BRAC UNIVERSITY

Inspiring Excellence

Department of Mathematics and Natural Sciences
MAT215 : Complex Variables and Laplace Transformations
Assignment 2
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Use this page as the cover page of your assignment. No late submission will be graded.

(1) Evaluate the following limits (any 2): [5 x 2 = 10]

a. $\lim_{z\to 0}(\cos z)^{1/z^2}$ b. $\lim_{z\to m\pi i}(z-m\pi i)\left(\frac{e^z}{\sin z}\right)$ c. $\lim_{z\to 0}\left(\frac{\sin z}{z}\right)^{1/z^2}$ d. $\lim_{z\to i}\frac{z^2-2iz-1}{z^4+2z^2+1}$

(2) Prove: (a) $\sin iz = i\sinh z$ (b) $\cos iz = \cosh z$ (c) $\sin(x+iy) = \sin x\cosh y + i\cos x\sinh y$ [5]

(3) Prove that [2.5 x 2 + 5 = 10]

a. $\mathrm{Re}\{z\} = (z+\bar{z})/2$, b. $\mathrm{Im}\{z\} = (z-\bar{z})/2i$. c. Suppose that $f(z) = x^2 - y^2 - 2y + i(2x-2xy)$, where $z = x+iy$. Use the above expressions to write $f(z)$ in terms of z and simplify the result.

(4) Check, whether $u(x,y)$ is harmonic function or not (any 2). [5 x 2 =10]

a. $u = 3x^2y + 2x^2 - y^3 - 2y^2$ b. $u = xe^x\cos y - ye^x\sin y$ c. $u = e^{-x}(x\sin y - y\cos y)$.

If u is a harmonic, find a function v such that $f(z) = u + iv$ is analytic [i.e., find the harmonic conjugate of u]. Express $f(z)$ in terms of z.

(5) Using the definitions, find the derivative of each function at the indicated point (any 1). [5]

(i) $f(z) = \frac{2z-i}{z+2i}$ at $z = -i$ (ii) $f(z) = 3z^{-2}$ at $z = 1+i$.

(6) Determine whether $|z|^2$ has a derivative anywhere. [5]

(7) Using the definition of derivative prove that $\frac{d}{dz}(z^2\bar{z})$ does not exist anywhere. [5]

(8) Using the definition of derivative prove that $\frac{d}{dz}(\bar{z})$ does not exist anywhere. [5]

(9) State the necessary and sufficient conditions for a function being analytic. Hence show that the functions $sin(z)$ and $cos(z)$ are both analytic. [5]

(10) Prove that: (i) $\sinh z = \sinh x\cos y + i\cosh x\sin y$ (ii) $\cosh z = \cosh x\cos y + i\sinh x\sin y$. Use this relations to show that $sinh 2z$ and $\cosh 3z$ are both analytic functions. [5]

(11) Show that the function $f(z) = \frac{1}{z-a}$ is analytic in $\mathbb{C}\backslash\{a\}$ for any arbitrary value of a. [5]

Ans to the Ques. no-1

c) $\lim_{z \to 0} \left(\frac{\sin z}{z}\right)^{1/z^2}$

let, $y = \lim_{z\to 0}\left(\frac{\sin z}{z}\right)^{1/z^2}$

$\Rightarrow \ln y = \lim_{z\to 0}\left(\frac{1}{z^2} \ln\left(\frac{\sin z}{z}\right)\right) = \lim_{z\to 0} \frac{\ln\left(\frac{\sin z}{z}\right)}{z^2}$

$$\Rightarrow \ln y = \lim_{z\to 0} \frac{\ln\left(\frac{\sin z}{z}\right)}{z^2}$$

$$= \lim_{z\to 0} \frac{\ln \sin z - \ln z}{z^2}$$

$$= \lim_{z\to 0} \frac{\frac{1}{\sin z}\cdot \cos z - \frac{1}{z}}{2z} \quad \text{[L' Hospital]}$$

$$= \lim_{z\to 0} \frac{z\cos z - \sin z}{2z^2 \sin z}$$

$$= \lim_{z\to 0} \frac{1\cdot \cos z + z(-\sin z) - \cos z}{4z\cdot \sin z + 2z^2 \cos z} \quad \text{[L' Hospital]}$$

