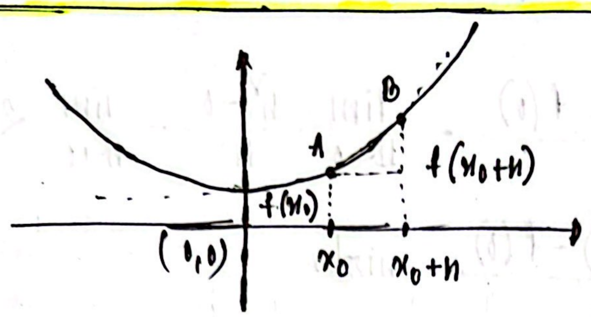
Lect-05: complex Differentiation & Cauchy- Reimann equations.

Differentiable Real valued function :



$\frac{f(x_0+h)-f(x_0)}{h}$ → slope of line segment joining A & B

$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$ → slope of tangent at x_0

if $\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$ exists then we say, $f(x)$ is differentiable at $x=x_0$ & this limit is called the derivative of f at x_0 (denoted by $f'(x_0)$)

Differentiability of Complex valued function :

$f(z)$ → Single valued in some region R of the z-plane

If $\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}$ exists

- f is said to be differentiable at $z=z_0$
- & this limit is called the derivative of f at z_0
- denoted y $f'(z_0)$

(*) $z_0+h = z \Rightarrow h = z-z_0$

equivalently $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists ⇒

Problem-01 : Using the definition check whether $f(z)=z^2$ is differentiable at $z=0$?

Soln :

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{h^2-0}{h} = \lim_{h\to 0} 2h = 0$$

$\Rightarrow \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$ exists

$\Rightarrow$ f is differentiable at $z=0$.

Problem-02 : Using the defn, find the derivative of

$$f(z)=\frac{2z-3i}{3z-2i} \text{ at } z=-i$$

Soln :

$$f'(-i)=\lim_{h\to 0}\frac{f(-i+h)-f(-i)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left\{\frac{2(-i+h)-3i}{3(-i+h)-2i}-\frac{-5i}{-5i}\right\}$$

$$=\lim_{h\to 0}\frac{1}{h}\left\{\frac{-5i+2h}{-5i+3h}-1\right\}$$

$$=\lim_{h\to 0}\frac{1}{h}\left\{\frac{-5i+2h+5i-3h}{-5i+3h}\right\}$$

$$=\lim_{h\to 0}\frac{1}{h}\frac{-h}{-5i+3h}$$

$$=\lim_{h\to 0}\frac{1}{5i-3h}$$

$$=\frac{1}{5i}=-i/5 \text{ (Ans.)}$$

Problem-03 : $f(z) = z^2$. Show that $f(z)$ is differentiable everywhere.

Soln : We show : $f(z)$ is differentiable for abitrary z_0.

$$\lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$$

$$= \lim_{h \to 0} \frac{(z_0+h)^2 - z_0^2}{h}$$

$$= \lim_{h \to 0} \frac{2z_0 h + h^2}{h}$$

$$= \lim_{h \to 0} 2z_0 + h$$

$$= 2z_0$$

$\therefore$ $f(z)$ is differentiable at $z = z_0$

$\because$ z_0 is arbitrary $\Rightarrow$ $f(z)$ is differentiable everywhere.

Remarks : Differentiable $\Rightarrow$ Continuous

$\not\Leftarrow$

Brain-teaser : Show that, if $f(z)$ is complex differentiable at $z = z_0$ $\Rightarrow$ it is continuous at $z = z_0$.

Remarks :

(i) $(x+iy)\overline{(x+iy)} = (x+iy)(x-iy) = x^2+y^2$

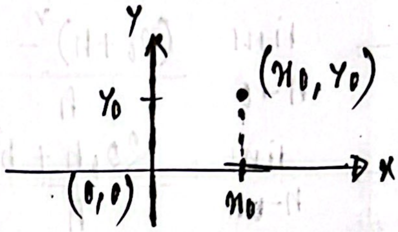
$\Rightarrow$ $z.\bar{z} = |z|^2$

Problem-04 : $f(z)=|z|^2$. Show that f is differentiable only at $z=0$.

Soln : Let, $z_0 = x_0+iy_0$ be an arbitrary point

We check: differentiability of $f(z)$ at $z=z_0$

$$L = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$$



$$= \lim_{z\to z_0} \frac{|z|^2-|z_0|^2}{z-z_0}$$

$$= \lim_{z\to z_0} \frac{z\bar{z} - z_0\bar{z_0}}{z-z_0}$$

~~Approach-01~~

$$= \lim_{z\to z_0} \frac{z\bar{z} - z\bar{z_0} + z\bar{z_0} - z_0\bar{z_0}}{z-z_0}$$

$$= \lim_{z\to z_0} \frac{z(\bar{z}-\bar{z_0}) + \bar{z_0}(z-z_0)}{z-z_0}$$

$$= \lim_{z\to z_0} \frac{z\,\overline{(z-z_0)}}{z-z_0} + \bar{z_0} \quad [\because \bar{a}\pm\bar{b} = \overline{a\pm b}]$$

$$= \lim_{x+iy\to x_0+iy_0} \frac{(x+iy)\,\overline{\{x+iy-(x_0+iy_0)\}}}{x+iy-(x_0+iy_0)} + \overline{x_0+iy_0}$$

$$\Rightarrow L = \lim_{\substack{x\to x_0\\ y\to y_0}} (x+iy)\frac{(x-x_0)-i(y-y_0)}{(x-x_0)+i(y-y_0)} + (x_0-iy_0)$$

