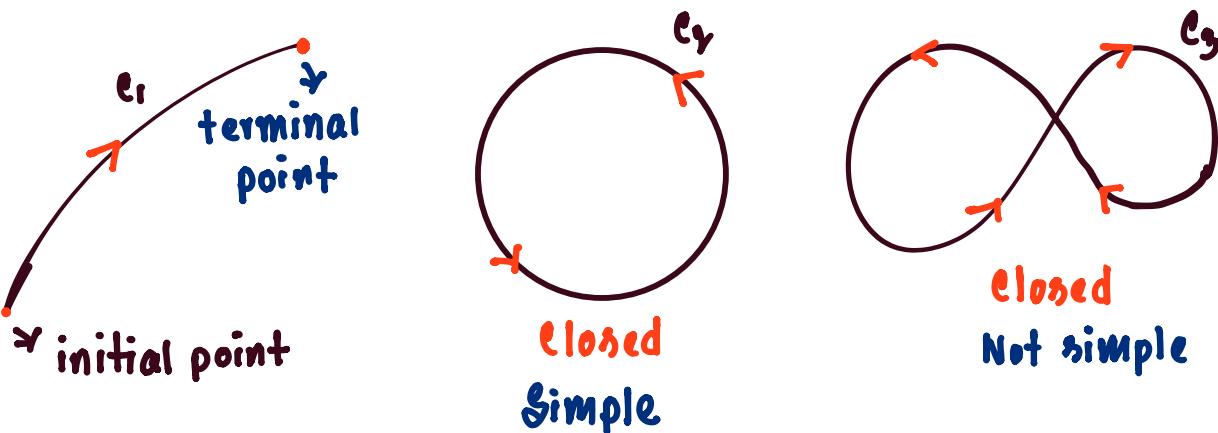
$$\int_{C_2} (12z^2 - 9iz) dz = -156 + 38i$$

$$\Rightarrow \int_C (12z^2 - 9iz) dz = -156 + 38i \quad (\text{Ans:})$$

(No self intersection)

## Closed curve and simple closed curve

(initial point = terminal point)



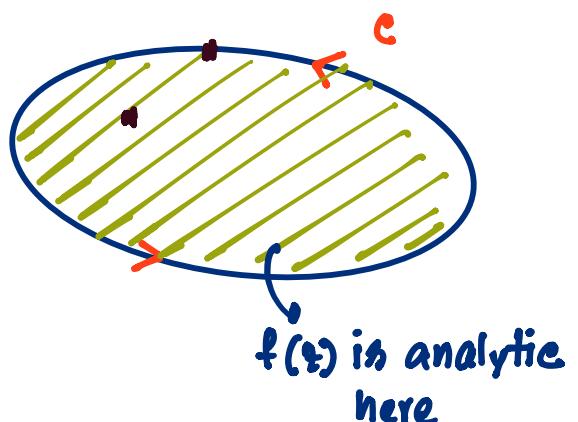
## Cauchy's theorem (Cauchy Goursat theorem)

$c$ : closed curve

$f(z) \rightarrow$  analytic in the region enclosed by  $c$  (inside & on  $c$ )

Then,

$$\int_c f(z) dz = 0$$



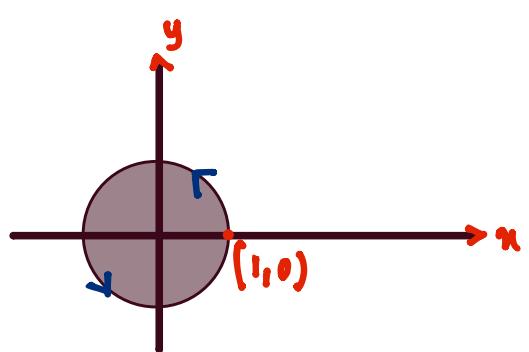
$$\text{Example: } f(z) = 12z^2 - 9z + 4i$$

$c$ : a circle with centre  $(0,0)$ , radius 1 (counter-clockwise)

$$\int_c f(z) dz = ?$$

$f(z)$  is analytic everywhere & so is on the region enclosed by  $c$

$\therefore \int_c f(z) dz = 0$  (Using Cauchy's theorem)

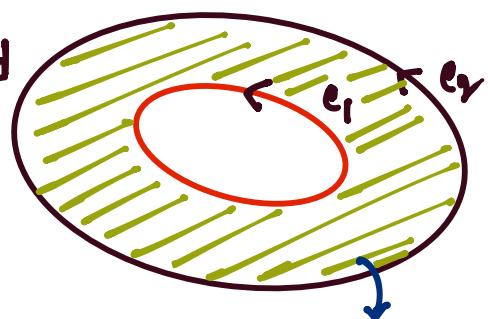


# A Result for region bounded by two simple closed curves

$\epsilon_1, \epsilon_2 \rightarrow$  Simple closed curves (with same direction)

$f(z) \rightarrow$  analytic in the region enclosed between  $\epsilon_1$  &  $\epsilon_2 \Rightarrow$

$$\int_{\epsilon_1} f(z) dz = \int_{\epsilon_2} f(z) dz$$



$f(z)$  is analytic here

## Parametrizing a circle

$|z| = r$  Mod ( $z$ ) is fixed  
We don't know about the argument  
Argument can be anything

$$\therefore z = re^{i\theta}; 0 \leq \theta \leq 2\pi$$

- circle with centre at  $z_0$ , radius  $r$

$$|z - z_0| = r$$

$$\Rightarrow z - z_0 = re^{i\theta}; 0 \leq \theta \leq 2\pi$$

$$\Rightarrow z = z_0 + re^{i\theta}; 0 \leq \theta \leq 2\pi$$

Parametric equation of a circle

Convention:

(i)  $\int_{\epsilon} f(z) dz \rightarrow \oint_{\epsilon} f(z) dz$  if  $\epsilon$  is closed

(ii) Assume counter-clockwise direction if not mentioned explicitly.

