Solutions to problem set - 1

Introduction to quantum computing using QSim

Problem 1: Polarizers, Photons and single qubits

The following questions are based on a single qubit representing the polarization state of a photon. The basis states are represented by $|0\rangle$ for vertically polarized light and $|1\rangle$ for horizontally polarized light respectively.

a) What is the angle made by the following polarization states with respect to the vertical axis?

i. $\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

ii. $\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$

Solution: The state of a photon with a polarization oriented at an angle θ to the vertical is given by:

$$|\psi\rangle = \cos\theta\ |0\rangle + \sin\theta\ |1\rangle$$

(explained in slide 11, Week 1 Session 2-2.)

Using the above relations the answers may be obtained as follows:

i. $\cos\theta = \sin\theta = \frac{1}{\sqrt{2}}$, this implies that the value of θ is 45° or $\frac{\pi}{4}$ radian.

ii. $\cos\theta = \frac{\sqrt{3}}{2}$; $\sin\theta = \frac{1}{2}$, this implies that the value of θ is 30° or $\frac{\pi}{6}$ radian.

b) The polarization state of a photon is given by $\frac{4}{5}|0\rangle + x|1\rangle$, what is the allowed value(s) of x?

Solution: A given vector $|\psi\rangle = a|0\rangle + b|1\rangle$ is a valid qubit state if and only if it obey the following relation:

$$\langle\psi|\psi\rangle = |a|^2 + |b|^2 = 1$$

(explained in slide 7, Week 1 Session 2-2 and many other places.)

This implies that the vector given in the question must obey the following relation:

$$\left|\frac{4}{5}\right|^2 + |x|^2 = 1$$

This means $|x|^2 = \frac{9}{25}$; at this point it should be remembered that **qubit states are complex vectors**. Therefore the allowed values x are given by:

$$x = e^{i\phi} \cdot \frac{3}{5} \;:\text{ where, } \phi \in \mathbb{R} \text{ and } i^2 = -1$$

It should be noted there are infinitely many allowed values of x. Using this, the allowed qubit states may be represented as:

$$|\psi\rangle = \frac{4}{5}\;|0\rangle + e^{i\phi} \cdot \frac{3}{5}\;|0\rangle$$

All values of ϕ give a qubit state with a different *relative phase.*

On a side note, all of the above states can be represented on the Bloch sphere by a circle. These states have been illustrated in Figure 1 as added trivia.

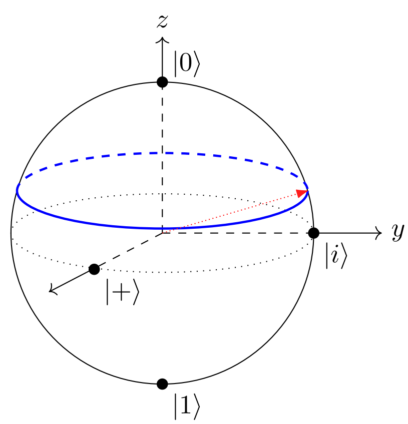

Figure 1: The allowed states from 1 b) lie on the blue circle, the red line makes an angle of 77.32° with the z-axis. Some other qubit states are also marked for reference.

c) Consider photons of an unknown polarization passing through a polarizer **A**, oriented along the vertical direction. The photons that are transmitted through **A** subsequently pass through a polarizer **B** oriented along the horizontal direction.

There is a minor omission in part ii. *this has been corrected here in red.*

i. How many photons will be transmitted through polarizer **B**?

ii. If a third polarizer **C** is inserted between **A** and **B**. What is the angle the orientation of C should make with the vertical axis such that the probability of photons being transmitted through polarizer **B** is highest.

iii. In the previous problem, maximum possible transmission probability for the three polarizer system?

Solution: The answers to these question are based on the ideas discussed in slides 12-13 of Week 1 Session 1-2.

i. Every photon transmitted through polarizer **A** is oriented at the same angle as polarizer **A**. This means that every photon transmitted through **A** is in the state $|0\rangle$. This implies that *none of the photons that were transmitted through **A** will be transmitted through polarizer **B***. The answer is therefore **zero**.

ii. Let us assume that the polarizer **C** is oriented at an angle α to the vertical.

This implies that the probability that a photon transmitted through **A** is also transmitted through **C** is given by $|\cos\alpha|^2$ as photons transmitted through **A** are vertically polarized.

Once again these photons will have a polarization oriented along the direction of polarizer **C**, an angle α to the vertical.

