1 LT
1 extra

31st January, 2024

Unit 2 Linear Independence basis and dimension

q: Check whether $\begin{bmatrix}1\\2\\1\\3\end{bmatrix}, \begin{bmatrix}1\\1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\3\\7\\1\end{bmatrix}$ are independent in R^4

Sol.

$$\begin{array}{c} \begin{matrix} V_1 & V_2 & V_3 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 7 \\ 3 & 3 & 1 \end{bmatrix} \end{array} \simeq \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - R_1$

$R_4 \to R_4 - 3R_1$

$R_3 \to R_3 - \left(\frac{1}{-1}\right) R_2$

$$\simeq \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{bmatrix} \simeq \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_4 \to R_4 - \left(\frac{-2}{7}\right) R_3$

$\rho(A) = 3 =$ no. of columns

$\therefore V_1\ V_2\ V_3$ are linearly independent

q. $\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}3\\1\\0\end{bmatrix}, \begin{bmatrix}4\\3\\1\end{bmatrix}$

Sol.

$$\begin{array}{c} \begin{matrix} V_1 & V_2 & V_3 \end{matrix} \\ \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix} \end{array} \simeq \begin{bmatrix} 1 & 3 & 4 \\ 0 & -5 & -5 \\ 0 & -3 & -3 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - R_1$

$R_3 \to R_3 - \left(\frac{-3}{-5}\right) R_2$

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 5 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$
$$\quad v_1 \quad v_2 \quad v_3$$

$\rho(A) = 2 \neq$ no. of columns

$\{v_1, v_2, v_3\}$ are not linearly independent in R^3
$\{v_1, v_2\}$ are linearly independent in R^3

Note : 1. The columns of a square invertible matrix are always independent.

2. The columns of a matrix A of order $m \times n$ with $m < n$ are always dependent.

3. The columns of A are independent exactly when $N(A) = Z$ (Z means 0 vector)
no of free var

4. The 'r' nonzero rows of an echelon matrix U and reduced matrix R are always independent and so are the 'r' columns that contains the pivots.

→ Read Span first → then basis

Span of a Set

Let $W = \{v_1, v_2, v_3 \ldots v_n\}$ be a set of vectors belonging to the vector space V, then span of W is the set of all linear combinations of vectors of W.

ie span of W = span(W) $= c_1v_1 + c_2v_2 + \cdots +$
$=$ subspace of V

q. What do these vectors $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}, \begin{bmatrix}3\\5\end{bmatrix}$ span

Sol. The given vectors span a 2D subspaces of R^2.

$$A = \begin{bmatrix}1 & 2 & 3\\2 & 4 & 5\end{bmatrix} \underset{R_2 \to R_2 - 2R_1}{\simeq} \begin{array}{ccc} v_1 & v_2 & v_3 \\ \end{array} \begin{bmatrix}1 & 2 & 3\\0 & 0 & -1\end{bmatrix}$$

$\{v_1\ v_3\}$ are L.I
$\{v_1\ v_2\ v_3\}$ are not LI

2-D subspaces of R^2 span whole of R^2

q. $\begin{array}{ccc} v_1 & v_2 & v_3 \end{array}$ $\begin{bmatrix}1 & 3 & 4\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$ 1D subspace (line) in R^3

q. $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ 2D subspaces of R^2
whole of R^2

q. $\begin{bmatrix}1 & 2 & 3\\0 & 1 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$ 2D subspaces of R^4

* for span the vectors need not be independent.

Basis

A vector $S = \{v_1, v_2, v_3 \dots v_n\}$ of a vector space is called a basis for vector space V if

i) S is linearly independent set.

ii) S spans the vector space V

* The dimensions of a vector space is the no. of basis vector.

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Note: 1. The vectors are linearly independent (not too many vectors)

2. They span the space V (not too few vectors)

3. Every vector in the space is a combination of the basis vectors because they span combination is unique

4. If $\{v_1, v_2, v_3 \dots v_n\}$ and $\{w_1, w_2, w_3 \dots w_n\}$ are both the bases for the same vector space then $m = n$, the no. of vectors is same

5. Basis is maximal independent set and minimal spanning set.

6. Any linearly independent set in V can be extended to a basis, by adding more vectors if necessary, any by spanning set in V can be reduced to a basis,

by discarding vectors if necessary.

7. There exists one and only one way to write any vector 'v' in vector space V as a combination of the bases vectors of the vector space.

→ Problems on Linear Dependence and Independence

q1. Decide the dependence/independence of

Not for isa

i) vectors (1, 3, 2), (2, 1, 3), (3, 2, 1)

Sol. Write the given set of vectors as columns of matrix A

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 3R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & -1 & -5 \end{bmatrix}$$

$R_3 \to R_3 - \left(\frac{1}{5}\right) R_2$

$$\simeq \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & 0 & -18/5 \end{bmatrix}$$

∵ All the columns of matrix A is with pivots

$R(A) = n = 3$

given vectors are linearly independent.

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q2: $(1, -3, 2)$, $(2, 1, -3)$, $(-3, 2, 1)$

Sol. $A = \begin{bmatrix} 1 & 2 & -3 \\ -3 & 1 & 2 \\ 2 & -3 & 1 \end{bmatrix}$ $\quad R_2 \to R_2 - (-3)R_1$, $R_3 \to R_3 - 2R_1$

$\simeq \begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & -7 & 7 \end{bmatrix}$ $\quad R_3 \to R_3 - (-1)R_2$

$\simeq \begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \end{bmatrix}$

$\exists$ columns without pivots also in A
given vectors are linearly dependent.

q3: Find the condition for 'a' for which the given vectors V_1, V_2, V_3 are linearly independent

ISC.

$V_1 = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$ $\quad V_2 = \begin{pmatrix} 2 \\ a \\ 3 \end{pmatrix}$ $\quad V_3 = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$

Sol. $A = \begin{array}{c} \begin{matrix} V_1 & V_2 & V_3 \end{matrix} \\ \begin{bmatrix} 1 & 2 & -3 \\ -3 & a & 2 \\ 2 & 3 & 1 \end{bmatrix} \end{array}$ $\quad R_2 \to R_2 - (-3)R_1$, $R_3 \to R_3 - 2R_1$

$\simeq \begin{bmatrix} 1 & 2 & -3 \\ 0 & a+6 & -7 \\ 0 & -1 & 7 \end{bmatrix}$

$a+6 \neq 0$

$R_3 \to R_3 - \left(\frac{-1}{a+6}\right) R_2$

$$\sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & a+6 & -7 \\ 0 & 0 & \frac{7a+35}{a+6} \end{bmatrix}$$

for V_1, V_2, V_3 to be linearly independent

$$a \neq -6 \quad , \quad \frac{7a+35}{a+6} \neq 0$$

$$a \neq -5$$

Q4. If w_1, w_2, w_3 are independent vectors, show that different vectors $V_1 = w_2 - w_3$, $V_2 = w_1 - w_3$, $V_3 = w_1 - w_2$ are dependent

Sol. Here we can't use vectors so for proof we use definition.

To Prove : V_1, V_2, V_3 are dependent
Given : w_1, w_2, w_3 are independent

By definition $c_1 w_1 + c_2 w_2 + c_3 w_3 = 0$ — (1)
$\Rightarrow c_1 = c_2 = c_3 = 0$

Now consider $\alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 = 0$

$$\Rightarrow \alpha_1 (w_2 - w_3) + \alpha_2 (w_1 - w_3) + \alpha_3 (w_1 - w_2) = 0 \quad \text{— (2)}$$

now to compare (2) with (1)

$$(\alpha_2 + \alpha_3) w_1 + (\alpha_1 - \alpha_3) w_2 + (-\alpha_1 - \alpha_2) w_3 = 0 \quad \text{— (4)}$$

Comparing ① & ④

$$\alpha_2 + \alpha_3 = c_1 = 0 \quad -⑤$$
$$\alpha_1 - \alpha_3 = c_2 = 0 \quad -⑥$$
$$-\alpha_1 - \alpha_2 = c_3 = 0 \quad -⑦$$

using gauss elimination

$$\begin{array}{ccc} \alpha_1 & \alpha_2 & \alpha_3 \end{array}$$
$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 \end{array}\right]$$

$R_2 \leftrightarrow R_1$

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right]$$

$R_3 \to R_3 + R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right] \quad \neq$$

$R_3 \to R_3 - (-1) R_2$

$$\simeq \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rho(A) = \rho(A^{-}|b) = 2 < 3$$

infinite soln.