## Problem

Evaluate  $\oint_C \frac{dz}{(z-a)}$  where  $C$  is any simple closed curve and

(a)  $z=a$  is outside  $C$ .

(b)  $z=a$  is inside  $C$ .

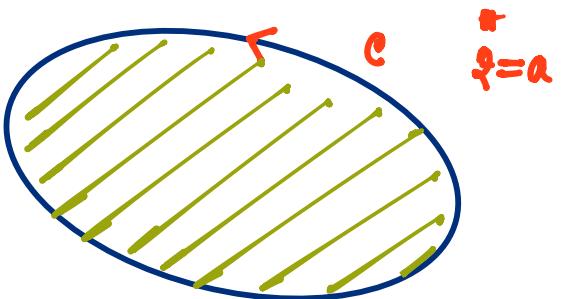
**Solution:** (a)  $f(z) = \frac{1}{z-a}$

$f(z)$  is analytic on  $\mathbb{C} \setminus \{a\}$

$\therefore z=a$  lies outside of  $C$

$\Rightarrow f(z)$  is analytic in the region enclosed by  $C$

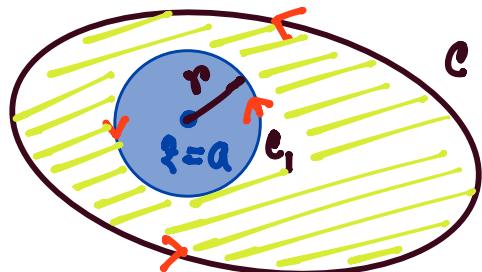
$\therefore \oint_C f(z) dz = 0$  (Using Cauchy's theorem)



(b)  $z=a$  is inside  $C$ :

**Step-01:** Reformulation

Construct a circle of radius  $r$ ,  
centred at  $z=a$  which lies within  $C$   
(& denote it by  $e_1$ )



$\therefore \oint_C f(z) dz = \oint_{e_1} f(z) dz$

**Step-02:**  $\oint_{e_1} f(z) dz$  computation

$e_1: z = a + re^{i\theta}; 0 \leq \theta \leq 2\pi$

$$\therefore \oint_{e_1} \frac{1}{z-a} dz = \int_0^{2\pi} \frac{1}{a+re^{i\theta}} re^{i\theta} \cdot i d\theta = \int_0^{2\pi} i d\theta = 2\pi i$$

$$\therefore \oint_{C_1} \frac{1}{z-a} dz = 2\pi i$$

$$\Rightarrow \oint_C \frac{1}{z-a} dz = 2\pi i \quad (\text{Ans.})$$

### ~~H.W.~~ Problem

Evaluate  $\oint_C \frac{dz}{(z-a)^n}$  where

- (a)  $z = a$  is outside  $C$ .
- (b)  $z = a$  is inside  $C$ .

for any simple closed curve  $C$  and  $n \geq 2$

### Problem

Evaluate  $\oint_C |z|^2 dz$  around the square with vertices at  $(0,0), (1,0), (1,1), (0,1)$ .

### Solution:

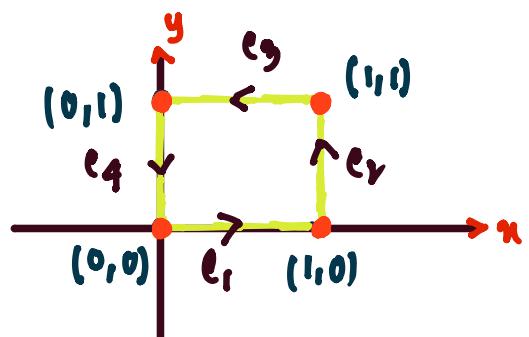
Step-01: Find parametric equations

$$C_1: z = t + i0 ; 0 \leq t \leq 1$$

$$C_2: z = 1 + it ; 0 \leq t \leq 1$$

$$C_3: z = (1-t) + i ; 0 \leq t \leq 1$$

$$C_4: z = (1-t)i ; 0 \leq t \leq 1$$



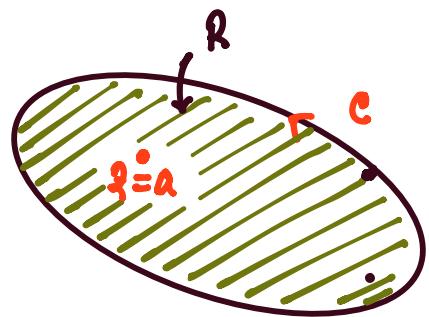
## Step-02: Apply the definition

$$\begin{aligned}\oint_C |z|^v dz &= \int_{C_1} |z|^v dz + \int_{C_2} |z|^v dz + \int_{C_3} |z|^v dz + \int_{C_4} |z|^v dz \\&= \int_0^1 |t|^v \cdot 1 dt + \int_0^1 |1+it|^v i dt + \int_0^1 |(1-t)+it|^v (-1) dt \\&\quad + \int_0^1 |(1-t)+it|^v (-i) dt \\&= \int_0^1 t^v dt + \int_0^1 (1+t^v) i dt + \int_0^1 \{(1-t)^v + 1\} (-i) dt \\&\quad + \int_0^1 (1-t)^v (-i) dt \\&= 1/3 + i 4/3 - 4/3 - i/3 = i - 1 \text{ (Ans.)}\end{aligned}$$

## Chapter 5: Cauchy's Integral Formula :

$c$ : Simple closed curve (anti-clockwise)

$f(z) \rightarrow$  analytic in the region enclosed by  $c$   
(inside & on  $c$ )



$z=a \rightarrow$  a point inside  $c$ . Then

$$(i) \oint_c \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

$$(ii) \oint_c \frac{f(z)}{(z-a)^n} dz = \frac{2\pi i f^{(n-1)}(a)}{(n-1)!} \quad \forall n \geq 2$$

### Problem

Evaluate  $\frac{1}{2\pi i} \oint_C \frac{e^z}{z-2} dz$  if  $C$  is:

- (a) the circle  $|z| = 3$ , (b) the circle  $|z| = 1$ .

