URBAN EDGE

30th December, 2022

UNIT-5

Special Functions

(PROOFS included : ☒)

→ Gamma function
→ Beta function
→ Bessel's function
→ Legendre's function

* Special functions : These help us evaluate certain definite integral which are either difficult / impossible to evaluate

* Gamma function and Beta function belong to the category of special transcendental functions and are defined in terms of improper definite integral

$$\int_{\infty}^{a} f(x)\,dx \quad (or) \quad \int_{a}^{\infty} f(x) \quad (or) \quad \int_{\infty}^{\infty} f(x)$$

Gamma Function

* Gamma function is an improper integral which is dependent on the parameter n and defined by

$$\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1}\,dx \quad (n>0)$$

* Introduced by Euler

* Also known as Euler's integral of second kind

eg: 1. $\int_0^{\infty} e^{-x} x^3\,dx = \Gamma(4)$ $\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1}$

$n-1 = 3$

2. $\int_0^{\infty} e^{-t} t^7\,dt = \Gamma(8)$

→ Another form of Gamma function

consider, $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx \quad (n>0)$

let $x = t^2$

Then $dx = 2t\, dt$

If $x \to 0$, $t \to 0$

$x \to \infty$, $t \to \infty$

$\therefore \Gamma(n) = \int_0^\infty e^{-t^2} (t^2)^{n-1} \cdot 2t\, dt$

$\Gamma(n) = 2\int_0^\infty e^{-t^2} \cdot t^{2n-1}\, dt$

→ Properties of Gamma Function

1. Recurrence Formula for Gamma Function : $\Gamma(n+1) = n\Gamma(n)$

PROOF : By definition

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx \quad -①$$

replacing n with $n+1$

$$\Gamma(n+1) = \int_0^\infty e^{-x} \cdot x^n \, dx \quad -②$$

using integration by parts

$$\Gamma(n+1) = x^n (-e^{-x})\Big|_0^\infty - \int_0^\infty (-e^{-x})\, n x^{n-1} dx$$

$$= 0 + n\int_0^\infty e^{-x} x^{n-1} dx$$

$$= n\Gamma(n) \quad \text{from ①}$$

$$= RHS$$

$(\infty \times 0) - 0$

$x^n e^{-x} = \frac{x^n}{e^x}$

LH rule

$= \frac{n!}{e^x} \to 0$ as $x \to \infty$

2. $\Gamma(n+1) = n!$ where n is an positive integer

PROOF : wkt $\Gamma(n+1) = n\Gamma(n) \quad -①$

replacing n by $n-1$ in ①

$$\Gamma(n) = (n-1)\Gamma(n-1) \quad -(2)$$

sub in (1)

$$\Gamma(n+1) = n(n-1)\Gamma(n-1) \quad -(3)$$

from (1) $\quad \Gamma(n-1) = (n-2)\Gamma(n-2)$

sub in (3)

$$\Gamma(n+1) = n(n-1)(n-2)\Gamma(n-2)$$

$$\Gamma(n+1) = n(n-1)(n-2)(n-3)\Gamma(n-3)$$

$$\vdots$$

$$\Gamma(n+1) = n(n-1)(n-2)(n-3)\dots\Gamma(1) \quad -(4)$$

By definition $\quad \Gamma(n) = \int_0^\infty e^{-x} x^{n-1}$

$$\Gamma(1) = \int_0^\infty e^{-x}\,dx = -e^{-x}\Big|_0^\infty = -e^{-\infty} + e^0 = 1$$

$$\Gamma(1) = 1$$

sub. this in (4)

$$\Gamma(n+1) = n(n-1)(n-2)(n-3)\dots(1)$$

$$\Gamma(n+1) = n!$$

eg: $\Gamma(4) = 3!$ $\quad \Gamma(n+1) = n!$

→ Standard Functions

1. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

PROOF: wkt $\quad \Gamma(n) = 2\int_0^\infty e^{-t^2} \cdot t^{2n-1}\,dt \quad (1)$

Taking $n = 1/2$ $\quad \Gamma(1/2) = 2\int_0^\infty e^{-t^2}\,dt \quad -(2)$

② can be written as

$$\Gamma(1/2) = 2\int_0^{\infty} e^{-x^2}\,dx \quad -③$$

and
$$\Gamma(1/2) = 2\int_0^{\infty} e^{-y^2}\,dy \quad -④$$

consider

$$\Gamma(1/2)\,\Gamma(1/2) = 2\cdot\int_0^{\infty} e^{-x^2}\,dx \cdot 2\int_0^{\infty} e^{-y^2}\,dy$$

$$= 4\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dx\,dy$$

$\longrightarrow$ double integral

take when limits are constants

$$\therefore [\Gamma(1/2)]^2 = 4\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dx\,dy$$

let us convert the above double integral to the Polar form by taking the substitution $x = r\cos\theta$, $y = r\sin\theta$

then $dx\,dy = r\,dr\,d\theta$

$dx\,dy = J\,dr\,d\theta$

Jacobian

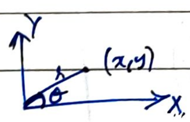
$J = x$

$x \rightarrow 0$ to ∞

$y \rightarrow 0$ to ∞

$r \rightarrow 0$ to ∞

$\theta \rightarrow 0$ to $\pi/2$



$$\therefore [\Gamma(1/2)]^2 = 4\int_0^{\pi/2}\left[\int_0^{\infty} e^{-r^2}\cdot r\,dr\right]d\theta$$

January, 2023

let $r^2 = t$

$2r\,dr = dt$

If $r \rightarrow 0$, $t \rightarrow 0$

$r \rightarrow \infty$, $t \rightarrow \infty$

$$\therefore [\Gamma(1/2)]^2 = 4\int_0^{\pi/2}\int_0^{\infty} e^{-t}\,\frac{dt}{2}\,d\theta$$

$$= 2\int_0^{\pi/2} \{-e^{-t}\}_0^{\infty} \, d\theta$$

$$= 2\int_0^{\pi/2} [0+1] \, d\theta$$

$$= 2\theta\Big|_0^{\pi/2} = 2\cdot\frac{\pi}{2} = \pi$$

$$[\Gamma(1/2)]^2 = \pi$$

$$\boxed{\Gamma(1/2) = \sqrt{\pi}}$$

Note 1. If n is a positive fraction,
$\Gamma(n) = (n-1)\Gamma(n-1)\ldots$ go on decreasing by 1. The series of factor being continued as long as factor remains +ve. the last factor being Γ(last factor)

eg: $\Gamma(9/2) = \frac{7}{2}\Gamma\left(\frac{7}{2}\right)$

$$= \frac{7}{2}\cdot\frac{5}{2}\cdot\Gamma\left(\frac{5}{2}\right)$$

$$= \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\Gamma\left(\frac{3}{2}\right)$$

$$= \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{105\sqrt{6}}{16}$$

(or) directly

$$\Gamma(9/2) = \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$$

$$\Gamma(15/2) = \frac{13}{2}\cdot\frac{11}{2}\cdot\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$$

2. If p is a positive integer, $\Gamma(p) = (p-1)!$

eg: $\Gamma(4) = 3!$

3. If p is 0 or a negative integer, $\Gamma(p)$ is not defined
Further $\Gamma(p) > 0$ for $p > 0$

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4. For a negative non-integer p, $\Gamma(p)$ is computed through the formula

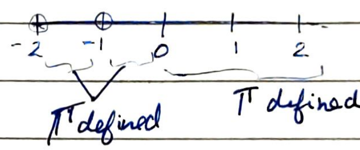
$$\Gamma(p+1) = (p)\,\Gamma(p) \quad \text{in the form} \quad \Gamma(p) = \frac{\Gamma(p+1)}{p}$$

eg: i) $\Gamma(-1/2) = \dfrac{\Gamma(-1/2+1)}{-1/2} = -2\,\Gamma(1/2) = -2\sqrt{\pi}$

ii) $\Gamma(-5/2) = \dfrac{\Gamma(-5/2+1)}{-5/2} = \dfrac{-2}{5}\,\Gamma(-3/2) = -\dfrac{2}{5}\,\dfrac{\Gamma(-3/2+1)}{-3/2}$

$= \dfrac{4}{15}\,\Gamma(-1/2) = \dfrac{4}{15}(-2\sqrt{\pi})$

* basic idea : to make everything inside Γ func. +ve.



q: Evaluate $\displaystyle\int_0^1 (x \log x)^4\, dx$

$= \displaystyle\int_{+\infty}^{0} x^4 (\log x)^4\, dx$

$= \displaystyle\int_{+\infty}^{0} x^4 \; t^4 \,(-e^{-t})\, dt$

$= -\displaystyle\int_{\infty}^{0} t^4 e^{-5t}\, dt$

$= \displaystyle\int_{0}^{\infty} t^4 e^{-5t}\, dt$

$\Gamma(n+1) = n!$ $\quad = 4!$

$\log x = -t$

$x = e^{-t}$

$dx = -e^{-t}\, dt$

x	0	1
t	$+\infty$	0

4th January, 2023

Beta Function

* The Beta function is denoted by $\beta(m,n)$ and is defined by

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad , (m>0,\ n>0)$$

is convergent for $m>0,\ n>0$

* This function is also known as Euler's integral of first kind

eg: i) $\int_0^1 x^{25}(1-x)^{16}\,dx = \beta(26,17)$

ii) $\int_0^1 \sqrt{x}\,(1-x)^3\,dx = \beta(3/2, 4)$

iii) $\int_0^1 x^{-3}(1-x)^5\,dx =$ is not a Beta func. as $m<0$

→ Properties of β function.

