12th May, 2023

UNIT-4

Fourier Series

Fourier Representative Types

- Periodic Signals
  - Continuous time → FS
  - Discrete Time → DTFS
- Aperiodic
  - Continuous time → FT
  - Discrete time → DTFT

Existence of Fourier series

* Fourier has shown that expansion of $f(x)$ as an infinite sum of sine and cosine waveforms is possible only if it satisfies Dirichlet conditions.

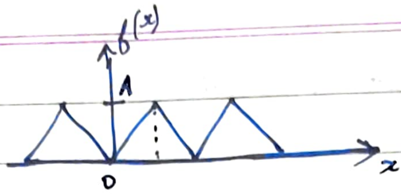
— Dirichlet's conditions :

Any func $f(x)$ defined in the interval $c < x < c+2\pi$ with period 2π is said to satisfy Dirichlet's conditions if :

i) $f(x)$ is periodic and single valued function.

ii) $f(x)$ is continuous or $f(x)$ has only a finite number of discontinuities in a period of 2π

iii) $f(x)$ has a finite no. of maxima and minima in a given period.



General form of Fourier series

1. Fourier series of a function $f(x)$ defined in period $(c, c+2\pi)$

* Consider a real valued function $f(x)$ defined in the interval $(c, c+2\pi)$ which satisfies Dirichlet's conditions. The the trignometric series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad -①$$

where the Fourier co-efficients a_0, a_n and b_n are given by

$$a_0 = \frac{1}{\pi} \int_c^{c+2\pi} f(x)\, dx \quad -②$$

$$a_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \cos nx\, dx \quad \text{for } n = 1, 2, 3 \ldots -③$$

$$b_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \sin nx\, dx \quad \text{for } n = 1, 2, 3 \ldots -④$$

is called Fourier series of $f(x)$ and the formulas ②, ③ and ④ are known as Euler formulae.

2. Fourier series of a function $f(x)$ defined in the arbitrary period $(c, c+2l)$

* Consider a real valued func $f(x)$ defined in the interval $(c, c+2l)$ and which satisfies Dirichlet's

conditions. Then the trigonometric series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right) \quad (1)$$

where the Fourier co-efficients a_0, a_n and b_n :

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx \quad -(2)$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx \quad \text{for } n=1,2,3 \quad -(3)$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx \quad \text{for } n=1,2,3 \quad -(4)$$

is called Fourier series of $f(x)$ and the formula (2), (3) and (4) are known as Euler formulae.

Note: 1. At the point of discontinuity x_0, the fourier series of $f(x)$ converges to the arithmetic mean of left and right hand limit of $f(x)$ at x_0

i.e, $f(x_0) = \frac{1}{2}\left[f(x_0-0) + f(x_0+0)\right]$

2. At the end point $f(x)$ converges to $\frac{1}{2}\left[f(c) + f(c+2\pi)\right]$

* Even function	Odd function
a. $f(-x) = f(x)$	$f(-x) = -f(x)$
b. graph of an even func. is symmetrical about y-axis	graph of an odd func. is symmetrical about origin

c) $\int_{-c}^{c} f(x)\,dx = 2\int_{0}^{c} f(x)\,dx$ | $\int_{-c}^{c} f(x)\,dx = 0$

d) Product of two even funcs. is even | Product of an even and odd func. is odd

Product of two odd funcs. is even

RESULT: If $f(x) = \begin{cases} \phi(x) & \text{in } -\pi < x < 0 \\ \varphi(x) & \text{in } 0 < x < \pi \end{cases}$ we say that

$f(x)$ is even if $\phi(-x) = \varphi(x)$ and
$f(x)$ is odd if $\phi(-x) = -\varphi(x)$

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Fourier series of even and odd function

DEFINITION: even function in $(-\pi, \pi)$ [Here $b_n = 0$]

Suppose $f(x)$ is an even func. in $(-\pi, \pi)$, then the Fourier series expansion contains only cosine terms and is known as Fourier cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad -②$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx, \quad n = 1, 2, 3 \ldots$$

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2. odd function in $(-\pi, \pi)$ $[a_0 = a_n = 0]$

Suppose $f(x)$ is an odd func. in $(-\pi, \pi)$, the Fourier series expansion contains only sine terms and is known as Fourier sine series.

$$f(x) = \sum_{b=1}^{\infty} b_n \sin nx \quad \text{—} ③$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{2}{\pi}\int_{0}^{\pi} f(x) \sin nx \, dx$$

$n = 1, 2, 3$

→ even func. defined in $(-l, l)$ [Here $b_n = 0$]

contains only cosine terms and is known as Fourier cosine series.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos \frac{n\pi x}{l}\right)$$

where $a_0 = \frac{1}{l}\int_{-l}^{l} f(x) \, dx = \frac{2}{l}\int_{0}^{l} f(x) \, dx$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} \, dx = \frac{2}{l}\int_{0}^{l} f(x) \cos \frac{n\pi x}{l} \, dx$$

$n = 1, 2, 3$

→ odd func. defined in $(-l, l)$ $[a_0 = a_n = 0]$

contains only sine terms and is known as Fourier sine series

$$f(x) = \sum_{n=1}^{\infty}\left(b_n \sin \frac{n\pi x}{l}\right)$$

where, $b_n = \frac{1}{l}\int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} dx = \frac{2}{l}\int_{0}^{l} f(x) \sin \frac{n\pi x}{l} dx$

$n = 1, 2, 3$

RESULTS:

$\cos n\pi = (-1)^n$

$\sin n\pi = 0$

$\cos \pi = -1$

$e^{-\infty} = 0$

$$\sin \frac{n\pi}{2} = \begin{cases} 0 & n \text{ is even} \\ 1 & n = 1, 5, 9, 13, \ldots \\ -1 & n = 3, 7, 11, \ldots \end{cases}$$

$$\cos \frac{n\pi}{2} = \begin{cases} 0 & n \text{ is odd} \\ 1 & n \text{ is } 0, 4, 8 \ldots \\ -1 & n \text{ is } 2, 6, 10, \ldots \end{cases}$$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}[a \sin bx - b \cos bx]$$

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}[a \cos bx + b \sin bx]$$

$$\int u v \, dx = uv_1 - \underbrace{u'}_{\text{differential}} v_2 + u'' v_3 - u''' v_4 + \ldots\ldots$$

integral part

q. Obtain Fourier series of the periodic function $f(x) = e^x$ in $-\pi < x < \pi$, $f(x+2\pi) = f(x)$

Hence find sum of series

$$\frac{1}{1+2^2} - \frac{1}{1+3^2} + \frac{1}{1+4^2} - \ldots + \frac{(-1)^n}{1+n^2}$$

Sol. All exponential functions are neither even nor odd.

The Fourier series of $f(x) = e^x$ in $(-\pi, \pi)$ is given by

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$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \quad -(1)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \, dx$$

$$= \frac{1}{\pi}\left[e^x\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[e^{\pi} - e^{-\pi}\right] = \frac{2\sinh\pi}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos nx \, dx$$

$$= \frac{1}{\pi}\left[\frac{e^x}{1^2+n^2}(\cos nx + n\sin nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{\pi}}{1+n^2}((-1)^n + 0) - \left\{\frac{e^{-\pi}}{1+n^2}((-1)^n + 0)\right\}\right]$$

$$= \frac{1}{\pi}\frac{(-1)^n(e^{\pi} - e^{-\pi})}{(1+n^2)}$$

$$a_n = \frac{(-1)^n 2\sinh\pi}{\pi(1+n^2)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin nx \, dx$$

$$= \frac{1}{\pi}\left[\frac{e^x}{1+n^2}(\sin nx - n\cos nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{\pi}}{1+n^2}(0 - n(-1)^n) - \left\{\frac{e^{-\pi}}{1+n^2}(0 - n(-1)^n)\right\}\right]$$

$$= \frac{-n(-1)^n(e^{\pi} - e^{-\pi})}{\pi(1+n^2)}$$

$$b_n = \frac{-n(-1)^n 2\sinh\pi}{\pi(1+n^2)}$$

(1) $\Rightarrow$

$$e^x = f(x) = \frac{\sinh\pi}{\pi} + \sum_{n=1}^{\infty} \frac{2(-1)^n \sinh\pi}{\pi(1+n^2)}\cos nx - \frac{2n(-1)^n \sinh\pi}{\pi(1+n^2)}\sin nx \quad -(2)$$

Deduction

put $x=0$ in ②

$$1 = \frac{\sinh\pi}{\pi} + \sum_{n=1}^{\infty} \frac{2(-1)^n \sinh\pi}{\pi(1+n^2)} \cdot 1$$

$$1 = \frac{\sinh\pi}{\pi}\left[1 + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}\right]$$

$$\frac{\pi}{\sinh\pi} = 1 + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}$$

$$\frac{\pi}{\sinh\pi} = 1 + 2\left[\frac{-1}{2} + \frac{1}{5} - \frac{1}{10} + \frac{1}{17} - \cdots\right]$$

$$\frac{\pi}{\sinh\pi} = \cancel{1} - \cancel{1} + \frac{2}{5} - \frac{2}{10} + \frac{2}{17} - \cdots$$

$$\frac{\pi}{\sinh\pi} = 2\left(\frac{1}{5} - \frac{1}{10} + \frac{1}{17} - \cdots\right)$$

$$\frac{\pi}{2\sinh\pi} = \frac{1}{1+2^2} - \frac{1}{1+3^2} + \frac{1}{1+4^2} - \cdots$$

