12th October, 2022

UNIT - 2

Ordinary Differential Equations

* LDE (Linear differential equation)

$$\frac{dy}{dx} + P(x)\,y = Q(x) \quad \text{(linear in } y\text{)}$$

Bernoullie's Differential Equation

(Reducible to Linear Differential Eqn.)

$$\frac{dy}{dx} + P(x)\,y = Q(x)\,y^n \quad -①$$

Procedure to solve the above eqn.

Step 1 Divide ① by y^n

$$\frac{1}{y^n}\frac{dy}{dx} + P\frac{1}{y^{n-1}} = Q$$

put $\frac{1}{y^{n-1}} = Z$

$$\frac{(1-n)}{y^n}\frac{dy}{dx} = \frac{dZ}{dx}$$

$$\frac{1}{y^n}\frac{dy}{dx} = \frac{1}{(1-n)}\frac{dZ}{dx}$$

$$\Rightarrow \frac{1}{(1-n)}\frac{dZ}{dx} + PZ = Q$$

$\times (1-n)$ on both sides

$$\frac{dZ}{dx} + [P(1-n)]Z = Q(1-n) \quad -② \quad \text{LDE in } Z$$

$$I.F = e^{\int P(1-n)\,dx}$$

solution to eqn. ② is

$$Z\cdot(IF) = \int Q(1-n)\, I.F\, dx + C$$ is the soln. to the given D.E

then substitute $Z = \frac{1}{y^{n-1}}$

1. $xy' + 2y = x^3 y^2$

$\div$ by term of the dependent variable

Sol. $\cancel{y' + \frac{2y}{x} = x^2 y^2}$

$$x\frac{dy'}{dx} + 2y' = x^3 y^2 \quad -①$$

$\div$ by y^2 on both sides

$$\frac{x}{y^2}\frac{dy}{dx} + \frac{2}{y} = x^3$$

$\div x$ to get rid of x

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{2}{yx} = x^2 \quad -②$$

Put $\frac{2}{y} = z \Rightarrow \frac{-2}{y^2}\frac{dy}{dx} = \frac{dz}{dx}$

$$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} = \frac{-1}{2}\frac{dz}{dx}$$

$$\frac{-1}{2}\frac{dz}{dx} + \frac{1}{x}z = x^2$$

$\times$ with -2

$$\frac{dz}{dx} - \left(\frac{2}{x}\right)z = -2x^2$$

a LDE in z

$$IF = e^{-\int \frac{2}{x}dx} = e^{-2\ln x} = e^{\ln x^{-2}} = \frac{1}{x^2}$$

$$z(IF) = \int Q(IF)\,dx + c$$

$$z\frac{1}{x^2} = -\int 2\cancel{x^2}\cdot\frac{1}{\cancel{x^2}}\,dx + c$$

$$\frac{z}{x^2} = -2x + c$$

$$z = -2x^3 + cx^2$$

$$\frac{2}{y} = -2x^3 + cx^2$$

is the soln. to the given D.E

Integration by parts

H/W q. Solve D.E $y' + 4xy + xy^3 = 0$

$y = \left(ce^{4x^2} - \frac{1}{4}\right)^{-1/2}$

Sol. $\frac{dy}{dx} + 4y.x = -xy^3$

find the dependent variable
÷ by highest power of that

÷ y^3

$\frac{1}{y^3}\frac{dy}{dx} + 4x\frac{1}{y^2} = -x$

put $\frac{1}{y^2} = Z$

$\frac{-2}{y^3}\frac{dy}{dx} = \frac{dz}{dx}$

$\Rightarrow \frac{-dz}{2dx} + 4xZ = -x$

$\frac{dz}{dx} - 8x(z) = 2x$ (LDE) ∵ co-efficient of $\frac{dz}{dx}$ is 1

$I.F = e^{\int -8x} = e^{-8\int x} = e^{-8\frac{x^2}{2}} = e^{-4x^2}$

dependent variable

Soln. to eqn. $Z(IF) = \int Q\, IF\, dx + c$

$Ze^{-4x^2} = \int 2x \cdot e^{-4x^2} + c$

$Ze^{-4x^2} = 2\int \underset{u}{x} \cdot \underset{v}{e^{-4x^2}} \cdot dx + c$

$Ze^{-4x^2} = 2\left[x \frac{e^{-4x^2}}{-4} - \int 1 . \frac{e^{-4x^2}}{-4}\right]$

$Ze^{-4x^2} = 2\left[\frac{x \cdot e^{-4x^2}}{-4}\right.$

$\int x \cdot e^{-4x^2} dx$

$x^2 = t$

$2x\, dx = dt$

$\frac{e^{-4t} dt}{2}$

$\int \frac{e^{-4t}}{2} dt$

$= \frac{e^{-4t}}{-8}$

$= \frac{e^{-4x^2}}{-8}$

$Ze^{-4x^2} = \cancel{2}\left[\frac{e^{-4x^2}}{\cancel{-8}\,4}\right] + c$

$Z = \frac{-1}{4} + \frac{c}{e^{4x^2}}$

$Z = \frac{-1}{4} + \frac{c}{e^{4x^2}}$

$\frac{1}{y^2} = \frac{-1}{4} + ce^{-4x^2}$

$$y = \left(ce^{-4x^2} - \frac{1}{4}\right)^{-1/2}$$

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* Integration By Parts

$$\int d(uv) = \int u\,dv + \int v\,du$$

$$uv = \int u\,dv + \int v\,du$$

$$\Rightarrow \int u\,dv = uv - \int v\,du$$

$$\Rightarrow \int uv = u\int v - \int\left(u' \int v\right)$$

q. Solve $y - \cos x \frac{dy}{dx} = y^2[1-x]$ (Right hand side of question wrong $\Rightarrow$ complex integral)

Sol.

$$\frac{dy}{dx} = y\sec x - y^2[1-x]\sec x$$

$$\frac{dy}{dx} + y^2(1-x)\sec x = y\sec x$$

$$\frac{dy}{dx} - y\sec x = y^2(x-1)\sec x$$

$$\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y}\sec x = (x-1)\sec x$$

let $-\frac{1}{y} = z$

$\frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}$

$$\frac{dz}{dx} + z\sec x = (x-1)\sec x$$

$$IF = e^{\int \sec x} = e^{\ln|\sec x + \tan x|} = \sec x + \tan x$$

$$z(IF) = \int Q(IF)\,dx + C$$

$$z(\sec x + \tan x) = \int (x-1)\sec x(\sec x + \tan x) + C$$

$$z(\sec x + \tan x) = \int (x-1)(\sec^2 x + \sec x \tan x) + c$$

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q: $y - \cos x \frac{dy}{dx} = y^2(1-\sin x)\cos x$ given $y=2$ when $x=0$