$r_3 \to$ free variable

$$\alpha_3 = k$$
$$\alpha_2 = -k$$
$$\alpha_1 = k$$

for all values of k, linear combination

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$$
$$\alpha_1 = \alpha_2 = \alpha_3 \neq 0$$

$\therefore$ Linearly Dependent

Q5. Find the basis and hence find the dimension of subspace of R^4

1) All vectors whose components are equal

$\because$ vectors are in R^4 it should contain 4 components

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in R^4 \sim \begin{pmatrix} x \\ x \\ x \\ x \end{pmatrix} = \left\{ x \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} ; x \in R \right\} \text{Subspace}$$

$\therefore$ Basis $(1\ 1\ 1\ 1)^T$; Dimension $=1$

2) All vectors whose components add upto 0

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \quad ST: x+y+z+t=0$$

whenever eqn. $\Rightarrow$ represent 1 var in form of other

Consider
$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} -y-z-t \\ y \\ z \\ t \end{pmatrix} = y\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + z\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{basis} = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

3D subspace of R^4

dimensions : 3

q6. Let V be a subspace of four dimensional space R^4

S.T $V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in R^4 \ ; \ x_1 - x_2 + x_3 - x_4 = 0 \right\}$

Find the Basis and Dimension of V

Sol $x_1 - x_2 + x_3 - x_4 = 0 \Rightarrow x_1 = x_2 - x_3 + x_4$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \sim \begin{bmatrix} x_2 - x_3 - x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\sim x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Any vector in subspace V of R^4 can be obtained as a linear combination of vectors (v_1, v_2, v_3)

Q7. Find a basis for each of the following subspaces of 2 by 2 matrices.

i) All diagonal matrices
ii) All symmetric matrices $(A^T = A)$
iii) All skew-symmetric matrices $(A^T = -A)$

Sol. Any 2×2 matrix is given by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$a, b, c, d \in R$

Basis of 2 by 2 matrix is

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$\because \exists$ 4 vectors in basis set

Dimension of 2x2 matrix subspace is 4

i) $V_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \middle| a, d \in R \right\}$

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$a, d \in R$

Basis $= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

dimension $= 2$

ii) $V_2 = \left\{ \begin{bmatrix} a & c \\ c & d \end{bmatrix} \middle| a, c, d \in R \right\}$

$$\left\{ \begin{bmatrix} a & c \\ c & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \middle| a, c, d \in R \right\}$$

Basis $= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

dimension $= 3$

iii) $V_3^T = -V_3$

$\left\{ V_3 = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} \middle| a \in R \right\}$

$\left\{ \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} = a \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \middle| a \in R \right\}$

Basis $= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ dim $= 1$

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q8. Find a basis for subspace of polynomials $p(x)$ of degree $= 3$

Sol. $\{(1, t, t^2, t^3)\}$ basis
Dimension $= 4$

q9. Describe the subspace of R^3 spanned by
i) two vectors $(1, 1, -1)$ & $(-1, -1, 1)$

Sol. Step 1: Check the independence of given vectors before drawing conclusions wrt subspace

$$\therefore \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 + R_1$

$$\simeq \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\because$ Only 1 column with pivot, these vectors are dependent
Hence, the spance of the given vectors is a line (1D)

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Pivot Variables and Free Variables

eg:
$$\begin{array}{c} \begin{array}{ccccc} x & y & z & w & t \end{array} \\ \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 6 & 7 & 8 \\ 0 & 0 & 0 & 8 & 7 \end{bmatrix} \end{array} = U$$

$Ax = 0$
(or) $Ux = 0$
(or) $Rx = 0$

y & t → free variables

y	1	0
t	0	1

The complete soln. $y\begin{pmatrix} \\ \end{pmatrix} + t\begin{pmatrix} \\ \end{pmatrix}$

q. For every c, find R and special solutions to $Ax = 0$ where

$$A = \begin{bmatrix} 1-c & 0 \\ 0 & 2-c \end{bmatrix}$$

Sol. If $c = 1$ then $A = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$ — pivot

$R_2 \to R_2 - 2R_1$

$$A \sim \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = U$$

$R_1 \to R_1/2$

$$R = \begin{array}{cc} x & y \\ \end{array}$$
$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

first reduce to Echloen form
then reduce to RREF

To get special soln.

$x \to$ free variable

$y \to$ pivot variable

1 special ~~summary~~ soln.

Consider $Rx = 0$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$y = 0$

let $x = 1$

The special soln. is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Complete soln. $\begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Case ii) $c = 2$ in ①

$$A = \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} = U$$

$R_1 \to R_1 / -1$

$$\simeq \begin{array}{c} \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \\ \;\; x \quad\; y \end{array} = R$$

To get special soln. $x \to$ pivot var

$y \to$ free var.

(1 special soln.)

$Rx = 0$

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x - 2y = 0$$

let $y = 1$

$\Rightarrow x = 2$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is a special soln.

Complete soln. $\begin{bmatrix} x \\ y \end{bmatrix} = y \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Case iii) $c \neq 1$ & $c \neq 2$ in ①

$$A = \begin{array}{c} \begin{bmatrix} a & 2 \\ 0 & b \end{bmatrix} \\ \begin{matrix} x & y \end{matrix} \end{array}$$

$(a \neq 0)$
$(b \neq 0)$

x & y → pivot var.
No free var.
No special soln.

q. For each of the following matrices find
1. U
2. R
3. Rank of a matrix
4. Special soln.

a) $A = \begin{bmatrix} 1 & 3 \\ -2 & 2 \end{bmatrix}$

$R_2 \to R_2 - \left(\frac{-2}{1}\right) R_1$

$\simeq \begin{bmatrix} 1 & 3 \\ 0 & 8 \end{bmatrix} = U$

$R_1 \to R_1 - \left(\frac{3}{8}\right) R_2$

$\simeq \begin{bmatrix} 1 & 0 \\ 0 & 8 \end{bmatrix}$

$R_2 \to R_2 / 8$

$$\simeq \begin{array}{c} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{matrix} x & y \end{matrix} \end{array} = R$$

$\rho(A) = 2$

No free variables
x & y pivot variables
No special soln.

b) $B = \begin{bmatrix} 2 & 6 & -2 \\ 3 & -2 & 8 \end{bmatrix}$

$R_2 \to R_2 - \left(\frac{3}{2}\right) R_1$

$$\simeq \begin{array}{c} \begin{bmatrix} 2 & 6 & -2 \\ 0 & -11 & 11 \end{bmatrix} \\ \begin{matrix} x & y & z \end{matrix} \end{array} = U$$

$R_1 \to R_1 - \left(\frac{6}{-11}\right) R_2$

$$\simeq \begin{bmatrix} 2 & 0 & 4 \\ 0 & -11 & 11 \end{bmatrix} \simeq \begin{array}{c} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix} \\ \begin{matrix} x & y & z \end{matrix} \end{array} = R$$

$R_1 \to R_1 / 2$

$R_2 \to R_1 / -11$

x & y → pivot var.

z → free var.

Let $z = 1$, $Rx = 0$

$$x + 0y + 2z = 0$$
$$0x + 1y - 1z = 0$$

$y = 1, x = -2$

special soln. $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

c) $C = \begin{bmatrix} 2 & -2 & 4 \\ 4 & 1 & -2 \\ 6 & -1 & 2 \end{bmatrix}$

$U = \begin{bmatrix} 2 & -2 & 4 \\ 0 & 5 & -10 \\ 0 & 0 & 0 \end{bmatrix}$

$R_1 \to R_1 - \left(\frac{-2}{5}\right) R_2$

$$\simeq \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & -10 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 / 2$

$R_2 \to R_2 / 5$

$$\sim \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \end{array} = R$$

$\rho(C) = 2$

To get special soln.

x & y → pivot var.

z → free var.

$$Rx = 0$$

$$x = 0$$

$$y - 2z = 0$$

let $z = 1$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$ special soln.

d) $D = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$

$R_2 \to R_2 - (3/2)R_1$

$R_3 \to R_3 - (1/-2)R_1$

$$\sim \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} = U$$

x

$R_1 \to R_1/-2$

$$\sim \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = R$$

$\rho(D) = 1$

x → pivot variable

no free variables

no special soln.

e) $E = \begin{bmatrix} -2 & 3 & 1 \end{bmatrix} = U$

$R_1 \to R_1/-2$

$\simeq \begin{bmatrix} 1 & -3/2 & -1/2 \end{bmatrix} = R$

$x \to$ pivot var.

y & $z \to$ free var.

To get special soln.

$Rx = 0$

$x - \frac{3}{2}y - \frac{1}{2}z = 0$

y	z	x
1	0	3/2
0	1	1/2

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}$

q. Solve the foll system of eqns.

$x + 2y + 3z = 9$

$2x - 2z = -2$

$3x + 2y + z = 7$

$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & 0 & -2 & -2 \\ 3 & 2 & 1 & 7 \end{array}\right]$

$R_2 \to R_2 -$

Q. Reduce $A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & c & 2 & 2 \end{bmatrix}$ to its U

and hence find special soln.

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - R_1$

$$\simeq \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & c-1 & 0 & 0 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\simeq \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & c-1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Case i) when $c = 1$

$$U = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$\quad\quad\quad\quad\quad\quad x \quad y \quad z \quad t$

$x \rightarrow$ pivot variable

$y, z, \& t \rightarrow$ free variable

special soln. $\quad Ux = 0$

$x + y + 2z + 2t = 0$

3 special soln.

y	z	t	x
1	0	0	-1
0	1	0	-2
0	0	1	-2

special soln.

$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Case ii) $c \neq 1$

$$U = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$a \neq 0$

$c - 1 \neq 0$

$x \& y \rightarrow$ pivot var.

$z \& t \rightarrow$ free var.

Special soln.

$$Ux = 0$$

$$x + y + 2z + 2x = 0$$

$$ay = 0, \quad y = 0$$

z	t	y	x
1	0	0	-2
0	1	0	-2

special solns. $\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

4 Fundamental subspaces

→ Column space : set of all linear combinations of the columns of $A_{m \times n}$ denoted by $C(A)$

$$C(A) = \{ b \in R^m \mid Ax = b \text{ is solvable} \}$$

linearly independent columns

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Note: $C(A)$ is a subspace of R^m

* linear combination of linearly independent vectors

* Let A be a matrix of order $m \times n$, the column space of A is the set of all linear combination of linearly independent vectors of A

1. The column space is denoted by $C(A)$

$C(A) \in \{ b \in R^m \mid Ax = b$ is solvable $\}$

Note: $C(A)$ is a subspace of R^m

2. $C(A)$ lies b/w Zero space (Z) and whole space of R^m

3. The system of linear eqns. $Ax = b$ is solvable iff vector b can be expressed as a combination of columns of A, then b is in $C(A)$.