1. Symmetry : $\beta(m,n) = \beta(n,m)$

PROOF: By definition : $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

$$= \int_0^1 (1-x)^{m-1}[1-(1-x)]^{n-1}\,dx \qquad \vdots\ \int_0^a f(x)\,dx = \int_0^a f(0+a-x)\,dx$$

KING'S RULE

$$= \int_0^1 x^{n-1}(1-x)^{m-1}\,dx$$

$$= \beta(n,m)$$

2. Beta function in terms of trignometric functions

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta$$

PROOF: By definition $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

Let $x = \sin^2\theta$

$1-x = \cos^2\theta$

$dx = 2\sin\theta\cos\theta\,d\theta$

x	0	1
θ	0	$\pi/2$

$$\beta(m,n) = \int_0^1 (\sin^2\theta)^{m-1}(\cos^2\theta)^{n-1} \cdot 2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^1 \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta\,d\theta$$

eg: i) $2\int_0^{\pi/2} \sin^6\theta\cos^4\theta\,d\theta = \beta(7/2, 5/2)$

$2m-1 = 6$

$m = 7/2$

$2n-1 = 4$

$n = 5/2$

Case i) $\int_0^{\pi/2} \sin^p\theta\cos^q\theta\,d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

$(p > -1,\ q > -1)$

$2m-1 = p$

$2n-1 = q$

$m > 0$

$2m > 0$

$2m-1 > -1$

$p > -1$

eg: $\int_0^{\pi/2} \sin^8\theta\cos^5\theta\,d\theta = \frac{1}{2}\beta\left(\frac{9}{2}, 3\right)$

Case ii) If $\cancel{p} \to q$ & $q \to 0$ (replacing) in case i

$$\int_0^{\pi/2} \sin^q\theta\,d\theta = \frac{1}{2}\beta\left(\frac{q+1}{2}, \frac{1}{2}\right)$$

$\beta(p,q) = 2\int_0^{\pi/2} \sin^{2p-1}\theta\cos^{2q-1}\theta\,d\theta$

$2\int_0^{\pi/2} \frac{\sin^{2p}\theta\cos^{2q}}{\sin\theta\cos\theta}$

$2\int_0^{\pi/2} \frac{\sin^p\theta\cos^q\theta}{\sin\theta\cos\theta}$

3. Beta function expressed as an improper integral

$$\beta(p,q) = \int_0^\infty \frac{y^{p-1}}{(1+y)^{p+q}}\,dy = \int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}}\,dy$$

Let $x = \dfrac{y}{1+y}$ $\qquad dx = \dfrac{(1+y) - y(1)}{(1+y)^2}dy = \dfrac{1}{(1+y)^2}dy$

$$1 - x = 1 - \frac{y}{1+y} = \frac{1}{1+y}$$

If $x=0$ $\quad y = 0$

$x = 1$ $\quad y = \infty$

$$\int_0^\infty \left(\frac{y}{1+y}\right)^{p-1}\left[1 - \frac{y}{1+y}\right]^{q-1} \cdot \frac{1}{(1+y)^2}\,dy$$

$$\int_0^\infty \left(\frac{y}{1+y}\right)^{p-1}\left(\frac{1}{1+y}\right)^{q-1} \cdot \frac{1}{(1+y)^2}\,dy$$

$$\int_0^\infty \frac{(y)^{p-1}}{(1+y)^{p-1+2}} \frac{1}{(1+y)^{q-1}}\,dy$$

$$\int_0^\infty \frac{y^{p-1}}{(1+y)^{p+q}}\,dy$$

$1 = \dfrac{y}{1+y}$

$\cancel{1+y = y}$

$x(1+y) = y$

$x + xy = y$

$x = y(1-x)$

$y = \dfrac{x}{1-x}$

using the symmetry property

$$\beta(p,q) = \int_0^\infty \frac{y^{p-1}}{(1+y)^{p+q}}\,dy = \int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}}\,dy$$

→ The relation between β and Γ function

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

PROOF: $\Gamma(m) = 2\int_0^\infty e^{-x^2} x^{2m-1}\, dx$ — ①

$\Gamma(n) = 2\int_0^\infty e^{-y^2} y^{2n-1}\, dy$ — ②

$\Gamma(m+n) = 2\int_0^\infty e^{-r^2} \cdot r^{2(m+n)-1}\, dr$ — ③

$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

Consider

$\Gamma(m)\Gamma(n)$

$$= 2\int_0^\infty e^{-x^2} x^{2m-1}\, dx \cdot 2\int_0^\infty e^{-y^2} y^{2n-1}\, dy$$

$$= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)} x^{2m-1} y^{2n-1}\, dx\, dy$$

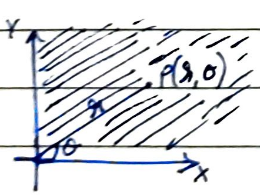
converting the double integral to polar form as it is hard to integrate $e^{-(x^2+y^2)}$

let $x = r\cos\theta$, $y = r\sin\theta$

Then $dx\,dy = r\,dr\,d\theta$

$x \to 0$ to ∞ also r varies from 0 to ∞

$y \to 0$ to ∞ θ varies from 0 to $\pi/2$



$$\therefore \Gamma(m)\Gamma(n) = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}(r\cos\theta)^{2m-1}(r\sin\theta)^{2n-1}\, r\,dr\,d\theta$$

$$= 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}\cdot r^{2(m+n)-1}\cdot \cos^{2m-1}\theta \sin^{2n-1}\theta\, dr\, d\theta$$

$$= 2\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta \cdot 2\int_0^\infty e^{-r^2} r^{2(m+n)-1}\, dr$$

$$\Gamma(m)\Gamma(n) = \beta(m,n)\cdot\Gamma(m+n)$$

$$\Rightarrow \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

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eg: evaluate $\int_0^{\pi/2} \sin^6\theta \cos^5\theta \, d\theta$

$$= \frac{1}{2} \beta\left(\frac{7}{2}, 3\right)$$

$$= \frac{1}{2} \cdot \frac{\Gamma(7/2)\Gamma(3)}{\Gamma\left(\frac{7}{2}+3\right)} = \frac{\frac{1}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi} \cdot 2!}{\Gamma\left(\frac{13}{2}\right)}$$

$$= \frac{\cancel{\frac{1}{2}} \cdot \cancel{\frac{5}{2}} \cdot \cancel{\frac{3}{2}} \cdot \frac{1}{2} \cdot \cancel{2} \cdot \cancel{\sqrt{\pi}}}{\frac{11}{2} \cdot \frac{9}{2} \cdot \frac{7}{2} \cdot \cancel{\frac{5}{2}} \cdot \cancel{\frac{3}{2}} \cdot \cancel{\frac{1}{2}} \cdot \cancel{\sqrt{\pi}}}$$

$$= \frac{8}{693}$$

→ Legendre's Duplication Formula

1. Duplication Formula in terms of Γ function

$$\sqrt{\pi}\,\Gamma(2p) = 2^{2p-1}\,\Gamma(p) \cdot \Gamma\left(p+\frac{1}{2}\right)$$

2. Duplication Formula in terms of Beta function

$$\beta\left(p, \frac{1}{2}\right) = 2^{2p-1}\,\beta(p, p)$$

PROOFS not included in syllabus

→ $\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin p\pi}$, $0 < p < 1$

eg: $\Gamma(3/4)\Gamma(1/4) = \frac{\pi}{\sin\frac{3\pi}{4}} = \frac{\pi}{\frac{1}{\sqrt{2}}} = \sqrt{2}\pi$

eg: $\Gamma(1/6)\Gamma(5/6) = \frac{\pi}{\sin\frac{\pi}{6}} = 2\pi$

q. Find the values of

i) $\Gamma(3/2)$ ii) $\Gamma(7/2)$ iii) $\Gamma(-1/2)$ iv) $\Gamma(6)$

v) $\Gamma(4)$ vi) $\Gamma(3.5)$

Sol. i) $\Gamma(3/2) = \frac{1}{2}\sqrt{\pi}$

ii) $\Gamma(7/2) = \frac{15}{8}\sqrt{\pi}$

iii) $\Gamma(-1/2) = \dfrac{\Gamma(1/2)}{-1/2} = -2\sqrt{\pi}$

iv) $\Gamma(6) = 5!$ v) $\Gamma(4) = 3!$

vi) $\Gamma(3.5) = \frac{15}{8}\sqrt{\pi}$

q2. Prove the following results

i) $\displaystyle\int_0^\infty x^p e^{-ax^q}\,dx = \frac{1}{q}\,\frac{\Gamma\left(\frac{p+1}{q}\right)}{a^{\frac{p+1}{q}}}$

