Lecture-02 : Root of a complex number

$2^2 = 4 \Rightarrow 2$ is a square root of 4

$3^3 = 27 \Rightarrow 3$ is a cube root of 27

$w^n = z \Rightarrow w$ is a n-th root of z

Defn (n-th root): A complex number w is called a n-th root of the complex number z if

$$w^n = z$$

& we write it as $w = z^{1/n}$ (There are n such w's)

Equality of two complex numbers:

$z_1 = x_1 + iy_1$

$z_2 = x_2 + iy_2$

$z_1 = z_2$ iff $x_1 = x_2$ & $y_1 = y_2$

$z_1 = r_1 e^{i\theta_1}$

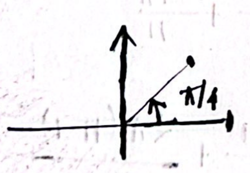
$z_2 = r_2 e^{i\theta_2}$

$z_1 = z_2$ iff

$r_1 = r_2,\ \theta_1 = \theta_2 + 2\pi k$;

$k = 0, \pm 1, \pm 2, \dots$

$r = 3,\ \theta = \pi/4$



$r = 3,\ \theta = \pi/4 + 2\pi$

$\theta = \pi/4 + 4\pi$

$\theta = \pi/4 + 6\pi$

$\vdots$

$r = 3,\ \theta = \pi/4 - 2\pi$

$\theta = \pi/4 - 4\pi$

$\theta = \pi/4 - 6\pi$

$\vdots$

Problem-01 : Find all n-th roots of a complex number of the form $z_0 = r_0 e^{i\theta_0}$

Soln : let,

$z = re^{i\theta}$ be a n-th root of $z_0 = r_0 e^{i\theta_0}$ — (i)

We need : r, θ

From (i),

$$z^n = z_0$$

$$\Rightarrow (re^{i\theta})^n = r_0 e^{i\theta_0}$$

$$\Rightarrow r^n e^{in\theta} = r_0 e^{i\theta_0}$$

Equality of two complex numbers in polar form implies

$r^n = r_0$ & $n\theta = \theta_0 + 2\pi k$; $k = 0, \pm 1, \pm 2, \ldots$

$\Rightarrow r = r_0^{1/n}$ & $\theta = \frac{\theta_0}{n} + \frac{2\pi k}{n}$; $k = 0, \pm 1, \pm 2, \ldots$

$k=0 \Rightarrow r = r_0^{1/n}$; $\theta = \frac{\theta_0}{n}$

$k=1 \Rightarrow r = r_0^{1/n}$; $\theta = \frac{\theta_0}{n} + \frac{2\pi}{n}$

$k=2 \Rightarrow r = r_0^{1/n}$; $\theta = \frac{\theta_0}{n} + \frac{4\pi}{n}$

$\vdots$

$k = n-1 \Rightarrow r = r_0^{1/n}$; $\theta = \frac{\theta_0}{n} + \frac{2\pi(n-1)}{n}$

$\frac{2\pi(n-1)}{n} < 2\pi \Rightarrow$ They are all Distinct

$k = n \Rightarrow r = r_0^{1/n}$; $\theta = \frac{\theta_0}{n} + \frac{2\pi n}{n} = \frac{\theta_0}{n} + 2\pi \rightarrow$ ($k=0$ Already obtained)

$= \frac{\theta_0}{n}$

$k = n+1 \Rightarrow r = r_0^{1/n}$; $\theta = \frac{\theta_0}{n} + \frac{2\pi(n+1)}{n}$

$= \frac{\theta_0}{n} + \frac{2\pi}{n} \rightarrow$ ($k=1$ — Already obtained

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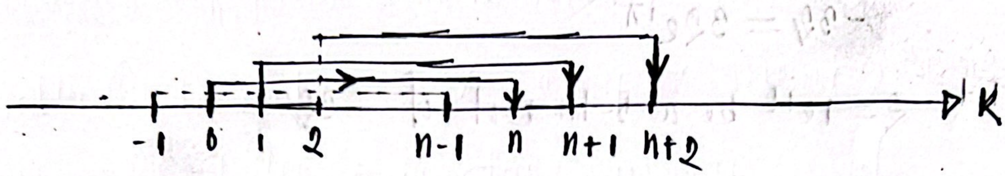
$k = n+2 \longrightarrow$ ($k=2$ Already obtained)

