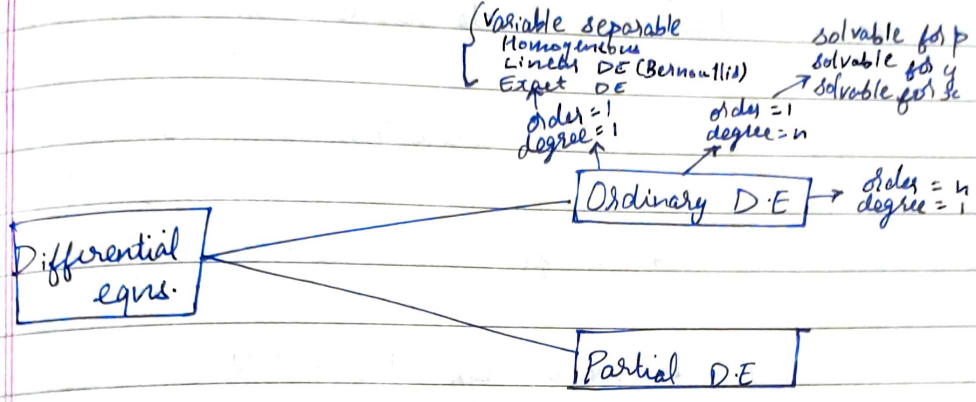
2nd December, 2022

UNIT-4

Partial Differential Equations



* Definition : An eqn. consisting x, y, z, p, q defines a first order partial D.E

That is $f(x,y,z,p,q) = 0$ —①

where $p = \dfrac{\partial z}{\partial x}$ & $q = \dfrac{\partial z}{\partial y}$

This eqn. is linear if it is linear in p, q

* Definition : An eqn. consisting x, y, z, p, q, r, s, t defines a second order P.D.E

i.e $g(x, y, z, p, q, r, s, t) = 0$ —②

where $p = \dfrac{\partial z}{\partial x} = z_x$, $q = \dfrac{\partial z}{\partial y} = z_y$, $r = \dfrac{\partial^2 z}{\partial x^2} = z_{xx}$,

$s = \dfrac{\partial^2 z}{\partial x \partial y} = z_{xy}$, $t = \dfrac{\partial^2 z}{\partial y^2} = z_{yy}$

* The order of a P.D.E is defined to be the order of the highest derivative in the eqn (basically same as ODE)

* The degree of P.D.E is the degree of the highest order partial derivative in the eqn.

$O=2, D=1$

examples: i) 1-D heat conduction eqn. : $U_t = U_x$ $\boxed{\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}}$

ii) Laplace eqn: $U_{xx} + U_{yy} = 0$ $\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$

$O=2, D=1$

iii) 1-D wave eqn. : $U_{tt} = c^2 U_{xx}$

$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}$ $O=2, D=1$

$\rightarrow \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} + 2\left(\frac{\partial u}{\partial x}\right)^2 = 0$; $O=2, D=1$

→ Formation of First & Second order PDE

WORKING RULE

* Given the func of the form $f(x,y,z,a,b) = 0$ ①
where z is a func of x & y and a, b are arbitrary constants

Step-1 Differentiate ① partially w.r.t x & y

Step-2 Eliminate the arbitrary constants a & b from the above eqn.

Self Learning Component : Formation of PDE by eliminating arbitrary functions.

Note * It may not always be possible to eliminate the arbitrary constants. In such cases obtain the second order PDE & then elimate them

⊗ This higher order PDE may not be unique

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* If we eliminate $n+1$ constants from the eqn. It may be possible to obtain an n^{th} order PDE

Q. Form the PDE by eliminating arbitrary constants from the foll. relations

1. $\underbrace{Z = (x-a)^2 + (y-b)^2}_{\text{arbitrary constants}}$ — ①

Sol. i) Diff partially wrt x

$$\underbrace{\frac{\partial Z}{\partial x}}_{p} = 2(x-a) = p \Rightarrow x-a = \frac{p}{2}$$

ii) Diff partially wrt y

$$\frac{\partial z}{\partial y} = q = 2(y-b) \Rightarrow y-b = \frac{q}{2}$$

iii) substitute in ①

$$Z = \frac{p^2}{4} + \frac{q^2}{4}$$

$$Z = \frac{1}{4}\left[\left(\frac{\partial Z}{\partial x}\right)^2 + \left(\frac{\partial Z}{\partial y}\right)^2\right]$$

O = 1
D = 2

2. $Z = (x+a)(y+b)$

$$\frac{\partial Z}{\partial x} = p = (y+b)(1)$$

$$\frac{\partial Z}{\partial y} = q = (x+a)(1)$$

$$\Rightarrow Z = pq$$

$$Z = \frac{\partial Z}{\partial x} \cdot \frac{\partial z}{\partial y}$$

3. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = Z$

$$\frac{\partial Z}{\partial x} = \frac{2x}{a^2} = p \Rightarrow \frac{1}{a^2} = \frac{p}{2x}$$

$$\frac{\partial Z}{\partial y} = \frac{2y}{b^2} = q \Rightarrow \frac{1}{b^2} = \frac{q}{2y}$$

$$\Rightarrow \frac{x^2(p)}{2x} + y^2\left(\frac{q}{2y}\right) = Z$$

$$\frac{xp}{2} + \frac{yq}{2} = Z$$

$$\frac{1}{2}[xp + yq] = Z$$

$$\frac{1}{2}\left[x\frac{\partial Z}{\partial x} + y\frac{\partial Z}{\partial y}\right] = Z$$

4. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

Z is considered as pure thing → dependent variable

$Z = f(x,y)$

$$\frac{\partial Z}{\partial x} = \frac{2x}{a^2} = p \Rightarrow \frac{1}{a^2} = \frac{p}{2x}$$

$$\frac{\partial Z}{\partial y} \qquad \frac{1}{b^2} = \frac{q}{2y}$$

$$\frac{\partial Z}{\partial z} \qquad \frac{1}{c^2} = \frac{r}{\cdot}$$

$$\frac{2x}{a^2} + \frac{2z}{c^2}\frac{\partial z}{\partial x} = 0 \quad -(2)$$

$$\frac{2y}{b^2} + \frac{2z}{c^2}\frac{\partial z}{\partial y} = 0 \quad -(3)$$

constant not eliminated so we differentiate again wrt x

$$\frac{2}{a^2} + \frac{2}{c^2}\frac{\partial z}{\partial x}\frac{\partial z}{\partial x} + \frac{2z}{c^2}\frac{\partial^2 z}{\partial x^2} = 0$$

$$\frac{1}{a^2} + \frac{1}{c^2}p^2 + \frac{z}{c^2}\cdot r = 0 \quad -④$$

From ② $\frac{1}{a^2} = -\frac{z}{c^2}\cdot\frac{p}{x}$

substitute this in ④

$$-\frac{z}{c^2}\frac{p}{x} + \frac{1}{c^2}p^2 + \frac{z}{c^2}r = 0$$

$$-\frac{zp}{x} + p^2 + zr = 0$$

$$\frac{zp}{x} = p^2 + zr$$

$$zp = x(p^2 + zr)$$

try diff wrt y & get PDE

$$\frac{2}{b^2} + \frac{2}{c^2}\frac{\partial z}{\partial y}\frac{\partial z}{\partial y} + \frac{2z}{c^2}\frac{\partial^2 z}{\partial y^2} = 0$$

$$\frac{1}{b^2} = \frac{1}{c^2}p^2 + \frac{z}{c^2}t = 0$$

From ③

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5. $z = ce^{-\omega t} \sin(\omega x)$

$$\frac{\partial z}{\partial x} = ce^{-\omega t} \cos(\omega x) \cdot \omega \quad -①$$

$$\frac{\partial z}{\partial t} = C \sin(\omega x) e^{-\omega t} (-\omega) \quad -②$$

$$\frac{\partial^2 z}{\partial x^2} = -Ce^{-\omega t} \omega^2 \sin(\omega x) \quad -③$$

$$\frac{\partial^2 z}{\partial t^2} = C\omega^2 e^{-\omega t} \sin(\omega x) \quad -④$$

from ③ & ④

$$\frac{\partial^2 z}{\partial x^2} = -\frac{\partial^2 z}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = 0$$

6. Family of all spheres whose centres lie on the XY plane and have the constant radius r

centre = $(a, b, 0)$

eqn. of sphere $\Rightarrow (x-a)^2 + (y-b)^2 + (z-0)^2 = r^2 \quad -①$

$$\frac{\partial}{\partial x}(\ ): 2(x-a) + 0 + 2z\frac{\partial z}{\partial x} = 0$$

$z \Rightarrow$ dependent variable

$$x - a = -zp \quad -②$$

$$\frac{\partial}{\partial x}(\ ): 2(y-b) + 2z\frac{\partial z}{\partial y} = 0$$

$$y - b = -zq \quad -③$$

② & ③ in ①

$$z^2p^2 + z^2q^2 + z^2 = r^2$$

$$z^2(p^2 + q^2 + 1) = r^2$$

Solution of a PDE

1. Complete integral / Complete solution:

Any relation of the form $f(x, y, z, a, b) = 0$ which contains two arbitrary constants and satisfies the PDE $f(x, y, z, p, q) = 0$ is called a complete integral / complete solution. Such a solution is also called an integral surface.

eg: $Z = px + py + pq \rightarrow$ PDE
$Z = ax + by + ab \rightarrow$ solution

2. Particular integral / Particular solution:

The solution obtained by determining the arbitrary constants in the complete integral / complete solution or the arbitrary func. in the general integral (general solution) by using some specified conditions is called as a particular integral soln. / particular integral.

eg: $Z = px + qy + pq \rightarrow$ PDE
$Z = ax + by + ab \rightarrow$ complete soln.
$Z = 4x + 5y + 20 \rightarrow$ particular soln. $\left(\begin{matrix} a=4 \\ b=5 \end{matrix}\right)$

3. General integral / General solution:

A relation of the form $\phi(u, v)$, where ϕ is an arbitrary function of $u = u(x, y, z)$ and $v = v(x, y, z)$ and satisfies $f(x, y, z, p, q) = 0$ is called a general integral / general solution of the PDE $f(x, y, z, p, q) = 0$

4. Singular integral/ singular solution

If $f(x,y,z,a,b)$ is the complete integral then the eqn. of the surface obtained by eliminating a and b from the eqns. $f=0$, $\frac{\partial f}{\partial a}=0$, $\frac{\partial f}{\partial b}=0$ is the singular solution.

