8th Jan, 2024

Linear Algebra

* A linear eqn. in n variables is of form
$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$
where $x_1, x_2 \ldots x_n$ are unknown/variables.
$x_i \ (i = 1, 2, \ldots n)$
$a_i \in R \ (i = 1, 2, \ldots n)$ are coefficients of variables

Equations

Algebraic → algebraic quantities

Transcendtal — hyperbolic, exponential, trig.

→ System Of Linear Equations

* set of linear equations involving same unknowns/variables.

→ Matrix Notation

* m eqns. with n unknowns

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad\qquad\qquad & \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \qquad a_{mn}x_n &= b_m
\end{aligned}$$

$$Ax = B$$

$$A = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} \cdots & a_{mn} \end{bmatrix}_{m \times n} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m \times 1}$$

$A_{m\times n}$: coefficient matrix
$x_{n\times 1}$: vector of unknowns
$b_{m\times 1}$: constant vector

* if $m=n$ $\Rightarrow$ Square matrix of order n
  n eqns & n variables

* if all b's are 0 $\Rightarrow$ Homogeneous $(Ax=0)$
  if atleast one b is non 0 $\Rightarrow$ non-Homogeneous

$[A:b]$ Augmented matrix $= \begin{bmatrix} a_{11} & a_{12} & a_{1n} & : & b_1 \\ a_{21} & a_{22} & a_{2n} & : & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{mn} & : & b_m \end{bmatrix}$

* Consistent $\rightarrow$ has a soln. (unique / ∞ many)
* Inconsistent $\rightarrow$ no soln.

$\rightarrow$ Elementary Row Operations

i) $R_i \rightarrow kR_i \quad (k \neq 0)$

ii) $R_i \rightarrow R_i + kR_j \quad (k \neq 0)$

iii) $R_i \leftrightarrow R_j$

* Elementary Matrix : Take Identity Matrix apply any row transformation

* Equivalent Matrix : apply definite no. of row transformations

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Echelon Matrix

Echelon Form (U)

* First non-zero element in every row is known as pivot

* Below pivot is a column of 0s

* Each pivot lies to right of pivot of above row (Staircase pattern)

* 0 rows (if exist) appear at bottom of matrix

→ eg: 1.
$$\begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & 6 & 7 & 8 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix} = U$$

Echelon form

eg 2.
$$\begin{bmatrix} 2 & 6 & 7 \\ 0 & 0 & 0 \\ 0 & 7 & 8 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 2 & 6 & 7 \\ 0 & 7 & 8 \\ 0 & 0 & 0 \end{bmatrix} = U$$

eg. 3.
$$\begin{bmatrix} 0 & 2 & 7 & 8 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

→ Row Reduced Echelon Form (RREF)

* Every row in echelon form must be divided by its pivot ⇒ first non-zero entry in every row is 1

* Using pivot rows produce 0s above first non-0 (ie)

$$U = \begin{bmatrix} a & b & c & d & e \\ 0 & 0 & f & g & h \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

starting i => make above & below i all 0

$$R_1 \rightarrow R_1 - \left(\frac{d}{i}\right) R_3$$

$$R_2 \rightarrow R_2 - \left(\frac{g}{i}\right) R_3$$

$$\simeq \begin{bmatrix} a & b & c & 0 & e \\ 0 & 0 & f & 0 & h \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

for f => make c 0

$$R_1 \rightarrow R_1 - \left(\frac{c}{f}\right) R_2$$

$$\simeq \begin{bmatrix} a & b & 0 & 0 & e-\left(\frac{c}{f}\right)h \\ 0 & 0 & f & 0 & h \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

making pivoted elements 1

$$R_1 \rightarrow R_1 / a$$

$$R_2 \rightarrow R_2 / f$$

$$R_3 \rightarrow R_3 / i$$

$$\simeq \begin{bmatrix} 1 & b/a & 0 & 0 & \frac{e-\left(\frac{c}{f}\right)h}{a} \\ 0 & 1 & 0 & 0 & h/f \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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→ Reduce the foll matrices into Row Echelon form

q1. $A = \begin{bmatrix} 1 & 1 & -2 & 3 & 4 \\ \boxed{2} & 3 & 3 & -1 & 3 \\ \boxed{5} & \boxed{7} & 4 & -1 & 5 \end{bmatrix}$

to find min. no. of elements to make it 0 take square matrix & consider lower ◺

$$R_2 \to R_2 - \left(\frac{2}{1}\right) R_1$$
$$R_3 \to R_3 - \left(\frac{5}{1}\right) R_1$$

$$\simeq \begin{bmatrix} 1 & 1 & -2 & 3 & 4 \\ 0 & 1 & 7 & -7 & -5 \\ 0 & 2 & 14 & -16 & -15 \end{bmatrix}$$

$$R_3 \to R_3 - \left(\frac{2}{1}\right) R_2$$

$$\simeq \begin{bmatrix} 1 & 1 & -2 & 3 & 4 \\ 0 & 1 & 7 & -7 & -5 \\ 0 & 0 & 0 & -2 & -5 \end{bmatrix} = U$$

q2.

$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 9 \\ 2 & 4 & 6 \\ 8 & 6 & 7 \end{bmatrix}$

can be 0 wrt R_1 $\quad R_2 \quad R_3$

$$R_2 \to R_2 - \left(\frac{4}{1}\right) R_1$$
$$R_3 \to R_3 - \left(\frac{2}{1}\right) R_1$$
$$R_4 \to R_4 - \left(\frac{8}{1}\right) R_1$$

$$\simeq \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \\ 0 & \underline{\underline{-10}} & -17 \end{bmatrix}$$

$$R_4 \to R_4 - \left(\frac{-10}{-3}\right) R_2$$

$$\simeq \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & -7 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$\simeq \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -3 \\ 0 & 0 & -7 \\ 0 & 0 & 0 \end{bmatrix} = U$$

q3. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 6 & 9 \end{bmatrix}$ $\quad R_2 \to R_2 - \left(\frac{4}{1}\right) R_1$, $\quad R_3 \to R_3 - \left(\frac{3}{1}\right) R_1$

$R_2 \to R_2 - 4R_1$
$R_3 \to R_3 - 3R_1$

$$\simeq \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} = U$$

Note: no. of pivots $\leq \min(m, n)$

→ Reduce matrix into row reduced Echelon

$$\begin{array}{ccccc} & & R_2\ \text{VI} & R_3\ \text{V} & R_4\ \text{IV} \\ \text{q1. } A = & \left[\begin{array}{c} 2 \\ 6 \\ 5 \\ 7 \end{array}\right. & \begin{array}{c} 4 \\ 8 \\ 7 \\ 6 \end{array} & \begin{array}{c} 6 \\ 2 \\ 8 \\ 8 \end{array} & \left.\begin{array}{c} 10 \\ 3 \\ 10 \\ 12 \end{array}\right] \\ & R_1\ \text{I} & R_2\ \text{II} & R_3\ \text{III} & \end{array}$$

