26th May, 2023

UNIT-5

Fourier Transforms

Definition: FT Let $f(t)$ be piecewise continuous on $(-\infty, \infty)$. Assume that $f(t)$ is absolutely convergent i.e $\int_{-\infty}^{\infty} |f(t)|\, dt$ converges. Then the

Fourier transform of $f(t)$ denoted by $F[f(t)]$ is defined as

$$F[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt = F(\omega)$$

$\omega \rightarrow$ freq. spectrum

IFT : Assume that $\int_{-\infty}^{\infty} [F(\omega)]\, d\omega$ converges. Then we define as

$$f(t) = F^{-1}[F(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, d\omega$$

Q: Find the F.T of the func. $f(t) = \begin{cases} 1, & |t| \le 1 \\ 0, & |t| > 1 \end{cases}$

and hence evaluate $\int_{0}^{\infty} \frac{\sin \omega}{\omega}\, d\omega$

Sol. By definition of Fourier Transform

$$F[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$|t| \le 1 \Rightarrow -1 \le t \le 1$

$|t| > 1 \Rightarrow t < -1 \;\&\; t > 1$

$$= \cancelto{0}{\int_{-\infty}^{-1} 0\, e^{-i\omega t}\, dt} + \int_{-1}^{1} e^{-i\omega t}\, dt + \cancelto{0}{\int_{1}^{\infty} 0 \cdot e^{-i\omega t}\, dt}$$

$$= \left[\frac{e^{-i\omega t}}{-i\omega}\right]_{-1}^{1}$$

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$$= \frac{-1}{i\omega}\left[e^{-i\omega} - e^{i\omega}\right]$$

$$= \frac{1}{i\omega}\left[e^{i\omega} - e^{-i\omega}\right]$$

$$\frac{e^{ix} - e^{-ix}}{2i} = \sin x$$

$$= \frac{1}{\cancel{i}\omega}\left[2\cancel{i}\sin\omega\right]$$

$$F[f(t)] = \frac{2\sin\omega}{\omega} = F(\omega)$$

use IFT for deductions

Deduction :

By Inverse Fourier Transform,

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, d\omega$$

$$f(t) = \frac{1}{\cancel{2}\pi}\int_{-\infty}^{\infty} \frac{\cancel{2}\sin\omega}{\omega}\, e^{i\omega t}\, d\omega$$

put $t = 0$

$\therefore f(0) = 1 \Rightarrow 1 = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin\omega}{\omega}\cdot 1\, d\omega$

$\sin\omega \times \frac{1}{\omega}$ = odd func. x odd func.

= even func.

$$\Rightarrow 1 = \frac{2}{\pi}\int_{0}^{\infty} \frac{\sin\omega}{\omega}\, d\omega$$

$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

if $f(x)$ is even

$$\Rightarrow \int_{0}^{\infty} \frac{\sin\omega}{\omega}\, d\omega = \frac{\pi}{2}$$

$-1 \le t \le 1$

q. Find the Fourier Transform of $f(t) = \begin{cases} 1 - t^2, & |t| \le 1 \\ 0, & |t| > 1 \end{cases}$

Hence evaluate ii) $\int_{0}^{\infty}\left[\frac{\sin\omega - \omega\cos\omega}{\omega^3}\right]\cos\frac{\omega}{2}\, d\omega$

i) $\int_{0}^{\infty} \frac{\sin\omega - \omega\cos\omega}{\omega^3}$

Sol. By definition ~~of~~ F.T

$$F[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$= \int_{-1}^{1} (1-t^2)\, e^{-i\omega t}\, dt$$

$$= \left[\cancel{(1-t^2)\frac{e^{-i\omega t}}{-i\omega}}^{\to 0} - (-2t)\frac{e^{-i\omega t}}{(-i\omega)^2} + (-2)\frac{e^{-i\omega t}}{(-i\omega)^3} \right]_{-1}^{1}$$

$$= \left[+\frac{2e^{-i\omega}}{-\omega^2} - \frac{2e^{-i\omega}}{i\omega^3} - \left(\frac{-2e^{i\omega}}{-\omega^2} - \frac{2e^{i\omega}}{i\omega^3} \right) \right]$$

$$= \frac{2}{\omega^2}\left[-e^{-i\omega} - e^{i\omega} \right] + \frac{2}{i\omega^3}\left[e^{i\omega} - e^{-i\omega} \right]$$

$$= \frac{-2}{\omega^2}\left(e^{i\omega} + e^{-i\omega} \right) + \frac{2}{i\omega^3}\left(e^{i\omega} - e^{-i\omega} \right)$$

$$= \frac{-2}{\omega^2}(2\cos\omega) + \frac{2}{i\omega^3}(2i\sin\omega)$$

$$= \frac{-4\cos\omega}{\omega^2} + \frac{4\sin\omega}{\omega^3}$$

$$F(\omega) = 4\left[\frac{\sin\omega - \omega\cos\omega}{\omega^3} \right]$$

Deduction

By IFT

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{-i\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} 4\left(\frac{\sin\omega - \omega\cos\omega}{\omega^3} \right) e^{i\omega t}\, d\omega$$

put $t=0$

wkt $f(t) = 1-t^2$

$f(0) = 1-0 = 1$

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$$\Rightarrow 1 = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\sin w - w\cos w}{w^3}\, dw$$

$$\frac{\sin w \; - \text{odd}}{w^3 \; - \text{odd}} \Rightarrow \text{even}$$

$$-\frac{\cos w \; - \text{even}}{w^2 \; - \text{even}} \Rightarrow \text{even}$$

$$\Rightarrow \frac{\pi}{2} = 2\int_0^{\infty} \frac{\sin w - w\cos w}{w^3}$$

$$\int_0^{\infty} \frac{\sin w - w\cos w}{w^3} = \frac{\pi}{4}$$

put $t = 1/2$

$$f(1/2) = 1 - \left(\frac{1}{2}\right)^2 = \frac{3}{4}$$

$$\Rightarrow \frac{3}{4} = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\sin w - w\cos w}{w^3}\, e^{+\frac{iw}{2}}$$

$$\frac{3\pi}{8} = \int_{-\infty}^{\infty} \frac{\sin w - w\cos w}{w^3}\left(\cos\frac{w}{2} + i\sin\frac{w}{2}\right) dw$$

equating real parts

$$\frac{3\pi}{8} = \int_{-\infty}^{\infty} \frac{\sin w - w\cos w}{w^3} \cdot \cos\frac{w}{2}\; dw$$

$$\frac{3\pi}{8} = 2\int_0^{\infty} \left(\frac{\sin w - w\cos w}{w^3}\right) \cdot \cos\frac{w}{2}$$

$$\therefore \int_0^{\infty} \left(\frac{\sin w - w\cos w}{w^3}\right) \cdot \cos\frac{w}{2} = \frac{3\pi}{16}$$

