Liouville's Extansion of Dirichlet's Integral

If the variables x, y, z are all positive such that $h_1 < (x+y+z) < h_2$ then

$$\iiint f(x+y+z)\, x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz$$

$$= \frac{\Gamma l\, \Gamma m\, \Gamma n}{\Gamma l+m+n} \int_{h_1}^{h_2} f(t)\, t^{l+m+n-1}\, dt.$$

Ques:- 2021 Evaluate $\iiint xyz \sin(x+y+z)\, dx\, dy\, dz$, the integral being extended to all positive values of the variables subject to the condition $x+y+z \le \frac{\pi}{2}$.

Soln:- For all positive values of variables

$$0 < x+y+z \le \frac{\pi}{2}$$

So

$$\iiint xyz \sin(x+y+z)\, dx\, dy\, dz$$

$$= \iiint x^{2-1} y^{2-1} z^{2-1} \sin(x+y+z)\, dx\, dy\, dz$$

$$= \frac{\Gamma 2\, \Gamma 2\, \Gamma 2}{\Gamma 2+2+2} \int_0^{\pi/2} (\sin t) \cdot t^{2+2+2-1}\, dt$$

(By Liouville's extansion)

$$= \frac{\Gamma 2 \, \Gamma 2 \, \Gamma 2}{\Gamma 6} \int_0^{\pi/2} \underset{\downarrow \atop u}{t^5} \, \underbrace{\sin t}_{\downarrow \atop v} \, dt$$

$$\boxed{\int uv\,dx = u(v_1) - (u')(v_2) + (u'')(v_3) - (u''')(v_4) + \cdots}$$

$$= \frac{1\times1\times1}{5!}\Big[t^5(-\cos t) - (5t^4)(-\sin t) + (20t^3)(\cos t) - (60t^2)(\sin t) + (120t)(-\cos t) - (120)(-\sin t)\Big]_0^{\pi/2}$$

$$\because \sin\frac{\pi}{2} = 1$$
$$\cos\frac{\pi}{2} = 0$$

$$= \frac{1}{5!}\left[0 - 5\left(\frac{\pi}{2}\right)^4(-1) + 0 - 60\left(\frac{\pi}{2}\right)^2(1) + 0 - (120)(-1)\right]$$

$$= \frac{1}{5!}\left[\frac{5\pi^4}{16} - \frac{60\pi^2}{4} + 120\right]$$

$$= \frac{1}{5!}\left[\frac{5\pi^4}{16} - 15\pi^2 + 120\right]$$

Ans

Ques:- Evaluate $\iiint_V e^{-(x+y+z)}\,dx\,dy\,dz$ where the region of integration is bounded by the planes $x=0,\ y=0,\ z=0$ & $x+y+z=a,\ a>0$

Sol^n:- Here $0 \le x+y+z \le a$

So

$$\iiint e^{-(x+y+z)}\,dx\,dy\,dz = \iiint x^{1-1}y^{1-1}z^{1-1}e^{-(x+y+z)}\,dx\,dy\,dz$$

$$= \frac{\Gamma 1\,\Gamma 1\,\Gamma 1}{\Gamma 1+1+1}\int_0^a e^{-t}\,t^{1+1+1-1}\,dt$$

[By Liouville's Extension]

$$= \frac{1\times 1\times 1}{2!}\int_0^a \underset{\downarrow}{t^2}\,\underset{\downarrow}{e^{-t}}\,dt$$
$$\qquad\qquad\quad u \quad\ v$$

$$\boxed{\int uv\,dx = u(v_1) - (u')(v_2) + (u'')(v_3) - (u''')(v_4) + \cdots}$$

$$= \frac{1}{2!}\left[t^2(-e^{-t}) - (2t)(e^{-t}) + (2)(-e^{-t})\right]_0^a$$

$$= \frac{1}{2!}\left[-t^2e^{-t} - 2te^{-t} - 2e^{-t}\right]_0^a$$

$$= \frac{1}{2}\left[\left\{-a^2e^{-a} - 2ae^{-a} - 2e^{-a}\right\} - \left\{0-0-2e^0\right\}\right]$$

$$= \frac{1}{2}\left[-a^2e^{-a} - 2ae^{-a} - 2e^{-a} + 2\right]$$

$$= \underbrace{-\frac{a^2}{2}e^{-a} - ae^{-a} - e^{-a}}+1$$

$$= 1 - e^{-a}\left[\frac{a^2}{2} + a + 1\right]$$ Ans

Ques: Evaluate $\iiint \log(x+y+z)\,dx\,dy\,dz$, the integral extending over all positive and zero values of x, y, z subject to $x+y+z<1$.