Note: Singular soln. may not exist for every PDE.

eg: PDE $z = px + qy + pq$

Soln $z = ax + by + ab$

$$\frac{\partial z}{\partial a} = x + b = 0 \Rightarrow b = -x$$

$$\frac{\partial z}{\partial b} = y + a = 0 \Rightarrow a = -y$$

$$\Rightarrow z = -xy - xy + xy$$

$$z = -xy \rightarrow \text{singular solution}$$

Solution of PDE by method of Direct Integration

WORKING RULE:

Step 1. Given a PDE, integrate it w.r.t the independent variable

Step 2. If the integration is w.r.t x then add a func. of y as arbitrary constant.

Step 3. If the integration is w.r.t y then add a func. of x as arbitrary constant.

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Q: Solve $\frac{\partial^2 z}{\partial x \partial y} = \frac{x}{y} + a$ using the method of direct integration.

Step 1: Integrating w.r.t y

$$\int \frac{\partial^2 z}{\partial x \partial y} = \int\left(\frac{x}{y} + a\right) dy + f(x)$$

$$\frac{\partial z}{\partial x} = x \ln y + ay + f(x)$$

$\int$ this eqn.

Step 2: Integrating w.r.t x

$$\cancel{\frac{\partial z}{\partial y}} \; z = \frac{x^2}{2} \ln y + axy + \int f(x)\,dx + g(y)$$

alternative:

Integrating w.r.t x

Step 1 $$\frac{\partial z}{\partial y} = \frac{x^2}{2y} + ax + f(y)$$

Step 2 w.r.t y $$z = \frac{x^2}{2} \ln y + axy + \int f(y)\,dy + g(x)$$

Q: Solve $\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x + 3y)$ using D.I

integrating w.r.t y $\Rightarrow$ $$\frac{\partial^2 z}{\partial x^2} = \frac{\sin(2x+3y)}{3} + f(x) \quad (1)$$

integrating w.r.t x $\Rightarrow$ $$\frac{\partial z}{\partial x} = \frac{-\cos(2x+3y)}{3(2)} + \int f(x) + g(y)$$

$$\frac{\partial z}{\partial x} = -\frac{\cos(2x+3y)}{6} + F(x) + g(y) \quad (2)$$

where $F(x) = \int f(x)$

integrating w.r.t x $\Rightarrow$ $$z = \frac{-\sin(2x+3y)}{12} + \int F(x)\,dx + x\,g(y) + h(y)$$

q. Solve : $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$ ~~when x = 0~~ given that $\frac{\partial z}{\partial y} = -2\sin y$ when $x = 0$

& $z = 0$ when y is an odd multiple of $\frac{\pi}{2}$

integrating wrt x

$$\frac{\partial z}{\partial y} = -\cos x (\sin y) + f(y) \quad -(1)$$

integrating wrt y

$F(y) = \int f(y)$

$$z = \cos x \cos y + F(y) + g(x) \quad -(2)$$

~~given $\Rightarrow -2\sin y = -\cos x (\sin y) + f(y)$, $x = 0$~~

~~$-2 = -1 + f(y)$~~

~~$f(y) = -1$~~

~~0~~ $x = 0$

$$-2\sin y = -\cos x (\sin y) + f(y)$$

$$-2\sin y = -\sin y + f(y)$$

$$-\sin y = f(y)$$

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$$\Rightarrow F(y) = \int f(y)\, dy$$

$$= \int -\sin y = \cos y$$

$z = 0$, $y = (2n+1)\frac{\pi}{2}$ in (2)

$\cos y = \cos (2n+1)\frac{\pi}{2} = 0$

$$0 = 0 + 0 + g(x)$$

$$g(x) = 0$$

Solution : substitute $g(x)$ & $f(y)$ in (2)

$$z = \cos x \cos y + \cos y + 0$$

$$z = \cos y (\cos x + 1)$$

$e^{-xy} = e^{-y}$ $\cdot$ ~~Dq = x~~ $q(D-x)$

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q: Solve the eqn. $\frac{\partial^2 z}{\partial x^2} = x + y$, given that $z = y^2$ when $x = 0$ & $\frac{\partial z}{\partial x} = 0$ when $x = 2$

integrating w.r.t x

$$\frac{\partial z}{\partial x} = \frac{x^2}{2} + xy + f(y) \quad -①$$

integrating w.r.t x

$$z = \frac{x^3}{6} + \frac{x^2}{2}y + x f(y) + g(y) \quad -②$$

$\frac{\partial z}{\partial x} = 0$ when $x = 2$ in ①

$$0 = 2 + 2y + f(y) \quad \Rightarrow f(y) = -2(y+1)$$

$z = y^2$ when $x = 0$

$$y^2 = 0 + 0 + 0 + g(y)$$

$$\Rightarrow g(y) = y^2$$

$$z = \frac{x^3}{6} + \frac{x^2 y}{2} - 2x(y+1) + y^2$$

q: Solve $\frac{\partial^2 z}{\partial y^2} - x\frac{\partial z}{\partial y} = -\sin y - x\cos y$

wkt $\frac{\partial z}{\partial y} = q \quad \therefore \frac{\partial q}{\partial y} - xq = -\sin y - x\cos y \quad -①$

If x is treated as a constant then ① is a LDE (first order) in q

$$\Rightarrow \frac{dq}{dy} - x\, q = -\sin y - x\cos y$$

$$IF = e^{\int -x\,dy} = e^{-xy}$$

solution: $e^{-xy} \cdot q = \int e^{-xy}[-\sin y - x\cos y]\,dy + f(x)$

$$= \int \frac{d}{dy}[e^{-xy}\cos y]\,dy + f(x)$$

$\frac{d}{dy}[e^{-xy}\cos y] = -xe^{-xy}\cos y - \sin y\, e^{-xy}$

$$e^{-xy}\cdot\frac{\partial z}{\partial y} = e^{-xy}\cos y + f(x)$$

$$\frac{\partial z}{\partial y} = \cos y + e^{xy} f(x)$$

integrating wrt y

$$z = \sin y + \frac{e^{xy}}{x}\cdot f(x) + g(x)$$

q. Solve $\frac{\partial^2 z}{\partial x\partial y} + 9x^2y^2 = \cos(2x-y)$ given that $z=0$ when $y=0$ & $\frac{\partial z}{\partial y} = 0$ when $x=0$

integrating wrt x

$$\frac{\partial z}{\partial y} + \cancel{9}^{3}y^2\frac{x^3}{\cancel{3}} = \frac{\sin(2x-y)}{2} + f(y) \quad (1)$$

integrating wrt y

$$z + x^3y^3 = \cancel{-}\frac{\cos(2x-y)}{2(\cancel{-1})} + \int f(y) + g(x) \quad (2)$$

$z=0$, $y=0$ $\quad 0+0 = \frac{\cos(2x)}{2} + \int f(y) + g(x)$

$\frac{\partial z}{\partial y}=0$, $x=0$

$$0+0 = \frac{\sin(-y)}{2} + f(y)$$

$1-\cos 2x = 2\sin^2 x$

$$0 = -\frac{\sin y}{2} + f(y) \Rightarrow f(y) = \frac{\sin y}{2}$$

$$\int f(y) = -\frac{\cos y}{2}$$

$z=0$, $y=0$ $\Rightarrow 0 = \frac{\cos 2x}{2} - \frac{\cos y}{2} + g(x) \Rightarrow 0 = \frac{\cos 2x}{2} - \frac{1}{2} + g(x)$

$$g(x) = \frac{1 - \cos 2x}{2} = \sin^2 x$$

$$Z + x^3y^3 = \frac{\cos(2x-y)}{2} - \frac{\cos y}{2} + \frac{\cos^2 y - \cos 2x}{2}$$

$$Z = -x^3y^3 + \frac{\sin(2x-y)}{2} + \frac{-\cos y}{2} + \sin^2 x$$

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Unit - 4

Partial Differential Equations (continued)