$R_2 \to R_2 - \left(\frac{6}{2}\right) R_1$
$R_3 \to R_3 - \left(\frac{5}{2}\right) R_1$
$R_4 \to R_4 - \left(\frac{7}{2}\right) R_1$

$$\simeq \begin{bmatrix} 2 & 4 & 6 & 10 \\ 0 & -4 & -16 & -27 \\ 0 & -3 & -7 & -7 \\ 0 & -8 & -13 & -23 \end{bmatrix}$$

$R_3 \to R_3 - \left(\frac{-3}{-4}\right) R_2$

$R_4 \to R_4 - \left(\frac{-8}{-4}\right) R_2$

$$\simeq \begin{bmatrix} 2 & 4 & 6 & 10 \\ 0 & -4 & -16 & -27 \\ 0 & 0 & 5 & -141/4 \\ 0 & 0 & 19 & 31 \end{bmatrix}$$

$R_4 \to R_4 - \left(\frac{19}{5}\right) R_3$

$$\simeq \begin{bmatrix} 2 & 4 & 6 & 10 \\ 0 & -4 & -16 & -27 \\ 0 & 0 & 5 & 21/4 \\ 0 & 0 & 0 & 221/20 \end{bmatrix}$$

$R_1 \to R_1 - \left(\frac{10}{\frac{221}{20}}\right) R_4$

$R_2 \to R_2 - \left(\frac{-27}{\frac{221}{20}}\right) R_4$

$R_3 \to R_3 - \left(\frac{\frac{21}{4}}{\frac{221}{20}}\right) R_4$

$$\simeq \begin{bmatrix} 2 & 4 & 6 & 0 \\ 0 & -4 & -16 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & \frac{221}{20} \end{bmatrix}$$

$R_1 \to R_1 - \left(\frac{6}{5}\right) R_3$

$R_2 \to R_2 - \left(\frac{-16}{5}\right) R_3$

$$\simeq \begin{bmatrix} 2 & 4 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & \frac{221}{20} \end{bmatrix} \simeq \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & \frac{221}{20} \end{bmatrix}$$

$R_1 \to R_1 - \left(\frac{4}{-4}\right) R_2$

$R_1 \to R_1/2$

$R_2 \to R_2/-4$

$R_3 \to R_3/5$

$R_4 \to R_4/\frac{221}{20}$

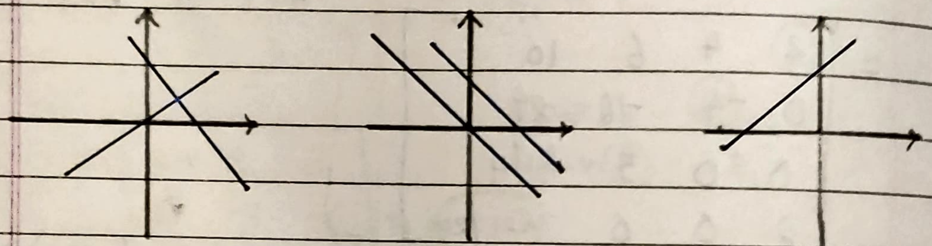
$$\simeq \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & & 1 \end{bmatrix}$$

Note : For the given system of eqn. $(A|b)$ if we reduce the matrix A, where A is a square matrix into row reduced Echelon, then the 'A' will be reduced to identity matrix

Geometry of Linear Equations

2 variables :

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$



intersecting lines — Unique soln.

// lines — No soln.

Overlapping lines — ∞ solns.

→ Row Picture

i) $x - y = -1$
$2x + y = 4$

Consider $x - y = -1$

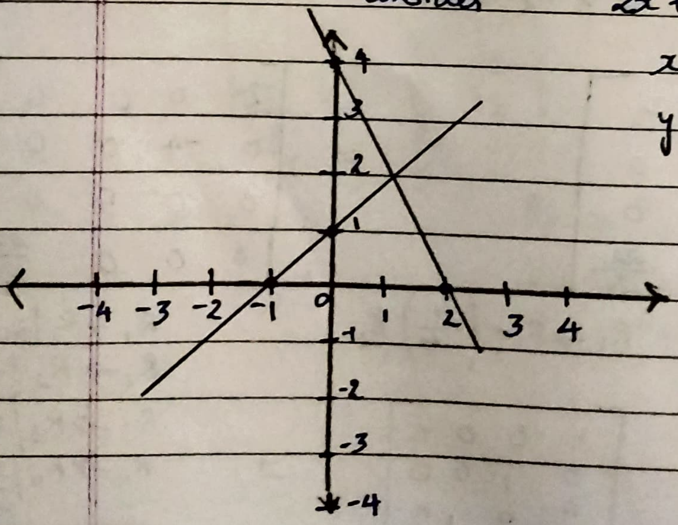
$x = 0 \Rightarrow y = 1$ $(0,1)$

$y = 0 \Rightarrow x = -1$ $(-1,0)$

Consider $2x + y = 4$

$x = 0 \Rightarrow y = 4$ $(0,4)$

$y = 0 \Rightarrow x = 2$ $(2,0)$

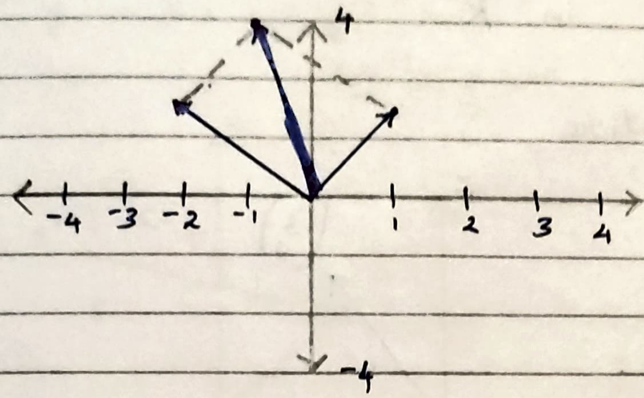


if det $\neq 0$ $\Rightarrow$ unique soln. co efficients $\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} \neq 0$

$\rightarrow$ Column Picture

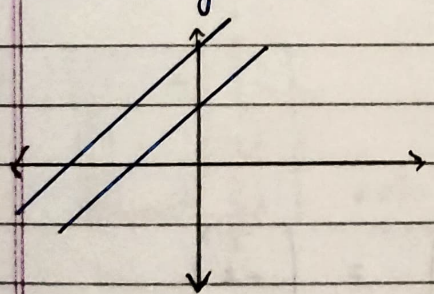
Linear Combination

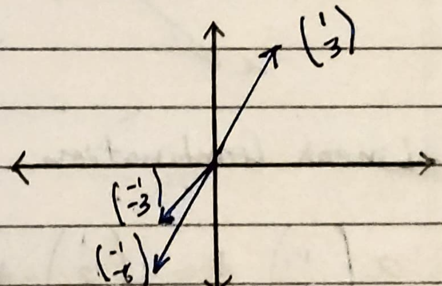
$$x\begin{pmatrix} 1 \\ 2 \end{pmatrix} + y\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$



eg: $x - y = -1$
$3x - 3y = -6$ } ∥ lines $\Rightarrow$ no soln.