$k = -1 \Rightarrow r = r_0^{1/n}$; $\theta = \frac{\theta_0}{n} - \frac{2\pi}{n} = \frac{\theta_0}{n} - \frac{2\pi}{n} + 2\pi$

$= \frac{\theta_0}{n} + \frac{2\pi(n-1)}{n} \longrightarrow$ ($k = n-1$ already obtained)



Distinct values of the n-th roots are

$$z = r_0^{1/n} \exp\left\{ i\left(\frac{\theta_0}{n} + \frac{2\pi k}{n}\right)\right\} \; ; \; k = 0, 1, \dots n-1$$

Remarks :

(i) Any n consecutive values of k will do

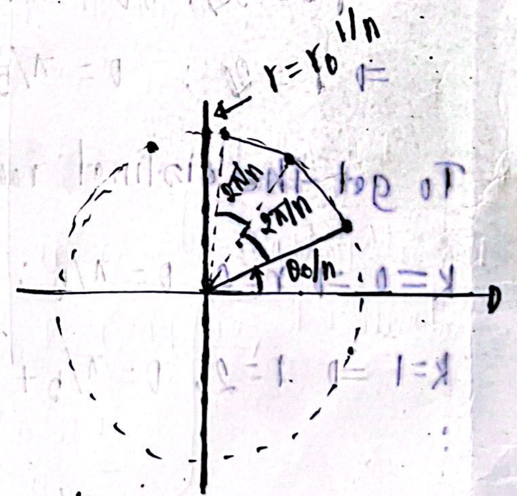
(ii) Distinct n-th roots are

$z_1 = (r_0^{1/n}, \theta_0/n)$

$z_2 = (r_0^{1/n}, \frac{\theta_0}{n} + 2\pi/n)$

$\vdots$

$z_n = \left(r_0^{1/n}, \frac{\theta_0}{n} + \frac{2\pi(n-1)}{n}\right)$



- They lie on a circle of radius $r_0^{1/n}$
- They form a regular n-polygon $\longrightarrow$ Why ?

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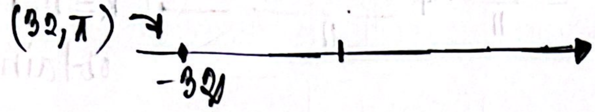
Problem-01 : (i) Find all values of z such that $z^5 = -32$

(ii) locate these values in the complex plane.

Soln. We want: 5-th roots of -32



$\therefore -32 = 32e^{i\pi}$

let, $z = re^{i\theta}$ be a 5-th root of -32

$\Rightarrow z^5 = -32$

$\Rightarrow (re^{i\theta})^5 = 32e^{i\pi}$

$\Rightarrow r^5 e^{i5\theta} = 32e^{i\pi}$

From equality of two complex numbers in polar form

$r^5 = 32; \; 5\theta = \pi + 2\pi k \;;\; k = 0, \pm1, \pm2, \dots$

$\Rightarrow r = 2; \quad \theta = \pi/5 + \frac{2\pi k}{5} \;;\; k \in \mathbb{Z}$

To get the distinct roots it is sufficient to put $k = \overline{0,4}$

$k = 0 \Rightarrow r = 2, \; \theta = \pi/5 \;;\; z_1 = 2\exp(i\pi/5)$

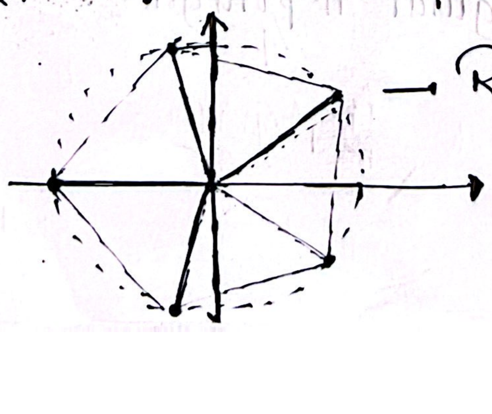
$k = 1 \Rightarrow r = 2, \; \theta = \pi/5 + 2\pi/5 \;;\; z_2 = 2\exp\{i(\pi/5 + 2\pi/5)\}$