Lagrange's Linear Equation

* The Linear first order P.D.E of the form

$$Pp + Qq = R \qquad p = \frac{\partial z}{\partial x}, \; q = \frac{\partial z}{\partial y}, \; z = f(x, y)$$

is called the Lagrange's equation in two independent variables x, y

* Theorem: The general solution of the equation $Pp + Qq = R$ where $P, Q, R \rightarrow f(x, y, z)$ where ϕ is an arbitrary constant and $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ are two linearly independent solutions of the equations $\underbrace{\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}}_{\text{auxillary eqn.}}$

$W \neq 0$ determinant

soln: $\phi(u, v) = 0$

WORKING RULE:

Step 1. Form the auxillary eqns. / subsidary eqns.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{Z} \quad -①$$

Step 2. Obtain the solutions $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ of eqn ① by taking any two pairs and using the methods of solution of Ordinary D.E

Step 3. Then $\phi(u, v) = 0$ forms the solution of the given Lagrange's Linear PDE

eg: $\frac{dx}{2xz} = \frac{dy}{2yz} = \frac{dz}{2z^2y}$

EDGE

① $\dfrac{dx}{-xz} = \dfrac{dy}{-yz} \Rightarrow \int \dfrac{dx}{x} = \int \dfrac{dy}{y}$

$\dfrac{x}{y} = c_1 \Rightarrow u = c_1$

② $\dfrac{dy}{-yz} = \dfrac{dz}{-z^2 y} \Rightarrow dy = \dfrac{dz}{z} \Rightarrow y = \log z + c_2$

$y - \log z = c_2$

$v = c_2$

Sol: $\phi(u, v) = 0 \Rightarrow \phi\left(\dfrac{x}{y}, y - \log x\right) = 0$

→ To obtain the solution of Auxillary equations:

Case 1: If x, y or z is not present or gets cancelled from one of the pairs in ① then the relation b/w other two variables forms one part of the solution and the other part can be obtained similarly.

Case 2: Sometimes when we are not able to get the solution as mentioned in case 1, we use multipliers a, b, c such that

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{a\,dx + b\,dy + c\,dz}{aP + bQ + cR} \quad —②$$

(choose a, b, c such that: $aP + bQ + cR = 0$)

Then integrating these we obtain the relations connecting x, y, z which constitute the solution.

Note: The multipliers a, b, c are generally chosen in such a way that $aP + bQ + cR = 0$

URBAN EDGE

so that we get $a\,dx + b\,dy + c\,dz = 0$ which can be easily integrated.

q. Solve $2yzp + zxq = 3xy$

Sol → $Pp + Qq = R$

1. $P = 2yz$ $Q = zx$ $R = 3xy$

2. write auxillary eqns. $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{2yz} = \frac{dy}{zx} = \frac{dz}{3xy}$$

3. consider

$$\frac{dx}{2y\cancel{z}} = \frac{dy}{\cancel{z}x}$$

$\int x\,dx = \int 2y\,dy$ → ODE (variable seperable)

$\frac{x^2}{2} = y^2 + c$

$x^2 - 2y^2 = c_1$

4. consider

$$\frac{dy}{\cancel{z}\cancel{x}} = \frac{dz}{3\cancel{x}y}$$

$\int y\,dy = \int \frac{z}{3}\,dz$

$\frac{y^2}{2} = \frac{z^2}{6} + c$

$3y^2 - z^2 = c_2$

soln: $\phi(x^2 - 2y^2, 3y^2 - z^2) = 0$

q. Solve $2p + q = \sin(x - 2y)$

→ $Pp + Qq = R$

Sol. 1. $P = 2$ $Q = 1$ $R = \sin(x - 2y)$

2. $\frac{dx}{2} = \frac{dy}{1} = \frac{dz}{\sin(x-2y)}$

3. consider $\frac{dx}{2} = \frac{dy}{1}$ ⇒ $x = 2y + c$

$x - 2y = c_1$

EDGE

4. Consider $\dfrac{dy}{} = \dfrac{dz}{\sin(x-2y)} = \dfrac{dz}{\sin(c_1)}$

$$y = \frac{z}{\sin(x-2y)} + c$$

~~(integrating wrt z, ... const which we obtained previously)~~

$$y - \frac{z}{\sin(x-2y)} = c_2$$

$\therefore$ soln. : $\phi\left(x-2y,\ y - \dfrac{z}{\sin(x-2y)}\right) = 0$

q. Solve : $(x+2z)p + (4zx-y)q = 2x^2+y$

Soln. $Pp + Qq = R$

$P = x+2z \qquad Q = 4zx - y \qquad R = 2x^2+y$

AE: $\dfrac{dx}{x+2z} = \dfrac{dy}{4zx-y} = \dfrac{dz}{2x^2+y}$ —(1)

using multipliers $-2x, +1, +1$ ✓ each ratio in (1)

or $2x, -1, -1$ ✓

$$= \frac{-2x\,dx + dy + dz}{\cancel{-2x^2} - \cancel{4xz} + \cancel{4zx} - \cancel{y} + \cancel{2x^2} + \cancel{y}} = k$$

$\therefore -2x\,dx + dy + dz = 0$

Integrating

$$-x^2 + y + z = c$$

$x^2 - y - z = c_1$ $\rightarrow$ one part of soln (u)

to find other part of soln (v) take new set of multipliers

$2xy + 4yz + 4zx - y \quad 2x^2 + y$

URBAN EDGE

new multipliers : $y, x, -2z$

$$\frac{y\,dx + x\,dy - 2z\,dz}{\cancel{xy} + \cancel{2yz} + \cancel{4zx^2} - \cancel{xy} - \cancel{4zx^2} - \cancel{2yz}}$$

$$y\,dx + x\,dy - 2z\,dz = 0.$$

$$d(xy) - 2z\,dz = 0$$

integrating

$$xy - z^2 = c_2$$

$\therefore$ soln: $\phi(x^2 - y - z,\ xy - z^2) = 0$

q. Solve : $\underbrace{(x^2 - y^2 - z^2)}_{P}p + \underbrace{2xy}_{Q}q = \underbrace{2xz}_{R}$

AE : $\dfrac{dx}{x^2 - y^2 - z^2} = \dfrac{dy}{2xy} = \dfrac{dz}{2xz}$ — (1)

consider $\dfrac{dy}{\cancel{2x}y} = \dfrac{dz}{\cancel{2x}z}$ $\Rightarrow \ln \dfrac{y}{z} = \ln c$

$$\frac{y}{z} = c_1 \rightarrow \text{dont substitue in (1)}$$

Here we cannot find multipliers

* Using the multipliers x, y, z each ratio in (1) is equal to

$$\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2xz} = \frac{x\,dx + y\,dy + z\,dz}{x(x^2 - y^2 - z^2) + y(2xy) + z(2xz)}$$

$x^3 - xy^2 - xz^2 + 2xy^2 + 2xz^2$

$x^3 + xy^2 + xz^2 \Rightarrow x(x^2 + y^2 + z^2)$

$$= \frac{x\,dx + y\,dy + z\,dz}{x(x^2 + y^2 + z^2)}$$

Consider

$$\frac{dz}{2\cancel{x}z} = \frac{x\,dx + y\,dy + z\,dz}{\cancel{x}(x^2+y^2+z^2)}$$

$$\frac{dz}{z} = \frac{2(x\,dx + y\,dy + z\,dz)}{x^2+y^2+z^2}$$

$$\frac{dz}{z} = \frac{d(x^2+y^2+z^2)}{x^2+y^2+z^2}$$

integrating

$$\log z = \log(x^2+y^2+z^2) + \log c_2$$

$$\Rightarrow \frac{z}{x^2+y^2+z^2} = c_2$$

$$\therefore \phi\left(\frac{y}{z}, \frac{z}{x^2+y^2+z^2}\right) = 0$$

if we consider

$$\frac{dy}{2\cancel{x}y} = \frac{x\,dx + y\,dy + z\,dz}{\cancel{x}(x^2+y^2+z^2)}$$

$$\frac{dy}{y} = \frac{d(x^2+y^2+z^2)}{x^2+y^2+z^2}$$

$$\log y = \log(x^2+y^2+z^2) + \log c_2$$

$$\frac{y}{x^2+y^2+z^2} = c_2 \qquad \text{also} \qquad \frac{x^2+y^2+z^2}{y} = c_2$$

$$\phi\left(\frac{y}{z}, \frac{y}{x^2+y^2+z^2}\right)$$

$$\phi\left(\frac{y}{z}, \frac{x^2+y^2+z^2}{y}\right)$$

Q. Solve: $\underbrace{y^2}_{P} p \underbrace{- xy}_{Q} q = \underbrace{x(z-2y)}_{R}$

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

$$\frac{dx}{y^{\cancel{z}}} = \frac{dy}{-xy}$$

$$-x\,dx = y\,dy$$

$$-\frac{x^2}{2} = \frac{y^2}{2} + c$$

$$x^2 + y^2 = c_1$$

$$x^3 - xy^2 - x^2y + y^3$$

$$x(x^2-y^2) - y(x^2-y^2)$$

$$(x^2-y^2)(x-y)$$

$$\frac{dy}{-\cancel{x}y} = \frac{dz}{\cancel{x}(z-2y)}$$

$$z\,dy - 2y\,dy = -y\,dz$$

$$z\,dy + y\,dz - 2y\,dy = 0$$

$$d(yz) - 2y\,dy = 0$$

integrating $$yz - y^2 = c_2$$

$\therefore$ soln: $\phi(x^2+y^2,\ yz-y^2)$

q. Solve: $\underbrace{(x^2-y^2-yz)}_{P}p + \underbrace{(x^2-y^2-zx)}_{Q}q = \underbrace{z(x-y)}_{R}$

