Department of Mathematics and Natural Sciences
MAT215 : Complex Variables and Laplace Transformations
Assignment 3
Date: August 13, 2025

Inspiring Excellence

Deadline : August 20,2025 | Summer 2025 | Total Marks: 120

Name: Priyanti Biswas

Section: 04

ID: 23201039

Use this page as the cover page of your assignment. No late submission will be graded. Answer any 4 questions from each part

Part A:

a. Evaluate $\int_{(0,1)}^{(2,5)}(3x+y)dx+(2y-x)dy$ along (a) the curve $y=x^2+1$, (b) the straight line joining $(0,1)$ and $(2,5)$, (c) the straight lines from $(0,1)$ to $(0,5)$ and then from $(0,5)$ to $(2,5)$, (d) the straight lines from $(0,1)$ to $(2,1)$ and then from $(2,1)$ to $(2,5)$.

b. (a) Evaluate $\oint_C (x+2y)dx+(y-2x)dy$ around the ellipse C defined by $x=4\cos\theta, y=3\sin\theta, 0\leq\theta<2\pi$ if C is described in a counterclockwise direction. (b) What is the answer to (a) if C is described in a clockwise direction?

c. Evaluate $\oint_C |z|^2 dz$ around the square with vertices at $(0,0),(1,0),(1,1),(0,1)$.

d. Evaluate $\int_i^{2-i}\left(3xy+iy^2\right)dz$ (a) along the straight line joining $z=i$ and $z=2-i$, (b) along the curve $x=2t-2, y=1+t-t^2$.

e. Evaluate $\oint_C \frac{dz}{(z-a)^n}, n=1,2,3,4,\ldots$ where (a) $z=a$ is outside the simple closed curve C, (b) $z=a$ is inside the simple closed curve C.

f. Evaluate $\oint_C(\bar{z})^2 dz$ around the circles (a) $|z|=1$ and (b) $|z-1|=1$

Part B:

a. Evaluate: (a) $\oint_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)}dz$ (b) $\oint_C \frac{e^{2z}}{(z+1)^4}dz$ where C is the circle $|z|=3$.

b. Evaluate $\frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}dz$ if C is: (a) the circle $|z|=3$, (b) the circle $|z|=1$.

c. Evaluate (a) $\oint_C \frac{\sin 3z}{z+\pi/2}dz$ if C is the circle $|z|=5$. (b) $\oint_C \frac{e^{iz}}{z^3}dz$ where C is the circle $|z|=2$.

d. (a) Let $F(z)$ be analytic inside and on a simple closed curve C except for a pole of order m at $z=a$

inside C. Prove that

$$\frac{1}{2\pi i}\oint_C F(z)dz = \lim_{z\to a}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m F(z)\}$$

(b) How would you modify the result in (a) if there were three poles inside C ?

e. Evaluate $\oint_C \frac{e^z}{(z^2+\pi^2)^2}dz$, where C is the circle $|z|=4$.

f. Given C is the circle $|z|=1$. Find the value of (a) $\oint_C \frac{\sin^6 z}{z-\pi/6}dz$ (b) $\oint_C \frac{\sin^6 z}{(z-\pi/6)^3}dz$

g. Evaluate (a) $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{(z^2+1)^2}dz$, (b) $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2+1}dz$ if $t>0$ and C is the circle $|z|=3$.

h. Evaluate $\oint_C \frac{e^{3z}}{z-\pi i}dz$ where C is the circle $|z-1|=4$.

i. Evaluate $\oint_C \frac{dz}{z-2}$ around (a) the circle $|z-2|=4$ (b) the circle $|z-1|=9$.

Part C:

a. Evaluate $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}dz$ around the circle C with equation $|z|=3$.

b. Let C be a square bounded by $x=\pm 2, y=\pm 2$. Evaluate $\oint_C \frac{\sinh 3z}{(z-\pi i/4)^3}dz$.

c. Evaluate $\oint_C \frac{2z^2+5}{(z+2)^3(z^2+4)z^2}dz$ where C is (a) $|z-2i|=6$, (b) the square with vertices at $1+i, 2+i$, $2+2i, 1+2i$.

d. Evaluate $\oint_C \frac{2+3\sin\pi z}{z(z-1)^2}dz$ where C is a square having vertices at $3+3i, 3-3i, -3+3i, -3-3i$.