4. The smallest column space $C(A) = Z$ (zero vector) and largest $C(A) = I$ (identity matrix)

Let $A = \begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 3 \end{bmatrix} \simeq \begin{bmatrix} 1 & 0 \\ 0 & 4 \\ 0 & 3 \end{bmatrix}$

$R_2 \to R_2 - 5R_1$
$R_3 \to R_3 - 2R_1$

$R_3 \to R_3 - \frac{3}{4} R_2$

$\simeq \begin{bmatrix} 1 & 0 \\ 0 & 4 \\ 0 & 0 \end{bmatrix}$

$$C(A) = \left\{ \alpha \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix} \middle| \alpha, \beta \in R \right\}$$

2-D plane in R^3

Let $B = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 3 & 5 \end{bmatrix}$ dependent column

$R_2 \to R_2 - 5R_1$
$R_3 \to R_3 - 2R_1$

$$\simeq \begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 4 \\ 0 & 3 & 3 \end{bmatrix}$$

$R_3 \to R_3 - \left(\frac{3}{4}\right) R_2$

$$\simeq \begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C(A) = \left\{ \overbrace{\alpha \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix}}^{\text{from original matrix}} \,\middle|\, \alpha, \beta \in \mathbb{R} \right\}$$

2D plane in $\mathbb{R}^3$

5. Whenever we add a dependend column to matrix the column space will not alter.

6. The basis of $C(A)$ corresponds to the columns having the pivots in the Echleon form of A.

7. The dimension of $C(A) = \rho$ (rank)

→ Row Space

* Let A be a matrix of order $m \times n$. the row space of A is set of all linear combination of independent rows of A

$$C(A^T) = R(A)$$

$$C(A^T) = \{ d \in \mathbb{R}^n \mid A^T y = d \text{ is solveable} \}$$

$C(A^T)$ is a subspace of R^n

2. $\dim C(A^T) = \text{rank}(\rho)$

3. Basis of $C(A^T)$ is the set of all independent rows of A or in the Echelon form of A.

Let $B = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 3 & 5 \end{bmatrix}$

$R_2 \to R_2 - 5R_1$

$R_3 \to R_3 - 2R_1$

$\simeq \begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 4 \\ 0 & 3 & 3 \end{bmatrix}$

$R_3 \to R_3 - \left(\frac{3}{4}\right) R_2$

$\simeq \begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix}$.

$C(A^T) = \left\{ \alpha \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 5 \\ 4 \\ 9 \end{pmatrix} \middle| \alpha, \beta \in R \right\}$

(or)

$= \left\{ \alpha \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix} \middle| \alpha, \beta \in R \right\}$

$\{ \text{2D plane in } R^3 \mid \alpha, \beta \in R \}$

→ Nullspace

Let A be the matrix of order $m \times n$, the nullspace of A is the set of all solutions of the homogeneous system of

equns $Ax = 0$

It is denoted by $N(A)$

mathematically

$$N(A) = \{ x \in R^n \mid Ax = 0 \}$$

Note 1. $N(A)$ is span by special soln. to $Ax = 0$ which is same for solving $Ux = 0$ & $Rx = 0$

2. dimension of $N(A) = n - r$

3. For a system to be non-singular

$$N(A) = Z \text{ (zero vector)}$$

4. $N(A)$ lies b/w Z and whole of R^n

5. $N(A)$ is a subspace of R^n.

6. Special solns. are basis of $N(A)$

7. The dimension of $N(A)$ is called Nullity.

Let $A = \begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 3 \end{bmatrix}$ $\quad R_2 \to R_2 - 5R_1$, $R_3 \to R_3 - 2R_1$

$\simeq \begin{bmatrix} 1 & 0 \\ 0 & 4 \\ 0 & 3 \end{bmatrix}$ $\quad R_3 \to R_3 - \left(\frac{3}{4}\right)R_1$

$\simeq \begin{bmatrix} 1 & 0 \\ 0 & 4 \\ 0 & 0 \end{bmatrix}$

no. of free var. $= n - r$
$= 2 - 2$
$= 0$

No special solns.

$N(A) = Z$ (zero vector) $= \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$N(A)$ is the system is R^2

Let $B = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 3 & 5 \end{bmatrix}$ $\quad R_2 \to R_2 - 5R_1$, $R_3 \to R_3 - 2R_1$

$\simeq \begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 4 \\ 0 & 3 & 3 \end{bmatrix}$ $\quad R_3 \to R_3 - \left(\frac{3}{4}\right)R_2$

$\simeq \begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} = U$

$$N(A) = \left\{ \alpha \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \;\middle|\; \alpha \in R \right\}$$

$N(A)$ is 1D line in R^3

* If we add a dependent column $N(A)$ will be different.

→ Left Null Space

Let A be a matrix of order $m \times n$ then the left null space is the set of all solutions of the homogeneous system of eqns.

$$A^T y = 0$$

It is denoted by $N(A^T)$

Mathematically, $N(A^T) = \{ y \in R^m \mid A^T y = 0 \}$

Note 1. $N(A^T)$ is a subsystem of R^m.

2. $\dim N(A^T) = m - r$

3. ~~Consider~~ Linear combination of rows which gives Zero ~~vector~~ rows will form the basis of left Null space

method Let $B = \left[\begin{array}{ccc|c} 1 & 0 & 1 & b_1 \\ 5 & 4 & 9 & b_2 \\ 2 & 3 & 5 & b_3 \end{array}\right]$

$R_2 \to R_2 - 5R_1$
$R_3 \to R_3 - 2R_1$

$\simeq \left[\begin{array}{ccc|c} 1 & 0 & 1 & b_1 \\ 0 & 4 & 4 & b_2 - 5b_1 \\ 0 & 3 & 3 & b_3 - 2b_1 \end{array}\right]$

$R_3 \to R_3 - \left(\frac{3}{4}\right) R_2$

$\simeq \left[\begin{array}{ccc|c} 1 & 0 & 1 & b_1 \\ 0 & 4 & 0 & b_2 - 5b_1 \\ 0 & 0 & 0 & b_3 - \frac{3}{4} b_2 + \frac{7}{4} b_1 \end{array}\right]$

$(b_3 - 2b_1) - \left(\frac{3}{4}\right)(b_2 - 5b_1)$

$b_3 - 2b_1 - \frac{3}{4} b_2 + \frac{15}{4} b_1$

$b_3 - \frac{3}{4} b_2 + \frac{7}{4} b_1$

method Let $B^T = \begin{bmatrix} 1 & 5 & 2 \\ 0 & 4 & 3 \\ 1 & 9 & 5 \end{bmatrix}$

$R_3 \to R_3 - R_1$

$\simeq \begin{bmatrix} 1 & 5 & 2 \\ 0 & 4 & 3 \\ 0 & 4 & 3 \end{bmatrix}$

$R_3 \to R_3 - R_2$

$\simeq \begin{bmatrix} 1 & 5 & 2 \\ 0 & 4 & 3 \\ 0 & 0 & 0 \end{bmatrix} = U$

no. of free variables = 1

1 special solns.

To get special solns:

$$Ux = 0$$

$$\begin{bmatrix} 1 & 5 & 2 \\ 0 & 4 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x + 5y + 2z = 0$, $z = 1$ (1 free var.)

$4y + 3z = 0$, $y = -3/4$

$x = 7/4$

$$N(B) = \left\{ \alpha \begin{pmatrix} +7/4 \\ -3/4 \end{pmatrix} \middle| \alpha \in R \right\}$$

1D line in R^3

Problems:

1. Obtain the basis of dimension of the 4 fundamental system.

i) $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 6 & 3 \\ 0 & 2 & 5 \end{bmatrix}$

Sol. Consider $\left[\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 2 & 6 & 3 & b_2 \\ 0 & 2 & 5 & b_3 \end{array}\right]$

$R_2 \to R_2 - 2R_1$

$\simeq \left[\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 0 & 2 & 1 & b_2 - 2b_1 \\ 0 & 2 & 5 & b_3 \end{array}\right]$

$R_3 \to R_3 - R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 0 & 2 & 1 & b_2 - 2b_1 \\ 0 & 0 & 4 & b_3 - b_2 + 2b_1 \end{array}\right]$$

i) $C(A) = \left\{ \alpha \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix} + \delta \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} \middle| \alpha, \beta, \delta \in R \right\}$

$$\text{Basis} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

dim = 3

3D plane in R^3 (whole of R^3)

ii) $C(A^T) = \left\{ \alpha \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix} + \delta \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix} \quad \alpha, \beta, \delta \in R \right\}$

iii) $N(A^T) = Z$ (zero vector) $= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

origin in R^3

(No combination getting rows)

$$\text{basis} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix}$$

dim = 3

3D plane in R^3

2. $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}$

$$(A|b) = \left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 1 & 2 & 4 & b_2 \\ 2 & 4 & 8 & b_3 \end{array}\right]$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - 2R_1$

$$\simeq \left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 1 & 3 & b_2 - b_1 \\ 0 & 2 & 6 & b_3 - 2b_1 \end{array}\right]$$

$R_3 \to R_3 - 2R_2$

$$\simeq \left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 1 & 3 & b_2 - b_1 \\ 0 & 0 & 0 & (b_3 - 2b_1) - 2(b_2 - b_1) \end{array}\right]$$

$= b_3 - 2b_2$

i) $C(A) = \left\{ \underbrace{\alpha \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}}_{\text{Basic vector}} \;\middle|\; \alpha, \beta \in R \right\}$

dim = 2

2D plane in R^3

ii) $C(A^T) = \left\{ \underbrace{\alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}}_{\text{Basic vector}} \;\middle|\; \alpha, \beta \in R \right\}$

2D plane in R^3 dim = 2

iii) $N(A^T) = \left\{ \alpha \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \;\middle|\; \alpha \in R \right\}$

Coefficient of $b_1\, b_2\, b_3$ in $b_3 - 2b_1$

1D line in R^3

iv) To find Null space ($Ax = 0$ or $Ux = 0$)

1 free variable $\Rightarrow$ 1 special soln.

