28th March, 2024

UNIT-4 (No Proofs)

Singular Value Decomposition

Diagonalization of a matrix

Statement: If A is a square matrix of order 'n' has 'n' linearly independent vectors, then a matrix 'S' can be found such that $S^T A S$ is a diagonal matrix.

Proof: Let A be a square matrix of order n(3) $\lambda_1, \lambda_2, \lambda_3$ be its Eigen value and

$$X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \text{ and } X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$$

be the corresponding Eigen vectors

Let square matrix $S = [X_1 \; X_2 \; X_3]$

i.e $$S = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

Multiplying by A, $AS = A[X_1 \; X_2 \; X_3]$

$$= [AX_1, AX_2, AX_3]$$

$$AS = [\lambda_1 X_1, \lambda_2 X_2, \lambda_3 X_3]$$

$$AS = \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \lambda_3 x_3 \\ \lambda_1 y_1 & \lambda_2 y_2 & \lambda_3 y_3 \\ \lambda_1 z_1 & \lambda_2 z_2 & \lambda_3 z_3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$\therefore AS = SD$

$\times$ both sides by S^{-1}

$\boxed{S^{-1}AS = D}$ or $\boxed{S^{-1}AS = \Lambda}$

or $\boxed{A = SDS^{-1}}$

S is invertible because its columns (the eigen vectors) were assumed to be independent.

Note: 1. Any matrix with distinct eigen values can be diagonalised.

2. Not all matrices has 'n' linear independent eigen vectors. $\therefore$ All vectors can't be diagonalized.

eg: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, here $\lambda_1 = \lambda_2 = 0$, there is only 1 independent vector, $\therefore$ cannot construct S.

3. If Eigen vectors $X_1, X_2, \dots X_{16}$ correspond to distinct Eigen Values $\lambda_1, \lambda_2, \dots \lambda_{16}$ then those Eigen Vectors are linearly independent.

4. Eigen vector matrix is not unique since if X is an Eigen ~~value~~ vector corresponding to λ then kX is also an Eigen vector

Problems 1. Factor the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ into $S\Lambda S^{-1}$ and also find SAS^{-1}

Sol. $|A - \lambda I| = 0$

$\lambda = 0, 2$

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 2\lambda + 0 = 0$$

$$\lambda(\lambda - 2) = 0$$

$\lambda = \underbrace{0, 2}$ when diagonalizing write in ascending order

To find Eigen vectors

$$(A - \lambda I)x = 0$$

$$(1-\lambda)x + y = 0$$

$$x + (1-\lambda)y = 0$$

Case i) $\lambda = 0$

$$\left.\begin{aligned} x + y &= 0 \\ x + y &= 0 \end{aligned}\right] \text{ same}$$

$y = 1$ (free variable)

$x = -1$

$X_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

ii) $\lambda = 2$

$$\left.\begin{aligned} -x + y &= 0 \\ x - y &= 0 \end{aligned}\right] \text{ same}$$

$y = 1$ (free var.)

$X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Date	
Page	

$$S = \begin{array}{cc} x_1 & x_2 \\ \end{array}$$
$$S = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \quad \Lambda = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \quad S^{-1} = \frac{\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}}{-2}$$

$$\therefore A = \underset{S}{\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}} \underset{\Lambda}{\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}} \underset{S^{-1}}{\begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}}$$

To find $SAS^{-1} = D$ of A

$$\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = \Lambda = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

isa 2. Find the matrix A where Eigen values are 2, 5 and Eigen vectors are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Sol. $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad S^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ $\quad$ & $\Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$

take in same order

$$\therefore A = S\Lambda S^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

3. Find the Eigen values and Eigen vectors of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and write 2 different diagonalising matrices

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 3\lambda^2 + 0\lambda - 0 = 0$$
$$\lambda^3 - 3\lambda^2 = 0$$
$$\lambda^2(\lambda - 3) = 0$$
$$\lambda = 0, 0, 3$$

To find Eigen vectors

$$(A - \lambda I)x = 0$$

$$(1-\lambda)x + y + z = 0$$
$$x + (1-\lambda)y + z = 0$$
$$x + y + (1-\lambda)z = 0$$

case i $\lambda = 0$

$$x + y + z = 0$$

y	z	x
1	0	-1
0	1	-1
1	1	-2

I combination

$$X_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad X_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

II combination

$$X_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad X_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

Case ii $\lambda = 3$

$$\left.\begin{aligned} -2x + y + z &= 0 \\ x - 2y + z &= 0 \end{aligned}\right] \text{diff}$$
$$x + y - 2z = 0$$

Date
Page

$$\frac{x}{\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix}}$$

$$\frac{x}{3} = \frac{-y}{-3} = \frac{z}{3}$$

$$X_3 = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} \text{ or } \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

I

$$\therefore \quad S = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

II

$$S = \begin{bmatrix} -1 & -2 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Power and Product of matrices

* Diagonization of a square matrix helps to find powers of A, A^2, A^3 etc.

we have $D = S^{-1} A S$

$$D^2 = (S^{-1}AS)(S^{-1}AS)$$

$$= S^{-1}ASS^{-1}AS$$

$$= S^{-1} A I A S$$

$$D^2 = S^{-1} A^2 S$$

pre multiply by S & post multiply S^{-1}

$$SD^2S^{-1} = SS^{-1}A^2SS^{-1}$$

$$SD^2S^{-1} = A^2$$

$$\therefore A^2 = SD^2S^{-1}$$

$$D^n = \begin{bmatrix} \lambda_1^n & & \\ & \lambda_2^n & \\ & & \ddots & \lambda_n^n \end{bmatrix}$$

$$A^n = SD^nS^{-1}$$

Q1. Diagonalize $A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ and hence find A^{100}. Show that $A^{100} = A$

Sol. $A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1/2 - \lambda & 1/2 \\ 1/2 & 1/2 - \lambda \end{vmatrix} = 0$$

$$\lambda^2 - \lambda + 0 = 0$$
$$\lambda(\lambda - 1) = 0$$
$$\lambda = 0, 1$$

To get Eigen vectors

$$(A - \lambda I)x = 0$$

$$(1/2 - \lambda)x + y = 0$$
$$x + (1/2 - \lambda)y = 0$$

i) when $\lambda = 0$

$$\frac{1}{2}x + \frac{y}{2} = 0$$
$$\frac{x}{2} + \frac{1}{2}y = 0$$
same

$y = 1$

$x = -1$

$\cancel{y = -\frac{1}{2}x}$

$\cancel{x + \frac{1}{2}\left(\frac{-1}{2}\right)x = 0}$

$$X_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

ii) when $\lambda = 1$

$$-\frac{1}{2}x + \frac{y}{2} = 0$$
$$\frac{x}{2} - \frac{y}{2} = 0$$
same

$y = 1, \ x = 1$ $\qquad X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$S = \begin{array}{cc} X_1 & X_2 \end{array}$$
$$S = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$S^{-1} = \frac{\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}}{-2} = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A = S \Lambda S^{-1}$$

$$A^{100} = S \Lambda^{100} S^{-1}$$

$$= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}^{100} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0^{100} & 0 \\ 0 & 1^{100} \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$A^{100} = A$$

q. Diagonalize the matrix $A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$ and find one of its square roots, how many square roots will be there.

