MAT215: Complex Variables & Laplace Transformations

Chapter : Laplace Transformations

Laplace Transformation introduction

Laplace transform of elementary functions

BRAC University
Department of Mathematics
Summer 2025

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Laplace transformation

Functional transformation

$$f(x) = x^2 \xrightarrow{d/dx} 2x$$ Functional transform of $f(x)$

$$f(x) = x^2 \xrightarrow{\int dx} \frac{x^3}{3} + c$$

input function

(Functional Transformation)

Integral transformation

(A special functional transformation)

$f(x) \rightarrow$ input function defined on $[a, b]$

$$F(y) = \int_a^b K(x,y) f(x)\, dx$$

Kernal of the integral transformation

$$f(x) \longrightarrow \int_a^b K(x,y) f(x)\, dx \longrightarrow F(y)$$ $\rightarrow$ Integral transform of $f(x)$

input

integral transformation

Function of time domain

$f(t)\,;\ t \geq 0$

Integral transformation (for a function of time domain)

For a function $f(t)$ defined on $t \geqslant 0$, an integral transformation has the general form

$$F(s) = \int_0^\infty K(s,t)\, f(t)\, dt$$

$f(t)$ → input function

$K(s,t)$ → Kernal

$F(s)$ → Integral transform of $f(t)$

Laplace transformation

A special integral transformation where $K(s,t) = e^{-st}$

$f(t)$ → function defined on $t \geqslant 0$. Then

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

$f(t)$ → input

e^{-st} → Kernal of the transformation

$F(s)$ → Laplace transform of $f(t)$ provided that the integral converges

Notation:

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

$$\mathcal{L}\{f(t)\} = F(s)$$ → {laplace transform of $f(t)$ is $F(s)$}

(i) $\int uv\,dx = u\int v\,dx - \int \left(u'\int v\,dx\right)dx$ (integration by parts formula)

choosing u, v : LIATE formula

(ii) Shortcut for u-v integration (Tabular method)

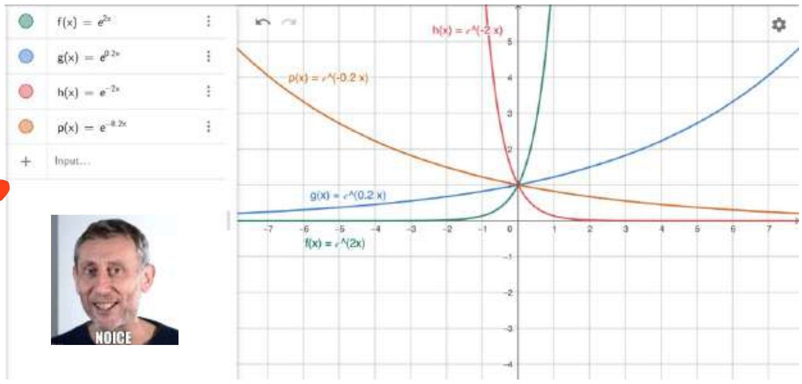
$$I = \int_a^b x^3 e^x\,dx$$

$$= \left[x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x\right]_a^b$$

Sign	D	I
+	x^3	e^x
−	$3x^2$	e^x
+	$6x$	e^x
−	6	e^x
+	0	e^x

(iii) $\lim_{x\to\infty} e^{mx}$ is finite if $m<0$

In fact $\lim_{x\to\infty} e^{mx} = 0$ if $m<0$



Laplace transformation of some elementary functions

Problem

Find $\mathscr{L}\{f(t)\}$ where $f(t) = 1$.

Solution: $\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$$= \int_0^\infty e^{-st}\,dt \rightarrow$$ (Improper integral of 1st kind)

$$= \lim_{p\to\infty} \int_0^p e^{-st}\,dt$$

$$= \lim_{p\to\infty} \left. \frac{e^{-st}}{-s} \right|_0^p$$

For the improper integral to converge we must have

$\downarrow$

$\lim_{p\to\infty} e^{-sp}$ to be finite

$\sim -s<0$ or $s>0$

$$= -\frac{1}{s} \lim_{p\to\infty} \left(e^{-sp} - 1\right)$$

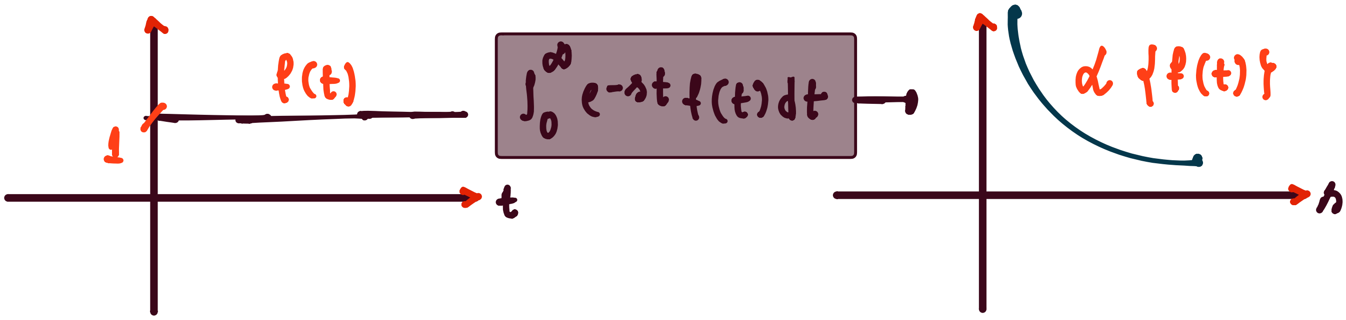
$$= -\frac{1}{s}\left(\lim_{p\to\infty} e^{-sp} - \lim_{p\to\infty} 1\right)$$

$$= -\frac{1}{s}\left(\lim_{p\to\infty} e^{-sp} - 1\right)$$

$$= -\frac{1}{s}(0-1) \quad \text{when } s>0$$

$$= \frac{1}{s} \quad \text{when } s>0$$

$$\mathcal{L}\{1\} = \frac{1}{s} \quad \text{when } s>0$$



Problem

Find $\mathcal{L}\{t\}$.

Solution:

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\infty e^{-st}\, t\,dt$$

$$= \lim_{p\to\infty} \int_0^p e^{-st}\, t \, dt$$

sign	D	I
$+$	t	e^{-st}
$-$	1	$\frac{e^{-st}}{-s}$
$+$	0	$\frac{e^{-st}}{s^2}$

$$= \lim_{p\to\infty} \left. \frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2} \right|_0^p$$

$$= -\lim_{p\to\infty} \left. \frac{te^{-st}}{s} + \frac{e^{-st}}{s^2} \right|_0^p$$

$$= -\lim_{p\to\infty} \left\{ \frac{pe^{-sp}}{s} + \frac{e^{-sp}}{s^2} - \frac{1}{s^2} \right\}$$

$$= -\left(0 + 0 - \frac{1}{s^2}\right) \text{ when } s>0$$

$$= \frac{1}{s^2} \text{ when } s>0$$

$$\mathcal{L}\{t\} = \frac{1}{s^2} \text{ when } s>0$$

Problem

Find $\mathscr{L}\{t^2\}$.

Ans: $\mathcal{L}\{t^2\} = \frac{2}{s^3}$; $s>0$

Problem

Find $\mathscr{L}\{t^3\}$.

Ans: $\mathcal{L}\{t^3\} = \frac{6}{s^4}$; $s>0$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}; \; s>0$$ where n is a positive integer

Problem

Find $\mathscr{L}\{t^n\}$ where n is a positive integer.

Solution:

$\int uv\,dx = u\int v\,dx - \int (u'\int v\,dx)\,dx$ (integration by parts formula)

choosing u, v : LIATE formula

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st} t^n \, dt$$

$u = t^n, \; v = e^{-st}$

$$= \lim_{p\to\infty} \int_0^p e^{-st} t^n \, dt$$

$$= \lim_{p\to\infty} \left\{ \left[t^n \frac{e^{-st}}{-s} \right]_0^p - \int_0^p n t^{n-1} \frac{e^{-st}}{-s} \, dt \right\}$$

$$= \lim_{p\to\infty} \left\{ \frac{p^n}{-s e^{sp}} + \frac{n}{s} \int_0^p e^{-st} t^{n-1} \, dt \right\}$$

$$= \lim_{p\to\infty} \frac{p^n}{-s e^{sp}} + \frac{n}{s} \lim_{p\to\infty} \int_0^p e^{-st} t^{n-1} \, dt$$

$$= I + \frac{n}{s} \int_0^\infty e^{-st} t^{n-1} \, dt$$

$$= I + \frac{n}{s} \mathcal{L}\{t^{n-1}\} \quad \text{—(i)}$$

computation of I:

$$I = \lim_{p\to\infty} \frac{p^n}{-s e^{sp}}$$

$\frac{\infty}{\infty}$ form $\rightarrow$ L'hospital Rule

$$= \lim_{p\to\infty} \frac{n p^{n-1}}{-s^2 e^{sp}}$$

$$= \lim_{p\to\infty} \frac{n(n-1) p^{n-2}}{-s^3 e^{sp}}$$

$$\vdots$$

$$= \lim_{p\to\infty} \frac{n(n-1)\cdots 1\, p^{n-n}}{-s^{n+1} e^{sp}}$$

$$= \lim_{p\to\infty} \frac{n!}{-s^{n+1} e^{sp}}$$

$$= \frac{n!}{-s^{n+1}} \lim_{p\to\infty} e^{-sp} = \frac{n!}{-s^{n+1}} \cdot 0 \text{ if } s>0 = 0 \text{ if } s>0$$

Substitute this value in (i) $\Rightarrow$

$$\mathcal{L}\{t^n\} = \frac{n}{s}\mathcal{L}\{t^{n-1}\}$$

$$\therefore \mathcal{L}\{t^{n-1}\} = \frac{n-1}{s}\mathcal{L}\{t^{n-2}\}$$

$$\begin{aligned}
\therefore \mathcal{L}\{t^n\} &= \frac{n}{s}\mathcal{L}\{t^{n-1}\} \\
&= \frac{n}{s}\,\frac{n-1}{s}\,\mathcal{L}\{t^{n-2}\} \\
&= \frac{n}{s}\,\frac{n-1}{s}\,\frac{n-2}{s}\,\mathcal{L}\{t^{n-3}\} \\
&\;\vdots \\
&= \frac{n(n-1)\cdots\cdots 1}{s^n}\mathcal{L}\{t^{n-n}\} \\
&= \frac{n!}{s^n}\mathcal{L}\{1\} \\
&= \frac{n!}{s^{n+1}} \text{ if } s>0
\end{aligned}$$

$$\boxed{\therefore \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \text{ if } s>0}$$ where $n \in \mathbb{Z}^+$

Gamma function:

$$\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1}\, dt$$

(i) $\Gamma(n) = (n-1)!$ if $n \in \mathbb{Z}^+$

$$\boxed{\therefore \mathcal{L}\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}} \text{ if } s>0}$$ where $n \in \mathbb{Z}^+$

Qustion: what if n is not an integer

$$\boxed{\therefore \mathcal{L}\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}} \text{ if } s>0}$$ where n is not an integer.

Hint: Utilize gamma function def^n

Problem

Find $\mathscr{L}\{e^{at}\}$.

Solution: $\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$$\mathscr{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\,dt$$

$$= \lim_{p\to\infty} \int_0^p e^{-(s-a)t}\,dt$$

$$= \lim_{p\to\infty} \left. \frac{e^{-(s-a)t}}{-(s-a)} \right|_0^p$$

$$= -\frac{1}{s-a} \lim_{p\to\infty} \left\{ e^{-(s-a)p} - 1 \right\}$$

$$= -\frac{1}{s-a} \left\{ \lim_{p\to\infty} e^{-(s-a)p} - 1 \right\}$$

$$= -\frac{1}{s-a}(0-1) \quad \text{when } s-a>0 \text{ or } s>a$$

$$= \frac{1}{s-a} \quad \text{when } s>a$$

$$\mathscr{L}\{e^{at}\} = \frac{1}{s-a} \quad \text{when } s>a$$

H.W

Problem

Find (i) $\mathscr{L}\{\sin at\}$ and (ii) $\mathscr{L}\{\cos at\}$.