$\vdots$

$k = 4 \Rightarrow r = 2, \; \theta = \pi/5 + 8\pi/5 \;;\; z_5 = 2\exp\{i(\pi/5 + 8\pi/5)\}$

$\therefore z_1, z_2, \dots, z_5$ are the required values of z.

(ii)



— Regular pentagon.

Problem-02 : Find $(-1+i)^{1/3}$

Sol^n : let, $z_0 = -1+i$

$$|z_0| = \sqrt{1+1} = \sqrt{2}$$

$$Arg(z_0) = \pi - \tan^{-1}(|1|/|1|)$$

$$= \pi - \tan^{-1}(1) = 3\pi/4$$

We want : z such that $z = (-1+i)^{1/3} \Rightarrow z^3 = -1+i$
(cube roots of $-1+i$)

let, $z = re^{i\theta}$ be a cube root of $-1+i$

$\Rightarrow z^3 = -1+i$

$\Rightarrow (re^{i\theta})^3 = \sqrt{2}e^{i\,3\pi/4}$

$\Rightarrow r^3 e^{i3\theta} = \sqrt{2}e^{i\,3\pi/4}$

$\Rightarrow r^3 = \sqrt{2}$, $3\theta = 3\pi/4 + 2\pi k$; $k = 0, \pm 1, \pm 2, \ldots$

$\Rightarrow r = 2^{1/6}$, $\theta = \pi/4 + 2\pi k/3$; $k \in \mathbb{Z}$

To get the distinct values, it's sufficient to put $k = \overline{0,2}$

$k=0 \Rightarrow r = 2^{1/6}, \theta = \pi/4$; $z_1 = 2^{1/6}\exp(i\,\pi/4)$

$k=1 \Rightarrow r = 2^{1/6}, \theta = \pi/4 + 2\pi/3$; $z_2 = 2^{1/6}\exp\{i(\pi/4 + 2\pi/3)\}$

$k=2 \Rightarrow r = 2^{1/6}, \theta = \pi/4 + 4\pi/3$; $z_3 = 2^{1/6}\exp\{i(\pi/4 + 4\pi/3)\}$

$\therefore$ z_1, z_2, z_3 are the required values.

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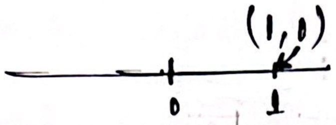
Problem-03: Find all 5-th roots of unity.

Soln:



$$z_0 = 1 = 1e^{i0}$$

let $z = re^{i\theta}$ be a 5-th root of z_0

$\Rightarrow z^5 = z_0$

$\Rightarrow (re^{i\theta})^5 = 1e^{i0}$

$\Rightarrow r^5 e^{i5\theta} = 1e^{i0}$ —(i)

(i) $\Rightarrow r^5 = 1$; $5\theta = 0 + 2\pi k,\ k \in \mathbb{Z}$

$\Rightarrow r = 1$; $\theta = \frac{2\pi k}{5},\ k \in \mathbb{Z}$

To get the distinct 5-th roots, it's sufficient to put $k = \overline{0,4}$.