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Q. Find the Fourier series of $f(x) = x + x^2$ for $-\pi < x < \pi$ and hence deduce a series expansion for $\frac{\pi^2}{6}$

Sol.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + b_n \sin nx \quad -①$$

$f(x) = x + x^2$

$f(-x) = -x + x^2 \neq f(x)$

$= -(x - x^2) \neq -f(x)$

neither even nor odd

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{\pi^2}{2} + \frac{\pi^3}{3}\right]$$

$$= \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{(x+x^2)}_{u} \underbrace{\cos nx}_{v}\, dx$$

$u = x + x^2$ $\quad$ $v = \cos nx$

$u' = 1 + 2x$ $\quad$ $v_1 = \frac{\sin nx}{n}$

$u'' = 2$ $\quad$ $v_{11} = -\frac{\cos nx}{n^2}$

$u''' = 0$ $\quad$ $v_{111} = -\frac{\sin nx}{n^3}$

$$\int uv\, dx = uv_1 - u'v_2 + u''v_3 - \ldots$$

$$a_n = \frac{1}{\pi}\left[\cancel{(x+x^2)\frac{\sin nx}{n}}^{0} - (1+2x)\left(-\frac{\cos nx}{n^2}\right) + \cancel{2\left(-\frac{\sin nx}{n^3}\right)}^{0}\right]_{-\pi}^{\pi}$$

$\sin n\pi = 0$

$$= \frac{1}{\pi}\left[(1+2x)\frac{\cos nx}{n^2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(1+2\pi)\frac{(-1)^n}{n^2} - \left((1-2\pi)\frac{(-1)^n}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\frac{(-1)^n}{n^2}(\cancel{1}+2\pi - \cancel{1}+2\pi)$$

$$a_n = \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{(x+x^2)}_{u} \underbrace{\sin nx}_{v}\, dx$$

$u = x + x^2$ $\quad$ $v = \sin nx$

$u' = 1 + 2x$ $\quad$ $v_1 = -\frac{\cos nx}{n}$

$u'' = 2$ $\quad$ $v_2 = -\frac{\sin nx}{n^2}$

$u''' = 0$ $\quad$ $v_3 = \frac{\cos nx}{n^3}$

$$b_n = \frac{1}{\pi}\left[(x+x^2)\left(-\frac{\cos nx}{n}\right) - (1+2x)\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(\pi+\pi^2)\left(-\frac{(-1)^n}{n}\right) + \frac{2(-1)^n}{n^2} - \left((-\pi+\pi^2)\left(\frac{-(-1)^n}{n}\right) + \frac{2(-1)^n}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left[\cancel{(\pi+\pi^2)}\left(-\frac{(-1)^n}{n}\right)\left[\pi + \cancel{\pi^2} + \pi - \cancel{\pi^2}\right]\right]$$

$$= 2\left(-\frac{(-1)^n}{n}\right)$$

$$b_n = \frac{-2(-1)^n}{n}$$

$\therefore$ ① $\rightarrow$

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx - \frac{2(-1)^n}{n}\sin nx \quad -②$$

Deduction:

Put $x=\pi$, since $x=\pi$ is an end point in the interval $(-\pi,\pi)$, the value of $f(x)$ at $x=\pi$ is given by

$$f(x) = \frac{1}{2}\left[f(-\pi) + f(\pi)\right]$$

$$= \frac{1}{2}\left[-\cancel{\pi} + \pi^2 + \cancel{\pi} + \pi^2\right]$$

$$= \pi^2$$

② $\rightarrow$

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n(-1)^n}{n^2}$$

$$\pi^2 - \frac{\pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2}$$

$$\frac{2\pi^2}{3\times 4} = \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2}$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

q. Obtain Fourier series for $f(x) = x^3$ for $-\pi < x < \pi$

Sol. odd func.

$$\therefore a_0 = a_n = 0$$

Required F.S is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad -①$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} x^3 \sin nx \, dx$$

$$= \frac{2}{\pi}\left[x^3\left(-\frac{\cos nx}{n}\right) \cancel{- 3x^2\left(-\frac{\sin nx}{n^2}\right)} \xrightarrow{0} + 6x\left(\frac{\cos nx}{n^3}\right) \cancel{- 6\left(\frac{\sin nx}{n^4}\right)} \xrightarrow{0} \right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi^3\left(-\frac{(-1)^n}{n}\right) + 6\pi\left(+\frac{(-1)^n}{n^3}\right) - 0\right]$$

$$= \frac{2\cancel{\pi}(-1)^n}{\cancel{\pi} n^3}\left[-\pi^2 n^2 + 6\right]$$

$$b_n = \frac{2(-1)^n(6 - n^2\pi^2)}{n^3}$$

① →

$$x^2 = \sum_{n=1}^{\infty} \frac{2(-1)^n(6 - n^2\pi^2)}{n^3} \sin nx$$

q. Denote $p_n = 1 - \cos n\pi$. Find the Fourier series of the function $f(x) = 1 - |x|$ on given interval $-\pi < x < \pi$

Sol. $f(x) = 1 - |x|$ is an even function.

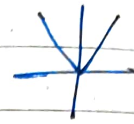
$$b_n = 0$$

Required Fourier Series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad -①$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} (1-|x|)\, dx$$

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$



$$= \frac{2}{\pi}\int_0^{\pi} 1 - x \, dx$$

$$= \frac{2}{\pi}\left[x - \frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi - \frac{\pi^2}{2}\right]$$

$$= 2 - \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} (1-|x|) \cos nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \underset{u}{(1-x)}\, \underset{v}{\cos nx}\, dx$$

$$= \frac{2}{\pi}\left[(1-x)\cancelto{0}{\frac{\sin nx}{n}} - (-1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\cos n\pi}{n^2} - \left(-\frac{1}{n^2}\right)\right]$$

$$a_n = \frac{2}{\pi n^2}(1 - \cos n\pi)$$

$$a_n = \frac{2}{\pi n^2} p_n$$

① →

$$1 - |x| = \frac{2-\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2} p_n \cos nx$$

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q. Obtain the Fourier Series of $f(x) = \frac{\pi - x}{2}$ in $0 < x < 2\pi$. Hence deduce that:

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$$

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Note: To check $f(x)$ is odd / even in $(0, 2\pi)$ or $(0, 2l)$

for $(0, 2\pi)$:
$$f(2\pi - x) = f(x) \rightarrow \text{even}$$
$$f(2\pi - x) = -f(x) \rightarrow \text{odd}$$

for $(0, 2l)$:
$$f(2l - x) = f(x) \rightarrow \text{even}$$
$$f(2l - x) = -f(x) \rightarrow \text{odd}$$

Sol: $f(2\pi - x) = \dfrac{\pi - (2\pi - x)}{2} = \dfrac{-\pi + x}{2} = \dfrac{-(\pi - x)}{2} = -f(x)$

$\Rightarrow$ odd func.

$\therefore a_0 = a_n = 0$

The required Fourier Series is $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ — ①

$$b_n = \frac{\cancel{2}}{\pi} \int_0^{\pi} \left(\frac{\pi - x}{\cancel{2}}\right) \sin nx \, dx$$

$$= \frac{1}{\pi} \left[(\pi - x)\left(\frac{-\cos nx}{n}\right) - (-1)\cancel{\left(\frac{-\sin nx}{n^2}\right)}^{0} \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[0 - \left(\frac{-\pi}{n}\right) \right]$$

$$b_n = \frac{1}{n}$$

① $\rightarrow$

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

Deduction:

put $x = \dfrac{\pi}{2}$ in ②

$$f(x) = \frac{\pi - x}{2} \qquad f\left(\frac{\pi}{2}\right) = \frac{\pi}{4}$$

$$\therefore \frac{\pi}{4} = \frac{1}{1} \sin\frac{\pi}{2} + \frac{1}{2} \sin\pi + \frac{1}{3} \sin\frac{3\pi}{2} + \frac{1}{4} \sin 2\pi$$

$$\therefore \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} + \cdots$$

q. Expand $f(x) = 2x - x^2$ as a Fourier Series in the interval $(0, 3)$

Sol. replace $(0, 3)$ with $(0, 2l)$

$$2l = 3$$
$$l = 3/2$$

$$f(3-x) = 2(3-x) - (3-x)^2$$
$$= 6 - 2x - 9 - x^2 + 6x$$
$$= -x^2 + 4x - 3 \neq f(x)$$
$$= -(x^2 - 4x + 3) \neq -f(x)$$

$f(x)$ is neither odd / even

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$l = 3/2$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2}{3} n\pi x\right) + b_n \sin\left(\frac{2}{3} n\pi x\right) \quad -(1)$$

$$a_0 = \frac{1}{l} \int^{3} 2x - x^2 \, dx$$

$$= \frac{2}{3} \left[x^2 - \frac{x^3}{3} \right]_0^3 = \frac{2}{3} \left[9 - 9 - 0 \right] = 0$$

$$a_0 = 0$$

$$a_n = \frac{1}{l} \int_0^{2l} f(x) \cos\left(\frac{n\pi x}{l}\right)$$

$$= \frac{2}{3} \int_0^3 (2x - x^2) \cos\left(\frac{2\pi n x}{3}\right) dx$$

$$- (2)\left(-\frac{9}{4\pi^2 n^2}\right)$$

$$= \frac{2}{3} \Bigg[(2x - x^2) \left(\sin\left(\frac{2\pi n x}{3}\right) \cdot \frac{3}{2\pi n}\right)^{\to 0} - (2 - 2x)\left(-\cos\left(\frac{2\pi n x}{3}\right) \cdot \frac{9}{4\pi^2 n^2}\right)$$
$$+ (-2)\left(-\sin\left(\frac{2\pi n x}{3}\right) \cdot \frac{27}{8\pi^3 n^3}\right)^{\to 0} \Bigg]_0^3$$