$$= \lim_{z\to 0} \frac{-z \sin z}{4z\sin z + 2z^2\cos z}$$

$$= \lim_{z\to 0} \frac{-1.\sin z - z\cos z}{4\sin z + 4z\cos z + 4z\cos z + 2z^2(-\sin z)}$$

$$= \lim_{z\to 0} \frac{-\sin z - z\cos z}{4\sin z + 8z\cos z - 2z^2\sin z} \quad \text{[L' Hospital]}$$

$$= \lim_{z\to 0} \frac{-\cos z - 1\cdot\cos z - z(-\sin z)}{4\cos z + 8\cos z + 8z(-\sin z) - 4z\sin z - 2z^2(\cos z)}$$

$$= \frac{-1-1}{4+8} = -1/6 \quad \text{[L' Hospital]}$$

Now,

$$\ln y = -1/6$$

$$\Rightarrow y = e^{-1/6}$$

$$\therefore \lim_{z \to 0} \left(\frac{\sin z}{z}\right)^{1/z^2} = e^{-1/6}$$

d) $$\lim_{z \to i} \frac{z^2 - 2iz - 1}{z^4 + 2z^2 + 1}$$

$$= \lim_{z \to i} \frac{2z - 2i}{4z^3 + 4z} \quad [\text{L' Hospital}]$$

$$= \lim_{z \to i} \frac{2}{12z^2 + 4} \quad [\text{L' Hospital}]$$

$$= \frac{2}{-12 + 4}$$

$$= \frac{2}{-8} = -1/4$$

Ans to the Ques. 2

a) $$\sin iz = \frac{1}{2i}\left(e^{i(iz)} - e^{-i(iz)}\right) \quad \left[\because \sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)\right]$$

$$= \frac{1}{2i}\left(e^{-z} - e^{z}\right)$$

$$= \frac{1}{2i} \times \frac{-i}{-i}\left(e^{-z} - e^{z}\right)$$

$$= \frac{-i}{2}\left\{-\left(e^{z} - e^{-z}\right)\right\}$$

$$= \frac{i}{2}\left(e^{z} - e^{-z}\right) \quad \text{isinhz}$$

$$\therefore \sin iz = i \sinh z \quad [\text{Proved}]$$

b) $\cos iz = \frac{1}{2}\left(e^{i(iz)} + e^{-i(iz)}\right)$ $\left[\because \cos z = \frac{1}{2}(e^{iz} + e^{-iz})\right]$

$= \frac{1}{2}(e^{-z} + e^{z}) = \cosh z$

$\therefore \cos iz = \cosh z$

[Proved]

c) $\sin(x+iy) = \sin x \cos iy + \cos x \sin iy$ [From 'a' and 'b']

$= \sin x \cosh z + \cos x \, i \sinh y$

$= \sin x \cosh z + i \cos x \sinh y$

$\therefore \sin(x+iy) = \sin x \cosh z + i \cos x \sinh y$

[Proved]

Ans to the Ques. no - 3

$[\because z = x+iy, \bar{z} = x-iy]$

a) $L.H.S = Re\{z\} = x$

$R.H.S = \frac{z+\bar{z}}{2} = \frac{x+iy+x-iy}{2} = x$

$\therefore L.H.S = R.H.S$ [Proved]

b) $L.H.S = Im\{z\} = y$

$R.H.S = \frac{z-\bar{z}}{2i} = \frac{x+iy-x+iy}{2i} = y$

$\therefore L.H.S = R.H.S$ [Proved]

c) $f(z) = x^2 - y^2 - 2y + i2x - 2ixy$

$= x^2 - 2ixy + i^2y^2 - 2y + i2x$

$= x^2 - 2ixy + i^2y^2 + 2i^2y + i2x$

$= (x - iy)^2 + 2i(x + iy)$

$\therefore f(z) = (\overline{z})^2 + 2iz \quad \left[\because z = x + iy,\ \overline{z} = x - iy\right]$

Ans to the Ques. no-4

a) $u(x,y) = 3x^2y + 2x^2 - y^3 - 2y^2$

$u_x = 6xy + 4x \qquad u_y = -3y^2 - 4y + 3x^2$

$u_{xx} = 6y + 4 \qquad u_{yy} = -6y - 4$

$\Delta(u) = u_{xx} + u_{yy} = 0$ on $\mathbb{R}^2$

$\therefore u$ is harmonic on $\mathbb{R}^2$.