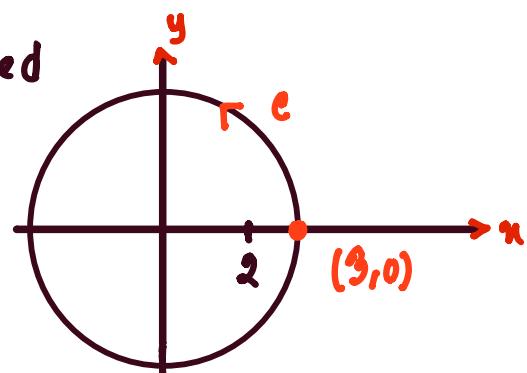
Solution: (a)  $f(z) = e^z$  is analytic everywhere

$\therefore f(z)$  is analytic in the region enclosed by  $c$

&  $z=2$  lies inside  $c$

$$\oint_c \frac{e^z}{z-2} dz = 2\pi i f(2) = 2\pi i e^2$$

$$\Rightarrow \frac{1}{2\pi i} \oint_c \frac{e^z}{z-2} dz = e^2 \quad (\text{Ans:})$$

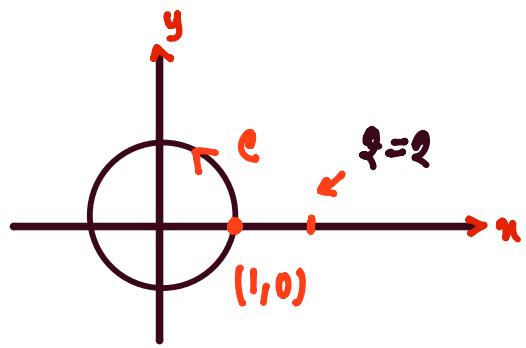


(b)  $\frac{e^z}{z-2}$  is analytic on  $\mathbb{C} \setminus \{2\}$

$\Rightarrow \frac{e^z}{z-2}$  is analytic in the region enclosed by  $C$

$$\Rightarrow \oint_C \frac{e^z}{z-2} dz = 0 \quad (\text{Using Cauchy's Theorem})$$

$$\Rightarrow \frac{1}{2\pi i} \oint_C \frac{e^z}{z-2} dz = 0 \quad (\text{Ans:})$$



### Problem

Evaluate  $\oint_C \frac{e^{2z}}{(z+1)^4} dz$  where  $C$  is the circle  $|z| = 3$ .

Solution:  $f(z) = e^{2z}$  is analytic everywhere

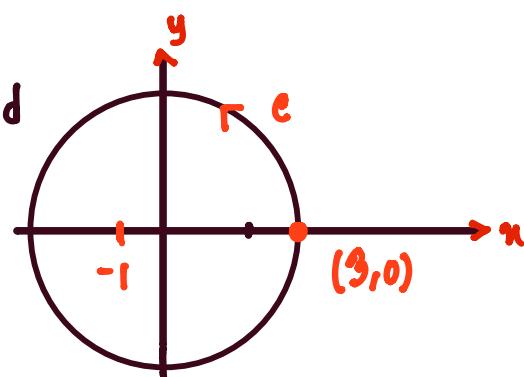
$\therefore f(z)$  is analytic in the region enclosed by  $C$

&  $z = -1$  lies inside  $C$

$$\therefore \oint_C \frac{e^{2z}}{(z+1)^4} dz = \frac{2\pi i f'''(-1)}{3!}$$

$$= \frac{2\pi i \cdot 8}{6 e^r}$$

$$= \frac{8\pi i}{3e^r} \quad (\text{Ans:})$$



$$f(z) = e^{2z}$$

$$f'(z) = 2e^{2z}$$

$$f'''(z) = 8e^{2z}$$

$$f'''(-1) = \frac{8}{e^r}$$

## Problem

Evaluate  $\oint_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)(z-2)} dz$  where  $C$  is the circle  $|z| = 3$ .

**Solution:** Step-01: Partial fraction decomposition.

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$\Rightarrow 1 = A(z-2) + B(z-1)$$

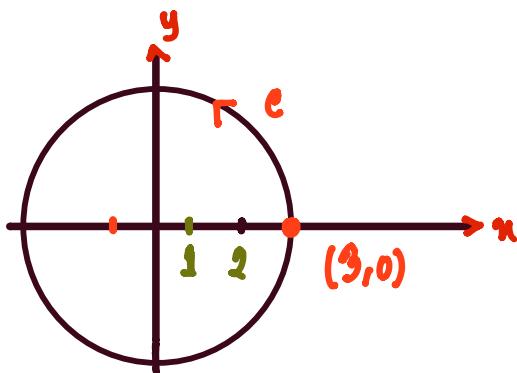
$$(i) z=1 \Rightarrow 1 = A(-1) \Rightarrow A = -1$$

$$(ii) z=2 \Rightarrow 1 = B(1) \Rightarrow B = 1$$

$$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$

$$\Rightarrow \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)(z-2)} = \frac{\sin(\pi z^2) + \cos(\pi z^2)}{z-2} - \frac{\sin(\pi z^2) + \cos(\pi z^2)}{z-1}$$

$$\Rightarrow \oint_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)(z-2)} dz = \oint_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{z-2} dz - \oint_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{z-1} dz$$



$$= 2\pi i \left\{ \sin(4\pi) + \cos(4\pi) \right\}$$

$$- 2\pi i \left\{ \sin(\pi) + \cos(\pi) \right\}$$

$$= 2\pi i \left\{ \cos(4\pi) - \cos(\pi) \right\}$$

$$= 2\pi i \left\{ 1 - (-1) \right\} = 4\pi i \quad (\text{Ans:})$$

Singular point (of  $f(z)$ ) : A point at which  $f(z)$  fails to be analytic

Pole :  $z = z_0 \rightarrow$  singular point of  $f(z)$

$$\lim_{z \rightarrow z_0} (z - z_0)^n f(z) = A \neq 0 \text{ for some } n \in \mathbb{Z}^+$$