Sol. $|A - \lambda I| = 0$

$$\begin{vmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{vmatrix} = 0$$

$+9$

$-9 \quad -1$

$$\lambda^2 - 10\lambda + 9 = 0$$
$$\lambda^2 - 9\lambda - \lambda + 9 = 0$$
$$\cancel{\lambda(\lambda-1) + -9(}$$
$$\lambda^2 - \lambda - 9\lambda + 9 = 0$$
$$\lambda(\lambda-1) - 9(\lambda-1) = 0$$
$$\lambda = 1, 9$$

i) $\lambda = 1$

$$\left.\begin{array}{l} 4x + 4y = 0 \\ 4x + 4y = 0 \end{array}\right\} \text{same}$$

$$X_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

ii) $\lambda = 9$

$$-4x + 4y = 0 \qquad x - y = 0$$
$$4x + -4y = 0 \qquad x - y = 0$$

$y = 1 \quad x = 1$

$$X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Date
Page

$$\therefore \quad S = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$S^{-1} = \frac{\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}}{-2} = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$\therefore \quad A = S \Lambda S^{-1}$$

$$= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$A^{1/2} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}^{1/2} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1^{1/2} & 0 \\ 0 & 9^{1/2} \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

4 possibilities

$$= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$ positive one

since $\Lambda^{1/2} = \begin{bmatrix} 1^{1/2} & 0 \\ 0 & 9^{1/2} \end{bmatrix} = \begin{bmatrix} \sqrt{1} & 0 \\ 0 & \sqrt{9} \end{bmatrix}$

$$\begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 3 \end{bmatrix}$$

1st April, 2024

Q. Check the matrix are orthogonally diagonalizable. If then orthogonally diagonalize as

$$A = S \Lambda S^{-1} = Q \Lambda Q^{-1} = Q \Lambda Q^{T}.$$

Which Q is an orthogonal matrix

P. Sol. $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ $\Rightarrow |A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 3\lambda^2 = 0$$
$$\lambda^2(\lambda - 3) = 0$$
$$\lambda = 0, 0, 3$$

consider $(A - \lambda I)x = 0$

$$(1-\lambda)x + y + z = 0$$
$$x + (1-\lambda)y + z = 0$$
$$x + y + (1-\lambda)z = 0$$

case i) $\lambda = 0$

$$\left.\begin{aligned} x + y + z &= 0 \\ x + y + z &= 0 \\ x + y + z &= 0 \end{aligned}\right\} \text{same}$$

y & z $\rightarrow$ free var.

y	z	x
1	0	-1
0	1	-1

$X_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ $X_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Date	
Page	

case ii) $\lambda = 3$

$$\left.\begin{array}{l} -2x + y + z = 0 \\ x - 2y + z = 0 \end{array}\right] \text{diff}$$

$$x + y - 2z = 0$$

$$\frac{x}{\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix}}$$

$$\frac{x}{3} = \frac{-y}{-3} = \frac{z}{3}$$

$$X_3 = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} \text{ or } \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\therefore\ S = \begin{array}{ccc} a & b & c \\ v_1 & v_2 & v_3 \end{array}$$
$$\begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$-x + y = 0$

$x + y + z = 0$

$z = 1$

$\Rightarrow x + y = -1$

$2x = 1$

$x = -1/2$

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

wkt $v_1 \not\perp v_2$, $v_2 \not\perp v_3$, $v_3 \not\perp v_1$ & $v_3 \perp v_2$

$\{v_1, v_2, v_3\}$ are not orthogonal vectors

use G.S process

$$q_1 = \frac{a}{\|a\|} = \frac{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}{\sqrt{(-1)^2 + (-1)^2 + 0^2}} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$q_2 = \frac{e}{\|e\|}, \quad e = b - (q_1^T b)\, q_1$$

$$e = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \left[\begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right] \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \left(\frac{1}{\sqrt{2}} \right) \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} -1/2 \\ 1/2 \\ 0 \end{pmatrix}$$

$$e = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}$$

$$\|e\| = \sqrt{\frac{1}{4} + \frac{1}{4} + 1} = \frac{\sqrt{6}}{2}$$

$$q_2 = \frac{e}{\|e\|} = \frac{2}{\sqrt{6}} \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix}$$

$$q_3 = \frac{E}{\|E\|} \qquad E = C - \left[(q_1^T c) q_1 \right] - \left[(q_2^T c) q_2 \right]$$

$$E = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \left[\begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \right]$$

$$- \left[\begin{pmatrix} \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 0 \end{pmatrix} \right]$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$q_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

$$Q = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$$

ascending order

$$A = Q \Lambda Q^T = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

Quadratic Form (Q.F)

A quadratic form on R^n is a function Q defined on R^n whose value at a vector x in R^n can be computed by an expression of the form $Q(x) = x^T A x$, where A is an $n \times n$ symmetric matrix and $x \in R^n$

* A is matrix of the quadratic form

* A homogeneous polynomial of 2nd degree in any no. of variables

eg: 1. $Q(x) = 10x_1^2 - 6x_1x_2 - 3x_2^2$

Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $A = \begin{bmatrix} 10 & -3 \\ -3 & -3 \end{bmatrix}$ $\quad x^T = (x_1 \ \ x_2)$

Consider $x^T A x = [x_1 \ x_2] \begin{bmatrix} 10 & -3 \\ -3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$= 10x_1^2 - 6x_1x_2 - 3x_2^2 = Q.F$$

2. $Q(x) = x^T I x = \|x\|^2$
is the simplest QF

Problems

Q1. Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ compute $x^T A x$

a) $A = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$

$$x^T A x = [x_1 \ \ x_2] \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x^T A x = \underbrace{4x_1^2 + 3x_2^2}_{\text{pivots / diagonal elements of } A}$$

Note: If A is a diagonal matrix, then the associated QF has no cross-product terms

b) $A = \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix}$

$$x^T A x = 3x_1^2 + 7x_2^2 - 4x_1x_2$$

c) $A = \begin{bmatrix} 7 & 2 & 3 \\ 2 & 6 & 1 \\ 3 & 1 & 5 \end{bmatrix}$

$$x^T A x = 7x_1^2 + 6x_2^2 + 5x_3^2 + \overset{2+2}{4x_1x_2} + \overset{3+3}{6x_1x_3} + \underset{1+1}{2x_3x_2}$$

q2.

a) $Q(x) = 5x_1^2 + 3x_2^2 + 2x_3^2 - x_1x_2 + 8x_2x_3$

$$A = \begin{bmatrix} 5 & -1/2 & 0 \\ -1/2 & 3 & 4 \\ 0 & 4 & 2 \end{bmatrix}$$

b) $Q(x) = 5x_1^2 - x_2^2 + 7x_3^2 + 5x_1x_2 - 3x_1x_3$

$$A = \begin{bmatrix} 5 & 5/2 & -3/2 \\ 5/2 & -1 & 0 \\ -3/2 & 0 & 7 \end{bmatrix}$$

c) $Q(x) = x_3^2 - 4x_1x_2 + 4x_2x_3$

$$A = \begin{bmatrix} 0 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

q3. Compute QF $x^T A x$

$$A = \begin{bmatrix} 4 & 3 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$ &

a) $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ b) $x = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$ c) $x = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

a) $Q(x) = x^T A x = 4x_1^2 + 3x_2^2 + x_3^2 + 6x_1x_3 + 2x_2x_3$

b) $Q\begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} = \cancel{26}\ 21$

c) $Q\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} = 15/3 = 5$

Classification of Quadratic Forms :-

A positive form Q is

a) Positive Definite if $Q(x) > 0$ for all $x \neq 0$

b) Negative Definite if $Q(x) < 0$ for all $x \neq 0$

c) Indefinite if $Q(x)$ assumes both +ve & -ve values.

d) Positive semi definite if $Q(x) \geq 0$ for all x

e) Negative semi definite if $Q(x) \leq 0$ for all x

Result : Quadratic Forms and Eigen Values

Let A be an $n \times n$ symmetric matrix, then a QF $x^T A x$ is

a) Positive definite iff all eigen values are +ve.

b) Negative definite iff all eigen values are -ve.

c) Indefinite iff both +ve & -ve eigen values

q: Is $Q(x) = 3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 + 4x_2x_3$ positive definite ?

Sol $A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{bmatrix}$

$$\lambda^3 - 6\lambda^2 + 3\lambda - 6 = 0$$

$$\lambda = 5, -1, 2$$

indefinite

3rd April 2024

Test for Positive Definitiveness

q. Which symmetric matrices have the property that $x^T A x > 0$ for all non-zero vectors x?