Solution:

$$\boxed{\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt}$$

$$\mathscr{L}\{\sin at\} = \int_0^\infty e^{-st} \sin at\,dt$$

$$\mathscr{L}\{\cos at\} = \int_0^\infty e^{-st} \cos at\,dt$$

$$\mathcal{L}\{e^{iat}\} = \int_0^\infty e^{-st} e^{iat}\, dt$$

$$= \lim_{p\to\infty} \int_0^p e^{-(s-ai)t}\, dt$$

$$= \lim_{p\to\infty} \left. \frac{e^{-(s-ai)t}}{-(s-ai)} \right|_0^p$$

$$= -\frac{1}{(s-ai)} \lim_{p\to\infty} \{e^{-(s-ai)p} - 1\}$$

$$= -\frac{1}{s\ \ ai} \lim_{p\to\infty} [e^{-sp}\{\cos(ap) + i\sin(ap)\} - 1]$$

$$= -\frac{1}{s-ai} [\lim_{p\to\infty} e^{-sp}\{\cos(ap) + i\sin(ap)\} - 1]$$

$$= -\frac{1}{s-ai} (0-1) \text{ if } s>0$$

$$= \frac{1}{s-ai} \text{ if } s>0$$

$$= \frac{s+ai}{(s-ai)(s+ai)} = \frac{s+ai}{s^2+a^2} = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}$$

So we have,

$$\int_0^\infty e^{-st} e^{iat}\, dt = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}$$

$$\Rightarrow \int_0^\infty e^{-st}\{\cos at + i\sin at\}\, dt = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}$$

$$\Rightarrow \int_0^\infty e^{-st}\cos at\, dt + i\int_0^\infty e^{-st}\sin at\, dt = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}$$

$$\int_0^\infty e^{-st}\cos at\, dt = \frac{s}{s^2+a^2};\ s>0 \quad \& \quad \int_0^\infty e^{-st}\sin at\, dt = \frac{a}{s^2+a^2};\ s>0$$

$$\Rightarrow \boxed{\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}};\ s>0 \quad \& \quad \boxed{\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}};\ s>0$$

MAT215: Complex Variables & Laplace Transformations

Chapter : Laplace Transformations

Linearity of Laplace transformation
1st Translation theorem
Multplication by t^n

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Linearity of Laplace transformation

(i) $\mathscr{L}\{f(t)+g(t)\}=\mathscr{L}\{f(t)\}+\mathscr{L}\{g(t)\}$

(ii) $\mathscr{L}\{\alpha f(t)\}=\alpha\mathscr{L}\{f(t)\}$

Reason:

(i) $$\mathcal{L}\{f(t)+g(t)\} = \int_0^\infty e^{-st}\{f(t)+g(t)\}\,dt$$

$$= \int_0^\infty e^{-st} f(t)\,dt + \int_0^\infty e^{-st} g(t)\,dt$$

$$= \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$$

(ii) $$\mathcal{L}\{\alpha f(t)\} = \int_0^\infty e^{-st}(\alpha f(t))\,dt$$

$$= \alpha \int_0^\infty e^{-st} f(t)\,dt = \alpha\,\mathcal{L}\{f(t)\}$$

Linearity of Laplace transformation (Compact form)

$\mathscr{L}\{\alpha f(t)+\beta g(t)\}=\alpha\mathscr{L}\{f(t)\}+\beta\mathscr{L}\{g(t)\}$

Problem

Find (i) $\mathscr{L}\{\sinh at\}$ and (ii) $\mathscr{L}\{\cosh at\}$.

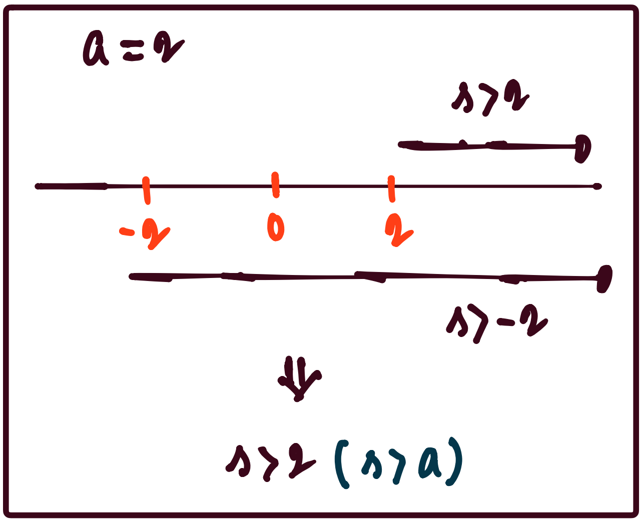
Solution: $$\sinh(at) = \frac{e^{at}-e^{-at}}{2}$$

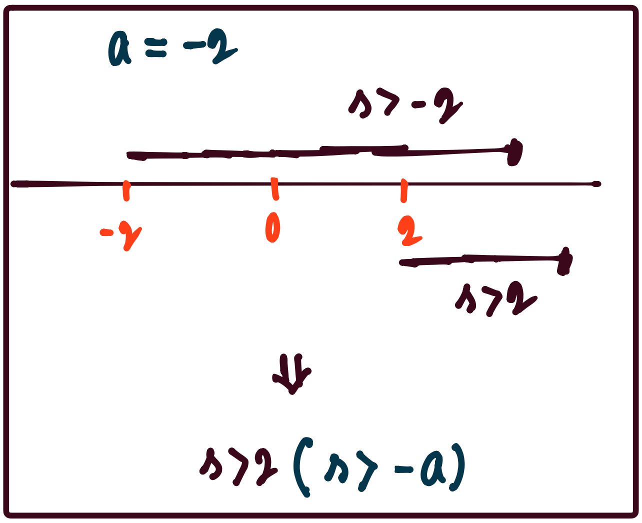
$$\mathcal{L}\{\sinh(at)\} = \mathcal{L}\left\{\frac{e^{at}-e^{-at}}{2}\right\}$$

$$= \frac{1}{2}\left(\mathcal{L}\{e^{at}\} - \mathcal{L}\{e^{-at}\}\right)$$

$s>a$ $\qquad$ $s>-a$

$$= \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) \; ; \; s>|a|$$





$s > |a|$

Therefore we have,

$$\mathcal{L}\{\sinh(at)\} = \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) \; ; \; s > |a|$$

$$= \frac{1}{2}\,\frac{s+a-s+a}{s^2-a^2} \; ; \; s > |a|$$

$$\mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2} \; ; \; s > |a| \quad \text{(Ans:)}$$

$$\mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2} \; ; \; s > |a| \quad \text{(Ans:)}$$

Laplace Transforms of Elementary Functions (Real s)

Function $f(t)$	$\mathscr{L}\{f(t)\} = F(s)$	Function $f(t)$	$\mathscr{L}\{f(t)\} = F(s)$
1	$\frac{1}{s}, \quad s > 0$	$\sin(at)$	$\frac{a}{s^2+a^2}, \quad s > 0$
$t^n,\ n \in \mathbb{Z}^+$	$\frac{n!}{s^{n+1}}, \quad s > 0$	$\cos(at)$	$\frac{s}{s^2+a^2}, \quad s > 0$
$t^n,\ n \notin \mathbb{Z}$	$\frac{\Gamma(n+1)}{s^{n+1}}, \quad s > 0$	$\sinh(at)$	$\frac{a}{s^2-a^2}, \quad s > \lvert a \rvert$
e^{at}	$\frac{1}{s-a}, \quad s > a$	$\cosh(at)$	$\frac{s}{s^2-a^2}, \quad s > \lvert a \rvert$

Problem

Find:

(i) $\mathscr{L}\{3e^{-2t}\}$ (ii) $\mathscr{L}\{4t^3 - e^{-t}\}$ (iii) $\mathscr{L}\{(t^2+1)^2\}$

(iv) $\mathscr{L}\{7\sin 2t - 3\cos 4t\}$ H.W. (v) $\mathscr{L}\{(4e^{2t}-2)^3\}$

Solution: (iii) $\mathscr{L}\{(t^2+1)^2\}$

$$= \mathcal{L}\{t^4 + 2t^2 + 1\}$$

$$= \mathcal{L}\{t^4\} + 2\mathcal{L}\{t^2\} + \mathcal{L}\{1\}$$

$$= \frac{4!}{s^5} + 2\frac{2!}{s^3} + \frac{1}{s} \; ; s>0$$

(iv) $\mathscr{L}\{7\sin 2t - 3\cos 4t\}$

$$= 7\mathcal{L}\{\sin 2t\} - 3\mathcal{L}\{\cos 4t\}$$

$$= 7\frac{2}{s^2+4} - 3\frac{s}{s^2+16} \; ; s>0$$

(Ans:)

(ii) $\mathscr{L}\{4t^3 - e^{-t}\}$

$$= 4\mathcal{L}\{t^3\} - \mathcal{L}\{e^{-t}\}$$

$s>0 \downarrow$ $\downarrow s>-1$

$$= 4\frac{3!}{s^4} - \frac{1}{s+1} \; ; s>0$$

(Ans:)

1st Translation theorem

$$\mathcal{L}\{f(t)\} = F(s) \Rightarrow \mathcal{L}\{e^{at} f(t)\} = F(s-a) \text{ for any } a \in \mathbb{R}$$

$$\mathcal{L}\{e^{at} f(t)\} = F(s)\big|_{s\to s-a} = \mathcal{L}\{f(t)\}\big|_{s\to s-a}$$

1st Translation theorem

$\mathscr{L}\{e^{at} f(t)\} = \mathscr{L}\{f(t)\}\big|_{s\to s-a}$ for any real number a.

Problem

Find: (i) $\mathscr{L}\{e^{-3t}t^3\}$ (ii) $\mathscr{L}\{5e^{3t}\sin 4t\}$

Solution:

$$\text{(i) } \mathscr{L}\{e^{-3t}t^3\} = \mathcal{L}\{t^3\}\Big|_{s\to s+3}$$

$$= \frac{3!}{s^4}\Big|_{s\to s+3} \; ; \; s>0\Big|_{s\to s+3}$$

$$= \frac{3!}{(s+3)^4} \; ; \; s+3>0 \quad \text{(Ans:)}$$

$$\text{(ii) } \mathscr{L}\{5e^{3t}\sin 4t\} = 5\,\mathcal{L}\{e^{3t}\sin 4t\}$$

$$= 5\,\mathcal{L}\{\sin 4t\}\Big|_{s\to s-3}$$

$$= 5\,\frac{4}{s^2+16}\Big|_{s\to s-3} \; ; \; s>0\Big|_{s\to s-3}$$

$$= \frac{20}{(s-3)^2+16} \; ; \; s>3 \quad \text{(Ans:)}$$

Problem

Find:

(i) $\mathscr{L}\{(t+2)^2e^t\}$ (ii) $\mathscr{L}\{e^{-}t(3\sinh 2t-5\cosh 3t)\}$

(iii) $\mathscr{L}\{e^{-4t}\cosh 2t\}$ (iv) $\mathscr{L}\{e^{2t}(3\sin 4t-4\cos 3t)\}$

Solution:

$$\text{(iv) } \mathscr{L}\{e^{2t}(3\sin 4t-4\cos 3t)\} = \mathcal{L}\{3\sin 4t - 4\cos 3t\}\Big|_{s\to s-2}$$

$$= 3\mathcal{L}\{\sin 4t\} - 4\mathcal{L}\{\cos 3t\}\Big|_{s\to s-2}$$

$s>0$ ↓ ↓ $s>0$

$$= 3\,\frac{4}{s^2+16} - 4\,\frac{s}{s^2+9}\Bigg|_{s\to s-2} \quad ; \quad s>0\Big|_{s\to s-2}$$

$$= \frac{12}{(s-2)^2+16} - \frac{4(s-2)}{(s-2)^2+9} \quad ; \quad s-2>0 \quad \text{(Ans:)}$$

Laplace transformation of the form $t^n f(t)$

If $\mathscr{L}\{f(t)\} = F(s)$, then $\mathscr{L}\{t^n f(t)\} = ?$

Derivation: $\mathscr{L}\{f(t)\} = F(s)$

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$\Rightarrow \frac{d}{ds}(F(s)) = \frac{d}{ds}\left(\int_0^\infty e^{-st} f(t)\,dt\right)$$

Assume that order of differentiation & integration can be interchanged

$$= \int_0^\infty \frac{\partial}{\partial s}\left(e^{-st} f(t)\right) dt$$

$$\frac{d}{ds}(F(s)) = \int_0^\infty f(t)\, e^{-st}\,(-t)\,dt$$

$$\Rightarrow -\frac{d}{ds}(F(s)) = \int_0^\infty e^{-st}\,(t f(t))\,dt$$

$$\mathscr{L}\{t f(t)\} = -\frac{d}{ds}\left(\mathscr{L}\{f(t)\}\right)$$

$$\mathscr{L}\{t^2 f(t)\} = \mathscr{L}\{t\,\underbrace{(t f(t))}_{g(t)}\}$$

$$= -\frac{d}{ds}\left(\mathcal{L}\{t\,f(t)\}\right)$$

$$= -\frac{d}{ds}\left(-\frac{d}{ds}\left(\mathcal{L}\{f(t)\}\right)\right)$$

$$\mathcal{L}\{t^2 f(t)\} = (-1)^2 \frac{d^2}{ds^2}\left(\mathcal{L}\{f(t)\}\right)$$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\left(\mathcal{L}\{f(t)\}\right)$$

Laplace transformation of the form $t^n f(t)$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[\mathcal{L}\{f(t)\}]$$

Problem

Find: (i) $\mathcal{L}\{t\sin t\}$ (ii) $\mathcal{L}\{t^2e^{2t}\}$ (iii) $\mathcal{L}\{te^{-3t}\sin t\}$

Solution:

(i) $\mathcal{L}\{t\sin t\}$ $= -\frac{d}{ds}\left(\mathcal{L}\{\sin t\}\right)$

$$= -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) \; ; \; s>0$$

$$= -\frac{(s^2+1).0 - 1.2s}{(s^2+1)^2}$$

$$= \frac{2s}{(s^2+1)^2} \; ; \; s>0 \quad \text{(Ans:)}$$

(ii) $\mathcal{L}\{t^2e^{2t}\}$ $= (-1)^2\frac{d^2}{ds^2}\left(\mathcal{L}\{e^{2t}\}\right)$

$$= \frac{d^2}{ds^2}\left(\frac{1}{s-2}\right) \; ; \; s>2$$

$$= (-1)(-2)(s-2)^{-3} ; \ s>2$$

$$= \frac{2}{(s-2)^3} ; \ s>2$$

(iii) $\mathscr{L}\{te^{-3t}\sin t\}$ $= \mathcal{L}\{e^{-3t}\, t\sin t\}$

$$= \mathcal{L}\{t\sin t\}\Big|_{s\to s+3}$$

$$= \frac{2s}{(s^2+1)^2}\Big|_{s\to s+3} ; \ s>0\Big|_{s\to s+3}$$

$$= \frac{2(s+3)}{\{(s+3)^2+1\}^2} ; \ s>-3 \ \text{(Ans:)}$$

~~H.W.~~

Brain Teaser

Find $\mathscr{L}\{te^{-3t}\cos t\sin t\}$.