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$$= \frac{2}{3}\left[-(-4)\left(-\cos 2n\pi\right)\frac{9}{4\pi^2 n^2} - \left(2\times 1\times \frac{9}{4\pi^2 n^2}\right)\right]$$

$$= \frac{2}{3}\left[\frac{-9}{\pi^2 n^2} - \frac{9}{2\pi^2 n^2}\right]$$

$$a_n = -\frac{\cancel{2}}{\cancel{3}} \times \frac{9}{n^2\pi^2}\left[\frac{\cancel{3}}{\cancel{2}}\right]$$

$$a_n = \frac{-9}{\pi^2 n^2}$$

$$b_n = \frac{2}{3}\int_0^3 (2x - x^2)\sin\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{2}{3}\left[(2x-x^2)\left(-\cos\left(\frac{2n\pi x}{3}\right)\cdot\frac{3}{2\pi n}\right) - \cancel{(2-2x)\left(-\sin\left(\frac{2\pi n x}{3}\right)\frac{9}{4n^2\pi^2}\right)}^{\,0} + (-2)\left(\cos\left(\frac{2n\pi x}{3}\right)\cdot\frac{27}{8\pi^3 n^3}\right)\right]_0^3$$

$$= \frac{2}{3}\left[-3\left(\frac{-3}{2n\pi}\right) - \frac{\cancel{2}\times 27}{\underset{4}{\cancel{8}}\pi^3 n^3} - \left(0 - \frac{\cancel{2}\times 27}{\underset{4}{\cancel{8}}\pi^3 n^3}\right)\right]$$

$$= \frac{2}{3}\left[\frac{9}{2n\pi} - \cancel{\frac{27}{4\pi^3 n^3}}^{\,0} + \cancel{\frac{27}{4\pi^3 n^3}}^{\,0}\right]$$

$$b_n = \frac{3}{n\pi}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{-9}{\pi^2 n^2}\cos\left(\frac{2}{3}n\pi x\right) + \frac{3}{n\pi}\sin\left(\frac{2}{3}n\pi x\right)$$

q. Obtain Fourier Series of $f(x) = x \sin x$ in $-\pi < x < \pi$

$x \downarrow$ odd $\times$ $\sin x \downarrow$ odd $=$ even

Sol. $f(x) = x\sin x$ is an even func.

here $b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad -①$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x \sin x \, dx$$

$$= \frac{2}{\pi}\left[x(-\cos x) - 1(-\sin x)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi(1) - 0\right]$$

$\cos n\pi = (-1)^n$

$\cos(n+1)\pi = (-1)^{n+1}$

$$a_0 = 2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos nx \, dx$$

$$= \frac{\cancel{2}}{\pi}\int_0^{\pi} \frac{x}{\cancel{2}}\left(\sin(1+n)x + \sin(1-n)x\right) dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x \sin(1+n)x \, dx + \int_0^{\pi} x \sin(1-n)x \, dx\right]$$

$$= \frac{1}{\pi}\left\{\left[x\left(-\frac{\cos(1+n)x}{1+n}\right) - \left(-\frac{\sin(1+n)x}{(1+n)^2}\right)\right]_0^{\pi}\right.$$

$$\left. + \left[x\left(-\frac{\cos(1-n)x}{1-n}\right) - \left(-\frac{\sin(1-n)x}{(1-n)^2}\right)\right]_0^{n}\right\}$$

$$= \frac{1}{\pi}\left[\frac{-\pi(-1)^{n+1}}{n+1} - 0 + \left(\frac{-\pi(-1)^{1-n}}{1-n} - 0\right)\right]$$

$$= \frac{-(-1)^n(-1)}{1+n} + \frac{-(-1)^1(-1)^{-n}}{1-n}$$

$$= \frac{(-1)^n}{1+n} + \frac{(-1)^{-n}}{1-n}$$

$(-1)^n(n-1)$

$$= (-1)^n\left(\frac{1}{1+n} - \frac{1}{n-1}\right)$$

$$= (-1)^n\left(\frac{n-1-n-1}{n^2-1}\right)$$

$$a_n = \frac{-2(-1)^n}{n^2-1}$$; is true when $n \neq 1$

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when $n = 1$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \frac{\sin 2x}{2} \, dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - \left(-\cancel{\frac{\sin 2x}{4}}^{\,0}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi}{2}\right]$$

$$a_1 = \frac{-1}{2}$$

① →

$$f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx$$

$$= 1 - \frac{\cos x}{2} + \sum_{n=2}^{\infty} \frac{-2(-1)^n}{n^2-1} \cos nx$$

Q. Find Fourier Series expansion of $f(x) = x(1-x)(2-x)$ in $(0, 2)$. Hence deduce the sum of series

$$\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \dots$$

Sol. $(0, 2) \longleftrightarrow (0, 2l)$

$$2 = 2l$$

$$l = 1$$

replace $x \to 2-x$ in $f(x) = x(1-x)(2-x)$

$$f(2-x) = (2-x)(1-(2-x))(2-(2-x))$$

$$= (2-x)(-1+x)x$$

$$= -x(1-x)(2-x) = -f(x)$$

$\Rightarrow$ odd func. $a_0 = a_n = 0$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{1} \quad -① $$

$$b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{2}{1}\int_0^1 x(1-x)(2-x) \sin \frac{n\pi x}{1} \, dx$$

$$= 2\int_0^1 (x^3 - 3x^2 + 2x) \sin \frac{n\pi x}{1} \, dx$$

$$= 2\Big[(x^3-3x^2+2x)\left(\frac{-\cos n\pi x}{n\pi}\right) - (3x^2-6x+2)\left(\frac{-\sin n\pi x}{n^2\pi^2}\right) + (6x-6)\left(\frac{\cos n\pi x}{n^3\pi^3}\right) - (6)\left(\frac{+\sin n\pi x}{n^4\pi^4}\right)\Big]_0^1$$

$$= 2\left[0 + 0 - \left(0 - \frac{6\cdot(1)}{n^3\pi^3}\right)\right]$$

$$b_n = \frac{12}{n^3\pi^3} \quad -②$$

① →

$$f(x) = \sum_{n=1}^{\infty} \frac{12}{n^3\pi^3} \sin n\pi x \quad -③$$

Deduction

put $x = \frac{1}{2}$

$$f\left(\frac{1}{2}\right) = \frac{1}{8} - \frac{3}{4} + 1 = \frac{3}{8}$$

$\sin \pi/2 = 1$

$\sin 3\pi/2 = -1$

$\sin 5\pi/2 = 1$

$\vdots$

$$\frac{3}{8} = \frac{12}{\pi}\sum_{n=1}^{\infty} \frac{1}{n^3} \sin \frac{n\pi x}{2}$$

$$\frac{3}{8} \times \frac{\pi^3}{12} = \frac{1}{1^3}\sin\frac{\pi}{2} + \frac{1}{3^3}\sin\frac{3\pi}{2} + \frac{1}{5^3}\sin\frac{5\pi}{2}$$

$$\frac{\pi^3}{32} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \dots\dots$$

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q. A periodic function of period 2 is defined as $f(x) = 1 + x$, $-1 < x < 1$. Obtain the Fourier series expansion of $f(x)$ and hence S.T $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{9} + \dots$

Sol. $(-1, 1) \longleftrightarrow (-l, l)$

$$l = 1$$

$$f(-x) = 1 - x \neq f(x)$$
$$= -(x-1) \neq -f(x)$$

$f(x)$ is neither odd nor even.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

put $l = 1$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos n\pi x + b_n \sin n\pi x\right) \quad -①$$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$$

$$= \int_{-1}^{1}(1+x)\,dx = 2$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}$$

$$= \int_{-1}^{1}(1+x)\cos n\pi x$$

$$= \left[(1+x)\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_{-1}^{1}$$

$$= \left[0 + \frac{(-1)^n}{n^2\pi^2} - 0 - \frac{(-1)^n}{n^2\pi^2}\right]$$

$$a_n = 0$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x) \sin \frac{n\pi x}{l}\, dx$$

$$= \int_{-1}^{1} (1+x) \sin n\pi x \, dx$$

$$= \left[(1+x)\left(-\frac{\cos n\pi x}{n\pi}\right) - \left(-\frac{\cancel{\sin n\pi x}}{n^2\pi^2}\right)\right]_{-1}^{1}$$

$$b_n = \frac{-2(-1)^n}{n\pi}$$

① →

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n\pi} \sin n\pi x \quad \text{—②}$$

Deduction

put $x = 1/2$

$$f(x) = 1 + x = \frac{3}{2}$$

$$\frac{3}{2} = 1 + \frac{-2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin n\pi x$$

$$\frac{1}{2}\left(\frac{\pi}{-2}\right) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin \frac{n\pi}{2}$$

$$-\frac{\pi}{4} = -1 + 0 + \frac{1}{3} + 0 - \frac{1}{5}$$

17th May, 2023

q. Obtain the Fourier series of expansion of

$$f(x) = \begin{cases} -a & \text{in } -\pi < x < 0 \\ a & \text{in } 0 < x < \pi \end{cases}$$

Deduce that $\frac{\pi}{4} = \sum_{0}^{\infty} \frac{(-1)^n}{2n+1}$

Sol. $f(x) = \begin{cases} \phi(x) & \text{in } -\pi < x < 0 \\ \psi(x) & \text{in } 0 < x < \pi \end{cases}$

if $\phi(-x) = \psi(x) \rightarrow$ even

$\phi(-x) = -\psi(x) \rightarrow$ odd

$\phi(a) = -a \qquad \psi(a) = a$

$\phi(-(-a)) = a = \psi(a)$

$f(x)$ is an even func.