(Some defined value)

$\left. \begin{array}{l} z = z_0 \text{ is a pole of order } n \end{array} \right\}$

(\*) if  $n=1 \rightarrow z_0 \rightarrow$  simple pole

Example : (i)  $f(z) = \frac{1}{(z-2)^3} \rightarrow z=2$

$\therefore z=2$  is a pole of order 3

$$\therefore \lim_{z \rightarrow 2} (z-2)^3 f(z) = 1$$

↳ Pole of order 2

(ii)  $f(z) = \frac{3z-2}{(z-1)^5 (z+1)^3 (z-4)^5}, z=1, -1, 4 \rightarrow$  Pole of order 5  
 ↳ Simple pole

(iii)  $z=a$  is a pole of order  $n$  of  $f(z)$

$$\therefore f(z) = \frac{g(z)}{(z-a)^n}$$

lemma :  $f(z) \rightarrow$  complex function

$f$  is analytic in a region  $R$

$\Rightarrow$  all its higher derivatives also exist in  $R$



**Question:** (i) Why → (Hint: Cauchy's Integral formula)

(ii) Is it true for real valued function?

**Problem-09:**

$C$ : closed curve (anti-clockwise)

$R$ : Region enclosed by  $C$

$F(z) \rightarrow$  Analytic on  $R \setminus \{a\}$

$z=a \rightarrow$  pole of order  $m$  inside  $C$  of  $F(z)$

$$\frac{1}{2\pi i} \oint_C F(z) dz = \lim_{z \rightarrow a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \{(z-a)^m F(z)\}$$

**Solution:**  $z=a \rightarrow$  pole of order  $m$  of  $F(z)$

$F(z) = \frac{f(z)}{(z-a)^m}$  on  $R$ ,  $f(z)$  is analytic on  $R$

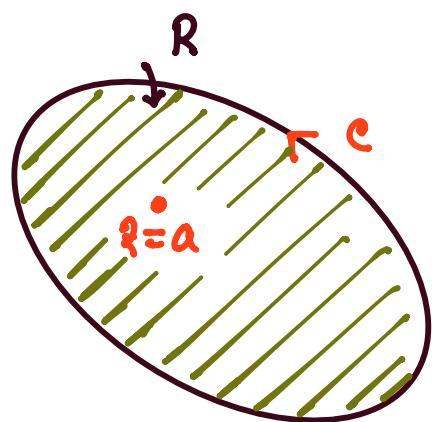
$$\frac{1}{2\pi i} \oint_C F(z) dz = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^m} dz$$

$$= \frac{1}{2\pi i} \frac{2\pi i f^{(m-1)}(a)}{(m-1)!}$$

$$= \frac{1}{(m-1)!} \left. \frac{d^{m-1}}{dz^{m-1}} (f(z)) \right|_{z=a}$$

$$= \frac{1}{(m-1)!} \lim_{z \rightarrow a} \frac{d^{m-1}}{dz^{m-1}} (f(z))$$

$$= \frac{1}{(m-1)!} \lim_{z \rightarrow a} \frac{d^{m-1}}{dz^{m-1}} (f(z)(z-a)^m)$$



Why?

$$\therefore \frac{1}{2\pi i} \oint_C F(z) dz = \lim_{z \rightarrow a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} (F(z)(z-a)^m)$$

$F(z)$  has a single pole inside  $C$

$C$  : Closed curve

$R$  : Region enclosed by  $C$

$z = a$  : a pole of order  $m$  inside  $C$  of  $F(z)$

$F(z)$  : analytic on  $R \setminus \{a\}$

Then,

$$\frac{1}{2\pi i} \oint_C F(z) dz = \lim_{z \rightarrow a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \{(z-a)^m F(z)\}$$

### Problem

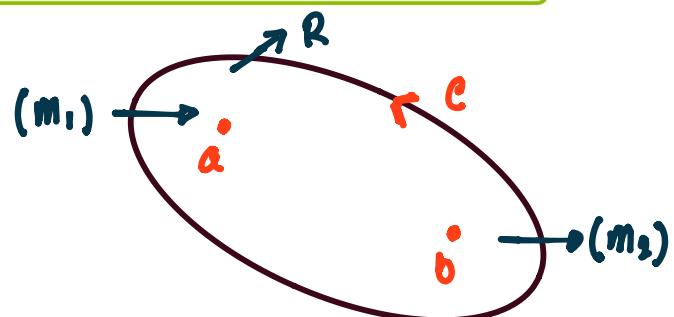
Modify the result of previous problem in case of multiple poles.

