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Unit - 1

Integral Calculus

Double integral

Consider a function of two variables $z = f(x, y)$

An integral $\iint_R f(x, y)\, dx\, dy$

determined by two integrals, one wrt x and the other wrt y is called double integral.

Note : 1. A double integral can be evaluated by successive single integrations.

2. Double integral gives the volume under the surface $z = f(x, y)$ and above the xy plane in the region of integration R.

3. If $f(x, y) = 1$, then $\iint_R dx\, dy$ = Area

co ordinates of x-axis

curve

$$\int_a^b f(x)\, dx = \text{area}$$

surface

$$\iint_R f(x, y)\, dx\, dy = \text{volume}$$

xy plane

→ Evaluation of double integral

* The method of evaluating the double integrals depend upon the nature of curves

bounding the region R

* Double integral of a function $f(x,y)$ over the region R can be evaluated by two successive integrations.

Two different types to evaluate a double integral.

Type 1. $\iint_R f(x,y)\,dx\,dy = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$

Here the order of integration is immaterial, provided the limits of integration are changed accordingly.
i.e when all the limits are constants the order of dx and dy determine the limit of the variable.

Note : 1. If all four limits are constants, then the function $f(x,y)$ can be integrated w.r.t any variable first.

2. If $f(x,y)$ has discontinuous within or on the boundary of the region of integration does not result into same integrals.

eg: $\int_0^1 \int_0^1 \frac{x-y}{(x+y)^3}\,dy\,dx \neq \int_0^1 \int_0^1 \frac{x-y}{(x+y)^3}\,dx\,dy$

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Type 2. i) $\iint_R f(x,y)\,dx\,dy = \int_a^b \left[\int_{y_1(x)}^{y_2(x)} f(x,y)\,dy \right] dx$

when variable limits are involved, we have to first integrate wrt the variable having variable limits and then wrt the variable having constant limits

ii) $\iint_R f(x,y)\,dx\,dy = \int_c^d \left[\int_{x_1(y)}^{x_2(y)} f(x,y)\,dx \right] dy$

eg: $\int_{y=1}^{y=2} \left[\int_{x=0}^{x=1-y} x^2 y^2\,dx \right] dy$, $\int_{x=1}^{x=2} \left[\int_{y=0}^{y=1-x} xy\,dy \right] dx$

$\int_0^1 \int_0^{1-y^2} xy\,dy\,dx = \int_0^1 \int_0^{1-y^2} xy\,dx\,dy$

c/w

1. Evaluate: $\int_0^1 \int_0^3 x^3 y^3\,dx\,dy$

Sol. $I = \int_0^1 \left[\int_0^3 x^3 y^3\,dx \right] dy$

$= \int_0^1 y^3 \left(\frac{x^4}{4} \right)_0^3 dy$

$= \frac{81}{4} \int_0^1 y^3\,dy = \frac{81}{4} \left(\frac{y^4}{4} \right)_0^1 = \frac{81}{4}\left(\frac{1}{4}\right) = \frac{81}{16}$

(OR) $I = \int_0^3 \left[\int_0^1 x^3 y^3\,dy \right] dx = \int_0^3 x^3 \left[\frac{y^4}{4} \right]_0^1 dx = \frac{1}{4} \left(\frac{x^4}{4} \right)_0^3$

$= \frac{81}{16}$

(OR) $I = \int_0^1 y^3\, dy \cdot \int_0^3 x^3\, dx$ (only when all limits are constants)

$$= \left(\frac{y^4}{4}\right)_0^1 \left(\frac{x^4}{4}\right)_0^3 = \frac{81}{16}$$

2. Evaluate $\int_0^1 \int_0^1 \frac{dx\, dy}{\sqrt{(1-x^2)}\sqrt{(1-y^2)}}$

$$\cancel{\int_0^1 \sin^{-1}\left(\frac{x}{1}\right)}$$

Sol. $I = \int_0^1 \frac{dx}{\sqrt{1-x^2}} \cdot \int_0^1 \frac{dy}{\sqrt{1-y^2}}$

$$= \left(\sin^{-1}(x)\right)_0^1 \cdot \left(\sin^{-1}(y)\right)_0^1$$

$$= \frac{\pi}{2} \cdot \frac{\pi}{2} = \frac{\pi^2}{4}$$

3. Evaluate $\int_1^4 \int_0^{\sqrt{4-x}} xy\, dx\, dy$

Sol. $I = \int_1^4 \int_0^{\sqrt{4-x}} x y\, dx\, dy$

$$= \int_1^4 \left(\int_0^{\sqrt{4-x}} x y\, dy \right) dx$$

$$= \int_1^4 x \left(\frac{y^2}{2}\right)_0^{\sqrt{4-x}} dx$$

$$= \frac{1}{2}\int_{1}^{4} x \cdot \left(\sqrt{4-x}\right)^2 dx$$

$$= \frac{1}{2}\int_{1}^{4} x(4-x)\, dx$$

$$= \frac{1}{2}\int_{1}^{4} (4x - x^2)\, dx$$

$$= \frac{1}{2}\left[\cancel{4}^{2}\cdot\frac{x^2}{\cancel{2}} - \frac{x^3}{3}\right]_1^4$$

$$= \frac{1}{2}\left[\left(32 - \frac{16}{3}\right) - \left(2 - \frac{1}{3}\right)\right]$$

q. Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} x^2 y \, dx\, dy$

Sol. $\int_0^a \left[\int_0^{\sqrt{a^2-x^2}} x^2 y \, dy\right] dx$

$$\int_0^a x^2 \left(\frac{y^2}{2}\right)_0^{\sqrt{a^2-x^2}} dx$$

$$\int_0^a x^2 \left(\frac{a^2-x^2}{2}\right) dx$$

$$\frac{1}{2}\int_0^a a^2x^2 - x^4 \, dx = \frac{1}{2}\left[a^2\frac{x^3}{3} - \frac{x^5}{5}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{a^5}{3} - \frac{a^5}{5}\right] = \frac{2a^5}{30} = \frac{a^5}{15}$$

q. Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$

$$\frac{1}{y^2+(1+x^2)}$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

Sol. $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy}{(\sqrt{1+x^2})^2+y^2}\,dx$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \left(\tan^{-1}\left(\frac{y}{\sqrt{1+x^2}}\right)\right)_0^{\sqrt{1+x^2}} dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \left[\tan^{-1}(1) - \tan^{-1}(0)\right] dx$$

$$= \frac{\pi}{4} \int_0^1 \frac{1}{\sqrt{1+x^2}}\,dx$$

H/W q. $\int_1^2 \int_3^4 (xy + e^y)\,dy\,dx$ ans: $\frac{21}{4} + e^4 - e^3$

q. $\int_{\frac{\pi}{2}}^{\pi} \int_0^{x^2} \frac{1}{x} \cos\left(\frac{y}{x}\right) dx\,dy$ ans: 1

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Evaluation of double integrals over a given region R

* Evaluation of $\iint_R f(x,y)\,\underset{\text{horizontal strip}}{dx}\,dy$ or $\iint_R f(x,y)\,dy\,dx$
* Always :
  over the specific region R
* Always inner integral → variable limit

1. inner limit or variable limit

If the inner integral is wrt x (dx) then consider the horizontal strip from left to right and from curve to curve.

If the inner integral is wrt y (dy) then consider the vertical strip and from curve to curve. from bottom to top

2. outer limit or constant limit

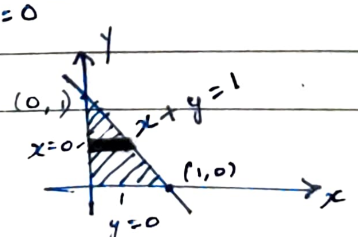
To find the constant limit for either x or y consider the coordinate axes (x axis limit for x and y axis limit for y) that forms the boundary for the region from lowest point to highest point (starting point to ending point).

Q. Evaluate $\iint_R xy\,dx\,dy$ where R is the region bounded by the coordinate axes and the line $x+y=1$ ans: $\frac{1}{24}$

axes ↓ $y=0;\ x=0$

$x+y=1$ ↓

x	0	1
y	1	0



$$I = \int_{y=0}^{y=1} \int_{x=0}^{x=1-y} x\, y\, dx\, dy$$

$$= \int_0^1 y \left(\frac{x^2}{2}\right)_0^{1-y} dy$$

$$= \int_0^1 y \frac{(1-y)^2}{2}\, dy$$ // calculator can be used

$$= \frac{1}{2} \int_0^1 y(1+y^2-2y)\, dy$$

$$= \frac{1}{2} \int_0^1 y + y^3 - 2y^2\, dy$$

$$= \frac{1}{2} \left[\frac{y^2}{2} + \frac{y^4}{4} - 2\frac{y^3}{3}\right]_0^1$$

$$= \frac{1}{2} \left[\frac{1}{2} + \frac{1}{4} - \frac{2}{3}\right]$$

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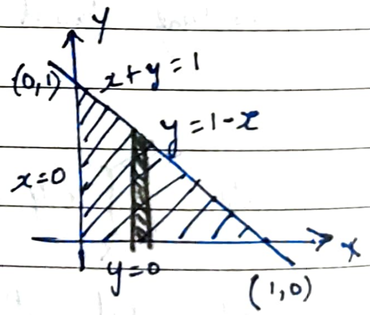
$$= \frac{1}{2}\left[\frac{3}{4} - \frac{2}{3}\right] = \frac{1}{2}\left[\frac{1}{12}\right] = \frac{1}{24}$$