Ans: $\frac{\sec x + \tan x}{y} = \sin x + \frac{1}{2}$

Sol. $\frac{dy}{dx} = \frac{y - y^2(1-\sin x)\cos x}{\cos x}$

$\frac{dy}{dx} = y\sec x - y^2(1-\sin x)$

$\frac{dy}{dx} - y\sec x = -y^2(1-\sin x)$

$= y^2(\sin x - 1)$

$\div y^2$

$\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y}\sec x = (\sin x - 1)$

let $\frac{-1}{y} = Z$

$\frac{1}{y^2}\frac{dy}{dx} = \frac{dZ}{dx}$

$\frac{dZ}{dx} + Z\sec x = (\sin x - 1)$

$I.F = e^{\int \sec x} = e^{\ln|\sec x + \tan x|} = \sec x + \tan x$

Soln. $Z(I.F) = \int Q(I.F) + c$

$Z(\sec x + \tan x) = \int(\sin x - 1)(\sec x + \tan x)$

$Z(\sec x + \tan x) = \int \cancel{\tan x} + \sin x \tan x - \sec x - \cancel{\tan x}$

$\frac{\sin^2 x}{\cos x} - \frac{1}{\cos x}$ $\quad Z(\sec x + \tan x) = \int \sin x \tan x - \sec x \, dx$

$\frac{\sin^2 x - 1}{\cos x}$ $\quad Z(\sec x + \tan x) = \int -\cos x + c$

$\frac{-\cos^{\cancel{2}} x}{\cancel{\cos x}}$ $\quad Z(\sec x + \tan x) = -\sin x + c$

$\frac{-1}{y}(\sec x + \tan x) = -\sin x + c$ (general solution)

$y=2$, $x=0$ $\quad \frac{-1}{2}(1+0) = 0 + c$

$c = \frac{-1}{2}$

$\Rightarrow \frac{-1}{y}(\sec x + \tan x) = -\sin x - \frac{1}{2}$

$\Rightarrow \frac{\sec x + \tan x}{y} = \sin x + \frac{1}{2}$ (Particular solution)

q. $\frac{dy}{dx} - y = y^2(\sin x + \cos x)$

ans: $y = \frac{1}{ce^{-x} - \sin x}$

$\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y} = \sin x + \cos x$

let $-\frac{1}{y} = z$

$\frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}$

$\frac{dz}{dx} + z = \sin x + \cos x$

$I.F = e^{\int 1} = e^x$

soln. $z(I.F) = \int Q(I.F) + c$

$z \cdot e^x = \int (\sin x + \cos x)\, e^x + c$

$= \int e^x \sin x\, dx + \int e^x \cos x$

$-\frac{1}{y} e^x = e^x \sin x + c$

$= \left\{ \frac{e^x}{2}[-\cancel{\cos x} + \sin x] + \frac{e^x}{2}[\sin x + \cancel{\cos x}] \right\}$

$\frac{e^x}{y} = -e^x \sin x - c$

$= \frac{e^x}{2}[2\sin x] = e^x \sin x$

$\frac{1}{y} = -\sin x + ce^{-x}$

$y = \frac{1}{+ce^{-x} - \sin x}$

Trick put a = 0

* $\int e^{ax}\cos(bx+c)\,dx = \frac{e^{ax}}{a^2+b^2}\left[b\sin(bx+c) + a\cos(bx+c)\right]$

opposite

* $\int e^{ax}\sin(bx+c)\,dx = \frac{e^{ax}}{a^2+b^2}\left[-b\cos(bx+c) + a\sin(bx+c)\right]$

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Differential Exact Equations

$Mdx + Ndy$ - standard form

A d.e $M(x,y)dx + N(x,y)dy = 0$ is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

solution is:

$$\int_{y \text{ constant}} M\,dx + \int (\text{Terms of } N \text{ not having } x)\,dy = c$$

i.e. y alone terms with constants

q. Check the eqn. $(3x^2+2e^y)dx + (3y^2+2xe^y)dy = 0$ for exactness. If it is exact, find the solution

Sol. $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(3x^2+2e^y) = 2e^y$

$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(3y^2+2xe^y) = 2e^y(1)$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ it is exact

$$\int_{y \text{ const}} M\,dx + \int (y \text{ alone terms of } N)\,dy \neq c$$

$$\int_{y \text{ const}} (3x^2+2e^y)\,dx + \int 3y^2\,dy \neq c$$

$$= \frac{3}{3}x^3 + 2e^y x + \frac{3}{3}y^3 \neq c$$

$$\therefore \quad x^3 + 2e^y x + y^3 \neq c$$

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q. Solve: $(2xy\cos x^2 - 2xy + 1)dx + (\sin x^2 - x^2 + 3)dy = 0$

$\frac{\partial M}{\partial y} = 2x\cos x^2 - 2x$

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$$\frac{\partial N}{\partial y} = 2x\cos x^2 - 2x$$

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y} \Rightarrow \text{Hence Exact}$$

$$\underset{y\ \text{const.}}{\int M\,dx} + \int (N \text{ terms having } y \text{ only})\,dy = c$$

$$\int (2xy\cos x^2 - 2xy + 1)\,dx + \int (\cancel{\sin x^2}\ \cancel{x^2} + 3)\,dy = c$$

$x^2 = t$
$2x\,dx = dt$

$$y\int 2x\cos x^2\,dx - 2y\int x\,dx + \int 1\,dx + 3y = c$$

$$y\sin x^2 - \frac{\cancel{2}yx^2}{\cancel{2}} + x + 3y = c$$

Q. Determine for what values of a & b the following D.E exact & obtain general solution of the exact eqn $(y + x^3)\,dx + (ax + by^3)\,dy = 0$

Sol. for exactness $\frac{\partial M}{\partial y} = \frac{\partial M}{\partial x}$

$$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = a$$

$a = 1$, b can be any real no. as it does not depend on b

Soln. to the given

$$\underset{y\ \text{const}}{\int (y + x^3)\,dx} + \int by^3\,dy = c$$

$$yx + \frac{x^4}{4} + \frac{by^4}{4} = c$$

q: Solve the D.E $(5x^3+12x^2+6y^2)dx + 6xy\,dy = 0$

$\frac{\partial M}{\partial y} = 12y$

$\frac{\partial N}{\partial x} = 6y$ ⇒ Not exact, factor which makes it integrable is missing

Reducible to Exact Form

given that $Mdx + Ndy = 0$ is not exact

i.e $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Step 1: Find the Integrating Factor (I.F)

Step 2: Multiply the given eqn. by I.F. This converts the d.e into an exact d.e of the form

$$M_1 dx + N_1 dy = 0$$

Step 3: General soln. is $\int_{y\text{ const.}} M_1\,dx + \int \left(\text{terms of } N_1 \text{ not having } x\right) dy = c$