* Q.F $Q(x) = x^T A x$

If $Q(x) > 0$ ⇒ positive definite

$r_1 c_1 > 0$ & det > 0

eg: Let $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is a symmetric matrix

methods: i) A is positive definite if $a > 0$ and $ac - b^2 > 0$ $(|A| > 0)$

ii) both Eigen values of A are positive

Product of eigen values of A $= \lambda_1 \lambda_2 > 0$

$\because \lambda_1 \lambda_2 = |A|$

$\lambda_1 \lambda_2 > 0$

iii) $Q(x) = x^T A x$

$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

$ax^2 + cy^2 + 2bxy$

$ax^2 + 2bxy + cy^2$

quadratic eqn. in x

$\cancel{+ c_4 - \left[\frac{1}{2}(2by)\right]^2}$

$\cancel{\Rightarrow ax^2 + 2bxy + (by)^2 - (by)^2 + cy^2}$

$\div a$

$a\left(x^2 + \frac{2b}{a}xy + \frac{c}{a}y^2\right)$

$$+ \text{ \& } - \left(\frac{1}{2}\frac{2by}{a}\right)^2$$

$$a\left(\underbrace{x^2 + \frac{2b}{a}xy + \left(\frac{b}{a}y\right)^2} - \left(\frac{b}{a}y\right)^2 + \frac{c}{a}y^2\right)$$

$$a\left(\left[x + \frac{b}{a}y\right]^2 - \left(\frac{b}{a}y\right)^2 + \frac{c}{a}y^2\right)$$

$$= a\left(\left[x + \frac{b}{a}y\right]^2 + \left[\frac{ac - b^2}{a^2}\right]y^2\right)$$

$$= a\left(x + \frac{b}{a}y\right)^2 + \left(\frac{ac-b^2}{a}\right)y^2$$

Pivots

> 0

= sum of squares

$\therefore \varphi(x) > 0 \quad \forall x \neq 0$

Note : a, $\frac{ac-b^2}{a}$ are pivots of matrix A

$x^T A x$ stays positive when n independent squares are multiplied by positive pivots.

iv) Upper left submatrices of a matrix

consider $A = \begin{bmatrix} \boxed{a_{11}} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$A_1 = [a_{11}] \quad A_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad A_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$|A_1|$ & $|A_2|$ & $|A_3| > 0$

Results: Tests for positive definiteness of a matrix

1. $x^TAx > 0$ for all non zero real vectors x
($\because x^TAx$ = sum of squares)

2. All the eigen values A satisfy $\lambda_i > 0$
are positive (Eigen value test)

easy 3. All the upper left submatrices A_k have
positive determinants (Determinants test)

4. All the pivots (without row exchanges)
satisfy $d_k > 0$ (are positive) (Pivot test)

Problems q. Test for positive definiteness:

a) $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$

i) sum of squares

$$Q(x) = x^TAx > 0 \quad \forall x \neq 0$$

let $x = \begin{pmatrix} x \\ y \end{pmatrix}$

$$x^TAx = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= 2x^2 + 5y^2 + 6xy$$

$$= 2\left[x^2 + \frac{5}{2}y^2 + 3xy\right]$$

$$+ \text{eq} - \left(\frac{1}{2} \text{ co-eff. of } x\right)^2$$

Date
Page

$$+ 4p - \left[\frac{1}{2} 3y\right]^2 = \frac{9}{4} y^2$$

$$= 2\left[x^2 + 3xy + \frac{9}{4}y^2 - \frac{9}{4}y^2 + \frac{5}{2}y^2\right]$$

$$= 2\left[\left(x + \frac{3}{2}y\right)^2 + \left(\frac{5}{2} - \frac{9}{4}\right)y^2\right]$$

$$= 2\left[\left(x + \frac{3}{2}y\right)^2 + \left(\frac{1}{4}\right)y^2\right]$$

$$= \underset{>0}{2}\left(x + \frac{3}{2}y\right)^2 + \underset{>0}{\frac{1}{2}}y^2$$

pivots

$$\phi(x) > 0 \quad \forall x \neq 0$$

ii) Eigen value test

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$$

$$\lambda^2 - 7\lambda + 1 = 0$$

$$\lambda = \frac{7 \pm \sqrt{45}}{2}$$

$$\lambda_1 = \frac{7 - \sqrt{45}}{2} > 0, \quad \lambda_2 = \frac{7 + \sqrt{45}}{2} > 0$$

$\therefore$ it is +ve definite

iii) determinant test

$$|2| > 0$$

$$\begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} > 0$$

$\therefore$ it is +ve definite

iv) Pivot test

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \underset{R_2 \to R_2 - \left(\frac{3}{2}\right)R_1}{\simeq} \begin{bmatrix} 2 & 3 \\ 0 & \frac{1}{2} \end{bmatrix}$$

>0

b) $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

i) sum of squares

Let $x = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$Q(x) = x^T A x$

$$= [x \ y \ z] \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \end{array} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$Q(x) = 2x^2 + 2y^2 + 2z^2 + 0xz - 2yz + 2xy$

Let $A = LDU$

since A is symmetric $U = L^T$

$A = LDL^T$

consider $x^T A x = x^T (LDL^T) x$

$= x^T L D L^T x$

$= (L^T x)^T D (L^T x)$

diagonal matrix with pivots

elimination matrix

lower Δ matrix where diagonal elements =

$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

$R_2 \to R_2 - \left(\frac{-1}{2}\right) R_1$

$\simeq \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1/2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

$R_3 \to R_3 - \left(\frac{-1}{\frac{3}{2}}\right) R_2$

$$\simeq \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & 4/3 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix} \qquad L^T = \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$L^T x = \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{bmatrix} x - y/2 + 0z \\ 0x + y - 2/3\, z \\ 0x + 0y + z \end{bmatrix}$$

$$Q(x) = x^T A x = (L^T x)^T D (L^T x)$$

$$= \begin{bmatrix} x - \frac{y}{2}, & y - \frac{2}{3}z, & z \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 4/3 \end{bmatrix} \begin{bmatrix} x - y/2 \\ y - \frac{2}{3}z \\ z \end{bmatrix}$$

$$= 2\left(x - \frac{y}{2}\right)^2 + \frac{3}{2}\left(y - \frac{2}{3}\right)^2 + \frac{4}{3}(z^2)$$

ii) Eigen value test

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\lambda^3 - 6\lambda^2 + 10\lambda - 4 = 0$$

$$\lambda = \underset{>0}{2},\ \underset{>0}{2+\sqrt{2}},\ \underset{>0}{2-\sqrt{2}}$$

it is +ve definite

iii) Determinant test

$$|2| > 0$$

$$\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4-1 = 3 > 0$$

$$|A| = 4 > 0$$

$\therefore$ it is +ve definitive

iv) Pivot test

$$\begin{matrix} 2 & 3/2 & 4/3 \\ >0 & >0 & >0 \end{matrix}$$

c) $A = \begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}$

$$a = 1 > 0$$

$$ac - b^2 = -4 < 0$$

$\therefore$ it is not positive definitive

it is indefinitive

d) $\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$

$-2+1$

$-1+2$

$$|2| > 0$$

$$\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4-1 = 3 > 0$$

$$\begin{vmatrix} & \\ & \end{vmatrix} = 2(3) + 1(-1) - 1(1) = 4 > 0$$

$+2+2=0$

e) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}^2$

let $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$

$+4$
λ

$$\lambda^3 - 0\lambda^2 - 6\lambda + 4 = 0$$
$$\lambda^3 - 6\lambda + 4 = 0$$
$$\lambda = -2.73, 2, 0.73$$

$$\lambda^2 = 7.4529, 4, 0.5329$$

q. For which 's' and 't' do A & B have all $\lambda > 0$

$A = \begin{bmatrix} s & -4 & -4 \\ -4 & s & -4 \\ -4 & -4 & s \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & t & 8 \\ 4 & 8 & 7 \end{bmatrix}$

start with determinant test then pivot test then eigen value.