$|A| = 0$





$$x\begin{pmatrix} 1 \\ 3 \end{pmatrix} + y\begin{pmatrix} -1 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 \\ -6 \end{pmatrix}$$

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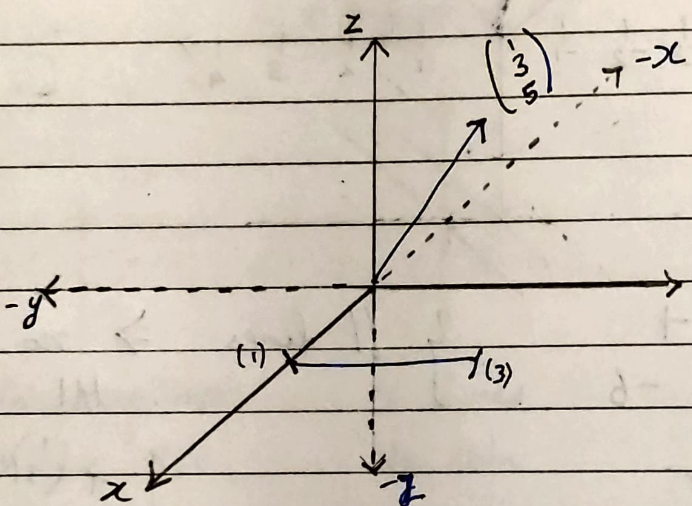
eg: $x - y = -1$
$3x - 3y = -3$ } overlapping lines
∞ solns

3 variables

$$x + y + 2z = 1$$
$$x + 2y - z = -2$$
$$x + 3y + z = 5$$

→ Row Picture : Each eqn. describes a 2-D place in R^3

→ Column Picture



$V = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} \begin{matrix} x \\ y \\ z \end{matrix}$

Linear Combination

$$x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + y \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + z \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$$

eg: $x + y + z = 8$ (plane) 2D

$x + y = 8$ (line) 1D

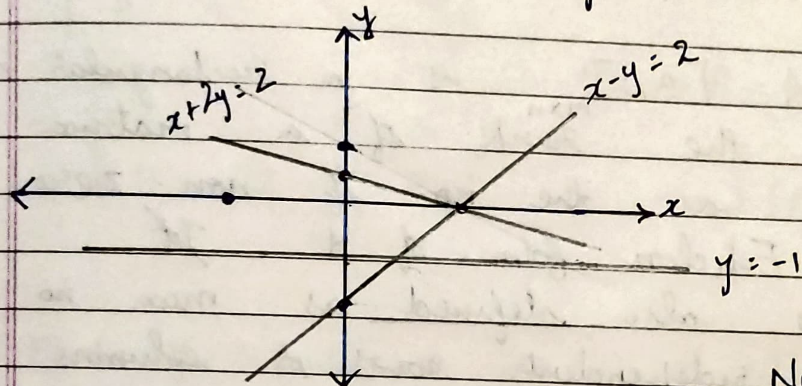
. (point) 0D

q. $x + 2y = 2$
$x - y = 2$
$-y = +1$

} determine if they are solvable ?
What happens if all RHS are 0 ?
Is there any non-zero choice of RHS that allow 3 lines to intersect at a common point of intersection.

Sol. $x + 2y = 2$; $x = 0, y = 1$ $(0,1)$, $y = 0, x = 2$ $(2,0)$
$x - y = 2$; $y = 0, x = 2$ $(2,0)$
$x = 0, y = -2$ $(0,-2)$

& $y = -1$



No soln.
(No common point of inters)

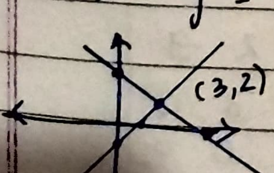
$$* \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 1 & -1 & 2 \\ 0 & -1 & 1 \end{array}\right]$$

make this 0

count non of non 0 rows
should be = full matrix non 0 rows
$\Rightarrow$ unique soln

q. draw row pic & column pic,
discuss consistency, singularity and existence of soln

1. $x + 2y = 7$ $\Rightarrow (0, 3.5)$ $(7, 0)$
$x - y = 1$ $\Rightarrow (0, -1)$ $(1, 0)$



$\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} \neq 0 \Rightarrow$ non singular

2. $2x + 3y = 6$
$4x + 6y = 12$

$\begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 0 \Rightarrow$ Singular

∞ solns.

3. $x - 3y = 5$
$x - 3y = -5$

$\begin{vmatrix} 1 & -3 \\ 1 & -3 \end{vmatrix} = 0 \Rightarrow$ Singular

No soln.

Gaussian Elimination

* If $A = [a_{ij}]_{m \times n}$ is a rectangular matrix then the rank of a matrix is defined as the no. of non-zero in the Echelon form of A. ~~It~~
It is also defined as max. no. of linearly independent rows or columns of matrix A.

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* i.e no. of non-zero rows after reducing to Echelon form

eg i) $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix} = U$

rank(A) = 2

ii) $A = \begin{bmatrix} 2 & 2 & -3 \\ -2 & -2 & 3 \\ 4 & 4 & 6 \end{bmatrix}$

$R_2 \to R_2 - \left(\frac{-2}{2}\right) R_1$

$R_3 \to R_3 - \left(\frac{4}{2}\right) R_1$

$\simeq \begin{bmatrix} 2 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 12 \end{bmatrix}$