$b_n = 0$

required Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad — ①$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} a\, dx$$

$$= \frac{2}{\pi} a \left[x\right]_0^{\pi}$$

$$= \frac{2a}{\pi}(\pi)$$

$$a_0 = 2a$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} a \cos nx\, dx = \frac{2a}{\pi} \sin nx$$

DIDN'T WORK

Sol. using general method

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \quad — ①$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -a\, dx + \int_0^{\pi} a\, dx\right]$$

$$= \frac{1}{\pi}\left[-\left[ax\right]_{-\pi}^{0} + \left[ax\right]_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-a\pi + a\pi\right] = 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} -a \cos nx \, dx + \int_{0}^{\pi} a \cos nx \, dx \right]$$

$$= \frac{1}{\pi} \left[-a \left[\frac{\sin nx}{n} \right]_{-\pi}^{0} + a \left[\frac{\sin nx}{n} \right]_{0}^{\pi} \right]$$

$$a_n = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} -a \sin nx \, dx + \int_{0}^{\pi} a \sin nx \, dx \right]$$

$$= \frac{1}{\pi} \left[a \left[\frac{\cos nx}{n} \right]_{-\pi}^{0} - a \left[\frac{\cos nx}{n} \right]_{0}^{\pi} \right]$$

$$= \frac{a}{n\pi} \left[1 - (-1)^n - (-1)^n + 1 \right]$$

$$b_n = \frac{2a}{n\pi} \left[1 - (-1)^n \right]$$

① $\rightarrow$

$$f(x) = \sum_{n=1}^{\infty} \frac{2a}{n\pi} \left[1 - (-1)^n \right] \sin nx \quad ②$$

$x = \pi/2$ in ②

$$a = \frac{2a}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left[1 - (-1)^n \right] \sin \frac{n\pi}{2}$$

$$\frac{\pi}{2} = 2 \sin \frac{\pi}{2} + 0 + \frac{2}{3} \sin \frac{3\pi}{2} + 0 + \cdots$$

$$= 2 \left[\sin \frac{\pi}{2} + \frac{1}{3} \sin \frac{3\pi}{2} + \frac{1}{5} \sin \frac{5\pi}{2} + \cdots \right]$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

Q. Find Fourier series for $f(x) = \begin{cases} -x, & -2 < x \le 0 \\ x, & 0 < x < 2 \end{cases}$

hence deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

Sol: wkt if $f(x) = \begin{cases} \phi(x) & -l < x \le 0 \\ \psi(x) & 0 < x < l \end{cases}$

then

$\phi(-x) = \psi(x) \rightarrow$ even

$\phi(-x) = -\psi(x) \rightarrow$ odd

$\phi(-x) = x \qquad \psi(x) = x$

$f(x)$ is an even func.

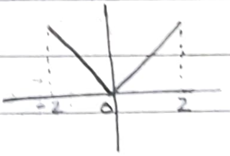
$b_n = 0$

Required Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$(-2, 2) \longleftrightarrow (-l, l)$

$l = 2$



O: point of discontinuity

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right) \quad -(1)$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{2}\int_0^2 x\,dx = \left[\frac{x^2}{2}\right]_0^2 = \frac{4}{2} = 2$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= 1\cdot\int_0^2 x\cdot\cos\frac{n\pi x}{2}\,dx$$

$$= \left[x\left(\sin\left(\frac{n\pi x}{2}\right) \cdot \frac{2}{n\pi} \right) - \left(-\cos\left(\frac{n\pi x}{2}\right) \cdot \frac{4}{n^2\pi^2} \right) \right]_0^2$$

$$= (-1)^n \cdot \frac{4}{n^2\pi^2} - \frac{4}{n^2\pi^2} = \left((-1)^n - 1\right) \frac{4}{n^2\pi^2}$$

$$a_n = \frac{4}{n^2\pi^2}\left((-1)^n - 1\right)$$

① →

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left((-1)^n - 1\right) \cos\left(\frac{n\pi x}{2}\right) \quad -②$$

Deduction

put $x = 0$ in ②

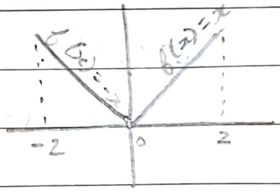
since $f(x)$ has a jump discontinuity at $x = 0$

$f(x) = 0$ is given by

$$f(x) = \frac{1}{2}\left[f(0^+) + f(0^-) \right]$$

$$= \frac{1}{2}[0 + 0]$$

$$= 0$$



② →

$$0 = 1 + \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2}\left((-1)^n - 1\right)$$

$$0 = 1 + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2}$$

$$0 = 1 + \frac{4}{\pi^2}\left[\frac{-2}{1^2} + 0 - \frac{2}{3^2} + 0 - \frac{2}{5^2} + 0 \dots \right]$$

$$0 = 1 - \frac{8}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right]$$

$$\therefore \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$$

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Q. Obtain Fourier series expansion of $f(x) = \sqrt{1-\cos x}$ in $(0, 2\pi)$. Deduce that

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}$$

Sol. $f(2\pi - x) = \sqrt{1-\cos(2\pi - x)}$

$= \sqrt{1-\cos x}$

$= f(x)$

$\therefore f(x)$ is an even func.

$b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{—} (1)$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sqrt{2\sin^2\frac{x}{2}}\,dx$$

$$= \frac{2\sqrt{2}}{\pi}\int_0^{\pi} \sin\frac{x}{2}\,dx$$

$$= \frac{2\sqrt{2}}{\pi}\left[-\cos\frac{x}{2}\cdot 2\right]_0^{\pi}$$

$$= \frac{2\sqrt{2}}{\pi}[0 + 2] = \frac{4\sqrt{2}}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sqrt{2}\sin\overset{A}{\frac{x}{2}}\cos \overset{B}{nx}\,dx$$

$$= \frac{2\sqrt{2}}{\pi}\cdot\frac{1}{2}\int_0^{\pi} \sin\left(n+\frac{1}{2}\right)x + \sin\left(\frac{1}{2}-n\right)x \quad dx$$

$$= \frac{\sqrt{2}}{\pi}\int_0^{\pi} \sin\left(n+\frac{1}{2}\right)x - \sin\left(n-\frac{1}{2}\right)x \quad dx$$

S | A
T | C

$\cos\left(\frac{\pi}{2} + n\pi\right) = -\sin n\pi$

$\cos\left(n - \frac{1}{2}\right)\pi = \cos\left(\frac{\pi}{2} - n\pi\right) = \sin n\pi$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{-\cos\left(n+\frac{1}{2}\right)x}{n+\frac{1}{2}} - \frac{\cos\left(n-\frac{1}{2}\right)x}{n-\frac{1}{2}}\right]_0^{\pi}$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{-2\cos\left(n+\frac{1}{2}\right)\pi}{2n+1} + \frac{2\cos\left(n-\frac{1}{2}\right)\pi}{2n-1} - \left(\frac{-2}{2n+1} + \frac{2}{2n-1}\right)\right]$$

$$= \frac{\sqrt{2}}{\pi}\left[\cancel{\frac{-2(-\sin n\pi)}{2n+1}}^{0} + \cancel{\frac{2(\sin n\pi)}{2n-1}}^{0} + \frac{2}{2n+1} - \frac{2}{2n-1}\right]$$

$$= 2\frac{\sqrt{2}}{\pi}\left[\frac{2n-1-2n-1}{4n^2-1}\right]$$

$$= \frac{-4\sqrt{2}}{\pi(4n^2-1)}$$

$$f(x) = \frac{2\sqrt{2}}{\pi} + \sum_{n=1}^{\infty} \frac{-4\sqrt{2}}{\pi(4n^2-1)}\cos nx \quad -(2)$$

Deduction

put $x = 0$

(1) →

$$\sqrt{1-\cos x} = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cdot 1$$

$$\frac{\cancel{4}^{2}\cancel{\sqrt{2}}}{\cancel{\pi}}\sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \frac{2\cancel{\sqrt{2}}}{\cancel{\pi}}$$

$$\sum_{1}^{\infty}\frac{1}{4n^2-1} = \frac{1}{2}$$

Q: ~~Obtain~~ A periodic func. of period 2π is defined as

$$f(x) = \begin{cases} 1, & -\pi/2 < x < \pi/2 \\ 0, & \pi/2 \le x < 3\pi/2 \end{cases}$$

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Obtain Fourier series of expansion of $f(x)$ and