Sol i) Determinant Test

$|s| > 0$ $(s > 0)$

$\begin{vmatrix} s & -4 \\ -4 & s \end{vmatrix}$

$$s^2 - 16 > 0$$
$$(s+4)(s-4) > 0$$
$$s > 4 \text{ or } s > -4$$

combining

$\Rightarrow s > 4$

$$\cancel{B =} \begin{vmatrix} s & -4 & -4 \\ -4 & s & -4 \\ -4 & -4 & s \end{vmatrix}$$

$$s^3 - 48s - 128 > 0$$

$$s > 8, -4, -4$$

$$\therefore s > 8$$

4th April, 2024

$$B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & t & 8 \\ 4 & 8 & 7 \end{bmatrix}$$

Sol i) $|1| > 0$

ii) $\begin{vmatrix} 1 & 2 \\ 2 & t \end{vmatrix} > 0$ $\quad t - 4 > 0 \;;\; t > 4$

iii) $\begin{vmatrix} 1 & 2 & 4 \\ 2 & t & 8 \\ 4 & 8 & 7 \end{vmatrix} > 0$

$$\begin{array}{r} 32 \\ -14 \\ \hline 18 \end{array}$$

$$1(7t - 64) - 2(-18) + 4(16 - 4t) > 0$$

$$7t - \cancel{64} + 36 + \cancel{64} - 16t > 0$$

$$-9t + 36 > 0$$

$$36 > +9t$$

$$4 > +t$$

$$t < 4$$

$\therefore$ No value of t exists

Date
Page

Pivot test

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & t & 8 \\ 4 & 8 & 7 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 4R_1$

$$\simeq \begin{bmatrix} 1 & 2 & 4 \\ 0 & t-4 & 0 \\ 0 & 0 & -9 \end{bmatrix}$$

All pivots are not positive

It is not +ve definite

Positive definite matrices and Least Squares

So far, we have connected positive definite matrices with pivots, determinants and eigen values.

Now, let us connect with Least Squares
Consider the least square problem in the form of

$$Rx = b$$

Here R has 'm' eqns. with $m \geq n$

R - not symmetric $\quad Rx = b, \; m \geq n$ will not have a soln.

The least-square choice/solution $\hat{x}$ is in the form :

$$R^T R \hat{x} = R^T b$$

$\left.\begin{matrix} A^T A \\ A A^T \end{matrix}\right\}$ always symmetric

Here $R^T R$ is not only symmetric but also is +ve definite – provided that 'n' columns of R are linearly independent

"The symmetric matrix A is +ve definite iff there is a matrix R with independent columns such that $A = R^T R$"

i.e $x^T A x = x^T (R^T R) x$
$= (x^T R^T)(Rx)$
$= (Rx)^T (Rx)$
$= \|Rx\|^2 > 0$

$\| \| \geq 0$ but x cannot be 0

To find R

1. $A = LDU$

since A is a symmetric matrix

$A = LDL^T$
$= L\sqrt{D}\sqrt{D}L^T$
$= (L\sqrt{D})(\sqrt{D}L^T)$

choose $R = \sqrt{D}L^T$
$R^T = (\sqrt{D}L^T)^T$
$= L\sqrt{D}^T$
$= L\sqrt{D}$ ($\because$ diagonal matrix)

$\therefore \boxed{A = R^T R, \quad R = \sqrt{D}L^T}$

Cholesky Decomposition

2. with Eigen values

diagonal matrix

$$A = Q \Lambda Q^T$$

symmetric $\therefore$ orthogonally

$$= (Q\sqrt{\Lambda})(\sqrt{\Lambda}\, Q^T)$$

$$= R^T R$$

Let $R = \sqrt{\Lambda}\, Q^T$

$$R^T = (\sqrt{\Lambda}\, Q^T)^T$$

$$= (Q^T)^T (\sqrt{\Lambda})^T$$

$$= Q\sqrt{\Lambda}$$

3. $A = QR$

$$R = Q^{-1} A$$

and since A is a symmetric matrix

$$Q^T = Q^{-1}$$ < square matrix

$$R = Q^T A$$

we have

$$A = QR$$

$$A^T A = (QR)^T (QR)$$

$$= R^T Q^T Q R$$

$$= R^T R = A$$

Result: The symmetric matrix A is positive definite iff there is a matrix R with independent columns such that $A = R^T R$.

Semi definite matrix

Necessary conditions :

* $xAx^T \geq 0 \quad \forall x$ (+ve semi definite)

* All eigen values $\lambda_i \geq 0$

* No principal submatrix has -ve determinants

* None of the pivots are -ve (can be 0)

* $A = R^T R$, if R has dependent columns $\Rightarrow$ semi definite

q: Test for +ve definite and semi +ve definite

a) $\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$

i) $Q(x) = x^T A x$ (sum of squares) let $x = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$x^T A x = (x \ y \ z) \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= 2x^2 + 2y^2 + 2z^2 - 2xy - 2xz + 2yz$$

hard to group

go for LDU

Date
Page

$$A = LDU$$
$$= LDL^T$$

$$x^T A x = x^T (L D L^T) x$$
$$= x^T L D L^T x$$
$$= (L^T x)^T D (L^T x)$$

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

$R_2 \to R_2 - \left(\frac{-1}{2}\right) R_1$
$R_3 \to R_3 - \left(\frac{-1}{2}\right) R_1$

$$\simeq \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3/2 & 1/2 \\ 0 & 1/2 & 3/2 \end{bmatrix}$$

$R_3 \to R_3 - \frac{(1/2)}{(3/2)} R_2$

$$\simeq \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3/2 & 1/2 \\ 0 & 0 & 4/3 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/2 & 1/3 & 1 \end{bmatrix} \qquad L^T = \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$L^T x = \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 1 & 1/3 \\ 0 & 0 & 01 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} x - y/2 - z/2 \\ y + z/3 \\ z \end{bmatrix}$$

$$A = (L^T x)^T D (L^T x)$$

$$= \begin{bmatrix} x - \frac{y}{2} - \frac{z}{2} & \frac{y+z}{3} & z \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 4/3 \end{bmatrix} \begin{bmatrix} x - y/2 - z/2 \\ y + z/3 \\ z \end{bmatrix}$$

$$= 2\left(x - \frac{y}{2} - \frac{z}{2}\right)^2 + \frac{3}{2}\left(y + \frac{z}{2}\right)^2 + \frac{4}{3} z^2$$

$$= > 0 \quad \forall x \neq 0$$

it will be +ve semi definite if any one of pivots is 0 (one term missing)

ii) Eigen value test

$$\lambda = 1, 1, 4 > 0$$

iii) determinant test

$$|2| > 0$$

$$\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} > 0$$

$$\begin{vmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{vmatrix} > 0$$

iv) pivot test

pivots $2, 3/2, 4/3$ all +ve

v) Least squares

$$A = R^T R, \quad R = \sqrt{D} L^T$$

method 1 to find R

$\sqrt{D}$ – diagonal matrix with pivots

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 4/3 \end{bmatrix}^{1/2} \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{3/2} & 0 \\ 0 & 0 & 2/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1 \end{bmatrix}$$

Date

Page

multiplication of 2 upper Δ matrix will be upper Δ matrix

$$= \begin{bmatrix} \sqrt{2} & -1/\sqrt{2} & -1/\sqrt{2} \\ 0 & \sqrt{3/2} & 1/\sqrt{6} \\ 0 & 0 & 2/\sqrt{3} \end{bmatrix}$$

-2 -1

independent columns

$\therefore$ +ve definite

1 4 2

b) $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ Σ minor $3 + (+3) + 3$

$|| = 2(3) - 1(-3) - 1(3) = 0$

$R_2 \to R_2 - \left(\frac{-1}{2}\right) R_1$

$R_3 \to R_3 - \left(\frac{-1}{2}\right) R_1$

$$\simeq \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3/2 & -3/2 \\ 0 & -3/2 & 3/2 \end{bmatrix}$$

$R_3 \to R_3 - (-R_2)$

$$\simeq \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3/2 & -3/2 \\ 0 & 0 & 0 \end{bmatrix} = U$$