31st May, 2023

q: Find the Fourier Transform for the function

$$f(t) = \begin{cases} 1-|t| , & |t| \le 1 \quad -1 \le t \le 1 \\ 0 , & |t| > 1 \end{cases}$$

and hence evaluate $\int_0^\infty \frac{\sin^2 t}{t}\, dt$

Sol: By definition of F.T

$$F[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$= \int_{-1}^{1} (1-|t|) e^{-i\omega t}\, dt$$

$$= \int_{-1}^{1} \underbrace{(1-|t|)}_{\text{even}} (\underbrace{\cos \omega t}_{\text{even}} - i \underbrace{\sin \omega t}_{\text{odd}})\, dt$$

$$= 2\int_0^1 (1-|t|) \cos \omega t\, dt - 0$$

$\int_{-a}^{a} f(x)\, dx = 0$ if $f(x)$ odd

$$= 2\int_0^1 (1-t) \cos \omega t\, dt$$

$$= 2\left[(1-t) \cancel{\frac{\sin \omega t}{\omega}} - (-1)\left(-\frac{\cos \omega t}{\omega^2}\right) \right]_0^1$$

$$= 2\left[-\frac{\cos \omega}{\omega^2} + \frac{1}{\omega^2} \right]$$

$$= \frac{+2}{\omega^2} (1-\cos\omega)$$

$$= \frac{2}{\omega^2} \cdot 2\sin^2\left(\frac{\omega}{2}\right)$$

$$\therefore F(\omega) = \frac{4\sin^2\left(\frac{\omega}{2}\right)}{\omega^2}$$

Deduction

By IFT

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, d\omega$$

put $t=0$

$f(t) = 1-0 = 1$

$$\Rightarrow 1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4\sin^2\left(\frac{\omega}{2}\right)}{\omega^2}\, d\omega$$

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$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4\sin^2\left(\frac{\omega}{2}\right)}{\omega^2} e^{i\omega t}\, d\omega$$

put $\frac{\omega}{2} = t$

$\omega = 2x$

$d\omega = 2dx$

when $\omega = -\infty \quad x = -\infty$

$\omega = \infty \quad x = \infty$

$$1 = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2\left(\frac{\omega}{2}\right)}{\omega^2}\, d\omega$$

$$\frac{\pi}{2} = \int_{-\infty}^{\infty} \frac{\sin^2\left(\frac{\omega}{2}\right)}{\omega^2}\, d\omega$$

put $\frac{\omega}{2} = x$

$\omega = 2x$

$d\omega = 2dx$

when $\omega = -\infty$; $x = -\infty$

$\omega = \infty$; $x = \infty$

$$\Rightarrow \frac{\pi}{\cancel{4}} = \int_{-\infty}^{\infty} \frac{\sin^2 x}{\cancel{4}\, x^2} \cdot \cancel{2}\, dx$$

$$\pi = \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\, dx$$

$$\Rightarrow \pi = 2\int_{0}^{\infty} \frac{\sin^2 t}{t^2}\, dt$$

$$\int_{0}^{\infty} \frac{\sin^2 t}{t^2}\, dt = \frac{\pi}{2}$$

Q: Find the Fourier Transform of ~~$f(x) = e^{-a|t-2|}$~~

$f(x) = e^{-a|t|}$

Sol. $f(t) = \begin{cases} e^{-at} & t > 0 \\ e^{at} & t < 0 \end{cases}$ q) $f(x) = e^{-a|t|}$

By definition of F.T

$$F[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{at}\, e^{-i\omega t}\, dt + \int_{0}^{\infty} e^{-at}\, e^{-i\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{(a-i\omega)t}\, dt + \int_{0}^{\infty} e^{-(a+i\omega)t}\, dt$$

$$= \left[\frac{e^{(a-i\omega)t}}{a-i\omega}\right]_{-\infty}^{0} + \left[\frac{e^{-(a+i\omega)t}}{-(a+i\omega)}\right]_{0}^{\infty}$$

$$= \left(\frac{1}{a-i\omega} - 0\right) + \left(0 - \frac{1}{-(a+i\omega)}\right)$$

$$= \frac{1}{a-i\omega} + \frac{1}{a+i\omega} = \frac{a+i\omega + a - i\omega}{a^2+\omega^2}$$

$$F(\omega) = \frac{2a}{a^2+\omega^2}$$

q: Find $F[H(t)\, e^{-2t}]$

$H(t)$ = unit step func

$$H(t) = \begin{cases} 0 & t<0 \\ 1 & t>0 \end{cases}$$

By definition

$$F[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} H(t)\, e^{-2t}\, e^{-i\omega t}\, dt$$

$$= \int_{-\infty}^{0} 0 \cdot dt + \int_{0}^{\infty} 1 \cdot e^{-2t}\, e^{-i\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-(i\omega+2)t}\, dt$$

$$= \left[\frac{e^{-(i\omega+2)t}}{-(i\omega+2)}\right]_0^{\infty}$$

$$= \frac{1}{2+i\omega}$$

$$= \frac{2-i\omega}{4+\omega^2}$$

Properties of Fourier Transforms

1. Linearity of F.T

$$F[af(t) + bg(t)] = aF[f(t)] + bF[g(t)]$$

provided F.T of $f(t)$ and $g(t)$ exist.

2. Shifting on t-axis

If $F[f(t)] = F(\omega)$ and t_0 is any real number
then $F[f(t-t_0)] = F(\omega)\, e^{-i\omega t_0}$

eg. Find F.T of $f(t) = e^{-a|t-2|}$

$$F\left[e^{-a|t-2|}\right] = F(\omega)\, e^{-i\omega 2}$$

where $F(\omega) = F[f(t)]$
$= F\left[e^{-a|t|}\right]$

$$= \frac{2a}{a^2+\omega^2}\, e^{-i\omega 2}$$

3. Frequency Shifting

If $F[f(t)] = F(\omega)$ and ω_0 is any real no.
then

$$F\left[e^{i\omega_0 t} f(t)\right] = F(\omega - \omega_0)$$

4. Modulation Theorem

If $F[f(t)] = F(\omega)$ and ω_0 is any real no.
then

$$F\left[f(t)\cos(\omega_0 t)\right] = \frac{1}{2}\left[F(\omega+\omega_0) + F(\omega-\omega_0)\right]$$

$$F\left[f(t)\sin(\omega_0 t)\right] = \frac{i}{2}\left[F(\omega+\omega_0) - F(\omega-\omega_0)\right]$$

→ Fourier Transform of derivatives

Let $f(t)$ be continuous and $f^{(k)}(t)$,
$k = 1, 2, \ldots n$

be piecewise continuous on every interval $[-l, l]$
and $\int_{-\infty}^{\infty} \left|f^{(k-1)}(t)\right| dt$, $k = 1, 2, \ldots n$ converges.