S.T $\quad \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\cdots$

Sol: General form of F.S of $f(x)$ in $(c, c+2l)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right) \quad (1)$$

$$(-\pi/2, 3\pi/2) \longleftrightarrow (c, c+2l)$$

$c = -\pi/2 \qquad c + 2l = \frac{3\pi}{2} \Rightarrow 2l = 2\pi$

$l = \pi$

(1) $\rightarrow$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \quad (2)$$

$$a_0 = \frac{1}{l}\int_{c}^{c+2l} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{3\pi/2} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi/2}^{\pi/2} 1\,dx + \cancel{\int_{\pi/2}^{3\pi/2} 0\,dx}\xrightarrow{} 0\right]$$

$$= \frac{1}{\pi}(\pi) = 1$$

$a_0 = 1$

$$a_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi/2}^{\pi/2} 1\cdot\cos nx + \cancel{\int_{\pi/2}^{3\pi/2} 0\cdot\cos nx}\xrightarrow{} 0\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{-\pi/2}^{\pi/2} = \frac{1}{\pi n}\left[2\sin\frac{n\pi}{2}\right]$$

$$a_n = \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\, \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi/2}^{\pi/2} 1\cdot \sin nx + \int_{\pi/2}^{3\pi/2} 0\cdot \sin nx\right] \quad \to 0$$

$$= \frac{1}{\pi}\left[\frac{-\cos nx}{n}\right]_{-\pi/2}^{\pi/2}$$

$$b_n = 0$$

② →

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cos nx \quad -③$$

Deduction

put $x=0$ in ③

$$01 = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n} \sin\frac{n\pi}{2}$$

$$\frac{1}{2}\left(\frac{\pi}{2}\right) = 1 + 0 - \frac{1}{3} + 0 + \frac{1}{5} + 0 - \ldots\ldots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots$$

18th May, 2023

Half Range Fourier Series

Half range cosine series ⇒ cosine formula ⇒ even func

Half range sine series ⇒ sine formula ⇒ odd func

eg: $f(x) = x$ in $(0, \pi)$

Case 1. Even extension

Case 2. Odd extension

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→ Half range Fourier series of $f(x)$ in $(0, \pi)$

Half range cosine series	Half range sine series
$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$	$f(x) = \sum_{b=1}^{\infty} b_n \sin nx$
$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$	$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx$
$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\, dx$	

→ Half range Fourier series of $f(x)$ in $(0, l)$

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$	$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$
$a_0 = \frac{2}{l}\int_0^{l} f(x)\, dx$	$b_n = \frac{2}{l}\int_0^{l} f(x) \sin \frac{n\pi x}{l}\, dx$
$a_n = \frac{2}{l}\int_0^{l} f(x) \cos \frac{n\pi x}{l}\, dx$	

Q. Obtain half range Fourier ~~Series~~ sine and cosine series for func. $f(x) = x^2$ in $(0, \pi)$

Sol. a) Half range cosine series (Even func. formula)

$b_n = 0$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\, dx \qquad a_0 = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx \, dx$$

$$= \frac{2}{\pi}\left[x^2 \frac{\sin nx}{n} - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{2\pi(-1)^n}{n^2}\right]$$

$$= \frac{4(-1)^n}{n^2}$$

$\therefore$ HRFCS is given by

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx$$

b) Half range Fourier sine series (odd func. formula)

$$a_0 = a_n = 0$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin x \quad -(1)$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx \, dx$$

$$= \frac{2}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right) - 2x\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-\pi^2(-1)^n}{n} + \frac{2(-1)^n}{n^3} - \left(0 + \frac{2}{n^3}\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{n^3}\left((-1)^n - 1\right) - \frac{\pi^2(-1)^n}{n}\right]$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{2}{\pi}\left[\frac{2}{n^3}\left((-1)^n - 1\right) - \frac{\pi^2(-1)^n}{n}\right] \sin nx$$

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q. Obtain Half range sine series for the func.

$$f(x) = \begin{cases} \frac{1}{4} - x & \text{in } \left(0, \frac{1}{2}\right) \\ x - \frac{3}{4} & \text{in } \left(\frac{1}{2}, 1\right) \end{cases}$$

Sol. $f(x) \rightarrow (0,1)$

HRFSS : $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$

$(0,1) \longleftrightarrow (0, l)$

$\therefore l = 1$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x \quad -①$$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= 2 \int_0^1 f(x) \sin n\pi x \, dx$$

$$= 2\left[\int_0^{1/2} \left(\frac{1}{4} - x\right) \sin n\pi x \, dx + \int_{1/2}^1 \left(x - \frac{3}{4}\right) \sin n\pi x \, dx\right]$$

$$= 2\left\{\left[\left(\frac{1}{4} - x\right)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_0^{1/2} + \left[\left(x - \frac{3}{4}\right)\left(-\frac{\cos n\pi x}{n\pi}\right) - 1\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_{1/2}^1\right\}$$

$$= 2\left\{\left[\frac{\cancel{\cos \frac{n\pi}{2}}}{4n\pi} - \frac{\sin \frac{n\pi}{2}}{n^2\pi^2} - \left(\frac{-1}{4n\pi} - 0\right)\right] + \left[\frac{1}{4}\frac{(-1)^n}{n\pi} + 0 - \left(\frac{\cancel{\cos \frac{n\pi}{2}}}{4n\pi} + \frac{\sin \frac{n\pi}{2}}{n^2\pi^2}\right)\right]\right\}$$

$$= 2\left[\frac{1 - (-1)^n}{4n\pi} - \frac{2\sin n\pi/2}{n^2\pi^2}\right]$$

① →

$$f(x) = \sum_{n=1}^{\infty} 2\left[\frac{1 - (-1)^n}{4n\pi} - \frac{2\sin n\pi/2}{n^2\pi^2}\right] \sin n\pi x$$

we have $(0, \pi)$ or $(0, l)$ & $(0, \pi/2)$

19th May, 2023

Q. Write the Fourier cosine series and sine series for the func in given interval (whenever possible)

$$f(x) = \cos x \quad \text{in} \quad 0 < x \, \pi/2$$

Sol. cosine & sine series $\Rightarrow$ Half range

$(0, \pi/2) \longleftrightarrow (0, l)$

a) HRFCS (even func formulae)

$b_n = 0$

$\therefore \; l = \pi/2$

Required FS is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos 2nx \quad -(1)$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{\pi/2}\int_0^{\pi/2} \cos x\,dx$$

$$= \frac{4}{\pi}\left[\sin x\right]_0^{\pi/2}$$

$$a_0 = \frac{4}{\pi}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{4}{\pi}\int_0^{\pi/2} \cos x \cos 2nx\,dx$$

$$= \frac{4}{\pi}\int_0^{\pi/2} \cos 2nx \cos x\,dx$$

$$= \frac{4}{\pi}\left(\frac{1}{2}\right)\int_0^{\pi/2} \cos(2n+1)x + \cos(2n-1)x\,dx$$

$$= \frac{2}{\pi}\left[\frac{\sin(2n+1)x}{2n+1} + \frac{\sin(2n-1)x}{2n-1}\right]_0^{\pi/2}$$

$$= \frac{2}{\pi}\left[\frac{\sin(2n+1)\pi/2}{2n+1} + \frac{\sin(2n-1)\pi/2}{2n-1}\right]$$

$$= \frac{2}{\pi}\left[\frac{\sin(\pi/2 + n\pi)}{2n+1} - \frac{\sin(\pi/2 - n\pi)}{2n-1}\right]$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi}{2n+1} - \frac{\cos n\pi}{2n-1}\right]$$

$$= \frac{2(-1)^n}{\pi}\left[\frac{\cancel{2n}-1-\cancel{2n}-1}{4n^2-1}\right]$$

$$a_n = \frac{-4(-1)^n}{\pi(4n^2-1)}$$

① →

$$f(x) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{-4(-1)^n}{\pi(4n^2-1)} \cos 2nx$$

b) HRFSS (odd func. formula)

$$a_0 = a_n = 0$$

Required FS

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \qquad l = \pi/2$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(2nx) \quad -①$$

$$b_n = \frac{2}{\pi}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{4}{\pi}\int_0^{\pi/2} \cos x \sin 2nx \, dx \qquad = \frac{4}{\pi}\int_0^{\pi/2} \sin 2nx \cos x \, dx$$

$$= \frac{4}{\pi(2)}\int_0^{\pi/2}\left[\sin(2n+1)x + \sin(2n-1)x\right]$$

$$= \frac{2}{\pi}\left[-\frac{\cos(2n+1)x}{2n+1} - \frac{\cos(2n-1)x}{2n-1}\right]_0^{\pi/2}$$

$$= -\frac{2}{\pi}\left[\frac{\cos(2n+1)\pi/2}{2n+1} + \frac{\cos(2n-1)\pi/2}{2n-1} - \left(\frac{1}{2n+1} + \frac{1}{2n-1}\right)\right]$$

$-\sin n\pi = 0$

$$= -\frac{2}{\pi}\left[\frac{\cos(\pi/2 + n\pi)}{2n+1} + \frac{\cos(\pi/2 - n\pi)}{2n-1} - \left(\frac{2n\cancel{-1} + 2n\cancel{+1}}{4n^2-1}\right)\right]$$

$$= -\frac{2}{\pi}\left[\frac{-4n}{4n^2-1}\right]$$

$$b_n = \frac{8n}{\pi(4n^2-1)}$$

① →

$$f(x) = \sum_{n=1}^{\infty} \frac{8n}{\pi(4n^2-1)} \sin(2nx)$$

q: $f(x) = \begin{cases} x & 0 < x < \pi/2 \\ \pi - x & \pi/2 < x < \pi \end{cases}$