PROOF: Consider $\displaystyle\int_0^\infty x^p e^{-ax^q}\,dx$

By definition

$\displaystyle\Gamma(p) = \int_0^\infty e^{-x} x^{p-1}\,dx$

let $ax^q = t \Rightarrow x = \left(\frac{t}{a}\right)^{1/q}$

$aqx^{q-1}\,dx = dt$

If $x = 0,\ t = \infty$

$x \to \infty,\ t \to \infty$

$$\int_0^\infty \cancel{x^p}\, e^{-t} \cdot \frac{dt}{aqx^{q-1}}$$

$$\int_0^\infty \left(\frac{t}{a}\right)^{p/q} \frac{e^{-t}}{aqx^{q-1}}\,dt = \int_0^\infty \left(\frac{t}{a}\right)^{p/q} \frac{e^{-t}}{aq\left(\frac{t}{a}\right)^{\frac{q-1}{q}}}\,dt$$

$a^{\frac{p}{q}}$

$$= \int_0^\infty \left(\frac{t}{a}\right)^{\frac{p-q+1}{q}} \frac{e^{-t}}{aq}\,dt$$

$$\int_0^\infty (t)^{\left(\frac{p+1}{q}\right)-1} \cdot \frac{e^{-t}}{a^{\frac{p-q+1}{q}+1}}$$

$$\frac{1}{a^{\frac{p+1}{q}}}\,\frac{1}{q}\int_0^\infty t^{\left(\frac{p+1}{q}-1\right)}\, e^{-t}$$

$$= \frac{1\cdot\Gamma\left(\frac{p+1}{q}\right)}{q\cdot a^{(p+1)/q}}$$

ii) $\displaystyle\int_0^1 x^m(\log x)^n\,dx = \frac{(-1)^n\,\Gamma(n+1)}{(m+1)^{n+1}}$, where n is a +ve integer and $m > -1$

PROOF: Consider LHS $= \displaystyle\int_0^1 x^m(\log x)^n\,dx$

let $\log x = t$ $\qquad x = e^{-t}$

$\qquad dx = -e^{-t}\,dt$

If $x \to 0$ $\quad t = +\infty$

$x = 1$ $\quad t = 0$

$$\int_\infty^0 e^{-mt}(-t)^n(-e^{-t})\,dt = -\int_\infty^0 (e^{-t})^{m+1}(-t)^n$$

$$= \int_0^\infty (e^{-t})^{m+1}\cdot(-1)^n\, t^n\,dt$$

let $(m+1)t = y$ $\qquad t = \frac{y}{m+1} \Rightarrow dt = \frac{dy}{m+1}$

$$= (-1)^n\int_0^\infty e^{-y}\cdot\left(\frac{y}{m+1}\right)^n\frac{dy}{(m+1)}$$

$$= \frac{(-1)^n}{(m+1)^{n+1}}\int_0^\infty e^{-y}\,y^n\,dy$$

$$= \frac{(-1)^n\,\Gamma(n+1)}{(m+1)^{n+1}}$$

5th January, 2023

iii) $\int_0^1 x^p (1-x^q)^r \, dx = \frac{1}{q} \beta\left(\frac{p+1}{q}, r+1\right)$

Sol. LHS: $\int_0^1 x^p (1-x^q)^r \, dx$

$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx$

let $x^q = t$

$x = t^{1/q}$

$dx = \frac{1}{q} t^{1/q - 1} \, dt$

If $x = 0$, $t = 0$

$x = 1$, $t = 1$

$$\int_0^1 t^{p/q} (1-t)^r \frac{1}{q} t^{1/q - 1} \, dt$$

$$\frac{1}{q} \int_0^1 t^{\left(\frac{p+1}{q} - 1\right)} (1-t)^r \, dt$$

$m \cancel{-1} = \frac{p+1}{q} \cancel{-1}$

$$= \frac{1}{q} \beta\left(\frac{p+1}{q}, r+1\right) = \text{RHS}$$

$n - 1 = r$

$n = r + 1$

iv) $\int_0^\infty e^{-ax} x^{n-1} \, dx = \frac{\Gamma(n)}{a^n}$ where a & n are +ve constants

$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}$

$$\int_0^\infty e^{-t} \left(\frac{t}{a}\right)^{n-1} \frac{dt}{a}$$

$e^{-ax} = e^{-t}$

$-ax = t$

$x = \frac{t}{a}$

$$\int_0^\infty \frac{e^{-t} \, t^{n-1}}{a^{n-1+1}} \, dt$$

$$\frac{1}{a^n} \int_0^\infty e^{-t} \, t^{n-1} \, dt$$

$dx = \frac{dt}{a}$

$$\frac{1}{a^n} \Gamma(n) = \text{RHS} \quad //$$

x	0	∞
t	0	∞

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q3. Evaluate the following integrals

i) $\int_0^\infty \sqrt{y}\, e^{-y^2}\, dy$

Let $y^2 = t$ $\quad y = \sqrt{t}$ $\quad \sqrt{y} = t^{1/4}$

$2y\, dy = dt$

$dy = \frac{dt}{2y} = \frac{dt}{2t^{1/2}}$

y	0	∞
t	0	∞

$$\int_0^\infty e^{-t}\, t^{1/4}\, \frac{dt}{2t^{1/2}}$$

$$\frac{1}{2}\int_0^\infty e^{-t}\, e^{-1/4}$$

$n-1 = \frac{-1}{4}$

$n = \frac{3}{4}$

$$\frac{1}{2}\, \Gamma\left(\frac{3}{4}\right)$$

ii) $\int_0^\infty 3^{-4x^2}\, dx$

$3^{-4x^2} = e^{\log[3^{-4x^2}]}$

$= e^{-4x^2 \ln 3}$

$$\int_0^\infty e^{-4x^2 \ln 3}\, dx$$

$4x^2 \ln 3 = t$

$8x \ln 3\, dx = dt$

$x = \sqrt{\frac{t}{4\ln 3}}$ $\quad dx = \frac{dt}{8x \ln 3}$

$dx = \frac{dt}{8\ln 3 \sqrt{\frac{t}{4\ln 3}}}$

$= \frac{dt}{4\sqrt{(\ln 3)t}}$

$$= \int_0^\infty e^{-t}\, \frac{dt}{4\sqrt{(\ln 3)t}}$$

$$= \frac{1}{4\sqrt{\ln 3}} \int_0^\infty e^{-t}\, t^{-1/2}$$

$n - 1 = -1/2$

$n = 1/2$

$$= \frac{1}{4\sqrt{\ln 3}}\, \Gamma(1/2) = \frac{\sqrt{\pi}}{4\sqrt{\ln 3}}$$

iii) $\displaystyle\int_0^2 \frac{x^2}{\sqrt{2-x}}\,dx$

$x^2 = t$
$2x\,dx = dt$
$dx = \frac{dt}{2x} = \frac{dt}{2\sqrt{t}}$

$\frac{1}{2}\displaystyle\int_0^2 \frac{dt}{\sqrt{t}\sqrt{2-\sqrt{t}}}$

$= \displaystyle\int_0^2 x^2 (2-x)^{-1/2}\,dx$ $\quad x^2 = t$, $2x\,dx = dt$

$= 2^{-1/2}\displaystyle\int_0^2 x^2 \left[1-\frac{x}{2}\right]^{-1/2} dx$ $\quad 2-x = t$, $x = t+2$

$\frac{x}{2} = t$

$dx = 2dt$

x	0	2
t	0	1

$\Rightarrow 2^{-1/2}\displaystyle\int_0^1 4t^2 \left[1-t\right]^{-1/2} 2dt$

$= 8 \cdot 2^{-1/2}\displaystyle\int_0^1 t^2 (1-t)^{-1/2} dt$

$= 4\sqrt{2}\ \beta(3, 1/2)$

$= 4\sqrt{2} \cdot \frac{\Gamma(3) \cdot \Gamma(1/2)}{\Gamma(3+1/2)}$

$= \frac{4\sqrt{2} \cdot 2! \sqrt{\pi}}{5/2 \cdot 3/2 \cdot 1/2 \sqrt{\pi}} = \frac{64\sqrt{2}}{15}$

iv) $\displaystyle\int_0^1 x^4 (1-x)^3\,dx$ $\qquad \beta(m,n) = \displaystyle\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