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Let $f^{(t)}(t) \longrightarrow 0$ as $t \to \infty$ for $k = 0, 1, 2 \ldots n-1$

* If $F[f(t)] = F(\omega)$, then $F[f^{(n)}(t)] = (i\omega)^n F(\omega)$

where $f(t)$ and all its derivatives vanish at infinity

1st June 2023

Q. Find the soln. of the differential eqn.

$$y' - 2y = H(t)e^{-2t}, \quad -\infty < t < \infty$$ using Fourier Transfo

Sol. $y'(t) - 2y(t) = H(t)e^{-2t}$

$F[H(t)e^{-at}] = \frac{1}{a+i\omega}$

$$F[y'(t)] - 2F[y(t)] = F[H(t)e^{-2t}]$$

$$i\omega F[y(t)] - 2F[y(t)] = \frac{1}{2+i\omega}$$

$$F[y(t)](i\omega - 2) = \frac{1}{2+i\omega}$$

$$F[y(t)] = \frac{-1}{(2+i\omega)(2-i\omega)}$$

$$F[y(t)] = \frac{-1}{4+\omega^2}$$

$$y(t) = -F^{-1}\left[\frac{1}{4+\omega^2}\right] \quad -①$$

wkt

$$F[e^{-a|t|}] = \frac{2a}{a^2+\omega^2}$$

$$F[e^{-2|t|}] = \frac{4}{4+\omega^2}$$

$$F^{-1}\left[\frac{1}{4+\omega^2}\right] = \frac{1}{4}e^{-2|t|}$$

$\textcircled{1} \rightarrow \quad y(t) = \dfrac{e^{-2|t|}}{4}$

$$y(t) = \begin{cases} -e^{-2t}/4 & t > 0 \\ -e^{2t}/4 & t < 0 \end{cases}$$

6: Symmetric Property of Fourier Transform

If $F[f(t)] = F(\omega)$ then $F[F(t)] = 2\pi f(-\omega)$

eg: Find F.T of $f(t) = \dfrac{1}{5+it}$

Sol. wkt

$$F[H(t)e^{-at}] = \frac{1}{a+i\omega}$$

$F[f(t)] = F(\omega)$

$F[b(t)] = b(\omega)$

$$F[H(t)e^{-5t}] = \frac{1}{5+i\omega}$$

$$F[b(t)] = F[H(t)e^{-5t}] = \frac{1}{5+i\omega} = B(\omega)$$

$b(t) = H(t)e^{-5t}$

$b(-\omega) = H(-\omega)e^{5\omega}$

Using the symmetric property

$$F[B(t)] = F\left[\frac{1}{5+it}\right] = 2\pi b(-\omega)$$

$$\therefore F\left[\frac{1}{5+it}\right] = 2\pi H(-\omega)e^{5\omega}$$

$$= \begin{cases} 2\pi e^{5\omega} & \omega < 0 \\ 0 & \omega > 0 \end{cases}$$

$$H(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

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q. Find the Fourier Transform of $e^{-a^2t^2}$, $a>0$. Hence deduce that $e^{-t^2/2}$ is self reciprocated in respect of the Fourier Transform.

Sol. By definition of Fourier Transform

$$F[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$F[e^{-a^2t^2}] = \int_{-\infty}^{\infty} e^{-a^2t^2}\, e^{-i\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-a^2\left(t^2 + \frac{i\omega t}{a^2}\right)}\, dt$$

$$= \quad a^2 + 2ab + b^2 - b^2$$

$$t^2 + 2t\cdot\frac{i\omega}{a^2(2)} + \left(\frac{i\omega}{2a^2}\right)^2 - \left(\frac{i\omega}{2a^2}\right)^2$$

$$= \int_{-\infty}^{\infty} e^{-a^2\left(t^2 + 2t\cdot\frac{i\omega}{2a^2} + \left(\frac{i\omega}{2a^2}\right)^2 - \left(\frac{i\omega}{2a^2}\right)^2\right)}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-a^2\left[\left(t + \frac{i\omega}{2a^2}\right)^2 + \frac{\omega^2}{4a^4}\right]}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-a^2\left(t + \frac{i\omega}{2a^2}\right)^2} \cdot e^{\frac{-\omega^2}{4a^2}}\, dt$$

$$= e^{\frac{-\omega^2}{4a^2}} \int_{-\infty}^{\infty} e^{-a^2\left(t + \frac{i\omega}{2a^2}\right)^2}\, dt$$

put $a\left(t + \frac{i\omega}{2a^2}\right) = x$

$a\,dt = dx$

$$F[e^{-a^2t^2}] = e^{\frac{-\omega^2}{4a^2}} \int_{-\infty}^{\infty} e^{-x^2}\, \frac{dx}{a}$$

$$= \frac{e^{-\omega^2/4a^2}}{a} 2\int_0^\infty e^{-x^2} dx$$

$$= \frac{2e^{-\omega^2/4a^2}}{a} \cdot \frac{\sqrt{\pi}}{2}$$

$$F[e^{-a^2t^2}] = \frac{\sqrt{\pi}}{a} e^{-\omega^2/4a^2}$$

put $a = \frac{1}{\sqrt{2}}$

$$F[e^{-t^2/2}] = \frac{\sqrt{\pi}}{\frac{1}{\sqrt{2}}} e^{-\omega^2/2}$$

$$F[e^{-t^2/2}] = \sqrt{2\pi}\ e^{-\omega^2/2}$$

2nd June, 2023

Fourier Sine Transform

The Fourier Sine Transform of $f(t)$ is defined as :-

$$F_s[f(t)] = \int_0^\infty f(t) \sin \omega t \ dt$$

Inverse Fourier Sine Transform

$$f(t) = \frac{2}{\pi} \int_0^\infty F_s(\omega) \sin \omega t \ d\omega$$

Fourier Cosine Transform

$$F_c[f(t)] = \int_0^\infty f(t) \cos \omega t \ dt$$

Inverse Fourier Cosine Transform

$$f(t) = \frac{2}{\pi} \int_0^\infty F_c(\omega) \cos \omega t \ d\omega$$

Q. Find the Fourier Sine Transform of : $e^{-|t|}$ and hence evaluate $\int_0^\infty \frac{\omega \sin m\omega}{1+\omega^2} d\omega$

