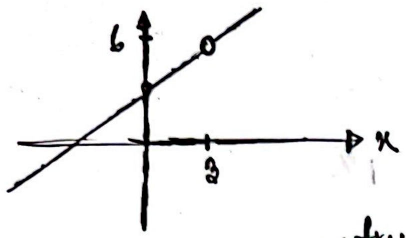
Lect-01: Limit & Continuity

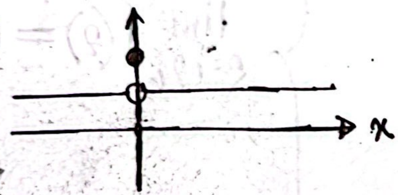
Limit Concept:

(i) $f(x) = \dfrac{x^2-9}{x-3} \longrightarrow f(3)$ is undefined



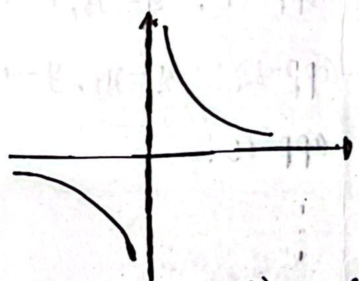
x approaches 3 $\begin{cases} \text{from left } (x \to 3^-) \longrightarrow f(x) \to 6 \\ \text{from right } (x \to 3^+) \longrightarrow f(x) \to 6 \end{cases}$ $\lim_{x \to 3} f(x) = 6$

(ii) $f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$



$\left.\begin{array}{l} \lim_{x \to 0^+} f(x) = 1 \\ \lim_{x \to 0^-} f(x) = 1 \end{array}\right\}$ $\lim_{x \to 0} f(x) = 1$

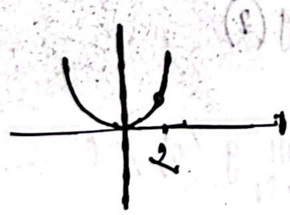
(iii) $f(x) = 1/x$



$\lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x)$

$\Rightarrow \lim_{x \to 0} f(x)$ DNE

(iv) $f(x) = x^2$



$\longrightarrow$ if $f(x)$ is continuous at $x=a$

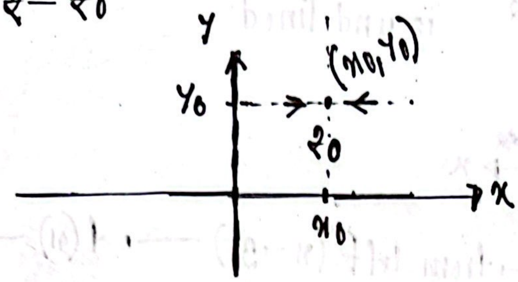
$\lim_{x \to a} f(x) = f(a)$

$\lim_{x \to 2} f(x) = \lim_{x \to 2} x^2 = 2^2 = 4$

Remarks: $f(x)$ need not be defined at $x=a$ for $\lim_{x \to a} f(x)$ to exist

limit (of complex valued function) :

$f(z) \to$ defined & single valued complex function in a neighbourhood of $z = z_0$ with possible exception at $z = z_0$



Q: $z \to z_0$ (z approaches z_0) possible in how many ways?

app-01 : $y = y_0,\ x \to x_0 \to f(z) \to L$
app-02 : $x = x_0,\ y \to y_0 \to f(z) \to L$
app-03 : .... $\to f(z) \to L$
:
.... $f(z) \to L$

$\Big\}\ \lim_{z \to z_0} f(z) = L$

Some properties of limit :

(i) $\lim_{z \to z_0} \{f(z) \pm g(z)\} = \lim_{z \to z_0} f(z) \pm \lim_{z \to z_0} g(z)$

(ii) $\lim_{z \to z_0} \{f(z)\, g(z)\} = \lim_{z \to z_0} f(z) \lim_{z \to z_0} g(z)$