MAT215: Complex Variables & Laplace Transformations

Chapter : Laplace Transformations

Unit Step function
Piecewise function representation in terms of Unit Step function

2nd Translation Theorem

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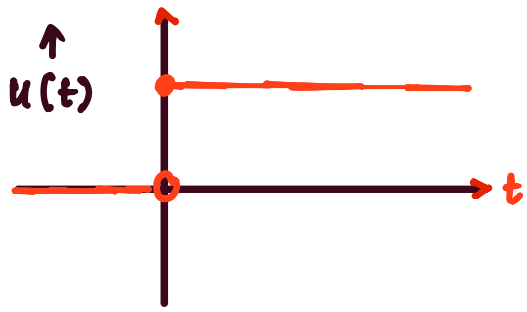
Unit Step Function

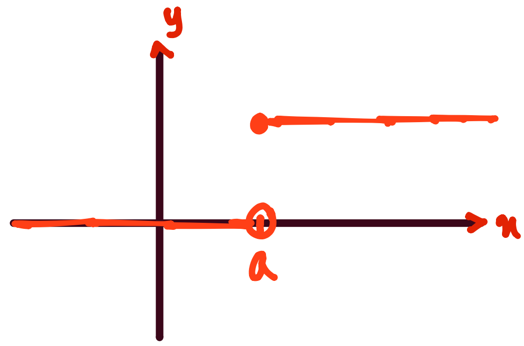
$$u(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$



a is any positive real number

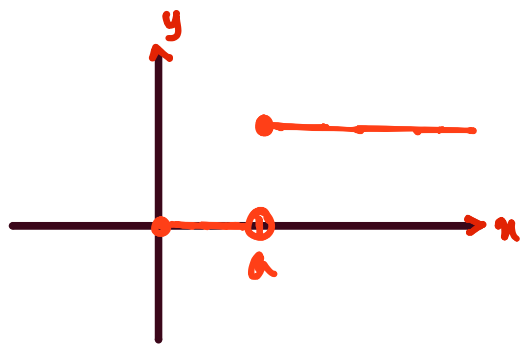
$$u(t-a) = \begin{cases} 1 & \text{if } t-a \geq 0 \\ 0 & \text{if } t-a < 0 \end{cases}$$

$$u(t-a) = \begin{cases} 1 & \text{if } t \geq a \\ 0 & \text{if } t < a \end{cases}$$



$\downarrow$

restrict this function in non-negative Domain



$$u(t-a) = \begin{cases} 1 & \text{if } t \geq a \\ 0 & \text{if } 0 \leq t < a \end{cases}$$

$\downarrow$

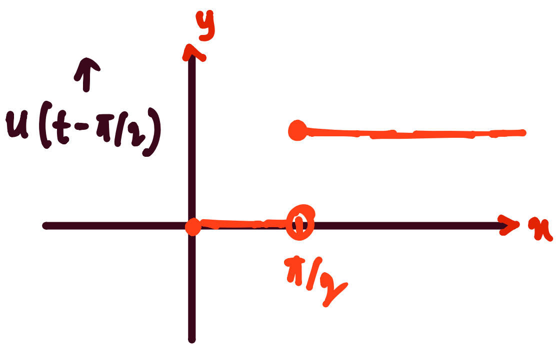
(Unit step Function at a)

Problem

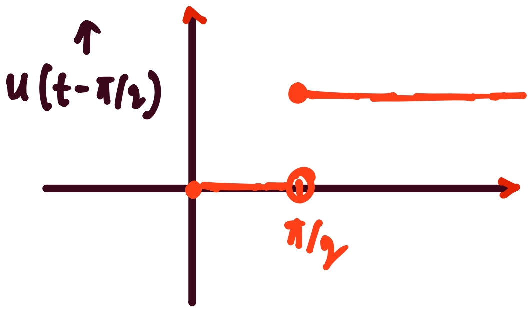
Plot the functions: (i) $u(t - \frac{\pi}{2})$ (ii) $1 - u(t - \frac{\pi}{2})$

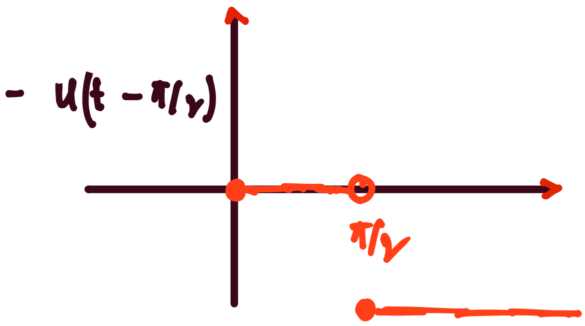
Solution:

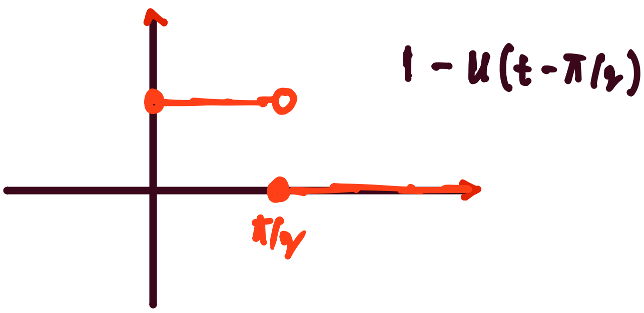
(i) $u(t - \frac{\pi}{2})$



(ii) $1 - u(t - \frac{\pi}{2})$







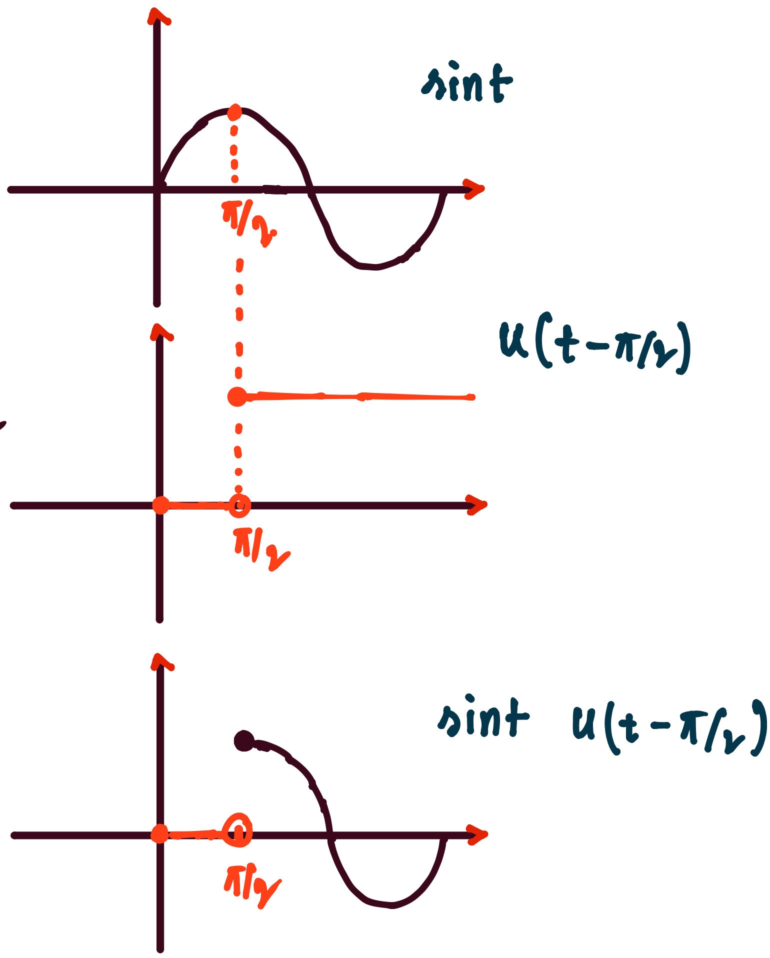
2 Geometric Properties of Unit Step Function

(i) Multiplication by $u(t-a)$:

$f(t) = \sin t$

$g(t) = u(t - \pi/2)$

$h(t) = \sin t \; u(t - \pi/2)$



$$\sin t \; u(t - \pi/2) = \begin{cases} \sin t \; ; & t \geq \pi/2 \\ 0 & ; 0 \leq t < \pi/2 \end{cases}$$

$$f(t)\, u(t-a) = \begin{cases} f(t) \; ; & t \geq a \\ 0 & ; 0 \leq t < a \end{cases}$$

$\downarrow$

(i) right portion to a (including a) $\longrightarrow$ as it is

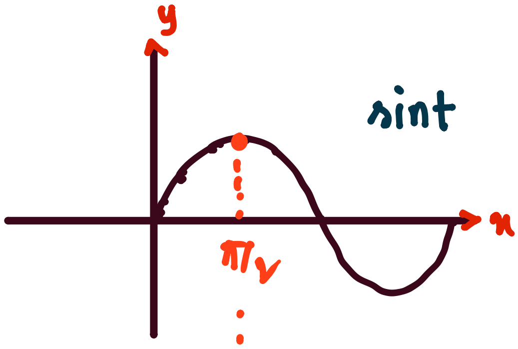
(ii) left portion to a (excluding a) $\longrightarrow$ nullifies

(ii) Multiplication by $1-u(t-a)$:

$f(t) = \sin t$

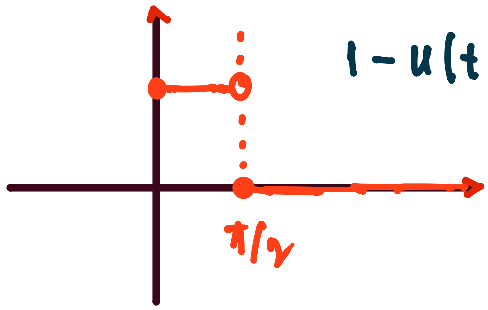
$g(t) = 1 - u(t-\pi/2)$

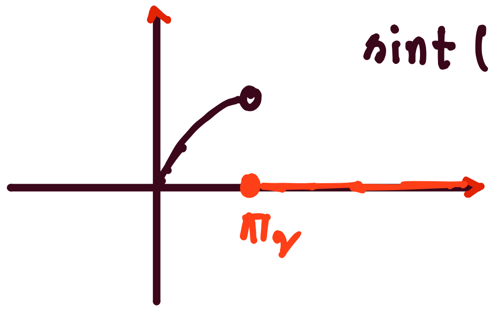
$h(t) = \sin t\{1 - u(t-\pi/2)\}$



$\sin t\{1 - u(t-\pi/2)\}$

$$= \begin{cases} \sin t & ; \; 0 \le t < \pi/2 \\ 0 & ; \; t \geq \pi/2 \end{cases}$$





$f(t)\ \{1 - u(t-a)\}$

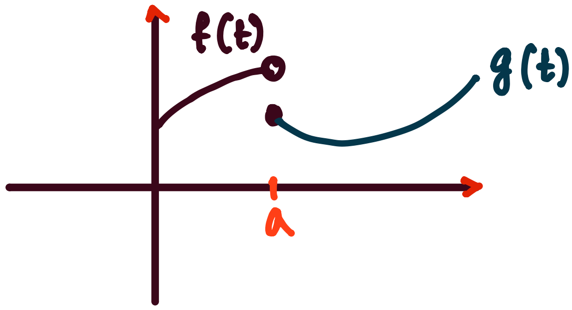
$$= \begin{cases} f(t) & ; \; 0 \le t < a \\ 0 & ; \; t \geq a \end{cases}$$

(i) left portion to a (excluding a) $\longrightarrow$ as it is

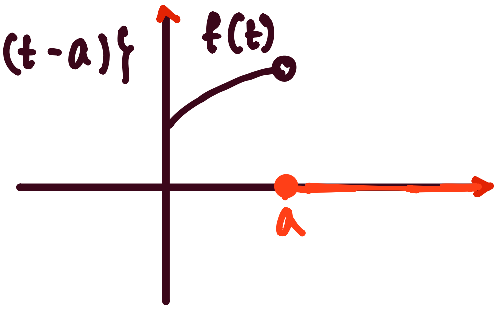
(ii) right portion to a (including a) $\longrightarrow$ nullifies

Piecewise Function in Terms of Unit Step Function

$$h(t) = \begin{cases} f(t); & 0 \leq t < a \\ g(t); & t \geq a \end{cases}$$

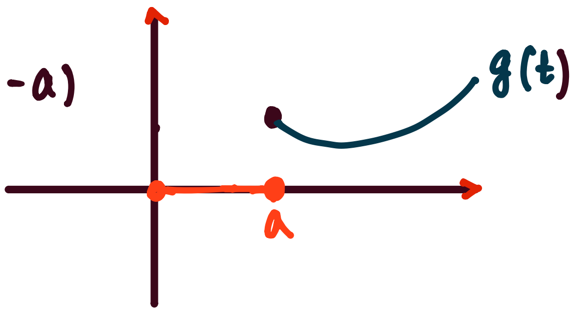


$f(t)\{1-u(t-a)\}$



$$\boxed{h(t) = f(t)\{1-u(t-a)\} + g(t)\,u(t-a)}$$

$g(t)\,u(t-a)$



Piecewise function representation through unit step function

Suppose that,

$$h(t) = \begin{cases} f(t); & 0 \leq t < a \\ g(t); & t \geq a \end{cases}$$

Then,

$$h(t) = f(t)\big(1 - u(t-a)\big) + g(t)u(t-a)$$

2nd Translation Theorem

$\mathcal{L}\{f(t)\} = F(s) \Rightarrow \mathcal{L}\{f(t-a)\,u(t-a)\} = F(s)e^{-as}$

for any positive value of a

$\mathcal{L}\{f(t-a)\,u(t-a)\} = \mathcal{L}\{f(t)\}e^{-as}$

2nd Translation Theorem

$\mathscr{L}\{f(t-a)u(t-a)\} = \mathscr{L}\{f(t)\}e^{-as}$ for any positive value of a.