Sol. $F_s[f(t)] = \int_0^\infty f(t) \sin \omega t \, dt$

$$= \int_0^\infty e^{-|t|} \sin \omega t \, dt$$

$$= \int_0^\infty e^{-t} \sin \omega t \, dt$$

$$= \left[\frac{e^{-t}}{1+\omega^2} (-\sin \omega t - \omega \cos \omega t) \right]_0^\infty$$

$$= 0 - \left(\frac{1}{4\omega^2} (0 - \omega) \right)$$

$$\therefore F_s(\omega) = \frac{\omega}{1+\omega^2}$$

Deduction,

By Inverse Fourier sine Transform

$$f(t) = \frac{2}{\pi} \int_0^\infty F_s(\omega) \sin \omega t \, d\omega$$

$$e^{-t} = \frac{2}{\pi} \int_0^\infty \frac{\omega}{1+\omega^2} \sin \omega t \, d\omega$$

Replace t by m

$$\int_0^\infty \frac{\omega}{1+\omega^2} \sin m\omega \, d\omega = e^{-m} \cdot \frac{\pi}{2}$$

Q. Find the Fourier sine transform of $\frac{e^{-at}}{t}$, $t \neq 0, a>0$

Sol. $F_s[f(t)] = \int_0^{\infty} f(t) \sin wt \, dt$

$$F_s(w) = \int_0^{\infty} \underbrace{\frac{e^{-at}}{t} \sin wt}_{\text{complex integral}} \, dt$$

Leibniz Rule of Integral

Differentiation under integral

* differentiate on both sides w.r.t w

$$\frac{d}{dw} F_s(w) = \int_0^{\infty} \frac{e^{-at}}{\cancel{t}} (\cancel{t} \cos wt) \, dt$$

$$= \int_0^{\infty} e^{-at} \cos wt \, dt$$

$$= \left[\frac{e^{-at}}{a^2 + w^2} (-a \cos wt + w \sin wt) \right]_0^{\infty}$$

$$= \left[0 - \left(\frac{1}{a^2 + w^2} (-a) \right) \right]$$

$$\frac{d}{dw} F_s(w) = \frac{a}{a^2 + w^2}$$

* integrate on both sides w.r.t w

$$F_s(w) = \int \frac{a}{a^2 + w^2} \, dw$$

$$= a \cdot \frac{1}{a} \tan^{-1} \frac{w}{a}$$

$$F_s(w) = \tan^{-1}\left(\frac{w}{a}\right) + c$$

Q: Find the Fourier sine transform of e^{-at} and hence

S.T $\int_0^{\infty} \frac{w \sin kw}{a^2 + w^2} \, dw = \frac{\pi}{2} e^{-ak}$

Sol. $F_s[f(t)] = \int_0^{\infty} e^{-at} \sin \omega t \, dt$

$$= \left[\frac{e^{-at}}{a^2+\omega^2} (-a \sin \omega t - \omega \cos \omega t) \right]_0^{\infty}$$

$$= 0 - \left(\frac{1}{a^2+\omega^2} (0 - \omega) \right)$$

$$\therefore F_s(\omega) = \frac{\omega}{a^2+\omega^2}$$

Deduction,

By Inverse Fourier sine Transform

$$f(t) = \frac{2}{\pi} \int_0^{\infty} F_s(\omega) \sin \omega t \, d\omega$$

$$e^{-at} \cdot \frac{\pi}{2} = \int_0^{\infty} \frac{\omega}{a^2+\omega^2} \sin \omega t \, d\omega$$

replace t by k

$$\int_0^{\infty} \frac{\omega}{a^2+\omega^2} \sin k\omega \, d\omega = \frac{\pi}{2} e^{-ak}$$

q. Find the Fourier cosine Transform of $\frac{1}{1+t^2}$

Sol. $F_c[f(t)] = \int_0^{\infty} f(t) \cos \omega t \, dt$

$$I = \int_0^{\infty} \frac{1}{1+t^2} \cos \omega t \, dt \quad -①$$

differentiate on both sides w.r.t ω

$$\frac{dI}{d\omega} = \int_0^{\infty} \frac{1}{1+t^2} (-\sin \omega t \cdot t) \, dt$$

$$= -\int_0^{\infty} \frac{t}{1+t^2} \sin \omega t \, dt$$

$\times$ & $\div$ by t $\quad = -\int_0^{\infty} \frac{t^2}{t(1+t^2)} \sin \omega t \, dt$

add & sub 1 $= -\int_0^\infty \frac{(1+t^2)-1}{t(1+t^2)} \sin \omega t \; dt$

$= -\int_0^\infty \frac{\sin \omega t}{t} dt + \int_0^\infty \frac{\sin \omega t}{t(1+t^2)} dt$

$$\frac{dI}{dt} = -\frac{\pi}{2} + \int_0^\infty \frac{\sin \omega t}{t(1+t^2)} dt \quad -(2)$$

- again differentiating w.r.t ω

$$\frac{d^2 I}{d\omega^2} = 0 + \int_0^\infty \frac{1}{\cancel{t}(1+t^2)} \cos \omega t \cdot \cancel{t} \; dt$$

$$\frac{d^2 I}{d\omega^2} = I$$

$(D^2 - 1) I = 0$ where $D = \frac{d}{d\omega}$

$m^2 - 1 = 0$

$m = \pm 1$

$$I = c_1 e^{\omega} + c_2 e^{-\omega} \quad -(3)$$

- to find c_1 & c_2

put $\omega = 0$ in (1) & (3)

$\int_0^\infty \frac{1}{1+t^2} dt = c_1 + c_2$

$(\tan^{-1} t)_0^\infty = c_1 + c_2$

$$c_1 + c_2 = \frac{\pi}{2} \quad -(4)$$

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- differentiate ③ w.r.t ω

$$\frac{dI}{d\omega} = C_1 e^{\omega} - C_2 e^{-\omega} \quad -⑤$$

put $\omega = 0$ in ② & ⑤

$$\frac{-\pi}{2} + 0 = C_1 - C_2 \quad -⑥$$

- solving ④ & ⑥

$$C_1 = 0 \; ; \; C_2 = \frac{\pi}{2}$$

∴ ③ →

$$I = \frac{\pi}{2} e^{-\omega}$$

$$\therefore F_c[f(t)] = F_c(\omega) = \frac{\pi}{2} e^{-\omega}$$

q. Find the Fourier cosine Transform of e^{-t^2}

Sol. $$F_c[f(t)] = \int_0^{\infty} f(t) \cos \omega t \, dt$$

$$I = \int_0^{\infty} e^{-t^2} \cos \omega t \, dt \quad -①$$

- differentiate on both sides wrt ω

$$\frac{dI}{d\omega} = \int_0^{\infty} e^{-t^2} (-\sin \omega t \cdot t) \, dt$$

$$= \int_0^{\infty} \sin \omega t \, (-t e^{-t^2} dt)$$

$$= \int_0^{\infty} \underbrace{\sin \omega t}_{u} \; \underbrace{d\left(\frac{e^{-t^2}}{2}\right)}_{dv}$$