$k=0 \Rightarrow r=1,\ \theta=0$; $z_1 = 1$

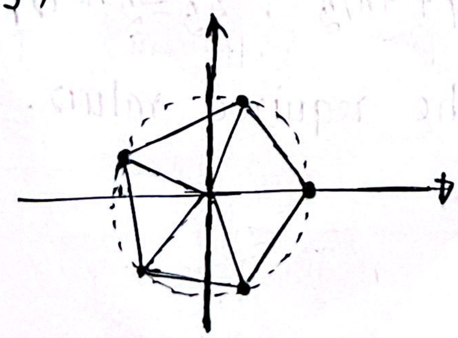
$k=1 \Rightarrow r=1,\ \theta = 2\pi/5$; $z_2 = \cos(2\pi/5) + i\sin(2\pi/5)$

$k=2 \Rightarrow r=1,\ \theta = 4\pi/5$; $z_3 = \cos(4\pi/5) + i\sin(4\pi/5)$

$k=3 \Rightarrow r=1,\ \theta = 6\pi/5$; $z_4 = \cos(6\pi/5) + i\sin(6\pi/5)$

$k=4 \Rightarrow r=1,\ \theta = 8\pi/5$; $z_5 = \cos(8\pi/5) + i\sin(8\pi/5)$

Graphically,



Problem-04 : Solve $z^2+(2i-3)z+5-i=0$.

Sol^n : $z=\dfrac{3-2i\pm\sqrt{(2i-3)^2-4(5-i)}}{2}$

$\Rightarrow z=\dfrac{3-2i\pm\sqrt{-15-8i}}{2}$

$=\dfrac{3-2i\pm\sqrt{-16-8i+1}}{2}$

$=\dfrac{3-2i\pm\sqrt{(4i)^2-2.1.4i+1^2}}{2}$

$=\dfrac{3-2i\pm\sqrt{(4i-1)^2}}{2}$

$=\dfrac{3-2i+4i-1}{2},\ \dfrac{3-2i-(4i-1)}{2}$

$\Rightarrow z = 1+i,\ 2-3i$ (Ans.)

Problem-05 : Shade the region described by,

(i) $\left|\dfrac{z-3}{z+3}\right| \overset{<}{\geq} 2$ (ii) $\operatorname{Im}\left(\dfrac{1}{z}\right)\leq 1/2$

(iii) $1<|z+i|\leq 2$ (iv) $\operatorname{Re}(z^2)>1$ (v) $\operatorname{Im}(z^2)=4$

(vi) $|z+1-i|\leq|z-1+i|$

(vii) $|z-i|=|z+i|$

(viii) $|z+2i|+|z-2i|=6$

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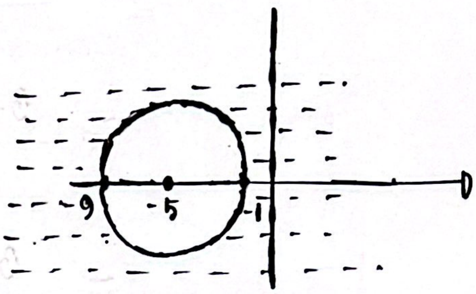
Soln: (i) $\left|\frac{z-3}{z+3}\right| < 2$

