12th September, 2022 Maths

# Partial Differentiation

dependent variable $y = f(x)$ ← independent variable (input) $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$f'(x) = \frac{dy}{dx}$ slope of tangent to curve : $y = f(x)$ at a point $(x, y)$

$z = f(x, y)$ : surface in 3-D

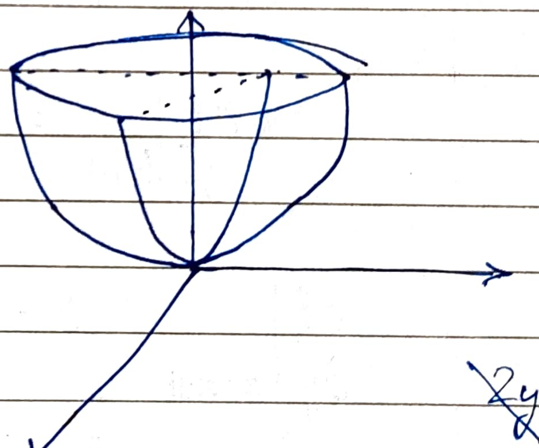
If $x$ is fixed or treated as a constant

$\partial$ - del operator

$$\left.\frac{\partial z}{\partial y}\right|_{x \text{ constant}} = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}$$

$$\left.\frac{\partial z}{\partial x}\right|_{y \text{ const.}} = \lim_{h \to 0} \frac{f(y, x+h) - f(x,y)}{h}$$

q. $z = x^2 + y^2$



$\frac{\partial z}{\partial x} = 2x$

$\frac{\partial z}{\partial y} = 2y$

~~2y~~

q. $z = \sin(xy) + \ln(x^2 + y^2)$

CW – QB – assignments, Test

$$\left.\frac{\partial z}{\partial y}\right|_{x \text{ const.}} = \cos(xy)x + \frac{(2y)}{x^2+y^2}$$

$$\left.\frac{\partial z}{\partial x}\right|_{y \text{ const.}} = \cos(xy).y + \frac{2x}{x^2+y^2}$$

q. $z = \sin^{-1}\left(\dfrac{x^2+y^2}{x^2-y^2}\right)$

$$z_x = \left.\frac{\partial z}{\partial x}\right|_{y \text{ const.}} : z_x = \frac{1}{\sqrt{1-\left(\frac{x^2+y^2}{x^2-y^2}\right)^2}} \cdot \frac{(x^2-y^2)(2x) - (x^2+y^2)(2x)}{(x^2-y^2)^2}$$

$$= \frac{1}{\sqrt{1-\left(\frac{x^2+y^2}{x^2-y^2}\right)^2}} \cdot \frac{2x\left[-2y^2\right]}{(x^2-y^2)^2}$$

~~q. Fin~~

13th September, 2022

q. Find the partial derivatives of

$$f(x,y) = x^4 - x^2y^2 + y^4 \quad \text{at } (-1,1)$$

$$f_x : \left.\frac{\partial f}{\partial x}\right|_{y \text{ const}} = 4x^3 - \left[x^2(0) + y^2(2x)\right].$$

$$= 4x^3 - 2xy^2$$

$$\left.\frac{\partial f}{\partial x}\right|_{(-1,1)} = -4+2 = -2$$

$$\left.\frac{\partial f}{\partial y}\right|_{x \text{ const.}} = 0 - x^2(2y) + 4y^3$$

$$= -2+4 = 2$$

Note : $f(x,y) = f(y,x) \rightarrow$ Symmetric function

$$\frac{\partial f^2}{\partial y \partial x} = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left[4x^3 - 2xy^2\right]$$

$$-2x(2y)$$

$$= 4$$

$$\frac{\partial f^2}{\partial x \partial y} = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left[4y^3 - 2x^2y\right]$$

$= -2y(2x)$

$= 4$

$f_{xy} = f_{yx}$ (for continuous fns)

$f(x,y) = ye^{-x}$

$\left.\frac{\partial f}{\partial x}\right|_{y\text{ const}} = ye^{-x}(-1)$

$\left.\frac{\partial f}{\partial y}\right|_{x\text{ const}} = e^{-x}$

$= \frac{\partial f^2}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \left.\frac{\partial}{\partial y}\left(-ye^{-x}\right)\right|_{x\text{ const.}} = (-e^{-x})1$

$x = \left.\frac{\partial}{\partial x}\left(e^{-x}\right)\right|_{y\text{ const.}} = \left(e^{-x}\right)(-1)$

$xx = \left.\frac{\partial^2 f}{\partial x^2}\right|_{y\text{ const.}} = ye^{-x}(-1)(-1) = ye^{-x}$

$f_{yy} = \left.\frac{\partial^2 f}{\partial y^2}\right|_{x\text{ const}} = 0$

q. $f(x,y) = \sin(3x+5y)$

$f_x = \left.\frac{\partial f}{\partial x}\right|_{y\text{ cost.}} = \cos(3x+5y)\cdot(3)$

$f_y = \frac{\partial f}{\partial y} = \cos(3x+5y)(5)$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left[3\cos(3x+5y)\right]$$

$$= -3\sin(3x+5y)(5)$$

$$f_{xx} = -3^2\sin(3x+5y)$$

$$f_{yy} = -25\sin(3x+5y)$$

$$f_{yx} = \frac{\partial}{\partial x}\left[5\cos(3x+5y)\right] = -5\sin(3x+5y)(3)$$

q. Show that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ for all $(x,y) \neq (0,0)$

given that

$$f(x,y) = x^y$$

$$\left.\frac{\partial f}{\partial x}\right|_{y\ \text{const}} = y x^{y-1}$$

$$\left.\frac{\partial f}{\partial y}\right|_{x\ \text{const}} = x^y \log x$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \left.\frac{\partial}{\partial x}\left[x^y . \log x\right]\right|_{y\ \text{const}}$$

$$= \cancel{y x^{y-1} \log x}\ x^y\left(\frac{1}{x}\right) + \log_e x\left[y x^{y-1}\right]$$

$$\frac{\partial^2 f}{\partial y \partial x} = \left.\frac{\partial}{\partial y}\left[y . x^{y-1}\right]\right|_{x\ \text{const}} = $$

$$y x^{y-1} \ln x + x^{y-1}(1)$$

$$f_{xx}\Big|_{y\ const.} = y(y-1)\,x^{y-2}$$

$$f_{yy}\Big|_{x\ const.} = x^{y}(\log x)(\log x)$$

14th September, 2022

Q. Find all the partial derivatives of $f(x,y) = \log_e\left(\frac{1}{x} - \frac{1}{y}\right)$ at $(1,2)$

$$f_x = \frac{\partial f}{\partial x}\Big|_{y\ const.} = \frac{1}{\frac{1}{x}-\frac{1}{y}}\left(-1\,x^{-2}\right) = \frac{-1}{1-\frac{1}{2}}(1) = -2$$

$$f_y = \frac{\partial f}{\partial y}\Big|_{x\ const.} = \frac{1}{\frac{1}{x}-\frac{1}{y}}\left(-1\left(-y^{-2}\right)\right) = \frac{1}{1-\frac{1}{2}}\left(2^{-2}\right) = \frac{2}{4} = \frac{1}{2}$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}\Big|_{y\ const.} = \frac{\partial}{\partial x}\left(\frac{-1}{\frac{1}{x}-\frac{1}{y}}\,x^{-2}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{-\cancel{x}y}{y-x}\times\frac{1}{x^{\cancel{2}}}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{-y}{x(y-x)}\right) = \frac{x(y-x)(0)-(-y)\left[x(-1)+(y-x)\right]}{\left[x(y-x)\right]^2}$$

$$= \frac{\cancel{+(2-1)}\ 0-(-2)\left[-1+1\right]}{\left[1(1)\right]^2}$$

$$= 0$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2}\Big|_{x\ const.} = \frac{\partial}{\partial y}\left(\frac{x\cancel{y}}{(y-x)}\times\frac{1}{y^{\cancel{2}}}\right) = \frac{\partial}{\partial y}\Big|_{x\ const}\left(\frac{x}{y^2-xy}\right)$$

$$= \cancel{y^2 4 y^2 - xy}\quad x\times\frac{-1}{\left(y^2-xy\right)^2}(2y-x)$$

$$= \frac{-1(3)}{(4-2)^2} = \frac{-3}{4}$$

$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\Big|_{y\ const.}$

q. $f(x,y,z) = e^{x^2+y^2+z^2}$ then find

Symmetric fn.