Problem

Find:

(i) $\mathscr{L}\{(t^2+1)u(t-1)\}$ (ii) $\mathscr{L}\{(t-1)u(t-1)\}$

(iii) $\mathscr{L}\{e^{-2t}u(t-1)\}$ (iv) $\mathscr{L}\{\cos 2t u(t-\pi)\}$

(v) $\mathscr{L}\{u(t-1)(t^2+1)\}$

Solution:

(i) $\mathscr{L}\{(t^2+1)u(t-1)\} = \mathcal{L}\{((t-1)^2+2t)\,u(t-1)\}$

$= \mathcal{L}\{((t-1)^2+2(t-1)+2)\,u(t-1)\}$

$f(t) = t^2+2t+2$

$= \mathcal{L}\{f(t-1)\,u(t-1)\}$

$= \mathcal{L}\{f(t)\}e^{-s}$

$= \mathcal{L}\{t^2+2t+2\}e^{-s}$

$= (\mathcal{L}\{t^2\} + 2\mathcal{L}\{t\} + 2\mathcal{L}\{1\})e^{-s}$

$= \left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{2}{s}\right)e^{-s}$; $s>0$ (Ans:)

(iv) $\mathscr{L}\{\cos 2t\,u(t-\pi)\}$ $= \mathscr{L}\{\cos(2(t-\pi)+2\pi)\,u(t-\pi)\}$

$f(t) = \cos(2t+2\pi) = \cos(2t)$

$= \mathscr{L}\{f(t-\pi)\,u(t-\pi)\}$

$= \mathscr{L}\{f(t)\}\,e^{-\pi s}$

$= \mathscr{L}\{\cos(2t)\}\,e^{-\pi s}$

$= \dfrac{s}{s^2+4}\,e^{-\pi s}$; $s>0$ (Ans:)

(ii) $\mathscr{L}\{(t-1)u(t-1)\}$ $= \mathscr{L}\{f(t-1)\,u(t-1)\}$

$= \mathscr{L}\{f(t)\}\,e^{-s}$ $\quad f(t) = t$

$= \mathscr{L}\{t\}\,e^{-s}$

$= \dfrac{1}{s^2}\,e^{-s}$; $s>0$ (Ans:)

(iii) $\mathscr{L}\{e^{-2t}u(t-1)\}$ $= \mathscr{L}\{e^{-2(t-1)-2}\,u(t-1)\}$

$f(t) = e^{-2t-2}$

$= \mathscr{L}\{f(t-1)\,u(t-1)\}$

$= \mathscr{L}\{f(t)\}\,e^{-s}$

$= \mathscr{L}\{e^{-2t}\,e^{-2}\}\,e^{-s}$

$$= e^{-2}\,\mathcal{L}\{e^{-2t}\}\,e^{-s}$$

$$= e^{-2-s}\,\frac{1}{s+2}\,; \quad s>-2 \quad \text{(Ans:)}$$

Problem

Find $\mathscr{L}\{f(t)\}$ where

$$\text{(i)} f(t) = \begin{cases} 2t+1; & 0 \le t < 2 \\ 3t; & t \ge 2 \end{cases} \qquad \text{(ii)}\ f(t) = \begin{cases} 0; & 0 \le t < \pi \\ \sin t; & t \ge \pi \end{cases}$$

$$\text{(iii)} f(t) = \begin{cases} \sin t; & 0 \le t < 2\pi \\ 0; & t \ge 2\pi \end{cases} \qquad \text{(iv)}\ f(t) = \begin{cases} 0; & 0 \le t < \pi \\ 3\cos t; & t \ge \pi \end{cases}$$

Solution:

$$\text{(i)} f(t) = \begin{cases} 2t+1; & 0 \le t < 2 \\ 3t; & t \ge 2 \end{cases}$$

$$f(t) = (2t+1)\,(1 - u(t-2)) + 3t\,u(t-2)$$

$$= 2t+1 - (2t+1)\,u(t-2) + 3t\,u(t-2)$$

$$= 2t+1 + u(t-2)\,(3t-2t-1)$$

$$f(t) = 2t+1 + (t-1)\,u(t-2)$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{2t+1 + (t-1)\,u(t-2)\}$$

$$= 2\,\mathcal{L}\{t\} + \mathcal{L}\{1\} + \mathcal{L}\{(t-1)\,u(t-2)\}$$

$$= 2\frac{1}{s^2} + \frac{1}{s} + \mathcal{L}\{(t-2+1)\,u(t-2)\}$$

$f(t) = t+1$

$$= \frac{2}{s^2} + \frac{1}{s} + \mathcal{L}\{f(t-2)\,u(t-2)\}$$

$$= \frac{2}{s^2} + \frac{1}{s} + \mathcal{L}\{t+1\}e^{-2s}$$

$$= \frac{2}{s^2} + \frac{1}{s} + \left(\frac{1}{s^2} + \frac{1}{s}\right)e^{-2s}\ ;\ s>0 \text{ (Ans:)}$$

$$\text{(iii)} f(t) = \begin{cases} \sin t; & 0 \le t < 2\pi \\ 0; & t \ge 2\pi \end{cases}$$

$$f(t) = \sin t\ (1 - u(t-2\pi))$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin t\ (1-u(t-2\pi))\}$$

$$= \mathcal{L}\{\sin t - \sin t\, u(t-2\pi)\}$$

$$= \mathcal{L}\{\sin t\} - \mathcal{L}\{\sin t\, u(t-2\pi)\}$$

$$= \mathcal{L}\{\sin t\} - \mathcal{L}\{\sin(t-2\pi+2\pi)\ u(t-2\pi)\}$$

$f(t) = \sin(t+2\pi) = \sin(t)$

$$= \mathcal{L}\{\sin t\} - \mathcal{L}\{f(t-2\pi)\,u(t-2\pi)\}$$

$$= \mathcal{L}\{\sin t\} - \mathcal{L}\{f(t)\}e^{-2\pi s}$$

$$= \mathcal{L}\{\sin t\} - \mathcal{L}\{\sin t\}e^{-2\pi s}$$

$$= \mathcal{L}\{\sin t\}(1-e^{-2\pi s})$$

$$= \frac{1}{s^2+1}(1-e^{-2\pi s})\ ;\ s>0 \text{ (Ans:)}$$

(iv) $f(t) = \begin{cases} 0; & 0 \le t < \pi \\ 3\cos t; & t \ge \pi \end{cases}$

$f(t) = 3\cos t \; u(t-\pi)$

$g(t) = \cos(t+\pi)$

$= -\cos t$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{3\cos t \; u(t-\pi)\}$

$= 3\mathcal{L}\{\cos(t-\pi+\pi)\, u(t-\pi)\}$

$= 3\mathcal{L}\{g(t-\pi)\, u(t-\pi)\}$

$= 3\mathcal{L}\{g(t)\}\, e^{-\pi s}$

$= 3\mathcal{L}\{-\cos t\}\, e^{-\pi s} = -\dfrac{3s}{s^2+1} e^{-\pi s}, \; s>0$ (Ans:)

Brain Teaser

Write the following function in terms of unit step functions.

$$h(t) = \begin{cases} f_1(t), & 0 \le t < t_1 \\ f_2(t), & t_1 \le t < t_2 \\ f_3(t), & t \ge t_2 \end{cases}$$

Brain Teaser

Find the Laplace transform of

$$h(t) = \begin{cases} 0, & 0 \le t < \pi \\ \sin t, & \pi \le t < 2\pi \\ 0, & t \ge 2\pi \end{cases}$$

MAT215: Complex Variables & Laplace Transformations

Chapter : Laplace Transformations

Partial Fraction Decomposition

BRAC University
Department of Mathematics
Summer 2025

Instructor: Kazi Hafizur Rahman

Partial Fraction Decomposition

$\dfrac{P(s)}{Q(s)}$ — $P(s)$ → polynomial, $Q(s)$ → polynomial & $\text{Deg}(Q(s)) > \text{Deg}(P(s))$

$Q(s)$ has distinct linear factors

$$\frac{P(s)}{(a_1s+b_1)(a_2s+b_2)\cdots(a_ns+b_n)} = \frac{A_1}{a_1s+b_1} + \frac{A_2}{a_2s+b_2} + \cdots + \frac{A_n}{a_ns+b_n}$$

-1

Problem

Partially decompose $\dfrac{s^2+6s+9}{(s+4)(s-1)(s-2)}$

Solution:

$$\frac{s^2+6s+9}{(s+4)(s-1)(s-2)} = \frac{A}{s+4} + \frac{B}{s-1} + \frac{C}{s-2}$$

$$\Rightarrow \quad s^2+6s+9 = A(s-1)(s-2) + B(s+4)(s-2) + C(s+4)(s-1)$$

↓

valid for any value of s

(i) $s=-4 \Rightarrow 16-24+9 = A(-5)(-6) \Rightarrow A = 1/30$

(ii) $s=1 \Rightarrow B = -16/5$

(iii) $s=2 \Rightarrow C = 25/6$

$$\frac{s^2+6s+9}{(s+4)(s-1)(s-2)} = \frac{1/30}{s+4} + \frac{-16/5}{s-1} + \frac{25/6}{s-2} \quad \text{(Ans!)}$$

$Q(s)$ has repeated linear factor

$$\frac{P(s)}{(as+b)^n} = \frac{A_1}{as+b} + \frac{A_2}{(as+b)^2} + \cdots + \frac{A_n}{(as+b)^n}$$

Problem

Partially decompose $\dfrac{2s+5}{(s-3)^2}$

Solution:

$$\frac{2s+5}{(s-3)^2} = \frac{A}{s-3} + \frac{B}{(s-3)^2}$$

$$\Rightarrow \quad 2s+5 = A(s-3) + B$$

(i) $s=3 \Rightarrow \quad 11 = B$

(ii) co-efficient of s: $2 = A$

$$\frac{2s+5}{(s-3)^2} = \frac{2}{s-3} + \frac{11}{(s-3)^2} \qquad \text{(Ans:)}$$

$Q(s)$ has irreducible quadratic factor(not repeated)

$$\frac{P(s)}{as^2+bs+c} = \frac{As+B}{as^2+bs+c}$$

-1

Problem

Partially decompose $\dfrac{6s^2+50}{(s+3)(s^2+4)}$

Solution:

$$\frac{6s^2+50}{(s+3)(s^2+4)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+4}$$

$$\Rightarrow \quad 6s^2+50 = A(s^2+4) + (Bs+C)(s+3)$$

(i) $s=-3 \Rightarrow 104 = 13A \Rightarrow A = 8$

(ii) s^2 : $6 = A + B \Rightarrow B = -2$

(iii) constant : $50 = 4A + 3C \Rightarrow C = 6$

$$\frac{6s^2+50}{(s+3)(s^2+4)} = \frac{8}{s+3} + \frac{-2s+6}{s^2+4} \quad \text{(Ans:)}$$

Problem

Partially decompose $\dfrac{2s+1}{s(s+1)(s^2+4s+6)}$

Solution:

$$\frac{2s+1}{s(s+1)(s^2+4s+6)} = \frac{A}{s} + \frac{B}{s+1} + \frac{Cs+D}{s^2+4s+6}$$

$$\Rightarrow 2s+1 = A(s+1)(s^2+4s+6) + Bs(s^2+4s+6) + (Cs+D)(s^2+s)$$

(i) $s=0 \Rightarrow 1 = A.6 \Rightarrow A = 1/6$

(ii) $s=-1 \Rightarrow -1 = -B(1-4+6) \Rightarrow B = 1/3$

(iii) s^3 : $0 = A + B + C \Rightarrow C = -(A+B) = -1/2$

(iv) s : $2 = 6A + 4A + 6B + D \Rightarrow D = -5/3$

$$\frac{2s+1}{s(s+1)(s^2+4s+6)} = \frac{1/6}{s} + \frac{1/3}{s+1} + \frac{-\frac{1}{2}s - 5/3}{s^2+4s+6} \quad \text{(Ans:)}$$

Problem

Partially decompose $\dfrac{3s}{(s+1)(s^2+1)}$

Solution:

$$\frac{3s}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}$$

$$\Rightarrow \quad 3s = A(s^2+1) + (Bs+C)(s+1)$$

(i) $s=-1 \Rightarrow -3 = 2A \Rightarrow A = -3/2$

(ii) s^2 : $0 = A + B \Rightarrow B = 3/2$

(iii) constant : $0 = A + C \Rightarrow C = 3/2$

$$\frac{3s}{(s+1)(s^2+1)} = \frac{-3/2}{s+1} + \frac{3/2\, s + 3/2}{s^2+1} \quad \text{(Ans:)}$$

Problem

Partially decompose $\dfrac{s}{(s^2+4)(s^2+9)}$

Solution: $$\frac{s}{(s^2+4)(s^2+9)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9}$$

$$\Rightarrow s = (As+B)(s^2+9) + (Cs+D)(s^2+4)$$

$$\Rightarrow s = As^3 + 9As + Bs^2 + 9B + Cs^3 + 4Cs + Ds^2 + 4D$$

$$\Rightarrow s = (A+C)s^3 + (B+D)s^2 + (9A+4C)s + 9B+4D$$

s^3: $\quad 0 = A+C$ — (i) $\Rightarrow A = -C \Rightarrow A = 1/5$

s^2: $\quad 0 = B+D$ — (ii) $\Rightarrow B = -D \Rightarrow B = 0$

s: $\quad 1 = 9A+4C$ — (iii) $\Rightarrow -9C+4C = 1 \Rightarrow C = -1/5$

constant: $\quad 0 = 9B+4D$ — (iv) $\Rightarrow -9D+4D = 0 \Rightarrow D = 0$

$$\frac{s}{(s^2+4)(s^2+9)} = \frac{1/5\, s}{s^2+4} + \frac{-1/5\, s}{s^2+9} \quad \text{(Ans:)}$$

$Q(s)$ has repeated irreducible quadratic factor

$$\frac{P(s)}{(as^2+bs+c)^n} = \frac{A_1 s + B_1}{as^2+bs+c} + \frac{A_2 s + B_2}{(as^2+bs+c)^2} + \cdots + \frac{A_n s + B_n}{(as^2+bs+c)^n}$$

Problem

H.W.