Q. Evaluate $\iint_R xy\, dy\, dx$ where R is the region bounded by coordinate axes & line $x+y=1$

Sol. $$I = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} x\, y\, dy\, dx$$



$$= \int_0^1 \int_0^{1-x} x\, y\, dy\, dx$$

$$= \frac{1}{24}$$

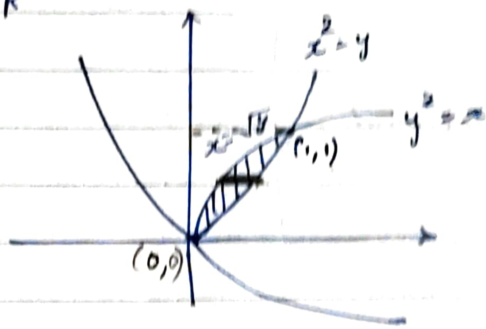
$x^2 = 4ay \quad y^2 = 4a$

POI $= (4a, 4a)$

Note: $\iint_R f(x,y)\,dx\,dy = \iint_R f(x,y)\,dy\,dx$

q. If R is the region bounded by the parabolas $y^2 = x$ and $x^2 = y$, then show that $\iint_R xy(x+y)\,dx\,dy$

$\iint_R (x^2y + xy^2)\,dx\,dy$



$$\int_0^1 \int_{x=y^2}^{x=\sqrt{y}} (x^2y + xy^2)\,dx\,dy$$ // lower limits from L to R

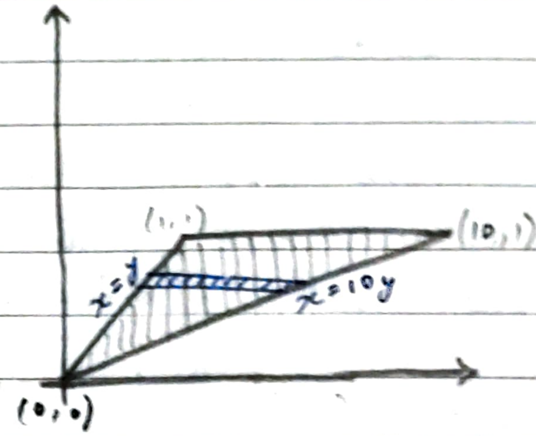
$$= \int_0^1 \left(y\frac{x^3}{3} + y^2\frac{x^2}{2} \right)_{y^2}^{\sqrt{y}} dy$$

$$= \int_0^1 \left(y\frac{y^{3/2}}{3} + \frac{y^2}{2}\cdot y \right) - \left(y\cdot\frac{y^6}{3} + \frac{y^2 y^4}{2} \right) dy$$

$$= \int_0^1 \frac{y^{5/2}}{3} + \frac{y^3}{2} - \left(\frac{y^7}{3}\right) - \frac{y^6}{2}\,dy$$

$$= \frac{3}{28}$$

q. Evaluate $\iint_R \sqrt{xy - y^2}\,dx\,dy$ over the triangle with vertices $(0,0)$, $(10,1)$, $(1,1)$



$\overset{x_1\ y_1}{(0,0)}$ & $\overset{x_2\ y_2}{(10,1)}$

eqn. of line $\dfrac{x-x_1}{x_2-x_1} = \dfrac{y-y_1}{y_2-y_1}$

$x = 10y$

$$I = \int_0^1 \int_y^{10y} \sqrt{xy - y^2}\, dx\, dy$$

$$= \int_0^1 \int_y^{10y} (xy - y^2)^{1/2}\, dx\, dy$$

$$\int_0^1 \left[\frac{(xy - y^2)^{3/2}}{\frac{3}{2} y} \right]_y^{10y} dy$$

$$\frac{2}{3} \int_0^1 \left[\frac{(10y^2 - y^2)^{3/2}}{y} - 0 \right] dy$$

$$\frac{2}{3} \int_0^1 \frac{(9y^2)^{3/2}}{y}\, dy$$

$$\frac{2}{3} \times 27 \int_0^1 \frac{y^3}{y}\, dy = 18 \int_0^1 y^2\, dy$$

$$= 18 \left(\frac{y^3}{3} \right)_0^1$$

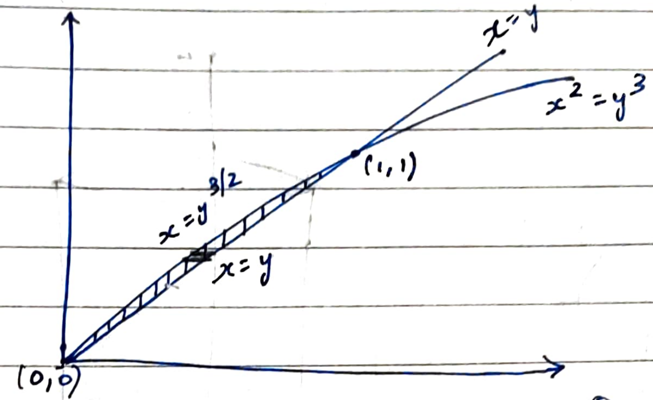
$$= 6$$

$(xy - y^2)^{1/2}$

$(y(x-y))^{1/2}$

$\int x^n \quad \dfrac{x^{n+1}}{n+1}$

Q. Find the area bounded by the curve $x^2 = y^3$ and $x = y$ using double integration



x	0	1	3	6
$y=x^{2/3}$	0	1	2.08	3.3

$$\iint_R f(x,y)\,dx\,dy = \text{volume}$$

$$\iint_R dx\,dy = \text{area}$$

$$\text{Area} = \iint_R dx\,dy$$

$$= \int_0^1 \int_{x=y^{3/2}}^{x=y} dx\,dy = \int_0^1 (x)_{y^{3/2}}^{y}\,dy$$

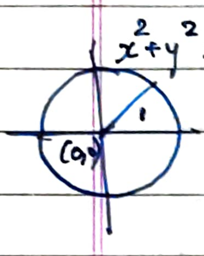
$$= \int_0^1 \left(y - y^{3/2}\right) dy$$

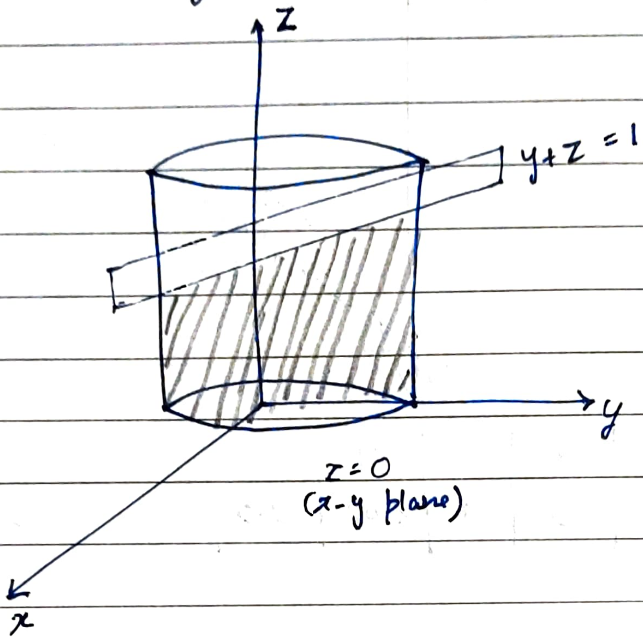
$$= \cancel{\left[\frac{2y^{5/2}}{5} - \frac{y^2}{2}\right]_0^1}$$

$$= \frac{1}{10} \text{ sq. units}$$

(OR) $$\text{Area} = \int_0^1 \int_{y=x}^{y=x^{2/3}} dy\,dx$$

Q. Find volume of solid which is bounded by cylinder $x^2+y^2=1$ and the planes $y+z=1$ and $z=0$ using double integration





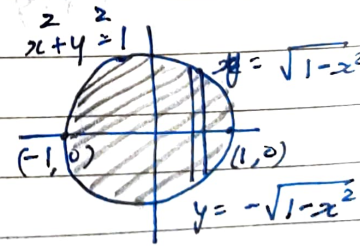
Volume $= \iint_R f(x,y)\,dy\,dx$

basically gives volume below surface, above ~~axes~~ plane.

$z=f(x,y)$

$z=1-y$

$$\int_{x=-1}^{x=1}\int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} (1-y)\,dy\,dx$$



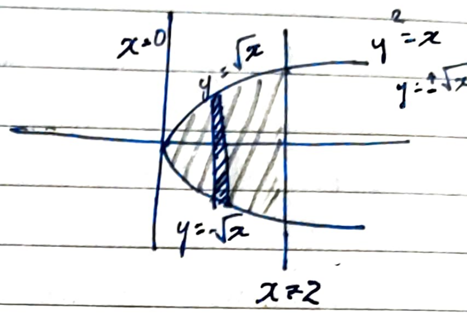
$$= \int_{-1}^{1}\left(y-\frac{y^2}{2}\right)_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx$$

$\sqrt{1-x^2}$ -

$=$ 3.14

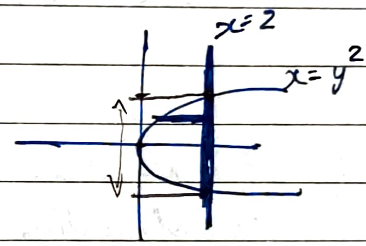
Q. Find the volume of solid which is below the plane $z = 2x + 3$ and above the x-y plane and bounded by $y^2 = x$, $x = 0$ and $x = 2$ using double integration.

$$V = \iint_R f(x, y)\, dy\, dx$$



$$= \int_{x=0}^{x=2} \int_{y=-\sqrt{x}}^{y=\sqrt{x}} (2x + 3)\, dy\, dx$$

$$\int_{y=-\sqrt{2}}^{y=\sqrt{2}} \int_{x=y^2}^{x=2} (2x+3)\, dx\, dy$$



$V = 20.36$

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Change of Variables in double integration : Cartesian to Polar coordinates