$f_{xx}$, $f_{yy}$, $f_{zz}$

$f_x = \frac{\partial f}{\partial x}\Big|_{y,z\ const.} = \frac{d}{dx}\Big|_{y,z\ const}\left(e^{x^2+y^2+z^2}\right)$

$= \left(e^{x^2+y^2+z^2}\right)(2x)$

$= 2x\left(e^{x^2+y^2+z^2}\right)$

$f_{xx}\Big|_{y,z\ const.} = 2x\left[e^{x^2+y^2+z^2}(2x)\right] + \left(e^{x^2+y^2+z^2}\right)(2)$

$= 4x^2\left[e^{x^2+y^2+z^2}\right] + 2\left[e^{x^2+y^2+z^2}\right]$

$f_{yy}\Big|_{x,z\ const.} = 4y^2\left[e^{x^2+y^2+z^2}\right] + 2\left[e^{x^2+y^2+z^2}\right]$

$= 2\left[\cancel{2y^2}\ e^{x^2+y^2+z^2}\left[\cancel{4y^2+2}\right]\right.$

$f\,[$

$= 2^2 x^2 \cdot e^{x^2+y^2+z^2} + 2\cdot e^{x^2+y^2+z^2}$

$= 2\cdot x\cdot f_x + 2\cdot f$

$= 2\left[x f_x + f\right]$

q. $f(x,y) = e^x \cdot \log y + \cos y \log x$

Find $f_{xxx}$, $f_{xxy}$, $f_{xyy}$, $f_{yyy}$, at $\left(1, \frac{\pi}{2}\right)$

$f_x\Big|_{y,z\ const.} = \log y \cdot e^x + \cos y \cdot \frac{1}{x}$

$x^{-2}$

$f_{xx}\Big|_{y,z\ const.} = e^x \log y + \cos y\left(\frac{-1}{x^2}\right)$

$f_{xxx}\Big|_{y,z\ const.} = e^x \log y + \cos y\left(\frac{2}{x^3}\right)$

$= e \log\frac{\pi}{2} + 0$

$y$ terms get differentiated in $x$

$$f_{xxy} = \frac{\partial}{\partial y}\Big|_{x,z\text{ const.}} \left(e^x \log y - \frac{\cos y}{x^2}\right)$$

$$= e^x\left(\frac{1}{y}\right) - \frac{1}{x^2}(-\sin y)$$

$$= \frac{2e}{\pi} + 1$$

$f_{x\cancel{y}xy} =$

$$f_{xy}\Big| = \frac{\partial}{\partial y}\Big|_{x\text{ const.}}\left(\log \cancel{y}\, e^x + \cos y \frac{1}{x}\right)$$

$$= e^x \frac{1}{y} + \frac{1}{x}(-\sin y)$$

$$f_{xyy}\Big|_{x\text{ const.}} = e^x\left(\frac{-1}{y^2}\right) + \frac{1}{x}(-\cos y)$$

$$= \frac{-e}{\pi^2}(4)$$

$f_{yyy} =$

$$f_y\Big|_{x,z\text{ const}} = e^x\left(\frac{1}{y}\right) + \log x(-\sin y)$$

~~$f_{yy} = e^x\left(\frac{-1}{y^2}\right) - \frac{\cos y}{x}$~~

~~$f_{yyy} = e^x\left(\frac{2}{y^3}\right) + \frac{\sin y}{x}$~~

~~$= e\frac{16}{\pi^3} + 1$~~

$$f_{yy} = e^x\left(\frac{-1}{y^2}\right) - \cos y \log x$$

$$f_{yyy} = e^x\left(\frac{2}{y^3}\right) + \sin y \log x$$

$$= \frac{16e}{\pi^3} + 0$$

q. $y = x^2$ $\quad x = t + t^2$ $\quad \frac{dy}{dt} = ?$

Sol. $\frac{dy}{dx} = 2x$ $\quad \frac{dx}{dt} = 1 + 2t$

$y = f(x(t))$

$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

$\cancel{\frac{dy}{dt}} = \frac{dy}{dt} = 2x(1+2t)$

$= 2(t+t^2)(1+2t) = 2t + 6t^2 + 4$

(OR)

$y = (t+t^2)^2 = t^2 + t^4 + 2t^3$

$\frac{dy}{dt} = 2t + 4t^3 + 6t^2$

# Total Derivative

* $Z = f(x(t), y(t))$ of $Z = Z(x, y)$

Chain rule kinda, using $\partial$ as two variables

$\frac{dZ}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$

this is the total derivative of Z wrt t

By Substitution

$Z = x^2 + y^2$

$x = t+1, \quad y = t^2$

$Z = (t+1)^2 + (t^2)^2$

$\frac{dz}{dt} = 2(t+1) + 4t^3$

$= 2(t + 1 + 2t^3)$

$\frac{dZ}{dt} = \frac{\partial Z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial Z}{\partial y} \frac{dy}{dt}$

$= 2x(1) + 2y(2t)$

$= 2(x + 2yt)$

$= 2(t + 1 + 2t^3)$

* $Z = z(x(r, \theta), y(r, \theta))$

no $\theta$ in denomi.

Chain rule

$\left.\frac{\partial Z}{\partial r}\right|_{\theta \text{ const.}} = \frac{\partial Z}{\partial x} \cdot \frac{dx}{dr} + \frac{\partial Z}{\partial y} \cdot \frac{dy}{dr}$

$\left.\frac{\partial Z}{\partial \theta}\right|_{r \text{ const.}} = \frac{\partial Z}{\partial x} \cdot \frac{dx}{d\theta} + \frac{\partial Z}{\partial y} \cdot \frac{dy}{d\theta}$

$z(x(r), y(r))$

$\frac{dZ}{dr} = \frac{\partial Z}{\partial x} \frac{dx}{dr} + \frac{\partial Z}{\partial y} \frac{dy}{dr}$

* If $z = z(x,y)$

$x = x(r,\theta)$, $y = y(r,\theta)$ then

$\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$ can be found

q: Find $\frac{du}{dt}$ if $u = \tan(x^2 + y^2)$ if $x = t^2 + 2$, $y = t^3$

Sol. given $u = u(x(t), y(t))$

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

$$= [\sec^2(x^2+y^2)(2x)](2t) + [\sec^2(x^2+y^2)(2y)](3t^2)$$

$$x^2 + y^2 = (t^2+2)^2 + (t^3)^2 = t^4 + 4 + 4t^2 + t^6$$

$$\frac{du}{dt} = 2[t^2+2]\sec^2(t^6 + t^4 + 4t^2 + 4)(2t) + 2(t^3)3t^2 \sec^2(t^6 + t^4 + 4 + 4t^2)$$

Note: Given $z = z(x,y)$; $x = x(x)$, $y = y(x)$

$x$ is independent variable

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial z}{\partial y} \frac{dy}{dx}$$

Given $z(x,y) = c$, $x = x(x)$, $y = y(x)$ c is constant

$$\rightarrow \frac{dz}{dx} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial z}{\partial y} \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{\partial z/\partial x}{\partial z/\partial y}$$

* in a fn. of $x$ & $y$ $\boxed{\frac{dy}{dx} = \frac{-Z_x}{Z_y}}$

$C = Z(x, y)$ $x = x(t)$

$y = y(t)$

$dz$ - differential of $z$

$$\frac{dz}{dt} = \frac{\partial Z}{\partial x}\frac{dx}{dt} + \frac{\partial Z}{\partial y}\frac{dy}{dt} = 0$$

doesn't matter

q. Find the total differential of i) $u = xy$

ii) $u = \frac{x}{y}$

$xy$

Sol. i) $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$

$du = y\,dx + x\,dy$

ii) $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$

$= y^{-1}dx + \frac{-x}{y^2}dy$

$= \frac{dx}{y} - \frac{x}{y^2}dy$

(OR) Quotient Rule

$du = \frac{y\,dx - x\,dy}{y^2}$

$= \frac{dx}{y} - \frac{x}{y^2}dy$

q. Find $\frac{du}{dx}$ if $u = \tan(x^2+y^2)$ & $x, y$ are connected by the relation $x^2 - y^2 = 2$.