Partially decompose $\dfrac{s+3}{(s-1)(s^2+1)^2}$

Solution:

$$\frac{s+3}{(s-1)(s^2+1)^2} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1} + \frac{Ds+E}{(s^2+1)^2}$$

MAT215: Complex Variables & Laplace Transformations

Chapter : Laplace Transformations

Inverse of Laplace transform meaning
Elementary inverse Laplace transforms & Linearity of Inverse Laplace transform
1st Translation theorem in inverse form
Inverse using Partial fraction decomposition

BRAC University
Department of Mathematics
Summer 2025

Instructor: Kazi Hafizur Rahman

Inverse Laplace Transformation

$f(t) \rightarrow$ input function

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt \Longleftrightarrow \mathcal{L}\{f(t)\} = F(s) \Longleftrightarrow \mathcal{L}^{-1}\{F(s)\} = f(t)$$

$F(s) \rightarrow$ laplace transform of $f(t)$

$f(t) \rightarrow$ Inverse laplace transform of $F(s)$

- $\mathcal{L}\{1\} = \frac{1}{s} \Longleftrightarrow \mathcal{L}^{-1}\{\frac{1}{s}\} = 1$

Inverse Laplace Transforms of some known expressions

$F(s)$	$\mathcal{L}^{-1}\{F(s)\} = f(t)$	$F(s)$	$\mathcal{L}^{-1}\{F(s)\} = f(t)$
$\frac{1}{s}$	1	$\frac{a}{s^2+a^2}$	$\sin(at)$
$\frac{n!}{s^{n+1}},\ n \in \mathbb{Z}^+$	t^n	$\frac{s}{s^2+a^2}$	$\cos(at)$
$\frac{\Gamma(n+1)}{s^{n+1}},\ n \notin \mathbb{Z}$	t^n	$\frac{a}{s^2-a^2}$	$\sinh(at)$
$\frac{1}{s-a}$	e^{at}	$\frac{s}{s^2-a^2}$	$\cosh(at)$

Linearity of Inverse Laplace transformation

$$\mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha \mathcal{L}^{-1}\{F(s)\} + \beta \mathcal{L}^{-1}\{G(s)\}$$

Problem

Find:

(i) $\mathscr{L}^{-1}\{\frac{1}{s^4}\}$ (ii) $\mathscr{L}^{-1}\{\frac{12}{4-3s}\}$ (iii) $\mathscr{L}^{-1}\{\frac{23s-15}{s^2+8}\}$
(iv) $\mathscr{L}^{-1}\{\frac{2s-5}{s^2-9}\}$ (v) $\mathscr{L}^{-1}\{\frac{1}{s^{3/2}}\}$

Solution: (i) $\mathscr{L}^{-1}\{\frac{1}{s^4}\}$

$$\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$$

$$\mathscr{L}^{-1}\{\frac{1}{s^4}\} = \mathcal{L}^{-1}\left\{\frac{3!}{s^4}\,\frac{1}{3!}\right\}$$

$$= \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\} = \frac{1}{6}t^3 \quad \text{(Ans:)}$$

Try to generate results of known laplace transform in the argument of $\mathcal{L}^{-1}$

(ii) $\mathscr{L}^{-1}\{\frac{12}{4-3s}\}$ $= \mathcal{L}^{-1}\left\{\frac{12}{-3(s-4/3)}\right\}$

$$= -4\mathcal{L}^{-1}\left\{\frac{1}{s-4/3}\right\}$$

$$= -4e^{4/3t} \quad \text{(Ans:)}$$

(iii) $\mathscr{L}^{-1}\{\frac{23s-15}{s^2+8}\}$ $= 23\mathcal{L}^{-1}\left\{\frac{s}{s^2+8}\right\} - 15\mathcal{L}^{-1}\left\{\frac{1}{s^2+8}\right\}$

$$= 23\mathcal{L}^{-1}\left\{\frac{s}{s^2+(\sqrt{8})^2}\right\} - 15\mathcal{L}^{-1}\left\{\frac{1}{s+(\sqrt{8})^2}\right\}$$

$$= 23\cos(\sqrt{8}t) - 15\,\mathcal{L}^{-1}\left\{\frac{\sqrt{8}}{s^2+(\sqrt{8})^2}\,\frac{1}{\sqrt{8}}\right\}$$

$$= 23\cos(\sqrt{8}t) - 15/\sqrt{8}\,\sin(\sqrt{8}t) \quad \text{(Ans:)}$$

(iv) $\mathscr{L}^{-1}\{\frac{2s-5}{s^2-9}\}$ $= 2\mathcal{L}^{-1}\left\{\frac{s}{s^2-9}\right\} - 5\,\mathcal{L}^{-1}\left\{\frac{1}{s^2-9}\right\}$

$$= 2\cosh(3t) - 5/3\,\sinh(3t) \quad \text{(Ans:)}$$

(v) $\mathscr{L}^{-1}\{\frac{1}{s^{3/2}}\}$ $= \mathcal{L}^{-1}\left\{\frac{\Gamma(1/2+1)}{s^{1/2+1}}\cdot\frac{1}{\Gamma(1/2+1)}\right\}$

$$= \frac{1}{\Gamma(1/2+1)}\mathcal{L}^{-1}\left\{\frac{\Gamma(1/2+1)}{s^{1/2+1}}\right\}$$

$$= \frac{1}{\Gamma(1/2+1)}\,t^{1/2} \quad \text{(Ans:)}$$

First Translation Theorem in inverse form

$$\mathcal{L}\{f(t)\} = F(s) \Rightarrow \mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\mathcal{L}^{-1}\{F(s)\} = f(t) \Rightarrow \mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$$

$$\mathcal{L}^{-1}\{F(s)|_{s\to s-a}\} = e^{at}\,\mathcal{L}^{-1}\{F(s)\}$$

First Translation Theorem in inverse form

$$\mathscr{L}^{-1}\{F(s)\big|_{s\to s-a}\} = e^{at}\mathscr{L}^{-1}\{F(s)\}$$

Problem

Find:

(i) $\mathscr{L}^{-1}\{\frac{1}{(s-4)^2}\}$ (ii) $\mathscr{L}^{-1}\{\frac{2}{(s-3)^5}\}$

(iii) $\mathscr{L}^{-1}\{\frac{s-3}{(s-3)^2+1}\}$ (iv) $\mathscr{L}^{-1}\{\frac{6s-4}{(s^2-4s+20)}\}$

Solution:

(i) $\mathscr{L}^{-1}\{\frac{1}{(s-4)^2}\} = \mathscr{L}^{-1}\left\{\frac{1}{s^2}\Big|_{s\to s-4}\right\}$

$= \mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} e^{4t}$

$= t e^{4t}$ (Ans:)

(ii) $\mathscr{L}^{-1}\{\frac{2}{(s-3)^5}\} = \mathscr{L}^{-1}\left\{\frac{2}{s^5}\Big|_{s\to s-3}\right\}$

$= \mathscr{L}^{-1}\left\{\frac{2}{s^5}\right\} e^{3t}$

$= \mathscr{L}^{-1}\left\{\frac{2.4!}{s^5} \cdot \frac{1}{4!}\right\} e^{3t}$

$= \frac{1}{12} t^4 e^{3t}$ (Ans:)

(iii) $\mathscr{L}^{-1}\{\frac{s-3}{(s-3)^2+1}\} = \mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\Big|_{s\to s-3}\right\}$

$= \mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\right\} e^{3t}$

$= \cos t \, e^{3t}$ (Ans:)

(iv) $\mathscr{L}^{-1}\{\frac{6s-4}{(s^2-4s+20)}\}$

Hint : completing the square

$$\mathscr{L}^{-1}\{\frac{6s-4}{(s^2-4s+20)}\} = \mathscr{L}^{-1}\left\{\frac{6s-4}{(s-2)^2+16}\right\}$$

$$= 6\,\mathscr{L}^{-1}\left\{\frac{s}{(s-2)^2+16}\right\} - 4\,\mathscr{L}^{-1}\left\{\frac{1}{(s-2)^2+16}\right\}$$

$$= 6\,\mathscr{L}^{-1}\left\{\frac{s-2+2}{(s-2)^2+16}\right\} - 4\,\mathscr{L}^{-1}\left\{\frac{1}{(s-2)^2+16}\right\}$$

$$= 6\,\mathscr{L}^{-1}\left\{\frac{s-2}{(s-2)^2+16}\right\} + 12\,\mathscr{L}^{-1}\left\{\frac{1}{(s-2)^2+16}\right\}$$
$$- 4\,\mathscr{L}^{-1}\left\{\frac{1}{(s-2)^2+16}\right\}$$

$$= 6\,\mathscr{L}^{-1}\left\{\frac{s}{s^2+16}\Big|_{s\to s-2}\right\} + 8\,\mathscr{L}^{-1}\left\{\frac{1}{(s-2)^2+16}\right\}$$

$$= 6\,\mathscr{L}^{-1}\left\{\frac{s}{s^2+16}\right\}e^{2t} + 8\,\mathscr{L}^{-1}\left\{\frac{1}{s^2+16}\Big|_{s\to s-2}\right\}$$

$$= 6\cos(4t)\,e^{2t} + 8\,\mathscr{L}^{-1}\left\{\frac{1}{s^2+16}\right\}e^{2t}$$

$$= 6\cos(4t)\,e^{2t} + 2\sin(4t)\,e^{2t}$$ (Ans:)

Problem

Find:

(i) $\mathscr{L}^{-1}\{\frac{s^2+6s+9}{(s+4)(s-1)(s-2)}\}$ (ii) $\mathscr{L}^{-1}\{\frac{2s+5}{(s-3)^2}\}$

(iii) $\mathscr{L}^{-1}\{\frac{6s^2+50}{(s^2+4)(s+3)}\}$ (iv) $\mathscr{L}^{-1}\{\frac{3s}{(s+1)(s^2+1)}\}$

(v) $\mathscr{L}^{-1}\{\frac{2s+1}{s(s+1)(s^2+4s+6)}\}$ (vi) $\mathscr{L}^{-1}\{\frac{s}{(s^2+4)(s^2+9)}\}$

Solution: (i) $\mathscr{L}^{-1}\{\frac{s^2+6s+9}{(s+4)(s-1)(s-2)}\}$

$$\frac{s^2+6s+9}{(s+4)(s-1)(s-2)} = \frac{A}{s+4} + \frac{B}{s-1} + \frac{C}{s-2}$$

$$\Rightarrow \quad s^2+6s+9 = A(s-1)(s-2) + B(s+4)(s-2) + C(s+4)(s-1)$$

valid for any value of s

(i) $s=-4 \Rightarrow 16-24+9 = A(-5)(-6) \Rightarrow A = 1/30$

(ii) $s=1 \Rightarrow B = -16/5$

(iii) $s=2 \Rightarrow C = 25/6$

$$\frac{s^2+6s+9}{(s+4)(s-1)(s-2)} = \frac{1/30}{s+4} + \frac{-16/5}{s-1} + \frac{25/6}{s-2}$$

$$\mathscr{L}^{-1}\left\{\frac{s^2+6s+9}{(s+4)(s-1)(s-2)}\right\} = \mathscr{L}^{-1}\left\{\frac{1/30}{s+4} + \frac{-16/5}{s-1} + \frac{25/6}{s-2}\right\}$$

$$= 1/30\,\mathscr{L}^{-1}\left\{\frac{1}{s+4}\right\} - 16/5\,\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\} + 25/6\,\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$= 1/30\, e^{-4t} - 16/5\, e^{t} + 25/6\, e^{2t} \text{ (Ans:)}$$

(ii) $\mathscr{L}^{-1}\{\frac{2s+5}{(s-3)^2}\}$

$$\frac{2s+5}{(s-3)^2} = \frac{A}{s-3} + \frac{B}{(s-3)^2}$$

$$\Rightarrow \quad 2s+5 = A(s-3) + B$$

(i) $s=3 \Rightarrow \quad 11 = B$

(ii) co-efficient of s : $2 = A$

$$\frac{2s+5}{(s-3)^2} = \frac{2}{s-3} + \frac{11}{(s-3)^2}$$

$$\mathscr{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\} = \mathscr{L}^{-1}\left\{\frac{2}{s-3} + \frac{11}{(s-3)^2}\right\}$$

$$= 2\,\mathscr{L}^{-1}\left\{\frac{1}{s-3}\right\} + 11\,\mathscr{L}^{-1}\left\{\frac{1}{s^2}\Big|_{s \to s-3}\right\}$$

$$= 2e^{3t} + 11\,\mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} e^{3t}$$

$$= 2e^{3t} + 11te^{3t}$$ (Ans:)