→ Jacobian

Let x and y be functions of two independent variables u and v. Then the Jacobian of x, y wrt u, v denoted by $J\left(\frac{x, y}{u, v}\right)$ or $\frac{\partial(x, y)}{\partial(u, v)}$

is a second order functional determinant defined by :

$$J\left(\frac{x, y}{u, v}\right) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

→ Change of variables from cartesian coordinates to

$\iint$ 1. Polar coordinates : $x = r\cos\theta$, $y = r\sin\theta$

$\iiint$ 2. Cylindrical coordinates : $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

3. Spherical coordinates : $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$
$z = r\cos\theta$

Q. If $x = r\cos\theta$, $y = r\sin\theta$ find the jacobian of transformation from cartesian to polar

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$J\left(\frac{x, y}{r, \theta}\right) = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

q. If $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ find jacobian of transformation from cartesian to polar.

$$J\left(\frac{x, y, z}{r, \theta, z}\right) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

H/w q. If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ find J from cartesian to spherical co ordinates

ans: $J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \quad J = r^2\sin\theta$

Transforming double integrals into polars

* For a double integration in cartesian coordinates (x, y) the change of variables to polar coordinates (r, θ) can be done through transformation $x = r\cos\theta$, $y = r\sin\theta$ as follows

$$\iint_R f(x,y)\,dx\,dy = \iint_{R_1} \phi(r,\theta)\,|J|\,dr\,d\theta \quad -①$$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = r$$

$$① \Rightarrow \iint_R f(x,y)\,dx\,dy = \iint_{R_1} \phi(r,\theta)\, r\, dr\, d\theta$$

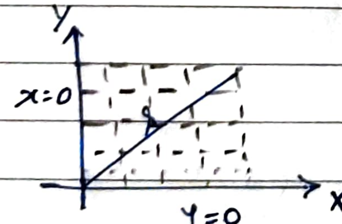
Q. change into Polar coordinates and evaluate

$$\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dy\,dx$$

Sol. $$I = \int_{x=0}^{x=\infty}\int_{y=0}^{y=\infty} e^{-(x^2+y^2)}\,dy\,dx$$

Step 1: Draw graph

The region is bounded by the curves $y=0;\ y=\infty$ & $x=0;\ x=\infty$



$\therefore$ Region of integration is I quadrant

$\therefore \theta \to 0$ to $\pi/2$

$r \to 0$ to ∞

By changing into polar coordinates we have

$x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$

$$\therefore x^2+y^2 = r^2$$

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$$I = \int_{\theta=0}^{\theta=\pi/2}\left(\int_{r=0}^{r=\infty} e^{-r^2}\, r\, dr\right) d\theta$$

let $r^2 = t$ $\quad 2r\,dr = dt$

$r=0 \quad t=0$
$r=\infty \quad t=\infty$

$$= \int_0^{\pi/2}\left(\frac{1}{2}\int_0^{\infty} e^{-t}\, dt\right) d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \left(-e^{-t}\right)_0^{\infty} d\theta$$

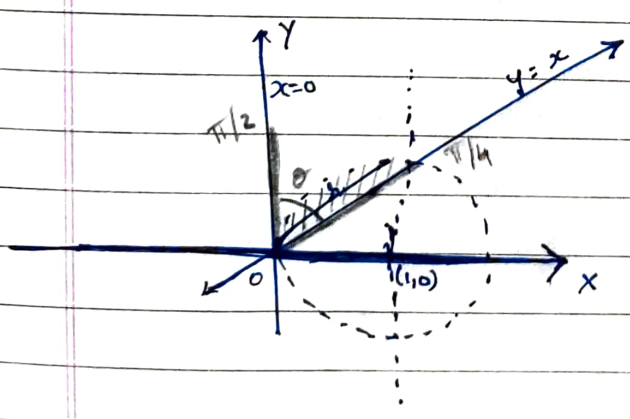
$$= \frac{1}{2}\int_0^{\pi/2} (-0-(-1))\, d\theta$$

$$= \frac{1}{2}(\theta)_0^{\pi/2} = \frac{1}{2}\left(\frac{\pi}{2}-0\right) = \frac{\pi}{4}$$

Q. Compute $\int_0^1\int_x^{\sqrt{2x-x^2}} (x^2+y^2)\, dy\, dx$ by changing into Polar co ordinates

Sol. $$I = \int_{x=0}^{x=1}\int_{y=x}^{y=\sqrt{2x-x^2}} (x^2+y^2)\, dy\, dx$$

The region is bounded by curves $y=x$; $y=\sqrt{2x-x^2}$
$x=1$; $x=1$



$y=\sqrt{2x-x^2}$

$x^2+y^2 = 2x$

$x^2-2x+y^2+1-1=0$

$(x-1)^2+(y-0)^2=1$

$C=(1,0)$ $r=1$

By changing into polar co ordinates $x = r\cos\theta$, $y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$$x^2+y^2=2x$$
$$r^2=2r\cos\theta \qquad x=r\cos\theta \quad y=r\sin\theta$$
$$r=2\cos\theta$$

$r \to 0$ to $2\cos\theta$

$\theta \to \frac{\pi}{4}$ to $\pi/2$

$$I=\int_{\pi/4}^{\pi/2}\int_{0}^{2\cos\theta} r^2 \cdot r \, dr \, d\theta$$
$$=\int_{\pi/4}^{\pi/2}\left(\frac{r^4}{4}\right)_0^{2\cos\theta} d\theta$$
$$=\frac{1}{4}\int_{\pi/4}^{\pi/2} 2^4\cos^4\theta \, d\theta$$
$$=4\int_{\pi/4}^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$
$$=\int_{\pi/4}^{\pi/2}\left(1+\cos^2 2\theta+2\cos 2\theta\right) d\theta$$
$$=\int_{\pi/4}^{\pi/2}\left(1+\frac{1+\cos 4\theta}{2}+2\cos 2\theta\right) d\theta$$

$$=\frac{3\pi}{8}-1$$

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q. Find the area inside the circle $r = 2a\cos\theta$ and outside the circle $r = a$.

Sol. $r = 2a\cos\theta$

$r^2 = 2ar\cos\theta$

$x^2 + y^2 = 2ax$

$\underbrace{x^2 - 2ax + a^2}_{} + y^2 - a^2 = 0$

$(x-a)^2 + (y-0)^2 = a^2$

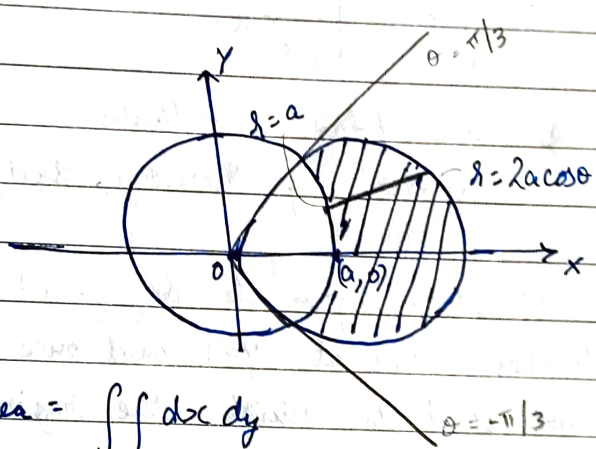
$C(a,0)$; $R = a$

$r = a$

$r^2 = a^2$

$x^2 + y^2 = a^2$

$C(0,0)$; $R = a$



$r = 2a\cos\theta$; $r = a$

$2a\cos\theta = a$

$\theta = \frac{\pi}{3}$

$$\text{Area} = \iint_R dx\,dy$$

$$= \int_{-\pi/3}^{\pi/3}\int_{a}^{2a\cos\theta} r\,dr\,d\theta$$

$$= \int_{-\pi/3}^{\pi/3}\left(\frac{r^2}{2}\right)_a^{2a\cos\theta} d\theta$$

$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3}(4a^2\cos^2\theta - a^2)\,d\theta$$

$$= \frac{a^2}{2}\int_{-\pi/3}^{\pi/3}(4\cos^2\theta - 1)\,d\theta$$

$$= \frac{a^2}{2}(3.83) = 1.91\,a^2 \text{ sq. units}$$

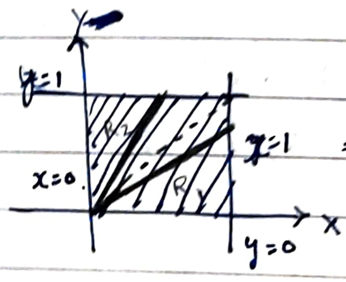
When all limits are constants region = rectangle

q: Transform the integral $\int_0^1 dx \int_0^1 f(x,y)\,dy$ into polar coordinates

$$I = \int_0^1\int_0^1 f(x,y)\,dx\,dy$$

The region is bounded by the curves $x=0,\ x=1$
$y=0,\ y=1$

$y=1$
$r\sin\theta = 1$
$r = \operatorname{cosec}\theta$

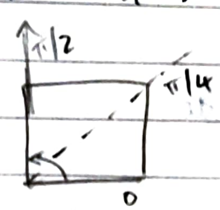


$\Rightarrow r\cos\theta = 1$
$r = \sec\theta$

By changing into polar coordinates
$x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$

$r \to$ dist. from origin to any point on the curve
here r touches once at $y=1$ and once at $x=1$
so we need to divide the region in 2 parts

In R_1 : $r \to 0$ to $\sec\theta$, $\theta \to 0$ to $\pi/4$
In R_2 $r \to 0$ to $\operatorname{cosec}\theta$, $\theta \to \pi/4$ to $\pi/2$



$$I = \int_0^{\pi/4}\int_0^{\sec\theta} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta + \int_{\pi/4}^{\pi/2}\int_0^{\operatorname{cosec}\theta} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