Rules for finding I.F: (any 1 will work)

easier & better

1. If $\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = f(x)$ or constant ÷N ⇒ IF wrt x

then $I.F = e^{\int f(x)\,dx}$

2. If $\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = g(y)$ or constant ÷M ⇒ IF wrt y

then $I.F = e^{\int g(y)\,dy}$

3. If M and N are homogeneous functions of same degree then:

$$IF = \frac{1}{Mx + Ny}$$

4. If $Mdx + Ndy$ is of the form : $y f(xy)dx + x g(xy) dy$ then :

$$IF = \frac{1}{Mx - Ny}$$

eg: i) x^3y^3 or $xy \sin xy$
every term of can (xy)
or a power of (xy)
(can be $(xy)^6$)

ii) $y[7xy + 3 + x^2y^2] + x[21x^3y^3 + 9] = 0$

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Solve the D.E $(5x^3 + 12x^2 + 6y^2)dx + 6xy\, dy = 0$ —①

$$\frac{\partial M}{\partial y} = 12y$$

$$\frac{\partial N}{\partial x} = 6y$$

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 6y \Rightarrow$ it is close $\frac{\partial N}{\partial x}$

$\Rightarrow \div N$ is better

$$\Rightarrow \frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{1}{6xy}(6y) = \frac{1}{x} = \lambda(x)$$

$$I.F = e^{\int \lambda(x)dx} = e^{\int \frac{1}{x}} = e^{\log x} = x$$

$\Rightarrow$ x^{ly} eqn ① with x

$[5x^4 + 12x^3 + 6xy^2]dx + [6x^2y]dy = 0$ is exact

$M_1 \qquad N_1$

$$\frac{\partial M_1}{\partial y} = 12xy = \frac{\partial N_1}{\partial x}$$

solution: $\int\limits_{y\ \text{const}} M_1\, dx + \int \left(\begin{matrix}\text{terms of } N_1 \\ \text{not having } x\end{matrix}\right) dy = c$

$x^5 + 3x^4 + 3y^2x^2 + 0 = c$

now to verify it is a solution, differentiate

$5x^4 dx + 12x^3 dx + 3[y^2 2x\, dx + x^2 2y\, dy] = 0$

$5x^4 dx + 12x^3 dx + 6xy^2 dx + 6x^2 y\, dy = 0$

$[5x^4 + 12x^3 + 6xy^2] dx + 6x^2 y\, dy = 0$

q. Solve: $(xy + y^2) dx + (x + 2y - 1) dy = 0$ — ①

Sol. $\frac{\partial M}{\partial y} = x + 2y$, $\frac{\partial N}{\partial x} = 1$

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = x + 2y - 1$

difference is close to N

$\Rightarrow \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{x+2y-1}(x+2y-1) = 1$

$I.F = e^{\int 1 dx} = e^x$

x^{ly} eqn ① with e^x

$\underbrace{e^x(xy + y^2)}_{M_1} dx + \underbrace{e^x(x + 2y - 1)}_{N_1} dy = 0$ is exact

$\frac{\partial M_1}{\partial y} = e^x[x + 2y]$, $\frac{\partial N_1}{\partial x} = e^x(x + 2y - 1) + e^x(1)$

$= e^x(x + 2y)$

solution $y\int\limits_{y\ \text{const}} e^x x\, dx + y^2 \int\limits_{y\ \text{const}} e^x dx + \int 0\, dy = c$

$y[xe^x - x] + e^x y^2 = c$

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H/W q. Solve $(3x^2y^3e^y + y^3 + y^2)dx + (x^3y^3e^y - xy)dy = 0$

Solve $(y^2e^x + 2xy)dx - x^2 dy = 0$

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Solve : $(\underbrace{y+x}_{M})dx + (\underbrace{y-x}_{N})dy = 0$

$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = -1$

homogeneous of same degree $\Rightarrow$ $IF = \frac{1}{Mx + Ny} = \cancel{\frac{1}{y+x+x-y}} = \cancel{1}$

$= \frac{1}{(y+x)x + (y-x)y} = \frac{1}{yx + x^2 + y^2 - xy} = \frac{1}{x^2+y^2}$

$\times^{ly}$ eqn: with $\frac{1}{x^2+y^2}$

$$\underbrace{\left(\frac{y+x}{x^2+y^2}\right)}_{M_1} dx + \underbrace{\left(\frac{y-x}{x^2+y^2}\right)}_{N_1} dy = 0$$

$$\frac{\partial M_1}{\partial y} = \frac{(x^2+y^2)(1) - (y+x)(2y)}{(x^2+y^2)^2} = \frac{x^2+y^2-2y^2-2xy}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2-2xy}{(x^2+y^2)^2}$$

equal hence exact

$$\frac{\partial N_1}{\partial x} = \frac{(x^2+y^2)(-1) - (y-x)(2x)}{(x^2+y^2)^2} = \frac{-x^2-y^2-2xy+2x^2}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2-2xy}{(x^2+y^2)^2}$$

Solution $\int_{y\ \text{const}} M_1\,dx + \int \left(\text{terms of } N_1 \text{ not having } x\right) dy = c$

$\int \frac{1}{x^2+a^2}$ as y is const

$= \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$

$$\int_{y\text{ const}} \frac{y+x}{y^2+x^2}\,dx + \int 0\,dy = C$$

let $x^2+y^2 = t$

$2x\,dx = dt$

$x\,dx = \frac{1}{2}dt$

$\Rightarrow \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\log(t)$

$$y\int_{y\text{ const}} \frac{1}{x^2+y^2}\,dx + \int_{y\text{ const}} \frac{x}{x^2+y^2}\,dx$$

$$y\left[\frac{1}{y}\tan^{-1}\left(\frac{x}{y}\right)\right] + \frac{1}{2}\log(x^2+y^2) = C \text{ is the req. soln.}$$

Alternative: $(y+x)dx + (y-x)dy = 0$

$$\frac{\partial M}{\partial x} = 1,\quad \frac{\partial N}{\partial y} = -1\quad ;\quad \underline{\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y}} = 2$$

if we $\div$ $\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y}$ with M & N

we wont get a fn. of x alone / y alone

$\therefore$ Not possible.