$c$  : closed curve (anti-clockwise)

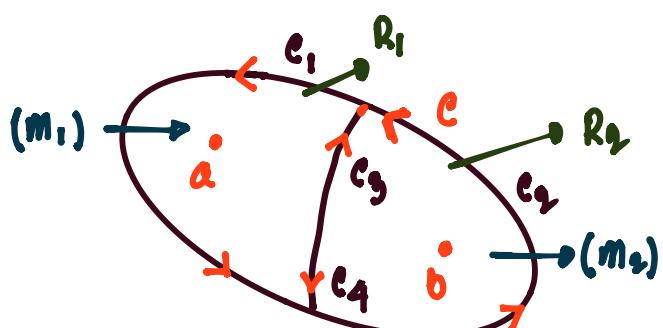
$R$  : region enclosed by  $c$

$F(z) \rightarrow$  Analytic on  $R \setminus \{a, b\}$

$z = a, b \rightarrow$  pole of order  $m_1, m_2$  inside of  $F(z)$



We need :  $\frac{1}{2\pi i} \oint_c F(z) dz$



$f(z)$  is analytic on  $\Omega_1 \setminus \{a\}$

$$\frac{1}{2\pi i} \oint_{C_1 \cup C_3} F(z) dz = \lim_{z \rightarrow a} \frac{1}{(m_1-1)!} \frac{d^{m_1-1}}{dz^{m_1-1}} \{ F(z) (z-a)^{m_1} \} \quad (i)$$

$f(z)$  is analytic on  $\Omega_2 \setminus \{b\}$

$$\frac{1}{2\pi i} \oint_{C_2 \cup C_4} F(z) dz = \lim_{z \rightarrow b} \frac{1}{(m_2-1)!} \frac{d^{m_2-1}}{dz^{m_2-1}} \{ F(z) (z-b)^{m_2} \} \quad (ii)$$

$$(i) + (ii) \Rightarrow$$

$$\frac{1}{2\pi i} \left( \oint_{C_1 \cup C_3} F(z) dz + \oint_{C_2 \cup C_4} F(z) dz \right) = M_1 + M_2$$

$$\Rightarrow \frac{1}{2\pi i} \left( \int_{C_1} F(z) dz + \int_{C_3} F(z) dz + \int_{C_2} F(z) dz + \int_{C_4} F(z) dz \right)$$

$$= M_1 + M_2$$

$$\Rightarrow \frac{1}{2\pi i} \left( \int_{C_1} F(z) dz + \int_{C_2} F(z) dz \right) = M_1 + M_2$$

$$\Rightarrow \frac{1}{2\pi i} \oint_{C_1 \cup C_2} F(z) dz = M_1 + M_2$$

$$\Rightarrow \boxed{\frac{1}{2\pi i} \oint_C F(z) dz = \lim_{z \rightarrow a} \frac{1}{(m_1-1)!} \frac{d^{m_1-1}}{dz^{m_1-1}} \{ F(z) (z-a)^{m_1} \} + \lim_{z \rightarrow b} \frac{1}{(m_2-1)!} \frac{d^{m_2-1}}{dz^{m_2-1}} \{ F(z) (z-b)^{m_2} \}}$$

$F(z)$  has a two poles inside  $C$

$C$  : Closed curve

$R$  : Region enclosed by  $C$

$z = a, b$  : poles order  $m_1, m_2$  inside  $C$  of  $F(z)$



$F(z)$  : analytic on  $R \setminus \{a, b\}$

Then,

$$\begin{aligned} \frac{1}{2\pi i} \oint_C F(z) dz &= \lim_{z \rightarrow a} \frac{1}{(m_1 - 1)!} \frac{d^{m_1-1}}{dz^{m_1-1}} \{(z-a)^{m_1} F(z)\} \\ &\quad + \lim_{z \rightarrow b} \frac{1}{(m_2 - 1)!} \frac{d^{m_2-1}}{dz^{m_2-1}} \{(z-b)^{m_2} F(z)\} \end{aligned}$$

$F(z)$  has n poles inside  $C$

$C$  : Closed curve

$R$  : Region enclosed by  $C$

$z = a_k$  : poles of order  $m_k$  inside  $C$  of  $F(z)$ , where  $k = 1, 2, \dots, n$

$F(z)$  : analytic on  $R \setminus \{a_1, a_2, \dots, a_n\}$



Then,

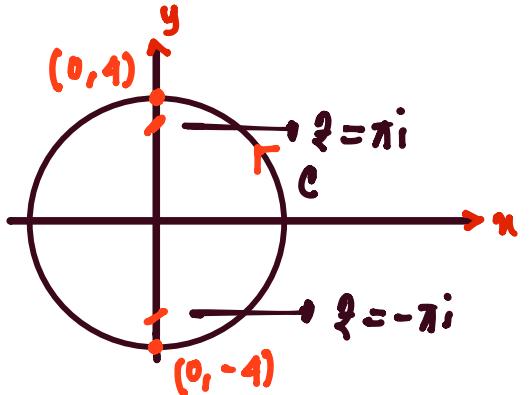
$$\frac{1}{2\pi i} \oint_C F(z) dz = \sum_{k=1}^n \lim_{z \rightarrow a_k} \frac{1}{(m_k - 1)!} \frac{d^{m_k-1}}{dz^{m_k-1}} \{(z-a_k)^{m_k} F(z)\}.$$

## Problem

Evaluate  $\oint_C \frac{e^z}{(z^2 + \pi^2)^2} dz$  where  $C$  is the circle  $|z| = 4$ .

$$\begin{aligned}
 \text{Solution: } f(z) &= \frac{e^z}{(z^r + \pi^r)^r} = \frac{e^z}{(z^r - i^r \pi^r)^r} \\
 &= \frac{e^z}{f(z + \pi i) (z - \pi i)^r} = \frac{e^z}{(z + \pi i)^r (z - \pi i)^r}
 \end{aligned}$$