0.7850^2

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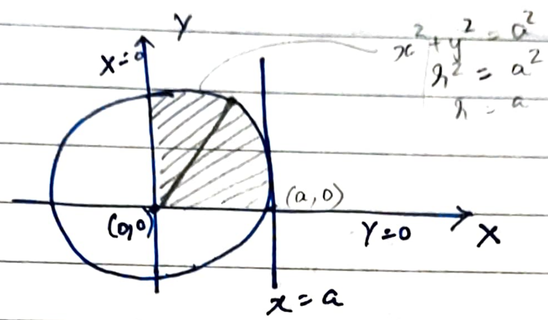
H|W q: Evaluate $\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} y^2\sqrt{x^2+y^2}\ dy\,dx$ by changing to polar coordinates

[ans: $\frac{\pi a^5}{20}$]

$y=0$, $y=\sqrt{a^2-x^2}$; $x=0$, $x=a$

$x^2+y^2=a^2$

$C(0,0)$, $r=a$

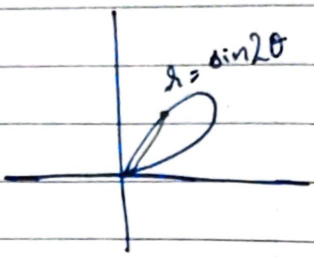


$$I=\int_{0}^{\pi/2}\int_{0}^{a} r^2\sin^2\theta\ (r)\ r\,dr\,d\theta$$

$r = \sin 2\theta$

H/W 9: Evaluate $\iint_R xy\, dx\, dy$ over the region (in polar coordinates)

$R: r = \sin 2\theta, \quad 0 \le \theta \le \frac{\pi}{2}$



ans: $\frac{1}{15}$

Changing the order of integration

In the given double integral, if the integration is w.r.t x and then w.r.t y, the process of converting the order of integration is called change of order of $\int$.

Change of order of $\int$ changes the limits of $\int$.

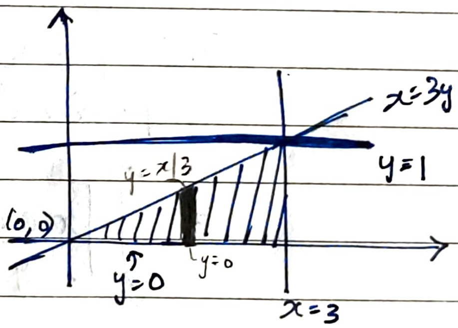
Q. Evaluate $\int_0^1 \int_{3y}^{3} e^{x^2}\,dx\,dy$ by changing the order of $\int$

$$I = \int_{y=0}^{y=1} \int_{x=3y}^{x=3} e^{x^2}\,dx\,dy$$

The region is bounded by the curves

$x = 3y \qquad x = y \qquad y = 0 \qquad y = 1$

x	0	3
$y = \frac{x}{3}$	0	1



By changing the order

$$I = \int_{x=0}^{x=3} \int_{y=0}^{y=x/3} e^{x^2}\,dy\,dx$$

* inner limit ⇒ variable
* dy ⇒ vertical strip from curve to curve

$$= \int_0^3 e^{x^2} (y)_0^{x/3} = \int_0^3 e^{x^2} \frac{x}{3}\,dx$$

$x^2 = t$ $\qquad$ $x = 0 \quad t = 0$

$2x\,dx = dt$ $\qquad$ $x = 3 \quad t = 9$

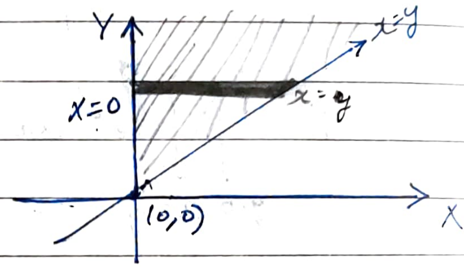
$$\Rightarrow \frac{1}{3}\int_0^9 e^t \frac{dt}{2}$$

$$= \frac{1}{6}\left[e^9 - 1\right]$$

13th March, 2023

Q $\displaystyle\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dx\,dy$ by changing order.

Sol $\displaystyle I = \int_{x=0}^{x=\infty} \int_{y=x}^{y=\infty} \frac{e^{-y}}{y}\,dy\,dx$



The region is bounded by curves $y = x$, $y = \infty$ & $x = 0$, $x = \infty$

By changing the order

$$I = \int_{y=0}^{y=\infty} \int_{x=0}^{x=y} \frac{e^{-y}}{y}\,dx\,dy$$

To find x inner limit consider horizontal strip curve to curve from L to R

$$= \int_0^\infty \left[x\frac{e^{-y}}{y}\right]_0^y$$

$$= \int_0^\infty \left[e^{-y}\right] dy \qquad = \left[-e^{-y}\right]_0^\infty$$

$$= 0 - (-1) = 1$$

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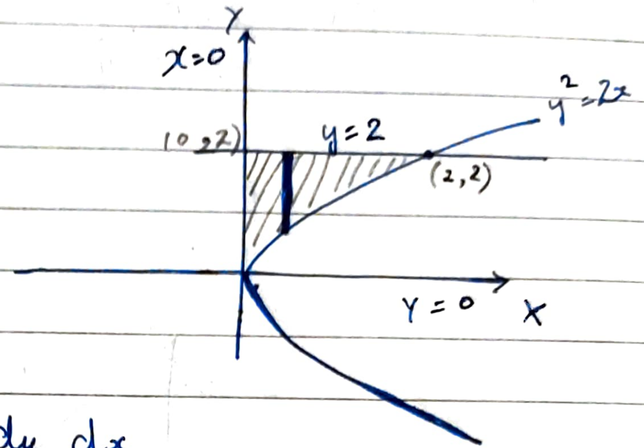
Q. Evaluate $\int_0^2 \int_0^{y^2/2} \frac{y}{\sqrt{x^2+y^2+1}}\, dx\, dy$ by changing order of integration

Region, $x=0$, $x=\frac{y^2}{2}$

$y=0$, $y=2$



By changing order

$$\int_{x=0}^{x=2} \int_{y=\sqrt{2x}}^{y=2} \frac{y}{\sqrt{x^2+y^2+1}}\, dy\, dx$$

To find y inner limit consider vertical strip bottom to top

put $x^2+y^2+1 = t$

$2y\, dy = dt$

$y\, dy = \frac{dt}{2}$

$y=\sqrt{2x}$ $\quad t = x^2+2x+1 = (x+1)^2$

$y = 2$ $\quad t = x^2+5$

$$I = \frac{1}{2}\int_0^2 \left(\int_{(x+1)^2}^{x^2+5} \frac{dt}{\sqrt{t}} \right) dx$$

$$I = \cancel{\frac{1}{2}} \int_0^2 \left(\frac{t^{1/2}}{\cancel{1/2}} \right)_{(x+1)^2}^{x^2+5} dx$$

$$I = \int_0^2 \left(\sqrt{x^2+5} - (x+1) \right) dx$$

$$I = 1.01$$

Q: $\displaystyle\int_{y=0}^{y=3}\int_{x=0}^{x=\sqrt{4-y}} (x+y)\,dx\,dy$ by change of order.

ans: 10.16

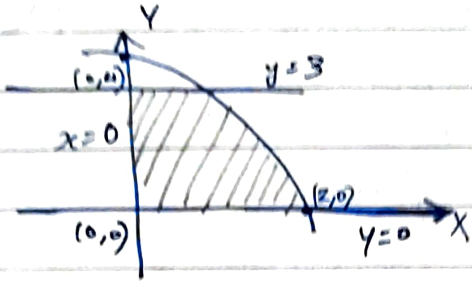
Sol. Region $x=0$

$x=\sqrt{4-y}$ $\quad x^2=4-y$

x	0	1	2	3
$y=4-x^2$	4	3	0	-5

$y=0$

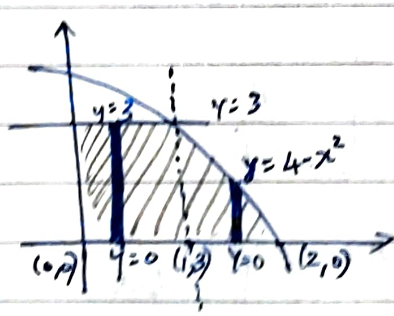
$y=3$



By changing order

$$I=\int_{x=}\int_{y=}(x+y)\,dy\,dx$$

here y touches at two differant points so,



$$\therefore I=\int_{x=0}^{x=1}\int_{y=0}^{y=3}(x+y)\,dy\,dx+\int_{x=1}^{x=2}\int_{y=0}^{y=4-x^2}(x+y)\,dy\,dx$$

$$=\int_0^1\left(xy+\frac{y^2}{2}\right)_0^3 dx+\int_1^2\left(xy+\frac{y^2}{2}\right)_0^{4-x^2}dx$$

$$=\int_0^1 3x+\frac{9}{2}\,dx+\cancel{\int_1^2 x+\frac{(4-3x)^2}{2}\,dx}\int_1^2 x(4-x^2)+\frac{(4-x^2)}{2}$$

$$=6+4.016=10.016$$

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q: $\int_0^1 \int_{x^2}^{2-x} xy \, dy \, dx$ By change of order