e. Evaluate $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z(z^2+1)}dz, t>0$ around the square with vertices at $2+2i, -2+2i, -2-2i, 2-2i$.

f. Find the residues of $f(z)=\frac{z^2-2z}{(z+1)^2(z^2+4)}$ at all its poles.

g. Evaluate $\oint_C \frac{z^2}{2z^2+5z+2}dz$ using the residue at the poles, where C is the unit circle $|z|=1$.

h. Evaluate $\oint_C \frac{z^2+4}{z^3+2z^2+2z}dz$ using the residue at the poles, around the circle $|z|=3$.

i. Evaluate $\oint_C \frac{ze^{i\pi z}}{(z^2+2z+5)(z^2+1)^2}dz$ using the residue at the poles, where C is the upper half circle of the equation $|z|=2$.

j. Evaluate $\frac{1}{2\pi i}\oint \frac{z^2-z+2}{z^4+10z^2+9}dz$ using the residue at the poles, around the circle C with the equation $|z|=4$.

Part A

a) along the parabola $y = x^2+1$

(x,y)	t
$(2,5)$	$t=2$
$(0,1)$	$t=0$

$x = t \rightarrow dx = dt$

$y = t^2+1 \rightarrow dy = 2t\,dt$

$$\int_{(0,1)}^{(2,5)} (3x+y)\,dx + (2y-x)\,dy$$

$$= \int_0^2 (3t+t^2+1)\,dt + (2t^2+2-t)\,2t\,dt$$

$$= \int_0^2 (3t+t^2+1+4t^3+4t-2t^2)\,dt$$

$$= \int_0^2 (4t^3-t^2+7t+1)\,dt$$

$$= \left[\frac{4t^4}{4} - \frac{t^3}{3} + \frac{7t^2}{2} + t\right]_0^2$$

$$= 16 - \frac{8}{3} + 14 + 2 - 0 = \frac{88}{3}$$

b) along the straight line from $(0,1)$ to $(2,5)$

$0+i$ $\quad$ $2+5i$

$$z = z_1 + (z_2-z_1)t$$
$$= i + (2+5i-i)t$$
$$= i + (2+4i)t$$
$$= i + 2t + 4it$$
$$= 2t + (1+4t)i$$

(x,y)	t
$(2,5)$	1
$(0,1)$	0

$x = 2t \quad dx = 2\,dt$

$y = 4t+1 \quad dy = 4\,dt$

$$\int_{(0,1)}^{(2,5)} (3x+y)\,dx + (2y-x)\,dy$$

$$= \int_0^1 (6t+4t+1)\cdot 2\,dt + (8t+2-2t)\,4\,dt$$

$$= \int_0^1 (12t + \cancel{8t} + 2 + 32t + 8 - \cancel{8t})\,dt$$

$$= \int_0^1 (44t+10)\,dt$$

$$= \left[\frac{44t^2}{2} + 10t\right]_0^1 = 22+10-0 = 32$$

c) From $(0,1)$ to $(0,5)$ and the from $(0,5)$ to $(2,5)$

$$\text{Integral} = \int_{(0,1)}^{(0,5)} (3x+y)\,dx + (2y-x)\,dy + \int_{(0,5)}^{(2,5)} (3x+y)\,dx + (2y-x)\,dy$$