Solⁿ:- For all positive and zero values of x, y, z

$$0 \leq x+y+z < 1$$

$\therefore$

$$\iiint \log(x+y+z)\,dx\,dy\,dz = \iiint x^{1-1}y^{1-1}z^{1-1}\log(x+y+z)\,dx\,dy\,dz$$

$$= \frac{\Gamma 1\,\Gamma 1\,\Gamma 1}{\Gamma 1+1+1}\int_0^1 \log t \cdot t^{1+1+1-1}\,dt$$

(By Liouville's Extension)

$$= \frac{\Gamma 1\,\Gamma 1\,\Gamma 1}{\Gamma 3}\int_0^1 \underset{\text{II}}{t^2}\,\underset{\text{I}}{\log t}\,dt$$

By Parts

$$\int \text{I}\cdot\text{II}\,dx = \text{I}\int \text{II}\,dx - \int\left\{\frac{d}{dx}(\text{I})\cdot\int(\text{II})\,dx\right\}dx$$

$$= \frac{1\times1\times1}{2!}\left[\log t\left(\frac{t^3}{3}\right) - \int \frac{1}{t}\cdot\frac{t^3}{3}\,dt\right]_0^1$$

$$= \frac{1}{2}\left[\frac{t^3}{3}\log t - \frac{1}{3}\int t^2\,dt\right]_0^1$$

$$= \frac{1}{2}\left[\frac{t^3}{3}\log t - \frac{1}{3}\left(\frac{t^3}{3}\right)\right]_0^1$$

$$= \frac{1}{2}\left[\left\{0-\frac{1}{9}\right\} - \{0\}\right]$$

$$= -\frac{1}{18}$$ Ans.

M. Imp

Ques :- 2015 Prove that $\iiint \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}} = \frac{\pi^2}{8}$, the integral being extended to all positive values of the variables for which the expression is real.

Sol^n :- The expression will be real if

$$1-x^2-y^2-z^2>0$$

or $$x^2+y^2+z^2<1$$

So the given integral is extended to all positive values of the variables such that

$$0<x^2+y^2+z^2<1$$

Put $x^2=u \Rightarrow \boxed{x=u^{\frac{1}{2}}}$ so $\boxed{dx=\frac{1}{2}u^{\frac{1}{2}-1}\,du}$

$y^2=v \Rightarrow \boxed{y=v^{\frac{1}{2}}}$ so $\boxed{dy=\frac{1}{2}v^{\frac{1}{2}-1}\,dv}$

$z^2=w \Rightarrow \boxed{z=w^{\frac{1}{2}}}$ so $\boxed{dz=\frac{1}{2}w^{\frac{1}{2}-1}\,dw}$.

Then $0<u+v+w<1$

$$\therefore \iiint \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}} = \iiint \frac{1}{\sqrt{1-u-v-w}}\left\{\frac{1}{2}u^{\frac{1}{2}-1}du\right\}\left\{\frac{1}{2}v^{\frac{1}{2}-1}dv\right\}\left\{\frac{1}{2}w^{\frac{1}{2}-1}dw\right\}$$

$$= \frac{1}{8}\iiint u^{\frac{1}{2}-1}v^{\frac{1}{2}-1}w^{\frac{1}{2}-1}\frac{1}{\sqrt{1-(u+v+w)}}\,du\,dv\,dw$$

$$= \frac{1}{8}\frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)}\int_0^1 \frac{1}{\sqrt{1-t}}\,t^{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}-1}\,dt$$

[By Liouville's extension]

$$= \frac{1}{8} \frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma\frac{3}{2}} \int_0^1 \frac{1}{\sqrt{1-t}}\, t^{\frac{3}{2}-1}\, dt$$

$$= \frac{1}{8} \frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\frac{1}{2}\Gamma\frac{1}{2}} \int_0^1 \frac{1}{\sqrt{1-t}}\, t^{\frac{1}{2}}\, dt$$

$$= \frac{1}{8} \frac{\sqrt{\pi}\times\sqrt{\pi}}{\left(\frac{1}{2}\right)} \int_0^1 \frac{1}{\sqrt{1-t}}\, t^{\frac{1}{2}}\, dt$$