$$\int uv = u \int v - \int u' \int v$$

* $\int u\,dv = uv - \int v\,du$

$u = \sin \omega t \quad ; \quad dv = d\left(\frac{e^{-t^2}}{2}\right)$

$du = \omega \cos \omega t \quad ; \quad v = \frac{e^{-t^2}}{2}$

$$= \left[\sin \omega t \left(\frac{e^{-t^2}}{2}\right)\right]_0^\infty - \int_0^\infty \frac{e^{-t^2}}{2} \cdot \omega \cos \omega t \, dt$$

$$\frac{dI}{d\omega} = 0 - \frac{\omega}{2} \int_0^\infty e^{-t^2} \cos \omega t \, dt$$

$$\frac{dI}{d\omega} = -\frac{\omega}{2}(I)$$

$$\frac{dI}{I} = -\frac{\omega}{2} d\omega$$

On integrating

$$\ln I = -\frac{\omega^2}{4} + c$$

$$I = e^{-\omega^2/4 + c}$$

$$I = k e^{-\omega^2/4} \quad \text{where } k = e^c \quad -②$$

To find k

From ① and ②

$$\int_0^\infty e^{-t^2} \cos \omega t \, dt = k e^{-\omega^2/4}$$

put $\omega = 0$

$$\int_0^\infty e^{-t^2} dt = k$$

$$k = \frac{\sqrt{\pi}}{2}$$

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② $\Rightarrow I = \frac{\sqrt{\pi}}{2} e^{-\omega^2/4}$

$\therefore F_c(\omega) = \frac{\sqrt{\pi}}{2} e^{-\omega^2/4}$

5th June, 2023

q. Find the Fourier cosine transforms of

$$f(t) = \begin{cases} t & 0 < t < 1 \\ 2-t & 1 < t < 2 \\ 0 & t > 2 \end{cases}$$

Sol. $F_c[f(t)] = \int_0^\infty f(t) \cos \omega t \, dt$

$$= \int_0^1 t \cos \omega t \, dt + \int_1^2 (2-t) \cos \omega t \, dt + \cancel{\int_2^\infty 0 \cdot \cos \omega t \, dt}$$

$$= \left[t \frac{\sin \omega t}{\omega} - \left(-\frac{\cos \omega t}{\omega^2} \right) \right]_0^1 + \left[(2-t) \frac{\sin \omega t}{\omega} - (-1) \left(-\frac{\cos \omega t}{\omega^2} \right) \right]_1^2$$

$$= \frac{\sin \omega}{\omega} + \frac{\cos \omega}{\omega^2} - \frac{1}{\omega^2} - \frac{\cos 2\omega}{\omega^2} - \left(\frac{\sin \omega}{\omega} - \frac{\cos \omega}{\omega^2} \right)$$

$$= \frac{2 \cos \omega}{\omega^2} - \frac{1}{\omega^2} - \frac{\cos 2\omega}{\omega^2}$$

$$F_c(\omega) = \frac{2 \cos \omega - 1 - \cos 2\omega}{\omega^2}$$

*q. Find the Fourier sine transform of

$$f(t) = \begin{cases} \sin t & 0 < t < a \\ 0 & t > a \end{cases}$$

for $\omega = 1$

$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

Sol. $F_s[f(t)] = \int_0^\infty f(t) \sin \omega t \, dt$

$$= \int_0^a \sin t \sin \omega t \, dt + \int_a^\infty \cancel{0 \sin \omega t} \, dt$$

$$= \frac{1}{2}\int_0^a \left[\cos(\omega-1)t - \cos(\omega+1)t\right] dt$$

$$F_s(\omega) = \frac{1}{2}\left[\frac{\sin(\omega-1)t}{\omega-1} - \frac{\sin(\omega+1)t}{\omega+1}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{\sin(\omega-1)a}{\omega-1} - \frac{\sin(\omega+1)a}{\omega+1}\right]$$

put $\omega = 1$

$$F_s(\omega) = \frac{1}{2}\left[\frac{0}{0} - \frac{\sin 2a}{2}\right]$$

$\frac{0}{0}$ indeterminant form

LH rule

$$\lim_{\omega \to 1} \frac{\sin((\omega-1)a)}{\omega-1}$$

$$\lim_{\omega \to 1} \frac{\cos(\omega-1)a \cdot a}{1} = a$$

$$F_s(\omega) = \frac{1}{2}\left[a - \frac{\sin 2a}{2}\right]$$

Change of Scale Property

If $F[f(t)] = F(\omega)$, then $F[f(at)] = \frac{1}{a} F\left(\frac{\omega}{a}\right)$

q. Find the F.T of

i) $e^{-2(t-3)^2}$

$F[e^{-a^2t^2}] = e^{-\omega^2/4a^2} \cdot \frac{\sqrt{\pi}}{a}$

Sol. Let $f(t) = e^{-t^2/2}$

$\therefore F[f(t)] = \sqrt{2\pi}\, e^{-\omega^2/2} = F(\omega)$

now

$F[e^{-2t^2}] = F[e^{-\frac{(2t)^2}{2}}] = F[2t] = \frac{1}{2} F\left(\frac{\omega}{2}\right)$

$= \frac{\sqrt{2\pi}}{2} e^{-\omega^2/8}$

$\therefore F[e^{-2t^2}] = \sqrt{\frac{\pi}{2}}\, e^{-\omega^2/8} = F(\omega)$

By Shifting Property
$F(f(t-t_0)) = F(\omega) e^{-i\omega t_0}$

$F[e^{-2(t-3)^2}] = \sqrt{\frac{\pi}{2}}\, e^{-\omega^2/8}\, e^{-i\omega 3}$

$= \sqrt{\frac{\pi}{2}}\, e^{-(3i\omega + \omega^2/8)}$

ii) $e^{-t^2} \cos 3t$

Sol. By modulation theorem

$F[f(t) \cos \omega_0 t] = \frac{1}{2}\left[F(\omega + \omega_0) + F(\omega - \omega_0)\right]$

$F[e^{-t^2}] = \sqrt{\pi}\, e^{-\omega^2/4}$

wkt,

$F[f(t)] = F[e^{-t^2}] = \sqrt{\pi}\, e^{-\omega^2/4}$
$= F(\omega)$

$\therefore F[e^{-t^2}\cos 3t] = \frac{1}{2}\left[\sqrt{\pi}\, e^{-(\omega+3)^2/4} + \sqrt{\pi}\, e^{-(\omega-3)^2/4}\right]$