~~q. Solve: $(2xy+y^2)\frac{dy}{dx} = 3y^2+2xy$~~

~~$(2xy+y^2)dy - (3y^2+2xy)dx = 0$~~

~~$(3y^2+2xy)dx - (2xy+y^2)dy = 0$~~

~~$M\,dx - N\,dy$~~

$y(3y+2x)$

M & N are homogeneous of same degree

~~$\frac{\partial M}{\partial y} = 6y+2x \qquad \frac{\partial N}{\partial x} = -(2y)$~~

~~IF~~ ~~$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 8y+2x$~~

~~$IF = \frac{1}{Mx+Ny} = \frac{1}{3xy^2+2x^2y-2xy^2-y^3} = \frac{1}{xy^2+2x^2}$~~

q. Solve: $(2xy + x^2)\frac{dy}{dx} = 3y^2 + 2xy$

Sol. $(3y^2 + 2xy)dx - (2xy + x^2)dy = 0$

$\frac{\partial M}{\partial y} = 6y + 2x$, $\frac{\partial N}{\partial x} = -2y - 2x$ Not Exact

M & N are homogeneous fn of degree 2

$$I.F. = \frac{1}{Mx + Ny} = \frac{1}{3y^2x + 2x^2y - 2xy^2 - x^2y} = \frac{1}{xy^2 + x^2y} = \frac{1}{xy(x+y)}$$

y^{ly} DE by IF

$$\frac{1}{xy(x+y)}(3y^2 + 2xy)dx - \frac{1}{xy(x+y)}(2xy + x^2)dy = 0$$

is exact

solution: $\int_{y\ \text{const}} M_1\,dx + \int \left(\begin{array}{c}\text{terms of } N_1 \\ \text{not having } x\end{array}\right) dy = c$

$$\frac{1}{y}\int_{y\ \text{const}} \frac{3y^2}{x(x+y)}dx + \frac{2y}{y}\int_{y\ \text{const}} \frac{x}{x(x+y)}dx + \int 0\,dy = c$$

Partial fraction

$$3y\int \frac{1}{x(x+y)}dx + 2\int_{y\ \text{const}} \frac{1}{x+y}dx = c$$

$$3y\int_{y\ \text{const}} \frac{1}{y}\left[\frac{1}{x} - \frac{1}{x+y}\right]dx + 2\int_{y\ \text{const}} \frac{1}{x+y}dx = c$$

$$3[\ln x - \ln(x+y)] + 2\ln(x+y) = \ln c$$

$$\log_e x^3 - \log_e (x+y)^3 + \log_e (x+y)^2 = \log_e c$$

$$\frac{x^3 (x+y)^2}{(x+y)^3} = c$$

$$x^3 = c(x+y)$$

q. solve $\left(\frac{y}{x} \sec y - \tan y\right) dx + (\sec y \log x - x) dy = 0$

Sol. $\left(\frac{y}{x}\frac{1}{\cos y} - \frac{\sin y}{\cos y}\right) dx + \left(\frac{1}{\cos y} \log x - x\right) dy = 0$

$\times x \cos y$ $\quad (\underbrace{y - x \sin y}_{M}) dx + (\underbrace{x \log x - x^2 \cos y}_{N}) dy = 0$

$$\frac{\partial M}{\partial y} = 1 - x\cos y \qquad \frac{\partial N}{\partial x} = x\left(\frac{1}{x}\right) + \log x - \cos y \, 2x$$

$$= 1 + \log x - 2x\cos y$$

$\times \cos y$ $\quad \left(\underbrace{\frac{y}{x} - \sin y}_{M}\right) dx + (\underbrace{\log x - x \cos y}_{N}) dy = 0$

$$\frac{\partial M}{\partial y} = \frac{1}{x} - \cos y \qquad \frac{\partial N}{\partial x} = \frac{1}{x} - \cos y$$

$\int \tan x = -\log \cos x$

solution $\displaystyle\int_{y\ \text{const}} \left(\frac{y}{x} - \sin y\right) dx + \int 0 \, dy = c$

$$y \log x - x \sin y = c$$

q. Solve : $\underbrace{[xy \sin(xy) + \cos(xy)] y}_{M} \, dx + [xy \sin(xy) - \cos(xy)] x \, dy = 0$

Sol $\quad y f(xy) dx + x g(xy) dy = c$

$$IF = \frac{1}{Mx - Ny}$$

$$IF = \frac{1}{x^2y^2\sin(xy) + xy\cos(xy) - x^2y^2\sin(xy) + xy\cos xy}$$

$$= \frac{1}{2xy\cos(xy)}$$

x'y IF with ques

$$\frac{1}{2xy\cos(xy)}[xy\sin(xy) + \cos(xy)]y\,dx + \frac{1}{2xy\cos(xy)}[xy\sin(xy) - \cos(xy)]x\,dy = 0$$

$$\cancel{\left[\frac{1}{2}\tan xy + \frac{1}{2xy}\right]y\,dx + \left[\frac{1}{2}\tan xy - \frac{1}{2xy}\right]x\,dy = 0}$$

solution $\Rightarrow \int_{y\ const.} \cancel{x}y\,\frac{\sin xy}{2\cancel{x}\cos xy}\,dx + \int_{y\ const} \frac{\cancel{\cos xy}}{2x\cancel{\cos xy}}\,dx + \int \frac{-1}{2y}\,dy = C$

$$\frac{-1}{2}y\,\frac{\log\cos(xy)}{\cancel{y}} + \frac{1}{2}\log x - \frac{1}{2}\log y = \frac{1}{2}\log_e c$$

$$\log\frac{1}{\cos xy} + \log x - \log y = \log C$$

$$\frac{x}{y\cos xy} = c$$

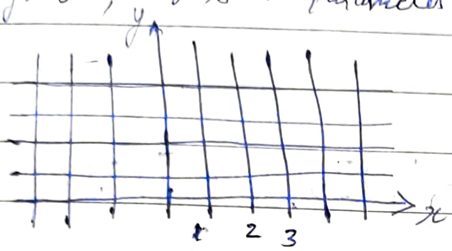
$$\frac{x}{y}\sec(xy) = C$$

27th October, 2022

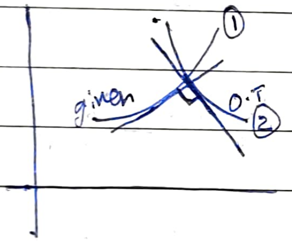
Orthogonal Trajectories

Given : A family of curves

eg: $x = c$, c is a parameter $c = 1, 2, 3, 4 \ldots\ldots$
$y = b$, b is a parameter $b = 1, 2, 3, 4$



given a fam of curves if ~~one~~ every member of one family of curves cuts every member of other family of curves orthogonally then we can say them as orthogonal Trajectories



$m_1 m_2 = -1$

$$\left(\frac{dy}{dx}\right)_1 = \frac{-1}{\left(\frac{dy}{dx}\right)_2}$$

fo $y = b$

$\frac{dy}{dx} = 0$, To move to the other curve

Replace $\frac{dy}{dx}$ by $\frac{-1}{\left(\frac{dy}{dx}\right)}$ $\Rightarrow$ $\frac{-1}{\frac{dy}{dx}} = 0$

$$\frac{dx}{dy} = 0$$

$$\int dx = \int 0$$

$$x = c$$

$\frac{dy}{dx}$ $\rightarrow$ should not have any parameters, we should eliminate them

28th October, 2022

~~Procedure~~

1. Given a family of curves : $f(\cancel{y}, \cancel{x}, \lambda) = 0$

λ is a parameter

Aim: To find another family of curves $g(\cancel{y}, \cancel{x}, a) = 0$

2. Differentiate eqn ① w.r.t x to get a new $F\left(y, x, \frac{dy}{dx}\right) = 0$

If there is a parameter $\Rightarrow$ It is not a D.E

$f(y, x, a)$ a D.E ~~should~~ not contain parameter

eg: $y = x^2 + ax$ (one parameter $\Rightarrow$ another curve should also have 1)

$\frac{dy}{dx} = 2x + a$ (This is not a D.E & we cannot differentiate again as 1 parameter $\Rightarrow$ order = 1)

To get DE substitute for a

$$\underbrace{\frac{dy}{dx} = 2x + \left[\frac{y - x^2}{x}\right]}_{F\left(y, x, \frac{dy}{dx}\right)} \text{ (It is a D.E)}$$

devoid of parameter λ.