ans: $\frac{3}{8}$

Region: $y = x^2$

$y = 2 - x \Rightarrow x + y = 2$

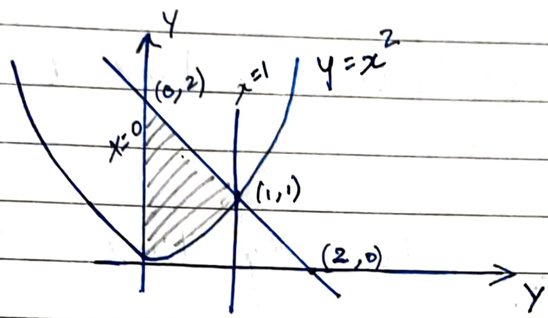
$x = 0$

$x = 1$

; $y = 2 - x$

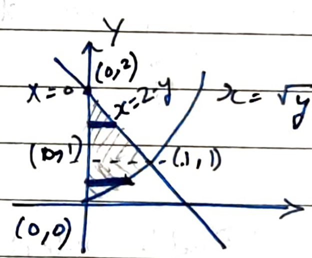
$x = 1$

$y = 1$



$\int \int xy \, dx \, dy$

$y =$ $x =$



$$I = \int_0^1 \int_0^{\sqrt{y}} xy \, dx \, dy + \int_1^2 \int_0^{2-y} xy \, dx \, dy$$

$$= \int_0^1 \left(\frac{x^2}{2} y\right)_0^{\sqrt{y}} dy + \int_1^2 \left(\frac{x^2}{2} y\right)_0^{2-y} dy$$

$$= \frac{1}{2} \int_0^1 y^2 \, dy + \frac{1}{2} \int_1^2 (2-y)^2 y \, dy$$

$$\frac{1}{6} \left(y^3\right)_0^1 + 0.208$$

$$= \frac{1}{6} + 0.208$$

$$= 0.375$$

✓ q.1 $\int_0^{\pi/2}\int_x^{\pi/2} \frac{\sin y}{y}\, dy\, dx$ By change of order

ans : 1

✓ q.2 $\int_0^{a}\int_y^{a} \frac{x}{x^2+y^2}\, dx\, dy$ By change of order

ans : $\frac{\pi a}{4}$

q.3 $\int_0^{1}\int_x^{1} \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx$ By change of order

ans : $\frac{\sqrt{2}-1}{2}$

q.4 $\int_0^{1}\int_x^{1} \sin(y^2)\, dy\, dx$ By change of order

ans : $\frac{1-\cos(1)}{2}$ $= 0.29$

✓ q.5 $\int_0^{1}\int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx$ By change of order

✓ q.6 $\int_0^{1}\int_{\sqrt{y}}^{2-y} xy\, dx\, dy$ By change of order

→ Double integral review

$$\iint_R f(x,y)\,dx\,dy = \text{Volume}$$

$$\iint_R dx\,dy = \text{Area}$$

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15th March, 2023

Triple integral

Triple $\int$ over the solid E

1. $\iiint_E f(x,y,z)\, \underbrace{dz\,dy\,dx}$ = mass of the solid

$f(x,y,z) \downarrow$ density, $dz\,dy\,dx \downarrow$ volume

2. $\iiint_E dz\,dy\,dx$ = Volume

* when all limits are constants in $\iiint$ ~~so~~ we get a cuboid

Q: Evaluate $\int_2^3\int_1^2\int_2^5 xy^2\, dz\,dy\,dx$

Sol. $I = \int_2^3\int_1^2\int_2^5 xy^2\, dz\,dy\,dx$

$= \int_2^3\int_1^2 \left(z\,xy^2\right)_2^5 dy\,dx$

$\int_2^3\int_1^2 3\,xy^2\, dy\,dx$

$\int_2^3 \left(\cancel{3}\,x\frac{y^3}{\cancel{3}}\right)_1^2 dx$

$\int_2^3 7x\, dx = \left(7\frac{x^2}{2}\right)_2^3 = \frac{7}{2}(5) = \frac{35}{2}$

(OR) $I = \int_2^3 x\,dx \cdot \int_1^2 y^2\,dy \cdot \int_2^5 dz$

$= \left(\frac{x^2}{2}\right)_2^3 \cdot \left(\frac{y^3}{3}\right)_1^2 \cdot (z)_2^5$

$= \left(\frac{9}{2}-\frac{4}{2}\right)\cdot\left(\frac{8}{3}-\frac{1}{3}\right)(3) = \frac{5}{2}\left(\frac{7}{\cancel{3}}\right)(\cancel{3}) = \frac{35}{2}$

H/W q. $I = \int_{x=0}^{x=1}\int_{y=0}^{y=1}\int_{z=0}^{z=1} (x+y+z)\, dz\, dy\, dx$

q. Evaluate $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z} (x+y+z)\, dy\, dx\, dz$

$$I = \int_{-1}^{1}\int_{0}^{z}\left(\int_{x-z}^{x+z}(x+y+z)\, dy\right) dx\, dz$$

$$\int_{-1}^{1}\int_{0}^{z}\left(xy + \frac{y^2}{2} + zy\right)_{x-z}^{x+z} dx\, dz$$

$$\int_{-1}^{1}\int_{0}^{z}\left[x(x+z) + \frac{(x+z)^2}{2} + z(x+z)\right] - \left[x(x-z) + \frac{(x-z)^2}{2} + z(x-z)\right] dx\, dz$$

$$\int_{-1}^{1}\int_{0}^{z} x(\cancel{x}+z-\cancel{x}+z) + \frac{1}{2}(4xz) + z(\cancel{x}+z-\cancel{x}+z)\, dx\, dz$$

$$\int_{-1}^{1}\int_{0}^{z} 2zx + 2zx + 2z^2\, dx\, dz$$

$$\int_{-1}^{1}\int_{0}^{z} 4zx + 2z^2\, dx\, dz$$

$$\int_{-1}^{1}\left(4z\frac{x^2}{\cancel{2}} + 2z^2x\right)_{0}^{z} dz$$

$$\int_{-1}^{1} 2z^3 + 2z^3\, dz = \int_{-1}^{1} 4z^3 = \left(z^4\right)_{-1}^{1}$$

$$= 0$$

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q. Find the volume of the solid bounded by the surfaces $z=0$, $z=1-x^2-y^2$, $y=0, y=1-x$, $x=0, x=1$

Volume $V = \int_{x=0}^{x=1}\int_{y=0}^{y=1-x}\int_{z=0}^{z=1-x^2-y^2} dz\, dy\, dx$

$= \int_0^1\int_0^{1-x} (z)_0^{1-x^2-y^2} dy\, dx$

$= \int_0^1\int_0^{1-x} 1-x^2-y^2 \, dy\, dx$

$= \int_0^1 \left(y - x^2 y - \frac{y^3}{3}\right)_0^{1-x} dx$

$= \int_0^1 (1-x) - x^2(1-x) - \frac{(1-x)^3}{3} \, dx$

$= 0.33$

~~Evaluation~~ Average value of a function of 3 variables

If $f(x,y,z)$ is integrable over a solid bounded region E with positive volume $V(E)$ then the average value of the function is given by

$$\frac{1}{V(E)} \iiint_E f(x,y,z)\, dv$$

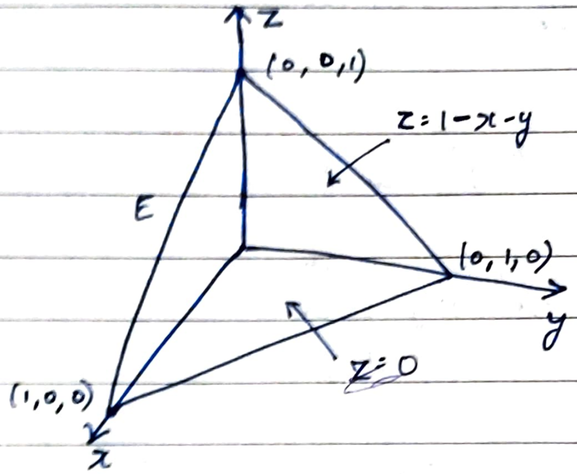
$V(E) = \iiint_E 1\, dv$

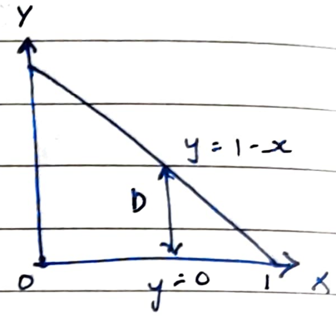
Note : Average $= \frac{1}{A} \iint_R f(x,y)\, dx\, dy$

$A = \iint dx\, dy$

q: The temperature at a point (x, y, z) of a solid E bounded by the planes $x=0, y=0, z=0$ and the plane $x+y+z=1$ is $\dfrac{1}{(1+x+y+z)^3}$ degree celcius.