$f(z)$  has poles at  $z = \pm \pi i$  both of order 2 lying inside  $C$



$$\begin{aligned}
 & \frac{1}{2\pi i} \oint_C f(z) dz \\
 &= \lim_{z \rightarrow -\pi i} \frac{1}{1!} \frac{d}{dz} \left( \frac{e^z}{(z - \pi i)^r} \right) + \lim_{z \rightarrow \pi i} \frac{1}{1!} \frac{d}{dz} \left( \frac{e^z}{(z + \pi i)^r} \right) \\
 &= \lim_{z \rightarrow -\pi i} \frac{(z - \pi i)^r e^z - e^z z (z - \pi i)}{(z - \pi i)^2} \\
 &\quad + \lim_{z \rightarrow \pi i} \frac{(z + \pi i)^r e^z - e^z z (z + \pi i)}{(z + \pi i)^2} \\
 &= \lim_{z \rightarrow -\pi i} \frac{f(z - \pi i) - 2 f(z)}{(z - \pi i)^3} + \lim_{z \rightarrow \pi i} \frac{f(z + \pi i) - 2 f(z)}{(z + \pi i)^3} \\
 &= \frac{(-2\pi i - 2)e^{-\pi i}}{(-2\pi i)^3} + \frac{(2\pi i - 2)e^{\pi i}}{(2\pi i)^3} = \dots = \frac{1}{2\pi^r}
 \end{aligned}$$

$$\frac{1}{2\pi i} \oint_C f(z) dz = \frac{1}{2\pi r}$$

$$\Rightarrow \oint_C f(z) dz = \frac{2\pi i}{2\pi r} = \frac{i}{r} \quad (\text{Ans:})$$

~~H.W.~~ Problem

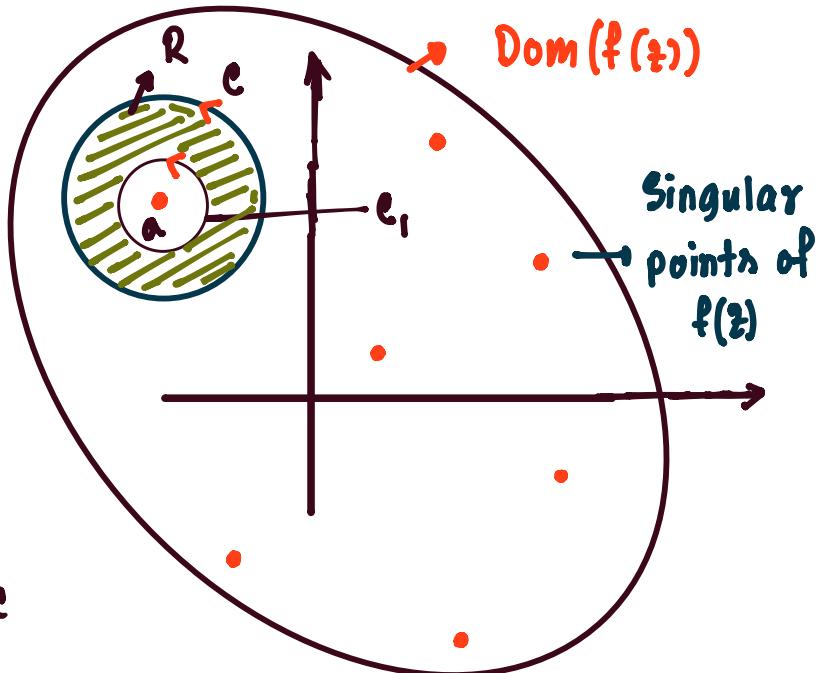
Evaluate  $\frac{1}{2\pi i} \oint_C \frac{e^{zt}}{(z^2 + 1)^2} dz$  when  $t > 0$  and  $C$  is the circle  $|z| = 3$ .

## Chapter 7: Cauchy's Residue Theorem

What is residue:

$f(z)$  is analytic on  $R$

$$\Rightarrow \oint_C f(z) dz = \oint_{\epsilon_1} f(z) dz$$



Also since  $f(z)$  is analytic

on  $R \Rightarrow$

$$\therefore f(z) = \dots + \frac{a_{-2}}{(z-a)^2} + \frac{a_{-1}}{(z-a)} + a_0 + a_1(z-a) + a_2(z-a)^2 + \dots$$

on  $R$  (Laurent series expansion about  $z=a$ )

$$\begin{aligned} \Rightarrow \oint_{\epsilon_1} f(z) dz &= \dots + a_{-2} \oint_{\epsilon_1} \frac{1}{(z-a)^2} dz + a_{-1} \oint_{\epsilon_1} \frac{1}{z-a} dz \\ &\quad + \oint_{\epsilon_1} \{a_0 + a_1(z-a) + \dots\} dz \\ &= 0 + \dots + 0 + a_{-1}(2\pi i) + 0 \\ &= a_{-1}(2\pi i) \end{aligned}$$

$$\oint_{\Gamma} f(z) dz = a_{-1} (2\pi i)$$

$$\oint_{\Gamma} f(z) dz = a_{-1} (2\pi i) \Rightarrow a_{-1} = \frac{1}{2\pi i} \oint_{\Gamma} f(z) dz$$

Thus  $a_{-1}$ : Residue of  $f(z)$  at  $z=a \rightarrow \text{Res}(f, a)$

Calculation of residue:

Residue of  $f(z)$  at  $z=a$  (singular point)

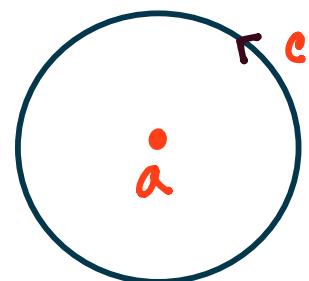
↓  
Laurent series expansion about  $z=a$   
↓

$a_{-1}$  (co-efficient of  $(z-a)^{-1}$ )

case:  $z=a$  is a pole of order  $m$

$$\text{Res}(f, a) = \frac{1}{2\pi i} \oint_{\Gamma} f(z) dz$$

$$\text{Res}(f, a) = \lim_{z \rightarrow a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \{ f(z) (z-a)^m \}$$