Now, the angle between the horizontally oriented **B** and the polarization of photons transmitted through **C** is given by $90^\circ - \alpha$. Therefore the probability that these photons are transmitted through polarizer **B** is given by:

$$|\cos(90^\circ - \alpha)|^2 = |\sin\alpha|^2$$

First, it is worth noting that eventhough the transmission probability for the two polarizer system was zero in part i. the introduction of a third polarizer has changed the transmission probability even though the orientation of **A** and **B** have been left unchanged.

The total transmission probability is given by the product of the two transmission probability values. This value is given by

$$\begin{aligned}
|\cos\alpha|^2 \cdot |\sin\alpha|^2 &= |\cos\alpha \cdot \sin\alpha|^2 \\
&= \left|\frac{\sin 2\alpha}{2}\right|^2 \\
&= \frac{1}{4}|\sin 2\alpha|^2
\end{aligned}$$

The highest value the above quantity can have is when $\sin 2\alpha$ is 1. The value of α can be found as"

$$\begin{aligned}
&\sin 2\alpha = 1 \\
\Rightarrow \quad &2\alpha = 90^\circ \\
\therefore \quad &\alpha = 45^\circ
\end{aligned}$$

Therefore, if the polarizer **C** is oriented along 45° to the vertical then the transmission probability of the three polarizer is the maximum.

iii. By observing the previous part, the maximum transmission probability is given by $\frac{1}{4}$.

Problem 2: Unitary transformations and orthonormal bases

The following questions are based on the properties of single qubit unitary transformations.

a) Given a single qubit unitary transformation U, such that

$$U|0\rangle = |a\rangle$$
$$U|1\rangle = |b\rangle$$

Show that $\{|a\rangle, |b\rangle\}$ is an orthonormal basis.

Solution: The important relation required for solving this problem, can be found in the definition of $\langle a|$ which is given in the following equation:

$$\begin{aligned} \langle a| &= |a\rangle^\dagger \\ &= (U|0\rangle)^\dagger \\ &= |0\rangle^\dagger U^\dagger \\ \therefore \quad \langle a| &= \langle 0| U^\dagger \end{aligned} \tag{1}$$

(This was also covered in slide 9, Week 4 Session 2. Refer to the corresponding section in the video for further explanations.)

Similarly, $\langle b| = \langle 1| U^\dagger$, both of these relations follow from the rules of matrix algebra. These rules will initially be used for verifying orthonormality.

Firstly. calculating the quantity $\langle a|a\rangle$ can be calculated as follows:

$$\begin{aligned} \langle a|a\rangle &= \langle 0| U^\dagger U |0\rangle \ \ ; \text{from (1)} \\ &= \langle 0|0\rangle \ \ ; U^\dagger U = I \text{ as } U \text{ is unitary} \\ &= 1 \ ; \{|0\rangle, |1\rangle\} \text{ is an orthonormal basis} \end{aligned}$$

Similarly it can be shown that $\langle b|b\rangle = \langle 1|1\rangle = 1$. The final step in proving $\{|a\rangle, |b\rangle\}$ is an orthonormal basis is evaluating $\langle a|b\rangle$, this can be evaluated as:

$$\begin{aligned} \langle a|b\rangle &= \langle 0| U^\dagger U |1\rangle \ \ ; \text{from (1)} \\ &= \langle 0|1\rangle \ \ ; U^\dagger U = I \text{ as } U \text{ is unitary} \\ &= 0 \ ; \{|0\rangle, |1\rangle\} \text{ is an orthonormal basis} \end{aligned}$$

This result also implies that $\langle b|a\rangle = \langle 1|0\rangle = 0$. It can therefore be seen that $\{|a\rangle, |b\rangle\}$ is indeed an orthonormal basis.

It should be stated here that, "*The action of of a unitary transformation any given orthonormal basis to another orthonormal basis.*", this result is also true for multi-qubit systems.

It is also possible to represent any state vector $|\psi\rangle$ in the $\{|a\rangle, |b\rangle\}$ basis in the following manner:

$$|\psi\rangle = \langle a|\psi\rangle |a\rangle + \langle b|\psi\rangle |b\rangle$$

(This is similar to the result found in slides 9-10, Week 1 Session 2-2.)

b) If a qubit in the state $|\psi\rangle$ is measured in the $\{|a\rangle, |b\rangle\}$ basis, what are the possible measurement outcomes and what are the corresponding probabilities?