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $z = 1$

$$x + y + z = 0$$
$$y + 3z = 0$$
$$y = -3$$
$$x = 2$$

$\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ is a special soln.

$$N(A) = \left\{ \alpha \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \;\middle|\; \alpha \in R^3 \right\}$$

1D line in R^3

3. $A = \left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & b_1 \\ 1 & 2 & 1 & 3 & b_2 \\ 3 & 6 & 3 & 7 & b_3 \end{array}\right]$

$R_2 \to R_2 - R_1 \quad R_3 \to R_3 - 3R_1$

$\simeq \left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & b_1 \\ 0 & 0 & 0 & 1 & b_2 - b_1 \\ 0 & 0 & 0 & 1 & b_3 - 3b_1 \end{array}\right]$

$R_3 \to R_3 - R_2$

$\simeq \left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & b_1 \\ 0 & 0 & 0 & 1 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & (b_3 - 3b_1) - (b_2 - b_1) \end{array}\right]$

$b_3 - b_2 - 2b_1$

i) $C(A) = \left\{ \alpha \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} \;\middle|\; \alpha, \beta \in R \right\}$

2D plane in R^3

ii) $C(A^T) = \left\{ \alpha \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix} \;\middle|\; \alpha, \beta \in R \right\}$

2D plane in R^4

iii) $N(A^T) = \left\{ \alpha \begin{pmatrix} -2 \\ -1 \\ +1 \end{pmatrix} \;\middle|\; \alpha \in R \right\}$ (coefficients in order of b_1, b_2, b_3

1D line in R^3

To get special soln. $Ax = 0$ or $Ux = 0$

free var

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

2 special soln.

$x + 2y + z + 2t = 0$

$t = 0$

	y	z	t	x
Special soln. I	1	0	0	-2
Special soln. II	0	1	0	-1

q4 Describe the column space & Null space

i) $[0]$

x - free var.

$C(A) =$ Zero Vector $= Z$ in R

$N(A) = \{\alpha(1) \mid \alpha \in R\} = R$

ii) $[0, -3]$

$$\begin{bmatrix} 0 \\ -3 \end{bmatrix} \simeq \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

$C(A) = \left\{ \alpha \begin{pmatrix} 0 \\ -3 \end{pmatrix} \middle| \alpha \in R \right\}$

y-axis in R^2

$N(A)$ = Zero vector (z) (no free var.)
= Origin in R

iii) $\begin{bmatrix} 0 & 3 \end{bmatrix} = U = A$

$C(A) = \{ \alpha(3) \mid \alpha \in R \} = R$

$N(A)$: $Ax = 0$ or $Ux = 0$

$$\begin{bmatrix} 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$3y = 0$ or $y = 0$

x free var. , $x = 1$

$N(A) = \left\{ \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} \middle| \alpha \in R \right\}$

x-axis in R^2

iv) $\begin{array}{cc} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \\ x \quad y \end{array} = U$

$C(A) = \left\{ \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} \middle| \alpha \in R \right\}$

x-axis in R^2

$N(A)$: y free var.
$Ax = 0$ or $Ux = 0$

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x - y = 0$, $y = 1$ (say)
$x = 1$

$$N(A) = \left\{ \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mid \alpha \in R \right\}$$

line $y = x$ in R^2

v) $$\begin{array}{ccc} & & \\ \end{array}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$\begin{array}{ccc} x & y & z \end{array}$$

$$C(A) = \left\{ \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mid \alpha \in R \right\}$$

x-axis in R^2

$x - y = 0$

	y	z	x
I	1	0	1
II	0	1	0

$$N(A) = \left\{ \alpha \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mid \alpha, \beta \in R \right\}$$

2D plane in R^3

Q5. Give one example of a matrix A of order 3×2 whose C(A) is same and the Nullspace is different if we add a new column to A

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}_{3 \times 2}$$

Q6. Find all the 4 fundamental subspaces of

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 3 & 6 & b_2 \end{array}\right]$$

$R_2 \to R_2 - 3R_1$

$$\left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & 0 & b_2 - 3b_1 \end{array}\right]$$

$$C(A) = \cancel{\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}}$$

$R_3 \to R_3 - 3R_1$

$$\sim \begin{array}{cc} x & y \\ \end{array}$$
$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$C(A) = \left\{ \alpha \begin{pmatrix} 1 \\ 3 \end{pmatrix} \middle| \alpha \in R \right\}$$

line in R^2

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2\times 2 \quad 2\times 1$

$x + 2y = 0$

let $y = 1$

$$C(A^T) = \left\{ \alpha \begin{pmatrix} 1 \\ 2 \end{pmatrix} \middle| \alpha \in R \right\}$$ line in R^2

$$N(A) = \left\{ \alpha \begin{pmatrix} -2 \\ 1 \end{pmatrix} \middle| \alpha \in R \right\}$$ line in R^2

$$N(A^T) = \left\{ \alpha \begin{pmatrix} -3 \\ +1 \end{pmatrix} \middle| \alpha \in R \right\}$$

Uniqueness, Existence of right and left inverse and Matrix of rank 1

Existence of inverse

1. Let $A_{m\times n}$ $(m \le n)$ such that rank $(A) = m$ then $Ax = b$ has at least one soln. x
In that case A has right inverse 'c' such that

$$\underset{m\times n}{A}\ \underset{n\times m}{C} = I_{m\times m}$$

2. Let $A_{m\times n}$ $(m \geq n)$. be a matrix such that rank of $A = n$ then $Ax = b$ has at most one soln. x. In that case A has left inverse B such that

$$B_{n\times m}\ A_{m\times n} = I_{n\times n}$$

3. When A is a square matrix, then it can have both left and right inverse.

12th February, 2024

* $A_{m\times n}$ $(m \le n)$
  rank of $A = m$

* $A_{m\times n}\, C_{n\times m} = I_{m\times m}$; $A_{m\times n}$ $(m \geq n)$
  rank $(A) = n$

* $B_{n\times m}\, A_{m\times n} = I_{n\times n}$

Note 1. Rank always satisfies $r \leq m$ & $r \leq n$

2. When $r = m$, then it is right inverse & $Ax = b$ always has a soln.

3. $r = n$, then it is a left inverse and the soln. (if it exists) is unique.

4. Square matrix has two sided inverse

Problems 1. Obtain left / right inverse (if it exists)

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}_{3 \times 2}$$

~~Do not use this method~~

Sol. rank $(A) = 2 = n$ (no. of cols.)

$\Rightarrow$ it will have left inverse

$$\underset{2 \times 3}{B} \; \underset{3 \times 2}{A_{m \times n}} = \underset{2 \times 2}{I_{n \times n}}$$

Let $B = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \underset{2 \times 2}{\begin{bmatrix} a & b \\ d & e \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2 \times 2}$$

$a = 1, \; b = 0, \; d = 0, \; e = 1$

c, f are free var. let $c = 1, \; f = 1$

then left inverse $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

* Since, the inverse is not unique we may have so many solutions, among that the best solution is given by

Use this method

$C = A^T (A A^T)^{-1}$ Best right inverse

$B = (A^T A)^{-1} A^T$ Best left inverse

→ Classwork Problem

1. Obtain the left / right inverse of

i) $A = \begin{bmatrix} 3 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ ii) $\begin{bmatrix} 3 & 1 \\ 0 & 0 \\ 1 & -2 \end{bmatrix}$

Sol i) $A = \begin{bmatrix} 3 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

$R_2 \to R_2 - \left(\frac{1}{3}\right) R_1$

$\cong \begin{bmatrix} 3 & -1 & 1 \\ 0 & 4/3 & -4/3 \end{bmatrix}$

$\text{rank}(A) = m = 2$

right inverse be C

$$C = A^T(AA^T)^{-1}$$

$$AA^T = \begin{bmatrix} 11 & 1 \\ 1 & 3 \end{bmatrix}$$

$$(AA^T)^{-1} = \begin{bmatrix} 11 & 1 \\ 1 & 3 \end{bmatrix}^{-1} = \frac{\text{adj}(A)}{|A|} = \frac{1}{32}\begin{bmatrix} 3 & -1 \\ -1 & 11 \end{bmatrix}$$

$$A^T(AA^T)^{-1} = \begin{bmatrix} 1/4 & 1/4 \\ -1/8 & 3/8 \\ 1/8 & -3/8 \end{bmatrix}$$

use calculator but show : AA^T
$(AA^T)^{-1}$
$A^T(AA^T)^{-1}$

Matrices of rank one (only concept)

When the rank of a matrix is as small as possible (1) a complicated system can be broken down into simple matrices

Those simple matrices are of rank 1

Such matrices can be written as column times row.

eg:
$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 6 & 3 & 3 \\ 8 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \end{bmatrix} \quad \text{(column x row)}$$