Now,

C_1 : straight line from i to $5i$

$$z = z_1 + (z_2 - z_1)t$$
$$= i + (5i - i)t$$
$$= i + 4it$$

$x = 0$

$y = 1+4t$

$dx = 0$

$dy = 4dt$

(x,y)	t
$(2,5)$	1
$(0,1)$	0

$$\int_{(0,1)}^{(0,5)} (3x+y)\,dx + (2y-x)\,dy$$

$$= \int_0^1 (3\cdot 0 + 4t + 1)\cdot 0 + (2+8t-0)\,4\,dt$$

$$= \int_0^1 (2+8t)\,dt$$

$$= \left[2t + \frac{8t^2}{2}\right]_0^1 = 2+4-0 = 6$$

Now,

$C_2 : 5i$ to $2+5i$

$$z = z_1 + (z_2 - z_1)t$$
$$= 5i + (2+5i-5i)t$$
$$= 5i + 2t$$
$$= 2t + 5i$$

$x = 2t$
$y = 5$
$dx = 2dt$
$dy = 0$

(x,y)	t
$(0,5)$	0
$(2,5)$	1

$$\int_{(0,5)}^{(0,1)} (3x+y)\,dx + (2y-x)\,dy$$

$$= \int_0^1 (3\cdot 2t + 5)\cdot 2dt + (10-2t)\cdot 0$$

$$= \int_0^1 (12t+10)\,dt$$

$$= \left[\frac{12t^2}{2} + 10t\right]_0^1$$

$$= 6 + 10 - 0$$

$$= 16$$

$$\therefore \int_{(0,1)}^{(0,5)} (3x+y)dx + (2y-x)dy + \int_{(0,5)}^{(2,5)} (3x+y)dx + (2y-x)dy$$

$$= 6 + 16$$

$$= 22$$

d) From $(0,1)$ to $(2,1)$ and the from $(2,1)$ to $(2,5)$

$$\text{Integral} = \int_{(0,1)}^{(2,1)} (3x+y)\,dx + (2y-x)\,dy + \int_{(2,1)}^{(2,5)} (3x+y)\,dx + (2y-x)\,dy$$

Now,

C_1 : Straight line from i to $2+i$.

$z = z_1 + (z_2 - z_1)t$

$= i + (2+i-i)t$

$= 2t + i$

$x = 2t$

$y = 1$

$dx = 2dt$

$dy = 0$

(x,y)	t
$(0,1)$	0
$(2,1)$	1

$$= \int_{(0,1)}^{(2,1)} (3x+y)\,dx + (2y-x)\,dy$$

$$= \int_0^1 (3\cdot 2t + 1)\cdot 2dt + (2-2t)\cdot 0$$

$$= \int_0^1 (12t + 2)\,dt$$

$$= \left[\frac{12t^2}{2} + 2t\right]_0^1$$

$$= 6 + 2 - 0$$

$$= 8$$

C_2 : $2+i$ to $2+5i$

$$z = z_1 + (z_2 - z_1)t$$
$$= 2+i+(\cancel{2}+5i-\cancel{2}-i)t$$
$$= 2+i+4it$$
$$= 2+(1+4t)i$$

$x = 2$
$y = 1+4t$
$dx = 0$
$dy = 4dt$

(x,y)	t
$(2,1)$	0
$(2,5)$	1

$$\int_{(2,1)}^{(2,5)} (3x+y)\,dx + (2y-x)\,dy$$
$$= \int_0^1 (3\cdot 2 + 1 + 4t)\cdot 0 + (\cancel{2}+8t-\cancel{2})\cdot 4dt$$
$$= \int_0^1 32t\,dt$$
$$= \left[\frac{32t^2}{2}\right]_0^1$$
$$= 16$$

$$\therefore \int_{(0,1)}^{(2,1)} (3x+y)\,dx + (2y-x)\,dy + \int_{(2,1)}^{(2,5)} (3x+y)\,dx + (2y-x)\,dx$$
$$= 8 + 16$$
$$= 24$$

b. (a) $x = 4\cos\theta$

$dx = -4\sin\theta\, d\theta$

$y = 3\sin\theta$

$dy = 3\cos\theta\, d\theta$

limit = from 0 to 2π

$$\oint_c (x+2y)\,dx + (y-2x)\,dy$$

$$= \int_0^{2\pi} (4\cos\theta + 6\sin\theta).(-4\sin\theta\, d\theta) + (3\sin\theta - 8\cos\theta).\ 3\cos\theta\, d\theta$$

$$= \int_0^{2\pi} (-16\cos\theta\sin\theta - 24\sin^2\theta + 9\sin\theta\cos\theta - 24\cos^2\theta)\,d\theta$$

$$= \int_0^{2\pi} (-7\cos\theta\sin\theta - 24)\,d\theta$$

$$= \int_0^{2\pi} (-7/2 \cdot 2\sin\theta\cos\theta - 24)\,d\theta$$

$$= \int_0^{2\pi} (-7/2 \cdot \sin 2\theta - 24)\,d\theta$$

$$= \left[7/2 \cdot \frac{\cos 2\theta}{2} - 24\theta \right]_0^{2\pi}$$

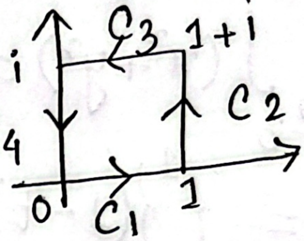
$$= \left(\frac{7}{4}\cos 4\pi - 24 . 2\pi\right) - (7/2 \cos 0 - 0)$$

$$= -48\pi$$

(b) When the curve is traversed clockwise instead of counterclockwise, the orientation is reversed. This changes the sign of the line integral.