Solution: When a qubit in state $|\psi\rangle$ is measured with respect to a basis $\{|a\rangle, |b\rangle\}$, *the final state of the qubit is either $|a\rangle$ or $|b\rangle$.* The probabilities of these outcomes are related to the inner product of $|\psi\rangle$ with the elements of the basis. the probability values are given by:

$$\Pr\{\text{final state} = |a\rangle\} = |\langle a|\psi\rangle|^2$$
$$\Pr\{\text{final state} = |b\rangle\} = |\langle b|\psi\rangle|^2$$

c) If a qubit in the state $U^\dagger |\psi\rangle$ is measured in the standard basis, what are the outcomes and the corresponding probabilities?

Solution: The answer to this part is straightforward; the final state after measurement in the standard basis will either be $|0\rangle$ or $|1\rangle$ and the probabilities of these outcomes are given by:

$$\Pr\{\text{final state} = |0\rangle\} = \left|\langle 0|U^\dagger|\psi\rangle\right|^2$$
$$\Pr\{\text{final state} = |1\rangle\} = \left|\langle 1|U^\dagger|\psi\rangle\right|^2$$

d) Are the answers for parts b) and c) related? If so, how are they related?

Solution: Using the result from (1) on the answer to part c), it can be seen that the probability values are exactly equal to that from part b).

The aim of this problem is to show how to perform measurements in a non-standard basis using standard basis measurements. In order to measure a qubit in state $|\psi\rangle$ with respect to the $\{|a\rangle, |b\rangle\}$ basis using standard basis measurements, one may do the following:

i. First, identify the transformation U that generates $\{|a\rangle, |b\rangle\}$ from the standard basis.

ii. Apply the transposed-conjugate of that transformation $U^\dagger$ on the qubit to transform it to state $U^\dagger |\psi\rangle$.

iii. Perform a standard basis measurement, treat the measurement outcome $|0\rangle$ as equivalent to $|a\rangle$ and the measurement outcome $|1\rangle$ as equivalent to $|b\rangle$

Once again these results can be generalised to multi-qubit states with appropriate extensions.

e) If a qubit in the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, is measured in the Hadamard basis. What are the possible measurement outcomes and their corresponding probabilities?

Solution: This problem can be solved by using the steps discussed in the previous part:

i. The unitary transformation that generates the Hadamard basis from the standard basis is the Hadamard gate H.

ii. $H^\dagger = H$, therefore H is applied on $|\psi\rangle$, this yields the resultant state:

$$\begin{aligned} H|\psi\rangle &= \alpha H|0\rangle + \beta H|1\rangle \\ &= \alpha|+\rangle + \beta|-\rangle \\ &= \alpha\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) + \beta\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \\ &= \frac{\alpha+\beta}{\sqrt{2}}|0\rangle + \frac{\alpha-\beta}{\sqrt{2}}|1\rangle \end{aligned}$$

iii. Performing a standard basis measurement and treating the measurement outcome $|0\rangle$ as equivalent to $|+\rangle$ and the measurement outcome $|1\rangle$ as equivalent to $|-\rangle$ gives the probabilities to be:

$$\begin{aligned} \Pr\{\text{final state} = |+\rangle\} &= \left|\frac{\alpha+\beta}{\sqrt{2}}\right|^2 \\ \Pr\{\text{final state} = |-\rangle\} &= \left|\frac{\alpha-\beta}{\sqrt{2}}\right|^2 \end{aligned}$$

Problem 3: The Bell basis

Consider the following quantum circuit:

Both $|q_0\rangle$ and $|q_1\rangle$ can either be $|0\rangle$ or $|1\rangle$. The measurement is done in the computational basis and the measurement outcome is stored in the classical bits c_0 and c_1

a) Find the output state (before measurement) of the above quantum circuit for the input combinations:

i. $|q_0\rangle = |0\rangle$ and $|q_1\rangle = |0\rangle$

ii. $|q_0\rangle = |1\rangle$ and $|q_1\rangle = |0\rangle$

iii. $|q_0\rangle = |0\rangle$ and $|q_1\rangle = |1\rangle$

iv. $|q_0\rangle = |1\rangle$ and $|q_1\rangle = |1\rangle$

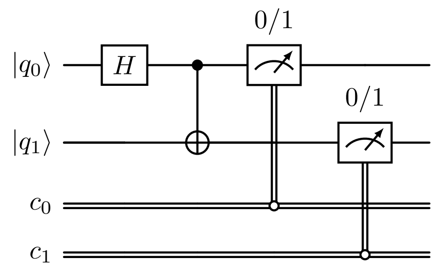

Figure 2: Circuit diagram for Problem 3

Solution: It is useful to remember the action of CNOT_1^0 on the two-qubit computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.

$$\begin{aligned}\mathrm{CNOT}_1^0 |00\rangle &= |00\rangle \\ \mathrm{CNOT}_1^0 |01\rangle &= |01\rangle \\ \mathrm{CNOT}_1^0 |10\rangle &= |11\rangle \\ \mathrm{CNOT}_1^0 |11\rangle &= |10\rangle\end{aligned}$$

(Explained in slide 8, Week 2 Session 2-1.)