$$\frac{dx}{x^2-y^2-yz} = \frac{dy}{x^2-y^2-zx} = \frac{dz}{z(x-y)}$$

$$zx - zy$$

using multipliers $1, -1, -1$ each ratio in ① =

$$\frac{+dx - dy - dz}{\cancel{x^2} \cancel{-y^2} - \cancel{yz} - \cancel{x^2} + \cancel{y^2} - \cancel{zx} - \cancel{zx} + \cancel{zy}} = \frac{dx - dy - dz}{0} \Rightarrow dx - dy - dz = 0$$

$$\Rightarrow x - y - z = 0$$

To find second soln: use $x, -y$ multipliers

$$\frac{x\,dx - y\,dy}{x^3 - xy^2 - \cancel{xyz} - x^2y + y^3 + \cancel{xyz}} = \frac{dz}{z(x-y)}$$

$$\frac{x\,dx - y\,dy}{(x^2-y^2)\cancel{(x-y)}} = \frac{dz}{z\cancel{(x-y)}}$$

$$\frac{2x\,dx - 2y\,dy}{x^2-y^2} = \frac{2dz}{z}$$

$$\frac{d(x^2-y^2)}{x^2-y^2} = \frac{2dz}{z}$$

integrating $\log(x^2-y^2) = 2\log z + \log c_2$

$$\frac{x^2-y^2}{z^2} = c_2$$

$\therefore$ Soln: $\phi\left(x-y-z, \frac{x^2-y^2}{z^2}\right) = 0$

Q. Solve: $\underbrace{(y+z)}_{P}p + \underbrace{(z+x)}_{Q}q = \underbrace{x+y}_{R}$

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} \quad -(1)$$

$$\cancel{\frac{x\,y}{y+z} = \frac{xy^2}{z+x} = \frac{}{x+y}}$$
$$\cancel{xy^2 + xzy \quad yz^2 + xyz}$$

$$\frac{dx+dy+dz}{2(x+y+z)} = \frac{dx-dy}{y-x} = \frac{dy-dz}{z-y} \quad -(2)$$

$$\frac{d(x+y+z)}{2(x+y+z)} = \frac{d(x-y)}{-(x-y)}$$

$$\frac{1}{2}\log(x+y+z) = -\log(x-y) + \log c_1$$

$$\frac{(x+y+z)(x-y)^2}{\cancel{(x-y)^2}} = c_1$$

$$\frac{dx - dy}{y - x} = \frac{dy - dz}{z - y}$$

$$\frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z}$$

$$\ln(x-y) = \ln(y-z) + \ln c.$$

$$\frac{x-y}{y-z} = c_2$$

soln $\phi\left(\frac{x+y+z(x-y)^2}{\cancel{\quad}}, \frac{x-y}{y-z}\right)$

q. $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$

multipliers won't work

16th December, 2022

Method of Separation of Variables

WORKING RULE : In the given DE, if x and y are the independent variables and z is the dependent variable then proceed as follows:

Step 1. Assume the solution of the given PDE to be $z = XY$ —① where $X = X(x)$ is a function of x alone and $Y = Y(y)$ is a function of y alone

Step 2. Substitute for z and its partial derivatives in the given PDE

Step 3. Rewrite the equation so that the LHS involves X and its derivatives and RHS involves Y and its derivatives

Step 4. Equate each side to a constant k

Step 5. Solve the resulting ODE for X and Y

Step 6. To obtain the soln. of the given PDE, substitute for X and Y in ①

q. Solve : $x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} = 0$ —①

Sol. Step 1 here $u \to (x, y)$

u dependent on x & y

Let $u = XY$ —② where $X = X(x)$ & $Y = Y(y)$ be the soln. of eqn ①

$X(0)^{10} + y x'$

URBAN EDGE __/__/__

Step 2 Substitute in ①

$y \to$ func of x only

$\therefore \frac{\partial X}{\partial x} = \frac{dX}{dx}$

$\frac{\partial (XY)}{\partial x} = Y\frac{\partial X}{\partial x} = Y\frac{dX}{dx}$

$$x^2 \frac{\partial (XY)}{\partial x} + y^2 \frac{\partial (XY)}{\partial y} = 0$$

$$x^2 Y \frac{dX}{dx} + y^2 X \frac{dY}{dy} = 0$$

Step 3 $\div XY$

$$x^2 \cdot \frac{1}{X} \frac{dX}{dx} + y^2 \cdot \frac{1}{Y} \frac{dY}{dy} = 0$$

$$\Rightarrow x^2 \frac{1}{X} \frac{dX}{dx} = - y^2 \frac{1}{Y} \frac{dY}{dy}$$

Step 4 $x^2 \frac{1}{X} \frac{dX}{dx} = k$

$$-y^2 \frac{1}{Y} \frac{dY}{dy} = k$$

$$\Rightarrow \frac{dX}{X} = \frac{k\,dx}{x^2} \qquad \Rightarrow \frac{dY}{Y} = \frac{-k\,dy}{y^2}$$

$$\ln X = -\frac{k}{x} + c_1 \qquad \ln Y = \frac{k}{y} + c_2$$

$$X = e^{-k/x + c_1} \qquad Y = e^{k/y + c_2}$$

Step 5 substitute in ②

$$u = XY$$

$$u = e^{-\frac{k}{x} + c_1} \cdot e^{\frac{k}{y} + c_2}$$

$$= e^{c_1 + c_2} \cdot e^{k\left(\frac{1}{y} - \frac{1}{x}\right)}$$

$$\boxed{u = C e^{k\left(\frac{1}{y} - \frac{1}{x}\right)}} \qquad C = e^{c_1 + c_2}$$

q: Solve $\frac{\partial u}{\partial x} = 2\frac{\partial u}{dt} + u$ —① where $u(x,0) = 6e^{-3x}$

Sol Step1 Let $U = XT$ where $X = X(x)$ & $T = T(t)$ be the soln. of ①

Step2 sub in ①

$$\frac{\partial (XT)}{\partial x} = \frac{2\partial (XT)}{dt} + XT$$

$$T\frac{\partial X}{\partial x} = 2X\frac{\partial T}{dt} + XT$$

$$T\frac{dX}{dx} = 2X\frac{dT}{dt} + XT$$

Step3 $\div XT$ $\frac{1}{X}\frac{dX}{dx} = \frac{2}{T}\frac{dT}{dt} + 1$

Step4 Equate each side to a constant

$$\frac{1}{X}\frac{dX}{dx} = k \quad ; \quad \frac{2}{T}\frac{dT}{dt} + 1 = k$$

$$\frac{dX}{X} = k\,dx \qquad \frac{dT}{T} = \frac{k-1}{2}dt$$

$$\ln X = kx + c_1 \qquad \ln T = \left(\frac{k-1}{2}\right)t + c_2$$

$$X = e^{kx+c_1} \quad ; \quad T = e^{\left(\frac{k-1}{2}\right)t + c_2}$$

Step 5 Substitute these in ②

$$u = XT$$

$$u = e^{kx + c_1 + \left(\frac{k-1}{2}\right)t + c_2}$$

$$= e^{c_1+c_2} \cdot e^{kx + \left(\frac{k-1}{2}\right)t}$$

$$u = Ce^{kx + \left(\frac{k-1}{2}\right)t} \quad —③$$

URBAN EDGE

Step 6 $u(x,0) = 6e^{-3x}$

$\because t=0,\ \cancel{x=?}\ u = 6e^{-3x}$

$6e^{-3x} = Ce^{kx}$

By comparing $C=6$ & $k=-3$

Step 7 substitute in ③

$$u = 6e^{-3x-2t}$$

H/W q. Solve : $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$ given $u(0,y) = 3e^{-y} - \cancel{e^{5y}}$

q. Solve : $u_{xy} = u \Rightarrow \frac{\partial^2 u}{\partial x \partial y} = u$ -①