$\oint_C = \pm 48\pi$, for clockwise orientation

c. $\oint_C |z|^2 \, dz$



$= \oint_C (x^2+y^2)\, dz$

$= \int_{C_1} (x^2+y^2) dz + \int_{C_2} (x^2+y^2) dz + \int_{C_3} (x^2+y^2) dz + \int_{C_4} (x^2+y^2) dz$

<u>For C_1:</u> From 0 to 1

$z = 0 + (1-0)t = t + 0i$

$x = t$ $dz = dt$
$y = 0$
$dx = dt$
$dy = 0$

$\int_{C_1} (x^2+y^2) \, dz$ $t \nearrow 0, \searrow 1$

$= \int_0^1 (t^2 + 0^2) \, dt$

$= \left[\frac{t^3}{3}\right]_0^1 = \frac{1}{3}$

<u>For C_2:</u> From 1 to 1+i

$z = 1 + (1+i-1)t = 1 + it$

$x = 1$ $dz = i\,dt$
$y = t$
$dx = 0$
$dy = dt$

$\int_{C_2} (x^2+y^2) \, dz$ $t \nearrow 0, \searrow 1$

$= \int_0^1 (1 + t^2) \cdot i \, dt$

$= i\left[t + \frac{t^3}{3}\right]_0^1 = \frac{4i}{3}$

For C_3: from $1+i$ to i

$$z = 1+i + (i-1-i)t$$
$$= 1+i - 1t$$
$$= (1-t) + i$$

$x = 1-t$

$y = 1$ $\quad t = 0, 1$

$dx = -dt$ $\quad dz = -dt$

$dy = 0$

$$\int_{C_3} (x^2+y^2)\,dz$$
$$= \int_0^1 \{(1-t)^2 + 1\} \cdot (-dt)$$
$$= \int_0^1 (1-2t+t^2+1)\,dt$$
$$= -\left[2t - \frac{2t^2}{2} + \frac{t^3}{3}\right]_0^1 = -\left(2 - 1 + \frac{1}{3}\right) = -\frac{4}{3}$$

For C_4: from i to 0.

$$z = i + (0-i)t$$
$$= 0 + (1-t)i$$

$x = 0$

$y = (1-t)$ $\quad t = 0, 1$

$dx = 0$ $\quad dz = i\,dt$

$dy = -dt$

$$\int_{C_4} (x^2+y^2)\,dz$$
$$= \int_0^1 \{0 + (1-t)^2\} \cdot (-i\,dt)$$
$$= (1-2t+t^2) \cdot (-i\,dt)$$
$$= -i/3$$

$$\oint_C (x^2+y^2)\,dz = \frac{1}{3} + \frac{4i}{3} - \frac{4}{3} - i/3$$
$$= -1+i$$

f. $\oint_c (\bar{z})^2 dz$

ⓐ $|z| = 1$ $\qquad dz = ie^{i\theta} d\theta$

$\Rightarrow z = 1.e^{i\theta}$ $\qquad \theta \to 0$ to 2π

$\Rightarrow z = e^{i\theta}$

$\Rightarrow \bar{z} = e^{-i\theta}$

$$\oint_c (\bar{z})^2 dz = \int_0^{2\pi} (e^{-i\theta})^2 ie^{i\theta} d\theta$$

$$= \int_0^{2\pi} (ie^{-2i\theta} . e^{i\theta}) d\theta$$

$$= \int_0^{2\pi} ie^{-i\theta} d\theta$$

$$= i\left[\frac{e^{-i\theta}}{-i}\right]_0^{2\pi}$$

$$= -\left[e^{-i\theta}\right]_0^{2\pi} = -\left[e^{-2\pi\theta} - e^0\right]$$

$$= -\left[\cos(-2\pi) + i\sin(-2\pi) - 1\right] = 1 + i.0 - 1 = 0$$

ⓑ $|z-1| = 1$ $\qquad dz = ie^{i\theta} d\theta$

$\Rightarrow z - 1 = e^{i\theta}$ $\qquad \theta \to 0$ to 2π

$\Rightarrow z = 1 + e^{i\theta}$

$\Rightarrow \bar{z} = 1 + e^{-i\theta}$

$$\oint_c (\bar{z})^2 dz = \int_0^{2\pi} (1+e^{-i\theta})^2 . ie^{i\theta} d\theta$$