Sol. $\frac{du}{dx} = \cancel{\sec^2(x^2+y^2)(2x)}$

means $y$ is a fn of $x$

Sol. $u = u(x(x), y(x))$

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 2x\sec^2(x^2+y^2) + \sec^2(x^2+y^2)\ldots$$

$$\frac{du}{dx} = 2x\sec^2(x^2+y^2) + 2y\sec^2(x^2+y^2)\frac{dy}{dx} \, ?$$

now, $x^2 - y^2 = 2$

$$\frac{dy}{dx} = -\frac{Z_x}{Z_y} = -\frac{2x}{-2y} = +\frac{x}{y}$$

$$\therefore \frac{du}{dx} = \sec^2(x^2+y^2)\left[2x + 2y\frac{dy}{dx}\right]$$

$$= \sec^2(x^2+y^2)\left[2x + 2\cancel{y}\left(\frac{x}{\cancel{y}}\right)\right] = 4x\sec^2(x^2+y^2)$$

q: Find $\frac{du}{dt}$ if $u = x^2 - y^2$

$x = e^t\cos t$, $y = e^t\sin t$ at $t = 0$

Sol. $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$

$= 2x\left[e^t(-\sin t + \cos t)\right] + \left[e^t(\cos t + \sin t)(-2y)\right]$

$= 2x \cdot e^t(\cos t - \sin t) - 2y \cdot e^t(\cos t + \sin t)$

$\left.\frac{du}{dx}\right|_{t=0} = 2xe^0(1) - 2ye^0(1+0)$

$= 2x - 2y$

$= 2(1) - 2(0) = 2$

$\frac{dx}{dt} = e^t\left[(-\sin t) + \cos t\right]$

$\frac{dy}{dt} = e^t\left[\cos t + \sin t\right]$

$x = e^0(1) = 1$

$y = 0$

q: Find the rate at which the area of a rectangle is increasing at a given instant when the sides of the rectangle are 5 ft and 4 ft and are increasing at a rate of 1.5 ft/sec and 0.5 ft/s respectively.

Sol. $\frac{dA}{dt} = ? \frac{\partial A}{\cancel{\partial}}$

area $= l \times b \Rightarrow A = A(l, b)$

$l = l(t)$, $b = b(t)$

$$\frac{dA}{dt} = \frac{\partial A}{\partial l}\frac{dl}{dt} + \frac{\partial A}{\partial b}\frac{db}{dt}$$

$A = l \times b$

$\frac{\partial A}{\partial l} = b$

$$= b\frac{dl}{dt} + l\frac{db}{dt}$$

$$= b(1.5) + l(0.5)$$

$$= 4(1.5) + 5(0.5)$$

$$= 8.5\ ft^2/s$$

Q. The altitude of a right circular cone is 15 cm and is increasing at 0.2 cm/s. The radius of the base is 10 cm and is decreasing at 0.3 cm/s. How fast is the vol. changing.

$V = \frac{1}{3}\pi r^2 h \qquad V = V(r(t), h(t))$

$\frac{dh}{dt} = 0.2$

$h = 15$

$r = 10$

$\frac{dr}{dt} = -0.3$

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\bigg|_{h\ const.}\cdot\frac{dr}{dt} + \frac{\partial V}{\partial h}\cdot\frac{dh}{dt}$$

$$= \frac{1}{3}h\pi(2r)(-0.3) + \frac{1}{3}\pi r^2(0.2)$$

$$= \frac{2}{3}\pi(10)(15)(-0.3) + \frac{1}{3}\pi(10)^2(0.2)$$

$$= -\frac{70}{3}\pi\ cm^3/s$$

Q. Find $\frac{dy}{dx}$ at $(1,1)$ for $e^y - e^{-x} + xy = 1$

$$\frac{dy}{dx} = -\frac{Z_x}{Z_y} = -\frac{(0 + e^{-x} + y)}{(e^y - 0 + x)}$$

$$= -\frac{(0 + e^{-1} + 1)}{e - 0 + 1}$$

$$\cancel{\frac{e + \frac{1}{e} + 1}{e - \frac{1}{e} + 1}} \quad = \quad \cancel{e^2 + 1}$$

$$-\frac{(e^{-1} + 1)}{e + 1} = \frac{-(\frac{1}{e} + 1)}{e + 1} = \frac{-(1+e)}{e(e+1)}$$

$$= \frac{-1}{e}$$

(or) $e^y y' + e^{-x} + xy' + y = 0$

$$ey' + e^{-1} + y' + 1 = 0$$

$$y'(e+1) = -(e^{-1} + 1)$$

$$y' = -\frac{(e^{-1}+1)}{e+1}$$

q. $z = \log(u^2 + v^2)$ $\quad u = e^{x+y^2}$, $\quad v = x + y^2$

Find $\quad 2y \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \qquad - \times 2y$$

$$= \frac{1(2u)}{u^2 + v^2}(e^{x+y^2}) + \frac{2v}{u^2+v^2}(1)$$

$$\frac{\partial z}{\partial y} = \frac{2u}{u^2+v^2}(e^{x+y^2})(2y) + \frac{2v}{u^2+v^2}(2y)$$

$$= 0$$

(or) Sol. $e^{z} = u^2 + v^2$ — ①

diff partially ① on both sides w.r.t $x$ ($y$ const.)

$$e^{z}\frac{\partial z}{\partial x} = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} \quad ②$$

× 2y

$$2y\, e^{z}\frac{\partial z}{\partial x} = 2y\cdot 2u\frac{\partial u}{\partial x} + 2y\, 2v\frac{\partial v}{\partial x} \quad ④$$

diff partially w.r.t. $y$ both sides ①

$$e^{z}\frac{\partial z}{\partial y} = 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} \quad ③$$

$$e^{z}\left[2y\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right] = 4uy\frac{\partial u}{\partial x} + 4yv\frac{\partial v}{\partial x} - \left(2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y}\right)$$

$u = e^{x+y^2} \quad v = x + y^2$

$$= 4e^{x+y^2}\cdot y\cdot e^{x+y^2} + 4y(x+y^2)\cdot 1 - \left[2e^{x+y^2}\cdot e^{x+y^2}\cdot 2y + 2(x+y^2)2y\right]$$

$$= 0$$

21st September, 2022

H/W Show that $(u_x)^2 + (u_y)^2 = (u_r)^2 + \frac{1}{r^2}(u_\theta)^2$

where $u = u(x,y)$, $x = r\cos\theta$, $y = r\sin\theta$

# Homogeneous Function of two or more variables.

(Equal degree of all terms):

eg 1: Consider $f(x,y) = x^2 + y^2 + 2gx + 2fy + c$ } Not a homogeneous function

Since the degree of each term is not the same.

eg 2: $x^2 + y^2 = f(x,y)$ is a homogeneous function

degree = 2

eg 3: $f(x,y) = x^3 + y^3 + x^2y + xy^2$ - homogeneous function

degree of all terms = 3

$$f(\lambda x, \lambda y) = \lambda^3 x^3 + \lambda^3 y^3 + \lambda^3 x^2 y + \lambda^3 xy^2 = \lambda^{3} f(x,y)$$

degree of homogeneous fn

* $f(x,y) = x^2 + x^2y \rightarrow$ Non-homogeneous

$f(\lambda x, \lambda y) = \lambda^2 x^2 + x^2 y \lambda^3$

eg: $f(x,y) = \sin^{-1}\left[\dfrac{x^3+y^3}{xy}\right]$

$f(\lambda x, \lambda y) = \sin^{-1}\left[\dfrac{\lambda^{\cancel{3}}(x^3+y^3)}{\cancel{\lambda^2}\, xy}\right] \rightarrow$ non-homogeneous $\because f(\lambda x, \lambda y) \neq \lambda^n f(x,y)$

eg: $f(x,y) = \sin^{-1}\left[\dfrac{x^3+y^3}{x^2y}\right] \rightarrow$ homogeneous $\lambda^0 \rightarrow$ degree = 0

* degree can be negative

# Alternate representations

eg: $f(x,y) = x^2 + xy$ is a homogeneous function of degree 2

degree

$= x^{2}\left[1 + \left(\dfrac{y}{x}\right)\right] = x^{2}\phi\left(\dfrac{y}{x}\right)$

eg 2: $f(x,y) = x^3 + x^2y + xy^2 + y^3$ is homogeneous of deg 3

$$f(x,y) = x^3\left[1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right)^3\right] = x^3 h\left(\frac{y}{x}\right)$$