$= \beta(5,4)$

$= \frac{\Gamma(5)\,\Gamma(4)}{\Gamma(9)} = \frac{4!\ 3!}{8!} = \frac{\cancel{4!}\ \cancel{(3\times 2)}}{8\times 7\times \cancel{6}\times 5\times \cancel{4!}}$

$= \frac{1}{280}$

URBAN EDGE / /

v) $\int_0^1 x^2 (1-x^5)^{-1/2} \, dx$

$x^5 = t \qquad x = t^{1/5}$

$5x^4 dx = dt$

$x^4 dx = \frac{dt}{5}$

$\int_0^1 x^2 (1-t)^{-1/2} \frac{dt}{5x^4}$

$\frac{1}{5}\int_0^1 x^{-2} (1-t)^{-1/2}$

$\frac{1}{2}\int_0^1 t^{-2/5} (1-t)^{-1/2} \, dt$

$-\frac{2}{5}+1$

$\frac{1}{\cancel{5}} \beta\left(\frac{3}{\cancel{5}}, \frac{1}{2}\right)$ ✓

$\frac{1}{\cancel{5}} \frac{\Gamma(3/5)\,\Gamma(1/2)}{\Gamma(2)}$

Formula : $\int_0^1 x^p (1-x^q)^r \, dx = \frac{1}{q} \beta\left(\frac{p+1}{q}, r+1\right)$

$= \frac{1}{5} \beta(3/5, 1/2)$

H/W (vi) $\int_0^1 x^2 (1-\frac{3}{x})^4 \, dx$

$x^3 = t$

$= \frac{1}{3} \beta\left(\frac{3}{2}, 5\right)$

$= \frac{1}{3} \frac{\Gamma(3/2)\,\Gamma(5)}{\Gamma(13/2)}$

$= \frac{1}{3} \frac{\cancel{\frac{1}{2}} \cdot \cancel{\sqrt{\pi}} \cdot 4!}{\frac{11}{2} \cdot \frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \cancel{\frac{1}{2}} \cdot \cancel{\sqrt{\pi}}}$

$= \frac{1}{3} \frac{(4 \times \cancel{3} \times 2) \times 2^5}{(11 \times 9 \times 7 \times 5 \times \cancel{3} \times 1)}$

vi) $\int_0^1 \frac{dx}{\sqrt{-\log x}}$ $\quad \int_0^1 (-\log x)^{-1/2}\, dx$

$\int_0^1 t^{-1/2} e^{-t}\, dt$

$\Gamma(1/2) = \sqrt{\pi}$

6th January, 2023

Q4. Evaluate $\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta \times \int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta}}\, d\theta$

$\int_0^{\pi/2} \sin^p\theta \cos^q\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

$$= \int_0^{\pi/2} \sin^{1/2}\theta \cos^0\theta\, d\theta \times \int_0^{\pi/2} \sin^{-1/2}\theta \cos^0\theta\, d\theta$$

$$= \frac{1}{2}\beta\left(\frac{1/2+1}{2}, \frac{0+1}{2}\right) \times \frac{1}{2}\beta\left(\frac{-1/2+1}{2}, \frac{0+1}{2}\right)$$

$$= \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{2}\right) \times \frac{1}{2}\beta\left(\frac{1}{4}, \frac{1}{2}\right)$$

$$= \frac{1}{4} \frac{\Gamma(3/4)\Gamma(1/2)}{\Gamma(3/4+1/2)} \cdot \frac{\Gamma(1/4)\Gamma(1/2)}{\Gamma(1/4+1/2)}$$

$$= \frac{1}{4} \frac{\cancel{\Gamma(3/4)}\, \Gamma(1/2)}{\Gamma(5/4)} \cdot \frac{\Gamma(1/4)\Gamma(1/2)}{\cancel{\Gamma(3/4)}}$$

$\Gamma(n+1) = n\Gamma(n)$

$$= \frac{\cancel{1/4}\, \sqrt{\pi}\sqrt{\pi}\, \Gamma(1/4)}{\cancel{1/4}\, \Gamma(1/4)} = \pi$$

Q5. Prove that $\int_0^\infty \sqrt{x}\, e^{-x^2} dx \times \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\, dx = \frac{\pi}{2\sqrt{2}}$

let $x^2 = t$ $\quad \sqrt{x} = t^{1/4}$

$2x\,dx = dt$

$x\,dx = \frac{dt}{2}$

$n - 1 = \frac{1}{4}$

$n = \frac{5}{4}$

$$\int_0^\infty t^{1/4} e^{-t} \frac{dt}{2} \times \int_0^\infty t^{-1/4} e^{-t} \frac{dt}{2} =$$

$$\frac{1}{2}\Gamma\left(\frac{5}{4}\right) \times \frac{1}{2}\Gamma\left(\frac{3}{4}\right)$$

$\frac{5\pi}{4}$

URBAN EDGE

* $\Gamma(n) = 2\int_0^\infty e^{-t^2} t^{2n-1} \, dt$

$2n-1 = \frac{1}{2}$ $\quad$ $2n-1 = -1/2$

$n = 3/4$ $\quad$ $n = 1/4$

$\Rightarrow \frac{1}{4} \Gamma(3/4)\Gamma(1/4)$

$= \frac{1}{4} \frac{\pi}{\sin \frac{3\pi}{4}} = \frac{1}{4} \frac{\pi}{\frac{1}{\sqrt{2}}}$ $\quad$ $\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin p\pi}$

$= \frac{1}{\cancel{4}} \frac{\pi}{2\sqrt{2}}$ = RHS

q6. ~~Show that~~ Evaluate $\int_0^\infty (x^2+4) e^{-2x^2} \, dx$

$\cancel{2x^2 = t}$ $\quad$ $x^2 + 4 = t$

$x^2 = t - 4$

$\int_0^\infty t \, e^{-(2t-8)}$ $\quad$ $2x\,dx = dt$ $\quad$ $2x^2 = 2t - 8$

$dx = \frac{dt}{2\sqrt{t-4}}$

$\int_0^\infty x^2 e^{-2x^2} dx + \int_0^\infty 4 e^{-2t^2} dx$

$\int_0^\infty \frac{t}{2} e^{-t} \frac{dt}{4}\sqrt{\frac{2}{t}} + 4\int_0^\infty e^{-t} \frac{dt}{4}\sqrt{\frac{2}{t}} \, dt$ $\quad$ let $2x^2 = t$

$4x \, dx = dt$

$dx = \frac{dt}{4x} = \frac{dt}{4\sqrt{\frac{t}{2}}}$ $\quad$ $x=0 \quad t=0$

$x=\infty \quad t=\infty$

$\frac{\sqrt{2}}{8}\int_0^\infty t^{+1/2} e^{-t} + \frac{\cancel{4}\sqrt{2}}{\cancel{4}} \int_0^\infty \cancel{t}^{-1/2} e^{-t} \, dt$

$\frac{\sqrt{2}}{8} \Gamma\left(\frac{3}{2}\right) + \sqrt{2}\,\Gamma\left(\frac{1}{2}\right)$

$\frac{\sqrt{2}}{8} \frac{1}{2}\sqrt{\pi} + \sqrt{2}\sqrt{\pi} = \sqrt{\pi}\left(\frac{17}{16}\right)$