$$= \int_0^{2\pi} (1 + 2e^{-i\theta} + e^{-2i\theta}) . ie^{i\theta} d\theta$$

$$= i\int_0^{2\pi} (e^{i\theta} + 2 + e^{-i\theta}) d\theta$$

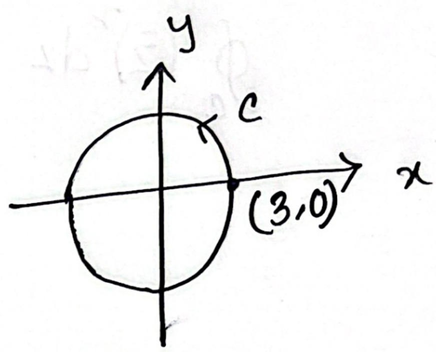
$$= i\left[\frac{e^{i\theta}}{i} + 2\theta + \frac{e^{-i\theta}}{-i}\right]_0^{2\pi}$$

$$= i\left[\frac{e^{i2\pi}}{i} + 2\cdot 2\pi + \frac{e^{-i2\pi}}{-i} - \frac{e^{0}}{i} - 0 - \frac{e^{0}}{-i}\right]$$

$$= i\left[\frac{e^{i2\pi} + 4\pi i - e^{-i2\pi}}{i}\right] = 4\pi i$$

Part B

a. (a) $\oint_c \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)}$



$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$1 \equiv A(z-2) + B(z-1)$$

$z = 2$; $B = 1$

$z = 1$; $A = -1$

$$\therefore \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$

$$\oint_c \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)} = \oint_c \frac{\sin\pi z^2 + \cos\pi z^2}{z-2} - \oint_c \frac{\sin\pi z^2 + \cos\pi z^2}{z-1}$$

$$= 2\pi i\{\sin(4\pi) + \cos(4\pi)\} - 2\pi i\{\sin(\pi) + \cos(\pi)\}$$

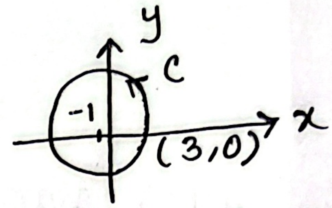
$$= 2\pi i\{\cos(4\pi) - \cos(\pi)\}$$

$$= 2\pi i\,(1 - (-1))$$

$$= 4\pi i$$

ⓑ $f(z) = e^{3z}$ and $a = -1$.

$f(z)$ is analytic inside and on C. Also $z = a = -1$ is inside c.



$$\oint_c \frac{e^{2z}}{(z+1)^4}\,dz$$

$$= \oint_c \frac{f(z)}{(z+1)^{3+1}}\,dz \quad \left[\because \oint_c \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^n(a)\right]$$

$f(z) = e^{2z}$

$f^3(z) = 8e^{2z}$

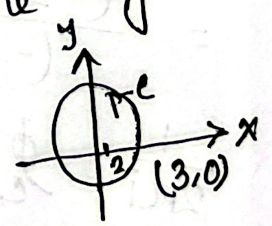
$$= \frac{2\pi i}{3!} f^3(-1)$$

$$= \frac{2\pi i}{6} \cdot 8e^{-2}$$

$$= \frac{8\pi i}{3e^2}$$

b ⓐ $f(z) = e^z$ is analytic everywhere

$\therefore f(z)$ ~~e^z~~ is analytic in the region enclosed by c.