$R_2 \leftrightarrow R_3$

$\simeq \begin{bmatrix} 2 & 2 & -3 \\ 0 & 0 & 12 \\ 0 & 0 & 0 \end{bmatrix}$

rank(A) = 2

iii) $A = \begin{bmatrix} 1 & 5 \\ -1 & 3 \\ 2 & 10 \end{bmatrix}$

$R_2 \to R_2 - \left(\frac{-1}{1}\right)R_1$

$R_3 \to R_3 - 2R_1$

$\simeq \begin{bmatrix} 1 & 5 \\ 0 & 8 \\ 0 & 0 \end{bmatrix}$

rank $(A) = 2$

iv) $\begin{bmatrix} 2 & -1 & 3 & 5 \\ 3 & 2 & 1 & 2 \end{bmatrix}$

$R_2 \to R_2 - \left(\frac{3}{2}\right)R_1$

$\simeq \begin{bmatrix} 2 & -1 & 3 & 5 \\ 0 & 7/2 & -7/2 & -11/2 \end{bmatrix}$

rank $(A) = 2$

v) Find the conditions of 'a' & 'b', such that rank of matrix is 1, 2, 3

$\begin{bmatrix} a & 1 & 2 \\ 0 & 2 & b \\ 1 & 3 & 6 \end{bmatrix}$

Soln. $(a \neq 0)$

$R_3 \to R_3 - \left(\frac{1}{a}\right)R_1$

$\simeq \begin{bmatrix} a & 1 & 2 \\ 0 & 2 & b \\ 0 & 3-\frac{1}{a} & 6-\frac{2}{a} \end{bmatrix}$

pivotal elements can't be 0

not simplifying further as calc. become complex

$\simeq \begin{bmatrix} a & 1 & 2 \\ 0 & 2 & b \\ 0 & 1\left(3-\frac{1}{a}\right) & 2\left(3-\frac{1}{a}\right) \end{bmatrix}$

rank = 1

i) For any values of 'a' & 'b' rank $\neq$ 1

ii) $a = 1/3$ & b can be any value
(or) $a \neq 1/3$ & $b = 4$ } rank = 2

iii) $a \neq 1/3$ & $b \neq 4$ rank = 3

→ Relationship b/w rank, consistency & solution

If rank = r, then the foll. conditions hold good

1. If $\boxed{\text{rank}(A) = \text{rank}(A|b)}$
then system is consistent and has a soln.

2. If $\boxed{\text{rank}(A) = \text{rank}(A|b) = n}$, n is unknowns
then system is consistent and has unique soln

3. If $\boxed{\text{rank}(A) = \text{rank}(A|b)} < n$
then system is consistent and has infinite soln

4. If $\boxed{\text{rank}(A) \neq \text{rank}(A|b)}$
then system is inconsistent and has no soln.

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→ Gauss Elimination

Consider $Ax = b$

In matrix form $[\underset{\uparrow\ \text{coefficient matrix}}{A} \mid \underset{\uparrow\ \text{constant vector}}{b}]$ complex system

↓ elementary row operation

Echelon Form $[U \mid c]$

constant vector

$Ux = c$ (Simple system)

then the soln. can be obtained by back substitution (last unknown is obtained first)

Note : No Exchange of Rows
1st row should not be altered

Q1. Check for Consistency and solve

i) $x_1 + x_2 - 2x_3 + 4x_4 = 5$
$2x_1 + 2x_2 - 3x_3 + x_4 = 3$
$3x_1 + 3x_2 - 4x_3 - 2x_4 = 1$

here rank can't be 4 ⇒ no unique soln.

$$(A|b) = \left[\begin{array}{cccc|c} 1 & 1 & -2 & 4 & 5 \\ 2 & 2 & -3 & 1 & 3 \\ 3 & 3 & -4 & -2 & 1 \end{array}\right]$$

$R_2 \to R_2 - \left(\frac{2}{1}\right) R_1$
$R_3 \to R_3 - \left(\frac{3}{1}\right) R_1$

$$\cong \left[\begin{array}{cccc|c} 1 & 1 & -2 & 4 & 5 \\ 0 & 0 & 1 & -7 & -7 \\ 0 & 0 & 2 & -14 & -14 \end{array}\right]$$

$R_3 \to R_3 - 2R_2$

$$\cong \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \underbrace{\left[\begin{array}{cccc|c} 1 & 1 & -2 & 4 & 5 \\ 0 & 0 & 1 & -7 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]}_{A|b} \end{array}$$

(the first four columns are labelled A)

$\text{rank}(A) = \text{rank}(A|b) = 2 < 4$

$\Rightarrow$ infinite no. of soln

To get soln.

consider $Ux = c$

$$x_1 + x_2 - 2x_3 + 4x_4 = 5$$
$$x_3 - 7x_4 = -7$$

The no. of unknowns $= 4$

no. of pivots $=$ rank $= 2$

no. of free variables $= 4 - 2 = 2$

Mark x_1, x_2, x_3, x_4 if in the column a pivot variable exists then it is not free var.

$\therefore$ x_2 and x_4 be free variables ($\because$ x_2 & x_4 columns doesn't contain pivot)

let $x_2 = k_2$, $x_4 = k_1$ (say)

now

(2) $\Rightarrow$ $\boxed{x_3 = 7k_1 - 7}$

(1) $\Rightarrow$ $\boxed{x_1 = 10k_1 - k_2 - 9}$

ii) $x_1 + x_2 - 2x_3 + 3x_4 = 4$
$2x_1 + 3x_2 + 3x_3 - x_4 = 3$
$5x_1 + 7x_2 + 4x_3 + x_4 = 5$

Sol. $(A|b) = \left[\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 4 \\ 2 & 3 & 3 & -1 & 3 \\ 5 & 7 & 4 & 1 & 5 \end{array}\right]$

$R_1 \quad R_2$

$R_2 \rightarrow R_2 - 2R_1$
$R_3 \rightarrow R_3 - 5R_1$

$\simeq \left[\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 4 \\ 0 & 1 & 7 & -7 & -5 \\ 0 & 2 & 14 & -14 & -15 \end{array}\right]$

$R_3 \rightarrow R_3 - 2R_2$

$\simeq \left[\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 4 \\ 0 & 1 & 7 & -7 & -5 \\ 0 & 0 & 0 & 0 & -5 \end{array}\right]$

rank $(A) = 2$
rank $(A|b) = 3$

$\Rightarrow$ No soln.

iii) $x_1 + 2x_2 + x_3 = 3$
$2x_1 + 5x_2 - x_3 = -4$
$3x_1 - 2x_2 - x_3 = 5$

$(A|b) = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 5 & -1 & -4 \\ 3 & -2 & -1 & 5 \end{array}\right]$

$R_2 \rightarrow R_2 - 2R_1$
$R_3 \rightarrow R_3 - 3R_1$

$\simeq \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & -3 & -10 \\ 0 & -8 & -4 & -4 \end{array}\right]$

$R_3 \rightarrow R_3 - (-8)R_2$

$$\simeq \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & -3 & -10 \\ 0 & 0 & -28 & -84 \end{array}\right]$$

Rank (A) = Rank (A|b) = 3 = n

$\Rightarrow$ unique soln.

$$1x_1 + 2x_2 + x_3 = 3$$
$$x_2 - 3x_3 = -10$$
$$-28x_3 = -84$$

$\boxed{x_3 = 3}$ $\boxed{x_2 = -1}$ $\boxed{x_1 = 2}$

iv) $2x - 3y + 2z = 1$

$5x - 8y + 7z = 1$

$y - 4z = 3$

$$(A|b) = \left[\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \\ 0 & 1 & -4 & 3 \end{array}\right]$$

$R_2 \to R_2 - \left(\frac{5}{2}\right)R_1$

$$\simeq \left[\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & -1/2 & 2 & -3/2 \\ 0 & 1 & -4 & 3 \end{array}\right]$$

$R_3 \to R_3 - (-2)R_2$

$$\simeq \left[\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & -1/2 & 2 & -3/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

rank (A) = rank (A|b) = 2 < 3

$\Rightarrow \infty$ solns.