URBAN EDGE

q.7. Evaluate $\int_0^{\infty} 2^{-3x^2} dx$

q.8. P.T $\int_0^{\infty} x e^{-ax} \sin bx \, dx = \frac{2ab}{(a^2+b^2)^2}$

$e^{ibx} = \cos bx + i \sin bx$

$\int_0^{\infty} x e^{-ax} \sin bx \, dx = Im \int_0^{\infty} x e^{-(a+ib)x} dx$

$\sin bx = Im[e^{ibx}]$

$x e^{-ax} \sin bx = Im\{x e^{-ax} \cdot e^{ibx}\}$

$= Im\{x e^{-(a+ib)x}\}$

$Im \int_0^{\infty} \frac{t}{a-ib} e^{-t} \cdot \frac{dt}{(a-ib)}$

let $(a-ib)x = t \Rightarrow x = \frac{t}{a-ib}$

$dx = \frac{dt}{a-ib}$

$x \to 0 \quad t \to 0$

$x \to \infty \quad t \to \infty$

$Im\left\{\frac{1}{(a-ib)^2} \Gamma(2)\right\}$

$Im\left\{\frac{1}{a^2-b^2-2iab} 1!\right\}$

$Im\left\{\frac{a^2-b^2+2iab}{(a^2-b^2)^2+4a^2b^2}\right\}$ rationalize

$\frac{1}{(a^2-b^2)-2abi} \times \frac{(a^2-b^2)+2aib}{(a^2-b^2)+2aib}$

$Im\left\{\frac{(a^2-b^2)+2iab}{a^4+b^4-2a^2b^2+4a^2b^2}\right\}$

$= Im\left\{\frac{(a^2-b^2)+2aib}{a^4+b^4+2a^2b^2}\right\}$

$= Im\left\{\frac{(a^2-b^2)+2iab}{(a^2+b^2)^2}\right\}$

$= \frac{2ab}{(a^2+b^2)^2}$

URBAN EDGE __/__/__

q $\int_0^\infty \frac{x^4}{1+x^6}\,dx = \int_0^\infty (1+x^6)^{-1} x^4\,dx$

$3x^2 dx = \cos\theta\, d\theta$

$x^3 = \sin\theta$

$x^2 = \sin^{2/3}\theta$

$x^3 = \sin^3\theta$

$(1+\sin^3\theta)^{-1} \sin^2\theta$

$\frac{1}{1+\sin^2\theta} \sin\theta\,(\sin\theta)^{1/3}$

$\cancel{\frac{1-\sin^2\theta}{1-\sin^4\theta}}$

$\frac{1}{1+\sin^2\theta} \sin^{4/3} \frac{\cos\theta\, d\theta}{3}$

$\cancel{x^4 = t}$

$x^4 = dt$

$x^5 = t$

$5x^4 dx = dt$

$x^4 dx = \frac{dt}{5}$

$\frac{1}{1+x^5x}$ $\frac{1}{1+t^{6/5}}$

$\frac{1}{1+t^{6/5}}$

$\int_0^\infty (1+t^{6/5})^{-1} \frac{dt}{5}$

* if $1+x^6$ take $x^6 = \tan^2\theta$

if $1-x^6$ take $x^6 = \sin^2\theta$

$x^6 = \tan^2\theta$

$x = \tan^{1/3}\theta$

$dx = \frac{1}{3}\tan^{-2/3}\theta \sec^2\theta\, d\theta$

$x \to 0, \ \theta \to 0$

$x \to \infty \ \ \theta \to \pi/2$

$\int_0^{\pi/2} \frac{\tan^{4/3}\theta}{\cancel{(1+\tan^2\theta)}} \frac{1}{3}\tan^{-2/3}\theta\, \cancel{\sec^2\theta}\, d\theta$

$\frac{1}{3}\int_0^{\pi/2} \tan^{2/3}\theta\, d\theta = \frac{1}{3}\int_0^{\pi/2} \sin^{2/3}\theta \cos^{-2/3}\theta\, d\theta$

$= \frac{1}{3}\cdot\frac{1}{2}\,\beta\left(\frac{5}{6}, \frac{1}{6}\right)$

URBAN EDGE

9th January, 2023

Bessel's Equation / Bessel's Function / Cylinder Function

→ Bessel's DE

* Any equation of the form

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \nu^2)y = 0 \quad \text{①}$$

is called as the Bessel's DE of order ν, where ν is a non-negative real no.

* Assume that the series soln. of ① as

$$y(x) = x^r(a_0 + a_1x + a_2x^2 + \dots) = \sum_{m=0}^{\infty} a_m x^{m+r} \quad \text{②}$$

where $a_0 \neq 0$ and $a_1, a_2 \dots$ are real constants to be determined

by solving ① & ②

$$y_1(x) = x^{\nu}\sum_{m=0}^{\infty} \frac{(-1)^m a_0 x^{2m}}{2^{2m}\cdot m!(\nu+1)(\nu+2)(\nu+3)\dots(\nu+m)}$$

$$y_2(x) = x^{-\nu}\sum_{m=0}^{\infty} \frac{(-1)^m a_0 x^{2m}}{2^{2m}\cdot m!(-\nu+1)(-\nu+2)(-\nu+3)\dots(-\nu+m)}$$

* complete soln: $y = c_1y_1 + c_2y_2$ — only when y_1, y_2 are linearly independent, $W \neq 0$

c_1, c_2 are arbitary constants

taking substitution $a_0 = \frac{1}{2^{\nu}\Gamma(\nu+1)}$

* Series solution : $J_v(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+v}}{2^{2m+v} m!\, \Gamma(v+m+1)}$

$$J_{-v}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m-v}}{2^{2m-v} m!\, \Gamma(-v+m+1)}$$

* Particular solution : $y(x) = c_1 J_v(x) + c_2 J_{-v}(x)$,

provided 'v' is not an integer

when v is integer J_v & J_{-v} become lineary dependent

→ Linear Dependence of Besseli Function

When v is an integer $J_v(x)$ and $J_{-v}(x)$ are lineary dependent and their relation is given by

$$J_{-n}(x) = (-1)^n J_n(x), \text{ where } n \text{ is a +ve } \mathbb{Z}$$

PROOF: Consider $J_v(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+v}}{2^{2m+v} m!\, \Gamma(v+m+1)}$

Then taking $v = n$ (+ve integer)

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+n}}{2^{2m+n} m!\, \Gamma(n+m+1)} \quad \text{--- (1)}$$

taking $v = -n$

$$J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m-n}}{2^{2m-n} m!\, \Gamma(m-n+1)} \quad \text{--- (2)}$$

when $m-n+1 \leq 0$ or $m \leq n-1$ the gamma func. $\Gamma(m-n+1)$ is not defined. ($\because \Gamma$(-ve integer) is not defined) $-[\Gamma(m-(n+1))]$

$\therefore$ (2) can be written as $J_{-n}(x) = \sum_{m=n}^{\infty} \frac{(-1)^m x^{2m-n}}{2^{2m-n} \cdot m!\, \Gamma(m-n+1)}$ --- (3)

Let $m-n = s$ $\qquad m: n$ to ∞

$m-n: 0$ to ∞

$$\therefore J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-1)^{s+n} \cdot x^{2(s+n)-n}}{2^{2(s+n)-n} \cdot (s+n)!\, \Gamma(s+1)}$$

$$= (-1)^n \sum_{s=0}^{\infty} \frac{(-1)^s\, x^{2s+n}}{2^{2s+n} \cdot s!\, \Gamma(s+n+1)} \quad -(4)$$

$(4) \equiv (1) \qquad \therefore J_{-n}(x) = (-1)^n J_n(x)$

→ Important Results, Recurrence Results

1. $x J'_v(x) = x J_{v-1}(x) - v J_v(x)$

2. $x \{ J'_v(x) \} = v J_v(x) - x J_{v+1}(x)$

3. $2 J'_v(x) = J_{v-1}(x)$

4. $2v J_v(x) = x [J_{v-1}(x) + J_{v+1}(x)]$

→ Derivatives of Bessel's Function

1. $\frac{d}{dx} \{ x^v J_v(x) \} = x^v J_{v-1}(x)$

2. $\frac{d}{dx} \{ x^{-v} J_v(x) \} = -x^{-v} J_{v+1}(x)$

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→ Integrals for Bessels' Function

1. $\int x^{p} J_{p-1}(x)\, dx = x^{p} J_p(x) + c$

2. $\int x^{-p} J_{p+1}(x)\, dx = -x^{-p} J_p(x) + c$

3. $\int J_{p+1}(x)\, dx = \int J_{p-1}(x)\, dx - 2J_p(x)$

11th January, 2023

Show that : $J_{1/2}(x) = \sqrt{\dfrac{2}{\pi x}} \sin x$ and hence find $\displaystyle\int_0^{\pi/2} \sqrt{x}\, J_{1/2}(x)\, dx$.

$$J_v(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+v}}{2^{2m+v} \cdot m!\, \Gamma(m+v+1)} \quad -①$$

$v \to 1/2$

$$J_{1/2}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+\frac{1}{2}}}{2^{2m+1/2} \cdot m!\, \Gamma(m+\frac{3}{2})} \quad -②$$