It is the D.E of the given family of curves.

Since $m_1 m_2 = -1 \Rightarrow \left(\frac{dy}{dx}\right)_1 = -\left(\frac{dx}{dy}\right)_2$

we replace $\left(\frac{dy}{dx}\right)_1$ in 2nd step by $-\left(\frac{dx}{dy}\right)_2$

to get a D.E of the O.T (Orthogonal Trajectory)

3. $F\left(y, x, \left(-\frac{dx}{dy}\right)\right) = 0$ is the D.E of O.T

4. Solve D.E in 3rd step to get the O.T's (family of curves)

q. Find the O.T^{s} of the family of hyperbole
$x^2 - y^2 = c$

$$2x - 2y\frac{dy}{dx} = 0$$

$$\frac{\cancel{2}x}{\cancel{2}y} = \frac{dy}{dx}$$

$$\left(\frac{dy}{dx}\right)_1 = \frac{x}{y}$$

$$\cancel{m_2 = \frac{-y}{x}} \qquad \cancel{\left(\frac{dy}{dx}\right)_1 \left(\frac{dy}{dx}\right)_2 = -1}$$

$$\cancel{-\left(\frac{dy}{dx}\right)_2 = \frac{y}{x}}$$

Replace $\frac{dy}{dx}$ with $-\frac{dx}{dy}$

$$-\frac{dx}{dy} = \frac{x}{y}$$

$$y\,dx + x\,dy = 0$$

$$\frac{1}{x}dx + \frac{1}{y}dy = 0$$

$$\ln(x) + \ln(y) = a$$

$\boxed{xy = a}$ are the O.Ts

31st October, 2022

16. Find the O.T of family of curves $\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1$ where λ is parameter

Sol. $$\frac{2x}{a^2} + \frac{2y\,y'}{b^2+\lambda} = 0$$

$$\frac{2x}{a^2} + \frac{2y\,y'}{\frac{-y^2a^2}{(x^2-a^2)}} = 0$$

$$x^2(b^2+\lambda) + y^2(a^2) = a^2(b^2+\lambda)$$

$$(x^2-a^2)(b^2+\lambda) + y^2a^2 = 0$$

$$\frac{2x}{a^2} + \frac{2(x^2-a^2)}{-y a^2}\frac{dy}{dx} = 0$$

$$\frac{\cancel{2}x}{\cancel{a^2}} = \frac{\cancel{2}(x^2-a^2)}{y\cancel{a^2}}\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{xy}{x^2-a^2}$$

replace $\frac{dy}{dx}$ with $-\frac{dx}{dy}$

$$-\frac{dx}{dy} = \frac{xy}{x^2-a^2}$$

$$\int \frac{-dx\,(x^2-a^2)}{x} = \int y\,dy$$

$$-\int\left(x - \frac{a^2}{x}\right)dx = \int y\,dy$$

$$-\frac{x^2}{2} + a^2 \ln x = \frac{y^2}{2} + c$$

$$\frac{x^2}{2} + \frac{y^2}{2} - a^2 \ln x = c$$ are O.Ts

q. Show that the one parameter family of curves $y^2 = 4c(c+x)$ are self orthogonal

$\downarrow$

if a given fam of curves cuts every member of family orthogonally.

$f(x,y,\lambda) = 0$

$\downarrow$

$F(x, y, \frac{dy}{dx}) = 0$

$\downarrow$

$F(x, y, -\frac{dx}{dy})$

$\downarrow$

$f(x,y,\lambda)$

The DE before replacing & after replacing is same $\Rightarrow$ self orthogonal

* A family of curves is said to be self orthogonal if every member of the given family intersects every member of the same family orthogonally.

* To ~~prov~~ establish self orthogonality show that the de of given family = de of O.T

$\Rightarrow$ de before replacing $\frac{dy}{dx}$ = de after replacing $\frac{dy}{dx}$

* Use the notation $\frac{dy}{dx} = p \Rightarrow -\frac{dx}{dy} = \frac{1}{p}$

$$y^2 = 4c(c+x)$$

$$\cancel{2}y\frac{dy}{dx} = \cancel{4}^{2}c$$

$$\frac{dy}{dx} = \frac{2c}{y}$$

$$p = \frac{2c}{y} \qquad 2c = py$$

$$\Rightarrow y^2 = 2py\left(\frac{py}{2} + x\right)$$

$$y^2 = p^2y^2 + pyx \quad -①$$

Replace p with $\frac{-1}{p}$

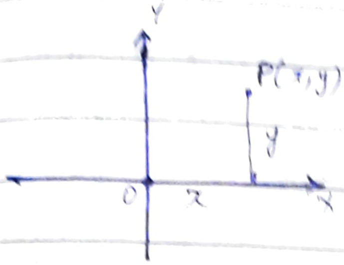
$$y^2 = \frac{1}{p^2}y^2 + \frac{-1}{p}yx$$

$$py^2 = y^2 - pyx$$

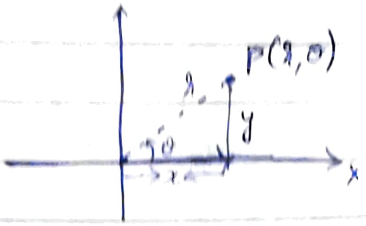
$$\Rightarrow y^2 = py^2 + pyx \quad -②$$

① = ② Hence self-orthogonal

Orthogonal Trajectories in Polar Form (r, θ)