* A function $f(x,y)$ is said to be homogeneous if

$$f(x,y) = \lambda^n f(x,y)$$

$$f(x,y) = x^n h\left(\frac{y}{x}\right)$$

$$f(x,y) = y^n k\left(\frac{x}{y}\right)$$

# Eulers theorem for homogeneous functions

* If $f(x,y)$ is a homogeneous function of degree $n$ then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x,y)$$

Proof: Suppose $f(x,y)$ is a homogeneous fn. of degree $n$

$$\therefore f(x,y) = x^n \phi\left(\frac{y}{x}\right)$$

$$\left.\frac{\partial f}{\partial x}\right|_{y\ const} = nx^{n-1}\phi\left(\frac{y}{x}\right) + x^n\phi'\left(\frac{y}{x}\right)\cdot\left(\frac{-y}{x^2}\right) \quad -(1)$$

$$\left.\frac{\partial f}{\partial y}\right|_{x\ const} = x^n\phi'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) \quad -(2)$$

(1) $\times x$ + (2) $\times y$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nx^n\phi\left(\frac{y}{x}\right) + \cancel{x^{n+1}\phi'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)}$$

$$+ \cancel{x^{n-1}\phi'\left(\frac{y}{x}\right)\cdot y}$$

$$x f_x + y f_y = nx^n\phi\left(\frac{y}{x}\right)$$

22nd September, 2022

q. $u = \tan^{-1}\left(\frac{x^2+y^2}{2x^3y}\right)$ i) is u a homogeneous function

ii) if u is not then can you find a homogeneous fn. containing u

i) u is not homogeneous $u = \tan^{-1}\left(\frac{\cancel{\lambda^2}(x^2+y^2)}{\lambda^{\cancel{4}2}(2x^3y)}\right)$

$$u = \tan^{-1}\left(\frac{x^2+y^2}{\lambda^2(2x^3 y)}\right)$$

ii) But $\tan u = Z = \frac{x^2+y^2}{2x^3y}$ is homogeneous fn. of degree -2

$$Z = \frac{x^2\left(1+\frac{y^2}{x^2}\right)}{x^4\left(2\frac{y}{x}\right)} = x^{-2}\phi\left(\frac{t}{x}\right)$$

# Euler's Theorem

$$xu_x + yu_y = \underset{\text{degree}}{n}u$$

eg. 1. If $u = \cos\left(\frac{y}{x}\right)$ then find the value of $xu_x + yu_y$

$u_x = -\sin\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) = \sin\left(\frac{y}{x}\right)\left(\frac{+y}{x^2}\right) \rightarrow x\,x = \sin\left(\frac{y}{x}\right)\left(\frac{y}{x}\right)$

$u_y = -\sin\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) = -\sin\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) \rightarrow x\,y = -\sin\left(\frac{y}{x}\right)\left(\frac{y}{x}\right)$ (+)

$= 0$

By Euler's Theorem: $xu_x + yu_y = \underset{\text{degree}}{n}u = 0\cdot u = 0$

Eg. 2. If $u = \tan^{-1}\left(\frac{x^2+y^2}{2x^3y}\right)$, find the value of

$$x u_x + y u_y$$

Sol. $u$ is not a homogeneous function
$\therefore$ Euler's theorem is not applicable for $u$.

But $\tan u = Z$ is a homogeneous function of degree $n = -2$
$\therefore$ Euler's theorem is applicable for $Z$

By Euler's theorem $x Z_x + y Z_y = n Z$

$Z_x = \frac{\partial Z}{\partial x}$

$Z = \tan u$

$$x \frac{\partial Z}{\partial x} + y \frac{\partial Z}{\partial x} = n Z$$

substitute for $Z$

$$x \cdot \frac{\partial}{\partial x}(\tan u) + y \frac{\partial}{\partial y}(\tan u) = (-2) \tan u$$

$$x \cdot \sec^2 u \cdot \frac{\partial u}{\partial x} + y \sec^2 u \cdot \frac{\partial u}{\partial y} = -2 \tan u$$

$\div$ by $\sec^2 u$:

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = -2 \cdot \frac{\tan u}{\sec^2 u} = -\sin 2u$$

q. 1. If $u = \frac{y^{1/3} - x^{1/3}}{x+y}$, find the values of

i) $x u_x + y u_y$

2. If $u = \tan^{-1}\left(\frac{x^4+y^4}{xy}\right)$, find the value of $x u_x + y u_y$

3. If $u = \left(\sqrt{y^2 - x^2}\right) \sin^{-1}\left(\frac{x}{y}\right)$, find $x u_x + y u_y$

Sol 1. $u = \frac{\lambda^{1/3}(y^{1/3} - x^{1/3})}{\lambda(x+y)} = \lambda^{-2/3} = nu$

$= -\frac{2}{3}u$

2. $u = \tan^{-1}\left(\frac{x^4+y^4}{xy}\right)$

$Z = \tan u = \frac{\lambda^4(x^4+y^4)}{\lambda^2(xy)} = \lambda^{+2} Z$

$$x\frac{\partial Z}{\partial x} + y\frac{\partial Z}{\partial y} = nZ$$

$$x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = +2\tan u$$

$$xu_x + yu_y = \frac{+2\tan u}{\sec^2 u} = \sin 2u$$

3. $u = \left(\sqrt{y^2-x^2}\right)\sin^{-1}\left(\frac{x}{y}\right)$

$= \lambda\sqrt{y^2-x^2}\sin^{-1}\left(\frac{\cancel{\lambda}x}{\cancel{\lambda}y}\right) = \lambda^1 u$

$u_x + u_y = 1u = \sqrt{y^2-x^2}\sin^{-1}\left(\frac{x}{y}\right)$

26th September, 2022

Questions:

1. Geometry of $Z = f(x,y)$

2. If $Z = x^3 + \frac{3x}{y} + e^{\log y}$. What is $\frac{\partial Z}{\partial y}, \frac{\partial Z}{\partial x}, \frac{\partial^2 Z}{\partial x\partial y}$ ✓

3. Is $\frac{\partial^2 Z}{\partial x\partial y} = \frac{\partial^2 Z}{\partial y\partial Z}$ for all $Z$ ? Give a reason.

4. $x^2 + y^3 = 4$ & $y = x^3$ Find $\frac{dy}{dx}$.

5. If $x^2+y^2 = Z$ & $x = r\cos\theta$, $y = r\sin\theta$ find $\frac{\partial Z}{\partial r}, \frac{\partial Z}{\partial \theta}, \frac{\partial Z}{\partial r\partial\theta}$ ?

6. Is $u = \sin^{-1}\left(\frac{x^3 + x^2y + xy^2 + y^3}{x^4 + y^4}\right)$ homogeneous ?

Find the value of $xu_x + yu_y$.

Extension of Euler's Theorem

* $u$ is a homogeneous function of $x$ & $y$. of degree $n$.

$\therefore$ Euler's theorem holds

$$xu_x + yu_y = nu \quad \text{—(1)}$$

differentiate partially (1) wrt $x$

$$xu_{xx} + 1\cdot u_x + yu_{xy} = nu_x \quad \text{—(2)} \quad \times x$$

$u_x = \frac{\partial}{\partial x}$

$u_{xx} = \frac{\partial}{\partial x}\left(\right.$

differentiate partially (1) wrt $y$

$$xu_{xy} + u_x(0) + 1\cdot u_y + y\cdot u_{yy} = nu_y \quad \text{—(3)} \quad \times y$$

(2) $\times x$ $\quad x^2u_{xx} + 1xu_x + xyu_{xy} = xnu_x$

(3) $\times y$ $\quad xyu_{xy} + yu_y + y^2u_{yy} = ynu_y$

add

$$x^2u_{xx} + y^2u_{yy} + 2xyu_{xy} + [xu_x + yu_y] = n[xu_x + yu_y]$$

$$\therefore x^2u_{xx} + y^2u_{yy} + 2xyu_{xy} = n^2u - nu$$

(like squaring $(x+y)^2$) $\quad \therefore x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = n[n-1]u$

(for third derivative $(x+y)^3$) $\quad x^3u_{xxx} + y^3u_{yyy} + \ldots\ldots\ldots = n[n-1][n-2]u$

1. If $u = \dfrac{y^{1/3} - x^{1/3}}{x+y}$ then find the value of $x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy}$

$n = -2/3$

$x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = n[n-1] u = -\frac{2}{3}\left[-\frac{2}{3} - 1\right] u$