Approach-01 : $x = x_0,\ y \to y_0$

$$l_1 = \lim_{y \to y_0} x_0 + iy \ \frac{-i(y-y_0)}{i(y-y_1)} + (x_0 - iy_0)$$

$$= \lim_{y \to y_0} -(x_0 + iy) + (x_0 - iy_0)$$

$$= -x_0 - iy_0 + x_0 - iy_0 = -2iy_0$$

Approach-02 : $y = y_0,\ x \to x_0$

$$l_2 = \lim_{x \to x_1} (x + iy_0) \frac{x - x_0}{x - x_0} + (x_0 - iy_0)$$

$$= \lim_{x \to x_0} (x + iy_0) + (x_0 - iy_0)$$

$$= x_0 + iy_0 + x_0 - iy_0 = 2x_0$$

Now, $l_1 = l_2$ **Lemma** : $P \Rightarrow Q,\ \sim Q \Rightarrow \sim P$

$$-2iy_0 = 2x_0$$

$$\Rightarrow 2(x_0 + iy_0) = 0$$

$$\Rightarrow x_0 + iy_0 = 0$$

$$\Rightarrow x_0 = 0 \ \& \ y_0 = 0 \Rightarrow z_0 = 0.$$

$l_1 = l_2 \Rightarrow z_0 = 0$

$\therefore z_1 \neq 0 \Rightarrow l_1 \neq l_2$

$\therefore z_0 \neq 0 \Rightarrow f(z)$ is not differentiable at z_0.

$z_0 = 0,$

$$L = \lim_{z \to z_0} \frac{f(z) - f(0)}{z - 0}$$

$$= \lim_{z \to 0} \frac{f(z)}{z}$$

$$= \lim_{z \to 0} \frac{z\bar{z}}{z}$$

$$= \lim_{z \to 0} \bar{z}$$

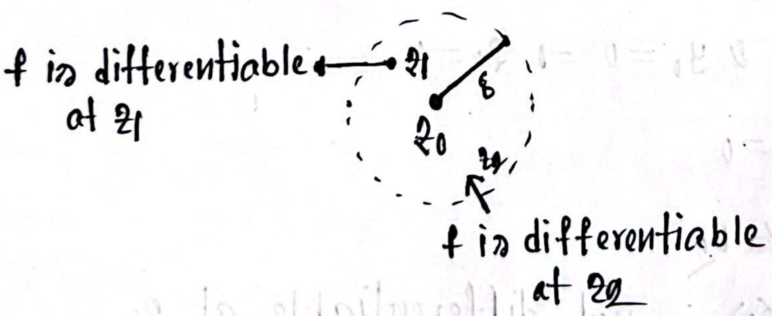
$$= \bar{0} = 0$$

$\Rightarrow$ $f(z)$ is differentiable at $z_0 = 0$.

$\Rightarrow$ $f(z)$ is differentiable only at $z_0 = 0$.

Analytic/ Holomorphic Complex function :

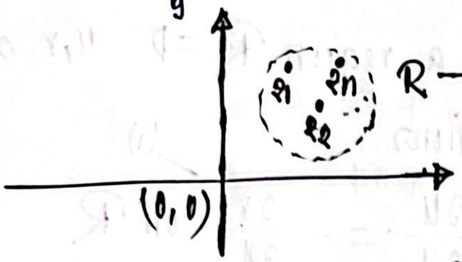
~~If~~ $f(z)$ is said to be analytic at a point z_0 if ~~$f'(z_0)$ exists~~ $\exists$ a neighbourhood $|z - z_0| < \delta$ at all points of which $f'(z)$ exists.



Analytic / Holomorphic complex function :

Analyticity in a region :

(i) $f(z) \rightarrow$ complex function



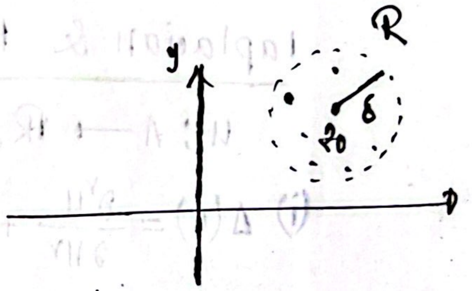
R ⟶ a region in the domain of f.

f is differentiable at z_1
f " " " z_2
⋮
f " " " z_n

} f is differentiable $\forall z \in R$
↓
$f(z)$ is analytic in R.