Sol. Step 1 Let $u = XY$ -② where $X = X(x)$ & $Y = Y(y)$ be soln. of ①

Step 2 Substitute ② in ①

$$\frac{\partial^2 (XY)}{\partial x \partial y} = XY$$

$$\frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}(XY)\right] = XY$$

$$\frac{\partial}{\partial x}\left[X\frac{dY}{dy}\right] = XY$$

$$\frac{dX}{dx}\cdot\frac{dY}{dy} = XY$$

$\frac{\partial (XY)}{\partial x} = YX'$

$\frac{\partial (YX')}{\partial y} = X'Y'$

$\frac{dX}{dx}\frac{dY}{dy}$

Step 3 $\frac{1}{X}\frac{dX}{dx} = Y\frac{dy}{dY}$

Step 4 equate each side to a constant k

$\frac{1}{X}\frac{dX}{dx} = k \quad ; \quad y\frac{dy}{dY} = k$

$\frac{dX}{X} = k\,dx \quad . \quad \frac{dY}{Y} = \frac{dy}{k}$

$\ln X = kx + c_1 \quad \ln Y = \frac{y}{k} + c_2$

$X = e^{kx+c_1} \quad Y = e^{y/k + c_2}$

$\therefore u = XY$

$u = e^{kx+c_1 + y/k + c_2}$

$u = e^{(c_1+c_2)}\, e^{(kx + \frac{y}{k})}$

$\boxed{u = C\, e^{kx + \frac{y}{k}}}$

9 Solve : $\frac{du}{dx} + \frac{du}{dy} = 2(x+y)u$

Let $u = XY$

$\frac{\partial(XY)}{\partial x} + \frac{\partial(XY)}{\partial y} = 2(x+y)(XY)$

$Y\frac{dX}{dx} + X\frac{dY}{dy} = 2(x+y)(XY)$

$Y\frac{dX}{dx} + X\frac{dY}{dy} = 2XYx + 2XYy$

$\div XY \quad \frac{1}{X}\frac{dX}{dx} + \frac{1}{Y}\frac{dY}{dy} = 2x + 2y$

$\frac{1}{X}\frac{dX}{dx} - 2x = 2y - \frac{1}{Y}\frac{dY}{dy}$

URBAN EDGE

$\frac{1}{X}\frac{dX}{dx} - 2x = k$; $2y - \frac{1}{Y}\frac{dY}{dy} = k$

$\frac{1}{X}\frac{dX}{dx} = k + 2x$

$\frac{1}{Y}\frac{dY}{dy} = 2y - k$

$\frac{1}{X}dX = (k+2x)dx$

$\frac{dY}{Y} = (2y-k)dy$

$\ln X = kx + x^2 + c_1$

$\ln Y = y^2 - ky + c_2$

$X = e^{kx+x^2+c_1}$

$Y = e^{y^2-ky+c_2}$

$u = XY$

$u = e^{\cancel{kx}+x^2+y^2-\cancel{kx}}$

$\cancel{u = e^{x^2+y^2}}$

$u = e^{c_1+c_2}\, e^{kx+x^2+y^2-ky}$

$\boxed{u = Ce^{x^2+y^2+k(x-y)}}$

Q. $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$

$u = XY$

$\Rightarrow 4\frac{\partial(XY)}{\partial x} + \frac{\partial(XY)}{\partial y} = 3XY$

$4Y\frac{dX}{dx} + X\frac{dY}{dy} = 3XY$

$\div XY$

$\frac{4}{X}\frac{dX}{dx} + \frac{dY}{Ydy} = 3$

$\frac{4}{X}\frac{dX}{dx} = 3 - \frac{dY}{Ydy}$

$\frac{4}{X}\frac{dX}{dx} = k$; $3 - \frac{dY}{Ydy} = k$

$\frac{4dX}{X} = kdx$

$(3-k)dy = \frac{dY}{Y}$

$4\ln X = kx$

$\ln X = \frac{kx}{4} + c_1$

$X = e^{\frac{kx}{4}+c_1}$

$3y - ky = \ln Y$

$Y = e^{3y-ky+c_2}$

$u = e^{(c_1+c_2)} \cdot e^{\left(\frac{kx}{4} - ky + 3y\right)}$

$u = C \cdot e^{\left(\frac{kx}{4} - ky + 3y\right)}$

$3e^{-y} = Ce^{(0-ky+3y)}$

$3e^{-y} = Ce^{-ky+3y}$

$\boxed{C=3}$

$-1 = -k+3$

$\boxed{k=4}$

19th December, 2022

Q. Solve the 1-D heat eqn: $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ (1) using the method of separation of variables

Sol Let $u = XT$ where $X = X(x)$ & $T = T(t)$

$$\frac{\partial (XT)}{dt} = c^2 \frac{\partial^2 (XT)}{\partial x^2}$$

$$X \frac{dT}{dt} = c^2 \, T \frac{d^2(X)}{dx^2}$$

$\div XT$

$$\frac{1}{T}\frac{dT}{dt} = c^2 \frac{1}{X}\frac{d^2X}{dx^2}$$

~~$\ln T = c^2$~~

$$\frac{1}{T}\frac{dT}{dt} = k \qquad c^2 \frac{1}{X}\frac{d^2X}{dx^2} = k$$

$$\frac{1}{T} dT = k\,dt \qquad \frac{1}{X} d^2X = \frac{k}{c^2} dx^2$$

$D = \frac{d}{dx}$

$$\ln T = kt + c_1 \qquad \frac{d^2X}{dx^2} - \frac{k}{c^2}X = 0$$

$T = e^{kt + c_1}$ order: 2, degree: 1

$$T = a e^{kt} \qquad \left(D^2 - \frac{k}{c^2}\right)X = 0$$

$$m^2 - \frac{k}{c^2} = 0 \Rightarrow m^2 = \frac{k}{c}$$

$$m = \pm\frac{\sqrt{k}}{c}$$

Since it is an application problem we need to take diff. cases

Case i) if k is +ve, say $k = p^2$ then $\left(m = \pm\frac{p}{c}\right)$

$$T = a e^{p^2 t} \qquad X = c_1 e^{\left(\frac{p}{c}\right)x} + c_2 e^{-\left(\frac{p}{c}\right)x}$$

URBAN EDGE

Case ii) if k is -ve, say $k = -p^2$ $\left(m = \pm \frac{ip}{c}\right)$

$T = ae^{-p^2 t}$ $\qquad X = c_1 \cos\left(\frac{p}{c}\right)x + c_2 \sin\left(\frac{p}{c}\right)x$

Case iii) if $k = 0$ $\qquad (m = 0, 0)$

$T = a$ $\qquad X = (c_1 + c_2 x)\cancel{e^{0x}}$ 1

∴ The various possible solutions of ① are

$u = XT$

~~ae^{p^2t}, c_1~~

$$u = ae^{p^2 t}\left(c_1 e^{(p/c)x} + c_2 e^{-(p/c)x}\right) \quad -(i)$$

$$u = ae^{-p^2 t}\left(c_1 \cos(p/c)x + c_2 \sin(p/c)x\right) \quad -(ii)$$

$$u = a\,(c_1 + c_2 x) \quad -(iii)$$

As we are dealing with the problem on heat conduction it must be a transient soln. i.e u must decrease with increase in time t

∴ The solution given by (ii) is the most suitable soln. of the heat eqn.

Q: Solve the Laplace eqn. : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ using the method of separation of variables

Sol Let $u = XY$

$$\frac{\partial^2 (XY)}{\partial x^2} + \frac{\partial^2 (XY)}{\partial y^2} = 0$$

$$Y\frac{d^2 X}{dx^2} + X\frac{d^2 Y}{dy^2} = 0$$

$\div XY$ $$\frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2} = 0$$

URBAN EDGE

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2}$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = k \quad ; \quad \frac{-1}{Y}\frac{d^2Y}{dy^2} = k$$

$$\frac{d^2X}{dx^2} - kX = 0 \qquad \frac{d^2Y}{dy^2} + kY = 0$$

$D = \frac{d}{dx}$

$$(D^2 - k)X = 0 \qquad (D^2 + k)Y = 0$$

$$m^2 - k = 0 \qquad m^2 + k = 0$$

$$m = \pm\sqrt{k} \qquad m = \pm i\sqrt{k}$$

Case i) if k is +ve, i.e $k = p^2$ $(m = \pm p \;,\; m = \pm ip)$

$\therefore X = c_1 e^{px} + c_2 e^{-px} \qquad Y = c_3 \cos py + c_4 \sin py$

Case ii) if k is -ve i.e $k = -p^2$ $(m = \pm ip \;,\; m = \pm p)$

$\therefore X = c_5 \cos px + c_6 \sin px \qquad Y = c_7 e^{py} + c_8 e^{-py}$

Case iii) if $k = 0$

$\therefore X = c_9 + c_{10}x \qquad Y = c_{11} + c_{12}y$

The various possible solutions are:

$$U = XY$$

$$U = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$$

$$U = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py})$$

$$U = (c_9 + c_{10}x)(c_{11} + c_{12}y)$$

out of these three solns. we take the soln. i.e consistent with the given boundary value condition

(here no conditions given)

H/W: Q. Solve the wave eqn. given by $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

~~Let u = XT~~

Q. $\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial y} + 2z$; $z(0,y) = 0$, $z_x(0,y) = e^{2y}$

Sol. Let $z = XY$

$$\frac{\partial^2 (XY)}{\partial x^2} = \frac{\partial (XY)}{\partial y} + 2(XY)$$

$$Y\frac{d^2X}{dx^2} = X\frac{dY}{dy} + 2XY$$

$\div XY$

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{Y}\frac{dY}{dy} + 2$$

$\frac{1}{X}\frac{d^2X}{dx^2} = k$; $\frac{1}{Y}\frac{dY}{dy} + 2 = k$

$\frac{d^2X}{dx^2} - kX = 0$ — $\frac{1}{Y}\frac{dY}{dy} = k-2$

$(D^2 - k)X = 0$ — $\frac{1}{Y}dY = (k-2)dy$

$m^2 - k = 0$

$m^2 = k$ — $\ln Y = ky - 2y + c_1$

$m = \pm\sqrt{k}$ — $Y = e^{ky - 2y + c_1}$

$X = c_1 e^{\sqrt{k}x} + c_2 e^{-\sqrt{k}x}$ — $Y = a e^{(k-2)y}$

~~Case i) if k is +ve ie $k = p^2$ ($m = \pm p^2$~~

$$u = XY = b_1 e^{\sqrt{k}x + (k-2)y} + b_2 e^{-\sqrt{k}x + (k-2)y}$$

21st December, 2022

Solution of Homogeneous Linear PDE

* Any eqn. of the form

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2} \partial y^2} + \dots a_n \frac{\partial^n z}{\partial y^n} = F(x,y) \quad -①$$

where $a_0, a_1, a_2 \dots a_n$ are all constants is called as Homogeneous Linear PDE with constant coefficients

Note : In all terms of ① the partial derivatives are of same order.

