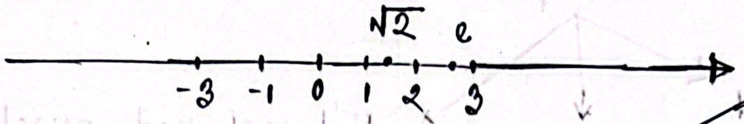
Lecture-01 : Introduction to Complex numbers

Real numbers :



$(x, y) \neq (y, x)$

Complex numbers : Ordered pairs of the form (x, y) where $x, y \in \mathbb{R} \longrightarrow$ 2 Dimensional points

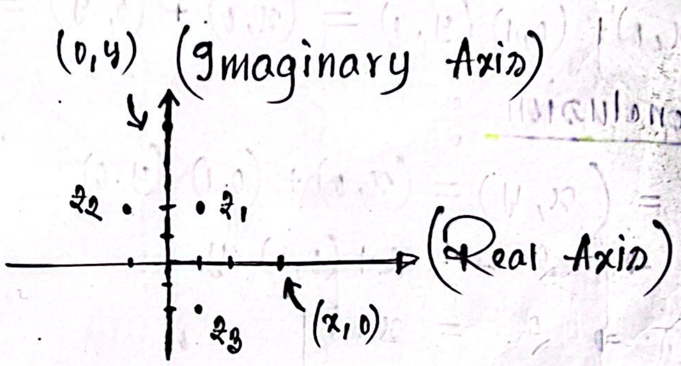
Notation : $z = (x, y)$ where $x, y \in \mathbb{R}$

Graphically :

$z_1 = (1, 2)$

$z_2 = (-1, 2)$

$z_3 = (1, -2)$



Remarks :

(i) $z = (x, y) \Rightarrow$

$x =$ Real Part of $z = Re(z)$

$y =$ Imaginary part of $z = Im(z)$

(ii) $z_1 = (x_1, y_1)$ & $z_2 = (x_2, y_2)$. Addition & multiplication are defined as,

$$z_1 + z_2 = (x_1 + x_2,\ y_1 + y_2)$$

$$z_1 z_2 = (x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2,\ x_1 y_2 + x_2 y_1)$$

(iii) $z = (x, y) \Rightarrow \bar{z} = (x, -y)$

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(iii) $z = (x, 0) \longrightarrow$ real number (with x written above the $(x, 0)$)

(iv) $z = (0, y) \longrightarrow$ purely imaginary numbers

Complex → Real; Purely imaginary; Not real, not purely imaginary

Task: Find (i) $(0,1)\,(y,0)$

Sol^n: (ii) $(x,0) + (0,1)(y,0)$

$$(x,0) + (0,1)(y,0) = (x,0) + (0,y) = (x,y)$$

Conclusion:

$$z = (x,y) = (x,0) + (0,1)(y,0)$$

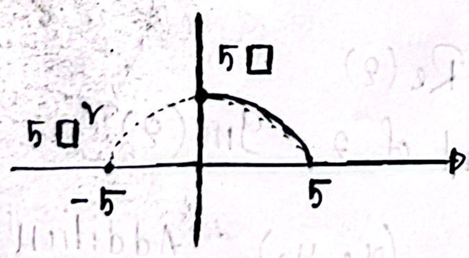
$$\Rightarrow z = x + (0,1)\,y$$

$$\Rightarrow z = x + iy$$

$(0,1) = i \rightarrow$ imaginary unit

$$i^2 = (0,1)(0,1) = (-1,0) = -1$$

What i does geometrically:



$$5\,\square^2 = -5$$

$$\Rightarrow \square^2 = -1$$

$3 \times \square \longrightarrow$ a outcome with 180° rotation ($\square = -1$)

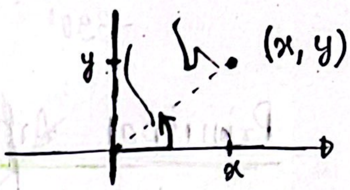
$5 \times \square \longrightarrow$ a outcome with 90° rotation (counter-clockwise) ($\square = i$)

Complex numbers (Redefinition) : Numbers of the form

$$z = x + iy, \quad x, y \in \mathbb{R}, \quad i^2 = -1$$

Modulous & Argument :