$2^{2m+1/2} = 2^{2m} \cdot 2^{1/2}$

$$J_{1/2}(x) = \sqrt{\frac{x}{2}} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} \cdot m!\, \Gamma(m+3/2)}$$

$$J_{1/2}(x) = \sqrt{\frac{x}{2}} \left\{ \frac{1}{\Gamma(3/2)} - \frac{x^2}{2^2 \cdot 1!\, \Gamma(5/2)} + \frac{x^4}{2^4 \cdot 2!\, \Gamma(7/2)} - \cdots\cdots \right\}$$

$$J_{1/2}(x) = \sqrt{\frac{x}{2}} \left\{ \frac{1}{\frac{1}{2}\sqrt{\pi}} - \frac{x^2}{2^2 \cdot 1 \cdot \frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi}} + \frac{x^4}{2^4 \cdot 2 \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi}} - \cdots\cdots \right\}$$

$$= \sqrt{\frac{x}{2\pi}} \left\{ 2 - \frac{x^2}{3} + \frac{x^4}{60} - \cdots \right\}$$

$$= \sqrt{\frac{x}{2\pi}} \cdot \frac{2}{x} \left\{ x - \frac{x^3}{3 \cdot 2} + \frac{x^5}{120} - \cdots \right\}$$

$$= \sqrt{\frac{2}{\pi x}} \cdot \left\{ x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\cdots \right\} = \sqrt{\frac{2}{\pi x}} \cdot \{\sin x\}.$$

now $\int_0^{\pi/2} \sqrt{x}\, J_{1/2}(x)\, dx = \int_0^{\pi/2} \cancel{\sqrt{x}} \cdot \sqrt{\frac{2}{\pi \cancel{x}}} \sin x \, dx$

$= \sqrt{\frac{2}{\pi}} \int_0^{\pi/2} \sin x \quad = \sqrt{\frac{2}{\pi}} \left[-\cos x\right]_0^{\pi/2} = \sqrt{\frac{2}{\pi}}$

q. Show that $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

$$J_{-1/2}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+v}}{2^{2m+v} \cdot m! \, \Gamma(m+v+1)} \quad -(1)$$

$v \to -1/2$

$$J_{-1/2}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m-1/2}}{2^{2m-1/2} m! \, \Gamma(m+\frac{1}{2})}$$

$$= \frac{x^{-1/2}}{2^{-1/2}} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} \cdot m! \, \Gamma(m+\frac{1}{2})}$$

$$= \sqrt{\frac{2}{x}} \left\{ \frac{1}{\Gamma(1/2)} - \frac{x^2}{2^2 \cdot 1! \, \Gamma(3/2)} + \frac{x^4 \cancel{(x^4)}}{2^4 \cdot 2! \, \Gamma(5/2)} - \cdots \right\}$$

$$= \sqrt{\frac{2}{x}} \left\{ \frac{1}{\sqrt{\pi}} - \frac{x^2}{2^2 \cdot \frac{1}{2} \cdot \sqrt{\pi}} + \frac{x^4}{2^4 \cdot 2 \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi}} - \cdots \right\}$$

$$= \sqrt{\frac{2}{x\pi}} \left\{ 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \right\}$$

$$= \sqrt{\frac{2}{\pi x}} \cos x$$

URBAN EDGE

q: Prove that $J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$

From recurrence relation

$$2v J_v(x) = x\left[J_{v-1}(x) + J_{v+1}(x)\right]$$

$$J_{v+1}(x) = \frac{2v J_v(x)}{x} - J_{v-1}(x)$$

let $v = 1/2$

$$J_{3/2}(x) = \frac{1}{x} J_{1/2}(x) - J_{-1/2}(x)$$

$$= \frac{1}{x}\sqrt{\frac{2}{\pi x}}\sin x - \sqrt{\frac{2}{\pi x}}\cos x$$

$$= \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$$

12th January, 2023

q: Prove that $J_{-\frac{3}{2}}(x) = -\sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x} + \sin x\right)$

wkt $2v J_v(x) = x\left[J_{v-1}(x) + J_{v+1}(x)\right]$

$$J_{v-1}(x) = \frac{2v J_v(x)}{x} - J_{v+1}(x)$$

$v = -1/2$

$$J_{-3/2}(x) = \frac{\cancel{2}(-1/\cancel{2}) J_{-1/2}(x)}{x} - J_{1/2}(x)$$

$$J_{-3/2}(x) = -\frac{1}{x}\left\{\sqrt{\frac{2}{\pi x}}\cos x\right\} - \left\{\sqrt{\frac{2}{\pi x}}\sin x\right\}$$

$$J_{-3/2}(x) = -\sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x} + \sin x\right)$$

q: Express $J_4(x)$ in terms of $J_0(x)$ and $J_1(x)$

$$2v J_v(x) = x\left[J_{v-1}(x) + J_{v+1}(x)\right]$$

$$J_{v+1}(x) = \frac{2v}{x} J_v(x) - J_{v-1}(x) \quad -(1)$$

put $v = 1, 2, 3$

$v=1 \quad J_2(x) = \frac{2}{x} J_1(x) - J_0(x) \quad -(2)$

$v=2 \quad J_3(x) = \frac{4}{x} J_2(x) - J_1(x) \quad -(3)$

$v=3 \quad J_4(x) = \frac{6}{x} J_3(x) - J_2(x) \quad -(4)$

$$= \frac{6}{x}\left[\frac{4}{x} J_2(x) - J_1(x)\right] - \left[\frac{2}{x} J_1(x) - J_0(x)\right]$$

$$= \frac{24}{x^2} J_2(x) - \frac{6}{x} J_1(x) - \frac{2}{x} J_1(x) + J_0(x)$$

$$= \frac{24}{x^2}\left[\frac{2}{x} J_1(x) - J_0(x)\right] - \frac{8}{x} J_1(x) + J_0(x)$$

$$= \frac{48}{x^3} J_1(x) - \frac{24}{x^2} J_0(x) - \frac{8}{x} J_1(x) + J_0(x)$$

$$= \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(x) + \left(1 - \frac{24}{x^2}\right) J_0(x)$$

q. Evaluate $\int x^4 J_1(x)\, dx$

$* \quad \int x^p J_{p-1}\, dx = x^p J_p(x) + c$

$p = 2$

$\int x^2 J_1(x) = x^2 J_2(x)$

$p = 3$

$\int x^3 J_2(x) = x^3 J_3(x)$

$$\int x^4 J_1(x)\, dx = \int \underbrace{x^2}_{u} \underbrace{[x^2 J_1(x)]}_{v}\, dx$$

$\int uv = u\int v - \int u' \int v$

$$\int x^4 J_1(x)\, dx = x^2[x^2 J_2(x)] - \int 2x[x^2 J_2(x)]\, dx$$

$$= x^4 J_2(x) - 2\int x^3 J_2(x)\, dx$$

$$= x^4 J_2(x) - 2x^3 J_3(x) + c$$

Express all ques in J_0 & J_1

$$x^4\left[\frac{2}{x}J_1(x) - J_0(x)\right] - 2x^3\left[\overset{J_2(x)}{\frac{4}{x}} - J_1(x)\right]$$

$$2x^3 J_1(x) - x^4 J_0(x) - 8x^2 J_2(x) + 2x^3 J_1(x)$$

$$2x^3 J_1(x) - x^4 J_0(x) - 8x^2\left[\frac{2}{x}J_1(x) - J_0(x)\right] + 2x^3 J_1(x)$$

$$4x^3 J_1(x) - x^4 J_0(x) - 16x J_1(x) + 8x^2 J_0(x)$$

$$(4x^3 - 16x)J_1 - (x^4 - 8x^2)J_0$$

q: $\int J_3(x)\,dx$

* $\int J_{p+1}(x)\,dx = \int J_{p-1}(x)\,dx -$ won't work

* $\int x^{-p} J_{p+1}\,dx = -x^{-p} J_p + c$ $p=2$

  $\int x^{-2} J_3\,dx = -x^{-2} J_2 + c$

$$\int J_3(x)\,dx = \int \underbrace{x^2}_{u}\,[\underbrace{x^{-2} J_3(x)}_{v}]\,dx$$

$$= x^2[-x^{-2}J_2] - \int -x^{-2}J_2 \cdot 2x\,dx$$

$$= -J_2 + 2\int x^{-1} J_2\,dx$$

$$= -J_2 + 2[-x^{-1}J_1]$$

$$= -\left\{\frac{2}{x}J_1 - J_0\right\} - \frac{2}{x}J_1$$

$$= -\frac{4}{x}J_1 + J_0$$

$\cancel{\int x^{-1} = x}$

th January, 2023

Generating Function

The generating function of a sequence of functions f_n is given by

$$G(f_n, x) = \sum_{n=0}^{\infty} f_n x^n$$

which generates $f_n(x)$

eg: 1, 1, 1, ...