* The operators D and D'

$$Z = f(x,y) \begin{cases} \partial z/\partial x & D = \frac{\partial}{\partial x} = D_x \\ \partial z/\partial y & D' = \frac{\partial}{\partial y} = D_y \end{cases}$$

we are following D & D' notation

eg: $5\frac{\partial z}{\partial x} + 6\frac{\partial z}{\partial y} = 0 \Rightarrow [5D + 6D']Z = 0$

* WORKING RULE to find Complimentary func. (CF)

Consider the PDE $a_0 \frac{\partial^2 z}{\partial x^2} + a_1 \frac{\partial^2 z}{\partial x \partial y} + a_2 \frac{\partial^2 z}{\partial y^2} = 0 \quad -①$

Step-1 Using $D = \frac{\partial}{\partial x}$ & $D' = \frac{\partial}{\partial y}$ write the given homogeneous LPDE in the symbolic form as:

$$[a_0 D^2 + a_1 DD' + a_2 (D')^2] Z = 0 \quad -②$$

Step-2 $\div$ ① by $(D')^2$ to obtain $\left[a_0 \left(\frac{D}{D_1}\right)^2 + a_1 \frac{D}{D'} + a_2\right] Z = 0 \quad -③$

Step-3 The Auxillary eqn. is given by $a_0 m^2 + a_1 m + a_2 = 0$
where $m = \frac{D}{D'}$

Step-4 Solve the A.E and obtain its roots

Step-5 Based on the nature of roots we write the C.F which itself is the general soln. of the D.E.

Case 1. Real and Distinct roots : m_1 and m_2
The general soln. is $Z = f_1(y + m_1 x) + f_2(y + m_2 x)$

Case 2. Real and repeated roots : $m = m_1 = m_2$
The general soln. is $Z = f_1(y + mx) + x f_2(y + mx)$

Q. Solve : $2\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0$

Sol.

Step-1 $[2D^2 + 5DD' + 2(D')^2]Z = 0$ where $D = \frac{\partial}{\partial x}$ $D' = \frac{\partial}{\partial y}$

Step-2 $\div (D')^2$ $\left[2\left(\frac{D}{D'}\right)^2 + 5\left(\frac{D}{D'}\right) + 2\right]Z = 0$

Step-3 AE : $2m^2 + 5m + 2 = 0$ where $\frac{D}{D'} = m$

4
$\wedge$
$-4 \quad -1$
$-2 \quad -1/2$

Step-4 roots : $-2, -1/2$

Step-5 $Z = f_1(y - 2x) + f_2(y - \frac{1}{2}x)$

Q. Solve: $r + 6s + 9t = 0$

Sol $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$, $t = \frac{\partial^2 z}{\partial y^2}$

Step-1 $\frac{\partial^2 z}{\partial x^2} + 6\frac{\partial^2 z}{\partial x \partial y} + 9\frac{\partial^2 z}{\partial y^2} = 0$

Step-2 Symbolic form : $(D^2 + 6DD' + 9D'^2)Z = 0$

Step-3 ~~A.E~~: $\left(\left(\frac{D}{D'}\right)^2 + 6\frac{D}{D'} + 9\right) Z = 0$

Step-4 A.E $m^2 + 6m + 9 = 0$

Step-5 roots : $-3, -3$

Step-6 $Z = f_1(y-3x) + x f_2(y-3x)$

q. Solve : $(D_x^3 - 6D_x^2 D_y + 11 D_x D_y^2 - 6D_y^3)Z = 0$

Sol $[D^3 - 6D^2 D' + 11DD' - 6D'^3]Z = 0$

$\div D_1^3 \Rightarrow \left[\left(\frac{D}{D'}\right)^3 - 6\left(\frac{D}{D'}\right)^2 + 11\left(\frac{D}{D'}\right) - 6\right] Z = 0$

A.E : $m^3 - 6m^2 + 11m - 6 = 0$

roots : $1, 2, 3$

solution $Z = f_1(y+x) + f_2(y+2x) + f_3(y+3x)$

→ Important points to remember :

1. If the linear factor is of the form $(aD + bD' + c)Z = 0$ then the G.S is $Z = e^{-\left(\frac{cx}{a}\right)} \phi(bx - ay)$

general soln.

2. If the linear factor is of the form $(aD + bD')Z = 0$ G.S is $Z = \phi(bx - ay)$

3. If the linear factor is of form $(bD' + c)Z = 0$

G.S is $Z = k_2 e^{-\left(\frac{cy}{b}\right)}$

4. If the linear factor is of form $(aD + c)Z = 0$

G.S is $Z = k_2 e^{-\left(\frac{cx}{a}\right)} \phi(ay)$

22nd December, 2022

#

Consider the PDE

$$a_0 \frac{\partial^2 z}{\partial x^2} + a_1 \frac{\partial^2 z}{\partial x \partial y} + a_2 \frac{\partial^2 z}{\partial y^2} = F(x,y) \quad -①$$

Step 1 write the given LPDE in symbolic form $f(D,D')z = F(x,y)$

Step 2 Form the A.E $f(m)=0$ and obtain its roots, $m = \frac{D}{D'}$

Step 3 Find the CF as discussed earlier.

Step 4 The particular integral is given by, $P.I = \frac{1}{f(D,D')} F(x,y)$

Step 5 Then $z = CF + PI$ is the complete soln. of the given LPDE.

WORKING RULE to find P.I

Case 1. When $F(x,y) = e^{ax+by}$

Then $P.I = \frac{1}{f(D,D')} e^{ax+by}$

replace $D \to a$ & $D' \to b$

$$P.I = \frac{1}{f(a,b)} e^{ax+by}$$, provided that $f(a,b) \neq 0$

Case 2: when $F(x,y) = \sin(mx+ny)$ or $\cos(mx+ny)$

Then $P.I = \dfrac{1}{f(D,D')} \sin(mx+ny)$ or $\cos(mx+ny)$

replace $D^2 \to -m^2$, $DD' \to -mn$, $D'^2 \to -n^2$

$$P.I = \frac{1}{\cancel{f(D^2, DD', D'^2)}} \sin(mx+ny) \text{ or } \cos(mx+ny)$$

$$\frac{1}{f(-m^2, -mn, -n^2)} \sin(mx+ny) \text{ or } \cos(mx+ny)$$

q: Solve : $(D_x^3 - 3D_x^2 D_y + 4D_y^3) Z = e^{x+2y}$

Sol. i) $(D^3 - 3D^2 D' + 4D'^3) Z = e^{x+2y}$

ii) $\div D'^3$ $\left[\left(\dfrac{D}{D'}\right)^3 - 3\left(\dfrac{D}{D'}\right)^2 + 4\right] Z = 0$

only to find CF

iii) A.E: $m^3 - 3m^2 + 4 = 0$

iv) roots : $-1, 2, 2$

v) C.F : $Z = f_1(y-x) + f_2(y+2x) + x f_3(y+2x)$

vi) $P.I = \dfrac{1}{D^3 - 3D^2D' + 4D'^3} e^{x+2y}$

$D \to 1 \quad D' \to 2$

$$P.I = \frac{1}{1 - 3(2) + 4(8)} e^{x+2y} = \frac{+e^{x+2y}}{27}$$

$\therefore$ General soln : $Z = CF + PI$

$$= f_1(y-x) + f_2(y+2x) + x f_3(y+2x) + \frac{e^{x+2y}}{27}$$

q: Solve: $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \cos(x+2y)$

Sol. $[D^2 - DD']z = \cos(x+2y)$

$\div D'^2 \Rightarrow \left[\left(\frac{D}{D'}\right)^2 - \left(\frac{D}{D'}\right)\right]z = 0$

$m^2 - m = 0$

$m(m-1) = 0$

$m = 0$ or $m = 1$

$\Rightarrow$ CF : $Z = f_1(y-0) + f_2(y+x)$

$P.I = \frac{1}{D^2 - DD'} \cos(x+2y)$

$D^2 \rightarrow -1^2 \quad DD' \rightarrow -2(1)$

$= \frac{1}{-1-(-2)} \cos(x+2y)$

$= \frac{1}{+1} \cos(x+2y) = \cos(x+2y)$

$=$

GS : $Z = CF + PI$

$= f_1(y) + f_2(y+x) + \cos(x+2y)$

Case 3. when $F(x,y) = x^m y^n$, m & n being constants

$P.I = \frac{1}{f(D,D')} x^m y^n = [f(D,D')]^{-1} x^m y^n$

If $n < m$, $\frac{1}{F(D,D')}$ is expanded in powers of $\frac{D'}{D}$

If $m < n$, $\frac{1}{F(D,D')}$ is expanded in powers of $\frac{D}{D'}$

Case 4. when $F(x,y) = e^{ax+by} V(x,y)$, where $V(x,y)$ is any func. of x & y.