Cauchy's Residue Theorem:

$\Gamma$ : Simple closed curve (anti-clockwise)

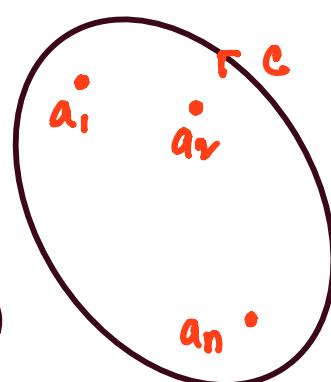
$R$ : region enclosed by  $\Gamma$

$a_1, a_2, \dots, a_n$ : singular points inside  $\Gamma$  of  $f(z)$

$f(z)$ : analytic on  $R \setminus \{a_1, a_2, \dots, a_n\}$ . Then,

$\oint_{\Gamma} f(z) dz = 2\pi i$  (sum of the residues of singularities inside  $\Gamma$ )

$$\oint_{\Gamma} f(z) dz = 2\pi i \sum_{i=1}^n \text{Res}(f, a_i)$$



## Cauchy's Residue Theorem

$C$  : Closed curve (anti-clockwise)

$R$  : Region enclosed by  $C$

$a_1, a_2, \dots, a_n$  : singularities inside  $C$  of  $F(z)$

$F(z)$  : analytic on  $R \setminus \{a_1, a_2, \dots, a_n\}$

Then,

$$\oint_C F(z) dz = 2\pi i (\text{Sum of residues of singularities of } f(z) \text{ inside } C)$$

$$= 2\pi i \sum_{i=1}^n \text{Res}(f, a_i)$$



## Residue of a pole of order m

If  $z=a$  is a pole of order  $m$  of a function  $f(z)$ , then

$$\text{Res}(f, a) = \lim_{z \rightarrow a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \{(z-a)^m f(z)\}$$

## Problem

Find residues of  $f(z) = \frac{z}{(z-1)^2(z+1)}$ .

## Solution:

$$f(z) = \frac{z}{(z-1)^2(z+1)}$$

Singular point :

$z=1 \rightarrow$  pole of order 2

$z=-1 \rightarrow$  simple pole

$$\begin{aligned}
 \text{Res}(f, 1) &= \lim_{z \rightarrow 1} \frac{1}{(z-1)!} \frac{d}{dz} \left( \frac{f(z)}{z+1} \right) \\
 &= \lim_{z \rightarrow 1} \frac{(z+1) \cdot 1 - z \cdot 1}{(z+1)^2} \\
 &= \lim_{z \rightarrow 1} \frac{1}{(z+1)^2} = \frac{1}{(1+1)^2} = \frac{1}{4} \quad (\text{Ans:})
 \end{aligned}$$

$$\begin{aligned}
 \text{Res}(f, -1) &= \lim_{z \rightarrow -1} \frac{1}{(1-1)!} \{ f(z) (z+1) \} \\
 &= \lim_{z \rightarrow -1} \frac{1}{0!} \frac{z}{(z-1)^2}; \quad 0! = 1 \\
 &= \lim_{z \rightarrow -1} \frac{z}{(z-1)^2} = \frac{-1}{(-1-1)^2} = -\frac{1}{4} \quad (\text{Ans:})
 \end{aligned}$$

### Problem

Using the previous result, find:

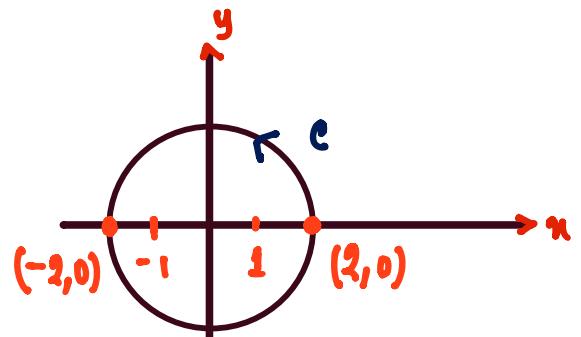
$$(i) \oint_C \frac{z}{(z-1)^2(z+1)} dz \text{ where } C: |z| = 2.$$

$$(ii) \oint_C \frac{z}{(z-1)^2(z+1)} dz \text{ where } C: |z+1| = \frac{3}{2}.$$

$$(iii) \oint_C \frac{z}{(z-1)^2(z+1)} dz \text{ where } C: |z-1| = \frac{3}{2}.$$

### Solution:

$$\begin{aligned}
 (i) \oint_C f(z) dz &= 2\pi i \{ \text{Res}(f, 1) + \text{Res}(f, -1) \} \\
 &= 2\pi i (1/4 - 1/4) = 0 \quad (\text{Ans:})
 \end{aligned}$$

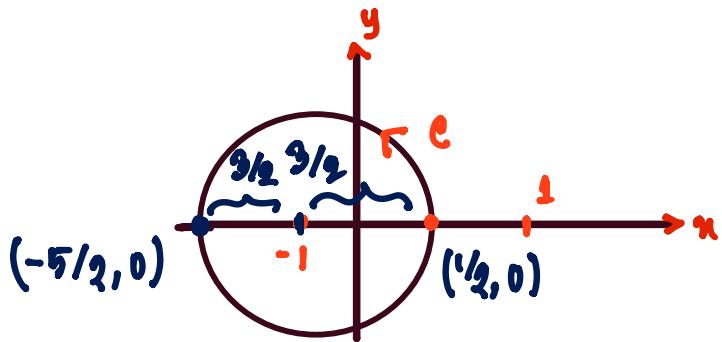


(ii)