c) $u(x,y) = e^{-x}(x\sin y - y\cos y)$

$= xe^{-x}\sin y - ye^{-x}\cos y$

$u_x = e^{-x}\sin y - xe^{-x}\sin y + e^{-x}y\cos y$

$u_{xx} = -e^{-x}\sin y - e^{-x}\sin y + xe^{-x}\sin y - e^{-x}y\cos y$

$= xe^{-x}\sin y - 2e^{-x}\sin y - e^{-x}y\cos y$

$u_y = xe^{-x}\cos y - e^{-x}(y(-\sin y) + 1\cos y)$

$= xe^{-x}\cos y + e^{-x}y\sin y - e^{-x}y\cos y$

$u_{yy} = -xe^{-x}\sin y + e^{-x}y\cos y + e^{-x}\sin y + e^{-x}y\sin y$

$= -xe^{-x}\sin y + 2e^{-x}\sin y + e^{-x}y\cos y$

$\Delta(u) = u_{xx} + u_{yy} = 0$ on $\mathbb{R}^2$

$\therefore$ u is harmonic on $\mathbb{R}^2$

<u>2nd part of 'a'</u>

$f(z) = u + iv$ is analytic.

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ —(I) $\quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ —(II)

from (I),

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

$$\Rightarrow \frac{\partial v}{\partial y} = 6xy + 4x$$

Integrate both sides w.r.t y,

$$v(x,y) = 6x \cdot y^2/2 + 4xy + c(x)$$

$$\Rightarrow v(x,y) = 3xy^2 + 4xy + c(x)$$

Differentiate both sides w.r.t. x,

$$\frac{\partial v}{\partial x} = 3y^2 + 4y + c'(x)$$

$$\Rightarrow -\frac{\partial u}{\partial y} = 3y^2 + 4y + c'(x)$$

$$\Rightarrow -3x^2 + 3y^2 + 4y = 3y^2 + 4y + c'(x)$$

$\Rightarrow c'(x) = -3x^2 \Rightarrow c(x) = -x^3 + c$ where c is any arbitary constant

$\therefore v(x,y) = 3xy^2 + 4xy - x^3 + c$ where c is arbitary constant

2nd part of 'c'

$f(z) = u + iv$ is analytic

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{(I)} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{(II)}$$

From (I) →

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

$$= e^{-x}\sin y - xe^{-x}\sin y + e^{-x}y\cos y$$

$$\Rightarrow \frac{\partial v}{\partial y}$$

$$v(x,y) = \int (e^{-x}\sin y - xe^{-x}\sin y + e^{-x}y\cos y)\,dy$$

$$= e^{-x}\int \sin y\,dy - xe^{-x}\int \sin y\,dy + e^{-x}\int y\cos y\,dy$$

$$= -e^{-x}\cos y + xe^{-x}\cos y + e^{-x}[y\sin y + \cos y] + g(x)$$

$$= -e^{-x}\cancel{\cos y} + xe^{-x}\cos y + e^{-x}y\sin y + e^{-x}\cancel{\cos y} + g(x)$$

$$\Rightarrow v(x,y) = xe^{-x}\cos y + e^{-x}y\sin y + g(x)$$

From (II) →

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\Rightarrow \frac{\partial}{\partial x}(xe^{-x}\cos y + e^{-x}y\sin y + g(x)) = -(xe^{-x}\cos y + e^{-x}y\sin y - e^{-x}\cos y)$$

$$\Rightarrow e^{-x}\cancel{\cos y} - xe^{-x}\cancel{\cos y} - e^{-x}\cancel{y}\sin y + g'(x)$$

$$= -xe^{-x}\cancel{\cos y} - e^{-x}\cancel{y}\cancel{\sin y} + e^{-x}\cancel{\cos y}$$

$\Rightarrow g'(x) = 0$

$\Rightarrow g(x) = \int 0 \, dx = c$

$\therefore v(x,y) = xe^{-x}\cos y + e^{-x} y \sin y + c$

where c is a constant.