$L = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/2 & -1 & 1 \end{bmatrix}$ $L^T = \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

$$L^T x = \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} x - y/2 - z/2 \\ y - z \\ z \end{bmatrix}$$

$$A = (L^T x)^T A (L^T x)$$

$$\begin{bmatrix} x-\frac{y}{2}-\frac{z}{2} & y-z & z \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x-y/2-z/2 \\ y-z \\ z \end{bmatrix}$$

$$= 2\left(x-\frac{y}{2}-\frac{z}{2}\right)^2 + \frac{3}{2}(y-z)^2 + 0$$

$\geq 0 \quad \forall x$ semi positive

iii) Eigen value test

$\lambda^3 - 6\lambda^2 + 9\lambda + 0 = 0$ $\qquad +9 \atop {-3 \; -3}$

$\lambda(\lambda^2 - 6\lambda + 9) = 0$

$\lambda = 0, 3, 3$

$\Rightarrow$ semi +ve definite

iv) determinant test

$|2| > 0$

$\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} > 0$

$\begin{vmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{vmatrix} = 0$

$\Rightarrow$ semi +ve definite

v) Pivot test

Pivots : $2, 3/2, 0$

$\Rightarrow$ semi +ve definite

vi) Least squares

$A = R^T R$, $R = \sqrt{D}\, L^T$

$\sqrt{\frac{3}{2}}\left(\frac{-1}{2}\right)$

$$R = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}^{1/2} \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{3/2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2} & -1/\sqrt{2} & -1/\sqrt{2} \\ 0 & -\frac{\sqrt{3}}{\sqrt{2}} & -\sqrt{\frac{3}{2}} \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2} & -1/\sqrt{2} & -1/\sqrt{2} \\ 0 & -\sqrt{\frac{3}{2}} & -\sqrt{\frac{3}{2}} \\ 0 & 0 & 0 \end{bmatrix}$$

not independent

$\therefore$ +ve semi definite

8th April, 2024 (absent)

Symmetric Matrices

A symmetric matrix is a matrix A such that $A^T = A$ such a matrix is necessarily square. Its main diagonal entries are arbitrary but its other entries are in pairs of opposite sides

eg: $A = \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix}$

* If A is symmetric, then any 2 eigen vectors from different eigen values are orthogonal.

* An $n \times n$ matrix A is originally diagonalizable iff A is symmetric matrix

$$\therefore A = S \Lambda S^{-1}$$
$$A = Q \Lambda S^{-1}$$

→ Spectral Theorem

An $n \times n$ symmetric matrix A has the foll properties

i) A has n real eigen values, counting multiplicity (how many times value is repeated).

ii) The dimension of the eigen space for each eigen value λ equals the multiplicity of λ as a root of the characteristic eqn.

iii) The eigen spaces are mutually orthogonal (ie the eigen vectors corresponding to different eigen values are orthogonal)

iv) A is orthogonally diagonalizable.

→ Spectral Decomposition Theorem

Let A be an $n \times n$ symmetric matrix, let $[u_1, u_2, \dots u_n]$ be an orthogonal basis for R^n consisting of eigen vectors A with corresponding eigen values $\lambda_1, \lambda_2, \dots \lambda_n$. Then there exists symmetric matrices $P_1, P_2 \dots P_n$ such that the foll results hold:

i) $A = \lambda_1 P_1 + \lambda_2 P_2 + \dots + \lambda_n P_n$

ii) Rank $P_i = 1 \quad \forall i$

iii) $P_i P_i = P_i \quad \forall i \quad \& \quad P_i P_j = 0$ for $i \neq j$

iv) $P_i u_i = u_i \quad \forall i \quad \& \quad P_i u_j = 0$ for $i \neq j$

Proj of b onto a $\quad P = \left(\frac{a^T b}{a^T a}\right) a$

matrix → $P = \left(\frac{a a^T}{a^T a}\right)$

$//^{ly}$ for $u_i \Rightarrow P = \frac{u_i u_i^T}{u_i^T u_i}$ → orthonormal $= 1$

q: $A = \begin{bmatrix} 3 & -4 \\ -4 & 3 \end{bmatrix}$

To find eigen values

$$A - \lambda I = 0$$

$$\begin{vmatrix} 3-\lambda & -4 \\ -4 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 6\lambda - 7 = 0$$

$$\lambda = -1, 7$$

$$(3-\lambda)x - 4y = 0.$$
$$-4x + (3-\lambda)y = 0$$

i) when $\lambda = -1$

$$\left.\begin{aligned} 4x - 4y &= 0 \\ -4x + 4y &= 0 \end{aligned}\right\} \text{same}$$

$y = 1$ (free var.)

$x = 1$

$$x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

ii) when $\lambda = 7$

$$\left.\begin{aligned} -4x - 4y &= 0 \\ -4x - 4y &= 0 \end{aligned}\right\} \text{same}$$

$$X_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$S = \begin{array}{cc} a & b \\ \end{array}$$
$$S = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}_{\text{orthogonal vectors}}$$

$$u_1 = \frac{a}{\|a\|} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$u_2 = \frac{b}{\|b\|} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Date
Page

$P_1 = u_1 u^T \Rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \times \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix} \Rightarrow \begin{pmatrix}1/2 & 1/2\\1/2 & 1/2\end{pmatrix}$

$P_2 = u_2 u_2^T \Rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix} \times \frac{1}{\sqrt{2}}\begin{pmatrix}-1 & 1\end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix} = \begin{pmatrix}1/2 & -1/2\\-1/2 & 1/2\end{pmatrix}$

$A = \lambda_1 P_1 + \lambda_2 P_2 = (-1)\begin{bmatrix}1/2 & 1/2\\1/2 & 1/2\end{bmatrix} + 7\begin{bmatrix}1/2 & -1/2\\-1/2 & 1/2\end{bmatrix}$

q. Find a spectral decomposition of matrix

$A = \begin{bmatrix}4 & 1 & -1\\1 & 4 & -1\\-1 & -1 & 4\end{bmatrix}$

$|A - \lambda I| = 0$

$\begin{vmatrix}4-\lambda & 1 & -1\\1 & 4-\lambda & -1\\-1 & -1 & 4-\lambda\end{vmatrix} = 0$

$\lambda^3 - 12\lambda^2 + 45\lambda - 54 = 0$

$\lambda = 3, 3, 6$

$(4-\lambda)x + y - z = 0$

$x + (4-\lambda)y - z = 0$

$-x - y + (4-\lambda)z = 0$

i) when $\lambda = 3$

$$\left.\begin{aligned} x + y - z &= 0\\ x + y - z &= 0\\ -x - y + z &= 0\end{aligned}\right\}\ \text{same}$$

y	z	x
1	0	-1
0	1	1

$X_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ $X_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\lambda = 6$

$$\left.\begin{aligned} -2x + y - z &= 0 \\ x - 2y - z &= 0 \end{aligned}\right\} \text{diff}$$

$$-x - y - 2z = 0$$

$$\frac{x}{\begin{vmatrix} 1 & -1 \\ -2 & -1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -2 & -1 \\ 1 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix}}$$

$$\frac{x}{-3} = \frac{-y}{3} = \frac{z}{3}$$

$$x_3 = \begin{pmatrix} -3 \\ -3 \\ 3 \end{pmatrix} \text{ or } \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$\therefore S = \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \end{array}$$

{a, b are not orthogonal as it is the same eigen value}

$a \not\perp b \quad a \perp c \quad b \perp c$

To get orthonormal vectors : GS process

$$u_1 = \frac{a}{\|a\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \|a\| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

$$u_2 = \frac{e}{\|e\|} \qquad e = b - (q_1^T b) q_1$$

$$e = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \left[\begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \right]$$

$$= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \left[\frac{-1}{\sqrt{2}} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \right]$$

$$= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 1 \end{pmatrix}$$

$$\|e\| = \sqrt{(1/2)^2 + (1/2)^2 + 1^2} = \frac{\sqrt{6}}{2}$$

$$u_2 = \frac{2}{\sqrt{6}} \begin{pmatrix} 1/2 \\ 1/2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix}$$

$$u_3 = \frac{E}{\|E\|} \qquad E = c - (q_1^T b) q_1 - (q_2^T c) q_2$$

$$E = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} - \left[\begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \right]$$

$$- \left[\begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix} \right]$$

$$\Rightarrow \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} - 0 - 0$$

without calculating we find 0 because $a \perp c \Downarrow q_1 \perp c$ $\quad b \perp c \Downarrow q_2 \perp c$

$$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \qquad \|E\| = \sqrt{3}$$

$$E = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{pmatrix}$$

$$P_1 = u_1 u_1^T = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$P_2 = u_2 u_2^T = \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} 1/6 & 1/6 & 2/6 \\ 1/6 & 1/6 & 2/6 \\ 2/6 & 2/6 & 4/6 \end{bmatrix}$$

$$P_3 = u_3 u_3^T = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 1/3 & -1/3 \\ 1/3 & 1/3 & -1/3 \\ -1/3 & -1/3 & 1/3 \end{bmatrix}$$

$$\therefore A = \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3$$

$$= 3\begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 3\begin{bmatrix} 1/6 & 1/6 & 2/6 \\ 1/6 & 1/6 & 2/6 \\ 2/6 & 2/6 & 4/6 \end{bmatrix} + 6\begin{bmatrix} 1/3 & 1/3 & -1/3 \\ 1/3 & 1/3 & -1/3 \\ -1/3 & -1/3 & 1/3 \end{bmatrix}$$

15th April, 2024

Singular Value Decomposition

Any m by n matrix A can be factored into
$A = U\Sigma V^T$
(orthogonal)(diagonal)(orthogonal)
The columns of U (m by m) are eigen vectors of AA^T and the columns of V (n by n) are eigen vectors of A^TA. The r singular values on the diagonal of Σ (m by n) are the square roots of non zero eigen values of both AA^T and A^TA.