(iii) $\mathscr{L}^{-1}\{\frac{6s^2+50}{(s^2+4)(s+3)}\}$

$$\frac{6s^2+50}{(s+3)(s^2+4)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+4}$$

$$\Rightarrow \quad 6s^2 + 50 = A(s^2+4) + (Bs + C)(s+3)$$

(i) $s = -3 \Rightarrow 104 = 13A \Rightarrow A = 8$

(ii) s^2 : $6 = A + B \Rightarrow B = -2$

(iii) constant : $50 = 4A + 3C \Rightarrow C = 6$

$$\frac{6s^2+50}{(s+3)(s^2+4)} = \frac{8}{s+3} + \frac{-2s+6}{s^2+4}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{6s^2+50}{(s+3)(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{8}{s+3} + \frac{-2s+6}{s^2+4}\right\}$$

$$= 8\,\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} - 2\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + 6\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= 8e^{-3t} - 2\cos(2t) + 3\sin(2t) \quad \text{(Ans:)}$$

(iv) $\mathcal{L}^{-1}\{\frac{3s}{(s+1)(s^2+1)}\}$

$$\frac{3s}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}$$

$$\Rightarrow \quad 3s = A(s^2+1) + (Bs+C)(s+1)$$

(i) $s = -1 \Rightarrow -3 = 2A \Rightarrow A = -3/2$

(ii) s^2 : $0 = A + B \Rightarrow B = 3/2$

(iii) constant : $0 = A + C \Rightarrow C = 3/2$

$$\frac{3s}{(s+1)(s^2+1)} = \frac{-3/2}{s+1} + \frac{3/2\,s + 3/2}{s^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{3s}{(s+1)(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{-3/2}{s+1} + \frac{3/2\, s + 3/2}{s^2+1}\right\}$$

$$= 3/2\left(-\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}\right)$$

$$= 3/2\left(-e^{-t} + \cos t + \sin t\right) \quad \text{(Ans!)}$$

(v) $\mathcal{L}^{-1}\{\frac{2s+1}{s(s+1)(s^2+4s+6)}\}$

$$\frac{2s+1}{s(s+1)(s^2+4s+6)} = \frac{A}{s} + \frac{B}{s+1} + \frac{Cs+D}{s^2+4s+6}$$

$$\Rightarrow 2s+1 = A(s+1)(s^2+4s+6) + Bs(s^2+4s+6) + (Cs+D)(s^2+s)$$

(i) $s=0 \Rightarrow 1 = A.6 \Rightarrow A = 1/6$

(ii) $s=-1 \Rightarrow -1 = -B(1-4+6) \Rightarrow B = 1/3$

(iii) $s^3: \quad 0 = A+B+C \Rightarrow C = -(A+B) = -1/2$

(iv) $s: \quad 2 = 6A+4A+6B+D \Rightarrow D = -5/3$

$$\frac{2s+1}{s(s+1)(s^2+4s+6)} = \frac{1/6}{s} + \frac{1/3}{s+1} + \frac{-1/2\, s - 5/3}{s^2+4s+6}$$

$$\mathcal{L}^{-1}\left\{\frac{2s+1}{s(s+1)(s^2+4s+6)}\right\} = \mathcal{L}^{-1}\left\{\frac{1/6}{s} + \frac{1/3}{s+1} + \frac{-1/2\, s - 5/3}{s^2+4s+6}\right\}$$

$$= 1/6\,\mathcal{L}^{-1}\{1/s\} + 1/3\,\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - 1/2\,\mathcal{L}^{-1}\left\{\frac{s}{(s+2)^2+2}\right\}$$

$$- 5/3\,\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+2}\right\}$$

$$= 1/6 + 1/3\, e^{-t} - 1/2\, \mathcal{L}^{-1}\left\{\frac{s+2-2}{(s+2)^2+2}\right\}$$
$$- 5/3\, \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+2}\right\}$$

$$= 1/6 + 1/3\, e^{-t} - 1/2\, \mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+(\sqrt{2})^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+2}\right\}$$
$$- 5/3\, \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+2}\right\}$$

$$= 1/6 + 1/3\, e^{-t} - 1/2\, \mathcal{L}^{-1}\left\{\frac{s}{s^2+(\sqrt{2})^2}\Big|_{s\to s+2}\right\}$$
$$- 2/3\, \mathcal{L}^{-1}\left\{\frac{1}{s^2+(\sqrt{2})^2}\Big|_{s\to s+2}\right\}$$

$$= 1/6 + 1/3\, e^{-t} - 1/2 \cos(\sqrt{2}t)\, e^{-2t} - \frac{2}{3\sqrt{2}} \sin(\sqrt{2}t)\, e^{-2t}$$

(Ans:)

(vi) $\mathcal{L}^{-1}\{\frac{s}{(s^2+4)(s^2+9)}\}$

$$\frac{s}{(s^2+4)(s^2+9)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9}$$

$$\Rightarrow s = (As+B)(s^2+9) + (Cs+D)(s^2+4)$$

$$\Rightarrow s = As^3 + 9As + Bs^2 + 9B + Cs^3 + 4Cs + Ds^2 + 4D$$

$$\Rightarrow s = (A+C)s^3 + (B+D)s^2 + (9A+4C)s + 9B+4D$$

$s^3: \quad 0 = A + C \longrightarrow \text{(i)} \Rightarrow A = -C \Rightarrow A = 1/5$

$s^2: \quad 0 = B + D \longrightarrow \text{(ii)} \Rightarrow B = -D \Rightarrow B = 0$

$s: \quad 1 = 9A + 4C \longrightarrow \text{(iii)} \Rightarrow -9C + 4C = 1 \Rightarrow C = -1/5$

constant: $\quad 0 = 9B + 4D \longrightarrow \text{(iv)} \Rightarrow -9D + 4D = 0 \Rightarrow D = 0$

$$\frac{s}{(s^2+4)(s^2+9)} = \frac{1/5\, s}{s^2+4} + \frac{-1/5\, s}{s^2+9}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)(s^2+9)}\right\} = \mathcal{L}^{-1}\left\{\frac{1/5\, s}{s^2+4} + \frac{-1/5\, s}{s^2+9}\right\}$$

$$= 1/5 \left\{\cos(2t) - \cos(3t)\right\}$$

(Ans:)

MAT215: Complex Variables & Laplace Transformations

Chapter : Laplace Transformations

2nd Translation theorem in inverse form
Laplace transform of derivatives
Solving Initial Value Problem (IVP) using Laplace transform

BRAC University
Department of Mathematics and Natural Sciences
Summer 2025

Instructor: Kazi Hafizur Rahman

Second Translation Theorem in inverse form

$\mathcal{L}\{f(t)\} = F(s) \Rightarrow \mathcal{L}\{f(t-a)\, u(t-a)\} = e^{-as} F(s)$

$\mathcal{L}^{-1}\{F(s)\} = f(t) \Rightarrow \mathcal{L}^{-1}\{e^{-as} F(s)\} = f(t-a)\, u(t-a)$

$\mathcal{L}^{-1}\{e^{-as} F(s)\} = \mathcal{L}^{-1}\{F(s)\}\Big|_{t \to t-a} u(t-a)$

Second Translation Theorem in inverse form

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = \mathscr{L}^{-1}\{F(s)\}\big|_{t\to t-a}u(t-a)$$

Problem

Find:

(i) $\mathscr{L}^{-1}\{\frac{1}{(s-4)}e^{-2s}\}$ (ii) $\mathscr{L}^{-1}\{\frac{3s}{(s+1)(s^2+1)}e^{-\pi s}\}$

Solution:

(i) $\mathscr{L}^{-1}\{\frac{1}{(s-4)}e^{-2s}\} = \mathcal{L}^{-1}\{\frac{1}{s-4}\}\Big|_{t\to t-2} u(t-2)$

$= e^{4t}\Big|_{t\to t-2} u(t-2)$

$= e^{4(t-2)} u(t-2)$ (Ans!)

(ii) $\mathscr{L}^{-1}\{\frac{3s}{(s+1)(s^2+1)}e^{-\pi s}\} = \mathcal{L}^{-1}\{\frac{3s}{(s+1)(s^2+1)}\}\Big|_{t-\pi} u(t-\pi)$

Finding $\mathcal{L}^{-1}\{\frac{3s}{(s+1)(s^2+1)}\}$:

$$\frac{3s}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}$$

$$\Rightarrow \quad 3s = A(s^2+1) + (Bs+C)(s+1)$$

(i) $s=-1 \Rightarrow \quad -3 = 2A \Rightarrow A = -3/2$

(ii) $s^2: \quad 0 = A+B \Rightarrow B = 3/2$

(iii) constant : $0 = A + C \Rightarrow C = 3/2$

$$\frac{3s}{(s+1)(s^2+1)} = \frac{-3/2}{s+1} + \frac{3/2\,s + 3/2}{s^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{3s}{(s+1)(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{-3/2}{s+1} + \frac{3/2\,s+3/2}{s^2+1}\right\}$$

$$= 3/2\left(-\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}\right)$$

$$= 3/2\left(-e^{-t} + \cos t + \sin t\right)$$

$$\mathcal{L}^{-1}\left\{\frac{3s}{(s+1)(s^2+1)}e^{-\pi s}\right\} = \mathcal{L}^{-1}\left\{\frac{3s}{(s+1)(s^2+1)}\right\}\Big|_{t-\pi} u(t-\pi)$$

$$= 3/2\left\{-e^{-(t-\pi)} + \sin(t-\pi) + \cos(t-\pi)\right\} u(t-\pi) \quad \text{(Ans:)}$$

Laplace transformation of derivatives

If $\mathscr{L}\{f(t)\} = F(s)$, then

(i) $\mathscr{L}\{f'(t)\} = ?$ (ii) $\mathscr{L}\{f''(t)\} = ?$ (iii) $\mathscr{L}\{f'''(t)\} = ?$ if $s > 0$

(iv)What is the generalization?

Solution: $\mathscr{L}\{f(t)\} = F(s)$

$$\mathscr{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= \lim_{p\to\infty} \int_0^p e^{-st} f'(t)\,dt$$

$\int uv\,dx = u\int v\,dx - \int(u'\int v\,dx)\,dx$ (integration by parts formula)

choosing u, v : LIATE formula

$u = e^{-st}$

$v = f'(t)$

$$= \lim_{p\to\infty} \left\{ e^{-st} f(t)\Big|_0^p - \int_0^p e^{-st}(-s) f(t)\,dt \right\}$$

$$= \lim_{p\to\infty} \left\{ e^{-sp} f(p) - f(0) + s\int_0^p e^{-st} f(t)\,dt \right\}$$

$$= \lim_{p\to\infty} e^{-sp} f(p) - f(0) + s \lim_{p\to\infty} \int_0^p e^{-st} f(t)\,dt$$

$$= 0 - f(0) + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= s\mathscr{L}\{f(t)\} - f(0)$$

$$\therefore \quad \mathscr{L}\{f'(t)\} = s\mathscr{L}\{f(t)\} - f(0)$$

$$\mathscr{L}\{f''(t)\} = \mathscr{L}\{(f'(t))'\}$$

$$= s\mathscr{L}\{f'(t)\} - f'(0)$$

$$= s\{s\mathcal{L}\{f(t)\} - f(0)\} - f'(0)$$

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

$$\mathcal{L}\{f'''(t)\} = s\mathcal{L}\{f''(t)\} - f''(0)$$

$$= s\{s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)\} - f''(0)$$

$$\mathcal{L}\{f'''(t)\} = s^3\mathcal{L}\{f(t)\} - s^2f(0) - sf'(0) - f''(0)$$

Laplace transformation of derivatives

If $\mathscr{L}\{f(t)\} = F(s)$, then

$$\mathscr{L}\{f'(t)\} = s \cdot F(s) - f(0)$$

$$\mathscr{L}\{f''(t)\} = s^2 \cdot F(s) - s \cdot f(0) - f'(0)$$

$$\mathscr{L}\{f'''(t)\} = s^3 \cdot F(s) - s^2 \cdot f(0) - s \cdot f'(0) - f''(0)$$

Provided that $s > 0$.