$= \frac{10}{9} u$

2. If $u = \tan^{-1}\left(\dfrac{x^4 + y^4}{xy}\right)$, Find $x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy}$

$Z = \tan u = \dfrac{x^4 + y^4}{x(y)} = \lambda^2 Z$ $\quad n = 2$

By Euler's theorem then for $Z$ we have

$x Z_x + y Z_y = 2Z$

$x \sec^2 u \, u_x + y \sec^2 u \, u_y = 2 \tan u$

$x u_x + y u_y = \sin 2u$ — (1)

diff (1) partially w.r.t $x$

$x u_{xx} + u_x + y u_{xy} = 2 \cos 2u \, u_x$

$xx$: $x^2 u_{xx} + x u_x + xy u_{xy} = 2\cos 2u \, x u_x$

diff (1) partially wrt $y$

$x u_{xy} + u_y + y u_{yy} = 2 \cos 2u \, u_y$

$xy$ $\quad xy u_{xy} + y u_y + y^2 u_{yy} = 2\cos 2u \, y u_y$

$[x u_x + y u_y] + x^2 u_{xx} + y^2 u_{yy} + 2xy u_{xy} = 2\cos 2u [x u_x + y u_y]$

$(\sin 2u) + a = 2 \cos 2u [\sin 2u]$

~~$a = 2\cos 2u [nu] - nu$~~

~~$a = nu[2\cos 2u - 1]$~~

~~$a = 2$~~

$a = 2\cos 2u \sin 2u - \sin 2u$

$= \sin 4u - \sin 2u$

q. When is Euler's theorem applicable ?

(given a homogeneous fn. of 2 or more variables) of degree n

q. Write Euler's theorem for a fn. of 2 variables of degree n

$$x u_x + y u_y = n u$$

q. What is the extension of Euler's theorem of 3 variables?

$$(x+y+z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$$

extension of Euler's theorem for $f(x, y, z)$

Euler's theorem: $x u_x + y u_y + z u_z = n(u)$

extension: $x^2 u_{xx} + y^2 u_{yy} + z^2 u_{zz} + 2xy\, u_{xy} + 2yz\, u_{yz} + 2zx\, u_{zx} = n(n-1)u$

# Given $u = f^{-1}(z)$ is a non-homogeneous function

But $z = f(u)$ is a homogeneous function of degree n

Euler's theorem for 'z' gives:

$$x z_x + y z_y = nz$$

$$x \cdot f'(u) u_x + y f'(u) u_y = n f(u)$$

$$x u_x + y u_y = n \frac{f(u)}{f'(u)}$$

q. $z = \tan^{-1}\left(\frac{x^4+y^4}{xy}\right) \rightarrow$ non-homogeneous

$u = \tan z \rightarrow$ homogeneous of degree $n = 2$

$$x z_x + y z_y = \frac{2f(z)}{f'(z)} = \frac{2\tan z}{\sec^2 z}$$

$= \sin 2z$

General result for non-homogeneous functions:

Given $u = f^{-1}(z)$ is non-homogeneous function

But $z = f(u)$ is a homogeneous function of degree n

Then $$\boxed{x u_x + y u_y = \frac{n f(u)}{f'(u)}} \; g(u)$$

degree of homogeneous fn.

diff eqn ① partially wrt $x$ & $y$ respectively

$$x u_{xx} + u_x + y u_{xy} = g'(u) u_x \quad \times x$$
$$x^2 u_{xx} + x u_x + xy u_{xy} = x g'(u) u_x$$

$$x u_{xy} + y u_{yy} + u_y = g'(u) u_y \quad \times y$$
$$xy u_{xy} + y^2 u_{yy} + y u_y = y g'(u) u_y$$

add $$\boxed{x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = g'(u)[g(u)] - g(u)}$$

Further: If $g(u) = \frac{n f(u)}{f'(u)}$ then:

$$x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = g(u)[g'(u) - 1]$$

q: If $u = x^3 \sin^{-1}\left(\frac{y}{x}\right) + y^3 \tan^{-1}\left(\frac{x}{y}\right)$ then evaluate

$x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy}$ by using the extension of Euler's theorem

Ans: $u(\lambda x, \lambda y) = \lambda^3 \{u(x,y)\}$ $\therefore$ u is a homogeneous function of degree $n = 3$.

$$x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = n(n-1) u$$
$$= 6u$$

28th September, 2022

q. If $u = x^3 \sin^{-1}\left(\frac{y}{x}\right) + y^{-3}\tan^{-1}\left(\frac{x}{y}\right)$, evaluate $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} + x u_x + y u_y$

Sol. Euler's Theorem can't be applied directly so

$$u = v + w \quad \text{—(1)}$$

$$v = x^3 \sin^{-1}\left(\frac{y}{x}\right) \qquad w = y^{-3}\tan^{-1}\left(\frac{x}{y}\right)$$

v is a homogeneous fn. of degree 3

$$x v_x + y v_y = 3v$$

$$x^2 v_{xx} + 2xy\, v_{xy} + y^2 v_{yy} = 6v$$

w is a homogeneous fn. of degree −3

$$x w_x + y w_y = -3w$$

$$x^2 w_{xx} + 2xy\, w_{xy} + y^2 w_{yy} = -3(-3-1)\, w = 12w$$

diff eqn. (1) partially wrt x & y respectively

$$u_x = v_x + w_x \quad ; \quad u_y = v_y + w_y$$

$$u_{xx} = v_{xx} + w_{xx} \qquad u_{yy} = v_{yy} + w_{yy}$$

$\times x^2$

$$u_{xy} = v_{xy} + w_{xy}$$

$$x v_x + y v_y = 3v$$

$$x w_x + y w_y = -3w$$

$\times y^2$

$$x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} + [x u_x + y u_y] \text{ } \} \text{ ques.}$$

$$= x^2 v_{xx} + x^2 w_{xx} + 2xy[v_{xy} + w_{xy}] + y^2[v_{yy} + w_{yy}] + [x u_x + y u_y]$$

$$= [x^2 v_{xx} + 2xy\, v_{xy} + y^2 v_{yy}] + [x^2 w_{xx} + 2xy\, w_{xy} + y^2 w_{yy}] + [x v_x + x w_x + y v_y + y w_y]$$

$$= [x^2 v_{xx} + 2xy\, v_{xy} + y^2 v_{yy}] + [x^2 w_{xx} + 2xy\, w_{xy} + y^2 w_{yy}] + [3v - 3w].$$

$= 6v + 12w + 3v - 3w$

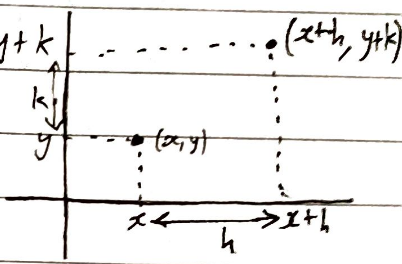
$= 9v + 9w$

$= 9u$ //

# Taylor's and Maclaurin's series expansion for a function of two variables $f(x, y)$:

$$f(x+h, y+k) = f(x,y) + \sum \frac{1}{r!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^r f(x,y)$$

neighbourhood of $x, y$




$r = 1, 2, 3, 4 \ldots\ldots$

(or) $x + h = a \qquad h = x - a$

$y + k = b \qquad k = y - b$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

29th September, 2022

acts on f

$$f(x+h, y+k) = f(x,y) + \sum_{r=1}^{\infty} \frac{1}{r!}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^r f(x,y)$$

$$= \underbrace{f(x,y)}_{\text{1st term}} + \underbrace{\frac{1}{1!}\left[h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right]}_{\text{2nd term}} + \frac{1}{2!}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial f}\right]\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right] f$$

$$= \text{"} + \text{"} + \frac{1}{2!}\left[h^2\frac{\partial^2}{\partial x^2} + 2hk\frac{\partial^2}{\partial x \partial y} + k^2\frac{\partial^2}{\partial y^2}\right] f$$

q. Write the 5th term in the taylor series expansion of $f(x, y)$, $h$ & $k$ are the increments as usual.

$$\frac{1}{4!}\left[h^4\frac{\partial^4}{\partial x^4} + 4hk^3\frac{\partial^4}{\partial x \partial y^3} + 4h^3k\frac{\partial^4}{\partial x^3 \partial y} + 6h^2k^2\frac{\partial^4}{\partial x^2 \partial y^2} + k^4\frac{\partial 4}{\partial y^4}\right]$$