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q2. Find all the values of 'a' for which $Ax = b$ has :

i) No soln.

ii) unique soln.

iii) infinite many soln.

$$x + y - z = 2$$
$$x + 2y + z = 3$$
$$x + y \; (a^2 - 5)z = a$$

Sol. $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 1 & 3 \\ 1 & 1 & a^2-5 & a \end{array}\right] \simeq \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & a^2-4 & a-2 \end{array}\right]$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

i) No soln.

$$\underset{(2)}{\rho(A)} \neq \underset{(3)}{\rho(A|b)}$$

$$a = -2$$

ii) unique soln.

$$\rho(A) = \rho(A|b) = 3$$
$$a \neq \pm 2$$

iii) infinite soln.

$$\rho(A) = \rho(A|b) < 3$$
$$a = 2$$

18th January, 2023

Q6. Finding the eqn. relating a, b & c so that the system is consistent and also find the soln. when $(a,b,c) = (2,3,4)$

$$2x - 3y + 2z = a$$
$$5x - 8y + 7z = b$$
$$0x + y - 4z = c$$

Sol. $(A|b) = \left[\begin{array}{ccc|c} 2 & -3 & 2 & a \\ 5 & -8 & 7 & b \\ 0 & 1 & -4 & c \end{array}\right]$

$R_2 \to R_2 - \left(\frac{5}{2}\right)R_1$

$$\simeq \left[\begin{array}{ccc|c} 2 & -3 & 2 & a \\ 0 & -1/2 & 2 & b - \frac{5}{2}a \\ 0 & 1 & 4 & c \end{array}\right]$$

$R_3 \to R_3 - \left(\frac{1}{-\frac{1}{2}}\right)R_2$

$$\simeq \underbrace{\left[\begin{array}{ccc|c} 2 & -3 & 2 & a \\ 0 & -1/2 & 2 & b - \frac{5}{2}a \\ 0 & 0 & 0 & c + 2b - 5a \end{array}\right]}_{A|b} \quad -(1)$$

(with A marked under the first three columns)

wkt $\rho(A) = 2$, for consistency $\rho(A|b) = 2$

$$\Rightarrow c + 2b - 5a = 0$$

if $(a,b,c) = (2,3,4)$

$$(1) \Rightarrow \begin{array}{c} \begin{matrix} x & y & z & \phantom{3 - \frac{5(2)}{2}} \end{matrix} \\ \left[\begin{array}{ccc|c} 2 & -3 & 2 & 2 \\ 0 & -1/2 & 2 & 3 - \frac{5(2)}{2} \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

rank (A) = rank $(A|b) = 2 < 3$

$\Rightarrow \infty$ soln.

To get soln.

$$2x - 3y + 2z = 2 \quad -(2)$$
$$-\frac{1}{2}y + 2z = -2 \quad -(3)$$

no. of free variables = unknowns − rank
= 3 − 2
= 1

z is free variable
as no pivot in z column

let $z = k$

(2) ⇒ $2x - 3y + 2k = 2$
(3) ⇒ $-\frac{1}{2}y + 2k = -2$

$$-y + 4k = -4$$
$$y = 4k + 4$$

$$2x - 12k - 12 + 2k = 2$$
$$2x - 10k = 14$$
$$x = 7 + 5k$$

Breakdown of gaussian Elimination

If 0 appears in pivot position
(i.e 1st row, 1st col.
2nd row, 2nd col.
⋮
)

We say that gauss elimination will breakdown

1. Temporary breakdown

eg: $\left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & 0 & 2 & 4 \\ 0 & 1 & 8 & 9 \end{array}\right]$ → breakdown happens at 2nd position

here we can't say 2 is pivot as there is a possibility of row exchange

$R_2 \leftrightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & 1 & 8 & 4 \\ 0 & 0 & 2 & 9 \end{array}\right]$$

unique soln.

2. Permanent breakdown

eg: i) $\left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 8 & 9 \end{array}\right]$

$R_3 \to R_3 - \left(\frac{8}{2}\right) R_2$

$$\simeq \left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & -9 \end{array}\right]$$

No solution

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ii) $\left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 8 & 16 \end{array}\right]$

$R_3 \to R_3 - \left(\frac{8}{2}\right) R_2$

$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$ infinite soln.

Case 1 Non - Singular and curable
$(|A| \neq 0)$

eg: $x + y + z = 6$
$x + y + 3z = 10$
$x + 2y + 4z = 12$

$(A \mid b) = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 1 & 3 & 10 \\ 1 & 2 & 4 & 12 \end{array}\right]$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

$\simeq \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & \boxed{0} & 2 & 4 \\ 0 & 1 & 3 & 6 \end{array}\right]$ 2nd pos breakdown

$R_2 \leftrightarrow R_3$

$\simeq \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 2 & 4 \end{array}\right]$

$x + y + z = 6$
$y + 3z = 6$
$2z = 4$

$(4, 0, 2)$

Case 2. Singular and incurable
($|A| = 0$)

$$x + y + z = 6$$
$$x + y + 3z = 10$$
$$x + y + 4z = 13$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 1 & 3 & 10 \\ 1 & 1 & 4 & 13 \end{array}\right]$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

$$\simeq \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & \boxed{0} & 2 & 4 \\ 0 & 0 & 3 & 7 \end{array}\right]$$

breakdown happen at 2nd pos
but it is permanent

$R_3 \to R_3 - (3/2) R_2$

$$\simeq \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$\rho(A) \neq \rho(A|b)$; No soln

Case 3. Singular & incurable (soln. will be there)

$$x + y + z = 6$$
$$x + y + 3z = 10$$
$$x + y + 4z = 12$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 1 & 3 & 10 \\ 1 & 1 & 4 & 12 \end{array}\right]$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

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$$\simeq \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & \boxed{0} & 2 & 4 \\ 0 & 0 & 3 & 6 \end{array}\right]$$ breakdown permanent

$R_3 \to R_3 - (3/2) R_2$

$$\simeq \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\rho(A) = \rho(A|b) = 2 < 3$

∞ soln.

y - free var

$\boxed{y = k}$

$x + y + z = 6$

$\boxed{z = 2}$

$\boxed{x = 4 - k}$

q1. Apply Gaussian elimination to solve $Ax = b$

$u + v + w = -2$

$3u + 3v - w = 6$

$u - v + w = -1$

i) Does elimination fail

ii) If it does then at what position

iii) Is it temporary or permanent

iv) What coefficient of v in 3rd eqn. should be replaced so that the breakdown is permanent.