Initial Value Problem

$x^2 + 2x + 1 = 0 \rightarrow$ unknown x $\rightarrow$ number

(Algebraic equation)

Initial Value Problem (IVP)

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + y + \sin t = 0 \quad \text{(i)}$$

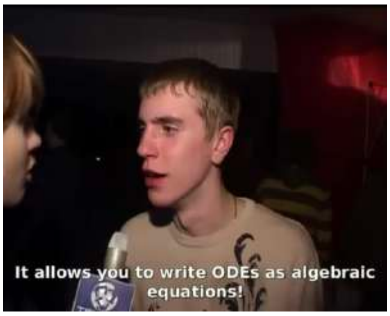
$$\left.\begin{array}{l} y(0) = 2 \\ y'(0) = 4 \end{array}\right\} \text{(ii)}$$

(initial conditions)

→ unknown $y(t)$

→ Differential equation for the unknown $y(t)$

Solving IVP Using Laplace Transformation



Step 1: Apply Laplace Transform (y(t) -> Y(s))

Step 2: Find Y(s)

Step 3: Apply Inverse Laplace Transform (Y(s) -> y(t))

Problem

Solve the following initial value problems:

(i) $\dfrac{dy}{dt} + 3y = 13\sin 2t, \quad y(0) = 6.$

(ii) $y'' - 3y' + 2y = e^{-4t}, \quad y(0) = 1, \quad y'(0) = 5.$

(iii) $y'' - 6y' + 9y = t^2 e^{3t}, \quad y(0) = 2, \quad y'(0) = 17$

(iv) $y'' + 4y' + 6y = 1 + e^{-t}, \quad y(0) = 0, \quad y'(0) = 0.$

(v) $y' + y = f(t), \quad y(0) = 5, \quad \text{where } f(t) = \begin{cases} 0, & 0 \le t < \pi \\ 3\cos t, & t \ge \pi \end{cases}$

(i) $\dfrac{dy}{dt}+3y=13\sin 2t, \quad y(0)=6.$

Solution:

$$\mathcal{L}\{y'(t)+3y\}=\mathcal{L}\{13\sin 2t\}$$

$$\Rightarrow \mathcal{L}\{y'(t)\}+3\mathcal{L}\{y(t)\}=13\,\frac{2}{s^2+4}$$

let,

$$\mathcal{L}\{y(t)\}=Y(s)$$

$$\mathcal{L}\{y'(t)\}=sY(s)-y(0)$$

$$sY(s)-y(0)+3Y(s)=\frac{26}{s^2+4}$$

$$\Rightarrow (s+3)Y(s)=6+\frac{26}{s^2+4}=\frac{6s^2+50}{s^2+4}$$

$$\Rightarrow Y(s)=\frac{6s^2+50}{(s+3)(s^2+4)}$$

$$\Rightarrow \mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{6s^2+50}{(s+3)(s^2+4)}\right\}$$

$$\Rightarrow y(t)=\mathcal{L}^{-1}\left\{\frac{6s^2+50}{(s+3)(s^2+4)}\right\}$$

Finding $\mathcal{L}^{-1}\left\{\dfrac{6s^2+50}{(s^2+4)(s+3)}\right\}$:

$$\frac{6s^2+50}{(s+3)(s^2+4)}=\frac{A}{s+3}+\frac{Bs+C}{s^2+4}$$

$$\Rightarrow \quad 6s^2+50 = A(s^2+4) + (Bs+C)(s+3)$$

(i) $s=-3 \Rightarrow 104 = 13A \Rightarrow A = 8$

(ii) s^2: $6 = A+B \Rightarrow B=-2$

(iii) constant : $50 = 4A+3C \Rightarrow C=6$

$$\frac{6s^2+50}{(s+3)(s^2+4)} = \frac{8}{s+3} + \frac{-2s+6}{s^2+4}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{6s^2+50}{(s+3)(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{8}{s+3} + \frac{-2s+6}{s^2+4}\right\}$$

$$= 8\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} - 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + 6\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= 8e^{-3t} - 2\cos(2t) + 3\sin(2t)$$

Therefore,

$$y(t) = 8e^{-3t} - 2\cos(2t) + 3\sin(2t) \quad \text{(Ans:)}$$

(ii) $y'' - 3y' + 2y = e^{-4t}, \quad y(0) = 1, \quad y'(0) = 5.$

Solution:

$$\mathcal{L}\{y'' - 3y' + 2y\} = \mathcal{L}\{e^{-4t}\}$$

$$\Rightarrow \mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \frac{1}{s+4}$$

let,

$$\mathcal{L}\{y(t)\} = Y(s)$$

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0) = sY(s) - 1$$

$$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s - 5$$

$$s^2Y(s) - s - 5 - 3\{sY(s) - 1\} + 2Y(s) = \frac{1}{s+4}$$

$$\Rightarrow (s^2 - 3s + 2)Y(s) = s + 5 - 3 + \frac{1}{s+4}$$

$$\Rightarrow (s-1)(s-2)Y(s) = \frac{(s+2)(s+4)+1}{s+4}$$

$$\Rightarrow Y(s) = \frac{s^2+6s+9}{(s+4)(s-1)(s-2)}$$

$$\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{s^2+6s+9}{(s+4)(s-1)(s-2)}\right\}$$

Finding $\mathcal{L}^{-1}\{\frac{s^2+6s+9}{(s+4)(s-1)(s-2)}\}$:

$$\frac{s^2+6s+9}{(s+4)(s-1)(s-2)} = \frac{A}{s+4} + \frac{B}{s-1} + \frac{C}{s-2}$$

$$\Rightarrow s^2 + 6s + 9 = A(s-1)(s-2) + B(s+4)(s-2) + C(s+4)(s-1)$$

valid for any value of s

(i) $s = -4 \Rightarrow 16 - 24 + 9 = A(-5)(-6) \Rightarrow A = 1/30$

(ii) $s = 1 \Rightarrow B = -16/5$

(iii) $s = 2 \Rightarrow C = 25/6$

$$\frac{s^2+6s+9}{(s+4)(s-1)(s-2)} = \frac{1/30}{s+4} + \frac{-16/5}{s-1} + \frac{25/6}{s-2}$$

$$\mathcal{L}^{-1}\left\{\frac{s^2+6s+9}{(s+4)(s-1)(s-2)}\right\} = \mathcal{L}^{-1}\left\{\frac{1/30}{s+4} + \frac{-16/5}{s-1} + \frac{25/6}{s-2}\right\}$$

$$= 1/30\,\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} - 16/5\,\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + 25/6\,\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$= 1/30\, e^{-4t} - 16/5\, e^{t} + 25/6\, e^{2t}$$

Therefore we have,

$$\Rightarrow \quad y(t) = \mathcal{L}^{-1}\left\{\frac{s^2+6s+9}{(s+4)(s-1)(s-2)}\right\}$$

$$\therefore y(t) = 1/30\, e^{-4t} - 16/5\, e^{t} + 25/6\, e^{2t} \quad \text{(Ans:)}$$

(iii) $y'' - 6y' + 9y = t^2 e^{3t}, \quad y(0) = 2, \quad y'(0) = 17$

Solution:

$$\mathcal{L}\{y'' - 6y' + 9y\} = \mathcal{L}\{t^2 e^{3t}\}$$

$$\Rightarrow \mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = \mathcal{L}\{t^2\}\big|_{s \to s-3}$$

let,

$$\mathcal{L}\{y(t)\} = Y(s)$$

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0) = sY(s) - 2$$

$$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 2s - 17$$

$$s^2Y(s) - 2s - 17 - 6\{sY(s) - 2\} + 9Y(s) = \frac{2}{(s-3)^3}$$

$$\Rightarrow (s^2 - 6s + 9)Y(s) - 2s - 17 + 12 = \frac{2}{(s-3)^3}$$

$$\Rightarrow \quad (s-3)^2 Y(s) = 2s+5+\frac{2}{(s-3)^3}$$

$$\Rightarrow \quad Y(s) = \frac{2s+5}{(s-3)^2}+\frac{2}{(s-3)^5}$$

$$\Rightarrow \quad \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}+\frac{2}{(s-3)^5}\right\}$$

$$\Rightarrow \quad y(t) = \mathcal{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\}+\mathcal{L}^{-1}\left\{\frac{2}{(s-3)^5}\right\}$$

Finding $\mathcal{L}^{-1}\{\frac{2}{(s-3)^5}\}$ & $\mathcal{L}^{-1}\{\frac{2s+5}{(s-3)^2}\}$:

(ii) $\mathcal{L}^{-1}\{\frac{2}{(s-3)^5}\}$

$$= \mathcal{L}^{-1}\left\{\left.\frac{2}{s^5}\right|_{s\to s-3}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2}{s^5}\right\}e^{3t}$$

$$= \mathcal{L}^{-1}\left\{\frac{2\cdot 4!}{s^5}\;\frac{1}{4!}\right\}e^{3t}$$

$$= \frac{1}{12}t^4e^{3t}$$

(ii) $\mathscr{L}^{-1}\{\frac{2s+5}{(s-3)^2}\}$

$$\frac{2s+5}{(s-3)^2} = \frac{A}{s-3} + \frac{B}{(s-3)^2}$$

$$\Rightarrow 2s+5 = A(s-3) + B$$

(i) $s=3 \Rightarrow 11 = B$

(ii) co-efficient of s: $2 = A$

$$\frac{2s+5}{(s-3)^2} = \frac{2}{s-3} + \frac{11}{(s-3)^2}$$

$$\mathscr{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\} = \mathscr{L}^{-1}\left\{\frac{2}{s-3} + \frac{11}{(s-3)^2}\right\}$$

$$= 2\mathscr{L}^{-1}\left\{\frac{1}{s-3}\right\} + 11\mathscr{L}^{-1}\left\{\frac{1}{s^2}\Big|_{s\to s-3}\right\}$$

$$= 2e^{3t} + 11\mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\}e^{3t}$$

$$= 2e^{3t} + 11te^{3t}$$

Therefore we have,

$$y(t) = 2e^{3t} + 11te^{3t} + \frac{1}{12}t^4e^{3t} \quad \text{(Ans:)}$$

(v) $y' + y = f(t), \quad y(0) = 5, \quad \text{where } f(t) = \begin{cases} 0, & 0 \le t < \pi \\ 3\cos t, & t \ge \pi \end{cases}$

$$\mathcal{L}\{y'+y\} = \mathcal{L}\{f(t)\}$$

$$\Rightarrow \mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{f(t)\}$$

let,

$$\mathcal{L}\{y(t)\} = Y(s)$$

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0) = sY(s) - 5$$

$$f(t) = 3\cos t \; u(t-\pi)$$

$$g(t) = \cos(t+\pi)$$
$$= -\cos t$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{3\cos t \; u(t-\pi)\}$$
$$= 3\mathcal{L}\{\cos(t-\pi+\pi)\, u(t-\pi)\}$$
$$= 3\mathcal{L}\{g(t-\pi)\, u(t-\pi)\}$$
$$= 3\mathcal{L}\{g(t)\} e^{-\pi s}$$
$$= 3\mathcal{L}\{-\cos t\} e^{-\pi s} = -\frac{3s}{s^2+1} e^{-\pi s}$$

$$sY(s) - 5 + Y(s) = -\frac{3s}{s^2+1} e^{-\pi s}$$

$$\Rightarrow (s+1)Y(s) = 5 - \frac{3s}{s^2+1} e^{-\pi s}$$

$$\Rightarrow Y(s) = \frac{5}{s+1} - \frac{3s}{(s+1)(s^2+1)} e^{-\pi s}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{5}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{3s}{(s+1)(s^2+1)} e^{-\pi s}\right\}$$

$$\Rightarrow y(t) = 5e^{-t} - \mathcal{L}^{-1}\left\{ \frac{3s}{(s+1)(s^2+1)} e^{-\pi s} \right\}$$

Finding $\mathcal{L}^{-1}\{\frac{3s}{(s+1)(s^2+1)}e^{-\pi s}\}$:

$$\frac{3s}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}$$

$$\Rightarrow \quad 3s = A(s^2+1) + (Bs+C)(s+1)$$

(i) $s=-1 \Rightarrow \quad -3 = 2A \Rightarrow A = -3/2$

(ii) $s^2 : \quad 0 = A + B \Rightarrow B = 3/2$

(iii) constant : $\quad 0 = A + C \Rightarrow C = 3/2$

$$\frac{3s}{(s+1)(s^2+1)} = \frac{-3/2}{s+1} + \frac{3/2\, s + 3/2}{s^2+1}$$

$$\mathcal{L}^{-1}\left\{ \frac{3s}{(s+1)(s^2+1)} \right\} = \mathcal{L}^{-1}\left\{ \frac{-3/2}{s+1} + \frac{3/2\, s + 3/2}{s^2+1} \right\}$$

$$= 3/2 \left(-\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \right)$$

$$= 3/2 \left(-e^{-t} + \cos t + \sin t \right)$$

$$\mathcal{L}^{-1}\left\{\frac{3s}{(s+1)(s^2+1)}e^{-\pi s}\right\} = \mathcal{L}^{-1}\left\{ \frac{3s}{(s+1)(s^2+1)} \right\}\Big|_{t-\pi} u(t-\pi)$$

$$= 3/2 \left\{ -e^{-(t-\pi)} + \sin(t-\pi) + \cos(t-\pi) \right\} u(t-\pi)$$

Therefore we have,

$$= 5e^{-t} - 3/2 \{-e^{-(t-\pi)} + \sin(t-\pi) + \cos(t-\pi)\} u(t-\pi)$$

(Ans:)

(iv) $y'' + 4y' + 6y = 1 + e^{-t}, \quad y(0) = 0, \quad y'(0) = 0.$

Solution:

$$\mathcal{L}\{y'' + 4y' + 6y\} = \mathcal{L}\{1 + e^{-t}\}$$

$$\Rightarrow \mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} + 6\mathcal{L}\{y\} = \frac{1}{s} + \frac{1}{s+1}$$

let,

$$\mathcal{L}\{y(t)\} = Y(s)$$

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0) = sY(s)$$

$$\mathcal{L}\{y''(t)\} = s^2 Y(s) - s y(0) - y'(0) = s^2 Y(s)$$

$$\therefore s^2 Y(s) + 4s Y(s) + 6Y(s) = \frac{1}{s} + \frac{1}{s+1}$$

$$\Rightarrow (s^2 + 4s + 6) Y(s) = \frac{2s+1}{s(s+1)}$$

$$\Rightarrow Y(s) = \frac{2s+1}{s(s+1)(s^2+4s+6)}$$

$$\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{2s+1}{s(s+1)(s^2+4s+6)}\right\}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{2s+1}{s(s+1)(s^2+4s+6)}\right\}$$