The transformation applied by the circuit (without the measurements) can be represented as: ($\circ$ represents matrix multiplication)

$$\mathrm{CNOT}_1^0 \circ (\mathrm{H} \otimes \mathrm{I}) |\mathrm{q}_0\rangle |\mathrm{q}_1\rangle$$

(This was discussed in slide 13, Week 2 Session 2-3.)

The answers to the questions are as follows:

i. $|q_0\rangle |q_1\rangle = |0\rangle |0\rangle$

$$|0\rangle |0\rangle \xrightarrow{H \otimes I} |+\rangle |0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$
$$\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \xrightarrow{\mathrm{CNOT}_1^0} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

therefore, the answer is: $|B_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

ii. $|q_0\rangle |q_1\rangle = |1\rangle |0\rangle$

$$|1\rangle |0\rangle \xrightarrow{H \otimes I} |-\rangle |0\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |10\rangle)$$
$$\frac{1}{\sqrt{2}}(|00\rangle - |10\rangle) \xrightarrow{\mathrm{CNOT}_1^0} \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

therefore, the answer is: $|B_{01}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$.

iii. $|q_0\rangle |q_1\rangle = |0\rangle |1\rangle$

$$|0\rangle |1\rangle \xrightarrow{H \otimes I} |+\rangle |1\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |11\rangle)$$
$$\frac{1}{\sqrt{2}}(|01\rangle + |11\rangle) \xrightarrow{\text{CNOT}_1^0} \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

therefore, the answer is: $|B_{10}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$.

iv. $|q_0\rangle |q_1\rangle = |1\rangle |1\rangle$

$$|1\rangle |1\rangle \xrightarrow{H \otimes I} |-\rangle |1\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |11\rangle)$$
$$\frac{1}{\sqrt{2}}(|01\rangle - |11\rangle) \xrightarrow{\text{CNOT}_1^0} \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

therefore, the answer is: $|B_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$.

It should be noted here that the four parts of this question correspond to the elements of the two-qubit standard basis. Additionally, each of the resultant vectors are entangled.

b) What are the possible measurement outcomes (c_0c_1 values) when the initial states are $|q_0\rangle = |0\rangle$ and $|q_1\rangle = |1\rangle$?

Solution: The state being measured (in the two-qubit standard basis) is

$$|B_{10}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

. Since this is an entangled state, the measurement outcomes (c_0c_1) are 01 or 10.

c) Show that the four vectors obtained from part a) form a two-qubit basis.

Solution: The four vectors obtained from part a) are obtained by applying the unitary transform $\text{CNOT}_1^0 \circ (\text{H} \otimes \text{I})$ on each element of the two-qubit computational basis.

By the same reasoning given in the solution to 2 a), these for vectors form a two-qubit basis.

This basis is referred to as the Bell basis.

d) If two-qubits in the state $|\Psi\rangle = |+\rangle |-\rangle$ is measured with the two qubit basis defined here, What are the possible measurement outcomes and their corresponding probabilities?

Solution: It should be remembered that measurement is done with respect to the Bell basis: $\{|B_{00}\rangle, |B_{01}\rangle, |B_{10}\rangle, |B_{11}\rangle\}$.

$|\Psi\rangle$ can be expanded and represented in the Bell basis as follows:

$$
\begin{aligned}
|\Psi\rangle &= |+\rangle\,|-\rangle \\
&= \frac{1}{2}(|0\rangle + |1\rangle) \otimes (|0\rangle - |1\rangle) \\
&= \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle) \\
&= \frac{1}{\sqrt{2}}\left(\frac{|00\rangle - |11\rangle}{\sqrt{2}} - \frac{|01\rangle - |10\rangle}{\sqrt{2}}\right) \\
&= \frac{1}{\sqrt{2}}(|B_{01}\rangle - |B_{11}\rangle)
\end{aligned}
$$

The possible measurement outcomes are $|B_{01}\rangle$ and $|B_{11}\rangle$ and the probability of each outcome is 0.5.

e) If two-qubits in the state $|\Phi\rangle = |0\rangle\,|0\rangle$ is measured with respect to the new basis. Answer the following questions with explanations.

   i. Is the initial state entangled?

   ii. Is the state of the system after measurement entangled?