Show that $f(x) = \frac{4}{\pi}\left[\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \dots\right]$

Sol. interval : $(0,\pi)$ ⇒ Half range

since $f(x)$ has sine terms ⇒ sine series

∴ we need HRFSS : odd func formula

$\Downarrow a_0 = a_n = 0$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{—①}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x \sin nx \, dx + \int_{\pi/2}^{\pi} (\pi - x) \sin nx \, dx\right]$$

$$= \frac{2}{\pi}\left[\left[x\left(\frac{-\cos nx}{n}\right) - 1\cdot\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi/2} + \left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\left[\cancel{\frac{-\pi}{2}\frac{\cos n\pi/2}{n}} + \frac{\sin n\pi/2}{n^2}\right] + \left[0 - \left(\cancel{-\frac{\pi}{2}\frac{\cos n\pi/2}{n}} - \frac{\sin n\pi/2}{n^2}\right)\right]\right]$$

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$$= \frac{2}{\pi}\left[\frac{2\sin\frac{n\pi}{2}}{n^2}\right]$$

$$b_n = \frac{4}{\pi n^2}\sin\left(\frac{n\pi}{2}\right)$$

(1) $\rightarrow$

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{\pi n^2}\sin\left(\frac{n\pi}{2}\right)\cdot\sin nx$$

$$= \frac{4}{\pi}\left[\frac{\sin \pi/2}{1^2}\sin x + 0 + \frac{\sin 3\pi/2}{3^2}\sin 3x + 0 + \frac{\sin 5\pi/2}{5^2}\sin 5x + \cdots\right]$$

$$= \frac{4}{\pi}\left[\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \cdots\right]$$

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Q. Find the Fourier half range a) cosine series
b) sine series

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 2-x & 1 < x < 2 \end{cases}$$

Sol. a)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

$$(0,2) \longleftrightarrow (0,2)$$

$$l = 2$$

$$a_0 = \frac{2}{l}\int_0^2 f(x)\, dx$$

$$= 1 \cdot \int_0^1 x\, dx + \int_1^2 2-x\, dx$$

$$= \frac{1}{2} + \left[2x - \frac{x^2}{2}\right]_1^2$$

$$= \frac{1}{2} + \left[4 - 2 - 2 + \frac{1}{2}\right]$$

$$= 1$$

$$a_n = \frac{2}{l}\int_0^2 f(x)\ \frac{n x\pi}{2}$$

$$= 1 \int_0^1 x\ \frac{n x \pi}{2} + \int_1^2 (2-x)\ \frac{n x \pi}{2}$$

$$= \left[x\left(-\frac{\cos n x\pi}{\pi n}\right) - 1\left(-\frac{\sin n x\pi}{\pi^2 n^2}\right)\right]_0^1 + \left[(2-x)\left(-\frac{\cos n x\pi}{\pi n}\right) - (-1)\left(-\frac{\sin n \pi x}{\pi^2 n^2}\right)\right]_1^2$$

$$= \left[-\frac{(-1)^n}{n\pi} - (0) + \left[0 - \left(1\left(-\frac{(-1)^n}{n\pi}\right.\right.\right.\right.$$

$$= \left[\frac{2x \sin n\pi x/2}{n\pi} + \frac{4\cos \frac{n\pi x}{2}}{n^2\pi^2} \right]_0^1 + \left[\frac{2(2-x)\sin n\pi x/2}{n\pi} - \frac{4x \cos n\pi x/2}{n^2\pi^2} \right]_1$$

$$= \left[\frac{4\cos \frac{n\pi}{2}}{n^2\pi^2} - \frac{4}{n^2\pi^2} \right] + \left[\frac{-8(-1)^n}{n^2\pi^2} + \frac{4\cos \frac{n\pi}{2}}{n^2\pi^2} \right]$$

$$= \frac{8\cos n\pi/2}{n^2\pi^2} - \frac{4}{n^2\pi^2}\left(1 + 2(-1)^n\right)$$

$$a_n = \frac{4}{n^2\pi^2}\left[2\cos\left(\frac{n\pi}{2}\right) - 1 - 2(-1)^n \right]$$

$$\therefore f(x) = \frac{1}{2} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}\left(2\cos\frac{n\pi}{2} - 1 - 2(-1)^n \right)\cos\frac{n\pi x}{2}$$

b)

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$b_n = 1. \left[\int_0^1 x \sin\frac{n\pi x}{2}\,dx + \int_1^2 (2-x)\sin\frac{n\pi x}{2}\,dx \right]$$

$$= \left[x\left(-\cos\frac{n\pi x}{2}\right)\left(\frac{2}{n\pi}\right) + \sin\frac{n\pi x}{2}\left(\frac{4}{n^2\pi^2}\right) \right]_1^2$$

$$+ \left[(2-x)\left(-\cos\frac{n\pi x}{2}\right)\left(\frac{2}{n\pi}\right) - \sin\left(\frac{n\pi x}{2}\right)\left(\frac{4}{n^2\pi^2}\right) \right]_1^2$$

$$= \left[\cancel{-\frac{2\cos n\pi/2}{n\pi}}^{\,0} + \frac{4\sin n\pi/2}{n^2\pi^2} \right] + \left[-\left(-\frac{2\cos n\pi/2}{n\pi} - \frac{4\sin\frac{n\pi}{2}}{n^2\pi^2} \right) \right]$$

$$b_n = \frac{8\sin n\pi/2}{n^2\pi^2}$$

$$f(x) = \frac{8}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{2}\right)$$

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22nd May 2023
(ABSENT)

Complex Form of the Fourier Series

* The complex exponential form of the Fourier series of a func. $f(x)$ defined in the interval $(c, c+2\pi)$ is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad \text{where } c_n = \frac{1}{2\pi}\int_c^{c+2\pi} f(x) e^{-inx}\, dx$$

* The complex exponential form of the Fourier series of a func. $f(x)$ in the interval $(c, c+2l)$ is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}, \quad \text{where } c_n = \frac{1}{2l}\int_c^{c+2l} f(x) e^{-\frac{in\pi x}{l}}\, dx$$

Note $e^{i\theta} = \cos\theta + i\sin\theta$, $\quad e^{-i\theta} = \cos\theta - i\sin\theta$

$$\frac{e^{i\theta} + e^{-i\theta}}{2} = \cos\theta, \quad \frac{e^{i\theta} - e^{-i\theta}}{2i} = \sin\theta$$

q. Find the complex Fourier series of $f(x) = \cos ax$ $(-\pi < x < \pi)$ where 'a' is not an integer.

Sol $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos ax \cdot e^{-inx}\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx} \cos ax\, dx$$

$$= \frac{1}{2\pi}\left[\frac{e^{-inx}}{a^2 - n^2}\left(-in\cos ax + a\sin ax\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-in\pi}}{a^2-n^2}(-in\cos a\pi + a\sin a\pi) - \left(\frac{e^{in\pi}}{a^2-n^2}(-in\cos a\pi - a\sin a\pi)\right)\right]$$

$$= \frac{1}{2\pi(a^2-n^2)}\left[e^{-in\pi}(-in\cos a\pi + a\sin a\pi) + e^{in\pi}(in\cos a\pi + a\sin a\pi)\right]$$

$$= \frac{1}{2\pi(a^2-n^2)}\left[in\cos a\pi\,(e^{in\pi} - e^{-in\pi}) + a\sin a\pi\,(e^{-in\pi} + e^{in\pi})\right]$$

$$= \frac{1}{2\pi(a^2-n^2)}\left[in\cos a\pi \cdot \cancel{2i\sin n\pi}^{\,0} + a\sin a\pi \cdot 2\cos n\pi\right]$$

$$c_n = \frac{a\sin a\pi \cdot \cos n\pi}{\pi(a^2-n^2)} = \frac{a\sin a\pi\,(-1)^n}{\pi(a^2-n^2)}$$

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{a\sin a\pi\,(-1)^n}{\pi(a^2-n^2)} \cdot e^{inx}$$

q. Find the complex form of the Fourier series of $f(x) = e^{-ax}$ in $-1 \le x \le 1$.