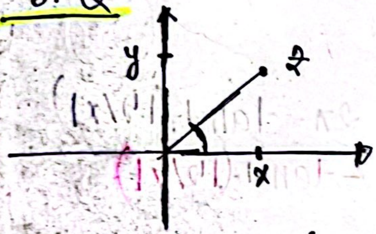
$z = (x, y)$



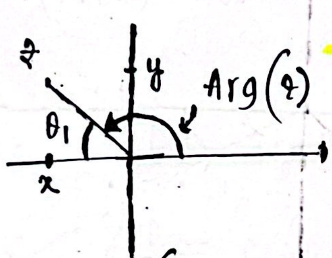
Modulous of $z = \sqrt{x^2 + y^2} = \mathrm{Mod}(z)$

Argument of $z = \underbrace{\tan^{-1}(y/x)}_{\text{Always true ?}} = \mathrm{Arg}(z)$

1st Q



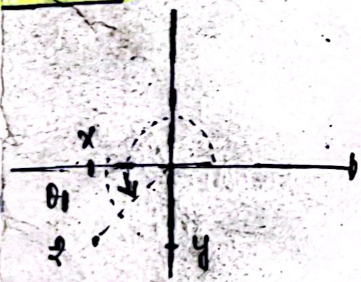
2nd Q



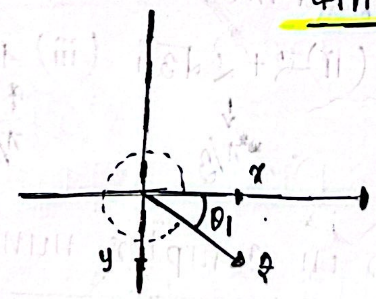
$\mathrm{Arg}(z) = \tan^{-1}(y/x)$
$= \tan^{-1}(|y|/|x|)$

$\mathrm{Arg}(z) = \pi - \tan^{-1}\left(\frac{y}{|x|}\right) = \pi - \tan^{-1}(|y|/|x|)$

3rd Q



4th Q



$\mathrm{Arg}(z) = \pi + \tan^{-1}\left(\frac{|y|}{|x|}\right)$
$= \pi + \tan^{-1}(|y|/|x|)$

$\mathrm{Arg}(z) = 2\pi - \theta_1 = 2\pi - \tan^{-1}\left(\frac{|y|}{x}\right)$
$= 2\pi - \tan^{-1}(|y|/|x|)$

Conclusion : 1st Q $\to \tan^{-1}(|y|/|x|)$

2nd Q $\to \pi - \tan^{-1}(|y|/|x|)$

3rd Q $\to \pi + \tan^{-1}(|y|/|x|)$

4th Q $\to 2\pi - \tan^{-1}\left(\left|\frac{y}{x}\right|\right)$

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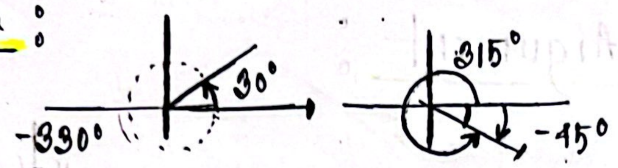
Problem: Find argument & modulous of

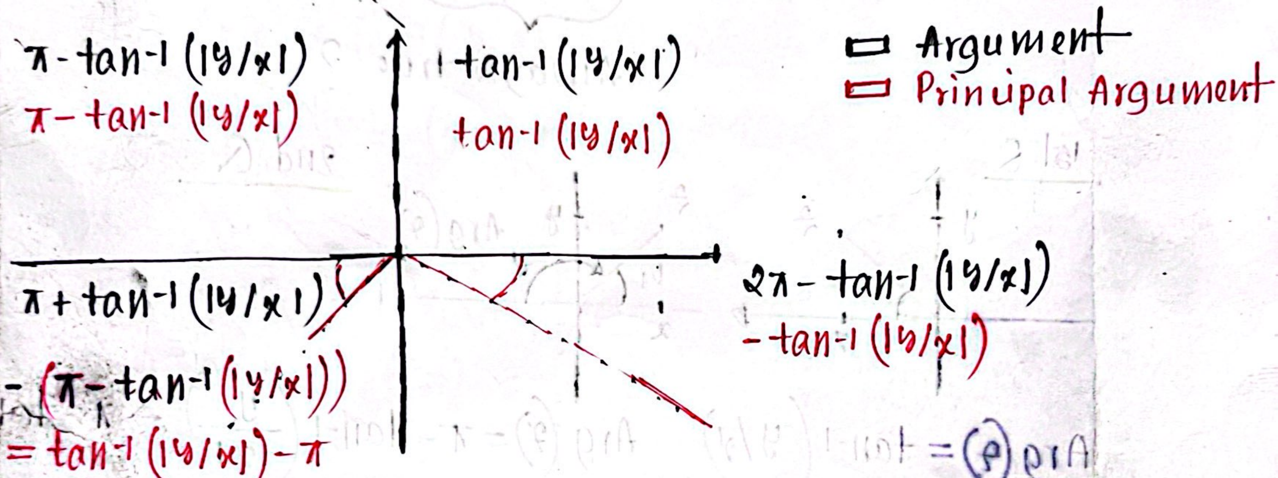
(i) $z = -2+5i$ (ii) $z = 1-4i$ (iii) $z = -4-3i$