Solution: It can be seen that the initial state $|\Phi\rangle = |00\rangle$ has the following representation in the Bell basis:

$$|00\rangle = \frac{1}{\sqrt{2}}(|B_{00}\rangle + |B_{01}\rangle)$$

   i. The initial state is separable. So the answer is **no**.

   ii. After measurement with respect to the Bell basis, the possible outcomes are $|B_{00}\rangle$ and $|B_{01}\rangle$. Both of these states are entangled, therefore the answer to this question is **yes**.

Problem 4: Multi-qubit entanglement

In the following multi-qubit states, identify which qubits are entangled to one another:

a) $\frac{1}{\sqrt{2}}\,|010\rangle + \frac{1}{\sqrt{2}}\,|111\rangle$

b) $\frac{1}{\sqrt{2}}\,|000\rangle + \frac{1}{\sqrt{2}}\,|111\rangle$

c) $\frac{1}{\sqrt{3}}\,|001\rangle + \frac{1}{\sqrt{3}}\,|010\rangle + \frac{1}{\sqrt{3}}\,|100\rangle$

d) $\frac{1}{2}\,|0000\rangle + \frac{1}{2}\,|0011\rangle + \frac{1}{2}\,|1100\rangle + \frac{1}{2}\,|1111\rangle$

e) $\frac{1}{2}\,|0000\rangle + \frac{1}{2}\,|0101\rangle + \frac{1}{2}\,|1010\rangle + \frac{1}{2}\,|1111\rangle$

Solution: The answer to these questions can be easily represented after setting up the following notation:

- The ordering of the qubits is a multi-qubit state is always fixed.
- In case the order in being changed, the qubit indices are reintroduced in order to keep track of the qubit ordering.
- Consider the following example involving three-qubit states.

$$|000\rangle \equiv |0\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3$$

this is the default ordering for the three-qubit states. Now, in the following example the rule for separating qubits out of order is shown.

$$\frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{2}}|010\rangle \equiv |00\rangle_{13} \otimes \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)_2$$

The final state still has the same ordering as the previous example and this is indicated through the presence of the indices.

a)
$$\frac{1}{\sqrt{2}}|010\rangle + \frac{1}{\sqrt{2}}|111\rangle = \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)_{13} \otimes |1\rangle_2$$

It can now be seen that qubits 1 and 3 form an entangled state and qubit 2 is separable form the other two qubits.

Note: In a multi- qubit. a particular qubit (or a subset of qubits) are separable if and only if it(they) are in the same state across all components of the multi-qubit state.

b)
$$\frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{2}}|111\rangle$$

It is impossible to factor out any single qubit from this state, therefore all three qubits are entangled to each other.

c)
$$\frac{1}{\sqrt{3}}|001\rangle + \frac{1}{\sqrt{3}}|010\rangle + \frac{1}{\sqrt{3}}|100\rangle$$

Factorization is not possible as none of the three qubits are in the same state across all components. Therefore, all three qubits are entangled to each other.

d) the initial state is $|\Psi\rangle = \frac{1}{2}|0000\rangle + \frac{1}{2}|0011\rangle + \frac{1}{2}|1100\rangle + \frac{1}{2}|1111\rangle$. The qubit states may be factorized as follows:

$$
\begin{aligned}
|\Psi\rangle &= \frac{|00\rangle_{12}}{\sqrt{2}} \otimes \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)_{34} + \frac{|11\rangle_{12}}{\sqrt{2}} \otimes \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)_{34} \\
&= \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)_{12} \otimes \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)_{34}
\end{aligned}
$$

Therefore, the qubits 1 & 2 are entangled and the qubits 3 & 4 are entangled.

e) the initial state is $|\Psi\rangle = \frac{1}{2}|0000\rangle + \frac{1}{2}|0101\rangle + \frac{1}{2}|1010\rangle + \frac{1}{2}|1111\rangle$. This qubit state may be factorized as follows:

$$
\begin{aligned}
|\Psi\rangle &= \frac{|00\rangle_{13}}{\sqrt{2}} \otimes \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)_{24} + \frac{|11\rangle_{13}}{\sqrt{2}} \otimes \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)_{24} \\
&= \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)_{13} \otimes \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)_{24}
\end{aligned}
$$

Therefore, the qubits 1 & 3 are entangled and the qubits 2 & 4 are entangled.

Observe that in order to find which qubits are entangled to each other, the coordinates of the multi-qubit states were never considered. It is possible to find out if a given multi-qubit state is entangled by just looking at the basis elements present in the state vector.