$\omega_0 = 3$

$= \frac{\sqrt{\pi}}{2}\left[e^{-(\omega+3)^2/4} + e^{-(\omega-3)^2/4}\right]$

6th June, 2023

q: Find the Fourier Transform of the foll elementary func:

i) Dirac Delta function

ii) Constant func $f(t) = k$

iii) e^{iat}

iv) cos at and sin at

Sol i) To Find : $F[\delta(t-a)]$

By definition of F.T

$$F[f(t)] = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt$$

i.e $$F[f(t)] = \int_{-\infty}^{\infty} \delta(t-a) e^{-i\omega t} \, dt \quad -①$$

By Filtering property of Dirac Delta Func.

$$\int_{-\infty}^{\infty} f(t) \delta(t-a) \, dt = f(a) \quad -②$$

$\Rightarrow$ ① $\rightarrow$ $$F[\delta(t-a)] = e^{-i\omega a}$$

put $a = 0$

$$F[\delta(t)] = 1$$

ii) To Find $F[f(t)] = 1$

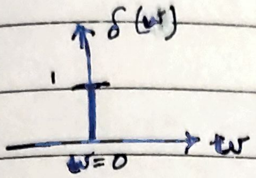
Fourier Transform of a constant func. cannot be found directly as it is not absolutely integrable.

So to find F.T of $f(t) = 1$, we use IFT of dirac delta function $\delta(\omega)$

By IFT,

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, d\omega \quad -①$$

here $F(\omega) = \delta(\omega)$ where



$$\delta(\omega) = \begin{cases} 1 & t=0 \\ 0 & t \neq 0 \end{cases}$$

now, $f(t) = F^{-1}[F(\omega)] = F^{-1}[\delta(\omega)]$

$$① \rightarrow \quad F^{-1}[\delta(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)\, e^{i\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} 1\cdot e^{i\omega t}\, d\omega$$

func. exists only for $\omega = 0$

$$= \frac{1}{2\pi}\, e^{i\omega t}\Big|_{\omega=0}$$

$$= \frac{1}{2\pi}$$

$$F^{-1}[2\pi\, \delta(\omega)] = 1$$

$$\therefore F[1] = 2\pi\, \delta(\omega)$$

In general, $\quad F[k] = 2k\pi\, \delta(\omega)$

iii) $F[e^{iat}]$

Sol. By Frequency Shifting Property

$$F[e^{i\omega_0 t} f(t)] = F[\omega - \omega_0]$$

here $f(t) = 1$

$$\therefore F[f(t)] = F[1] = 2\pi\,\delta(\omega) = F(\omega)$$

now

$$F[e^{iat}\cdot 1] = 2\pi\,\delta(\omega - a)$$

iv) $F[\cos at]$ and $F[\sin at]$

Sol By modulation property

$$F[f(t)\cos\omega_0 t] = \frac{1}{2}\left[F(\omega+\omega_0) + F(\omega-\omega_0)\right]$$

$$F[f(t)\sin\omega_0 t] = \frac{i}{2}\left[F(\omega+\omega_0) - F(\omega-\omega_0)\right]$$

here $f(t) = 1$

$$F[f(t)] = F[1] = 2\pi\,\delta(\omega) = F(\omega)$$

now,

$$F[1\cdot\cos at] = \frac{1}{2}\left[2\pi\,\delta(\omega+a) + 2\pi\,\delta(\omega-a)\right]$$

$$\therefore F[\cos at] = \pi\left[\delta(\omega+a) + \delta(\omega-a)\right]$$

$||^{ly}$ $$F[\sin at] = \frac{i}{2}\left[2\pi\,\delta(\omega+a) - 2\pi\,\delta(\omega-a)\right]$$

$$F[\sin at] = i\pi\left[\delta(\omega+a) - \delta(\omega-a)\right]$$

* q. If the Fourier sine transform of $f(t)$ is given by

$F_s(\omega) = \frac{\pi}{2}e^{-2\omega}$, find the func. $f(t)$

Sol. By Inverse Fourier Sine transform

$$f(t) = \frac{2}{\pi}\int_0^\infty F_s(\omega)\sin\omega t\, d\omega$$

$$= \frac{\cancel{2}}{\cancel{\pi}}\int_0^\infty \frac{\cancel{\pi}}{\cancel{2}} e^{-2\omega}\sin\omega t\, d\omega$$

$$= \int_0^\infty e^{-2\omega}\sin\omega t\, d\omega$$

$a = -2 \quad b = t$

$$= \left[\frac{e^{-2\omega}}{4+t^2}\left(-2\sin\omega t - t\cos\omega t\right)\right]_0^\infty$$

$$= 0 - \left(\frac{1}{4+t^2}(0 - t)\right)$$

$$f(t) = \frac{t}{4+t^2}$$

q. Find the function $f(t)$ if its cosine transform is given by $\frac{\sin a\omega}{\omega}$

Sol. By inverse Fourier cosine transform

$$f(t) = \frac{2}{\pi}\int_0^\infty F_c(\omega)\cos\omega t\, d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\sin a\omega}{\omega}\cos\omega t\, d\omega$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\sin(a+t)\omega + \sin(a-t)\omega}{\omega}\, d\omega$$

$$= \frac{1}{\pi}\left[\int_0^\infty \frac{\sin(a+t)\omega}{\omega}\, d\omega + \int_0^\infty \frac{\sin(a-t)\omega}{\omega}\, d\omega\right]$$

$$* \int_0^\infty \frac{\sin a\omega}{\omega}\, d\omega = \frac{\pi}{2}$$

$a > 0$

$\Rightarrow$ In the interval $0 < t < \infty$, $a + t > 0$

$$= \begin{cases} \frac{1}{\pi}\left(\frac{\pi}{2} + \frac{\pi}{2}\right), & a - t > 0 \text{ or } t < a \\ \frac{1}{\pi}\left(\frac{\pi}{2} - \frac{\pi}{2}\right), & a - t < 0 \text{ or } t > a \end{cases}$$

$$f(t) = \begin{cases} 1 & t < a \\ 0 & t > a \end{cases}$$

7th June, 2023

Inverse Fourier Transform

Results

1. wkt, $F[e^{-a|t|}] = \frac{2a}{a^2 + \omega^2}$

$$\therefore F^{-1}\left[\frac{2a}{a^2 + \omega^2}\right] = e^{-a|t|}$$

2. wkt, $F[e^{-at} u(t)] = \frac{1}{a + i\omega}$ $\qquad u(t) / H(t)$

$$F^{-1}\left[\frac{1}{a + i\omega}\right] = e^{-at} u(t)$$

q: Find IFT of the foll. funcs.