Ans to the Ques. no-5

1) $f(z) = \dfrac{2z-i}{z+2i}$ at $z=-i$ $\quad \left[\because \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}\right]$

$x \to z$

$\Delta x = h \to \Delta z$

$f'(z) = \lim_{\Delta z\to 0} \dfrac{f(z+\Delta z)-f(z)}{\Delta z}$

$\therefore f'(-i) = \lim_{\Delta z\to 0} \dfrac{1}{\Delta z}\left[f(-i+\Delta z)-f(-i)\right]$ $\quad \left[\because f(-i+\Delta z) = \dfrac{2(-i+\Delta z)-i}{(-i+\Delta z)+2i} \;\&\; f(-i) = \dfrac{2(-i)-i}{(-i)+2i}\right]$

$= \lim_{\Delta z\to 0} \dfrac{1}{\Delta z}\left[\dfrac{2(-i+\Delta z)-i}{-i+\Delta z+2i} - \dfrac{2(-i)-i}{-i+2i}\right]$

$= \lim_{\Delta z\to 0} \dfrac{1}{\Delta z}\left[\dfrac{-3i+2\Delta z}{i+\Delta z} + \dfrac{3\cancel{i}}{\cancel{i}}\right]$

$= \lim_{\Delta z\to 0} \dfrac{1}{\Delta z}\left[\dfrac{-3i+2\Delta z+3(i+\Delta z)}{(i+\Delta z)}\right]$

$= \lim_{\Delta z\to 0} \dfrac{1}{\Delta z}\left[\dfrac{-3\cancel{i}+2\Delta z+3\cancel{i}+3\Delta z}{(i+\Delta z)}\right]$

$= \lim_{\Delta z\to 0} \dfrac{1}{\cancel{\Delta z}} \cdot \dfrac{5\cancel{\Delta z}}{(i+\Delta z)}$

$= \dfrac{5}{i} = \dfrac{5}{i} \cdot \dfrac{(-i)}{(-i)} = -5i$

Ans to the Ques. no-6

$f(z) = |z|^2$

$f(z_0+\Delta z) = |z_0+\Delta z|^2$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{|z_0+\Delta z|^2 - |z_0|^2}{\Delta z} \qquad \left[\begin{array}{l} |z|^2 = x^2+y^2 \\ \because |z|^2 = z\bar{z} \end{array} \right]$$

$$= \lim_{\Delta z \to 0} \frac{(z_0+\Delta z)\overline{(z_0+\Delta z)} - z_0\bar{z}_0}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z_0+\Delta z)(\bar{z}_0+\Delta \bar{z}) - z_0\bar{z}_0}{\Delta z} \qquad \left[\because \overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2\right]$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{z_0\bar{z}_0} + \bar{z}_0\Delta z + z_0\Delta\bar{z} + \Delta z\Delta\bar{z} - \cancel{z_0\bar{z}_0}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\bar{z}_0\Delta z + z_0\Delta\bar{z} + \Delta z\Delta\bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left(\bar{z}_0 + z_0\frac{\Delta\bar{z}}{\Delta z} + \Delta\bar{z}\right)$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left(\bar{z}_0 + z_0 \cdot \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} + \Delta x - i\Delta y\right)$$

In $\Delta x = 0$ direction,

$$f'(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left(\bar{z}_0 + z_0 \cdot \frac{-i\Delta y}{i\Delta y} - i\Delta y\right)$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left(-z_0 + \bar{z}_0 - i\Delta y\right)$$

$$= -z_0 + \bar{z}_0$$

In $\Delta y = 0$ direction,

$$f'(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left(z_0 \cdot \frac{\Delta x}{\Delta x} + \bar{z}_0 + \Delta x \right)$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left(z_0 + \bar{z}_0 + \Delta x \right) = z_0 + \bar{z}_0$$

Since, $-z_0 + \bar{z}_0 \neq z_0 + \bar{z}_0$

for all $z_0 \neq 0$

$\Rightarrow f(z) = |z|^2$ is not differentiable other than 0.