Rank Nullity Theorem

1. Let A be a matrix of order $m \times n$, then

$$\dim(C(A)) + \dim(N(A^T)) = \text{no. of columns}$$

$$\Downarrow$$

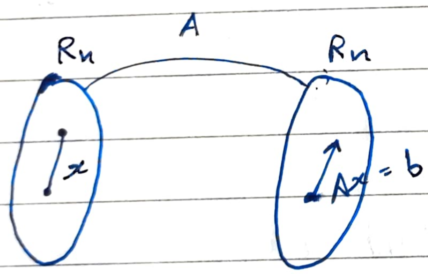
$$r + n - r = n$$

2. $$\dim(C(A^T)) + \dim(N(A^T)) = \text{no. of rows}$$

$$\Downarrow$$

$$r + m - r = m$$

Linear Transformations



$$T : A \to B$$

or

$$T : R^n \to R^n$$

Note: A transformation can now be understood

14th February, 2024

Linear Transformation

Definition: A transformation T on R^n is said to be linear if it satisfies the rule of linearity

$$T(cx + dy) = c(Tx) + d(Ty)$$

for all scalars c, d & vectors x, y

Alternatively: i) $T(x+y) = T(x) + T(y)$
ii) $T(cx) = cT(x)$

x: vector, c is scalar

Note: 1. If T is linear then $T(0) = 0$ i.e T preserves origin. The converse may / may not be true.

eg: $T(x) = x^2$

2. If

i) $T(0) = 0 \Rightarrow$ it preserves origin

ii) To check linear / not
$x, y \in T$

LHS : $T(x+y) = (x+y)^2$
RHS : $T(x) + T(y) = x^2 + y^2$

LHS $\neq$ RHS

$\therefore$ It is not linear

2. If A is a matrix of order $m \times n$ then A induced a transformation from R^n to R^m.

→ Few egs:

Let $v = (v_1, v_2)$. Then,

1. $T(v) = (v_2, v_1)$ is linear

or $T(v_1, v_2) = (v_2, v_1)$

Let $v = (v_1, v_2)$, $w = (w_1, w_2)$

$$T[av + bw] = T\left[a\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + b\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right]$$

$$= T\begin{bmatrix} av_1 + bw_1 \\ av_2 + bw_2 \end{bmatrix}$$

$$= \begin{bmatrix} av_2 + bw_2 \\ av_1 + bw_1 \end{bmatrix}$$

$$= a\begin{pmatrix} v_2 \\ v_1 \end{pmatrix} + b\begin{pmatrix} w_2 \\ w_1 \end{pmatrix}$$

$$= a\,T\underset{\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}}{(v)} + b\,T\underset{\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}}{(w)}$$

$\therefore$ T is linear

2. $T(v) = (v_1, v_1)$ is not linear

Let $v = (v_1, v_2)$, $w = (w_1, w_2)$

$$T[av + bw] = T\left[a\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + b\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right]$$

$$= T\begin{bmatrix} av_1 + bw_1 \\ av_2 + bw_2 \end{bmatrix}$$

$$= \begin{bmatrix} av_1 + bw_1 \\ av_1 + bw_1 \end{bmatrix}$$

$$= a\begin{pmatrix} v_1 \\ v_1 \end{pmatrix} + b\begin{pmatrix} w_1 \\ w_1 \end{pmatrix}$$

$$= aT(v_1) + bT(w_1)$$

$\therefore$ T is linear

3. $T(v) = (0, v_1)$ is linear

$$T(av + bw) = T\left[a\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + b\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right]$$

$$= T\begin{bmatrix} av_1 + bw_1 \\ av_2 + bw_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ av_1 + bw_1 \end{bmatrix}$$

$$[(u_1+u_2), (v_1+v_2)] = [(u_1, v_1) + (u_2, v_2)]$$

5. $T(v) = (v_1, v_2)$ is linear

Class Work Problems

1. Which of these transformations are not linear? Give reasons.

i) $T(x, y, z) = (x + y + z,\ 2x - 3y + 4z)$

Let $x, y \in T$ $\quad x = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \quad y = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$

$T : R^3 \to R^2$

$$T[ax + by] = T\left[a\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + b\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}\right]$$

$$= T\begin{bmatrix} ax_1 + bx_2 \\ ay_1 + by_2 \\ az_1 + bz_2 \end{bmatrix}$$

$$= \begin{bmatrix} a(x_1 + y_1 + z_1) + b(x_2 + y_2 + z_2), \\ 2(ax_1 + bx_2) - 3(ay_1 + by_2) + 4(az_1 + bz_2) \end{bmatrix}$$

$$= \begin{bmatrix} [a(x_1 + y_1 + z_1), 2(ax_1) - 3(ay_1) + 4(az_1)] \\ + [b(x_2 + y_2 + z_2), 2(bx_2) - 3(by_2) + 4(bz_2)] \end{bmatrix}$$

$$= a[(x_1 + y_1 + z_1), 2x_1 - 3y_1 + 4z_1] + b[(x_2 + y_2 + z_2), 2x_2 - 3y_2 + 4z_2]$$

$$= a\,\underset{x_1, y_1, z_1}{T(x)} + b\,\underset{x_2, y_2, z_2}{T(y)}$$

$\therefore$ Linear

ii) $T(x,y) = (x+3, 2y, x+y)$

here $T(0,0) = (3, 0, 0)$

$T(0) \neq 0$

$\therefore$ It is not linear

iii) $T(x,y) = (xy, x)$

Let $x = (x_1, x_2)$

$$T[cx] = T[c(x_1, x_2)]$$
$$= T[cx_1, cx_2]$$
$$= [cx_1 cx_2, x_1]$$
$$= c[cx_1x_2, x_1]$$
$$\neq cT(x)$$

$$\left[\begin{array}{l} cT(x) = c(x_1x_2, x_1) = (cx_1x_2, cx_1) \\ (x_1, x_2) \end{array} \right]$$

$\therefore$ It is not linear

Note : Consider a transformation $T: A \rightarrow B$ where A and B are subspaces

1. A is the domain of the transformation

2. B is the codomain of the transformation

3. For any x in A, there exist Tx in B, here Tx is the image of T and x is preimage of Tx

4. The set of all images is the subset of B is called range of transformation. (Column Space)

3

5. For all x in A such that $Tx = 0$ is called Kernel of the transformation.

6. Dimension of the range is called rank and dimension of kernel is called nullity.

Definition: The space of all polynomials in t of degree n is a vector space called the polynomial space.

eg1. The operation of differentiation is linear. It takes P_{n+1} to P_n. The column space is the whole of P_n and the null space is P_0, the 1-D space of all constraints

$$- P_{n+1} = P_5$$

$$2 + 3x + 4x^2 + 5x^3 + 6x^4 + 7x^5$$

$$\Big\downarrow \frac{d}{dy}$$

$$\underbrace{3 + 8x + 15x^2 + 24x^3 + 35x^4}_{\text{5 basis vectors can cover whole of } P_4} \quad P_4$$

Null space

$$P_{n+1} \xrightarrow{\text{diff}} P_n$$

eg2. The operation of integration is linear. It takes P_n to P_{n+1}. The column space is a subspace of P_{n+1} and the null space is just the zero vector

$$- 1 + 4x + 3x^2$$

$$\downarrow$$

$$x + \frac{4x^2}{2} + \frac{3x^3}{3} = P_3$$

eg 3. Multiplication by a fixed polynomial say $3+4t$ is also a linear transformation

$$P_2 : (2+3t+4t^2)$$

$$P_2(3+4t) : (3+4t)(2+3t+4t^2)$$

This A sends P_n to P_{n+1}

Transformations Represented by Matrices

The matrix of a linear transformation is a matrix for which $T(x) = Ax$, for every 'x' in the domain T. Such matrix is called standard matrix for the transformation

$$T : \underset{R^n}{R^2} \longrightarrow \underset{R^m}{R^3}$$

such that $T(x,y) = (x,y,x)$

apply the effect of transformation on basis vector

$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix}$$

$$T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$A_{3\times 2} = \begin{bmatrix}1 & 0\\0 & 1\\1 & 0\end{bmatrix}$$

Q1. Find the matrix of the

given, $T\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}x_1 - x_2\\2x_3\end{pmatrix}$

Sol. $\underset{2\times 3}{T}: R^3 \longrightarrow R^2$

standard of R^3: $\begin{pmatrix}1\\0\\0\end{pmatrix} \begin{pmatrix}0\\1\\0\end{pmatrix} \begin{pmatrix}0\\0\\1\end{pmatrix}$

standard of R^2: $\begin{pmatrix}1\\0\end{pmatrix} \begin{pmatrix}0\\1\end{pmatrix}$

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

$$= 1\begin{pmatrix}1\\0\end{pmatrix} + 0\begin{pmatrix}0\\1\end{pmatrix}$$

$$T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix}$$

$$= -1\begin{pmatrix}1\\0\end{pmatrix} + 0\begin{pmatrix}0\\1\end{pmatrix}$$

$$T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\2\end{pmatrix}$$

$$= 0\begin{pmatrix}1\\0\end{pmatrix} + 2\begin{pmatrix}0\\1\end{pmatrix}$$

$$A = \begin{bmatrix}1 & -1 & 0\\ 0 & 0 & 2\end{bmatrix}$$

15th February, 2024

→ Differentiation Matrix

$\frac{d}{dt}: T: \underset{4}{P_3} \rightarrow \underset{3}{P_2}$

we get matrix A of the transformations T of order 3×4

$T_{3\times4}$

Basis vector of P_3 : $\{1, t, t^2, t^3\}$
Basis vector of P_2 : $\{1, t, t^2\}$

$T[1] = \frac{d}{dt}(1) = 0 = 0(1) + 0(t) + (0)t^2 \quad c_1$

$T[t] = \frac{d}{dt}(t) = 1 = (1)1 + (0)t + (0)t^2 \quad c_2$

$T[t^2] = \frac{d}{dt}(t^2) = 2t = (0)1 + (2)t + (0)t^2 \quad c_3$

$T[t^3] = \frac{d}{dt}(t^3) = 3t^2 = (0)1 + (0)t + (3)t^2 \quad c_4$

$$A_{3\times4} = \begin{array}{c} \begin{array}{cccc} c_1 & c_2 & c_3 & c_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}_{3\times4} \end{array}$$