i) $F^{-1}\left[\frac{1}{(9+\omega^2)(4+\omega^2)}\right]$

Sol P.F $\frac{1}{(9+\omega^2)(4+\omega^2)} = \frac{A}{9+\omega^2} + \frac{B}{4+\omega^2}$

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$$\therefore \quad 1 = A(4+\omega^2) + B(9+\omega^2)$$

$\omega^2 = -9$

$\Rightarrow 1 = A(-5)$

$A = \frac{-1}{5}$

$\omega^2 = -4$

$1 = 5B$

$B = \frac{1}{5}$

$$F^{-1}\left[\frac{1}{(9+\omega^2)(4+\omega^2)}\right] = \frac{1}{5}\left[F^{-1}\left(\frac{-1}{9+\omega^2} + \frac{1}{4+\omega^2}\right)\right]$$

$$= \frac{1}{5}\left[-F^{-1}\left(\frac{1}{3^2+\omega^2}\right) + F^{-1}\left(\frac{1}{2^2+\omega^2}\right)\right]$$

$$= \frac{1}{5}\left[\frac{-1}{6} F^{-1}\left(\frac{2\cdot 3}{3^2+\omega^2}\right) + \frac{1}{4} F^{-1}\left(\frac{2\cdot 2}{2^2+\omega^2}\right)\right]$$

$$= \frac{-1}{30} e^{-3|t|} + \frac{1}{20} e^{-2|t|}$$

ii) $F^{-1}\left[\frac{1}{6+5i\omega-\omega^2}\right]$

Sol. $6+5i\omega-\omega^2 = 6 + 3i\omega + 2i\omega - \omega^2$

$= 3(2+i\omega) + i\omega(2+i\omega)$

$= (2+i\omega)(3+i\omega)$

$$\frac{1}{6+5i\omega-\omega^2} = \frac{1}{(2+i\omega)(3+i\omega)}$$

$$\frac{1}{6+5i\omega-\omega^2} = \frac{A}{2+i\omega} + \frac{B}{3+i\omega}$$

$$1 = A(3+i\omega) + B(2+i\omega)$$

put $i\omega = -2$ put $i\omega = -3$

$A = 1$ $-B = 1$

$B = -1$

$$\therefore F^{-1}\left[\frac{1}{(2+i\omega)(3+i\omega)}\right] = F^{-1}\left(\frac{1}{2+i\omega}\right) + F^{-1}\left(\frac{1}{3+i\omega}\right)$$

$$= e^{-2t}u(t) - e^{-3t}u(t)$$

$$= \begin{cases} e^{-2t} - e^{-3t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

iii) $F^{-1}\left[\frac{e^{i\omega}}{2(1+i\omega)}\right]$

Sol. By shifting on t-axis property

$$F[f(t-t_0)] = F(\omega)e^{-i\omega t_0}$$

$$\therefore F^{-1}\left[e^{-i\omega t_0}F(\omega)\right] = f(t-t_0)$$

now $t_0 = 1$

$$F^{-1}\left[e^{-i\omega}\cdot\frac{1}{2(1+i\omega)}\right] = \left\{F^{-1}\left[\frac{1}{2(1+i\omega)}\right]\right\}_{t\to t-1}$$

$$= \frac{1}{2}\left[e^{-1\cdot t}u(t)\right]_{t\to t-1}$$

$$= \frac{1}{2}\left[e^{-(t-1)}\cdot u(t-1)\right]$$

q. Using Fourier Transforms, find the solution of the foll. differential eqns.

i) $y' - 4y = H(t)e^{-4t}$, $-\infty < t < \infty$

ii) $y' + 3y = H(t)e^{-2t}$, $-\infty < t < \infty$

iii) $y'' + 5y' + 4y = \delta(t-2)$

iv) $y'' + 3y' + 2y = \delta(t-3)$

Sol.

i) $y'(t) - 4y(t) = H(t)\,e^{-4t}$

$$F[y'(t)] - 4F[y(t)] = F[H(t)e^{-4t}]$$

$$i\omega F[y(t)] - 4F[y(t)] = \frac{1}{4+i\omega}$$

$$F[y(t)] = \frac{1}{-(4-i\omega)(4+i\omega)}$$

$$y(t) = -F^{-1}\left[\frac{1}{(4+i\omega)(4-i\omega)}\right]$$

$$= -F^{-1}\left[\frac{1}{16+\omega^2}\right]$$

$$y(t) = \frac{-e^{-4|t|}}{8}$$

ii) $y'(t) + 3y(t) = H(t)e^{-2t}$

$$F[y'(t)] + 3F[y(t)] = F[H(t)e^{-2t}]$$

$$i\omega F[y(t)] + 3F[y(t)] = \frac{1}{2+i\omega}$$

$$F[y(t)] = \frac{1}{(2+i\omega)(3+i\omega)}$$

$$y(t) = F^{-1}\left[\frac{1}{(2+i\omega)(3+i\omega)}\right] = e^{-2t}u(t) - e^{-3t}u(t)$$

$$= e^{-2t}(1-e^{-t})u(t)$$

iii) $y''(t) + 5y'(t) + 4y(t) = \delta(t-2)$

$$F[y''(t)] + 5F[y'(t)] + 4F[y(t)] = F[\delta(t-2)]$$

$$(i\omega)^2 F[y(t)] + 5(i\omega) F[y(t)] + 4F[y(t)] = e^{-2i\omega}$$

$$F[y(t)](-\omega^2 + 5i\omega + 4) = e^{-2i\omega}$$

$$F[y(t)] = \frac{e^{-2i\omega}}{-\omega^2 + 5i\omega + 4}$$

$$F[y(t)] = \frac{e^{-2i\omega}}{(1+i\omega)(4+i\omega)}$$

$$y(t) = F^{-1}\left[\frac{e^{-2i\omega}}{(1+i\omega)(4+i\omega)}\right]$$

$$\frac{1}{(1+i\omega)(4+i\omega)} = \frac{A}{1+i\omega} + \frac{B}{4+i\omega}$$

$$1 = A(4+i\omega) + B(1+i\omega)$$

put $i\omega = -1$ put $i\omega = -4$

$3A = 1$ $-3B = 1$

$A = 1/3$ $B = -1/3$

① →

$$y(t) = F^{-1}\left[e^{-2i\omega}\left(\frac{1}{3}\cdot\frac{1}{1+i\omega} - \frac{1}{3}\cdot\frac{1}{4+i\omega}\right)\right]$$

$$= \frac{1}{3}\left[F^{-1}\left(e^{-2i\omega}\cdot\frac{1}{1+i\omega}\right) - F^{-1}\left(e^{-2i\omega}\cdot\frac{1}{4+i\omega}\right)\right]$$

$$= \frac{1}{3}\left[\left[e^{-t}u(t)\right]_{t\to t-2} - \left[e^{-4t}u(t)\right]_{t\to t-2}\right]$$

$$= \frac{1}{3}\left[e^{-(t-2)}u(t-2) - e^{-4(t-2)}u(t-2)\right]$$

Finite Fourier cosine transform

Let the func. $f(t)$ be piecewise continuous on $[0, \pi]$. Then the finite fourier cosine transform of $f(t)$ is defined by

$$F_c(n) = \int_0^{\pi} f(t) \cos nt \, dt$$

where n is a non-negative integer, $n = 0, 1, 2 \ldots$
This transform is also denoted by $C_n[f(t)]$