$z = 2$ lies inside c

$$\oint_c \frac{e^z}{z-2}\,dz = 2\pi i\, f(2) = 2\pi i e^2$$

$$\Rightarrow \frac{1}{2\pi i}\oint_c \frac{e^z}{z-2}\,dz = e^2$$

ⓑ the circle $|z| = 1$

$\frac{e^z}{z-2}$ is analytic on $\mathbb{C}\setminus\{2\}$

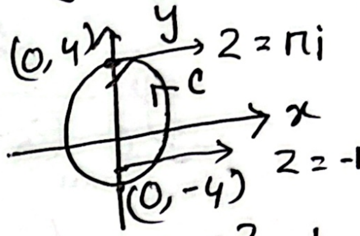
$\Rightarrow \frac{e^z}{z-2}$ is analytic in the region enclosed by c

$$\Rightarrow \oint_c \frac{e^z}{z-2}\,dz = 0 \quad \text{[using cauchy]}$$

$$\Rightarrow \frac{1}{2\pi i}\oint_c \frac{e^z}{z-2}\,dz = 0$$

e. $\cancel{(z^2+\pi^2)^2 = 0}$ $\quad f(z) = \dfrac{e^z}{(z^2+\pi^2)^2} = \dfrac{e^z}{(z^2-i^2\pi^2)^2}$

$$= \frac{e^z}{\{(z+i\pi)(z-i\pi)\}^2} = \frac{e^z}{(z+i\pi)^2(z-i\pi)^2}$$



$\therefore f(z)$ has poles at $z = \pm\pi i$ both of order 2 lying inside c

$$\frac{1}{2\pi i}\oint_c f(z)\,dz = \lim_{z\to-\pi i}\frac{1}{1!}\frac{d}{dz}\left\{\frac{e^z}{(z-i\pi)^2}\right\} + \lim_{z\to\pi i}\frac{1}{1!}\frac{d}{dz}\left\{\frac{e^z}{(z+i\pi)^2}\right\}$$

$$= \lim_{z\to-\pi i}\frac{(z-i\pi)^2e^z - e^z\,2(z-i\pi)}{(z-i\pi)^4} + \lim_{z\to\pi i}\frac{(z+i\pi)^2e^z - e^z.\,2(z+i\pi)}{(z+i\pi)^4}$$

$$= \lim_{z\to-\pi i}\frac{\{(z-i\pi)-2\}e^z}{(z-i\pi)^3} + \lim_{z\to\pi i}\frac{\{(z+i\pi)-2\}e^z}{(z+i\pi)^3}$$

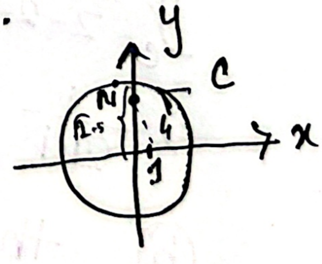
$$= \frac{(-2\pi i-2)e^{-\pi i}}{(-2\pi i)^3} + \frac{(2\pi i-2)e^{\pi i}}{(2\pi i)^3} = \frac{1}{2\pi^2}$$

$$\frac{1}{2\pi i}\oint_c f(z)\,dz = \frac{1}{2\pi^2}$$

$$\Rightarrow \oint_c f(z)\,dz = \frac{2\pi i}{2\pi^2} = \frac{i}{\pi}$$

h. $f(z) = e^{3z}$ $\quad a = \pi i$

$f(z)$ is analytic inside and on c.



Also $z = a = \pi i$ inside c

$$\oint_c \frac{f(z)}{z-\pi i} = 2\pi i\, f(\pi i)$$

$$= 2\pi i\, e^{3\pi i}$$

$$= 2\pi i(\cos 3\pi + i\sin 3\pi)$$

$$= -2\pi i$$

Part C

a. $f(z) = \dfrac{e^{2t}}{z^2(z^2+2z+2)}$

$= \dfrac{e^{2t}}{z^2(z+1-i)(z+1+i)}$

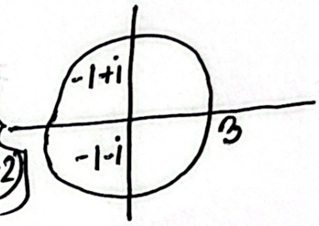
$z^2+2z+2=0$

$z^2+2.z+1^2+1=0$

$=(z+1)^2-i^2=0 \quad \} \; i^2=-1$

$(z+1-i)(z+1+i)=0$

$f(z)$ has a double pole at $z=0$ and two simple poles at $z=-1-i$, $z=-1+i$. all these poles are inside 'c'.