Sol.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 3 & 3 & -1 & 6 \\ 1 & -1 & 1 & -1 \end{array}\right]$$

$R_2 \to R_2 - 3R_1$

$R_3 \to R_3 - R_1$

breakdown (fails)
2nd pos
tempdary

$$\simeq \left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & \boxed{0} & -4 & 12 \\ 0 & -2 & 0 & 1 \end{array}\right]$$

$R_2 \leftrightarrow R_3$

$$\simeq \left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & -4 & 12 \end{array}\right]$$

make co efficient of v=1 for permanent breakdown

q2. For what values of 'a' will elimination breakdown not give 3 pivots

$$\left[\begin{array}{ccc|c} a & 2 & 3 & b1 \\ a & a & 4 & b2 \\ a & a & a & b3 \end{array}\right]$$

Sol. $(a \neq 0)$ $R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc|c} a & 2 & 3 & b1 \\ 0 & a-2 & 1 & b2-b1 \\ 0 & a-2 & a-3 & b3-b1 \end{array}\right]$$

$(a-2 \neq 0)$ $R_3 \to R_3 - R_2$

$$\left[\begin{array}{ccc|c} a & 2 & 3 & b1 \\ 0 & a-2 & 1 & b2-b1 \\ 0 & 0 & a-4 & (b_3-b_1)-(b_2-b_1) \end{array}\right]$$

~~(a-4~~

for elimination to fail
a) $a=0$
b) $a=2$
c) $a=4$

q3. For what values of 'a' and 'b' does the foll. system

$$x + 2y + 3z = 2$$
$$-x - 2y + az = 2$$
$$2x + by + 6z = 5$$

i) No soln.
ii) unique soln.
iii) ∞ soln.

Elementary Matrices

* Elementary matrix E_{ij} is obtained from I by using row transformation

$$R_i - l_{ij} R_j$$

multiplier

$$I \rightarrow E_{ij}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R_2 \rightarrow R_2 - R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_{21}$$

q1. Which Elementary matrices put A into upper Δ^{lar} matrix U

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$

$R_2 \to R_2 - (2/3)R_1$
$R_3 \to R_3 - (1/3)R_1$

$$\simeq \begin{bmatrix} 3 & 1 & 2 \\ 0 & -11/3 & -7/3 \\ 0 & 5/3 & 1/3 \end{bmatrix}$$

$R_3 \to R_3 - \left(\frac{5/3}{-11/3}\right) R_2$

$$\simeq \begin{bmatrix} 3 & 1 & 2 \\ 0 & -11/3 & -7/3 \\ 0 & 0 & -8/11 \end{bmatrix} = U$$

(i) $E = E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(ii) $G = E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix}$

(iii) $F = E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5/11 & 1 \end{bmatrix}$

$FGEA = U$

Q2. $\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$

$$R_2 \to R_2 - \left(-1/2\right) R_1$$

$$\simeq \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$R_3 \to R_3 - \left(-1/\tfrac{3}{2}\right) R_2$$

$$\simeq \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$R_4 \to R_4 - \left(-1/\tfrac{4}{3}\right) R_3$$

$$\simeq \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix}$$

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22nd January, 2024

Triangular factors

$$A = L\ U$$

square/rect matrix (A): $n \times n$, $m \times n$

L: square matrix, $m \times m$, $n \times n$

U: Echelon matrix / upper triangular matrix, $m \times n$, $n \times n$

where

$L \rightarrow$ lower Δ matrix with 1s on the main diagonal and having multipliers below the diagonal

$U \rightarrow$ Echelon form & upper Δ matrix where diagonal can contain pivots.

we have, $E_{32} E_{31} E_{21} A = U$

from this we have

$$L = E_{21}^{-1} E_{31}^{-1} E_{32}^{-1}$$

or

$$L = (E_{32} E_{31} E_{21})^{-1}$$

or

$$L^{-1} = E_{32} E_{31} E_{21}$$

Note: 1. It is not a symmetric factor system
if method fails - if 0 appears in pivot position (row exchange)

1. Factorise $A = LU$ given

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$

$R_2 \to R_2 - (2/3)R_1$

$R_3 \to R_3 - (1/3)R_1$

$$\simeq \begin{bmatrix} 3 & 1 & 2 \\ 0 & -11/3 & -7/3 \\ 0 & 5/3 & 1/3 \end{bmatrix}$$

$R_3 \to R_3 - \left(\frac{5/3}{-11/3}\right) R_2$

$$\simeq \begin{bmatrix} 3 & 1 & 2 \\ 0 & -11/3 & -7/3 \\ 0 & 0 & -8/11 \end{bmatrix} = U$$

$$\underset{3\times3}{A} = \underset{3\times3}{L}\ \underset{3\times3}{U}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & -5/11 & 1 \end{bmatrix}$$

2. $$A = \begin{bmatrix} 0 & 2 & -6 & -2 & 4 \\ 0 & -1 & 3 & 3 & 2 \\ 0 & -1 & 3 & 7 & 10 \end{bmatrix}$$

$$\underset{3\times5}{A} = \underset{3\times3}{L}\ \underset{3\times5}{U}$$

$$U = \begin{bmatrix} 0 & 2 & -6 & -2 & 4 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/2 & 3 & 1 \end{bmatrix}$$

* $A = L\ D\ U$ form (Symmetric decomposition)

L → lower Δ^{lar} matrix, same as $A = LU$

D → diagonal matrix → diagonal elements contain pivots

U → Echelon form matrix → it is same as $A = LU$ where in U each row has to be divided by pivotal element

q1. Factorize $A = LU$ & $A = LDU$ form

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$R_2 \to R_2 - (-1/2) R_1$

$$\simeq \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$R_3 \to R_3 - \left(-1/_{3/2}\right) R_2$

$$\simeq \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$R_4 \to R_4 - \left(-1/_{4/3}\right) R_3$

$$\simeq \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1/3 \\ 0 & 0 & 0 & 5/4 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{bmatrix} \quad 4\times4$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3/2 & 0 & 0 \\ 0 & 0 & 4/3 & 0 \\ 0 & 0 & 0 & 5/4 \end{bmatrix} \begin{bmatrix} 1 & -1/2 & 0 & 0 \\ 0 & 1 & -2/3 & 0 \\ 0 & 0 & 1 & -3/4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

diagonals are pivots rest 0

divide the entire row by pivot.

Note: Using Triangular factors solving system of eqns. is not in syllabus.