(iii) $\lim_{z \to z_0} \dfrac{f(z)}{g(z)} = \dfrac{\lim_{z \to z_0} f(z)}{\lim_{z \to z_0} g(z)}$ if $\lim_{z \to z_0} g(z) \neq 0$

(iv) if $f(z)$ is continuous $\Rightarrow \lim_{z \to z_0} f(z) = f(z_0)$

Example

Remarks : $f(z) = \dfrac{P(z)}{Q(z)}$ $\to$ then discontinuities at $Q(z) = 0$

$P(z) \to$ Poly.
$Q(z) \to$ polynomial

Problem-01:

(i) $\lim_{z \to 1+i} z^2 - 5z + 10$ (ii) $\lim_{z \to -2i} 2z + 3$ (iii) $\lim_{z \to 2e^{i\pi/3}} \frac{z^3+8}{z^4+4z^2+16}$.

Soln: (iii) $\lim_{z \to 2e^{i\pi/3}} \frac{z^3+8}{z^4+4z^2+16}$

Hint: $z^6 - 64$

$= z^6 - 4z^4 + 4z^4 - 16z^2 + 16z^2 - 64$

$= z^4(z^2-4) + 4z^2(z^2-4) + 16(z^2-4)$

$= (z^2-4)(z^4+4z^2+16)$

$\Rightarrow z^4 + 4z^2 + 16 = \frac{z^6-64}{z^2-4}$

$\lim_{z \to 2e^{i\pi/3}} \frac{(z^3+8)(z^2-4)}{(z^6-64)}$

$= \lim_{z \to 2e^{i\pi/3}} \frac{z^2-4}{z^3-8}$

$= \frac{4e^{i2\pi/3} - 4}{8e^{i\pi} - 8}$

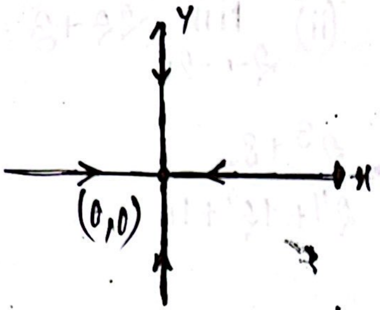
$= \frac{4(e^{i2\pi/3} - 1)}{-16}$

$= -\frac{1}{4}\left(e^{i2\pi/3} - 1\right)$

$= 3/8 - \sqrt{3}/8\, i$ (Ans.)

Problem-02 : Prove that, $\lim\limits_{z\to 0}\frac{\bar{z}}{z}$ does not exist.

Soln :



approach-01 : Along y axis

$$\lim_{\substack{z\to 0\\(\text{along } y=0)}}\frac{\bar{z}}{z}$$

$$=\lim_{\substack{x+iy\to 0\\(\text{along } y=0)}}\frac{\overline{x+iy}}{x+iy}$$

$$=\lim_{x\to 0}\frac{x}{x}$$

$$=\lim_{x\to 0}1=1$$

approach-02 : Along y axis

$$\lim_{\substack{z\to 0\\(\text{along } x=0)}}\frac{\bar{z}}{z}$$

$$=\lim_{\substack{(x,y)\to(0,0)\\(\text{along } x=0)}}\frac{x-iy}{x+iy}$$

$$=\lim_{\substack{x+iy\to 0\\(\text{along } x=0)}}\frac{x-iy}{x+iy}$$

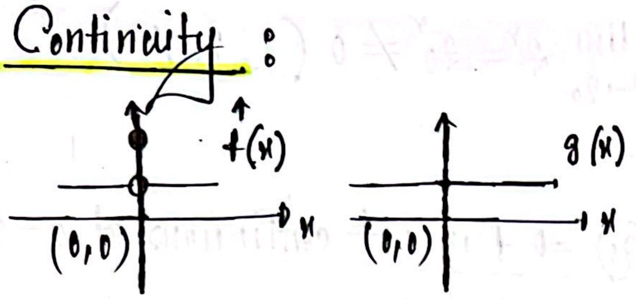
$$=\lim_{y\to 0}\frac{-iy}{iy}=\lim_{y\to 0}-1=-1$$

$$\therefore \lim_{\substack{z\to 0\\(\text{along } y=0)}}\frac{\bar{z}}{z}\neq\lim_{\substack{z\to 0\\(\text{along } x=0)}}\frac{\bar{z}}{z}\Rightarrow\lim_{z\to 0}\frac{\bar{z}}{z}\text{ DNE.}$$