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

$$\Rightarrow \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$\frac{1}{1-x}$ is the GF of $\{1, 1, 1, \ldots\}$

* The generating function for Bessel's function of integral order n is $e^{\frac{x}{2}\left(t - \frac{1}{t}\right)}$

i.e $$e^{\frac{x}{2}\left(t - \frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n$$ (PROOF not included)

Jacobi series

* $$\cos(x \overset{\sin\theta}{\cancel{\cos\theta}}) = J_0 + 2J_2 \cos 2\theta + 2J_4 \cos 4\theta + \cdots$$

* $$\sin(x \overset{\sin\theta}{\cancel{\cos\theta}}) = 2\left[J_1 \cos\theta + J_3 \cos 3\theta + J_5 \cos 5\theta + \cdots\right]$$

PROOF:

20th January, 2023

Jacobi Series

* $\cos(x\sin\theta) = J_0 + 2J_2\cos 2\theta + 2J_4\cos 4\theta + \cdots$
* $\sin(x\sin\theta) = 2[J_1\sin\theta + J_3\sin 3\theta + J_5\sin 5\theta + \cdots]$

Proof: Consider the generating function

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)t^n$$

$$= \cdots + J_{-3}(x)t^{-3} + J_{-2}(x)t^{-2} + J_{-1}(x)t^{-1} + J_0(x)\cancel{t^0} + J_1(x)t + J_2(x)t^2 + J_3(x)t^3 + \cdots \quad -①$$

wkt $J_{-n}(x) = (-1)^n J_n(x)$ $\therefore$

$$J_{-3}(x) = -J_3(x)$$
$$J_{-2}(x) = J_2(x)$$
$$J_{-1}(x) = -J_1(x)$$

substituting these in ①

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \cdots - J_3(x)t^{-3} + J_2(x)t^{-2} - J_1(x)t^{-1} + J_0(x) + J_1(x) + J_2(x)t^2 + \cdots$$

$$= J_0(x) + J_1(x)[t - t^{-1}] + J_2(x)[t^2 + t^{-2}] + J_3[t^3 - t^{-3}] \quad -②$$

Let $t = e^{i\theta}$ then $\frac{1}{t} = t^{-1} = e^{-i\theta}$

$t^2 = e^{2i\theta}$, $t^3 = e^{3i\theta}$

$\therefore$ $t - t^{-1} = e^{i\theta} - e^{-i\theta} = 2i\sin\theta$; $t^2 + t^{-2} = 2\cos 2\theta$

$t^3 - t^{-3} = 2i\sin 3\theta$

substituting these in ②

$$e^{\frac{x}{\cancel{2}}(\cancel{2}i\sin\theta)} = J_0(x) + J_1(x)[2i\sin\theta] + J_2(x)[2\cos 2\theta] + J_3(x)[2i\sin\theta] + \cdots$$

$$\cos\left(4\cdot\frac{\pi}{2}+4\theta\right) = \cos(2\pi+4\theta) = \cos 4\theta$$

$$\sin\left(3\cdot\frac{\pi}{2}+3\theta\right) = -\cos 3\theta$$

$$e^{i(x\sin\theta)} = \cos(x\sin\theta) + i\sin(x\sin\theta)$$

$$\cos(x\sin\theta) + i\sin(x\sin\theta) = [J_0(x) + 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta + \dots] + i[2J_1(x)\sin\theta + 2J_3(x)\sin 3\theta + \dots]$$

equating the real & imaginary parts

Jacobi Series

$$\cos(x\sin\theta) = J_0 + 2J_2\cos 2\theta + 2J_4\cos 4\theta + \dots \quad (3)$$

$$\sin(x\sin\theta) = 2[J_1\sin\theta + J_3\sin 3\theta + \dots] \quad (4)$$

replacing θ by $\pi/2+\theta$ in (3) & (4)

$$\cos[x\sin(\pi/2+\theta)] = J_0 + 2J_2\cos 2(\pi/2+\theta) + 2J_4\cos 4(\pi/2+\theta) + \dots$$

$$\cos[x\cos\theta] = J_0 - 2J_2\cos 2\theta + 2J_4\cos 4\theta + \dots$$

alternate form of Jacobi series

$$\sin[x\sin(\pi/2+\theta)] = 2[J_1\sin(\pi/2+\theta) + J_3\,3(\pi/2+\theta) + \dots]$$

$$\sin[x\cos\theta] = 2J_1\cos\theta - 2J_3\cos 3\theta + 2J_5\cos 5\theta + \dots$$

orthogonality

P.T $\displaystyle\int_0^a xJ_\nu(\alpha x)J_\nu(\beta x)\,dx = \frac{a^2}{2}J_{\nu-1}^2(\alpha\alpha)$

$$\int_0^a xJ_\nu(\alpha x)J_\nu(\beta x)\,dx = \frac{a^2}{2}[J_{\nu+1}(a\alpha)]^2$$

recurrence relation $2\nu J_\nu(x) = x[J_{\nu-1}(x) + J_{\nu+1}(x)]$

let $x = a\alpha$

$$\cancel{2\nu J(a\alpha)}^{0} = a\alpha[J_{\nu-1}(a\alpha) + J_{\nu+1}(a\alpha)]$$

$$0 = a\alpha[J_{\nu-1}(a\alpha) + J_{\nu+1}(a\alpha)]$$

$$\cancel{a\alpha}J_{\nu-1}(a\alpha) = -\cancel{a\alpha}J_{\nu+1}(a\alpha)$$

$$[J_{\nu-1}(a\alpha)]^2 = [-J_{\nu+1}(a\alpha)]^2 = [J_{\nu+1}(a\alpha)]^2$$

substitute in q

q. Show that $J_n(x)$ is an even function if n is even & is an odd function if n is odd

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \, x^{2m+n}}{2^{2m+n} \cdot m! \, \Gamma(m+n+1)}$$

$J_n(-x) = J_n(x) \rightarrow$ even

$J_n(-x) = -J_n(x) \rightarrow$ odd

$$J_n(-x) = \frac{\sum (-1)^m \, (-x)^{2m+n}}{2^{2m+n} \cdot m! \, \Gamma(m+n+1)}$$

$$\frac{(-x)^{2m} (-x)^n}{\Gamma(m+n+1)}$$

$\Gamma(m+3)$

Q: Prove $J_0^2 + 2J_1^2 + 2J_2^2 + 2J_3^2 + \cdots = 1$ (Solved later)

$$\cos(x\cos\theta) = J_0 + 2J_2\cos 2\theta + 2J_4\cos 4\theta + \cdots\cdots$$

$$\sin(x\cos\theta) = 2[J_1\sin\theta + J_3\sin 3\theta + J_5\sin 5\theta + \cdots]$$

important result $2\int_0^{\pi/2} \sin m\theta \cos n\theta \, d\theta = \begin{cases} 0 & , \text{ if } m \neq n \\ \pi & , \text{ if } m = n \end{cases}$

18th January, 2023

Remember :

1. $\int_0^{\pi} 2\sin^2(n\theta)\, d\theta = \pi$

2. $\int_0^{\pi} 2\sin(n\theta)\sin(m\theta)\, d\theta = 0$

3. $\int_0^{\pi} 2\cos^2(n\theta)\, d\theta = \pi$

4. $\int_0^{\pi} 2\cos(n\theta)\cos(m\theta)\, d\theta = 0$

Bessel's Integral

$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)\, d\theta$, where n is a positive integer

PROOF: Consider the Jacobi series

URBAN EDGE

* $\cos(x\sin\theta) = J_0 + 2J_2\cos 2\theta + 2J_4\cos 4\theta + \ldots$ — ①

* $\sin(x\sin\theta) = 2J_1\sin\theta + 2J_3\sin 3\theta + 2J_5\sin 5\theta + \ldots$ — ②

① × $\cos n\theta$ and integrating wrt θ b/w 0 and π

$$\int_0^\pi \cos n\theta\cos(x\sin\theta)\,d\theta = \int_0^\pi J_0\cos n\theta + 2J_2\cos n\theta\cos 2\theta + 2J_4\cos n\theta\cos 4\theta + \ldots\,d\theta$$

lets say $n=0$, $J_0\pi$

$n=2$, $J_2\pi$

$n=4$, $J_4\pi$

$\therefore$ n is even, πJ_n

n is odd, 0

$$\therefore \int_0^\pi \cos n\theta\cos(x\sin\theta)\,d\theta = \begin{cases}\pi J_n, & \text{if } n \text{ is even} \quad -③\\ 0, & \text{if } n \text{ is odd} \quad -④\end{cases}$$

② × $\sin n\theta$ and integrating wrt θ b/w 0 and π

$$\int_0^\pi \sin n\theta\sin(x\sin\theta)\,d\theta = \int_0^\pi 2J_1\sin n\theta\sin\theta + 2J_3\sin n\theta\sin 3\theta + \ldots\,d\theta$$

$$= \begin{cases}0, & \text{if } n \text{ is even} \quad -⑤\\ \pi J_n, & \text{if } n \text{ is odd} \quad -⑥\end{cases}$$

adding eqns. ③ & ⑤ (or) ④ & ⑥ we get

$$\int_0^\pi \cos n\theta\cos(x\sin\theta) + \sin n\theta\sin(x\sin\theta)\,d\theta = \pi J_n$$

$$\Rightarrow \int_0^\pi \cos(n\theta - x\sin\theta)\,d\theta = \pi J_n$$

$$J_n = \frac{1}{\pi}\int_0^\pi \cos(n\theta - x\sin\theta)\,d\theta$$

//

Q: Establish the Jacobi series and hence Prove that
$$J_0^2 + 2J_1^2 + 2J_2^2 + 2J_3^2 + \cdots = 1$$