2. $$A = \begin{bmatrix} 6 & -2 & -4 & 4 \\ 3 & -3 & -6 & 1 \\ -12 & 8 & 21 & -8 \\ -6 & 0 & -10 & 7 \end{bmatrix}$$

Row Exchanges and Permutation Matrices

* Permutation matrix is an identity matrix but not necessarily with same order

eg: $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

(interchange in identity matrix whatever you want to interchange in the matrix)

* Product of 2 Permutation matrix is also a Permutation matrix.

* Inverse of a PM is also a PM

* P^{-1} is always same as P^T

* PM of any order : $2 \rightarrow 2! = 2$ in no.
  $3 \rightarrow 3! = 6$ in no.
  $4 \rightarrow 4! = 24$ in no.

q1. $A = \begin{bmatrix} 2 & 3 & 3 \\ 6 & 9 & 8 \\ 0 & 5 & 7 \end{bmatrix} \simeq \begin{bmatrix} 2 & 3 & 3 \\ 0 & \underline{0} & -1 \\ 0 & 5 & 7 \end{bmatrix}$ fails at 2nd pos

$R_2 \rightarrow R_2 - 6/2 R_1$

$R_2 \leftrightarrow R_3$

$\simeq \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix}$

$LU = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix} \neq A$

Using Permutation Matrix ($P_{23}A$)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 & 3 \\ 6 & 9 & 8 \\ 0 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix}$$

$R_3 \to R_3 - 3R_1$

$$\simeq \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix} = U$$

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix} \to P_{23}A$$

24 th January, 2024

q2. Check whether $A = LU$ if not $PA = LU$ & find all the permutation matrices that makes $PA = LU$

$$A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & 1 & 4 & 10 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 6 \end{bmatrix}$$

Sol. $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, $R_4 \to R_4 - R_1$

$$\simeq \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 0 & 0 & 6 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 3 & 2 \end{bmatrix}$$

($A \neq LU$)

Zero appears in pivot position & we need to interchange

$R_2 \leftrightarrow R_3$

$$P_{23} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

now $B = P_{23} A$

$$= \begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 10 \\ 1 & 3 & 5 & 6 \end{bmatrix}$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, $R_4 \to R_4 - R_1$

$$\simeq \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 2 & 3 & 2 \end{bmatrix}$$

$R_4 \to R_4 - 2R_2$

$$\simeq \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

zero appears in pivot pos. $B \neq LU$

so we need to interchange $R_3 \leftrightarrow R_4$

$$P_{34} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$C = P_{34} B = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 6 \\ 1 & 1 & 2 & 10 \end{bmatrix}$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, $R_4 \to R_4 - R_1$

$$\simeq \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 6 \end{bmatrix} = U$$

$\Rightarrow C = LU$

$\Rightarrow P_{34} B = LU$

$$\Rightarrow P_{34} P_{23} A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

q3. Check $A = LU$, if not $PA = LU$

$$A = \begin{bmatrix} 0 & 2 & 4 \\ 0 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

zero appears in pivot part $A \neq LU$
& we need to interchange $R_1 \leftrightarrow R_3$

$$P_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$BP_{13} = A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 2 & 4 \end{bmatrix}$$

$R_3 \to R_3 - \left(\frac{2}{5}\right) R_2$

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$$\simeq \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 8/5 \end{bmatrix} = U$$

$\Rightarrow B = LU$

$$P_{13}A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2/5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 8/5 \end{bmatrix}$$

q4. Check $A = LU$, if not $PA = LU$ & $PA = LDU$
Explain why?

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -4 & 5 \\ -2 & 5 & -4 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 + 2R_1$

$$\simeq \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

zero appears in pivot pos

$A \neq LU$, interchange $R_2 \leftrightarrow R_3$

$$P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P_{23}A = B = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 5 & -4 \\ 2 & -4 & 5 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$
$R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = U$$

$$B = LU$$

$$P_{23}A = LU$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_{23}A = LDU$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= LDL^T$$

The relationship b/w U & L is $U = L^T$ or $U^T = L$
(B is symmetric matrix)

Note: If $A \neq LU$ and if $PA = LDU$, where PA is a symmetric matrix
then

$$\boxed{PA = LDL^T}$$

Inverses and Transposes

→ Gauss Jordan Method of computing A^{-1} :

diagonal matrix

$$[A : I] \rightarrow [U : C] \rightarrow [D : F] \rightarrow [I : A^{-1}]$$

Q1. Compute A^{-1} using Gauss Jordan Method

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 4 \\ -2 & 2 & 2 \end{bmatrix}$$

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & 4 & 0 & 1 & 0 \\ -2 & 2 & 2 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \rightarrow R_2 - 2R_1$

$R_3 \rightarrow R_3 + R_1$

$$\cong \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 3 & 3 & 1 & 0 & 1 \end{array}\right]$$

$R_3 \rightarrow R_3 - 3R_2$

$$\cong \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & -3 & 7 & -3 & 1 \end{array}\right] = [U \mid C]$$

$R_1 \rightarrow R_1 - (1/-3)R_1$

$R_2 \rightarrow R_2 - (2/-3)R_3$

$$\cong \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 10/3 & -1 & 1/3 \\ 0 & 1 & 0 & 8/3 & -1 & 2/3 \\ 0 & 0 & -3 & 7 & -3 & 1 \end{array}\right]$$

$R_1 \rightarrow R_1 - R_2$

$$\cong \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 2/3 & 0 & -1/3 \\ 0 & 1 & 0 & 8/3 & -1 & 2/3 \\ 0 & 0 & -3 & 7 & -3 & 1 \end{array}\right] = [D \mid F]$$

$R_1 \rightarrow R_1/2, \quad R_3 \rightarrow R_3/-3$

$$\cong \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 0 & -1/6 \\ 0 & 1 & 0 & 8/3 & -1 & 2/3 \\ 0 & 0 & 1 & -7/3 & 1 & -1/3 \end{array}\right] = [I \mid A^{-1}]$$

$$A^{-1} = \begin{bmatrix} 1/3 & 0 & -1/6 \\ 8/3 & -1 & 2/3 \\ -7/3 & 1 & -1/3 \end{bmatrix}$$

Q2. $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 3 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

$$\simeq \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 + 2R_2$

$$\simeq \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & -3 & 2 & 1 \end{array}\right] = [U|C]$$

$R_1 \to R_1 - (1/2)R_3$
$R_2 \to R_2 - (1/2)R_3$

$$\simeq \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 5/2 & -1 & -1/2 \\ 0 & 1 & 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & 2 & -3 & 2 & 1 \end{array}\right]$$

$R_1 \to R_1 - 2R_2$

$$\simeq \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/2 & -1 & 1/2 \\ 0 & 1 & 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & 2 & -3 & 2 & 1 \end{array}\right] = [D|F]$$