$\underset{z=0}{R_1} f(z) = \lim_{z\to 0} \dfrac{1}{1!}\dfrac{d}{dz}\left\{z^2 \cdot \dfrac{e^{2t}}{z^2(z^2+2z+2)}\right\}$

$= \lim_{z\to 0} \dfrac{d}{dz}\left(\dfrac{e^{2t}}{z^2+2z+2}\right)$

$= \lim_{z\to 0} \dfrac{(z^2+2z+2)e^{2t}(\cancel{1}) - e^{2t}(2z+2)}{(z^2+2z+2)^2}$

$= \dfrac{2t-2}{2^2} = \dfrac{t-1}{2}$

$\underset{z=-1+i}{R_2} f(z) = \lim_{z\to -1+i}\left\{(z+1-i)\dfrac{e^{2t}}{z^2(z^2+2z+2)}\right\}$

$= \lim_{z\to -1+i}\left\{(z+1-i)\dfrac{e^{2t}}{z^2(z+1-i)(z+1+i)}\right\}$

$= \lim_{z\to -1+i} \dfrac{e^{2t}}{z^2(z+1+i)}$

$= \dfrac{e^{(-1+i)t}}{(-1+i)^2(-1+i+1+i)}$

$= \dfrac{e^{(-1+i)t}}{(1-2i+i^2)(2i)}$

$= \dfrac{e^{(-1+i)t}}{(-2i)(2i)}$

$= \dfrac{e^{(-1+i)t}}{4}$

$$\underset{z=-1-i}{R_3} \quad f(z) = \lim_{z\to -1-i} \left\{ (z+1+i) \frac{e^{zt}}{z^2(z^2+2z+2)} \right\}$$

$$= \lim_{z\to -1-i} \left\{ (z+1+i) \frac{e^{zt}}{z^2(z+1-i)(z+1+i)} \right\}$$

$$= \lim_{z\to -1-i} \frac{e^{zt}}{z^2(z+1-i)}$$

$$= \frac{e^{(-1-i)t}}{(-1-i)^2(-1-i+1-i)}$$

$$= \frac{e^{(-1-i)t}}{(1+2i+i^2)(-2i)}$$

$$= \frac{e^{(-1-i)t}}{2i\cdot(-2i)} = \frac{e^{(-1-i)t}}{4}$$

$$\oint_c \frac{e^{zt}}{z^2(z^2+2z+2)}\,dz = 2\pi i\left[R_1+R_2+R_3\right]$$

$$= 2\pi i\left[\frac{t-1}{2} + \frac{e^{(-1+i)t}}{4} + \frac{e^{(-1-i)t}}{4}\right]$$

$$\therefore \frac{1}{2\pi i}\oint_c \frac{e^{zt}}{z^2(z^2+2z+2)}\,dz = \frac{t-1}{2} + \frac{1}{4}\left[e^{-t}e^{it} + e^{-t}e^{-it}\right.$$

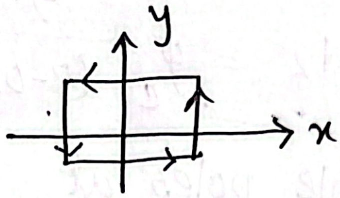
$$= \frac{t-1}{2} + \frac{e^{-t}}{4}\left[\cos t + i\sin t + \cos(-t) + i\sin(-t)\right]$$

$$= \frac{t-1}{2} + \frac{e^{-t}}{4}\left[\cos t + i\sin t + \cos t - i\sin t\right]$$

$$= \frac{t-1}{2} + \frac{e^{-t}}{4}\left[2\cos t\right]$$

$$= \frac{t-1}{2} + \frac{1}{2}e^{-t}\cos t$$

d. $f(z) = \frac{2+3\sin\pi z}{z(z-1)^2}$ has singularities at $z(z-1)^2 = 0 \Rightarrow z=0,\ z=1$



$$\operatorname{Res}(f, z=0) = \lim_{z\to 0} (z-0)\cdot f(z)$$

$$= \lim_{z\to 0} \frac{2+3\sin\pi z}{(z-1)^2}$$

$$= \frac{2}{1} = 2$$

$$\operatorname{Res}(f, z=1) = \frac{1}{(2-1)!} \lim_{z\to 1} \frac{d}{dz}\left((z-1)^2 f(z)\right)$$