Verification : Let $p(t) = 2 + 3t + 4t^2 + t^3$

$$A P_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix} \begin{array}{c} 1 \\ t \\ t^2 \\ t^3 \end{array}$$

$$= \begin{pmatrix} 3 \\ 8 \\ 3 \end{pmatrix} \begin{array}{c} 1 \\ t \\ t^2 \end{array}$$

3

→ Generalization

$$A_{m\times n} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & & m \end{bmatrix}_{m\times n}$$

→ Integration matrix

$\int_0^t dt = T : P_2 \longrightarrow P_3$

$\quad\quad\quad\quad\quad\quad\quad 3 \quad\quad\quad 4$

we get matrix A of the transformations T of order 4 × 3

Basis vector of $P_3 : \{1, t, t^2, t^3\}$

Basis vector of $P_2 : \{1, t, t^2\}$

$$T[1] = \int_0^t (1)\,dt = [t]_0^t = t = (0)1 + (1)t + (0)t^2 + (1)t^3$$

$$T[t] = \int_0^t (t)\,dt = \frac{t^2}{2} = (0)1 + (0)t + \left(\frac{1}{2}\right)t^2 + (0)t^3$$

$$T[t^2] = \int_0^t t^2\,dt = \frac{t^3}{3} = (0)1 + (0)t + (0)t^2 + \left(\frac{1}{3}\right)t^3$$

$$A_{4\times3} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}_{4\times3}$$

Generalization

$$A_{m\times n} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & 1/2 & 0 & \cdots & 0 \\ \vdots & \vdots & 1/3 & & \vdots \\ 0 & 0 & 0 & & 1/n \end{bmatrix}_{m\times n}$$

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Note: 1. $A_{diff} \times A_{int} = I$

2. Differentiation is a left inverse of integration

3. Integration is a right inverse of differentiation

4. Column space is range of A_{int} is subspace of P_3

5. Kernel is Null space $= \{\vec{0} \in P_2\}$

Problems 1. For the space of all 2x2 matrices find the standard basis. For the linear transformation of transposing, find matrix A wrt to this basis. Why is $A^2 = I$?

Sol. $M_{2\times2} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; a, b, c, d \in R \right\}$

$$\text{Basis} = \left\{ \overset{M_1}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} \overset{M_2}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} \overset{M_3}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}} \overset{M_4}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} \right\}$$

4

$T: M_{2\times2} \longrightarrow (M_{2\times2})^T$

① $T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$= 1M_1 + 0M_2 + 0M_3 + 0M_4$

② $T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$= 0M_1 + 0M_2 + 1M_3 + 0M_4$

(3) $T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$= 0M_1 + 1M_2 + 0M_3 + 0M_4$

(4) $T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}^T = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$= 0M_1 + 0M_2 + 0M_3 + 1M_4$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4 \times 4}$$

2. From the cubics P_3 to the 4th degree polynomial P_4. What matrix represents multiplication by $2+3t$?

$T: \quad P_3 \xrightarrow{\times(2+3t)} P_4$

(4) (5)

A will be of order 5×4

$P_3 : 1, t, t^2, t^3$

$P_4 : 1, t, t^2, t^3, t^4$

i) $(2+3t)(1) = 2+3t$

$= (2)1 + (3)t + (0)t^2 + (0)t^3 + (0)t^4$

ii) $(2+3t)(t) = 2t + 3t^2$

$= (0)1 + (2)t + (3)t^2 + (0)t^3 + (0)t^4$

iii) $(2+3t)(t^2) = 2t^2 + 3t^3$

$= (0)1 + (0)t + (2)t^2 + (3)t^3 + (0)t^4$

iv) $(2+3t)(t^3) = 2t^3 + 3t^4$
$= (0)1 + (0)t + (0)t^2 + (2)t^3 + (3)t^4$

v) $(2+3t)(t^4) = 2t^4 + 3t^5$
$= (0)1 + (0)t + (0)t^2 + (0)t^3 + (2)t^4 + (3)t^5$

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}_{5\times 4}$$

Class Work 2. For each of the foll. linear transformation T find a basis & dimension of the range & kernel of T

i) $T: R^3 \to R^2$; $T(x,y,z) = (2x+z,\ x+y)$
ii) $T: R^3 \to R^3$; $T(x,y,z) = (x+2y+3z,\ -3x-2y-z,\ -2x+2z)$

* only for Polynomial basis vector = $(n+1)$

Sol ii) $T: R^3 \to R^3$
(3) (3)

A is 3×3

Basis vector of R^3 : $\begin{pmatrix}1\\0\\0\end{pmatrix} \begin{pmatrix}0\\1\\0\end{pmatrix} \begin{pmatrix}0\\0\\1\end{pmatrix}$

$T(x,y,z) = (x+2y+3z,\ -3x-2y-z,\ -2x+2z)$

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\-3\\-2\end{pmatrix}$$

$$= 1\begin{pmatrix}1\\0\\0\end{pmatrix} - 3\begin{pmatrix}0\\1\\0\end{pmatrix} - 2\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}2\\-2\\0\end{pmatrix}$$
$$= 2\begin{pmatrix}1\\0\\0\end{pmatrix} - 2\begin{pmatrix}0\\1\\0\end{pmatrix} + 0\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}3\\-1\\2\end{pmatrix}$$
$$= 3\begin{pmatrix}1\\0\\0\end{pmatrix} - 1\begin{pmatrix}0\\1\\0\end{pmatrix} + 2\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$A = \begin{bmatrix}1 & 2 & 3\\ -3 & -2 & -1\\ -2 & 0 & 2\end{bmatrix}$$

$$\simeq \begin{bmatrix}1 & 2 & 3\\ 0 & 4 & 8\\ 0 & 0 & 0\end{bmatrix}$$

range =
column space

$$\left\{ \alpha\begin{pmatrix}1\\-3\\-2\end{pmatrix} + \beta\begin{pmatrix}2\\-2\\0\end{pmatrix} \;\middle|\; \alpha, \beta \in \mathbb{R} \right\}$$

Basis of dim 2

$Ux = 0$ (finding Null space (Kernel))

$$x + 2y + 3z = 0, \quad 4y + 8z = 0$$

$z \to$ free variable let $z = 1$

$\Rightarrow y = -2, \; x = 1$

$$\text{Kernel} = \left\{ \alpha\begin{pmatrix}1\\-2\\1\end{pmatrix} \;\middle|\; \alpha \in \mathbb{R} \right\}$$

Basis dim = 1

Find the Matrix of the linear transformation T on R^3 defined by

$$T(x,y,z) = \begin{pmatrix} 2y+z \\ x-4y \\ 3x \end{pmatrix} \text{ w.r.t.}$$

i) $(1,0,0)$ $(0,1,0)$ $(0,0,1)$

ii) $(1,1,1)$ $(1,1,0)$ $(1,0,0)$

sol i) standard $\Rightarrow$ $A = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -4 & 0 \\ 3 & 0 & 0 \end{bmatrix}$

ii) $T: R^3 \to R^3$

$\Rightarrow A_{3\times 3}$

i) $T\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

ii) $T\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

iii) $T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

case i) $c_1 + c_2 + c_3 = 3$

$c_1 + c_2 = -3$

$c_1 = 3, \; c_2 = -6, \; c_3 = 6$

case ii) $c_1 + c_2 + c_3 = 2$

$c_1 + c_2 = -3$

$c_1 = 3, \; c_2 = -6, \; c_3 = 5$

Case iii) $c_1 + c_2 + c_3 = 0$

$c_1 + c_2 = 1$

$c_1 = 3, \quad c_2 = -2, \quad c_3 = -1$

$$A = \begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix}$$

4. Find the lineal mapping $T: R^4 \to R^3$ whose kernel is spanned by $(-2,1,0,0)$ & $(1,0,-1,1)$

Sol. we have $Ax = 0$ (Null space)

order of A is 3×4

$$\underset{3\times4}{\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ 0 & 0 & 0 & 0 \end{bmatrix}} \underset{4\times2}{\begin{bmatrix} -2 & 1 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

3rd row should not contain pivot coz we need 2 special soln.

$-2a + b = 0$ $\quad\quad$ $a - c + d = 0$

$b = 2a$ $\quad\quad$ $c = a + d$

$-2e + f = 0$ $\quad\quad$ $e - g + h = 0$

$f = 2e$ $\quad\quad$ $g = e + h$

Let $a = 1, \; d = 0, \; b = 2, \; c = 1$

$e = 0, \; h = 1, \; f = 0, \; g = 1$

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 01 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

∴ Linear mapping is Ax

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$$

$$= (x + 2y + z, z + t, 0)$$

...th February, 2023

Sum of Subspaces

* The sum of two subspaces U and W of a vector V is defined as

$$U + W = \{u \in U, w \in W\}$$

* Direct sum ⇒ $V = U \oplus W$

Properties :

1. The zero vector '0' of V is in U + W.
2. For any u, w $\in U + W$, we have $u + v \in U + W$
3. For any $v \in U + W$ and $\alpha \in R$, we have $\alpha x \in V \in U + W$
4. $v = u + w$ must be unique

* If each and every column of A has pivot
$\Rightarrow$ One-to-one , rank $A = n$

* If each and every row of A has pivot
$\Rightarrow$ onto , rank $A = m$

i) If $T: R^2 \to R^3$ is a L.T defined by $T(x,y) = (3x+y, 5x+7y, x+3y)$

S.T: T is one-to-one. Is T onto?