Cartesian Form



Polar Form

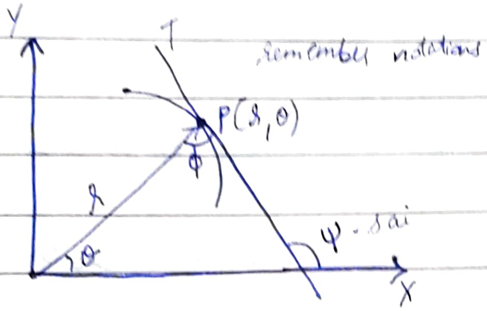
$x = r\cos\theta$

$y = r\sin\theta$

$r^2 = x^2 + y^2$

$\tan\theta = \dfrac{y}{x}$

remember notations



$$\boxed{\tan\phi = r\frac{d\theta}{dr}}$$

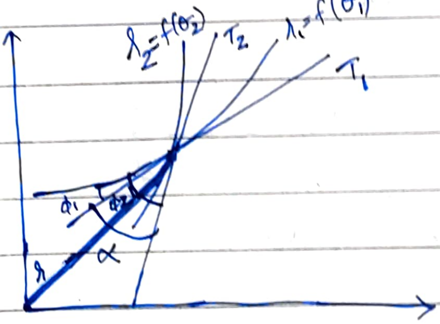
$\psi = \theta + \phi$

$\tan\psi = \tan(\theta + \phi)$

$\tan\psi = \dfrac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$

2nd November, 2022

$\phi \rightarrow$ angle b/w tangent & radius (r)



$\alpha = |\phi_1 - \phi_2|$

Let $\alpha = 90^\circ = \dfrac{\pi}{2}$

$\tan\alpha = \tan|\phi_1 - \phi_2|$

$\infty = \left|\dfrac{\tan\phi_1 - \tan\phi_2}{1 + \tan\phi_1\tan\phi_2}\right| \iff$ iff $\tan\phi_1 \tan\phi_2 = -1$

$r\left(\dfrac{d\theta}{dr}\right)_1 \cdot r\left(\dfrac{d\theta}{dr}\right)_2 = -1$

AIM: Given $r = f(\theta, \lambda)$ to find their O.Ts

1. $r = f(\theta, \lambda)$ is a family of curves in polar form

2. Differentiate w.r.t θ and eliminate λ to get a DE $F = \left(r, \theta, \frac{dr}{d\theta}\right) = 0$ the de of given family of curves

3. Replace $\frac{dr}{d\theta}$ by $-r^2 \frac{d\theta}{dr}$ in step 2 to get DE of O.T

4. Solve step 3 to get O.T^s

Q: Find the O.Ts of the family of curves

i) $r^2 = c\sin(2\theta)$. ii) $r = c(\sec\theta + \tan\theta)$

Sol i) $\cancel{2} r \frac{dr}{d\theta} = \cancel{2} c\cos 2\theta$

replace $\frac{dr}{d\theta} \rightarrow -r^2 \frac{d\theta}{dr}$

$r\left(-r^2 \frac{d\theta}{dr}\right) = c\cos 2\theta = \frac{r^2}{\sin 2\theta}\cos 2\theta$

$-\int \frac{1}{r^3} dr = c\int \sec 2\theta \, d\theta$

r^{-3}

$-3r^{-4}$

$\frac{3}{r^4} = \frac{c}{2}\log|\sec 2\theta + \tan 2\theta| + a$

$-r\frac{d\theta}{dr} = \cot 2\theta$

$\int \tan 2\theta \, d\theta = \int \frac{1}{r} d\theta$

$\frac{1}{2}\ln|\sec 2\theta| = -\ln r + a$

$\cancel{(\sqrt{\sec 2\theta + \tan 2\theta})\, r = a}$

$\cancel{r(\sqrt{\sec 2\theta + \tan 2\theta}) = a}$

$(\sqrt{\sec 2\theta})\, r = a$

$\sec 2\theta\, r^2 = a^2$

$r^2 = \cos 2\theta\, a^2$

ii) $r = c(\sec\theta + \tan\theta)$

$\frac{dr}{d\theta} = c(\sec\theta\tan\theta + \sec^2\theta)$

$\frac{dr}{d\theta} = \frac{r(\sec\theta\tan\theta + \sec^2\theta)}{(\sec\theta + \tan\theta)}$

$\frac{dr}{d\theta} = \frac{r(\sec\theta\tan\theta + \sec^2\theta)}{\sec\theta + \tan\theta)} = r\sec\theta$

$\cancel{\int \frac{1}{r}\, d\theta = \int \sec\theta\, d\theta}$

$\cancel{\ln r = \ln|\sec\theta + \tan\theta| + a}$

$\cancel{\frac{r}{\sec\theta + \tan\theta} = a}$

$\frac{dr}{d\theta} \to -\frac{r^2 d\theta}{dr}$

$\frac{r\, d\theta}{dr} = r\sec\theta$

$\int \cos\theta\, d\theta = -\int \frac{1}{r}\, dr$

$\sin\theta = -\ln r + a$

$\sin\theta + \ln r = a$

q. $r^n \cos n\theta = a^n$ $\quad a \to$ parameter find O.T

3rd November, 2022

Sol. $r^n \cos n\theta = a^n$

$\log r^n + \log \cos n\theta = \log a^n$

$n \log r + \log \cos n\theta = n \log a$

$\frac{n}{r}\frac{dr}{d\theta} + \frac{-n\sin n\theta}{\cos n\theta} = 0$

$\frac{\cancel{n}}{r}\frac{dr}{d\theta} = \cancel{n} \tan n\theta$

$\frac{\cos n\theta}{\sin n\theta}$

$\frac{dr}{d\theta} \rightarrow -r^2 \frac{d\theta}{dr}$

$\frac{-r^2}{r}\frac{d\theta}{dr} = \tan n\theta$

$-\int \frac{1}{r} dr = \int \cot n\theta \, d\theta$

$-\ln r = \frac{1}{n} \ln|\sin n\theta| + c$

$-\ln b - \ln r = \frac{1}{n} \ln|\sin n\theta|$

$(br)^n = \operatorname{cosec}(n\theta)$

Q: Find the O.Ts of the family of curves $r = \frac{2a}{1+\cos\theta}$

Sol. $r + r\cos\theta = 2a$

$\frac{dr}{d\theta} + r(-\sin\theta) + \cos\theta \frac{dr}{d\theta} = 0$

$\frac{dr}{d\theta}(1+\cos\theta) = + r\sin\theta$

$\frac{dr}{d\theta} = + \frac{r\sin\theta}{1+\cos\theta}$

Replacing

$-r^2 \frac{d\theta}{dr} = \frac{r\sin\theta}{1+\cos\theta}$

Date

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$$-\int \frac{1}{r}\,dr = \int \frac{1+\cos\theta}{\sin\theta}\,d\theta$$

~~$\sin\theta = t$~~

~~$1+\cos\theta = t$~~

$$\int \frac{1}{r}\,dr = \int \operatorname{cosec}\theta + \int \cot\theta\,d\theta$$

$\boxed{\begin{array}{l} 1-\cos\theta = 2\sin^2\frac{\theta}{2} \\ 1+\cos\theta = 2\cos^2\frac{\theta}{2} \\ \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \end{array}}$

$$-\ln r = \ln|\operatorname{cosec}\theta - \cot\theta| + \ln|\sin\theta| + c$$

$$r^{-1} = c\sin\theta(\operatorname{cosec}\theta - \cot\theta)$$

$$\frac{1}{r} = c - c\cos\theta$$

$$\frac{1}{r} = c(1-\cos\theta)$$

$$b = r(1-\cos\theta)$$

3rd November, 2022

Solvable for $p = \frac{dy}{dx}$ (for DE ~~degree~~!)