$R_3 \to R_3 / 2$

$$\simeq \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/2 & -1 & 1/2 \\ 0 & 1 & 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & 1 & -3/2 & 1 & 1/2 \end{array}\right] = [I|A^{-1}]$$

A^{-1}

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$2-\frac{1}{2}$ $\quad 2-\frac{2}{3}$

$\frac{3}{2}-\frac{1}{4}$ $\quad \frac{1}{2}+\frac{1}{4}=\frac{3}{4}$

q3. $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

$$[A|I] = \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 + \left(\frac{1}{2}\right) R_1$

$$\simeq \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & -1 & 1/2 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 + \frac{2}{3} R_2$

$$\simeq \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & -1 & 1/2 & 1 & 0 \\ 0 & 0 & 4/3 & 1/3 & 2/3 & 1 \end{array}\right] = [U|C]$$

$R_2 \to R_2 + \left(\frac{3}{4}\right) R_3$

$$\simeq \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 0 & 3/4 & 3/2 & 3/4 \\ 0 & 0 & 4/3 & 1/3 & 2/3 & 1 \end{array}\right]$$

$R_1 \to R_1 + (2/3) R_2$

$$\simeq \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 3/2 & 1 & 1/2 \\ 0 & 3/2 & 0 & 3/4 & 3/2 & 3/4 \\ 0 & 0 & 4/3 & 1/3 & 2/3 & 1 \end{array}\right] = [D|F]$$

$R_2 \to R_2/(3/2),\ R_1 \to R_1/2,\ R_3 \to R_3/(4/3)$

$$\simeq \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & 1/2 & 1/4 \\ 0 & 1 & 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & 1 & 1/4 & 1/2 & 3/4 \end{array}\right] = [I : A^{-1}]$$

Transpose of a Matrix

* Transpose of a lower triangular matrix is an upper triangular matrix.

→ $(A^T)^T = A$
→ $(AB)^T = B^T A^T$
→ $(A^{-1})^T = (A^T)^{-1}$
→ $(A \pm B)^T = A^T \pm B^T$
→ $(A^{-1})^T A^T = (AA^{-1})^T = I$

→ Symmetric Matrices

* Symmetric matrix $\Rightarrow$ $A^T = A$

* If A is symmetric matrix A^{-1} may / may not exist

* If for a SM A^{-1} exists then A^{-1} is also symmetric

* For a SM A, $(A^{-1})^T = (A)^{-1}$

$l_{11} = \sqrt{a_{11}}$	$l_{21} = \dfrac{a_{21}}{l_{11}}$	$l_{31} = \dfrac{a_{31}}{l_{11}}$ $l_{32} = \dfrac{a_{32} - l_{31} l_{21}}{l_{22}}$
	$l_{22} = \sqrt{a_{22} - l_{21}^2}$	$l_{33} = \sqrt{a_{33} - l_{31}^2 - l_{32}^2}$

q. For which three numbers "c" is matrix not invertible and why not?

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}$$

$R_2 \to R_2 - (c/2)R_1$

$R_3 \to R_3 - (8/2)R_1$

$$\simeq \begin{bmatrix} 2 & c & c \\ 0 & c - \frac{c^2}{2} & c - \frac{c^2}{2} \\ 0 & 7-4c & c-4c \end{bmatrix}$$

$R_3 \to R_3 - \left(\frac{7-4c}{c - \frac{c^2}{2}}\right) R_2$

$$\simeq \begin{bmatrix} 2 & c & c \\ 0 & c - c^2/2 & c - c^2/2 \\ 0 & 0 & c-7 \end{bmatrix}$$

for not invertible

$$c - \frac{c^2}{2} = 0$$

$$c - 7 = 0$$

$c = 7$, $\quad 2c - c^2 = 0$

$$c(c-2) = 0$$

$$c = 0, \ c = 2$$

Cholesky decomposition / Factorization

$$A = LL^T \to \underset{A}{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{22} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}} = \underset{L}{\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}} \underset{L^T}{\begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}}$$

q1. Factorize $A = \begin{bmatrix} 4 & 2 & 6 \\ 2 & 2 & 5 \\ 6 & 5 & 22 \end{bmatrix}$ using cholesky decomp.

Sol. $A = LL^T$

$$\begin{bmatrix} 4 & 2 & 6 \\ 2 & 2 & 5 \\ 6 & 5 & 22 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{12} & l_{13} \\ 0 & l_{22} & l_{23} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$l_{11} = \sqrt{a_{11}} = \sqrt{4} = 2$$

$$l_{12} = l_{21} = \frac{a_{21}}{l_{11}} = \frac{2}{2} = 1$$

$$l_{22} = \sqrt{a_{22} - l_{21}^2} = \sqrt{2-1} = 1$$

$$l_{13} = l_{31} = \frac{a_{31}}{l_{11}} = \frac{6}{2} = 3$$

$$l_{32} = l_{23} = \frac{a_{32} - l_{31} l_{21}}{l_{22}} = \frac{5 - 3(1)}{1} = 2$$

$$l_{33} = \sqrt{a_{33} - l_{31}^2 - l_{32}^2} = \sqrt{22-9-4} = \sqrt{9} = 3$$

$$A = \begin{bmatrix} 4 & 2 & 6 \\ 2 & 2 & 5 \\ 6 & 5 & 22 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 2 & 3 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}}_{L^T}$$

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q2. Factorize $A = \begin{bmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{bmatrix}$ using cholesky factorization

ans: $L = \begin{bmatrix} 2 & 0 & 0 \\ 6 & 1 & 0 \\ -8 & 5 & 3 \end{bmatrix}$

$$A = \begin{bmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$l_{11} = \sqrt{a_{11}} = 2$

$l_{21} = \dfrac{a_{21}}{l_{11}} = \dfrac{12}{2} = 6$

$l_{22} = \sqrt{a_{22} - l_{21}^2} = \sqrt{37 - 36} = 1$

$l_{31} = \dfrac{a_{31}}{l_{11}} = \dfrac{-16}{2} = -8$

$l_{32} = \dfrac{a_{32} - l_{31} l_{21}}{l_{22}} = \dfrac{-43 - (-8)(6)}{1} = 5$

$l_{33} = \sqrt{a_{33} - l_{31}^2 - l_{32}^2} = \sqrt{98 - 64 - 25} = 3$

$$A = \begin{bmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{bmatrix} = \underset{L}{\begin{bmatrix} 2 & 0 & 0 \\ 6 & 1 & 0 \\ -8 & 5 & 3 \end{bmatrix}} \underset{L^T}{\begin{bmatrix} 2 & 6 & -8 \\ 0 & 1 & 5 \\ 0 & 0 & 3 \end{bmatrix}}$$