$$P.I = \frac{1}{f(D,D')} e^{ax+by} V(x,y)$$

replace $D \to D+a$, $D' \to D'+b$

$$P.I = e^{ax+by} \frac{1}{f(D+a, D'+b)} V(x,y)$$

Case 5. when $F(x,y)$ is any func. of x & y.

$$P.I = \frac{1}{f(D,D')} F(x,y)$$

resolve $\frac{1}{f(D,D')}$ into partial fractions treating $f(D,D')$ as a func. of D alone.

operate each partial fraction on $F(x,y)$ by using

$$\frac{1}{D-mD'} F(x,y) = \int F(x, \underbrace{c-mx}_{y}) \, dx$$

replace $c \to y + mx$ after integration

$$\frac{1}{D+mD'} F(x,y) = \int F(x, \underbrace{c+mx}_{y}) \, dx$$

$c \to y - mx$

Q. Solve : $\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = e^{2x+y}$

$$(D^2 - 4DD' + 4D'^2) z = e^{2x+y}$$

$\div D^2$

$$(m^2 - 4m + 4) = 0$$

$+4$; -2 -2

$m = 2, 2$

CF : $Z = f_1(y+2x) + x f_2(y+2x)$

$$P.I = \frac{1}{D^2 - 4DD' + 4D'^2} e^{2x+y}$$

$\to 4 - 8 + 4 = 0$ use case 5

URBAN EDGE

$\frac{1}{(m-2)^2}$ → resolve partial fractions

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$m = \frac{D}{D'}$

$\Rightarrow \frac{1}{(D-2D')(D-2D')} e^{2x+y}$

$= \frac{1}{(D-2D')}\left[\frac{1}{D-2D'} e^{2x+y}\right]$

$= \frac{1}{(D-2D')} \cdot \int e^{\cancel{2x}+c-\cancel{2x}} \cdot dx$

$= \frac{1}{(D-2D')} x e^{c} = \frac{1}{(D-2D')} x e^{2x+y}$

$= \int x e^{\cancel{2x}+c-\cancel{2x}} dx$

$= \int x e^{c} dx$

$P.I = e^{c}\frac{x^2}{2} = \frac{x^2}{2} e^{2x+y}$

$\left(\frac{D}{D'}\right)^2 - 4\left(\frac{D}{D'}\right) + 4 = 0$

$\left(\frac{D}{D'} - 2\right)\left(\frac{D}{D'} - 2\right) = 0$

$(D - 2D')(D - 2D') = 0$

$\frac{1}{D - mD'} F(x,y) = \int F(x, c - mx)\, dx$

$\int F(x, \underbrace{c-2x}_{y})$

$y = c - 2x$

$c = 2x + y$

$Z = C.F + P.I$

$= f_1(y+2x) + x f_2(y+2x) + \frac{e^{2x+y} x^2}{2}$

q. Solve $4\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 16 \log(x+2y)$

Case. 5

$(4D^2 - 4DD' + D'^2) z = 0$

$\div D'^2$

4

-2 -2

$AE: 4m^2 - 4m + 1 = 0$

$m = \frac{D}{D'}$

$4m^2 - 2m - 2m + 1 = 0$

$2m(2m-1) - 1(2m-1) = 0$

$(2m-1)(2m-1) = 0$

$m = 1/2, 1/2$

$CF: = f_1(y + \frac{1}{2}x) + x f_2(y + \frac{1}{2}x)$

URBAN EDGE

$(2m-1)^2 = 0 \Rightarrow \left(2\frac{D}{D'} - 1\right)^2 = 0 \quad = \left(\frac{2D - D'}{D'}\right)^2 = 0$

$\Rightarrow (2D - D')^2 = 0$ $\times$ not in $(D - mD')$ form

$$P.I = \frac{1}{4D^2 - 4DD' + D'^2} 16 \log(x+2y)$$

$$= \frac{1}{(2D-D')^2} 16 \log(x+2y)$$

$$= \frac{1}{4\left(D - \frac{1}{2}D'\right)^2} 16 \log(x+2y)$$

$y = c - mx$

$y = c - \frac{1}{2}x$

$2y = 2c - x$

$2c = 2y + x$

$$= \frac{4}{\left(D - \frac{1}{2}D'\right)} \left\{ \frac{1}{\left(D - \frac{1}{2}D'\right)} \log(x+2y) \right\}$$

$$= \frac{4}{\left(D - \frac{1}{2}D'\right)} \int \log(\cancel{x} + 2c - \cancel{x})\, dx$$

$$= \frac{4}{\left(D - \frac{1}{2}D'\right)} x \log 2c \quad = \frac{4}{\left(D - \frac{1}{2}D'\right)} x \log(2y + x)$$

$$= 4 \int x \log(2c - x + x)\, dx$$

$$= 4 \int x \log 2c\, dx$$

$$= 4 \frac{x^2}{2} \log 2c$$

$$= \frac{4x^2}{2} \log(x+2y)$$

$$= 2x^2 \log(x+2y)$$

$$Z = CF + PI = f_1\left(y + \frac{1}{2}x\right) + x f_2\left(y + \frac{1}{2}x\right) + 2x^2 \log(x+2y)$$

q. Solve : $(D_x^3 - 4D_x^2 D_y + 4D_x D_y^2)z = 2\sin(3x+2y)$

$m^3 - 4m^2 + 4m = 0$ $\qquad m = \frac{D_x}{D_y}$ or $\frac{D}{D'}$

$m(m^2 - 4m + 4) = 0$

$m = 0$ or $(m-2)^2 = 0$

$m = 2, 2$

CF : $f_1(\cancel{y}) + f_2(y+2x) + x f_3(y+2x)$

PI : $\frac{1}{D_x^3 - 4D_x^2 D_y + 4D_x D_y^2} \; 2\sin(3x+2y)$

$D_x^2 \to -9 \quad D_x D_y \to -6 \quad D_y^2 \to -$

$\frac{1}{-9D_x - 4(-6)D_x + 4(-4)D_x} \left(2\sin(3x+2y) \right)$

$\frac{1}{D_x(-9+24-16)} \; 2\sin(3x+2y)$

$\frac{1}{D_x(-1)} \; 2\sin(3x+2y)$

$\frac{-1}{D_x}\left(2\sin(3x+2y)\right)$

$= -\int 2\sin(3x+2y)\,dx$

$= -2\frac{\left(-\cos(3x+2y)\right)}{3} = \frac{2}{3}\cos(3x+2y)$

(OR) × Nr ÷ D_x

(OR)

$y = c - mx$

$y = c - 0$

$2y = 2c$

$2\int \sin(3x+2c)$

$\frac{2\cos(3x+2y)}{3}$ ✓

$Z = CF + PI$

$= f_1(y) + f_2(y+2x) + x f_3(y+2x) + \frac{2}{3}\cos(3x+2y)$

Q. Solve : $(D_x^3 - 2D_x^2 D_y)Z = 3x^2y$

$\div D_y^2$ $\quad (m^3 - 2m^2) = 0$

$m = \frac{D_x}{D_y}$ $\quad m^2(m-2) = 0$

$m = 0, 0, 2$

CF : $f_1(y) + x f_2(y) + f_3(y+2x)$

$$PI = \frac{1}{D_x^3 - 2D_x^2 D_y} 3x^2y^2$$

Case 3

$$\frac{1}{D_x^3\left[1 - 2\frac{D_y}{D_x}\right]} \cdot 3x^2y^2$$

$$\frac{1}{D_x^3}\left[1 - 2\frac{D_y}{D_x}\right]^{-1} \cdot 3x^2y$$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$

$(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$

$$\frac{1}{D_x^3}\left[1 + 2\frac{D_y}{D_x} + 2\left(\frac{D_y}{D_x}\right)^2 + \dots\right] 3x^2y$$

$$\frac{1}{D_x^3}\left[3x^2y + 6\frac{D_y}{D_x}(x^2y) + 12\left(\frac{D_y}{D_x}\right)^2(x^2y) + \dots\right]$$