Note: 1. For positive definite matrices Σ is Λ (square symmetric)
$U\Sigma V^T \rightarrow Q\Lambda Q^T$

2. U and V are orthonormal bases for all 4 fundamental subspaces:

first r columns of U : column space of A
last m-r columns of U : left null space of A
first r columns of V : row space of A
last n-r columns of V : null space of A

REMARK 1. Eigen vectors of AA^T and A^TA must go into columns of U and V :

$$\begin{aligned} AA^T &= (U\Sigma V^T)(U\Sigma V^T)^T \\ &= (U\Sigma V^T)(V^{T^T}\Sigma^T U^T) \\ &= U\Sigma V^T V \Sigma^T U^T \\ &= U\Sigma\Sigma^T U^T \end{aligned}$$

$A^TA \rightarrow$ square symmetric

→ Different cases of SVD (3) (1, 2 (x))

Case 1: Let A be the short matrix say order 2x3

$$A_{2\times3} = U_{2\times2}\,\Sigma_{2\times3}\,V^T_{3\times3}$$

Step 1. Find AA^T

AA^T	A^TA
2x2 easier	3x3

Step 2. Find Eigen values of AA^T say λ_1, λ_2

Step 3. Find corresponding Eigen vectors say x_1, x_2

Step 4. Normalize x_1, x_2 to u_1, u_2 such that $U = [u_1, u_2]_{2\times2}$

Step 5. Singular values are $\sqrt{\lambda_1}, \sqrt{\lambda_2} = \sigma_1, \sigma_2$

such that

$$\Sigma = \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 \end{bmatrix}_{2\times3}$$

Note: $\lambda_1 > \lambda_2$, σ_1, σ_2 ascending order

Step 6. No need to find A^TA, Eigen value of A^TA will be $\lambda_1, \lambda_2, 0$

* eigen values of A^TA ($3\times3 \Rightarrow 3$ values) will be same as eigen values of AA^T ($2\times2 \Rightarrow 2$ values) and extra value will be 0.

Step 7. Use formula $v_i = \dfrac{A^Tu_i}{\sigma_i}$ (or) $v_i^T = \dfrac{u_i^TA}{\sigma_i}$ Find v_1, v_2

Step 8. Find v_3 using Grams Schmidt process.

Step 9. Matrix $V = [v_1, v_2, v_3]_{2\times3}$

Step 10. $A_{2\times3} = U_{2\times2}\,\Sigma_{2\times3}\,V_{3\times3}$

Date
Page

Q1. Find the SVD of $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}_{2\times3}$

Sol. $AA^T = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$

$$\lambda^2 - 34\lambda + 225 = 0$$

Eigen Values : $\lambda_1 = 25 \quad \lambda_2 = 9$ (always take $\lambda_1 > \lambda_2$)

$\lambda = 25$

$$(17-25)x + 8y = 0$$
$$8x + (17-25)y = 0$$

$$\left.\begin{aligned} -8x + 8y &= 0 \\ 8x + -8y &= 0 \end{aligned}\right\} \text{same}$$

$y \to$ free variable

$y = 1 \Rightarrow x = 1 \qquad X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = 9$

$$\left.\begin{aligned} 8x + 8y &= 0 \\ 8x + 8y &= 0 \end{aligned}\right\} \text{same}$$

$y \to$ free variable $\qquad X_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$y = 1 \Rightarrow x = -1$

$$u_1 = \frac{x_1}{\|x_1\|} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$u_2 = \frac{x_2}{\|x_2\|} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$U = [u_1, u_2] = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Eigen values of A^TA : 25, 9, 0

$$v_1^T = \frac{u_1^T A}{\sigma_1} = \frac{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}_{1\times2} \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}_{2\times3}}{5} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}^T$$

$$v_2^T = \frac{u_2^T A}{\sigma_2} = \frac{\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}}{3} = \begin{bmatrix} -1/3\sqrt{2} \\ 1/3\sqrt{2} \\ -4/3\sqrt{2} \end{bmatrix}^T$$

to find v_3,

v_3 should be orthogonal to v_2

$$v_3 \perp v_1 \Rightarrow \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y + 0z = 0 \Rightarrow x + y = 0$$

$$v_3 \perp v_2 \Rightarrow \frac{-1}{3\sqrt{2}}x + \frac{1}{3\sqrt{2}}y - \frac{4}{3\sqrt{2}}z = 0 \Rightarrow -x + y - 4z = 0$$

$z \rightarrow$ free variable $\quad z = 1$

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -1 & 1 & -4 & 0 \end{array}\right]$$

$R_2 \rightarrow R_2 + R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & -4 & 0 \end{array}\right]$$

$$2y - 4z = 0$$

$$\boxed{y = 2}$$

$$x + y = 0$$

$$\boxed{x = -2}$$

$$\therefore v_3 = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$$\begin{array}{cccc} & v_1 & v_2 & v_3 \\ V^T = & \left[\begin{array}{c} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{array}\right. & \begin{array}{c} -1/3\sqrt{2} \\ 1/3\sqrt{2} \\ -4/3\sqrt{2} \end{array} & \left.\begin{array}{c} -2/3 \\ 2/3 \\ 1/3 \end{array}\right] \end{array}$$

$A = U \Sigma V^T$

$$= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/3\sqrt{2} & -2/3 \\ 1/\sqrt{2} & 1/3\sqrt{2} & 2/3 \\ 0 & -4/3\sqrt{2} & 1/3 \end{bmatrix}$$

Case 2:

Q2. $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & 2 \end{bmatrix}_{3\times2}$

Sol. $A^TA = \begin{bmatrix} 9 & -1 \\ -1 & 9 \end{bmatrix}_{2\times2}$

eigen values: $\lambda^2 - 18\lambda + 80 = 0$

$\lambda_1 = 10 \quad \lambda_2 = 8$

$\lambda = 10$

$(9-10)x - y = 0 \quad \Rightarrow -x - y = 0$

$-x + (9-10)y = 0 \quad \Rightarrow -x - y = 0$ } same

$y = 1 \quad x = -1$

$(-1, 1)$

or $(1, -1)$

$\lambda = 8$

$x - y = 0$

$-x + y = 0$

$y = 1 \quad x = 1$

$(1, 1)$

$V_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \quad V_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \quad V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

singular values of A are:

$\sigma_1 = \sqrt{\lambda_1} = \sqrt{10} \quad \sigma_2 = \sqrt{\lambda_2} = \sqrt{8}$

Eigen values of AA^T : 10, 8, 0

$$u_1 = \frac{Av_1}{\sigma_1} = \frac{\begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & 2 \end{bmatrix}_{3\times 2} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}_{2\times 1}}{\sqrt{10}} = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \\ 0 \end{bmatrix}_{3\times 1}$$

$$u_2 = \frac{Av_2}{\sigma_2} = \frac{\begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}}{\sqrt{10}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$u_3 \Rightarrow u_1 \perp u_3 \Rightarrow x - 2y + 0z = 0$