let, $z = x+iy$

$\Rightarrow \frac{|x+iy-3|}{|x+iy+3|} < 2$

$\Rightarrow |x-3+iy| < 2\,|x+3+iy|$

$\Rightarrow \sqrt{(x-3)^2+y^2} < 2\sqrt{(x+3)^2+y^2}$



$\Rightarrow (x-3)^2+y^2 < 4\{(x+3)^2+y^2\}$

$\Rightarrow x^2+10x+9+y^2 > 0$

$\Rightarrow x^2+2.5x+25-16+y^2 > 0$

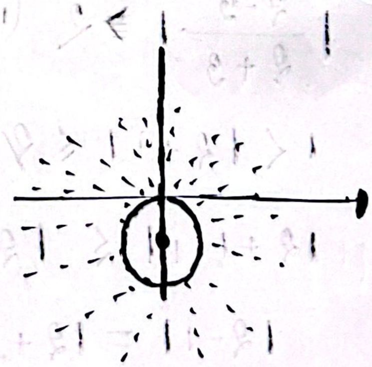
$\Rightarrow (x+5)^2+(y-0)^2 > 4^2$

lemma: $(x-a)^2+(y-b)^2 \; \substack{= \\ > \\ <} \; r^2$

(ii) $Im\left(\frac{1}{z}\right) \leq 1/2$

$Im\left(\frac{1}{x+iy}\right) \leq \frac{1}{2}$

$Im\left(\frac{x-iy}{x^2+y^2}\right) \leq 1/2$



$\Rightarrow \frac{-y}{x^2+y^2} \leq 1/2$

$\Rightarrow -2y \leq x^2+y^2$

$\Rightarrow x^2+y^2+2y \geq 0$

$\Rightarrow x^2+(y+1)^2 \geq 1$

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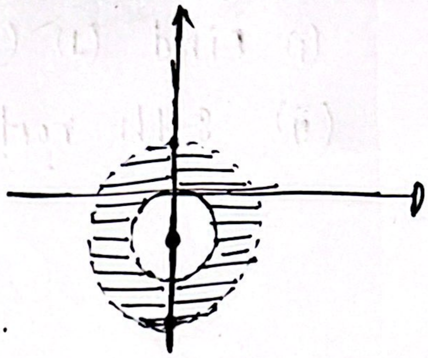
(iii) $1 < |z+i| < 2$

$\Rightarrow 1 < |x+iy+i| < 2$

$\Rightarrow 1 < \sqrt{x^2+(y+1)^2} < 2$

$\Rightarrow 1^2 < x^2+(y+1)^2 < 2^2$

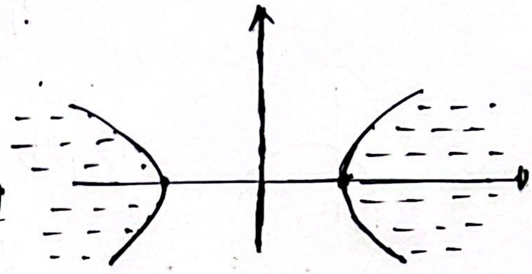


(iv) $\text{Re}(z^2) > 1$

$\Rightarrow \text{Re}((x+iy)^2) > 1$

$\Rightarrow \text{Re}(x^2-y^2+2xyi) > 1$

$\Rightarrow x^2-y^2 > 1$



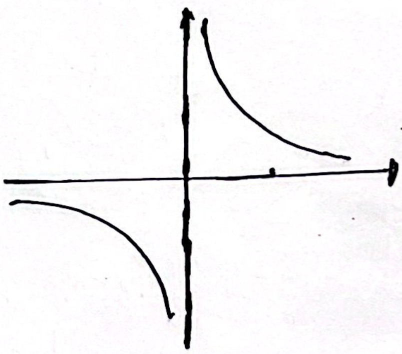
(v) $\text{Im}(z^2) = 4$

$\Rightarrow \text{Im}(x^2-y^2+i2xy) = 4$

$\Rightarrow 2xy = 4$

$\Rightarrow xy = 2$

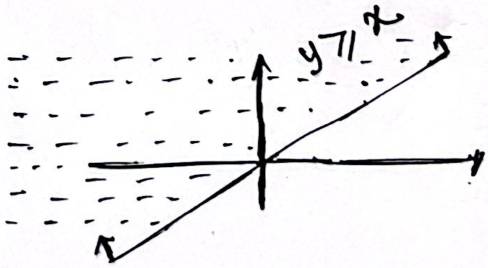
$\Rightarrow y = 2/x$



(vi) $|z+1-i| \leq |z-1+i|$

$\vdots$

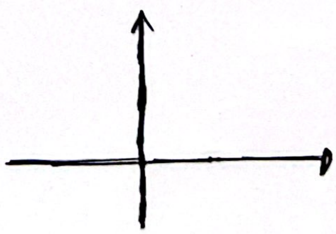
$y \geq x$



(vii) $|z-i| = |z+i|$

$\vdots$

$y = 0$



Problems:

(i) Find (1) $(1+i)^{1/4}$ (2) $(-8i)^{1/3}$ (3) $(-2\sqrt{3}-2i)^{1/4}$

(ii) 8-th roots of unity.