Find the avg. temp over the solid.





avg. value of the function $= \dfrac{1}{V} \iiint_E f(x,y,z)\, dz\, dy\, dx$

$$= \frac{1}{V} \iiint_E \frac{1}{(1+x+y+z)^3}\, dz\, dy\, dx$$

volume $V = \displaystyle\int_{x=} \int_{y=} \int_{z=} dz\, dy\, dx$

to find z limit : $z=0$ to $z=1-x-y$

to find y limit : put $z=0$ in $x+y+z=1$

$\therefore y=0$ to $y=1-x$

to find x limit : put $z=y=0$

$\therefore x=0$ to $x=1$

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$$V = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\, dy\, dx$$

9
44 33

$$\int_0^1 \int_0^{1-x} (z)_0^{1-x-y}\, dy\, dx$$

$$\int_0^1 \int_0^{1-x} (1-x-y)\, dy\, dx$$

$$\int_0^1 \left(y - xy - \frac{y^2}{2}\right)_0^{1-x} dx$$

$$\int_0^1 (1-x) - x(1-x) - \frac{(1-x)^2}{2}\, dx$$

$$= \frac{1}{6}$$

$$\int (\cancel{x}+\cancel{x})^{-3} dz$$

$$\text{avg} = \cancel{6} \int_0^1 \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} \frac{1}{(1+x+y+z)^3}\, dz\, dy\, dx$$

$$6 \int_0^1 \int_0^{1-x} \left(\frac{-1\,(1+x+y+z)^{-2}}{2}\right)_0^{1-x-y} dy\, dx$$

$$-3 \int_0^1 \int_0^{1-x} (1+\cancel{x}+\cancel{y}+1-\cancel{x}-\cancel{y})^{-2} - (1+x+y)^{-2}$$

$$-3 \int_0^1 \int_0^{1-x} \cancel{1+x+y}\ \frac{1}{4} - \frac{1}{(1+x+y)^2}\, dy\, dx$$

$$-3 \int_0^1 \left(\frac{y}{4} + \frac{1}{(1+x+y)}\right)_0^{1-x} dx$$

$$-3 \int_0^1 \frac{1-x}{4} + \frac{1}{(1+\cancel{x}+1-\cancel{x})} \cancel{dx} - \left(0 + \frac{1}{1+x}\right) dx$$

$$= \cancel{-3(0.625) = -1.875} = -3(-0.068) = 0.204$$

Change of Variables in Triple integrals
- Cylindrical and Spherical co-ordinates

→ Triple integral in cylindrical coordinates

cylinders & paraboloids

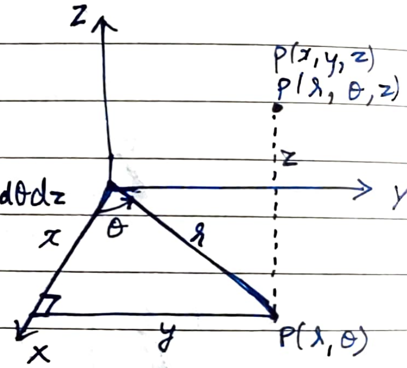
To change cartesian (rectangular) coordinates (x, y, z) to (x, y, z) to cylindrical coordinates (r, θ, z) we have

$$x = r\cos\theta \ , \ y = r\sin\theta \ , \ z = z$$

and Jacobian $J = \dfrac{\partial(x, y, z)}{\partial(r, \theta, z)} = r$

$$\iiint_V f(x, y, z)\, dx\, dy\, dz$$

$$= \iiint_{V'} f(r\cos\theta, r\sin\theta, z)\, r\, dr\, d\theta\, dz$$



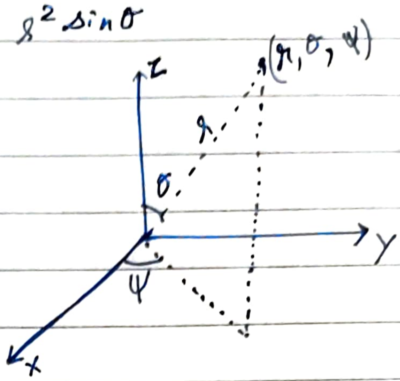
Note: Cylindrical coordinates (r, θ, z) are used to evaluate the integral in the regions which are bounded by cylinders along z-axis, planes through z-axis, planes perpendicular to z-axis and paraboloids

→ Triple integral in spherical coordinates (r, θ, ϕ)

To change rectangular coordinates (x, y, z) to spherical polar coordinates (r, θ, ϕ) we have

$x = r\sin\theta\cos\psi$, $y = r\sin\theta\sin\psi$, $z = r\cos\theta$

Jacobian $J = \dfrac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2\sin\theta$



Then,

$$\iiint_{R_{xyz}} f(x, y, z)\, dz\, dy\, dx$$

$$= \iiint_{R(r,\theta,\phi)} f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\, r^2\sin\theta\, dr\, d\theta\, d\phi$$

θ varies from $+z$ to $-z \Rightarrow \pi$

ϕ varies from 0 to 2π

Note 1. Spherical coordinates (r, θ, ϕ) are used to evaluate the integral in the regions which are bounded by sphere with centre at origin, cone with vertices at origin and z-axis.

2. If region of integration is a sphere $x^2 + y^2 + z^2 = a^2$ with centre $(0,0,0)$ and radius a, limits of r, θ, ϕ are

a) For positive octant of sphere

r: $r = 0$ to $r = a$

θ: $\theta = 0$ to $\theta = \pi/2$

ϕ: $\phi = 0$ to $\phi = \pi/2$

b) For hemisphere

r: $r = 0$ to $r = a$

θ: $\theta = 0$ to $\theta = \pi/2$

ϕ: $\phi = 0$ to $\phi = 2\pi$

c) For complete sphere

r: $r = 0$ to $r = a$

θ: $\theta = 0$ to $\theta = \pi$

ϕ: $\phi = 0$ to $\phi = 2\pi$

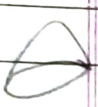
17th March, 2023

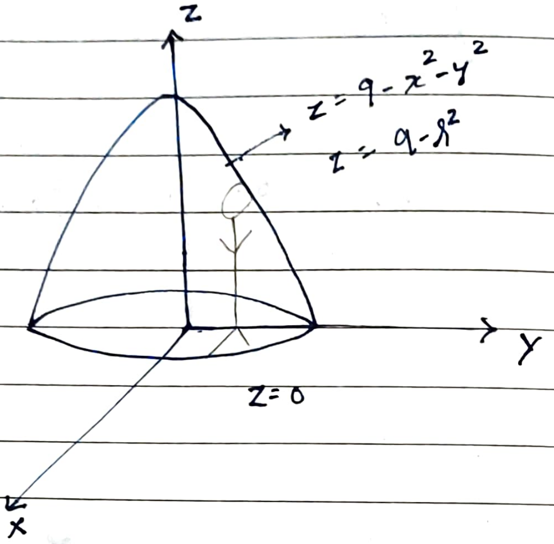
Q: using cylindrical co-ordinates evaluate

$$\iiint_V (x^2+y^2)\,dx\,dy\,dz$$

taken over the region V bounded by the paraboloid $z = 9-x^2-y^2$ taken over the plane $z=0$

Sol.





By changing into cylindrical co-ordinate system,

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$, $dz\,dy\,dx = r\,dz\,dr\,d\theta$

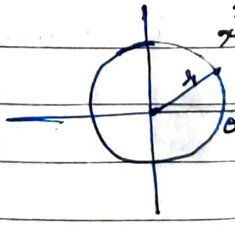
$$I = \iiint_V x^2+y^2 \; dz\,dy\,dx$$

$$\int_{\theta=0}^{2\pi}\int_{r=0}^{3}\int_{z=0}^{9-r^2} r^2 \cdot r\,dz\,dr\,d\theta$$

$$\int_0^{2\pi}\int_0^{3}\left(\frac{r^4}{4}\right)_0^{9-r^2} dr\,d\theta$$

$$\frac{1}{4}\int_0^{2\pi}\int_0^{3}(9-r^2)^4\,dr\,d\theta$$

Finding r limit
⇓ 2D
⇓
projection of solid on x-y plane.
$\Rightarrow z=0$
$\Rightarrow x^2+y^2=9$



$x^2+y^2=9$
$r^2=3$
$r=\pm 3$
r can't be -ve

cylinder : $x^2+y^2=a^2$
cone : $z^2=x^2+y^2$

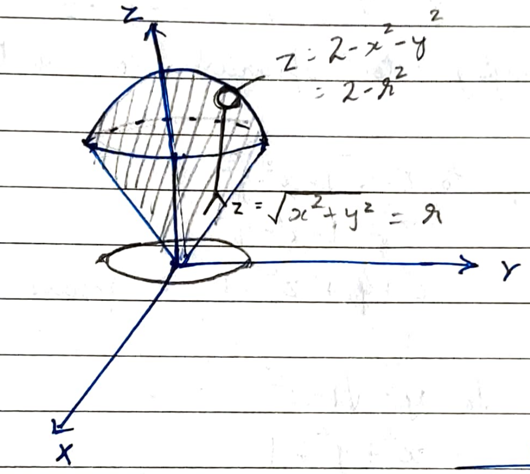
$$= \int_0^{2\pi}\int_0^{3} r^3 \,(z)_0^{9-r^2}\, dr\, d\theta$$

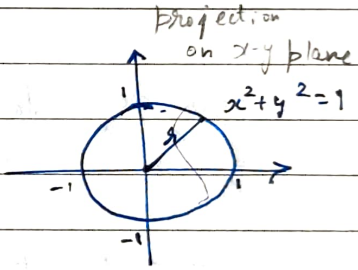
$$= \int_0^{2\pi}\int_0^{3} r^3(9-r^2)\, dr\, d\theta$$

$$= 381.70$$

q: Calculate the volume of solid bounded by the paraboloid $z = 2-x^2-y^2$ and conic surface $z=\sqrt{x^2+y^2}$

inverted tea cup





By cylindrical co ordinates we have

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$dz\,dy\,dx = r\,dz\,dr\,d\theta$

$z \rightarrow r$ to $2-r^2$

$r \rightarrow 0$ to 1

$\theta \rightarrow 0$ to 2π

to find radius of O^{le}

solve $z = 2-r^2$, $z = r$

to find inter

$r = 2-r^2$

$r^2+r-2=0$

$r = 1, -2$

Volume $V = \iiint_V dz\, dy\, dx$

$$= \int_0^{2\pi}\int_0^{1}\int_r^{2-r^2} r\, dz\, dr\, d\theta$$

$$\int_0^{2\pi}\int_0^1 r\,(z)_r^{2-r^2}\,dr\,d\theta$$

$$\int_0^{2\pi}\int_0^1 r\,(2-r^2-r)\,dr\,d\theta$$

$$= (\theta)_0^{2\pi}\int_0^1 r(2-r^2-r)\,dr$$

$$= 2.61$$

q: Evaluate $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}$

by changing into spherical co-ordinates.