eg: $p^2 + 2p = 3$, $\left(\frac{dy}{dx}\right)^2 + 2\left(\frac{dy}{dx}\right) = c$

$$(p-1)(p+3) = 0$$

$p = 1$ or $p = -3$

$\frac{dy}{dx} = 1$ $\qquad$ $\frac{dy}{dx} = -3$

$\int dy = \int dx$ $\qquad$ $\int dy = -\int 3\,dx$

$y = x + c_1$ $\qquad$ $y = -3x + c_2$

General soln: $(y - x - c_1)(y + 3x - c_2) = 0$

q. Solve : $p(p+y) = (x+y)x$

$$p^2 + py - x^2 - yx = 0$$

~~$p^2 + (p-1)y - x = 0$~~

$$p^2 - x^2 + y(p-x) = 0$$

$$(p+x)(p-x) + y(p-x) = 0$$

$$(p-x)(p+x+y) = 0$$

$p - x = 0$ or $p + x + y = 0$

$p = x$

$p = -(x+y)$

$\frac{dy}{dx} = x$

$\frac{dy}{dx} = -(x+y)$

$\frac{dy}{dx} + Py$

$\int dy = \int x\,dx$

$\frac{dy}{dx} + y = -x$

$y = \frac{x^2}{2} + c_1$

$IF = e^{\int 1} = e^x$

$y - \frac{x^2}{2} - c_1 = 0$

$y(IF) = \int Q \cdot IF$

$ye^x = -\int xe^x$

$ye^x = -[xe^x - \int e^x]$

$ye^x = -(e^x(x-1)) + c_2$

$ye^x = e^x(1-x) + c_2$

$ye^x - e^x(1-x) - c_2 = 0$

$e^x(y + x - 1) - c_2 = 0$

general soln: $\left(y - \frac{x^2}{2} - c\right)\left(e^x(y+x-1) - c\right) = 0$

order = 1 ⇒ constant = 1

q. $x^2\left(\frac{dy}{dx}\right)^4 + 2x\frac{dy}{dx} - y = 0$

$x^2p^4 + 2xp - y = 0$

This is difficult to solve for p so we can solve for y

$y = x^2p^4 + 2xp$

$y = f(x, p)$

$\frac{dy}{dx} =$

$y = f(x, c)$ is soln.

⇒ p, is a parameter & should be eliminated (but not a constant)

diff wrt x

we will get $\frac{dy}{dx}$ which is $= p$

we will get $F\left(x, p, \frac{dp}{dx}\right) = 0$ → is a D.E in p

Solve the above eqn. in p to get p, subst. for p in the given eqn. to get general soln.

singular soln: ~~particular~~ so'n → without parameters

we ignore this

Solvable for y

q. $x^2p^4 + 2xp - y = 0$

$y = x^2p^4 + 2xp \quad = f(x, p)$, eliminate p

$\frac{dy}{dx} = x^2 4p^3 \frac{dp}{dx} + p^4(2x) + 2x\frac{dp}{dx} + 2p$

$p = 4x^2p^3 \frac{dp}{dx} + 2xp^4 + 2x\frac{dp}{dx} + 2p$

$(4x^2p^3 + 2x)\frac{dp}{dx} = -p - 2xp^4$

$2x[2xp^3+1]\frac{dp}{dx} = -p[1 + 2xp^3]$

$[2xp^3+1]\cdot\left[2x\frac{dp}{dx} + p\right] = 0$

if $2xp^3 + 1 = 0$

$p^3 = \frac{-1}{2x}$

no arbitary constant

$\Rightarrow$ Singular soln.

we ignore

$\frac{dp}{dx} = \frac{-p}{2x}$

$-\int \frac{1}{p}\,dp = \int \frac{1}{2x}\,dx$

$-\ln p = \frac{1}{2}\ln x + c$

$\sqrt{x}(p) = c$

$p = \frac{c}{\sqrt{x}}$

substituting for p in eqn.

$x^2p^4 + 2xp - y = 0$

$\cancel{x^2}\frac{c^4}{\cancel{x^2}} + 2x\frac{c}{\sqrt{x}} = y$

$y = c^4 + 2\sqrt{x}\,c$

q: Solve $y = 2px + p^n$

$$\frac{dy}{dx} = 2p + 2x\frac{dp}{dx} + np^{n-1}\frac{dp}{dx}$$

$$p = 2p + 2x\frac{dp}{dx} + n\frac{p^n}{p}\frac{dp}{dx}$$

$$-p = \left(2x + \frac{np^n}{p}\right)\frac{dp}{dx}$$

$$-p^2 = (2xp + np^n)\frac{dp}{dx}$$

$$\frac{dp}{dx} = \frac{-p^2}{2xp + np^n}$$

$\frac{dy}{dx} + Py = Q$ LDE $\quad IF = e^{\int P dx}$

$$\frac{dx}{dp} = \frac{2xp + np^n}{-p^2}$$

$$\frac{dx}{dp} = -\frac{2x}{p} - np^{n-2} \Rightarrow \frac{dx}{dp} + \left(\frac{2}{p}\right)x = -np^{n-2}$$

dependent variable

$$I.F = e^{\int \frac{2}{p}dp} = e^{2\ln p} = p^2$$

dependent variable

Soln: $xp^2 = \int (R.H.S)\, IF\, dp + c$

$$xp^2 = -\int np^{n-\cancel{2}}\cancel{p^2} + c$$

$$xp^2 = \frac{-np^{n+1}}{n+1} + c$$

7th November, 2022

Solve $y = 2px + p^n$ ①

solvable for y

diff wrt x

$$\frac{dy}{dx} = p = 2\left[p + x\frac{dp}{dx}\right] + np^{n-1}\frac{dp}{dx}$$

$$-p = \frac{dp}{dx}\left\{2x + np^{n-1}\right\}$$

$$\underbrace{[2x + np^{n-1}]}_{M} \underbrace{dp}_{x} + \underbrace{[+p]}_{N} \underbrace{dx}_{y} = 0 \quad \text{--- (2)}$$

$$\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial p} \Rightarrow 2 \neq 1 \quad \text{, Not exact}$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial p}\right) = \frac{1}{p}$$

$$I.F = e^{\int \frac{1}{p} dp} = e^{\ln p} = p$$

x^{ly} (2) by I.F → to make it exact

$p[2x + np^{n-1}]dp + p^2 dx$ is exact

solution
$$\int_{x \text{ const}} M\,dp + \int \left(\begin{matrix}\text{terms of } \text{N} \\ \text{not having } p\end{matrix}\right) dx = c$$

$$\int_{x \text{ const.}} [2xp + np^n]\,dp + 0 = c$$

$$xp^2 + \frac{np^{n+1}}{n+1} = c \quad \text{(same old thing)}$$

P cannot be found explicitly in terms of x.