$\frac{D_y}{D_x}(x^2y) = \frac{1}{D_x}x^2 = \frac{x^3}{3}$

$$\frac{1}{D_x^3}\left[3x^2y + 6 \cdot \frac{x^3}{3} + 12(0) + \dots\right]$$

$$\frac{1}{D_x^3}\left[3x^2y + 2x^3\right] = \frac{1}{D_x^2}\left[\frac{\cancel{3}x^3}{\cancel{3}}y + \frac{x^4}{2}\right] = \frac{1}{D_x}\left[\frac{x^4}{4}y + \frac{x^5}{10}\right]$$

$$= \frac{x^5}{20}y + \frac{x^6}{60}$$

$Z = CF + PI$

Q. Solve : $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2} = (y-1)e^x$

-2
$-2+1$

$(m^2 - m - 2) = 0$

$m^2 - 2m + m - 2 = 0$

$m(m-2) + 1(m-2) = 0$

$m = 2$ or $m = -1$

CF : $f_1(y+2x) + f_2(y-x)$

29th December, 2022

P.I. $= \frac{1}{(D-2D')(D+D')} e^{x}(y-1)$

$D \to D+1$, $D' \to D'+0$

$e^{x} \frac{1}{(D+1-2D')(D+1+D')} (y-1)$

$(D+1-2D')(D+1+D')$
$D^2+D+DD'+D+1+D'$
$+ -2D'D-2D'-2D'^2$
$D^2+2D-DD'+1-D'-2D'^2$
$m^2 -$

$e^{x}\left\{\frac{(y-1)}{D^2+2D+1-DD'-D'-2D'^2}\right\}$ case 3

$e^{x} \frac{1}{1+(D^2+2D-DD'-D'-2D'^2)} (y-1)$

$e^{x}\left[1+(D^2+2D-DD'-D'-2D'^2)\right]^{-1}(y-1)$

$(1+x)^{-1}$

$e^{x}\left[1-(D^2+2D-DD'-D'-2D'^2)+\cdots\right](y-1)$

$e^{x}\left[1(y-1)-(D^2+2D-DD'-D'-2D'^2)(y-1)+\cdots\right]$

0 0 0 -1 0

$e^{x}\left[y-1-(-1)\right] = e^{x}y$

$Z = CF + PI$
$= f_1(y+2x) + f_2(y-x) + e^{x}y$

q. Solve: $\left[2D_x^2-5D_xD_y+2D_y^2\right]Z = 5\sin(2x+y)$

AE: $2m^2-5m+2=0$
$2m^2-4m-m+2=0$
$2m(m-2)-1(m-2)=0$
$m=2$ & $m=1/2$

CF: $f_1(y+2x) + f_2\left(y+\frac{x}{2}\right)$

4
$\wedge$
-4 -1

$\frac{1}{2D_x^2-5D_xD_y+2D_y^2}(5\sin(2x+y))$

$D_x^2 \to -4$
$D_xD_y \to -2$
$D_y^2 \to -1$

?deno = 0
$\therefore$ case 5

$$(2m-1)(m-2) = 0$$

$$(2D - D')(D - 2D') = 0$$

$m = \frac{D}{D'}$

P.I: $\dfrac{1}{(2D-D')(D-2D')} 5\sin(2x+y)$

$$\frac{1}{2\left(D-\frac{D'}{2}\right)(D-2D')} 5\sin(2x+y)$$

$$= \frac{5}{2\left(D-\frac{D'}{2}\right)}\left\{\frac{1}{(D-2D')}\sin(2x+y)\right\}$$

$y = c - mx$

$m = 2$

$y = c - 2x$

$c = y + 2x$

$$= \frac{5}{2\left(D-\frac{D'}{2}\right)}\left\{\int \sin(\cancel{2x} + c - \cancel{2x})\,dx\right\}$$

$$= \frac{5}{2\left(D-\frac{D'}{2}\right)} x \sin c$$

$$= \frac{5}{2\left(D-\frac{D'}{2}\right)} x \sin(y+2x)$$

$y = c - \frac{1}{2}x$

$$= \frac{5}{2}\int x \sin\left(c - \frac{1}{2}x + 2x\right)dx$$

$$= \frac{5}{2}\int \underset{u}{x}\, \underset{v}{\sin}\left(\frac{3}{2}x + c\right)dx$$

ILATE

$$\cancel{= \frac{5}{2}\left[\cos\left(\frac{3}{2}x+c\right)\frac{x^2}{2} - \int\right.}$$

$$\frac{5}{2}\left[-x\,\frac{2}{3}\cos\left(\frac{3}{2}x+c\right) - \int 1\left(-\cos\left(\frac{3}{2}x+c\right)\frac{2}{3}\right)\right]$$

$$\frac{5}{2}\left[\frac{-2x}{3}\cos\left(\frac{3}{2}x+c\right) + \sin\left(\frac{3}{2}x+c\right)\frac{4}{9}\right]$$

$$= -\frac{5}{3}x\cos\left(\frac{3x}{2}+c\right) + \frac{10}{9}\sin\left(\frac{3x}{2}+c\right)$$

$c = y + \frac{1}{2}x$

$$PI = -\frac{5}{3}x\cos(2x+y) + \frac{10}{9}\sin(2x+y)$$

$$Z = CF + PI = f_1(y+2x) + f_2\left(y+\frac{x}{2}\right) - \frac{5}{2}x\cos(2x+y) + \frac{10}{9}\sin(2x+y)$$

URBAN EDGE

Q. Solve: $[D_x^3 + D_x^2 D_y - D_x D_y^2 - D_y^3] z = e^x \cos 2y$

AE: $(m^3 + m^2 - m - 1) = 0$

$m^3 - m + m^2 - 1 = 0$

$m(m^2-1) + (m^2-1) = 0$

$(m-1)(m+1)(m+1) = 0$

$m = 1, -1, -1$

CF: $f_1(y+x) + f_2(y-x) + x f_3(y-x)$

P.I: $\dfrac{e^x \cos 2y}{(D-D')(D+D')(D+D')}$ case 4

$D \to D+1$

$(D-D')(D+D')(D+D') = 0$

$\cancel{(D+1)}$

$(D+1-D')(D+1+D')(D+1+D') = 0$

$(D+1-D')(D+1+D')^2$

$(D+1-D')((D+1)^2 + D'^2 + 2(D+1)(D'))$

$(D+1-D')\ (D^2+1+2D+D'^2+2DD'+2D')$

$\cancel{(D^2+2D+2DD'+3D')}$

$D^3 + D + 2D' + D'^2 D + 2D^2 D' + \cancel{2DD'}$

$+ \ D^2 + 1 + 2D + D'^2 + \cancel{2DD'} + \cancel{2D'}$

$- D'D^2 - D' - \cancel{2DD'} - D'^3 - \cancel{2DD'^2} - \cancel{2D'^2}$

$D^3 + D' + 2D' + D^2 D' + D'^2 D + D^2 + 1 + 2D + D'^2 - D'$

$\cancel{D^3 + 3D + D' + D^2 D'} + D'^2 D + D^2 + 1$

$D^2 \to 0 \quad D'^2 \to -4 \quad D_x D_y = 0$

$\cancel{0-4}\ 0 + 3D + D' + 0 - 4D + 0 = 0$

$D' - D$

$= \dfrac{e^x \cos 2y}{3D_x + 5D_y + 5}$

$= e^x \dfrac{D_y \cos 2y}{3D_x D_y + 5D_y^2 + 5D_y}$

$= \dfrac{-2e^x}{5}\left\{\dfrac{\sin 2y}{(D_y - 4)}\right\}$

$= \dfrac{-2}{5} e^x \dfrac{(D_y+4)\sin 2y}{D_y^2 - 16}$

$= \dfrac{-2}{5} e^x \dfrac{(D_y+4)\sin 2y}{(-20)}$

$= \dfrac{1}{50} e^x [2\cos 2y + 4\sin 2y]$

$3D + 5D' + 5$

$3x + 5y + 5$

q: $(D_x^2 + D_y^2) z = x^2 y^2$

$m^2 + 1 = 0$

$m^2 = -1$

$m = \pm i$

Note: If $m = a \pm ib$ are the complex roots then,

$CF = f_1(y + a + ib) + f_1(y + a - ib) + i[f_2(y + a + ib) + f_2(y + a - ib)]$

$CF = f_1(y+i) + f_1(y-i) + i[f_2(y+i) + f_2(y-i)]$

PI : $\frac{1}{D^2 + D'^2} x^2 y^2$

$\frac{1}{D^2\left(1 + \left(\frac{D'}{D}\right)^2\right)} x^2 y^2$

$\frac{1}{D^2}\left[1 + \left(\frac{D'}{D}\right)^2\right]^{-1} x^2 y^2$

$\frac{1}{D^2}\left[1 + \left(\frac{D'}{D}\right)^2 + \left(\frac{D'}{D}\right)^4 + \dots\right] x^2 y^2$

$\frac{1}{D^2}\left[x^2y^2 + \left(\frac{D'}{D}\right)^2 x^2 y^2 + \left(\frac{D'}{D}\right)^4 x^2 y^2 + \dots\right]$

$\frac{1}{D^2}\left[x^2 y^2\right] + \frac{x^6}{180}$

$\frac{x^4}{12} y^2 + \frac{x^6}{180}$

$\frac{D'}{D}$

$\Downarrow$

diff twice y

& int twice x

$x^2 y^2$

D' $2y$

$\frac{1}{D} \rightarrow \frac{2x^3}{3} y$

$\frac{4}{9} x^4 y^2$

$\frac{4y}{9} \frac{x^5}{5}$

$\frac{24 y^2 x^6}{9(5)(6)}$

$\int 2x^2$

$\int \frac{2x^3}{3}$

$\frac{2}{3} \frac{x^4}{4} = \frac{x^4}{6}$

$\int \frac{x^4}{6} = \frac{x^5}{6(5)}$

$= \frac{x^6}{30(6)} = \frac{x^6}{180}$