Sol: Given, $T(x,y) = (3x+y, 5x+7y, x+3y)$

Basis of R^2 : $\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$

R^3 : $\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}$

$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}3\\5\\1\end{pmatrix} = 3\begin{pmatrix}1\\0\\0\end{pmatrix} + 5\begin{pmatrix}0\\1\\0\end{pmatrix} + 1\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\7\\3\end{pmatrix} = 1\begin{pmatrix}1\\0\\0\end{pmatrix} + 7\begin{pmatrix}0\\1\\0\end{pmatrix} + 3\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$A = \begin{bmatrix}3 & 1\\5 & 7\\1 & 3\end{bmatrix}$$

$$\simeq \begin{bmatrix}3 & 1\\0 & 16/3\\0 & 8/3\end{bmatrix}$$

$$\simeq \begin{bmatrix}3 & 1\\0 & 16/3\\0 & 0\end{bmatrix}$$

$\text{rank}(A) = 2 = n \Rightarrow$ One-to-One

$\text{rank}(A) \neq 3 = m \Rightarrow$ Not onto

ii) Is T invertible? Find a rule of T^{-1} like the one which defines T

where $T(x,y,z) = (3x, x-y, 2x+y+z)$

is a transformation in R^3

Sol.

$T: R^3 \to R^3$

Basis vectors of R^3 : $\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}$

$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}3\\1\\2\end{pmatrix}$; $T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\-1\\1\end{pmatrix}$; $T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$$A = \begin{bmatrix}3 & 0 & 0\\1 & -1 & 0\\2 & 1 & 1\end{bmatrix} \sim \begin{bmatrix}3 & 0 & 0\\0 & -1 & 0\\0 & 1 & 1\end{bmatrix} \sim \begin{bmatrix}3 & 0 & 0\\0 & -1 & 0\\0 & 0 & 1\end{bmatrix}$$

Let $3x = e_1$ — pre images $\Rightarrow \boxed{x = \frac{e_1}{3}}$

$x - y = e_2$ $\Rightarrow \frac{e_1}{3} - y = e_2$

$2x + y + z = e_3$ $\boxed{y = \frac{e_1}{3} - e_2}$

$\boxed{z = e_3 - e_1 + e_2}$

$$T^{-1} \Rightarrow y(e_1, e_2, e_3) = \left[\frac{e_1}{3}, \frac{e_1}{3} - e_2, -e_1 + e_2 + e_3\right]$$

$$T^{-1}(x, y, z) = \left[\frac{x}{3}, \frac{x}{3} - y, -x + y + z\right]$$

14. If the column space of A is spanned by the vectors $(1, 2, 7, 5)$, $(-2, -1, -8, -7)$, $(-1, 3, 3, 0)$ find all vectors that span the Null space A What are the basis and dimension of $C(A^T)$ and $N(A^T)$

Sol. Let $A = \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix}1 & -2 & -1\\2 & -1 & 3\\7 & -8 & 3\\5 & -7 & 0\end{bmatrix} \end{array}$

$$(A|b) = \left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 2 & -1 & 3 & b_2 \\ 7 & -8 & 3 & b_3 \\ 5 & -7 & 0 & b_4 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 7R_1$
$R_4 \to R_4 - 5R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 0 & 3 & 5 & b_2 - 2b_1 \\ 0 & 6 & 10 & b_3 - 7b_1 \\ 0 & 3 & 5 & b_4 - 5b_1 \end{array}\right]$$

$R_3 \to R_3 - \left(\frac{6}{3}\right) R_2$

$R_4 \to R_4 - R_2$

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \sim \left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 0 & 3 & 5 & b_2 - 2b_1 \\ 0 & 0 & 0 & (b_3 - 7b_1) - 2(b_2 - 2b_1) \\ 0 & 0 & 0 & (b_1 - 5b_1) - (b_2 - 2b_1) \end{array}\right] \end{array}$$

To find $N(A)$

$$Ux = 0$$

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & 3 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$z \to$ free variable

$$x - 2y - z = 0$$
$$3y + 5z = 0$$

$$z = 1,\ y = -5/3,\ x = -7/3$$

$$\text{Basis of } N(A) = (-7/3, -5/3, 1)$$

$$C(A^T) = \left\{ \alpha \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \middle| \alpha, \beta \in R \right\}$$ dim = 2

$$N(A^T) = \left\{ \alpha \begin{pmatrix} -3 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -3 \\ -1 \\ 0 \\ 1 \end{pmatrix} \middle| \alpha, \beta \in R \right\}$$ dim = 2

13. $$U = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \;\middle|\; \begin{matrix} a, b, c, d \in \mathbb{R} \\ b - 2c + d = 0 \end{matrix} \right\}$$

$$= \left\{ \begin{pmatrix} a \\ 2c-d \\ c \\ d \end{pmatrix} \right\} \quad a, c, d \in \mathbb{R}$$

$$= \left\{ a \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; a, b, c, d \in \mathbb{R} \right\}$$

dim = 3

$$V = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \;\middle|\; a = d \;;\; b = 2c \right\}$$

$$= \left\{ \begin{pmatrix} d \\ 2c \\ c \\ d \end{pmatrix} \;\middle|\; c, d \in \mathbb{R} \right\}$$

$$= \left\{ c \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; c, d \in \mathbb{R} \right\}$$

dim = 2

$$U \cap V = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \middle| \begin{array}{c} b-2c+d=0 \\ a=d \\ b=2c \\ a,b,c,d \in R \end{array} \right\}$$

$$b - 2c + d = 0$$
$$2c - 2c + d = 0$$
$$\boxed{d=0}$$
$$\boxed{a=0} \qquad b = 2c$$

$$U \cap V = \left\{ c \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} \middle| c \in R \right\}$$

$$= \left\{ \begin{pmatrix} 0 \\ 2c \\ c \\ 0 \end{pmatrix} \middle| c \in R \right\}$$

1st February, 2024

Q12. Find basis and dim. of the subspace W of $V = M_{2\times 2}$ by $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Sol.

$$\text{Let } A = \begin{array}{c} \begin{array}{ccccc} M_1 & M_2 & M_3 & M_4 & M_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 2 & 1 \\ 2 & 0 & 2 & 4 & 0 \\ 1 & 1 & 0 & 2 & 0 \\ 2 & 1 & 1 & 4 & 1 \end{bmatrix} \end{array}$$

$$\cong \begin{bmatrix} 1 & 1 & 0 & 2 & 1 \\ 0 & -2 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 & -1 \end{bmatrix} \cong \begin{bmatrix} 1 & 1 & 0 & 2 & 1 \\ 0 & -2 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = 0 \quad 3$$

Basis = $\{M_1, M_2, M_5\}$

dim = 3

q11.

Given $u\ v\ w$ are linearly independent

$$\Downarrow$$

$$c_1 u + c_2 v + c_3 w = 0 \quad -(1)$$

$$\Rightarrow \quad c_1 = c_2 = c_3 = 0 \quad -(2)$$

PT
$$\left.\begin{array}{l} v_1 = u + v - 2w \\ v_2 = u - v - w \\ v_3 = u + w \end{array}\right\} \text{ is linearly independent}$$

$$\alpha v_1 + \beta v_2 + \gamma v_3 = 0$$

$$\alpha[u + v - 2w] + \beta[u - v - w] + \gamma[u + w] = 0$$

$$(\alpha + \beta + \gamma)u + (\alpha - \beta)v + (-2\alpha - \beta + \gamma)w = 0 \quad -(3)$$

comparing (3) & (1)

$$\alpha + \beta + \gamma = c_1 = 0$$
$$\alpha - \beta = c_2 = 0$$
$$-2\alpha - \beta + \gamma = c_3 = 0$$

by solving we get

$$\alpha, \beta, \gamma = 0$$

$\Rightarrow$ Linearly independant

q10. $(Ax \mid b)$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & b_1 \\ 1 & 9 & -1 & b_2 \\ -3 & 8 & 3 & b_3 \\ -2 & 3 & 2 & b_4 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & b_1 \\ 0 & 7 & 0 & b_2 - b_1 \\ 0 & 14 & 0 & b_3 + 3b_1 \\ 0 & 7 & 0 & b_4 + 2b_1 \end{array}\right]$$

$R_3 \to R_3 - 2R_2$
$R_4 \to R_4 - 2R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & b_1 \\ 0 & 7 & 0 & b_2 - b_1 \\ 0 & 0 & 0 & (b_3 + 3b_1) - 2(b_2 - b_1) \\ 0 & 0 & 0 & (b_4 + 2b_1) - (b_2 - b_1 \end{array}\right]$$

if $Ax = b$ is solvable $\quad \rho(A) = \rho(A|b)$

$\Rightarrow b_3 - 2b_2 + 5b_1 = 0$
$b_4 - b_2 + 3b_1 = 0$

q.9 The plane $x + y - 3z = 0$
$(Ax = 0)$

$$\begin{bmatrix} 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

let $A = \begin{array}{ccc} x & y & z \\ \hline \end{array}$ $\begin{bmatrix} 1 & 1 & -3 \end{bmatrix}$

free var

y	z	x
1	0	-1
0	1	3

basis $\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

q5. $U = \{(a,b,c) \mid a+b+c = 0\}$
$V = \{(0,0,c)\}$

$$U + V = \left\{ \begin{pmatrix} a \\ b \\ 2c \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \,\middle|\, a,b,c \in \mathbb{R} \right\}$$

while adding don't take cond $a+b+c=0$ only while $\cap$

3 basis vectors
3D subspace whole of $\mathbb{R}^3$

Let $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cap \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}$

$a+b+c=0$

$$\begin{pmatrix} -b-c \\ b \\ c \end{pmatrix} \cap \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$a = -b-c = 0$

$\& \; b = 0$

$\Rightarrow c = 0$

94. :)

Sol. A vector lies in the span if it is can be written as linear combination of the vectors computing that span.