(ii) **Analyticity at a point :**

$f(z)$ is said to be analytic at z_0 if $\exists$ a neighbourhood $|z - z_0| < \delta$ at all points of which $f'(z)$ exists



f is differentiable $\forall z \in R$
↓
f is analytic at z_0

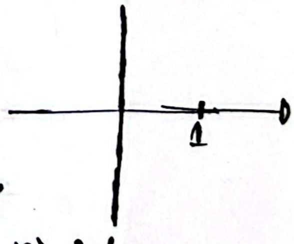
Examples :

(i) $f(z) = z^2$
$f'(z) = 2z$

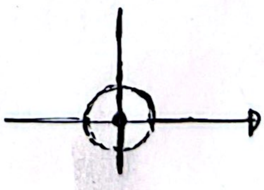
$\therefore f'(z)$ exists $\forall z \in \mathbb{C}$
$f(z)$ is analytic on $\mathbb{C}$

(ii) $f(z) = \dfrac{1}{z-1}$
$f'(z) = -\dfrac{1}{(z-1)^2}$

$f'(z)$ exists $\forall z \in \mathbb{C} \setminus \{1\}$
$\therefore f(z)$ is analytic on $\mathbb{C} \setminus \{1\}$



(iii) $f(z) = |z|^2$
f is differentiable at $z = 0$
But f is not analytic at $z = 0$



Cauchy-Riemann equations :

$f(z) = u(x,y) + iv(x,y)$

(i) If $f(z)$ is analytic in a region $R \Rightarrow u, v$ satisfy the Cauchy Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ on } R \quad \text{(i)}$$

(ii) u, v satisfy C-R equations on R & partial derivatives in (i) are continuous in R $\Rightarrow f(z)$ is analytic in R

Laplacian & Harmonic function :

$u : A \to \mathbb{R},\ A \subseteq \mathbb{R}^2$ (bi-variable function)

(i) $\Delta(u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \to$ laplacian of u

(ii) if $\Delta(u) = 0$ on $A' \subseteq A \to u$ is harmonic on A'

Example : (i) $u(x,y) = x^2 + 2y^4 \ (u : \mathbb{R}^2 \to \mathbb{R})$

$\Delta(u) = u_{xx} + u_{yy} = 2 + 24y^2$

(ii) $u(x,y) = x^2 - y^2$

$\Delta(u) = u_{xx} + u_{yy} = 2 - 2 = 0$

$\Rightarrow \Delta(u) = 0 \ \forall (x,y) \in \mathbb{R}^2$

$\Rightarrow u$ is harmonic on $\mathbb{R}^2$

Problem : $f(z) = u + iv$ ~~($f: A \to \mathbb{C}$)~~ is analytic on A. If second order partial derivatives are continuous on A, prove that, u & v are harmonic on A.

Soln : f is analytic on A

$\Rightarrow$ f satisfies C-R eqns on A

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \;\text{(i)} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \;\text{(ii)} \quad \text{on } A$$

(i) **u is harmonic** :

$(i)_x + (ii)_y$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x}$$

$\Rightarrow$ $\Delta(u) = v_{yx} - v_{xy} = 0$ (Using Clairaut's Thm)

$\Rightarrow$ $\Delta(u) = 0$ on A.

$\Rightarrow$ u is harmonic on A

Similarly, v is harmonic on A.

Clairaut's Thm : If the second order partials of a function are continuous $\Rightarrow$ Order of differentiation is immaterial.

Harmonic Conjugate (Defn) :

$f = u + iv \longrightarrow$ analytic on A

Then u, v are called harmonic conjugates on A

Problem-02: $u(x,y) = 3x^2y + 2x^2 - y^3 - 2y^2$

(i) Show that, u is harmonic

(ii) Find $v(x,y)$ such that $f(z) = u(x,y) + iv(x,y)$ is analytic

Soln: (i) $u(x,y) = 3x^2y + 2x^2 - y^3 - 2y^2$

$u_x = 6xy + 4x$; $u_{xx} = 6y + 4$

$u_y = 3x^2 - 3y^2 - 4y$; $u_{yy} = -6y - 4$

$\therefore \Delta(u) = u_{xx} + u_{yy} = 0 \;\forall (x,y) \in \mathbb{R}^2$

$\Rightarrow$ u is analytic everywhere.

(ii) $f(z) = u + iv$ is analytic

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

$$\Rightarrow \frac{\partial v}{\partial y} = 6xy + 4x$$

$$\Rightarrow v(x,y) = 6x\, y^2/2 + 4xy + c(x)$$

$$\Rightarrow v(x,y) = 3xy^2 + 4xy + c(x)$$

$$\Rightarrow \frac{\partial v}{\partial x} = 3y^2 + 4y + c'(x)$$

$$\Rightarrow -\frac{\partial u}{\partial y} = 3y^2 + 4y + c'(x)$$

$$\Rightarrow -3x^2 + 3y^2 + 4y = 3y^2 + 4y + c'(x)$$

$$\Rightarrow c'(x) = -3x^2 \Rightarrow c(x) = -x^3 + c$$

$\therefore v(x,y) = 3xy^2 + 4xy - x^3 + c$ where c is any arbitiary constant.