Put $t = \sin^2\theta$

$dt = 2\sin\theta\cos\theta\, d\theta$

$$= \frac{2}{8}\pi \int_0^{\pi/2} \frac{1}{\sqrt{1-\sin^2\theta}} (\sin^2\theta)^{\frac{1}{2}} \{2\sin\theta\cos\theta\, d\theta)$$

$$= \frac{\pi}{4} \int_0^{\pi/2} \frac{1}{\cancel{\sqrt{\cos^2\theta}}} \sin\theta \{2\sin\theta \cancel{\cos\theta}\, d\theta\}$$

$$= \frac{\pi}{4} \int_0^{\pi/2} 2\sin^2\theta\, d\theta$$

$$= \frac{\pi}{2} \int_0^{\pi/2} \sin^2\theta \cos^0\theta\, d\theta$$ [Note this point]

$$= \frac{\pi}{2} \frac{\Gamma\frac{2+1}{2}\,\Gamma\frac{0+1}{2}}{2\,\Gamma\frac{2+0+2}{2}} \quad \left[\because \int_0^{\pi/2} \sin^m\theta\cos^n\theta\, d\theta = \frac{\Gamma\frac{m+1}{2}\,\Gamma\frac{n+1}{2}}{2\,\Gamma\frac{m+n+2}{2}}\right]$$

$$= \frac{\pi}{2} \frac{\Gamma\frac{3}{2}\,\Gamma\frac{1}{2}}{2\,\Gamma 2} = \frac{\pi}{2} \frac{\{\frac{1}{2}\Gamma\frac{1}{2}\}\,\Gamma\frac{1}{2}}{2\times 1}$$

$$= \frac{\pi}{8} \frac{\sqrt{\pi}\times\sqrt{\pi}}{1} = \frac{\pi^2}{8}$$

$$\Rightarrow \boxed{\iiint \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}} = \frac{\pi^2}{8}}$$ Ans.

Qus:- Show that $\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3} = \frac{1}{2}\log 2 - \frac{5}{16}$, the integral being taken throughout the volume bounded by planes $x=0$, $y=0$, $z=0$ and $x+y+z=1$

Soln:- Here $0 \le x+y+z \le 1$

So

$$\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3} = \iiint x^{1-1} y^{1-1} z^{1-1} \frac{1}{(x+y+z+1)^3}\,dx\,dy\,dz$$

$$= \frac{\Gamma 1\,\Gamma 1\,\Gamma 1}{\Gamma 1+1+1} \int_0^1 \frac{1}{(t+1)^3} t^{1+1+1-1}\,dt$$

$$= \frac{1\times1\times1}{2!} \int_0^1 \frac{t^2}{(t+1)^3}\,dt$$

Put $t+1 = u$

$dt = du$

$$= \frac{1}{2}\int_1^2 \frac{(u-1)^2}{u^3}\,du \qquad \left[\begin{array}{l} \because t+1=u \\ \therefore t = u-1 \end{array}\right]$$

$$= \frac{1}{2}\int_1^2 \frac{u^2+1-2u}{u^3}\,du$$

$$= \frac{1}{2}\int_1^2 \left(\frac{1}{u} + \frac{1}{u^3} - \frac{2}{u^2}\right)du$$

$$= \frac{1}{2} \int_1^2 \left(\frac{1}{u} + u^{-3} - 2u^{-2} \right) du$$

$$= \frac{1}{2} \left[\log u + \left(\frac{u^{-2}}{-2} \right) - 2 \left(\frac{u^{-1}}{-1} \right) \right]_1^2$$

$$= \frac{1}{2} \left[\log u - \frac{1}{2u^2} + \frac{2}{u} \right]_1^2$$

$$= \frac{1}{2} \left[\left\{ \log 2 - \frac{1}{8} + 1 \right\} - \left\{ \log 1 - \frac{1}{2} + 2 \right\} \right]$$

$$= \frac{1}{2} \left[\log 2 - \frac{1}{8} + 1 + \frac{1}{2} - 2 \right] \quad \because \log 1 = 0$$

$$= \frac{1}{2} \left[\log 2 + \left(\frac{1}{2} - \frac{1}{8} - 1 \right) \right]$$

$$= \frac{1}{2} \left[\log 2 + \left[\frac{4 - 1 - 8}{8} \right] \right]$$

$$= \frac{1}{2} \left[\log 2 - \frac{5}{8} \right]$$

$$= \frac{1}{2} \log 2 - \frac{5}{16}$$

<u>Ans</u>