$(a+b)^4 = (a^3 + b^3 + 3a^2b + 3ab^2)(a+b)$

$= a^4 + ab^3 + 3a^3b + 3a^2b^2$

$+ a^3b + b^4 + 3a^2b^2 + 3ab^3$

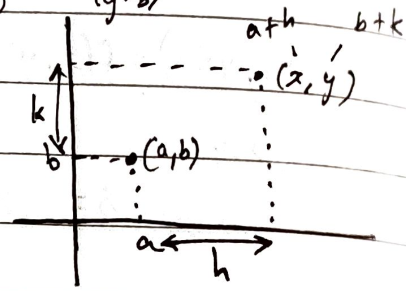
$= a^4 + 4ab^3 + 4a^3b + 6a^2b^2 + b^4$

4th term

$$\frac{1}{3!}\left[h^3\frac{\partial^3 f}{\partial x^3} + 3h^2 k\frac{\partial^3 f}{\partial x^2\partial y} + 3hk^2\frac{\partial^3 f}{\partial x\partial y^2} + k^3\frac{\partial^3 f}{\partial y^3}\right]$$

* Alternate form (Standard:)

$$f(\underset{x}{a+h}, \underset{y}{b+k}) = f(a,b) + \sum_{r=1}^{\infty}\frac{1}{r!}\left[\underset{(x-a)}{h} f_x + \underset{(y-b)}{k} f_y\right]^r (a,b)$$




$$f(x,y) = f(a,b) + \sum_{r=1}^{\infty}\frac{1}{r!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^r f(a,b)$$

* When $a=0$, $b=0$, the Taylor Series expansion is called Maclaurian's series

* Geometry of Taylor series expansion for one variable $y=f(x)$ about a point a

$$f(a+h) = f(a) + \sum_{r=1}^{\infty}\frac{1}{r!}\left(h\frac{d}{dx}\right)^r f$$

$$f(a+h) = f(a) + \frac{1}{1!}hf'(a) + \frac{1}{2!}h^2 f''(a) + \ldots$$

if $a+h = x$

$$f(x) = f(a) + \frac{1}{1!}(x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots$$

6th Oday

1. Write Taylor series for $f(x, y)$ about point $(a, b)$

2. Can you write the Taylor series for $f(x)$ about $a$? If Yes, find the Taylor series for $f(x) = e^x$ about $x = 0$

3. Find Maclaurians series for $f(x, y) = e^{xy}$

Sol. 1. $f(x,y) = f(a,b) + \sum_{r=1}^{\infty} \frac{1}{r!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^r f(a,b)$

q. Compute $\tan^{-1}\left(\frac{0.9}{1.1}\right)$ approx.

Sol. $$f(a+h, b+k) = f(a,b) + \sum_{r=1}^{\infty} \frac{1}{r!}\left\{h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right\}^{r} f(a,b)$$

$$0.9 = x = a+h \quad , \quad 1.1 = y = b+k$$

$$(a,b) = (1,1) \qquad h = -0.1 \; , \; k = 0.1$$

$$f(x,y) = \tan^{-1}\left(\frac{x}{y}\right) \; ; \qquad \text{at } (1,1) \quad f(1,1) = \tan^{-1}(1) = \pi/4$$

$$f_x = \frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\frac{1}{y} \qquad f_x(1,1) = \frac{1}{1+1} = \frac{1}{2}$$

$$= \frac{y}{x^2+y^2} = y(x^2+y^2)^{-1}$$

$$f_y = \frac{1}{1+\left(\frac{x}{y}\right)^2}\;\frac{-x}{y^2} \qquad f_y(1,1) = -\frac{1}{2}$$

$$= \frac{-x}{x^2+y^2}$$

$$f_{xx} = y\left(\frac{-1(2x)}{(x^2+y^2)^2}\right) \qquad f_{xx}(1,1) = -\frac{1}{2}$$

(y const.)

$$f_{yy} = x\left\{\frac{2y}{(x^2+y^2)^2}\right\} \qquad f_{yy}(1,1) = \frac{1}{2}$$

$$f_{xy} = \frac{1}{x^2+y^2}\cdot 1 + \frac{y(-1)2y}{(x^2+y^2)^2} \qquad f_{xy}(1,1) = \frac{1}{2} - \frac{2}{4} = 0$$

x const

$$\tan^{-1}\left(\frac{x}{y}\right) = f(1,1) = \frac{1}{1!}\left\{h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right\}(1,1) +$$

$$\frac{1}{2!}\left\{h^2 f_{xx} + 2hk\, f_{xy} + k^2 f_{yy}\right\}(1,1) + \cdots$$

$$= \frac{\pi}{4} + \frac{1}{1!}\left\{(-0.1)\frac{1}{2} - (0.1)\frac{1}{2}\right\} + \frac{1}{2!}\left\{(-0.1)^2\left(\frac{-1}{2}\right) + 2(-0.1)(0\ldots\right.$$

$$\left. + (0.1)^2\frac{1}{2}\right\} + \cdots$$

$\approx 0.6857$

$2^{3/2}$ $= \frac{1}{2^2\sqrt{2}}$ 4 $(x)^{1/2} = \frac{1}{2\sqrt{x}}$ $= \frac{-1}{2}(\ )^{-3/2}$

actual value of $\tan^{-1}\left(\frac{0.9}{1.1}\right) = 0.6857$

q: Find Taylor's expansion of $f(x,y) = \sqrt{1+x+y^2}$ in powers of $(x-1)$ and $(y-0)$

$a=1$, $b=0$ $\quad f(x,y) = f(a,b) + \sum_{r=1}^{\infty} \frac{1}{r!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^r f(a,b)$

at $(1,0)$

$f_x = \frac{1}{2\sqrt{1+x+y^2}}(1) \qquad \frac{1}{2\sqrt{2}}$

$f_y = \frac{1}{\cancel{2}\sqrt{1+x+y^2}}(\cancel{2}y) \qquad 0$

$f_{xy} = \frac{1}{\cancel{2}}(1+x+y^2)^{-3/2}(\cancel{2}y) \qquad 0$

$f_{xx} = \frac{1}{2}(1+x+y^2)^{-3/2}(1) = \frac{1}{2}(2)^{-3/2} = 2^{-5/2} = \frac{1}{2^{5/2}} = \frac{1}{4\sqrt{2}}$

$f_{yy} = \frac{1}{2}(1+x+y^2)^{-3/2}(2y) = \frac{1}{2}(\ )0 = 0 \qquad \frac{1}{4\sqrt{2}}$

$f(1,0) = \sqrt{2}$

$\Rightarrow \sqrt{2} + \frac{1}{1!}\left[(x-1)\frac{1}{2\sqrt{2}} + (y-0)(0)\right] + \frac{1}{2!}\left[(x-1)^2\frac{1}{2^2\sqrt{2}} + 0 + 0\right]$