Sol. $$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}$$

$$(-1, 1) \longleftrightarrow (-l, l)$$

$$\Rightarrow l = 1$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}$$

$$c_n = \frac{1}{2l}\int_{-1}^{1} e^{-ax} \cdot e^{-in\pi x}\,dx$$

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$$= \frac{1}{2l}\int_{-1}^{1} e^{-x(a+in\pi)}\,dx$$

$$= \frac{1}{2l}\left[\frac{e^{-x(a+in\pi)}}{-(a+in\pi)}\right]_{-1}^{1}$$

$$= \frac{1}{2l}\left[\frac{-e^{-(a+in\pi)}}{a+in\pi} - \left(\frac{-e^{(a+in\pi)}}{a+in\pi}\right)\right]$$

$$= \frac{1}{2(a+in\pi)}\left[e^{(a+in\pi)} - e^{-(a+in\pi)}\right]$$

$$= \frac{1}{2(a+in\pi)}\left[e^{a}\cdot e^{in\pi} - e^{-a}\cdot e^{-in\pi}\right]$$

$$= \frac{1}{2(a+in\pi)}\left[e^{a}(\cos n\pi + i\overset{0}{\cancel{\sin n\pi}}) - e^{-a}(\cos n\pi - i\overset{0}{\cancel{\sin n\pi}})\right]$$

$$= \frac{(-1)^n}{2(a+in\pi)}\left[e^{a} - e^{-a}\right]$$

$$= \frac{(-1)^n}{a+in\pi}\sinh a$$

$$= (-1)^n \sinh a \cdot \frac{(a-in\pi)}{(a+in\pi)(a-in\pi)}$$

$$c_n = \frac{(-1)^n \sinh a\cdot(a-in\pi)}{a^2+n^2\pi^2}$$

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n \sinh a\,(a-in\pi)}{a^2+n^2\pi^2}\, e^{in\pi x}$$

q. Find the complex form of the Fourier Series of the periodic func. $f(x) = \sin x$, $0 < x < \pi$
$f(\pi + x) = f(x)$

Sol. $(0, \pi) \longleftrightarrow (c + c+2l)$

$c = 0$ $\qquad c + 2l = \pi$

$l = \pi/2$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2inx}$$

$$c_n = \frac{1}{2l} \int_c^{c+2l} f(x)\, e^{-2inx}\, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} \sin x \cdot e^{-2inx}\, dx$$

$$= \frac{1}{\pi} \left[\frac{e^{-2inx}}{1-4n^2} (-2in \sin x - \cos x) \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[\frac{e^{-2in\pi}(-(-1))}{1-4n^2} - \frac{1(-1)}{1-4n^2} \right]$$

$$= \frac{1}{\pi(1-4n^2)} \left[e^{-2in\pi} + 1 \right]$$

$$= \frac{1}{\pi(1-4n^2)} \left[\cos 2n\pi - \cancel{i \sin 2n\pi}^{\,0} + 1 \right]$$

q. Obtain the complex form of the Fourier series of func.

$$f(x) = \begin{cases} 0 & : -\pi \le x < 0 \\ 1 & \quad 0 < x \le \pi \end{cases}$$

Sol $$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

$$= \frac{1}{2\pi} \int_0^{\pi} e^{-inx}\, dx$$

$$= \frac{1}{2\pi(+in)} \left[-e^{-inx} \right]_0^{\pi}$$

$$= \frac{1}{2in\pi} \left[-e^{-in\pi} + 1 \right]$$

$$= \frac{1}{2in\pi} \left[-\cos n\pi + \cancel{i \sin n\pi}^{\,0} + 1 \right]$$

$$c_n = \frac{1}{2in\pi}\left[1 - (-1)^n\right] \quad \text{when } n \neq 0$$

$$\cancel{f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{2in\pi}\left[1-(-1)^n\right] e^{inx}}$$

when $n = 0$

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0 + \int_{0}^{\pi} 1\right]$$

$$= \frac{1}{2\pi}[\pi]$$

$$= \frac{1}{2}$$

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$$

$$= \sum_{-\infty}^{-1} \frac{1}{2in\pi}\left[1-(-1)^n\right] e^{inx} + \frac{a_0}{2} + \sum_{1}^{\infty} \frac{1}{2in\pi}\left[1-(-1)^n\right] e^{inx}$$

Σ

24th May, 2023

q. Obtain the complex form of Fourier series for the function $f(x) = \begin{cases} -k & \text{in } -\pi < x < 0 \\ k & \text{in } 0 < x < \pi \end{cases}$

Sol. $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

$$= \frac{1}{2\pi} \left[\int_{-\pi}^{0} -k e^{-inx} + \int_{0}^{\pi} k e^{-inx} \right]$$

$$= \frac{1}{2\pi} \left[\left[\frac{k e^{-inx}}{in} \right]_{-\pi}^{0} + \left[\frac{-k e^{-inx}}{in} \right]_{0}^{\pi} \right]$$

$$= \frac{1}{2\pi} \left[\left[\frac{k}{in} - \frac{k e^{in\pi}}{in} \right] + - \left[\frac{k e^{-in\pi}}{in} - \frac{k}{in} \right] \right]$$

$$= \frac{1}{2\pi} \left[\frac{2k}{in} - \frac{k}{in} \left[e^{in\pi} + e^{-in\pi} \right] \right]$$

$$= \frac{1}{2\pi} \left[\frac{2k}{in} - \frac{k}{in} (2\cos n\pi) \right]$$

$$= \frac{1}{\cancel{2}\pi} \left[\frac{\cancel{2}k}{in} \left[1 - (-1)^n \right] \right]$$

$$= \frac{k}{in\pi} \left(1 - (-1)^n \right)$$

$$= \frac{k}{in\pi} \left(1 - (-1)^n \right)$$

$$= \frac{ki}{-n\pi} \left(1 - (-1)^n \right)$$

$c_n = \frac{ki}{n\pi} \left((-1)^n - 1 \right)$ is true only when $n \neq 0$

$\therefore$ when $n = 0$

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

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$$= \frac{1}{2\pi}\left[\int_{\pi}^{0} -k\,dx + \int_{0}^{\pi} k\,dx\right]$$

$$= 0$$

$$f(x) = \sum_{-\infty}^{-1} \frac{ki}{n\pi}\left((-1)^n - 1\right)e^{inx} + 0 + \sum_{1}^{\infty} \frac{ki}{n\pi}\left((-1)^n - 1\right)e^{inx}$$

Practical Harmonic Analysis

The Fourier series of a periodic function $f(x)$ having period 2π will be of the form

$$f(x) = \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + (a_3 \cos 3x + b_3 \sin 3x) + \dots$$

The term $(a_1 \cos x + b_1 \sin x)$ is called First Harmonic
$(a_2 \cos x + b_2 \sin x)$ is called Second Harmonic
$(a_3 \cos x + b_3 \sin x)$ is called Third Harmonic

Suppose we have a set of N values of $y = f(x)$ having period 2π or $2l$ at equidistant points of x in the interval $(0, 2\pi)$ or $(0, 2l)$ then the Fourier coefficients a_0, a_n and b_n are calculated using the relations given below

Period	Fourier Co-efficients		
2π $(0, 2\pi)$	$a_0 = \frac{2}{N}\sum y$	$a_n = \frac{2}{N}\sum y \cos nx$	$b_n = \frac{2}{N}\sum y \sin nx$
$2l$ $(0, 2l)$	$a_0 = \frac{2}{N}\sum y$	$a_n = \frac{2}{N}\sum y \cos\left(\frac{n\pi x}{l}\right)$	$b_n = \frac{2}{N}\sum y \sin\frac{n\pi x}{l}$

Note : 1. Here $N \to$ no. of observations

2. If the values of y at $x=0$ and $x=2\pi$ are given we must omit one of them ($y_{x=c} = y_{x=c+2\pi}$) by periodic property $f(x) = f(x+2\pi)$

3. The amplitude of the first harmonic is given by $\sqrt{a_1^2+b_1^2}$.
Similarly amplitude of second and third harmonics are given by $\sqrt{a_2^2+b_2^2}$, $\sqrt{a_3^2+b_3^2}$

q. Find the first two harmonics for the function $f(x)$ given by the foll. table

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
$f(x)$	1.0	1.4	1.9	1.7	1.5	1.2	1.0

Sol. Here we should omit either value at $x=0$ or $x=2\pi$
(Note 2) we are omitting 2π
Here the interval of x is $0 \le x < 2\pi$ (not considering 2π)
Length of interval is 2π

$N = 6$

Fourier expansion upto second harmonics is given by

$$f(x) = \frac{a_0}{2} + (a_1\cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) \quad ①$$

$$a_0 = \frac{2}{N}\Sigma y$$

x	$y=f(x)$	$y\cos x$	$y\cos 2x$	$y\sin x$	$y\sin 2x$
0	1		1	0	0
60	1.4	0.7	-0.7	1.2124	1.2124
120	1.9	-0.95	-0.95	1.6454	-1.6454
180	1.7	-1.7	1.7	0	0
240	1.5	-0.75	-0.75	-1.2990	1.2990
300	1.2	0.6	-0.6	-1.0392	-1.0392
Total	8.7	-1.1	-0.3	0.5196	-0.1732

$$a_0 = \frac{2}{6}\Sigma y = \frac{1}{3}(8.7) = 2.9$$

$$a_n = \frac{2}{6}\Sigma y \cos nx \qquad b_n = \frac{2}{6}\Sigma y \sin nx$$

$$a_1 = \frac{1}{3}\Sigma y\cos x \qquad b_1 = \frac{1}{3}\Sigma y \sin x$$

$$= \frac{1}{3}(-1.1) \qquad = \frac{1}{3}(\overset{0.5196}{\cancel{-1.0392}})$$

$$= -0.367 \qquad = 0.1732$$

$$a_2 = \frac{1}{3}\Sigma y\cos 2x \qquad b_2 = \frac{1}{3}\Sigma y\sin 2x$$

$$= \frac{1}{3}(-0.3) \qquad = \frac{1}{3}(-0.1732)$$

$$= -0.1 \qquad = -0.057$$

$$f(x) = \frac{2.9}{2} + (-0.367\cos x + 0.1732\sin x) + (-0.1\cos 2x + -0.057\sin 2x)$$

Q: Find the first two harmonics for the function $f(\theta)$ given by

θ	0	60	120	180	240	300	~~360~~
$f(\theta)$	0.8	0.6	0.4	0.7	0.9	1.1	~~0.8~~

Sol interval is $0 \leq \theta < 2\pi$

length of interval is 2π

$N = 6$

$$f(\theta) = \frac{a_0}{2} + (a_1 \cos\theta + b_1 \sin\theta) + (a_2 \cos 2\theta + b_2 \sin 2\theta)$$