Brain-teaser : $f(z)=\frac{xy}{x^2+y^2}$ or $f(z)=\frac{Re(z)\,Im(z)}{Re(z)^2+Im(z)^2}$.

Show that, $\lim\limits_{z\to 0}f(z)$ DNE.

Continuity :



$\lim_{x\to 0} f(x) \neq f(0)$ $\qquad$ $\lim_{x\to 0} g(x) = g(0)$

- A function $f(z)$ is continuous at z_0 if $\lim_{z\to z_0} f(z) = f(z_0)$
- 3 conditions must be met for $f(z)$ to be continuous at $z=z_0$
  - $\lim_{z\to z_0} f(z)$ must exist
  - $f(z_0)$ must be defined
  - They are equal.

Problem-01 : $f(z)=\begin{cases} z^2 & \text{if } z\neq i \\ 1 & \text{if } z=i \end{cases}$. Is f continuous at $z=i$?

Soln : $\lim_{z\to i} f(z) = \lim_{z\to i} z^2 = -1$ & $f(i)=0$

$\Rightarrow \lim_{z\to i} f(z) \neq f(i) \Rightarrow f$ is not continuous at i.

Problem-02 : $f(z)=z^2$ is continuous at $z=z_0$.

Soln : $\lim_{z\to z_0} f(z) = \lim_{z\to z_0} z^2 = z_0^2 = f(z_0)$

$\Rightarrow \lim_{z\to z_0} f(z) = f(z_0) \Rightarrow f$ is continuous at $z=z_0$.

Problem-03 : $f(z)=\begin{cases} z^2 \,; & z\neq z_0 \\ 0 \,; & z=z_0 \end{cases}$ where $z_0\neq 0$. Is f continuous at $z=z_0$?

Soln : $\lim_{z\to z_0} f(z) = \lim_{z\to z_0} z^2 = z_0^2 \neq 0$ $(\because z_0 \neq 0)$

$f(z_0) = 0$

$\Rightarrow \lim_{z\to z_0} f(z) \neq f(z_0) \Rightarrow f$ is not continuous at $z = z_0$.

L'hospital Rule :

$$\lim_{z\to a} \frac{f(z)}{g(z)} = \lim_{z\to a} \frac{f'(z)}{g'(z)} \quad \left(\text{if } 0/0 \text{ or } \infty/\infty \text{ form}\right)$$

Problem-04 : $\lim_{z\to i} \frac{z^2+1}{z^6+1}$

Soln : $\lim_{z\to i} \frac{z^2+1}{z^6+1} = \lim_{z\to i} \frac{2z}{6z^5} = \lim_{z\to i} \frac{1}{3z^4} = 1/3$ (Ans.)

Problem-05 : $f(z) = \frac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z-i}$

(i) Is f continuous at $z = i$?

(ii) If f is not continuous at $z = i$, redefine $f(z)$ such that it becomes continuous at $z = i$.

Soln (i) $f(i)$ is not defined $\Rightarrow f(z)$ is not continuous at $z = i$.

(ii) To make $f(z)$ continuous at $z = i$,

$$f(i) = \lim_{z\to i} f(z)$$

$$= \lim_{z\to i} \frac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z-i}$$

$$= \lim_{z\to i} 12z^3 - 6z^2 + 16z - 2$$

$$= 4 + 4i$$

Redefinition :

$$f(z) = \begin{cases} \frac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z-i} & \text{if } z \neq i \\ 4 + 4i & \text{if } z = i. \end{cases} \quad \text{(Ans)}$$