$+ \dots$

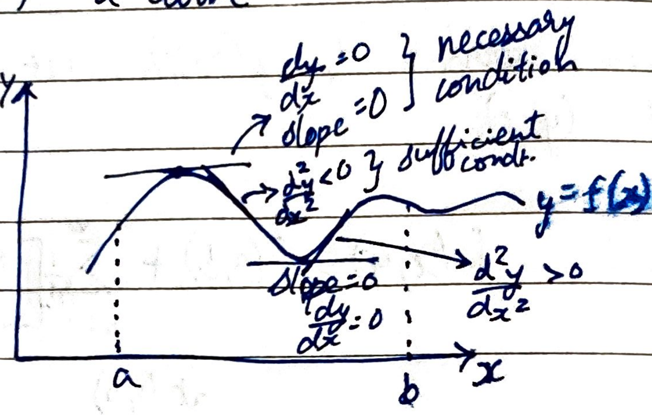
X

$= \sqrt{2}\left[1 + \frac{x-1}{4} + \frac{(x-1)^2}{16}\right.$

7th October, 2022

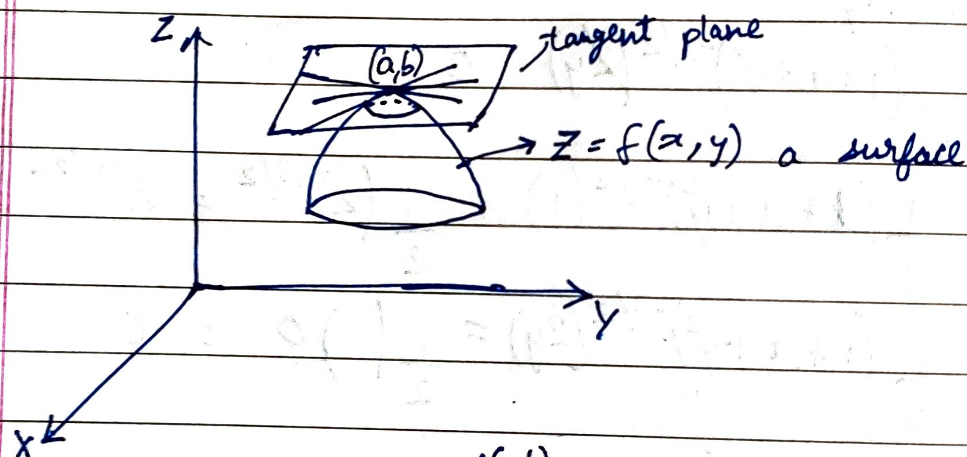
# MAXIMA AND MINIMA

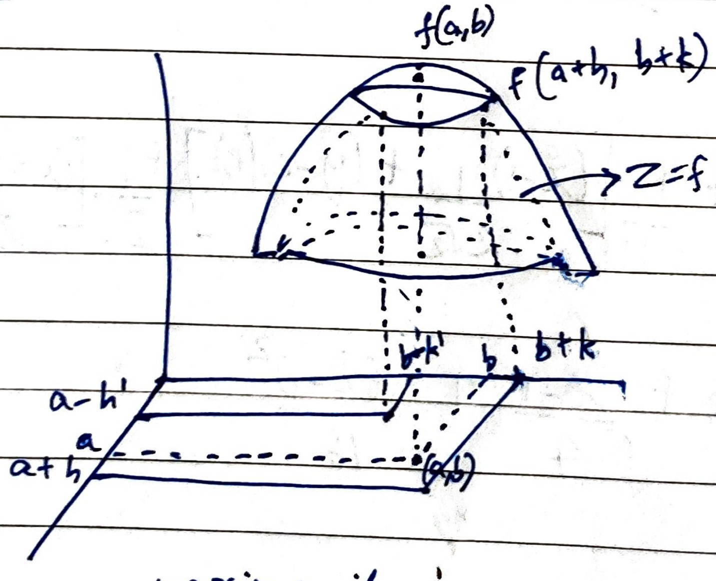
## # If $y=f(x)$ a curve




## # Maxima and Minima for a function of 2 variables

here point of maxima or minima lies on a plane // to x-y p

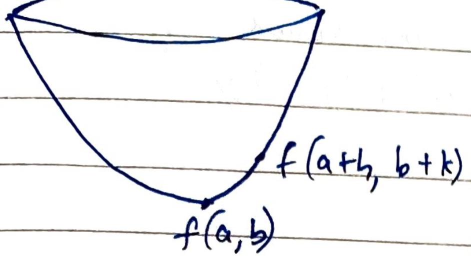



maxima if

$$f(a,b)-f(a+h,\ b+k)>0$$

Maxima point




for minima

$f(a,b) - f(a+h, b+k) < 0$

Minima point

* here $\frac{dy}{dx} = 0$, necessary condition will transform to $f_x = 0$ & $f_y$
for either maxima/minima

$\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y}$

Second Derivative Test : Suppose the second partial derivatives of f are continuous on a disk with centre (a,b) & suppose that $f_x(a,b) = 0$ & $f_y(a,b) = 0$ Let

$$D = \underbrace{f_{xx}(a,b)}_{} \underbrace{f_{yy}(a,b)}_{} - (f_{xy}(a,b))^2$$

$$= \underset{f_{xx}}{r} \quad \underset{f_{yy}}{t} \quad - \quad \underset{f_{xy}}{s^2}$$

$r = f_{xx}$
$t = f_{yy}$
$s = f_{xy}$

$D > 0$, $r < 0$ ⇒ minima
$D > 0$, $r < 0$ ⇒ maxima
$D < 0$, neither max nor min → Saddle point
$D = 0$, inconclusive

→ Necessary conditions for extremum $f_x = 0$ & $f_y = 0$
(x, y) are called critical points

→ Sufficient conditions

$$D = rt - s^2$$

$D > 0$, $r > 0$, (x, y) is minima
$D > 0$, $r < 0$, (x, y) is maxima
$D < 0$ saddle point
$D = 0$, No condition can be drawn

1. Find the maximum ... 
$$f(x,y) = x^3 - 12x + y^3 + 3y^2 - 9y$$

Sol. * i) Find critical points
solve $f_x = 0$ & $f_y = 0$

$f_x = 3x^2 - 12 = 0$
$x = +2, -2$

$f_y = 0 = 3y^2 + 6y - 9 = 0$
$y^2 + 2y - 3 = 0$
$y = 1, -3$

$-3$
$\wedge$
$+3 -1$

$(x,y)$: $(2,1)$, $(2,-3)$, $(-2,1)$, $(-2,-3)$ are critical points.

$r = f_{xx} = 6x$, $t = f_{yy} = 6y + 6$, $s = f_{xy} = 0$

$D = rt - s^2$
$D = 6x(6y+6) - 0$
$D = 36(xy + x)$

| critical points | D | r | conclusion |
|---|---|---|---|
| (2,1) | 144 > 0 | 12 > 0 | minima |
| (2,-3) | -144 < 0 | 12 > 0 | saddle |
| (-2,1) | -144 < 0 | | saddle |
| (-2,-3) | 144 > 0 | -12 < 0 | maxima |

6+2

q. Find max & min for $f(x,y) = 4x^2 + 2y^2 + 4xy - 10x - 2y - 3$

Sol. $f_x = 8x + 4y - 10 = 0$
$4x + 2y = 5$
$4x + 1 - 2x = 5$
$2x = 4$
$x = 2$

$f_y = 4y + 4x - 2 = 0$
$2x + 2y = 1$

$2y = -3$
$y = \frac{-3}{2}$

$(x,y)$ : $\left(2, \frac{-3}{2}\right)$

$r = f_{xx} = 8$ $t = f_{yy} = 4$ $s = f_{xy} = 4$

$D = rt - s^2 = 32 - 16 = \cancel{28}\ 16$

$(2, -3/2)$ $D > 0$ $r > 0$ $\Rightarrow$ minima

$\text{Minima} = 4(2)^2 + 2\left(\frac{-3}{2}\right)^2 + 4(\cancel{2})\left(\frac{-3}{\cancel{2}}\right) - 10(2) - 2\left(\frac{-3}{2}\right) - 3$

$= 16 + \frac{9}{2} - 12 - 20 + \cancel{3} - \cancel{3}$

$= 4 - 20 + 4.5$

$= -16 + 4.5 = -11.5$

10th October, 2022

q: Locate the stationary points

$f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

discuss the nature

$f_x = 3x^2 + 3y^2 - 30x + 72 = 0$

$x^2 + y^2 - 10x + 24 = 0$

$f_y = 0 = 6xy - 30y = 0$

$y(6x - 30) = 0$

$y = 0$ or $x = 5$

$y = 0$

$x^2 - 10x + 24 = 0$

$x = 4$ or $x = 6$

$24$, $-6\ -4$

$x = 5$

$25 + y^2 - 50 + 24 = 0$

$y^2 = 1$

$y = \pm 1$

critical points

$(x,y)$ : for $y = 0$ $(4,0)$ $(6,0)$

for $x = 5$ $(5,1)$ $(5,-1)$

$r = f_{xx} = 6x - 30$

$t = f_{yy} = 6x - 30$

$s = f_{xy} = 6y$

$$D = rt - s^2$$
$$= (6x - 30)^2 - (6y)^2$$

| | $D$ | $r$ | |
|---|---|---|---|
| $(4, 0)$ | $36 > 0$ | $< 0$ | Maxima |
| $(6, 0)$ | $36 > 0$ | $> 0$ | minima |
| $(5, 1)$ | $-36 < 0$ | $= 0$ | saddle point |
| $(5, -1)$ | $-36 < 0$ | $= 0$ | saddle point |

# Method of Lagrange Multipliers

Working rule: Given $f(x, y, z)$ a function to be maximised or minimised subject to a constraint $\phi(x, y, z) = 0$