Ans to the Ques. no - 7

$$f(z) = z^2 \bar{z}$$

$$f(z_0+\Delta z) = (z_0+\Delta z)^2 \cdot (\overline{z_0+\Delta z})$$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z_0+\Delta z)^2 \cdot (\overline{z_0+\Delta z}) - z_0^2 \overline{\Delta z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z_0^2 + 2z_0\Delta z + \Delta z^2) \cdot (\bar{z}_0 + \overline{\Delta z}) - z_0^2 \overline{\Delta z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{z_0^2 \bar{z}_0} + 2z_0\bar{z}_0\Delta z + \Delta z^2 \bar{z}_0 + z_0^2 \overline{\Delta z} + 2z_0 \Delta z \overline{\Delta z} + \Delta z^2 \overline{\Delta z} - \cancel{z_0^2} \overline{\Delta z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left(2z_0 \bar{z}_0 + \Delta z \bar{z}_0 + z_0^2 \cdot \frac{\overline{\Delta z}}{\Delta z} + 2z_0 \overline{\Delta z} + \Delta z \overline{\Delta z} \right)$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left(2z_0 \bar{z}_0 + (\Delta x + i\Delta y) \bar{z}_0 + z_0^2 \cdot \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} + 2z_0 (\Delta x - i\Delta y) + (\Delta x - i\Delta y)(\Delta x + i\Delta y) \right)$$

In $\Delta x = 0$ direction,

$$f'(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left(2z_0\bar{z}_0 + i\Delta y\,\bar{z}_0 + z_0^2\cdot(-1) + 2z_0(-i\Delta y) - (i\Delta y)^2\right.$$

$$= \quad 2z_0\bar{z}_0 - z_0^2$$

In $\Delta y = 0$ direction,

$$f'(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left(2z_0\bar{z}_0 + \Delta x\, z_0 + z_0^2 + 2z_0\Delta x + (\Delta x)^2\right.$$

$$= 2z_0\bar{z}_0 + z_0^2$$

since, $2z_0\bar{z}_0 - z_0^2 \neq 2z_0\bar{z}_0 + \bar{z}_0$

$\Rightarrow f(z) = z^2\bar{z}$ doesnot exit anywhere

[Proved]

Ans to the Ques. no-8

$$f(z) = \bar{z}$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\overline{z+\Delta z} - \bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\overline{\Delta x + iy}}{\Delta x + iy}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - iy}{\Delta x + iy}$$

In real axis, $\Delta y = 0$

$$\lim_{\Delta x \to 0} \frac{x}{x} = 1$$

imaginary, $\Delta x = 0$

$$\lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = -1$$

Then since the limit depends on the manner for which $\Delta z \to 0$, so does not exist.

<u>Ans to the Ques. no - 9</u>

$f(z) = \sin z$

$= \sin(x+iy)$

$= \sin x \cos iy + \cos x \sin(iy)$

$\cos iy = \dfrac{e^{i^2 y} + e^{-i^2 y}}{2}$ $\quad$ $\sin iy = \dfrac{e^{i^2 y} - e^{-i^2 y}}{2i}$

$= \dfrac{e^{-y} + e^{y}}{2}$ $\quad$ $= -\frac{1}{i}\left(\dfrac{e^{y} - e^{-y}}{2}\right)$

$= \cosh(y)$ $\quad$ $= i \sinh(y)$

$\therefore f(z) = \sin x \cosh y + i \cos x \sinh(y)$

$u = \sin x \cosh y$ $\quad$ $v = \cos x \sinh y$

$u_x = \cos x \cosh y$ $\quad$ $v_x = -\sin x \sinh y$

$u_y = \sin x \sinh y$ $\quad$ $v_y = \cos x \cosh y$

$\therefore \quad u_x = v_y \quad u_y = -v_x$

$\therefore$ Cauchy - Riemann equations hold $\forall (x,y) \in \mathbb{R}^2$

u_x, u_y, v_x, v_y are all continuous functions on $\mathbb{R}^2$

$\therefore f(z)$ is analytic on $\mathbb{R}^2 / \mathbb{C}$

$f(z) = \cos z$

$= \cos(x+iy)$

$= \cos x \cos iy - \sin x \sin iy$

$\cos iy = \cosh y$ $\quad$ $\sin iy = i \sinh y$

$\therefore f(z) = \cos x \cosh y - i \sin x \sinh y$

$u = \cos x \cosh y$ $\quad$ $v = -\sin x \sinh y$

$u_x = -\sin x \cosh y$ $\qquad$ $v_x = -\cos x \sinh y$

$u_y = \cos x \sinh y$ $\qquad$ $v_y = -\sin x \cosh y$

$u_x = v_y$ $\qquad$ $u_y = -v_x$

So, Cauchy Riemann equations hold $\forall (x, y) \in \mathbb{R}^2$

u_x, u_y, v_x, v_y are all continuous functions on $\mathbb{R}^2$.