Sol. $z \to 0$ to $z=\sqrt{1-x^2-y^2}$

$z^2 = 1-x^2-y^2$

$x^2+y^2+z^2=1 \to$ sphere centre $(0,0)$, $r=1$

$y=0$ to $y=\sqrt{1-x^2}$

$x^2+y^2=1$

$x=0$ to 1

Sphere that lies in I octant

The region integration is the sphere in I octant

By changing into spherical co-ordinates

$x = r\sin\theta\cos\phi$; $y = r\sin\theta\sin\phi$; $z = r\cos\theta$

$dz\,dy\,dx = r^2\sin\theta\,dr\,d\theta\,d\phi$

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For +ve octant of sphere

$r : 0$ to 1
$\theta : 0$ to $\pi/2$
$\phi : 0$ to $\pi/2$

$$I = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{1} \frac{r^2 \sin\theta \, dr \, d\theta \, d\phi}{\sqrt{1-r^2}}$$

$$= \int_0^{\pi/2} d\phi \int_0^{\pi/2} \sin\theta \, d\theta \int_0^{1} \frac{r^2}{\sqrt{1-r^2}} \, dr$$

$$(\phi)_0^{\pi/2} \, (-\cos\theta)_0^{\pi/2} \int_0^{\pi/2} \frac{\sin^2\theta \, \cancel{\cos\theta} \, d\theta}{\cancel{\cos\theta}}$$

$r = \sin\theta$
$dr = \cos\theta \, d\theta$
$r = 0 ; \quad \theta = 0$
$r = 1 ; \quad \theta = \pi/2$

$$\frac{\pi}{2}(1) \; \frac{1}{2} \beta\left(\frac{3}{2}, \frac{1}{2}\right)$$

$$\frac{\pi}{4} \cdot \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)}$$

$$\frac{\pi}{4}\left(\frac{1}{2}\right) \sqrt{\pi}\sqrt{\pi} = \frac{\pi^2}{8}$$

q. Evaluate $\iiint_V xyz\,(x^2+y^2+z^2)^{n/2} \, dx \, dy \, dz$ taken through the +ve octant sphere $x^2+y^2+z^2 = b^2$ provided $n+5>0$

Sol. By spherical coordinate system
$x = r\sin\theta\cos\phi$, $y = r\sin\phi \sin\phi$, $z = r\cos\theta$
$dz \, dy \, dx = r^2 \sin\theta \, dr \, d\theta \, d\phi$

$r \to 0$ to b

$\theta \to 0$ to $\pi/2$

$\phi \to 0$ to $\pi/2$

$$I = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{b} (r\sin\theta\cos\phi \; r\sin\theta\sin\phi \; r\cos\theta)\, r^{\cancel{2}\times n/\cancel{2}} \, r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{b} r^{n+5} \sin^3 \cos\theta \; \sin\phi\cos\phi \; dr\, d\theta\, d\phi$$

$$= \int_0^{\pi/2} \sin\phi\cos\phi \, d\phi \cdot \int_0^{\pi/2} \sin^3\theta\cos\theta \, d\theta \cdot \int_0^{b} r^{n+5}\, dr$$

$$= \int_0^{\pi/2} \frac{\sin 2\phi}{2} \times \frac{1}{2}\beta(2,1) \times \left(\frac{r^{n+6}}{n+6}\right)_0^b$$

$$= \frac{1}{2}\left(\frac{-\cos 2\phi}{2}\right)_0^{\pi/2} \times \frac{1}{2}\frac{\Gamma(2)\Gamma(1)}{\Gamma(3)} \times \frac{b^{n+6}}{n+6}$$

$$= \frac{1}{4}\left(-(-1)+1\right)\frac{1}{2} \cdot \frac{b^{n+6}}{n+6}$$

$$= \frac{1}{8}\frac{b^{n+6}}{(n+6)}, \text{ provided } n+5>0$$

q. Find the volume of that portion of ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ which lies in first octant

ans: $\frac{\pi abc}{6}$ cubic units

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substitution

Variable transformation — change of co ordinates

* Here we use variable transformation

put $\frac{x}{a} = u$; $\frac{y}{b} = v$; $\frac{z}{c} = w$

$dx = a\,du$; $dy = b\,dv$; $dz = c\,dw$

now the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ becomes

$u^2 + v^2 + w^2 = 1$ which is the sphere with centre at $(0,0,0)$, radius $= 1$

in the new variables u, v, w

$$\text{volume } V = \iiint_{V \text{ ellipsoid}} dz\,dy\,dx$$

$$= \iiint_{V^* \text{ sphere - } u^2+v^2+w^2=1} abc\,dw\,dv\,du$$

For +ve octant of sphere

Here we use spherical co ordinate system

$u = r\sin\theta\cos\phi$, $v = r\sin\theta\sin\phi$, $w = r\cos\theta$

$dw\,dv\,du = r^2\sin\theta\,dr\,d\theta\,d\phi$

$r : 0$ to 1

$\theta : 0$ to $\pi/2$

$\phi : 0$ to $\pi/2$

$$V = abc\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 r^2\sin\theta\,dr\,d\theta\,d\phi$$

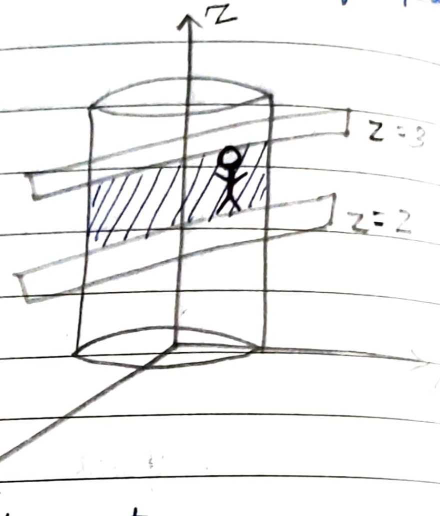
$$\int_0^{\pi/2}\int_0^{\pi/2}\left(\frac{r^3}{3}\sin\theta\right)_0^1 \sin\theta\,d\theta\,d\phi$$

$$\frac{abc}{3}\int_0^{\pi/2}(-\cos\theta)_0^{\pi/2}\,d\phi$$

$$= \frac{abc}{3}\left(\frac{\pi}{2}\right) = \frac{\pi abc}{6}$$

q. Evaluate $\iiint_V z(x^2+y^2)\,dx\,dy\,dz$ over the volume of the cylinder $x^2+y^2=1$ intercepted by planes $z=2$ and $z=3$.

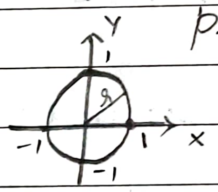
Sol.



cylinder $\Rightarrow$ cylindrical coordinate system

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$, $dz\,dy\,dx = r\,dz\,dr\,d\theta$

$z : 0$ to 3
$r : 0$ to 1
$\theta : 0$ to 2π



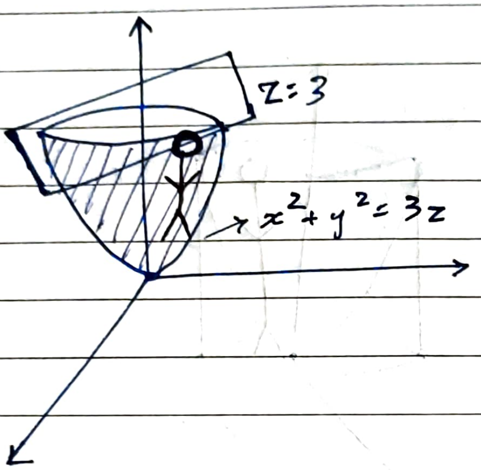
projection of cylinder on x-y plane

$$I = \int_0^{2\pi}\int_0^{1}\int_2^{3} z\cdot r^2\cdot r\,dz\,dr\,d\theta$$

$$\int_0^{2\pi}\int_0^{1} r^3\left(\frac{z^2}{2}\right)_2^3 dr\,d\theta$$

$$\frac{5}{2}\int_0^{2\pi}\left(\frac{r^4}{4}\right)_0^1 d\theta = \frac{5}{\cancel{2}}\left(\frac{1}{4}\right)(\cancel{2}\pi) = \frac{5\pi}{4}$$

Q. Evaluate $\iiint (x^2+y^2)\,dx\,dy\,dz$ taken over region bounded by paraboloid $x^2+y^2=3z$ and plane $z=3$.



$x^2+y^2=3z \Rightarrow r^2=3z$

$z=\frac{r^2}{3}$

$x^2+y^2=3z$ & $z=3$

$\Rightarrow x^2+y^2=9$

$r=3$

$z: \frac{r^2}{3}$ to 3

$r: 0$ to 3

$\theta: 0$ to 2π

$$I=\int_0^{2\pi}\int_0^{3}\int_{\frac{r^2}{3}}^{3}(x^2+y^2)\cdot r\,dz\,dr\,d\theta = \iiint r^3\,dz\,dr\,d\theta$$

27(3) = 51

$$\int_0^{2\pi}\int_0^{3}(z)_{\frac{r^2}{3}}^{3}\,r^3$$

$$\int_0^{3}\left(3-\frac{r^2}{3}\right)r^3\cdot\int_0^{2\pi}d\theta$$

$$2\pi\left(\frac{243}{4}-\frac{729}{18}\right) = \left(\frac{3r^4}{4}-\frac{r^6}{18}\right)_0^3\cdot(2\pi)$$

$$=2\pi(60.75-40.5) = 20.25(2\pi) = 40.5\pi$$

$\frac{2\sqrt{3}}{4} - \frac{1}{2}$

20th March, 2023

q. Evaluate $\iiint z^2 \, dx\, dy\, dz$ taken over the volume bounded by the surfaces