Step 1: Construct $F(x, y, z) = f(x, y, z) + \lambda \phi(x, y, z)$

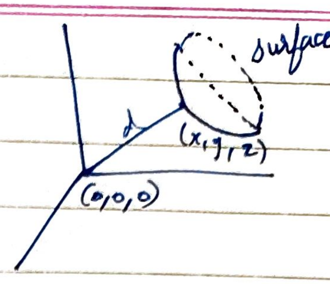
Step 2: Solve $F_x = 0$, $F_y = 0$ and $F_z = 0$ along with $\phi(x, y, z) = 0$ to get critical values

Step 3: Extreme values of the function depends on the nature of the problem

Disadvantage: Points of maxima/minima cannot be found specifically.

eg 1. Find the shortest distance from origin to the surface $xy(z)^2 = 2$

Sol. distance function : $d(x,y,z)$




$$d(x,y,z) = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2}$$

is to be minimised

we minimise $d^2$ instead (coz easier)

$$f(x,y,z) = d^2 = x^2 + y^2 + z^2$$

subject to the condition that the $(x,y,z)$ on $xyz^2 - 2 = 0$

$$\phi(x,y,z) = xyz^2 - 2 = 0$$

$\cancel{\phi_x = yz^2 = 0}$
$\cancel{\phi_y = xz^2 = 0}$
$\cancel{\phi_z = 2xyz = 0}$

let $F(x,y,z) = D + \lambda\phi \quad , \lambda \neq 0$

$$F = (x^2 + y^2 + z^2) + \lambda(xyz^2 - 2)$$

solve $F_x = 0,\ F_y = 0,\ F_z = 0$ & $\phi = 0$

$F_x = 2x + yz^2\lambda = 0 \Rightarrow \lambda = \dfrac{-2x}{yz^2}$ — ①

$F_y = 2y + xz^2\lambda = 0 \Rightarrow \lambda = \dfrac{-2y}{xz^2}$ — ②

$F_z = 2z + 2xyz\lambda = 0 \Rightarrow z(1 + xy\lambda) = 0$

$\Rightarrow z = 0 \ , x = \pm y$

$z = 0 \ , \lambda = -\dfrac{1}{xy}$ — ③

From ① & ②

$$\frac{-\cancel{2}x}{yz^2} = \frac{-\cancel{2}y}{xz^2}$$

$$x^2z^2 - y^2z^2 = 0$$

$$z^2(x^2 - y^2) = 0$$

$$z = 0 \ , \boxed{x = \pm y}$$

From ① & ③

$$\frac{-2y}{\cancel{y}z^2} = \frac{-1}{xy}$$

$$2xy^2 = yz^2$$

$$y(2x^2 - z^2) = 0$$

$$2x^2 = z^2$$

$$z = \pm x\sqrt{2}$$

now constraint $xyz^2 - 2 = 0$ — ④

$x = 0$ $\quad\quad$ $x = y$ $\quad\quad$ $x = -y$

$xyz^2 = 0$ gives $\quad\quad$ $y^2z^2 - 2 = 0$

$-2 = 0$

absurd result

$\Rightarrow$ $z$ cannot be 0

neither of $x, y, z$ can be 0

From ② & ③

$$\frac{-2y}{xz^2} = -\frac{1}{xy}$$

$$2xy^2 = xz^2$$

$$x[2y^2 - z^2] = 0 \quad\quad 2y^2 = z^2$$

$$\boxed{z = \pm\sqrt{2}y}$$

$$xyz^2 = 2$$

$$yy\cancel{2}y^2 = \cancel{2}$$

$$y^4 = 1$$

$$y = \pm 1$$

$$\Rightarrow x = \pm 1 \quad \& \quad z = \pm\sqrt{2}$$

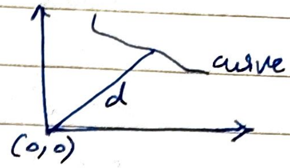
$$\min d^2 = 1 + 1 + 2 = 4$$

$$d = 2$$

11th October, 2022

H/W q: Find the maximum and minimum distances of the point $(3, 4, 12)$ from the unit sphere with centre at origin

q. Find the max. & min. distances from the origin to the curve $5x^2+6xy+5y^2-8=0$




Sol. $D = d^2 = x^2+y^2$ is to be minimised / maximised subject to the condition that $(x,y)$ lies on the given curve

$$\phi = 5x^2+6xy+5y^2-8=0$$

$$F = D + \lambda\phi$$

$$F = x^2+y^2 + \lambda(5x^2+6xy+5y^2-8)$$

$$F_x = 2x + \lambda(10x+6y) = 0$$

$$F_y = 2y + \lambda(10y+6x) = 0$$

$$\phi = 0$$

$$\lambda = \frac{-2x}{10x+6y}$$

$$\lambda = \frac{-2y}{10y+6x}$$

$$\Rightarrow \frac{\cancel{2}x}{5x+3y} = \frac{\cancel{2}y}{5y+3x}$$

$$\cancel{5xy} + 3x^2 = \cancel{5xy} + 3y^2$$

$$3x^2 = 3y^2$$

$$x = \pm y$$

$\phi = 0$

for $x = y$

$$5x^2+6x^2+5x^2-8=0$$

$$16x^2 = 8$$

$$x = \pm\frac{1}{\sqrt{2}}$$

$x = -y$

$$5x^2-6x^2+5x^2-8=0$$

$$4x^2 = 8$$

$$x = \pm\sqrt{2}$$

Critical points : $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$ ; $(\sqrt{2}, \sqrt{2}), (-\sqrt{2}, -\sqrt{2})$

$$d = \sqrt{x^2+y^2}$$

$$= \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{\pm 1}{\sqrt{2}}\right)^2} = 1$$

$$d = \sqrt{x^2+y^2}$$

$$= \sqrt{(\sqrt{2})^2 + (\pm\sqrt{2})^2} = 2$$

$d_{\text{minima}} = 1$

$d_{\text{maxima}} = 2$

q: Find the extreme values of $a^3x^2 + b^3y^2 + c^3z^2$ subject to the condition $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1.0$

$(a, b, c > 0)$

$D = a^3x^2 + b^3y^2 + c^3z^2$

$\phi = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1.0$

$F = D + \lambda\phi$

$= a^3x^2 + b^3y^2 + c^3z^2 + \lambda\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1\right)$

$F_x = 0 \Rightarrow 2a^3x + \lambda\left(\frac{-1}{x^2}\right) = 0$ $\qquad \frac{\lambda}{x^2} = 2a^3x$

$\lambda = 2a^3x^3$

$F_y = 0 \Rightarrow 2b^3y + \lambda\left(\frac{-1}{y^2}\right) = 0$ $\qquad \lambda = 2b^3y^3$

$F_z = 0 \Rightarrow 2c^3z + \lambda\left(\frac{-1}{z^2}\right) = 0$ $\qquad \lambda = 2c^3z^3$

$\cancel{2}a^3x^3 = \cancel{2}b^3y^3, \quad b^3y^3 = c^3z^3, \quad a^3z^3 = c^3z^3$

$x = \frac{by}{a} \qquad y = \frac{cz}{b} \qquad x = \frac{cz}{a}, \quad z = \frac{ax}{c}$

$= \frac{a}{c}\frac{by}{a}$

$\cancel{D = a^3\frac{b^2y^2}{a^2} + b^3\frac{c^2z^2}{b^2} + c^3\frac{x^2a^2}{c^2}}$

$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$

$\frac{a}{by} + \frac{1}{y} + \frac{c}{by} = 1$

$\frac{a+b+c}{by} = 1$

$z = \frac{by}{c} = \frac{b}{c}\left(\frac{a+b+c}{b}\right) = \frac{a+b+c}{c}$

$\text{ly } x = \frac{a+b+c}{a}$

$(x, y, z) = \left\{\left(\frac{a+b+c}{a}\right), \left(\frac{a+b+c}{b}\right), \left(\frac{a+b+c}{c}\right)\right\}$

Extreme value of $a^3x^2+b^3y^2+c^3z^2$ is

$$a^3\left[\frac{(a+b+c)^2}{a^2}\right]+b^3\left[\frac{(a+b+c)^2}{b^2}\right]+c^3\left[\frac{(a+b+c)^2}{c^2}\right]$$

$$=(a+b+c)^2(a+b+c)$$

$$=(a+b+c)^3$$

H/W q. Find the extreme value of $x^3+8y^3+64z^3$ when $xyz=1$