[illegible] analytic on $\mathbb{R}^2 / \mathbb{C}$

Ans to Ques. no - 10

i) $\sinh z$

$$= \frac{e^{z} - e^{-z}}{2}$$

$$= \frac{e^{x+iy} - e^{-(x+iy)}}{2}$$

$$= \frac{e^{x} \cdot e^{iy} - e^{-x} \cdot e^{-iy}}{2}$$

$$= \frac{e^{x} \cdot \cancel{e^{iy}} (\cos y + i \sin y) - e^{-x} (\cos y - i \sin y)}{2}$$

$$= \frac{e^{x} \cos y + i e^{x} \sin y - e^{-x} \cos y + i e^{-x} \sin y}{2}$$

$$= \frac{(e^{x} - e^{-x}) \cos y}{2} + i \frac{(e^{x} - e^{-x}) \sin y}{2}$$

$$= \sinh x \cos y + i \sinh x \sin y$$

ii) $\cosh z$

$$= \frac{e^{z} + e^{-z}}{2}$$

$$= \frac{e^{x+iy} + e^{-(x+iy)}}{2}$$

$$= \frac{e^{x} \cdot e^{iy} + e^{-x} \cdot e^{-iy}}{2}$$

$$= \frac{e^{x} \cdot (\cos y + i \sin y) + e^{-x} \cdot (\cos y - i \sin y)}{2}$$

$$= \frac{e^x \cos y + i e^x \sin y + e^{-x} \cos y - i e^{-x} \sin y}{2}$$

$$= \frac{(e^x + e^{-x})}{2} \cos y + i \frac{e^x - e^{-x}}{2} \cdot \sin y$$

$$= \cosh x \cos y + i \sinh x \sin y$$

[Proved]

$$\cosh 3z = \frac{e^{3z} + e^{-3z}}{2}$$

Since, e^{3z} and e^{-3z} are both analytic everywhere. So their sum and scalar multiples are also analytic. So, $\cosh(3z)$ is an analytic function.

$$\sinh 2z = \frac{e^{2z} - e^{-2z}}{2}$$

Since, e^{2z} and e^{-2z} are both analytic and sum also. So $\sinh 2z$ is an analytic function.

Ans to the Ques. no - 11

$$f(z) = \frac{1}{z-a}$$

$$= \frac{1}{(x+iy)-a}$$

$$= \frac{1}{(x-a)+iy}$$

$$= \frac{(x-a)-iy}{(x-a)^2+y^2}$$

$$= \frac{x-a}{(x-a)^2+y^2} + \frac{-iy}{(x-a)^2+y^2}$$

$$u = \frac{x-a}{(x-a)^2+y^2} \qquad v = \frac{-y}{(x-a)^2+y^2}$$

$$u_x = \frac{[(x-a)^2+y^2]\cdot 1 - (x-a)\cdot 2(x-a)}{[(x-a)^2+y^2]^2} = \frac{-(x-a)^2+y^2}{[(x-a)^2+y^2]^2}$$

$$u_y = \frac{[(x-a)^2+y^2]\cdot 0 - (x-a)\cdot 2y}{[(x-a)^2+y^2]^2} = \frac{-2(x-a)y}{[(x-a)^2+y^2]^2}$$

$$v_x = \frac{[(x-a)^2+y^2]\cdot 0 - (-y)\cdot 2(x-a)}{[(x-a)^2+y^2]^2} = \frac{2(x-a)y}{[(x-a)^2+y^2]^2}$$

$$v_y = \frac{[(x-a)^2+y^2]\cdot(-1) - (-y)\,2y}{[(x-a)^2+y^2]^2} = \frac{-(x-a)^2+y^2}{[(x-a)^2+y^2]^2}$$

$$u_x = v_y \qquad u_y = -v_x$$

CR cauchy and satisfied.

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ continuous except $(x, y) = a, 0$

$z = a + 0 \Rightarrow z = a$ $f = \frac{1}{z-a}$ is analytical in $\mathbb{C} \setminus \{a\}$