(i.e $Ax=b$ is solvable)

b is span of A

Consider $(A \mid b) =$

$$\begin{array}{cccc} v_1 & v_2 & v_3 & b \end{array}$$
$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right]$$

$R_1 \leftrightarrow R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

$R_2 \leftrightarrow R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$\rho(A) = \rho(A \mid b) = 3$

$\Rightarrow Ax = b$ is solvable

$\Rightarrow b$ will be in span of A

$$[A|b] = \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & u \end{array} \\ \left[\begin{array}{cccc|c} 2 & 0 & 2 & -2 & 2 \\ 1 & -2 & -5 & -3 & 1 \\ 2 & 1 & -5 & -1 & 1 \end{array}\right] \end{array}$$

$R_2 \to R_2 - \left(\frac{1}{2}\right)R_1$
$R_3 \to R_3 - R_1 - R_1$

$$\cong \left[\begin{array}{cccc|c} 2 & 0 & 2 & -2 & 2 \\ 0 & -2 & -6 & -2 & 0 \\ 0 & 1 & 3 & 1 & -1 \end{array}\right]$$

$R_3 \to R_3 - \left(\frac{1}{-2}\right)R_2$

$$\cong \left[\begin{array}{cccc|c} 2 & 0 & 2 & -2 & 2 \\ 0 & -2 & -6 & -2 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right]$$

$\rho(A) \neq \rho(A|b)$

$u \notin \text{span}(v_1\ v_2\ v_3\ v_4)$

Let S 5 $\begin{array}{c} \begin{bmatrix} 1 & -3 & 2 & -3 \\ 2 & 0 & 1 & 3 \\ -2 & -4 & 1 & -9 \\ 1 & 3 & -1 & 6 \end{bmatrix} \\ \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \end{array}$

$$\cong \begin{bmatrix} 1 & -3 & 2 & -3 \\ 0 & 6 & -3 & 9 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

They do not form basis of R^4

To make the basis of R^4 we can add the vector

$$\Rightarrow \begin{bmatrix} 1 & -3 & 2 & -3 & 0 & 0 \\ 2 & 0 & 1 & 3 & 0 & 0 \\ -2 & -4 & 1 & -9 & 1 & 0 \\ 1 & 3 & -1 & 6 & 0 & 1 \end{bmatrix}$$

will form basis of R^4

q1.b $\{\sin t, e^t, t^2\}$ & $\{\sin t, \cos t, t\}$
& $\{e^t, e^{2t}, e^{3t}\}$

Let $c_1 \sin t + c_2 e^t + c_3 t^2 = 0$

$t = 0$, $c_2 = 0$
$t = 1$, $c_1 \sin 1 + c_2 e^1 + c_3 1 = 0$
$t = 2$, $c_1 \sin 2 + c_2 e^2 + c_3 4 = 0$

$$\begin{array}{ccc} \begin{bmatrix} 0 & 1 & 0 \\ \sin 1 & e & 1 \\ \sin 2 & e^2 & 4 \end{bmatrix} \\ c_1 \quad c_2 \quad c_3 \end{array}$$

$R_2 \leftrightarrow R_1$

$$= \begin{bmatrix} \sin 1 & e & 1 \\ 0 & 1 & 0 \\ \sin 2 & e^2 & 4 \end{bmatrix}$$

$R_3 \to R_3 - \left(\frac{\sin 2}{\sin 1}\right) R_1$

$$= \begin{bmatrix} \sin 1 & e & 1 \\ 0 & 1 & 0 \\ 0 & e^2 - \left(\frac{\sin 2}{\sin 1}\right) e & 4 - \frac{\sin 2 (1)}{\sin 1} \end{bmatrix}$$

$R_3 \to R_3 - \left[e^2 - \left(\frac{\sin 2}{\sin 1}\right)\right] R_2$

$$\sim \begin{array}{c} \begin{bmatrix} \sin 1 & e & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 4 - \left(\frac{\sin 2}{\sin 1}\right) \end{bmatrix} \\ c_1 \quad c_2 \quad c_3 \end{array}$$

c_1 c_2 c_3 columns are independent

$\therefore \{\sin t, e^t, t^2\}$ are linearly independent

22nd February, 2024

a. $\{(4,2,-1,3),\ (6,5,-5,1),\ (2,-1,3,5)\}$

$$\begin{bmatrix} 4 & 6 & 2 \\ 2 & 5 & -1 \\ -1 & -5 & 3 \\ 3 & 1 & 5 \end{bmatrix}$$

$$\begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \simeq \begin{bmatrix} 4 & 6 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \quad U$$

$V_1\ V_2\ V_3$ are not L.I

$$c_1V_1 + c_2V_2 + c_3V_3 = 0$$

$$4c_1 + 6c_2 + 2c_3 = 0$$
$$2c_2 - 2c_3 = 0$$

c_3 (free varicble) $= 1$

$c_2 = 1,\quad c_1 = -2$

$$-2V_1 + V_2 + V_3 = 0$$

c. $\{t^2+t+2,\ 2t^2,\ 3t^2+2t+2\}$

$$A = \begin{array}{cc} \begin{array}{ccc} P_1 & P_2 & P_3 \end{array} & \\ \begin{bmatrix} 2 & 0 & 2 \\ 1 & 0 & 2 \\ 1 & 2 & 3 \end{bmatrix} & \begin{array}{c} 1 \\ t \\ t^2 \end{array} \end{array}$$

$R_2 \to R_2 - \frac{1}{2}R_1$

$R_3 \to R_3 - \frac{1}{2}R_1$

$$\simeq \begin{bmatrix} 2 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix} \quad \simeq \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$\Rightarrow \{(P_1, P_2, P_3)$ are L.I$\}$

d. $\left\{ \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} \right\}$ in $M_{2\times 2}$

$$A = \begin{array}{c} \begin{array}{ccc} M_1 & M_2 & M_3 \end{array} \\ \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & -1 & 2 \\ 0 & 2 & 2 \end{bmatrix} \end{array}$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - R_1$

$$\simeq \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{bmatrix} \simeq \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{bmatrix} \simeq \begin{array}{c} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \end{array}$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - R_1$

$R_3 \to R_3 - \left(\frac{-1}{5}\right) R_2$
$R_4 \to R_4 - \left(\frac{2}{-5}\right) R_2$

M_1, M_2, M_3 are L.I

$$c_1 M_1 + c_2 M_2 + c_3 M_3 = 0$$

$$c_1 + 2c_2 + 3c_3 = 0$$
$$-5c_2 - 5c_3 = 0$$
$c_3 = 1$ (free var.)

$c_2 = -1$ $\quad c_1 = -1$

$$-M_1 - M_2 + M_3 = 0$$

Note * $\boxed{\dim(U+V) = \dim(U) + \dim(V) - \dim(U \cap V)}$

* if $U+V$ is a direct sum

$\Rightarrow \dim(U+V) = \dim(U) + \dim(V)$

Q6 $U = \text{span}\{(1,-1,-1,-2), (1,-2,-2,0), (1,-1,-2,-2)\}$

$V = \text{span}\{(1,-2,-3,0), (1,-1,-3,2), (1,-1,-2,2)\}$

Sol.

$$U+V = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -2 & -1 & -2 & -1 & -1 \\ -1 & -2 & -2 & -3 & -3 & -2 \\ -2 & 0 & -2 & 0 & 2 & 2 \end{array}\right]$$

$R_2 \to R_2 - (-1)R_1$

$R_3 \to R_3 - (-1)R_1$

$R_4 \to R_4 - (-2)R_1$

$$\cong \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & -1 & -2 & -2 & -1 \\ 0 & 2 & 0 & 2 & 4 & 4 \end{bmatrix}$$

$R_3 \to R_3 - \left(\frac{-1}{-1}\right)R_2$

$R_4 \to R_4 - \left(\frac{2}{-1}\right)R_2$

$$\cong \begin{array}{c} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 4 & 4 \end{bmatrix} \\ \begin{matrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{matrix} \end{array}$$

$\{v_1, v_2, v_3, v_5\}$ = Basis of $U+V$, $\dim = 4$

$$U = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -1 \\ -1 & -2 & -2 \\ -2 & 0 & -2 \end{bmatrix} \cong \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\dim = 3$

$$V = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -1 & -1 \\ -3 & -3 & -2 \\ 0 & 2 & 2 \end{bmatrix} \cong \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix} \cong \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\dim = 3$

$4 = 3 + 3 - \dim(U \cap V)$

$4 = 3 + 3 - \dim(U \cap V)$

$\dim(U \cap V) = 2$

Q23 The rule for T^{-1} is Gauss Jordan Method

$$T^{-1} = A^{-1}x$$

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$$

rank $(A) = m =$ rank $(A) = n$

$\Rightarrow$ it is one-to-one and onto

Since it is invertible find A^{-1}