PROOF : Consider the Jacobi series

$$\cos(x\sin\theta) = J_0 + 2J_2\cos 2\theta + 2J_4\cos 4\theta + \cdots \quad (1)$$
$$\sin(x\sin\theta) = 2J_1\sin\theta + 2J_3\sin\theta + 2J_5\sin 5\theta + \cdots \quad (2)$$

squaring (1) and integrating w.r.t. θ b/w 0 to π

$$\int_0^{\pi} \cos^2(x\sin\theta)\,d\theta = \int_0^{\pi} J_0^2 + 4J_2^2\cos^2 2\theta + 4J_4^2\cos^2 4\theta + \underbrace{\cdots\cdots}_{\text{contains the 2ab terms which dont matter}} d\theta$$

$$= \pi J_0^2 + 2J_2^2\pi .$$

$$\int_0^{\pi} \cos 2\theta \cos 4\theta\, d\theta = 0 \quad (m \neq n)$$

squaring (2) and integrating w.r.t. θ b/w 0 to π

$$\int_0^{\pi} \sin^2(x\sin\theta)\,d\theta = \int_0^{\pi} 4J_1^2\sin^2\theta + 4J_3^2\sin^2 3\theta + 4J_5^2\sin^2 5\theta + \cdots\cdots d\theta$$

$$= 2\pi J_1^2 + 2\pi J_3^2 + 2\pi J_5^2 + \cdots \quad (4)$$

adding (3) and (4)

$$\int_0^{\pi} \cos^2(x\sin\theta) + \sin^2(x\sin\theta)\,d\theta = (\pi J_0^2 + 2\pi J_2^2 + 2\pi J_4^2 + \cdots) + (2\pi J_1^2 + 2\pi J_3^2 + 2\pi J_5^2 + \cdots)$$

$$\Rightarrow \cancel{\pi} = \cancel{\pi}\left[J_0^2 + 2J_1^2 + 2J_2^2 + 2J_3^2 + \cdots\right]$$

$$\Rightarrow J_0^2 + 2J_1^2 + 2J_2^2 + 2J_3^2 + \cdots = 1$$

Equations reducible to Bessel's DE

* The DEs of the form $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (k^2x^2 - v^2)y = 0$

is reducible to the Bessel's DE

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - v^2)y = 0$$

by using the substitution $kx = t$

* wkt the solution to Bessel's DE is given by
$y = c_1 J_v(t) + c_2 J_{-v}(t)$

* $\therefore$ The soln. to eqn. ① is given by $y = c_1 J_v(kx) + c_2 J_{-v}(kx)$

Orthogonality of two functions

* Two functions $f(x)$ and $g(x)$ are said to be orthogonal in an interval $[a, b]$ if

$$\int_a^b \phi(x)[f(x)g(x)\,dx] = 0$$

for some appropriate function $\phi(x)$ called as weight function

11th January, 2023

* The orthogonality condition for Bessel's function is given by

weighted func

$$\int_0^a x J_v(\alpha x) J_v(\beta x)\,dx = \begin{cases} 0 & , \text{if } \alpha \neq \beta \\ \frac{a^2}{2} J_{n+1}^2(a\alpha) & , \text{if } \alpha = \beta \end{cases}$$

where α, β are the roots of $J_v(ax) = 0$

URBAN EDGE

PROOF: From given α, β are roots of $J_v(ax) = 0$

$\therefore J_v(a\alpha) = 0$ & $J_v(a\beta) = 0$

Consider the Bessels DE in the form

$$x^2 y'' + xy' + (\alpha^2 x^2 - v^2)y = 0 \quad -①$$

$$\& \; x^2 z'' + xz' + (\beta^2 x^2 - v^2)z = 0 \quad -②$$

Let $y = J_v(\alpha x)$ and $z = J_v(\beta x)$ be the solutions of ① & ② respectively

y & z are functions of x

multiplying ① by $\frac{z}{x}$ and ② by $\frac{y}{x}$

$$xzy'' + zy' + \alpha^2 zxy - \frac{v^2 zy}{x} = 0 \quad -③$$

$$xyz'' + yz' + \beta^2 xyz - \frac{v^2 zy}{x} = 0 \quad -④$$

consider ③ − ④

$$\underline{x(zy'' - yz'') + (zy' - yz')} + xyz(\alpha^2 - \beta^2) = 0$$

$$\Rightarrow \frac{d}{dx}\left[x(zy' - yz')\right] = (\beta^2 - \alpha^2)xyz$$

$\frac{d}{dx}\left[x(zy' - yz')\right]$

$= (zy' - yz') + x(zy'' + \cancel{y'z'} - yz'' - \cancel{y'z'})$

$= \underline{(zy' - yz') + x(zy'' - yz'')}$

integrating wrt x b/w 0 and a

$$\int_0^a \frac{d}{dx}\left[x(zy' - yz')\right]dx = (\beta^2 - \alpha^2)\int_0^a xyz \, dx$$

$$x\left[zy' - yz'\right]\Big|_0^a = (\beta^2 - \alpha^2)\int_0^a xyz \, dx$$

$$\int_0^a xyz \, dx = \frac{1}{\beta^2 - \alpha^2}\left[x(zy' - yz')\right]\Big|_0^a$$

$y = J_v(\alpha x)$ $\quad z = J_v(\beta x)$

$y' = \alpha J_v'(\alpha x)$ $\quad z' = \beta J_v'(\beta x)$.

$$\int_0^a x J_v(\alpha x) J_v(\beta x)\, dx = \frac{1}{\beta^2-\alpha^2}\left\{ x\left[J_v(\beta x)\cdot \alpha J_v'(\alpha x) - J_v(\alpha x)\cdot J_v'(\beta x)\right]\right\}_0^a$$

$$= \frac{1}{\beta^2-\alpha^2}\left\{ a\left[J_v(a\beta)\cdot \alpha J_v'(a\alpha) - \beta J_v(a\alpha)\cdot J_v'(\alpha\beta)\right]\right\} \quad -(5)$$

Case 1. If $\alpha \neq \beta$, then $\int_0^a x J_v(a\alpha) J_v(\beta x)\, dx = 0 \quad -(6)$ since $J_v(a\alpha)=0$ & $J_v(a\beta)=0$

Case 2. If $\alpha = \beta$, then RHS of eqn. (5) takes the indeterminate form $\left(\frac{0}{0}\right)$

$\therefore$ Assume 'α' as a root $J_n(ax)=0$ and evaluate the RHS of (5) by taking the limit $\beta \to \alpha$

$$\lim_{\beta\to\alpha} \int_0^a x J_v(\alpha x) J_v(\beta x)\, dx = \lim_{\beta\to\alpha} \frac{\left\{ a\left[J_v(a\beta)\cdot\alpha\cdot J_v'(a\alpha) - \overset{0}{\boxed{J_v(a\alpha)}}\cdot \beta J_v'(a\beta)\right]\right\}}{\beta^2-\alpha^2}$$

constant

using LH rule

$$\int_0^a x\left[J_v(\alpha x)\right]^2 dx = \lim_{\beta\to\alpha} \frac{a\cdot\alpha\cdot a\cdot J_v'(a\beta)\, J_v'(a\alpha)}{2\beta} = \frac{a^2 \cancel{\alpha}\cdot J_v'(a\alpha)\cdot J_v'(a\alpha)}{2\cancel{\alpha}}$$

$$= \frac{a^2}{2}\left[J_v'(a\alpha)\right]^2 \quad -(7)$$

Consider the recurrance relation

$$x J_v'(x) = v J_v(x) - x J_{v+1}(x)$$

replace x by $a\alpha$

$$a\alpha\, J_v'(a\alpha) = v \overset{0\ (\text{root})}{J_v(a\alpha)} - \cancel{a\alpha} J_{v+1}(a\alpha)$$

$\therefore \cancel{a\alpha}\, J_v'(a\alpha) = -\cancel{a\alpha}\, J_{v+1}(a\alpha) \quad \Rightarrow J_v'(a\alpha) = -J_{v+1}(a\alpha)$

SBS

$$\left[J_v'(a\alpha)\right]^2 = \left[J_{v+1}(a\alpha)\right]^2$$

sub in (7)

$$\Rightarrow \int_0^a x\left[J_v(\alpha x)\right]^2 dx = \frac{a^2}{2}\left[J_{v+1}(a\alpha)\right]^2 \quad //$$