→ Inverse finite Fourier cosine transform

$$f(t) = \frac{1}{\pi}\left[F_c(0) + 2\sum_{n=1}^{\infty} F_c(n) \cos nt\right]$$

Finite Fourier sine transform

Let the func. $f(t)$ be piecewise continuous on $[0, \pi]$. Then the finite fourier sine transform of $f(t)$ is given by

$$F_s(n) = \int_0^{\pi} f(t) \sin nt \, dt$$

where n is a positive integer, $n = 1, 2, 3 \ldots$
This transform is also denoted by $S_n[f(t)]$

→ Inverse finite Fourier sine transform

$$f(t) = \frac{2}{\pi} \sum_{n=1}^{\infty} F_s(n) \sin nt$$

q. Find the finite Fourier sine transform of

$$f(x) = \begin{cases} -t & \text{for } 0<t<c \\ \pi - t & \text{for } c<t<\pi \end{cases}$$

Sol. $F_s(n) = \int_0^{\pi} f(t) \sin nt \, dt$

$$= \int_0^{c} -t \sin nt \, dt + \int_c^{\pi} (\pi - t) \sin nt \, dt$$

$$= -\left[t\left(-\frac{\cos nt}{n}\right) - \left(-\frac{\sin nt}{n^2}\right) \right]_0^c + \left[(\pi - t)\left(-\frac{\cos nt}{n}\right) - (-1)\left(-\frac{\sin nt}{n^2}\right) \right]_c^{\pi}$$

$$= \frac{c \cos nc}{n} - \frac{\sin nc}{n^2} + 0 - \left[-\frac{(\pi - c)\cos nc}{n} - \frac{\sin nc}{n^2} \right]$$

$$= \cancel{\frac{c \cos nc}{n}} - \cancel{\frac{\sin nc}{n^2}} + \frac{\pi \cos nc}{n} - \cancel{\frac{c \cos nc}{n}} + \cancel{\frac{\sin nc}{n^2}}$$

$$F_s(n) = \frac{\pi \cos nc}{n}$$

q. Find the finite Fourier sine transform of

$$f(x) = x(\pi - x) \quad \text{in} \quad 0<x<\pi$$

Sol. $F_s(n) = \int_0^{\pi} t(\pi - t) \sin nt \, dt$

$$= \int_0^{\pi} (\pi t - t^2) \sin nt \, dt$$

$$= \left[(\pi t - t^2)\left(-\frac{\cos nt}{n}\right) - (\pi - 2t)\left(-\frac{\sin nt}{n^2}\right) + (-2)\left(\frac{\cos nt}{n^3}\right) \right]_0^{\pi}$$

$$= 0 - 0 - \frac{2(-1)^n}{n^3} - \left(-\frac{2}{n^3}\right)$$

$$= \frac{2}{n^3} - \frac{2}{n^3}(-1)^n$$

$$F_s(n) = \frac{2}{n^3}\left(1 - (-1)^n\right)$$

q. Find the finite Fourier cosine transform of $f(t) = t(\pi - t)$ in $0 < t < \pi$

Sol. $$F_c(n) = \int_0^{\pi} t(\pi - t) \cos nt \, dt$$

$$= \left[\cancel{(\pi t - t^2)\left(\frac{\sin nt}{n}\right)} - (\pi - 2t)\left(-\frac{\cos nt}{n^2}\right) + \cancel{(-2)\left(-\frac{\sin nt}{n^3}\right)} \right]_0^{\pi}$$

$$= \left[(\pi - 2t)\frac{\cos nt}{n^2} \right]_0^{\pi}$$

$$= \frac{-\pi(-1)^n}{n^2} - \frac{\pi}{n^2}$$

$$= -\frac{\pi}{n^2}\left((-1)^n + 1\right)$$

q. Find the finite Fourier cosine transform of $f(t) = 2t$ in $0 < x < 4$.

Sol. $$F_c(n) = \int_0^4 f(t) \cos nt \, dt$$

$$= \int_0^4 2t \cos nt \, dt$$

$$= 2\left[t\frac{\sin nt}{n} - (1)\left(-\frac{\cos nt}{n^2}\right) \right]_0^4$$

$$= 2\left[4\frac{\sin 4n}{n} + \frac{\cos 4n}{n^2} - \right.$$

$$F_c(n) = \int_0^4 f(t)\cos\left(\frac{n\pi t}{4}\right) dt$$

$$= \int_0^4 2t \cos\left(\frac{n\pi t}{4}\right) dt$$

$$= \left[2t\left(\sin\left(\frac{n\pi t}{4}\right)\cdot\frac{4}{n\pi}\right) - 2\left(-\cos\left(\frac{n\pi t}{4}\right)\cdot\frac{16}{n^2\pi^2}\right)\right]_0^4$$

$$= \frac{32(-1)^n}{n^2\pi^2} - \frac{32}{n^2\pi^2}$$

$$= \frac{32}{n^2\pi^2}\left((-1)^n - 1\right)$$

q. Find the finite Fourier cosine transform of $f(t) = e^{at}$ in $0 < t < l$

Sol. $$F_c(n) = \int_0^l f(t)\cos\left(\frac{n\pi t}{l}\right) dt$$

$$= \int_0^l e^{at}\cos\left(\frac{n\pi t}{l}\right) dt$$

$$= \left[\frac{e^{at}}{a^2 + \frac{n^2\pi^2}{l^2}}\left(a\cos\left(\frac{n\pi t}{l}\right) + \frac{n\pi}{l}\sin\left(\frac{n\pi t}{l}\right)\right)\right]_0^l$$

$$= \frac{l^2 e^{at}}{a^2 l^2 + n^2\pi^2}\left(a(-1)^n - 0\right) - \frac{l^2 a}{a^2 l^2 + n^2\pi^2}$$

$$= \frac{a l^2\left(e^{al}(-1)^n - 1\right)}{a^2 l^2 + n^2\pi^2}$$

q. Find the inverse finite sine transform of $F_s(n) = \frac{1-\cos n\pi}{n^2\pi^2}$

$$f(t) = \frac{4}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin(2n-1)t}{(2n-1)^2}$$