$x^2 + y^2 = z$, $x^2 + y^2 = a^2$ and $z = 0$

ans: $\frac{a^8 \pi}{12}$

Sol. By cylindrical coordinate system

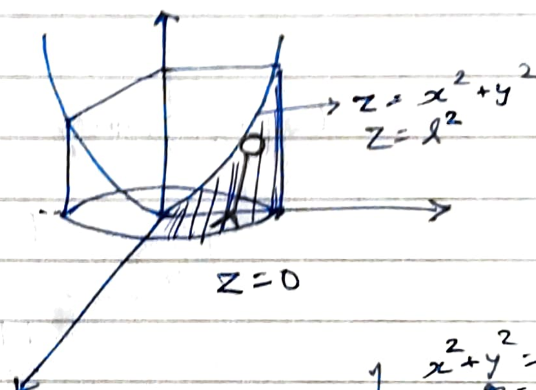
$x = r\cos\theta, \; y = r\sin\theta, \; z = z$

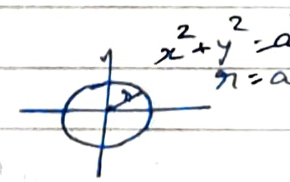
$dz\,dy\,dx = r\,dz\,dr\,d\theta$

$z : 0$ to r^2

$r : 0$ to a

$\theta : 0$ to 2π





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Applications of double integrals

→ Mass contained in a plane region D

Let $f(x,y)$ be the surface density (mass per unit area) of a given plane region D. Then the amount (quantity) of mass M contained in the plane region D is given by,

$$\text{Mass } M = \iint_D f(x,y)\,dx\,dy$$

→ Centre of gravity (Centroid) of a plane region D

$$x_c = \frac{\iint_D x f(x,y)\,dx\,dy}{M}, \quad y_c = \frac{\iint_D y f(x,y)\,dx\,dy}{M}$$

→ Moment of Inertia of a Plane region D

$$I_x = \iint_D y^2 f(x,y)\,dx\,dy$$

$$I_y = \iint_D x^2 f(x,y)\,dx\,dy$$

$$I_0 = I_x + I_y = \iint_D (x^2+y^2) f(x,y)\,dx\,dy$$

Applications of triple integrals

→ Mass of a solid

If $\rho = f(x, y, z)$ is the volume density (mass per unit volume) at any point (x, y, z) of the region V, then mass M of region is given as

$$\text{Mass } M = \iiint_V \rho \, dx\, dy\, dz$$

→ Moment of inertia of a solid

$$I_{zz} = \iiint_V \rho (x^2 + y^2) \, dx\, dy\, dz$$

density of substance

$$I_{xx} = \iiint_V \rho (y^2 + z^2) \, dx\, dy\, dz \qquad I_{yy} = \iiint_V \rho (x^2 + z^2) \, dx\, dy\, dz$$

→ Centre of gravity of a solid :

$$x_c = \frac{\iiint_V x \rho \, dV}{\iiint_V \rho \, dV}, \quad y_c = \frac{\iiint_V y \rho \, dV}{\iiint_V \rho \, dV}, \quad z_c = \frac{\iiint_V z \rho \, dV}{\iiint_V \rho \, dV}$$

Q. Find the total mass of the region in the cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$ whit density at any point given by xyz.

Sol. $\rho(x,y,z) = xyz$

mass of solid $= \iiint f(x,y,z)\, dz\, dy\, dx$

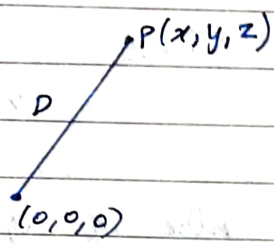
$$= \int_0^1\int_0^1\int_0^1 xyz\, dz\, dy\, dx$$

$$= \int_0^1 x\,dx \int_0^1 y\, dy \int_0^1 z\, dz$$

$$= \left(\frac{x^2}{2}\right)_0^1 \left(\frac{y^2}{2}\right)_0^1 \left(\frac{z^2}{2}\right)_0^1 = \frac{1}{8}$$

Q: Compute the mass of a sphere of radius 6 if the density varies inversely as the square of distance from the centre.

Sol. $\rho \propto \dfrac{1}{x^2+y^2+z^2}$



$\rho = k \cdot \dfrac{1}{x^2+y^2+z^2}$

$D = \sqrt{x^2+y^2+z^2}$

$D^2 = x^2+y^2+z^2$

mass of sphere $= \iiint_V \rho\, dz\, dy\, dx$

$$= \iiint \frac{k}{x^2+y^2+z^2}\, dz\, dy\, dx$$

By spherical coordinate system

$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$

$dz\, dy\, dx = r^2 \sin\theta\, dr\, d\theta\, d\phi$

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For a complete sphere

$r : 0 \to b$

$\theta : 0 \to \pi$

$\phi : 0 \to 2\pi$

$$\text{mass} = k \int_0^{2\pi}\int_0^{\pi}\int_0^{b} \frac{1}{\cancel{r^2}} \cdot \cancel{r^2} \sin\theta \, dr \, d\theta \, d\phi$$

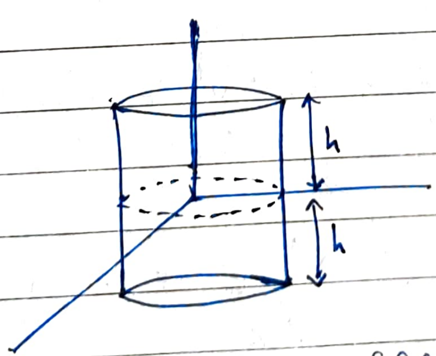
$$= k \int_0^b dr \cdot \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi$$

$$= k \left(b \cdot (-\cos\theta)_0^{\pi} \; (2\pi) \right)$$

$$= 4\pi b k$$

q. Calculate MoI of a right circular cylinder of altitude $2h$ and radius b, relative to the diameter of its median section with density $= k$, constant.

cutting cylinder in middle



Sol. MoI relative to x-axis $= \iiint \rho (y^2 + z^2) \, dz \, dy \, dx$

$$= k \iiint (y^2 + z^2) \, dz \, dy \, dx$$

STDOS 8K006

By cylindrical co-ordinate system

$x = r\cos\theta \, , \; y = r\sin\theta \, , \; z = z \, , \; dz\,dy\,dx = r\,dz\,dr\,d\theta$

$z : -h \text{ to } h$

$r : 0 \text{ to } b$

$\theta : 0 \text{ to } 2\pi$

MoI relative to x-axis $= k \int_0^{2\pi}\int_0^{b}\int_{-h}^{h} (r^2\sin^2\theta + z^2)\, r\, dz\, dr\, d\theta$

$$= k \int_0^{2\pi}\int_0^{b} \left(r^3\sin^2\theta\, z + \frac{r z^3}{3} \right)_{-h}^{h} dr\, d\theta$$

$$= k \int_0^{2\pi}\int_0^{b} \left(r^3\sin^2\theta\, h + \frac{r h^3}{3} \right) - \left(r^3\sin^2(\theta)(-h) - \frac{r h^3}{3} \right) dr\, d\theta$$

$$= k \int_0^{2\pi}\int_0^{b} \left(2h r^3 \sin^2\theta + \frac{2}{3} r h^3 \right) dr\, d\theta$$

$$= k \int_0^{2\pi} \left(2h\sin^2\theta \frac{r^4}{4} + \frac{2}{3} h^3 \frac{r^2}{2} \right)_0^{b} d\theta$$

$$= k \int_0^{2\pi} 2h\sin^2\theta \frac{b^4}{4} + \frac{\cancel{2}}{3} h^3 \frac{b^2}{\cancel{2}} \, d\theta$$

$$= \frac{\cancel{2} h k b^4 \pi}{\cancel{4}\, 2} + \frac{2\pi h^3 b^2 k}{3}$$

$$= hk\pi b^2 \left(\frac{b^2}{2} + \frac{2h^2}{3} \right)$$

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q. Find the centre of gravity of a plate whose density $\rho(x, y)$ is constant and is bounded by curves $y = x^2$ & $y = x + 2$.
Also find MoI about the axes.

Sol. COG : (x_c, y_c)

let $\rho = k$

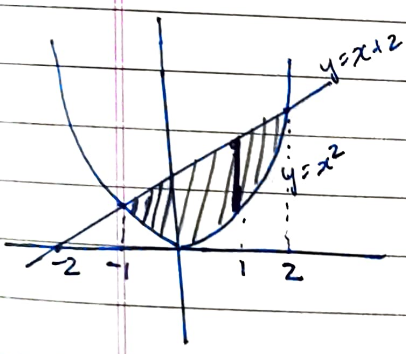
$$x_c = \frac{1}{M} \iint x \cdot f(x, y)\, dx\, dy$$

$$y_c = \frac{1}{M} \iint y \cdot f(x, y)\, dx\, dy$$

$x^2 = x + 2$
$(-1)^2 = -1 + 2$
$1 = 1$
$2^2 = 2 + 2$

mass of plate $= \iint f(x, y)\, dx\, dy$

$$= \iint k\, dx\, dy$$



$$= \int_{-1}^{2} \int_{y=x^2}^{y=x+2} k\, dy\, dx$$

$$= k \int_{-1}^{2} (y)_{x^2}^{x+2}\, dx$$

$$= k \int_{-1}^{2} x + 2 - x^2$$

$$= k \left(\frac{x^2}{2} + 2x - \frac{x^3}{3} \right)_{-1}^{2}$$

$$= k \left(2 + 4 - \frac{8}{3} - \left(\frac{1}{2} - 2 - \frac{1}{3} \right) \right)$$