Finding $\mathscr{L}^{-1}\{\frac{2s+1}{s(s+1)(s^2+4s+6)}\}$:

$$\frac{2s+1}{s(s+1)(s^2+4s+6)} = \frac{A}{s} + \frac{B}{s+1} + \frac{Cs+D}{s^2+4s+6}$$

$$\Rightarrow 2s+1 = A(s+1)(s^2+4s+6) + Bs(s^2+4s+6) + (Cs+D)(s^2+s)$$

(i) $s=0 \Rightarrow 1 = A.6 \Rightarrow A = 1/6$

(ii) $s=-1 \Rightarrow -1 = -B(1-4+6) \Rightarrow B = 1/3$

(iii) s^3: $0 = A+B+C \Rightarrow C = -(A+B) = -1/2$

(iv) s : $2 = 6A+4A+6B+D \Rightarrow D = -5/3$

$$\frac{2s+1}{s(s+1)(s^2+4s+6)} = \frac{1/6}{s} + \frac{1/3}{s+1} + \frac{-1/2\,s - 5/3}{s^2+4s+6}$$

$$\mathscr{L}^{-1}\left\{\frac{2s+1}{s(s+1)(s^2+4s+6)}\right\} = \mathscr{L}^{-1}\left\{\frac{1/6}{s} + \frac{1/3}{s+1} + \frac{-1/2\,s-5/3}{s^2+4s+6}\right\}$$

$$= 1/6\,\mathscr{L}^{-1}\{1/s\} + 1/3\,\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} - 1/2\,\mathscr{L}^{-1}\left\{\frac{s}{(s+2)^2+2}\right\}$$

$$- 5/3\,\mathscr{L}^{-1}\left\{\frac{1}{(s+2)^2+2}\right\}$$

$$= 1/6 + 1/3\,e^{-t} - 1/2\,\mathscr{L}^{-1}\left\{\frac{s+2-2}{(s+2)^2+2}\right\}$$

$$- 5/3\,\mathscr{L}^{-1}\left\{\frac{1}{(s+2)^2+2}\right\}$$

$$= 1/6 + 1/3\, e^{-t} - 1/2\, \mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2 + (\sqrt{2})^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+2}\right\}$$

$$- 5/3\, \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+2}\right\}$$

$$= 1/6 + 1/3\, e^{-t} - 1/2\, \mathcal{L}^{-1}\left\{\frac{s}{s^2+(\sqrt{2})^2}\Big|_{s \to s+2}\right\}$$

$$- 2/3\, \mathcal{L}^{-1}\left\{\frac{1}{s^2+(\sqrt{2})^2}\Big|_{s \to s+2}\right\}$$

$$= 1/6 + 1/3\, e^{-t} - 1/2 \cos(\sqrt{2}t)\, e^{-2t} - \frac{2}{3\sqrt{2}} \sin(\sqrt{2}t)\, e^{-2t}$$

Therefore we have,

$$y(t) = 1/6 + 1/3\, e^{-t} - 1/2 \cos(\sqrt{2}t)\, e^{-2t} - \frac{2}{3\sqrt{2}} \sin(\sqrt{2}t)\, e^{-2t}$$

(Ans:)

MAT215: Complex Variables & Laplace Transformations

Chapter : Laplace Transformations

Solving Boundary Value Problem (BVP) using Laplace transform

Solving System of equaions using Laplace transform

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Summer 2025

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Solving BVP Using Laplace Transformation

Step 1: Apply Laplace Transform (y(t) -> Y(s))

Step 2: Find Y(s)

Step 3: Apply Inverse Laplace Transform (Y(s) -> y(t))

Step 4: Find the unknown constant using boundary condition

Problem

Solve the BVP $y'' + 9y = \cos 2t$; $y(0) = 1$, $y(\pi/2) = -1$.

Solution:

$$\mathcal{L}\{y''+9y\} = \mathcal{L}\{\cos 2t\}$$

$$\Rightarrow \mathcal{L}\{y''\} + 9\mathcal{L}\{y\} = \frac{s}{s^2+4}$$

let,

$$\mathcal{L}\{y(t)\} = Y(s)$$

$$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s - y'(0)$$

$$\therefore s^2Y(s) - s - y'(0) + 9Y(s) = \frac{s}{s^2+4}$$

$$\Rightarrow (s^2+9)Y(s) = s + y'(0) + \frac{s}{s^2+4}$$

$$\Rightarrow Y(s) = \frac{s}{s^2+9} + \frac{y'(0)}{s^2+9} + \frac{s}{(s^2+4)(s^2+9)}$$

$$\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+9} + \frac{y'(0)}{s^2+9} + \frac{s}{(s^2+4)(s^2+9)}\right\}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + y'(0)\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)(s^2+9)}\right\}$$

$$\therefore y(t) = \cos(3t) + \frac{y'(0)}{3}\sin(3t) + \mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)(s^2+9)}\right\}$$

$$\therefore y(t) = \cos(3t) + A\sin(3t) + \mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)(s^2+9)}\right\}$$

Finding $\mathscr{L}^{-1}\{\frac{s}{(s^2+4)(s^2+9)}\}$:

$$\frac{s}{(s^2+4)(s^2+9)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9}$$

$$\Rightarrow s = (As+B)(s^2+9) + (Cs+D)(s^2+4)$$

$$\Rightarrow s = As^3 + 9As + Bs^2 + 9B + Cs^3 + 4Cs + Ds^2 + 4D$$

$$\Rightarrow s = (A+C)s^3 + (B+D)s^2 + (9A+4C)s + 9B+4D$$

s^3 : $0 = A + C \text{ --- (i)} \Rightarrow A = -C \Rightarrow A = 1/5$

s^2 : $0 = B + D \text{ --- (ii)} \Rightarrow B = -D \Rightarrow B = 0$

s : $1 = 9A + 4C \text{ --- (iii)} \Rightarrow -9C + 4C = 1 \Rightarrow C = -1/5$

constant : $0 = 9B + 4D \text{ --- (iv)} \Rightarrow -9D + 4D = 0 \Rightarrow D = 0$

$$\frac{s}{(s^2+4)(s^2+9)} = \frac{1/5\, s}{s^2+4} + \frac{-1/5\, s}{s^2+9}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)(s^2+9)}\right\} = \mathcal{L}^{-1}\left\{\frac{1/5\,s}{s^2+4} + \frac{-1/5\,s}{s^2+9}\right\}$$
$$= 1/5\,\{\cos(2t) - \cos(3t)\}$$

$$y(t) = \cos(3t) + A\sin(3t) + 1/5\,\{\cos(2t) - \cos(3t)\}$$

$$y(\pi/2) = -1$$

$$\cos(3\pi/2) + A\sin(3\pi/2) + 1/5\,\{\cos(2\pi/2) - \cos(3\pi/2)\} = -1$$

$$0 - A + 1/5\,(-1-0) = -1$$

$$\Rightarrow \quad -A - 1/5 = -1 \Rightarrow A = 4/5$$

Therefore we have,

$$y(t) = \cos(3t) + \frac{4}{5}\sin(3t) + 1/5\,\{\cos(2t) - \cos(3t)\}$$

(Ans:)

Problem

(i) $\mathscr{L}^{-1}\{\frac{2s}{(s+1)(s-1)}\}$ (ii) $\mathscr{L}^{-1}\{\frac{2}{(s+1)(s-1)(s^2+1)}\}$

Ans: (i) $e^{-t} + e^{t}$

(ii) $1/2\,(e^{t} - e^{-t}) - \sin t$

Shortcut: Inverse of a 2×2 Matrix

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{\mathrm{Det}(A)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ provided $\mathrm{Det}(A) \neq 0$.

Solving System of equations Using Laplace Transformation

Step 1: Apply Laplace Transform (y(t) -> Y(s), x(t) -> X(s))

Step 2: Find Y(s), X(s)

Step 3: Apply Inverse Laplace Transform (Y(s) -> y(t), X(s) -> x(t))

Problem

Solve the following system:

$$\begin{cases} \dfrac{dx}{dt} + y = \sin t \\ \dfrac{dy}{dt} + x = \cos t \end{cases} \quad ; \quad x(0) = 2,\ y(0) = 0.$$

Solution:

$$\mathcal{L}\{x'(t) + y\} = \mathcal{L}\{\sin t\}$$

$$\mathcal{L}\{y'(t) + x\} = \mathcal{L}\{\cos t\}$$

let,

$$\mathcal{L}\{y(t)\} = Y(s), \quad \mathcal{L}\{x(t)\} = X(s)$$

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0) = sY(s)$$

$$\mathcal{L}\{x'(t)\} = sX(s) - x(0) = sX(s) - 2$$

$$\therefore \quad sX(s) - 2 + Y(s) = \frac{1}{s^2+1}$$

$$sY(s) + X(s) = \frac{s}{s^2+1}$$

$$sX(s) + Y(s) = 2 + \frac{1}{s^2+1}$$

$$\Rightarrow \quad X(s) + sY(s) = \frac{s}{s^2+1}$$

$$\Rightarrow \begin{pmatrix} s & 1 \\ 1 & s \end{pmatrix} \begin{pmatrix} X(s) \\ Y(s) \end{pmatrix} = \begin{pmatrix} \frac{2s^2+3}{s^2+1} \\ \frac{s}{s^2+1} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} X(s) \\ Y(s) \end{pmatrix} = \begin{pmatrix} s & 1 \\ 1 & s \end{pmatrix}^{-1} \begin{pmatrix} \frac{2s^2+3}{s^2+1} \\ \frac{s}{s^2+1} \end{pmatrix}$$

$$= \frac{1}{s^2-1} \begin{pmatrix} s & -1 \\ -1 & s \end{pmatrix} \begin{pmatrix} \frac{2s^2+3}{s^2+1} \\ \frac{s}{s^2+1} \end{pmatrix}$$

$$\begin{pmatrix} X(s) \\ Y(s) \end{pmatrix} = \begin{pmatrix} \frac{2s(s^2+1)}{(s^2+1)(s^2-1)} \\ \frac{-(s^2+3)}{(s^2+1)(s^2-1)} \end{pmatrix}$$

$$\Rightarrow \begin{aligned} X(s) &= \frac{2s(s^2+1)}{(s^2+1)(s^2-1)} \Rightarrow x(t) = \mathcal{L}^{-1}\left\{ \frac{2s(s^2+1)}{(s^2+1)(s^2-1)} \right\} \\ Y(s) &= \frac{-(s^2+3)}{(s^2+1)(s^2-1)} \Rightarrow y(t) = \mathcal{L}^{-1}\left\{ \frac{-(s^2+3)}{(s^2+1)(s^2-1)} \right\} \end{aligned}$$

$$\Rightarrow x(t) = e^{-t} + e^{t}$$

$$\& \quad y(t) = \tfrac{1}{2}\left(e^{t} - e^{-t}\right) - \sin t \qquad \text{(Ans:)}$$

Department of Mathematics and Natural Sciences
MAT215 : Complex Variables and Laplace Transformations
Laplace Transformation Formula Sheet

Laplace Transform definition

$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

Linearity of Laplace transformation

$\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha\mathscr{L}\{f(t)\} + \beta\mathscr{L}\{g(t)\}$

Laplace Transforms of Elementary Functions (Real s)

Function $f(t)$	$\mathscr{L}\{f(t)\} = F(s)$	Function $f(t)$	$\mathscr{L}\{f(t)\} = F(s)$
1	$\frac{1}{s}, \quad s > 0$	$\sin(at)$	$\frac{a}{s^2+a^2}, \quad s > 0$
$t^n,\ n \in \mathbb{Z}^+$	$\frac{n!}{s^{n+1}}, \quad s > 0$	$\cos(at)$	$\frac{s}{s^2+a^2}, \quad s > 0$
$t^n, n \notin \mathbb{Z}$	$\frac{\Gamma(n+1)}{s^{n+1}}, \quad s > 0$	$\sinh(at)$	$\frac{a}{s^2-a^2}, \quad s > \lvert a\rvert$
e^{at}	$\frac{1}{s-a}, \quad s > a$	$\cosh(at)$	$\frac{s}{s^2-a^2}, \quad s > \lvert a\rvert$

1st Translation theorem

$\mathscr{L}\{e^{at} f(t)\} = \mathscr{L}\{f(t)\}\big|_{s\to s-a}$ for any real number a.

Laplace transformation of the form $t^n f(t)$

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[\mathscr{L}\{f(t)\}]$$

Piecewise function representation through unit step function

Suppose that,

$$h(t) = \begin{cases} f(t); & 0 \le t < a \\ g(t); & t \ge a \end{cases}$$

Then,

$$h(t) = f(t)\big(1 - u(t-a)\big) + g(t)u(t-a)$$

2nd Translation Theorem

$\mathscr{L}\{f(t-a)u(t-a)\} = \mathscr{L}\{f(t)\}e^{-as}$ for any positive value of a.

Inverse Laplace Transforms of some known expressions

$F(s)$	$\mathscr{L}^{-1}\{F(s)\} = f(t)$	$F(s)$	$\mathscr{L}^{-1}\{F(s)\} = f(t)$
$\frac{1}{s}$	1	$\frac{a}{s^2+a^2}$	$\sin(at)$
$\frac{n!}{s^{n+1}}$, $n \in \mathbb{Z}^+$	t^n	$\frac{s}{s^2+a^2}$	$\cos(at)$
$\frac{\Gamma(n+1)}{s^{n+1}}$, $n \notin \mathbb{Z}$	t^n	$\frac{a}{s^2-a^2}$	$\sinh(at)$
$\frac{1}{s-a}$	e^{at}	$\frac{s}{s^2-a^2}$	$\cosh(at)$

Linearity of Inverse Laplace transformation

$\mathscr{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha\mathscr{L}^{-1}\{F(s)\} + \beta\mathscr{L}^{-1}\{G(s)\}$

First Translation Theorem in inverse form

$\mathscr{L}^{-1}\{F(s)\big|_{s\to s-a}\} = e^{at}\mathscr{L}^{-1}\{F(s)\}$

Second Translation Theorem in inverse form

$\mathscr{L}^{-1}\{e^{-as}F(s)\} = \mathscr{L}^{-1}\{F(s)\}\big|_{t\to t-a}u(t-a)$

Laplace transformation of derivatives

If $\mathscr{L}\{f(t)\} = F(s)$, then

$$\mathscr{L}\left\{f'(t)\right\} = s \cdot F(s) - f(0)$$

$$\mathscr{L}\left\{f''(t)\right\} = s^2 \cdot F(s) - s \cdot f(0) - f'(0)$$

$$\mathscr{L}\left\{f'''(t)\right\} = s^3 \cdot F(s) - s^2 \cdot f(0) - s \cdot f'(0) - f''(0)$$