lemma:



(*) ACA $- 2\pi =$ CA

Principal Argument: Argument of the complex number lying in $(-\pi, \pi]$.



Problem: Find Principal Argument of

(i) $1-i$ (ii) $2+2\sqrt{3}i$ (iii) $4i$.

$\downarrow$ $-\pi/4$ $\quad$ $\downarrow$ $\pi/3$ $\quad$ $\downarrow$ $\pi/2$

Operations on complex numbers:

(i) Addition: $(a_1+ib_1) + (a_2+ib_2) = a_1+a_2 + i(b_1+b_2)$

(ii) Subtraction: $(a_1+ib_1) - (a_2+ib_2) = a_1-a_2 + i(b_1-b_2)$

(iii) Multiplication: $(a_1+ib_1)(a_2+ib_2) = a_1a_2 - b_1b_2 + i(a_1b_2 + a_2b_1)$

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(iv) Division:

$$\frac{a_1+ib_1}{a_2+ib_2} = \frac{(a_1+ib_1)(a_2-ib_2)}{(a_2+ib_2)(a_2-ib_2)} = \frac{(a_1a_2+b_1b_2)+i(a_2b_1-a_1b_2)}{a_2^2+b_2^2}$$

$$= \frac{a_1a_2+b_1b_2}{a_2^2+b_2^2} + i\,\frac{(a_2b_1-a_1b_2)}{a_2^2+b_2^2}$$

Problem-01: $z_1 = 2+3i,\ z_2 = 1-2i$

(i) Plot z_1, z_2

(ii) Find $|z_1|, |z_2|, Arg(z_1), Arg(z_2)$

(iii) Show that $|z_1| = |\bar{z}_1|$ where z_1 is the complex conjugate of z_1

(iv) Find $z_1+z_2,\ z_1-z_2,\ z_1\cdot z_2,\ z_1/z_2$

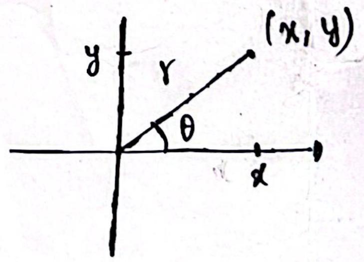
Euler's Formula: $e^{i\theta} = \cos\theta + i\sin\theta$

Polar form of a complex number:

$z = x+iy$

$x = r\cos\theta$

$y = r\sin\theta$



$z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$

$\Rightarrow z = re^{i\theta} \longrightarrow Arg(z) \longrightarrow$ Polar form of Complex Number.

$\quad \hookrightarrow Mod(z)$

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Problem-02 : Express $z = 3e^{i\pi/6}$ in cartesian form.

Soln : $r = 3$

$\theta = \pi/6$

$x = r\cos\theta \Rightarrow x = 3\cos(\pi/6) = 3\sqrt{3}/2$

$y = r\sin\theta \Rightarrow y = 3\sin(\pi/6) = 3/2$

$\Rightarrow z = \frac{3\sqrt{3}}{2} + i\frac{3}{2}$ (Ans:)

Problem-03 : Convert $z = \frac{3\sqrt{3}}{2} + i\,3/2$ into polar form.

Soln : $r = \sqrt{x^2+y^2} = 3$

$\theta = \tan^{-1}(y/x) = \pi/6$

$\therefore z = 3e^{i\pi/6}$ (Ans:)

De Moivre's Thm :

$(\cos\theta + i\sin\theta)^P = \cos(P\theta) + i\sin(P\theta)$; P is a rational number