$u_2 \perp u_3 \Rightarrow 0x + 0y + z = 0$

$z = 0$

$$\begin{array}{ccc} x & y & z \end{array}$$
$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$x = 2y$

free var, $y = 1$

$\Rightarrow x = 2$

$$u_3 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$$

$$\therefore A = U\Sigma V^T = \begin{bmatrix} 1/\sqrt{5} & 0 & 2/\sqrt{5} \\ -2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{8} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

17th April, 2024

$U_{2\times 2}$ $\overset{?}{V}_{3\times 3}$

$Q^{-1}Q = I$

$Q^TQ = I$

(orthonormal)

$A = U\Sigma V^T$

pre multiply by U^T

$U^TA = U^TU\Sigma V^T$

$U^TA = \Sigma U^T$

$U_i^TA = \sigma_i V_i^T$

$$\boxed{v_i^T = \frac{U_i^TA}{\sigma_i}}$$

Date
Page

$\overset{?}{U_{3\times3}}$ $V_{2\times2}$

$A = U\Sigma V^T$

post multiply V

$AV = U\Sigma V^T V$

$AV = U\Sigma$

$Av_i = u_i \sigma_i$

$$\boxed{u_i = \frac{Av_i}{\sigma_i}}$$

q3: Find SVD of $\begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix}_{3\times3} = A$

* whenever 3×3 cross verify for symmetric & positive definite

Sol. i) $A^T = A$ (symmetric)

ii) A is positive definite

|

determinant test

$|6| > 0$

$\begin{vmatrix} 6 & -2 \\ -2 & 6 \end{vmatrix} = 32 > 0$

$\begin{vmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{vmatrix} = 144 > 0$

$\therefore$ A is symmetric & +ve definite

$\Rightarrow$ SVD $\Rightarrow$ orthogonally diagonalizable

$$\boxed{A = U\Sigma V^T = Q\Lambda Q^T}$$

To find eigen values — Q

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 6-\lambda & -2 & -1 \\ -2 & 6-\lambda & -1 \\ -1 & -1 & 5-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 17\lambda^2 + 90\lambda - 144 = 0$$

$$\lambda = 8, 6, 3 \quad (\lambda_1 > \lambda_2 > \lambda_3)$$

To find eigen vectors

$$(6-\lambda)x - 2y - 1z = 0$$
$$-2x + (6-\lambda)y - 1z = 0$$
$$-x - y + (5-\lambda)z = 0$$

i) $\lambda = 8$ $\quad X_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

ii) $\lambda = 6$ $\quad X_2 = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$

iii) $\quad X_3 = \begin{pmatrix} +1 \\ +1 \\ +1 \end{pmatrix}$

symmetric matrix with diff eigen values, eigen vectors are orthogonal

$$q_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} \quad q_2 = \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} \quad q_3 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$A = Q\Lambda Q^T = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 8 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

Date
Page

Covariance Matrix (PCA → Principle Component Analysis)
dimensionality reduction of data set

Consider a 2-D data given by a set of weights and heights of N college students.
Let X_i denote the observation vector in R^2 that lists the weight and height of the j^{th} student. If w denotes weight and h denotes height then the matrix of observations has the form

$$\begin{bmatrix} w_1 & w_2 & \cdots & w_n \\ h_1 & h_2 & \cdots & h_n \end{bmatrix}$$

set of obs. vectors can be visualized as 2D scatter plot.

Data is around mean ⇒ sample mean = 0

→ Mean and Covariance

Let $[x_1 \dots x_n]$ be a $p * N$ matrix of obs. such as described above
sample mean M of observation vectors $X_1, X_2, \dots X_n$

$$M = \frac{1}{N}[X_1 + X_2 + \dots + X_n]$$

sample mean is the point in the centre of the scatter plot for $x = 1, 2 \dots N$

$$\hat{x}_k = x_k - M$$

Cols. of $p * N$ matrix $B = [\hat{x}_1, \hat{x}_2, \hat{x}_3]$

sample mean = 0 is said to be mean deviation form.

Now sample mean is from

$$S = \frac{1}{N-1} B B^T$$

$\frac{1}{N-1} \longrightarrow$ mean deviation

$\downarrow$ +ve semi definite

deviation $\downarrow$ relationship $\uparrow$

sample mean $\rightarrow$ Mean Deviation $\rightarrow$ Covariance matrix

$\downarrow$

gives relation

since any matrix of form BB^T is +ve semi definite $\Rightarrow$ S is also +ve semi definite

Date
Page

eg. 3 measurements are made on each of 4 individuals in a random sample from a population. The observation vectors are

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad X_2 = \begin{bmatrix} 4 \\ 2 \\ 13 \end{bmatrix} \quad X_3 = \begin{bmatrix} 7 \\ 8 \\ 1 \end{bmatrix} \quad X_4 = \begin{bmatrix} 8 \\ 4 \\ 5 \end{bmatrix}$$

Compute sample mean and covariance matrix

sol. Sample mean

$$M = \frac{1}{4}\left[X_1 + X_2 + X_3 + X_4\right]$$

$$= \begin{bmatrix} 5 \\ 4 \\ 5 \end{bmatrix}$$

Mean deviation $\hat{x}_k = x_k - M$

$$\hat{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 5 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ -4 \end{bmatrix}$$

$$\hat{x}_2 = \begin{bmatrix} 4 \\ 2 \\ 13 \end{bmatrix} - \begin{bmatrix} 5 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 8 \end{bmatrix}$$

$$\hat{x}_3 = \begin{bmatrix} 7 \\ 8 \\ 1 \end{bmatrix} - \begin{bmatrix} 5 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -4 \end{bmatrix}$$

$$\hat{x}_4 = \begin{bmatrix} 8 \\ 4 \\ 5 \end{bmatrix} - \begin{bmatrix} 5 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore B = \begin{array}{c} \begin{array}{cccc} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 & \hat{x}_4 \end{array} \\ \begin{bmatrix} -4 & -1 & 2 & 3 \\ -2 & -2 & 4 & 0 \\ -4 & 8 & -4 & 0 \end{bmatrix} \end{array} \quad 3 \times 4$$

Parameters, no. of students

$N = 4$

Covariance matrix

$$S = \frac{1}{N-1} BB^T$$

$$= \frac{1}{3} \begin{bmatrix} -4 & -1 & 2 & 3 \\ -2 & -2 & 4 & 0 \\ -4 & 8 & -4 & 0 \end{bmatrix} \begin{bmatrix} -4 & -2 & -4 \\ -1 & -2 & 8 \\ 2 & 4 & -4 \\ 3 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 6 & 0 \\ 6 & 8 & -8 \\ 0 & -8 & 32 \end{bmatrix}$$

The diagonal entry S_{ii} in S is called variance of x_{ii}.

Variance of $x_1 = 10$

Variance of $x_3 = 32$

The variance of 10 indicates that the set of third entries in the response vectors contains a

Pseudo - inverse

The pseudo-inverse A^+ of an $m \times n$ matrix A is an extension of the inverse of a square matrix to non-square matrices and to singular (non-invertible) square matrices.

The pseudo-inverse matrix A^+ is an $n \times m$ matrix with the foll. properties

* If $m \geq n$, then $A^T A$ is invertible and $A^+ = (A^T A)^{-1} A^T$ and so $A^+ A = I$, A^+ is a left inverse of A

* If $m \leq n$, then $A^T A$ is invertible and $A^+ = A^T (A A^T)^{-1}$ and so $A A^+ = I$, A^+ is a right inverse of A

18th April, 2024

→ Least Squares

For a rectangular system $Ax = b$ the least squares solution comes from the normal eqn.

$$A^T A \hat{x} = A^T b$$

Here in solving $Ax = b$ has 2 possibilities

Dependent rows,

i) With dependent rows, $Ax = b$ may have no solution

then instead of $Ax = b$ we can solve $A^T A \hat{x} = A^T b$

$$\hat{x} = \underbrace{(A^T A)^{-1} A^T}_{A^+} b$$

ii) If A has dependent columns then $\hat{x}$ will not be unique.

$$A^T A \hat{x} = A^T b$$

min len sol. $= x^+$

x^+ is the best soln. of $Ax = b$ (with no soln.)