$$= \lim_{z\to 1} \frac{d}{dz}\left(\frac{2+3\sin\pi z}{z}\right)$$

$$= \lim_{z\to 1} \frac{z(3\pi\cos z)(-2\cancel{z}+3\sin\pi z)\cdot 1}{z^2}$$

$$= \frac{1(-3\pi)-2}{1} = -2-3\pi$$

$$\oint_c \frac{2+3\sin\pi z}{z(z-1)^2}\,dz = 2\pi i\left(2+(-2-3\pi)\right)$$

$$= -6\pi^2 i$$

h. z^3+2z^2+2z

$= z(z^2+2z+2) = z\left[(z+1+i)(z+1-i)\right]$

$$\oint_c \frac{z^2+4}{z^3+2z^2+2z}\,dz = \oint_c \frac{z^2+4}{(z-0)\{z-(-1-i)\}\{z-(-1+i)\}}\,dz$$

$f(z)$ has simple poles at $z=0, -1-i, -1+i$ all inside the region

$$R_1 = \lim_{z\to 0} z.f(z) = \lim_{z\to 0} \frac{z^2+4}{(z+1+i)(z+1-i)} = \frac{4}{1-i^2} = 2$$

$$R_2 = \lim_{z\to -1-i} (z+1+i) = \frac{z^2+4}{z(z+1+i)(z+1-i)} = \lim_{z\to -1-i} \frac{z^2+4}{z(z+1-i)}$$

$$= \frac{(-1-i)^2+4}{(-1-i)(-1-i+1-i)} = \frac{1+2(-1)(-i)+(-i)^2+4}{(-1-i)(-2i)}$$

$$= -\tfrac{1}{2} - \tfrac{3}{2}i$$

$$R_3 = \lim_{z\to -1+i} (z+1-i) f(z)$$

$$= \lim_{z\to -1+i} \frac{z^2+4}{z(z+1+i)}$$

$$= \frac{(-1+i)^2+4}{(-1+i)(-1+i+1+i)}$$

$$= \frac{-2i+4}{-2i-2} = \frac{-i^2+2i+i-2}{1-i^2} = -1/2 + 3/2\, i$$

$$\oint_c \frac{z^2+4}{z^3+2z^2+2z}\,dz = 2\pi i(R_1+R_2+R_3)$$

$$= 2\pi i\left(2 - \tfrac{1}{2} - \tfrac{3i}{2} - \tfrac{1}{2} + \tfrac{3i}{2}\right)$$

$$= 2\pi i\,(2-1)$$

$$= 2\pi i$$

∴ As singularities at $z^4+10z^2+9=0$

$\Rightarrow z^4+9z^2+z^2+9=0$

$\Rightarrow (z^2+9)(z^2+1)=0$

$\Rightarrow z=\pm 3i, \pm i$

$$f(z) = \frac{z^2-z+2}{z^4+10z^2+9} = \frac{z^2-z+2}{(z-i)(z+i)(z-3i)(z+3i)}$$

$$\text{Res}(f, z=i) = \lim_{z\to i} (z-i)\, f(z)$$

$$= \lim_{z\to i} \frac{z^2-z+2}{(z+i)(z-3i)(z+3i)}$$

$$= \frac{-1-i+2}{2i(-2i)(4i)} = -\tfrac{1}{16} - \tfrac{i}{16}$$

$$\text{Res}(f, z=-i) = \lim_{z\to -i} (z-(-i))\, f(z)$$

$$= \lim_{z\to -i} (z+i) \frac{z^2-z+2}{z^4+10z^2+9}$$

$$= -\tfrac{1}{16} + \tfrac{i}{16}$$

$$\text{Res}(f, z=3i) = \lim_{z\to 3i} (z-3i)\, f(z)$$

$$= \tfrac{1}{16} - \tfrac{7i}{48}$$

$$\text{Res}(f, z=-3i) = \lim_{z\to -3i} (z+3i)\, f(z)$$

$$= \tfrac{1}{16} + \tfrac{7i}{48}$$

$$\oint_C \frac{z^2-z+2}{z^4+10z^2+9}\,dz = 2\pi i\left(\cancel{-\tfrac{1}{16}} \cancel{-\tfrac{i}{16}} \cancel{-\tfrac{1}{16}} \cancel{+\tfrac{i}{16}} \cancel{+\tfrac{1}{16}} \cancel{-\tfrac{7i}{48}} \cancel{+\tfrac{1}{16}} \cancel{+\tfrac{7i}{48}}\right)$$

$$= 0$$