θ	$y = f(\theta)$	$y\cos\theta$	$y\cos 2\theta$	$y\sin\theta$	$y\sin 2\theta$
0	0.8	0.8	0.8	0	0
60	0.6	0.3	-0.3	0.5196	0.5196
120	0.4	-0.2	-0.2	0.3464	-0.3464
180	0.7	-0.7	0.7	0	0
240	0.9	-0.45	-0.45	-0.7794	0.7794
300	1.1	0.5500	-0.5500	-0.9526	-0.9526
Total	4.5	0.3	0	-0.866	0

$$a_0 = \frac{2}{N}\sum y = \frac{2}{6}(4.5) = 1.5$$

$a_1 = \frac{1}{3}\sum y\cos\theta$

$= \frac{1}{3}(0.3)$

$= 0.1$

$b_1 = \frac{1}{3}\sum y\sin\theta$

$= \frac{1}{3}(-0.866)$

$= -0.2887$

$a_2 = \frac{1}{3} \Sigma y \cos 2\theta$ $\qquad$ $b_2 = \frac{1}{3} \Sigma y \sin 2\theta$

$= 0$ $\qquad$ $= 0$

$$f(x) = \frac{1.5}{2} + (0.1)\cos x - (0.2887)\sin x + 0$$

q. Find Fourier series upto the first harmonic

θ	0	30	60	90	120	150	180	210	240	270	300	330
u	298	356	373	337	254	155	80	51	60	93	147	221

Sol. $0 \le \theta < 2\pi$ $\qquad$ interval is 2π

$N = 12$

$$u = f(\theta) = \frac{a_0}{2} + (a_1 \cos\theta + b_1 \sin\theta)$$

(UNNECESSARY)

θ	$y = f(\theta)$	$y\cos\theta$	$y\cos 2\theta$	$y\sin\theta$	$y\sin 2\theta$
0	298	298	298	0	0
30	356	308.3050	178	178	308.3050
60	373	186.5	-186.5	323.0275	323.0275
90	337	0	-337	337	0
120	254	-127		219.9705	
150	155	-134.2339		77.5	
180	80	-80		0	
210	51	-44.1673		-25.5	
240	60	-30		-51.9615	
270	93	0		93	
300	147	73.5		-127.3057	
330	221	191.3916		-110.5	
Total	2425	642.2954		727.2308	

$$a_0 = \frac{2}{12} \times 2425$$

$$a_0 = 404.1667$$

$$a_1 = \frac{2}{12} \times 642.2945$$

$$a_1 = 107.0492$$

$$b_1 = \frac{2}{12} \times 727.2308$$

$$b_1 = 121.2051$$

$$f(x) = \frac{404.1667}{2} + (107.0492 \cos\theta + 121.2051 \sin\theta)$$

q. Obtain the constant term and the co-efficients of first cosine and sine terms in the Fourier expansion of y from the table.

x	0	1	2	3	4	5
y	9	18	24	28	26	20

Sol interval $0 \le x < 6$

$(0, 6) \longleftrightarrow (0, 2l)$

$l = 3$

$$f(x) = \frac{a_0}{2} + a_1 \cos\frac{\pi x}{l} + b_1 \sin\frac{\pi x}{l}$$

$$= \frac{a_0}{2} + a_1 \cos\frac{\pi x}{3} + b_1 \sin\frac{\pi x}{3} \quad (1)$$

$$\theta = \frac{\pi x}{3}$$

x	$\theta = \frac{\pi x}{3}$	y	$y\cos\theta$	$y\sin\theta$
0	0	9	9	0
1	$\pi/3$	18	9	15.5885
2	$2\pi/3$	24	-12	20.7846
3	π	28	-28	0
4	$4\pi/3$	26	-13	-22.5167
5	$5\pi/3$	20	10	-17.3205
Total :		125	-25	-3.4641

$$a_0 = \frac{1}{3}\Sigma y = \frac{1}{3}(125) = 41.67$$

$$a_1 = \frac{1}{3}\Sigma y\cos\theta = \frac{1}{3}(-25) = -8.33$$

$$b_1 = \frac{1}{3}\Sigma y\sin\theta = \frac{1}{3}(-3.4641) = -1.1547$$

25th May, 2023

Q. The foll. table gives the variations of the periodic current A over a period T

t (sec)	0	T/6	T/3	T/2	2T/3	5T/6	T
A (amps)	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

Show that there is a constant part of 0.75 amp in the current A and obtain

Sol. $t \in 0 \le t < T$

length of interval is T

$$(0, T) \leftrightarrow (0, 2l)$$

$$l = T/2$$

we observe that the values of A at $t = 0$ and $t = T$ are same. Hence we omit the last value.

$N = 6$

Fourier series upto first harmonics is given by:

$$A = f(t) = \frac{a_0}{2} + \left(a_1 \cos \frac{\pi t}{l} + b_1 \sin \frac{\pi t}{l} \right)$$

$$= \frac{a_0}{2} + a_1 \cos \frac{2\pi t}{T} + b_1 \sin \frac{2\pi t}{T}$$

$$f(t) = \frac{a_0}{2} + a_1 \cos\theta + b_1 \sin\theta \quad -(1)$$

where $\theta = \frac{2\pi t}{T}$

t	$y = A$	$\theta = \frac{2\pi t}{T}$	$y\cos\theta$	$y\sin\theta$
0	1.98	0	1.98	0
$T/6$	1.30	60	0.65	1.1258
$T/3$	1.05	120	−0.525	0.9093
$T/2$	1.30	180	−1.3	0
$2T/3$	−0.88	240	0.44	0.7621
$5T/6$	−0.25	300	−0.1250	0.2165
Total	4.5		1.12	3.0137

$$a_0 = \frac{2\sum y}{N}$$

$$= \frac{1}{3}(4.5) = 1.5$$

$$a_1 = \frac{2}{N}\sum y\cos\theta$$

$$= \frac{1}{3}(1.12) = 0.3733$$

$$b_1 = \frac{2}{N}\sum y\sin\theta$$

$$= \frac{1}{3}(3.0137) = 1.0046$$

$\therefore$ The constant part is $\frac{a_0}{2} = 0.75$

amplitude of the first harmonic $= \sqrt{a_1^2 + b_1^2}$

$= \sqrt{(0.3733)^2 + (1.0046)^2}$

$= 1.0717$

q.1. The following values of y and x are given. Find the Fourier series upto second harmonics.

x	0	2	4	6	8	10	12
y	9	18.2	24.4	27.8	27.5	22.0	9.0

q.2. Obtain the constant term and the co-efficients of $\sin\theta$ and $\sin 2\theta$ in the Fourier expansion of y given foll. data

θ	0	60	120	180	240	300	360
y	0	9.2	14.4	17.8	17.3	11.7	0

Sol 1. $0 \leq x < 12$

$(0, 12) \longleftrightarrow (0, 2l)$

$l = 6$

$$f(x) = \frac{a_0}{2} + \left(a_1 \cos\frac{\pi x}{l} + b_1 \sin\frac{\pi x}{l}\right) + \left(a_2 \cos\frac{2\pi x}{l} + b_2 \sin\frac{2\pi x}{l}\right)$$

$$f(x) = \frac{a_0}{2} + \left(a_1 \cos\theta + b_1 \sin\theta\right) + \left(a_2 \cos 2\theta + b_2 \sin 2\theta\right)$$

where $\theta = \frac{\pi x}{6}$

x	y	$\theta = \frac{\pi x}{6}$	$y\cos\theta$	$y\sin\theta$	$y\cos 2\theta$	$y\sin 2\theta$
0	9	0	9	0	9	0
2	18.2	60	9.1	15.7617	-9.1	15.7617
4	24.4	120	-12.2	21.1310	-12.2	-21.1310
6	27.8	180	-27.8	0	27.8	0
8	27.5	240	-13.75	-23.8157	-13.75	23.8157
10	22	300	11	-19.0526	-11	-19.0526
	128.9		-24.65	-5.9456	-9.25	-0.6062

$$a_0 = \frac{1}{3}(128.9) = 42.9667$$

$$a_1 = \frac{1}{3}(-24.65) = -8.2167 \qquad b_1 = \frac{1}{3}(-5.9456) = -1.9819$$

$$a_2 = \frac{1}{3}(-9.25) = 3.0833 \qquad b_2 = \frac{1}{3}(-0.6062) = -0.2021$$

$$f(x) = 21.4834 + -8.2167\cos\left(\frac{\pi x}{6}\right) - 1.9819\sin\left(\frac{\pi x}{6}\right)$$
$$+ 3.0833\cos\left(\frac{2\pi x}{6}\right) - 0.2021\sin\left(\frac{2\pi x}{6}\right)$$

Sol 2.

θ	y	$y\sin\theta$	$y\sin 2\theta$
0	0	0	0
60	9.2	7.9674	7.9674
120	14.4	12.4708	-12.4708
180	17.8	0	0
240	17.3	-14.9822	14.9822
300	11.7	-10.1325	-10.1325
	70.4	-4.6765	0.3463

$$a_0 = \frac{1}{3}(70.4) = 23.4667$$

$$a_1 = \frac{1}{3}(-4.6765) = -1.5588$$

$$a_2 = \frac{1}{3}(0.3463) = 0.1154$$