$$\oint_C f(z) dz$$

$$= 2\pi i \cdot \operatorname{Res}(f, -1)$$

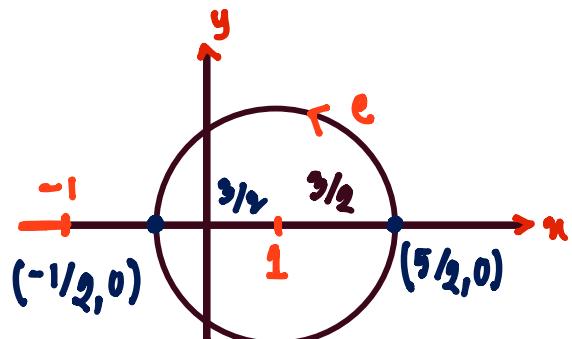
$$= 2\pi i \cdot (-1/4) = -\pi i/2 \quad (\text{Ans.})$$



$$(iii) \quad \oint_C f(z) dz$$

$$= 2\pi i \cdot \operatorname{Res}(f, 1)$$

$$= 2\pi i \cdot (1/4) = \pi i/2 \quad (\text{Ans.})$$



### H.W. Problem

$$f(z) = \frac{1}{z^2 + 1}. \text{ Find}$$

$$(i) \oint_C f(z) dz \text{ where } C: |z - i| = 1.$$

$$(i) \oint_C f(z) dz \text{ where } C: |z + i| = 1.$$

$$(i) \oint_C f(z) dz \text{ where } C: \text{any path enclosing } z = i \text{ and } z = -i.$$

### Problem

Evaluate  $\oint_C \frac{e^{zt}}{z^2(z^2 + 2z + 2)} dz$  around the circle  $C$  with equation  $|z| = 3$ .

**Solution:**  $f(z) = \frac{e^{zt}}{z^2(z^2 + 2z + 2)}$

singular point:  $z = 0 \rightarrow$  pole of order 2

$$z^2 + 2z + 2 = 0 \Rightarrow z = \frac{-2 \pm \sqrt{4 - 4 \cdot 2}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

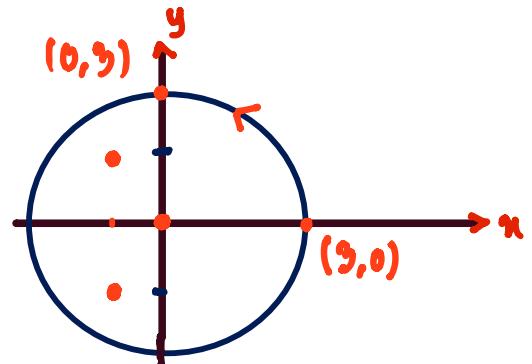
$$f(z) = \frac{e^{\frac{1}{2}z}}{z^2 \{z - (-1+i)\} \{z - (-1-i)\}}$$

$z = -1+i \rightarrow$  simple pole

$z = -1-i \rightarrow$  simple pole

$$\oint_C f(z) dz$$

$$= 2\pi i \{ \operatorname{Res}(f, 0) + \operatorname{Res}(f, -1+i) \\ + \operatorname{Res}(f, -1-i) \}$$



$$\operatorname{Res}(f, 0) = \lim_{z \rightarrow 0} \frac{1}{1!} \frac{d}{dz} \left( \frac{e^{\frac{1}{2}z}}{z^2 (z+1-i)(z-1+i)} \right)$$

$$= \lim_{z \rightarrow 0} \frac{(z^2 + 2z + 2) e^{\frac{1}{2}z} \cdot z - e^{\frac{1}{2}z} (2z + 2)}{(z^2 + 2z + 2)^2}$$

$$= \frac{2z + 2}{2^2} = \frac{z + 1}{2}$$

$$\operatorname{Res}(f, -1-i) = \lim_{z \rightarrow -1-i} \frac{1}{(-1-i)!} \frac{e^{\frac{1}{2}z}}{z^2 \{z - (-1+i)\}}$$

$$= \lim_{z \rightarrow -1-i} \frac{e^{\frac{1}{2}z}}{z^2 \{z - (-1+i)\}}$$

$$= \frac{e^{(-1-i)t}}{(-1-i)^2 \{-1-i - (-1+i)\}}$$

$$= \frac{e^{(-1-i)t}}{(1+2i-1)(-2i)} = \frac{e^{(-1-i)t}}{4}$$

$$\begin{aligned}
 \operatorname{Res}(f, -1+i) &= \lim_{z \rightarrow -1+i} \frac{1}{(1-i)!} \frac{e^{zt}}{z^{\nu}\{z - (-1-i)\}} \\
 &= \lim_{z \rightarrow -1+i} \frac{e^{zt}}{z^{\nu}\{z - (-1-i)\}} \\
 &= \frac{e^{(-1+i)t}}{(-1+i)^{\nu}\{(-1+i) - (-1-i)\}} \\
 &= \frac{e^{(-1+i)t}}{4}
 \end{aligned}$$

$$\begin{aligned}
 \oint_C f(z) dz &= 2\pi i \{ \operatorname{Res}(f, 0) + \operatorname{Res}(f, -1+i) + \operatorname{Res}(f, -1-i) \} \\
 &= 2\pi i \left\{ \frac{t-1}{2} + \frac{e^{(-1+i)t}}{4} + \frac{e^{(-1-i)t}}{4} \right\} \\
 &= 2\pi i \left\{ \frac{t-1}{2} + \frac{e^{-t}}{4} (e^{it} + e^{-it}) \right\} \\
 &= 2\pi i \left\{ \frac{t-1}{2} + \frac{e^{-t}}{2} \left( \frac{e^{it} + e^{-it}}{2} \right) \right\} \\
 &= 2\pi i \left\{ \frac{t-1}{2} + \frac{e^{-t}}{2} \cos t \right\} \text{ (Ans)}
 \end{aligned}$$