$x^+ = A^+ B$

pseudo inverse

- Using SVD

Suppose $A = U \Sigma V^T$ is SVD of A. Then the pseudo inverse $A^+ = U \Sigma^+ U^T$ where U^+ is diagonal matrix formed by taking reciprocal of each non zero element in Σ, leaving the 0s in place

q. Using SVD to find least square soln. of the system.

$$x_2 + 2x_3 = 5$$
$$x_1 + x_3 = 1$$

we have $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}_{2\times 3}$ $b = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

using SVD $\underset{2\times3}{A} = \underset{2\times2}{U}\,\underset{2\times3}{\Sigma}\,\underset{3\times3}{V^T}$

$\underset{3\times2}{A^T}\;\underset{2\times3}{A}$ $\quad$ $\underset{2\times3}{A}\;\underset{3\times2}{A^T}$

2×2 ✓

$$AA^T = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}$$

$$\lambda^2 - 7\lambda + 6 = 0$$
$$\lambda = 6, 1 \quad (\lambda_1 > \lambda_2)$$

$\lambda = 6$ $\quad (5-6)x + 2y = 0$
$\quad 2x + (2-6)y = 0$

$$\Rightarrow \left.\begin{aligned} -x + 2y &= 0 \\ 2x - 4y &= 0 \\ x - 2y &= 0 \end{aligned}\right] \text{same}$$

$y = 1 \quad x = 2 \quad X_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\lambda = 1$ $\quad 4x + 2y = 0$
$\quad 2x + y = 0$

$$\Rightarrow \left.\begin{aligned} 2x + y &= 0 \\ 2x + y &= 0 \end{aligned}\right] \text{same}$$

let $y = 2 \quad x = -1$

$$\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$U = [u_1 \quad u_2]$

$$u_1 = \frac{x_1}{\|x_1\|} \qquad u_2 = \frac{x_2}{\|x_2\|}$$

Date
Page

$$U = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

To find Σ $\quad \sigma_1 = \sqrt{\lambda_1} = \sqrt{6}$

$\sigma_2 = \sqrt{\lambda_2} = \sqrt{1} = 1$

$$\underset{2\times 3}{\Sigma} = \begin{bmatrix} \sqrt{6} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

To find V $\quad v_i^T = \dfrac{u_i^T A}{\sigma_i}$

$$v_1^T = \frac{u_1^T A}{\sigma_1} = \begin{bmatrix} \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & \frac{5}{\sqrt{30}} \end{bmatrix}$$

$$v_2^T = \frac{u_2^T A}{\sigma_2} = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & 0 \end{bmatrix}$$

To find v_3 $\quad v_3^T \perp v_1^T$, $v_3^T \perp v_2^T$, $v_3^T = (x\ y\ z)$

$v_3 \cdot v_1^T = 0 \qquad x + 2y + 5z = 0$

$v_3 \cdot v_2^T = 0 \qquad 2x - y = 0$

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 2 & -1 & 0 & 0 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$$\simeq \begin{array}{c} \left[\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 0 & -5 & -10 & 0 \end{array}\right] \\ \begin{array}{ccc} x & y & z \end{array} \end{array}$$

let $z = 1 \qquad y = -2 \qquad x = -1$

$$v_3^T = \begin{bmatrix} \frac{-1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \end{bmatrix}$$

$$V^T = \begin{bmatrix} v_1^T \\ v_2^T \\ v_3^T \end{bmatrix} = \begin{bmatrix} 1/\sqrt{30} & 2/\sqrt{30} & 5/\sqrt{30} \\ 2/\sqrt{5} & -1/\sqrt{5} & 0 \\ 1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \end{bmatrix}$$

now $\quad A^+ = V \Sigma^+ U^T$

$$= \begin{bmatrix} 1/\sqrt{30} & 2/\sqrt{5} & 1/\sqrt{6} \\ 2/\sqrt{30} & -1/\sqrt{5} & 2/\sqrt{6} \\ 5/\sqrt{30} & 0 & -1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

$$x^T = A^+ b =$$

$$= \begin{bmatrix} -5/6 \\ 8/6 \\ 11/6 \end{bmatrix}$$

CWP

Q.11 Find SVD

i) $\begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}$

$\downarrow$

Not symmetric

then procedure (tall) & (short matrix)
3×2 $\quad$ 2×3

Sol. $A^TA = \begin{bmatrix} 8 & 2 \\ 2 & 5 \end{bmatrix}$

To find eigen values

$$\lambda^2 - 13\lambda + 36 = 0$$

$$\lambda = 9, 4 \quad (\lambda_1 > \lambda_2)$$

To find eigen vectors

$$(A^TA - \lambda I)x = 0$$

$$(8-\lambda)x + 2y = 0$$
$$2x + (5-\lambda)y = 0$$

$\lambda = 9$
$$\left.\begin{aligned} -x + 2y &= 0 \\ 2x - 4y &= 0 \end{aligned}\right] \text{same}$$

let $y = 1$ $\quad$ $X_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$x = 2$

$\lambda = 4$
$$\left.\begin{aligned} 4x + 2y &= 0 \\ 2x + y &= 0 \end{aligned}\right] \text{same}$$

let $y = 2$ $\quad$ $x = 1$ $\quad$ $X_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$V = [V_1 \;\; V_2]$

$V_1 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$ $\quad$ $V_2 = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$

$$V = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

To find Σ

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{9} = 3$$

$$\sigma_2 = \sqrt{\lambda_2} = \sqrt{4} = 2$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}_{2\times 2}$$

To find U

$$u_1 = \frac{Av_1}{\sigma_1} \qquad u_2 = \frac{Av_2}{\sigma_2}$$

Date
Page

ii) $A = \begin{bmatrix} 4 & 2 & 0 \\ -2 & -1 & 0 \end{bmatrix}$

$A = \underset{2\times2}{U} \underset{2\times3}{\Sigma} \underset{3\times3}{V^T}$

V: $\underset{3\times2}{A^T} \underset{2\times3}{A}$, 3×3

U: $\underset{2\times3}{A} \underset{3\times2}{A^T}$, 2×2

$$AA^T = \begin{bmatrix} 4 & 2 & 0 \\ -2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 2 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 20 & -10 \\ -10 & 5 \end{bmatrix}$$

To find eigen values

$$\lambda^2 - 25\lambda = 0$$

$$\lambda = 25, 0 \quad (\lambda_1 > \lambda_2)$$

To find eigen vectors

$$(20-\lambda)x - 10y = 0$$

$$10x + (5-\lambda)y = 0$$

when $\lambda = 25$

$$\left.\begin{aligned} -5x - 10y &= 0 \\ -10x - 20y &= 0 \end{aligned}\right] \text{same}$$

$$X_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

when $\lambda = 0$

$$X_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$U = [u_1 \quad u_2]$

$$u_1 = \frac{x_1}{||x_1||} = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$$u_2 = \frac{x_2}{||x_2||} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

To find Σ

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{25} = 5$$

$$\underset{2\times3}{\Sigma} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

To find $V_{3\times3}$

eigen values 25, 0, 0

To find eigen vector $v_i = \frac{u_i^T A}{\sigma_i}$

$$v_1^T = \frac{u_1^T A}{\sigma_1} = \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & 0 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} -2/\sqrt{5} \\ -1/\sqrt{5} \\ 0 \end{bmatrix}$$

To find v_2, v_3

v_2 & $v_3 \perp v_1$

$$-2x - y + 0z = 0$$

$$\begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix} \perp \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

y	z	x	
2	0	-1	v_2'
0	2	0	v_3'

} they are already orthogonal else go for GS process

now $v_2' \perp v_3'$

$$v_2 = \frac{v_2'}{\|v_2\|} = \cancel{\begin{pmatrix} 2/\sqrt{5} \\ 0 \\ -1/\sqrt{5} \end{pmatrix}} \begin{pmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{pmatrix}$$

$$v_3 = \frac{v_3'}{\|v_3\|} = \cancel{\begin{pmatrix} 0 \\ 2/\sqrt{5} \\ 0 \end{pmatrix}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$A = U\Sigma V^T = \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} & 0 \\ -1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$