~~$y = xp + p^n$~~ Then we can express the general solution in the form as

$$x = x(c), \quad y = y(c, x)$$

✓ general soln: $[x(p,c), y(p,c)] = 0$

$$x = \frac{1}{p^2}\left[c - \frac{np^{n+1}}{n+1}\right]$$

$$y = 2xp + p^n$$

$$= 2\left[\frac{1}{p^2}\left(c - \frac{np^{n+1}}{n+1}\right)\right]p + p^n$$

$$y = \frac{2c}{p} - \frac{np^{n-1}}{n+1} + p^n$$

$$(x, y) = \frac{1}{p^2}\left[c - \frac{np^{n+1}}{n+1}\right], \frac{2c}{p} - \frac{np^{n-1}}{n+1} + p^n$$

~~$xp + \frac{np^n}{n+1} = \frac{c}{p}$~~

~~$xp(n+1) + np^n = \frac{c(n+1)}{p}$~~

~~$-xp(n+1) + \frac{c}{p}(n+1) = np^n$~~

~~$\frac{(n+1)}{n}\left(-xp + \frac{c}{p}\right) = p^n$~~

~~$\frac{(n+1)}{n}\left(\frac{np^n}{(n+1)} - \frac{c}{p} + \frac{c}{p}\right) = p^n$~~

lol

q. Solve: $xyp^2 + p(3x^2 - 2y^2) - 6xy = 0$

ans: $(y - cx^2)(y^2 + 3x^2 - c) = 0$

q. Solve: $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$

Sol. $p^2(p - y^2) + p\,2x(p - y^2) = 0$

$(p - y^2)(p^2 + 2px) = 0$

factorisable ⇒ solvable for p

$p(p - y^2)(p + 2x) = 0$

$p = 0$	$p = y^2$	$p + 2x = 0$
$\frac{dy}{dx} = 0$	$\frac{dy}{dx} = y^2$	$\frac{dy}{dx} = -2x$
$y = \cancel{0} + c_1$	$\int \frac{1}{y^2} dy = \int dx$	$\int dy = -2\int x\,dx$
$y - c_1 = 0$	$\frac{-1}{y} = x + c_2$	$y = -\cancel{2}\frac{x^2}{\cancel{2}} + c_3$
	$y = \frac{-1}{x + c_2}$	$y + x^2 - c_3 = 0$
	$\frac{-1}{y} - x - c_2 = 0$	

Solution: $\cancel{c_1\left(\frac{-1}{x + c_2}\right)(-x^2 + c_3) = 0}$

$$\left(y - c\right)\left(\frac{-1}{y} - x - c\right)\left(y + x^2 - c\right) = 0$$

one constant as order = 1

Newton's Law of Cooling

T : Temperature of an object ; t → time

T_0 : surrounding temperature (ambient temperature)

$$\frac{dT}{dt} = \text{Rate of change in } T \text{ with } t$$

$$\frac{dT}{dt} \propto (T - T_0) \quad \text{is Newton's Law of cooling}$$

8th November, 2022

$$\frac{dT}{dt} = k(T - T_0)$$

$$\int \frac{dT}{T - T_0} = \int k\,dt$$

$$\log_e (T - T_0) = kt + c$$

$$T - T_0 = e^{kt + c}$$

$$T - T_0 = e^{kt} \cdot e^{c}$$

$$\boxed{T = T_0 + C_1 e^{kt}}$$

Q. A thermometer is removed from a room where the air temp. is 70°F to the outside, where the temp is 10°F. After 1/2 minute thermometer reads 50°F. What is the reading of the thermometer at t = 1 min? How long will it take for the thermometer to reach 15°F.

ans: i) at t = 1 min reading = 36.666

ii) when T = 15°F, t = 3.0642 min.

Sol. $T = 70°F$

$T_0 = 10°F$

given: when $t = 1/2$ min, $T = 50°F$

when $t = 1$ min, $T = ?$

$t = ?$ when $T = 15°C$

$~~T = 10 + e^{-0.81(1)}~~$

$kt + \frac{k}{2}$

$kt - \frac{k}{2}$

$k\left(t - \frac{1}{2}\right)$

$\frac{3}{2} = \frac{e^{kt}}{e^{k/2}} = 3/2$

$$T = T_0 + C_1 e^{kt}$$

$$70 = 10 + C_1 e^{kt}$$

$$60 = C_1 e^{kt}$$

$$50 = 10 + C_1 e^{k(1/2)}$$

$$40 = C_1 e^{k/2}$$

$$\frac{60}{40} = \frac{\cancel{C_1} e^{kt}}{\cancel{C_1} e^{k/2}} \Rightarrow \frac{3}{2} = e^{k/2} \qquad k = 0.81$$

$$C_1 = 26.66$$

$T = T_0 + c_1 e^{kt}$

$T = 10 + c_1 e^{(0.81)1}$

$\cancel{T = 10 + 2666(2.247)}$

at $t = 0$, $T = 70^\circ F$, $T_0 = 10^\circ F$

$70 = 10 + c_1 e^{0t}$

$\boxed{c_1 = 60}$

$T = T_0 + 60 e^{kt}$

when $t = 1/2$, $T = 50^\circ F$

$50 - 10 = 60 e^{k/2}$ $\quad k = -0.811$

when $T = 15^\circ F$, $t = ?$

$T = T_0 + 60 e^{-0.811t}$

$15 - 10 = 60 e^{-0.811t}$

$t = 3.06$

Q: If temp of air is $30^\circ C$ and metal ball cools from $100^\circ C$ to $70^\circ C$ in 15 min, find how long will it take for the metal ball to reach a temp of $40^\circ C$

ans: $k \approx 0.0373$, $t = 52.17$, thus we conclude that it will take 52.2 min for metal ball to reach $40^\circ C$

Sol. $T_0 = 30^\circ C$, at $t = 0$, $T = 100^\circ C$

$t = 15$ min $\quad T = 70^\circ C$

Find t when $T = 40^\circ C$

$T = T_0 + c_1 e^{kt}$

$100 = 30 + c_1$ $\quad \Rightarrow c_1 = 70$

$T = T_0 + 70 e^{kt}$

at $t = 15$, $T = 70$

$70 - 30 = 70 e^{k(15)}$

$\frac{40}{70} = e^{kt}$

$k = -0.037$

$T = 30 + 70e^{-0.037t}$

$t = ?$ when $T = 40$

$40 = 30 + 70e^{-0.037t}$

$\frac{10}{70} = e^{-0.037t}$

$t = 52.17$

Q. water at temp 10°C takes 5min to warm up to 20°C in R.T 40°C. Find T after 20min ; after ½ hours

when will T be 25°C

ans: a) 34.1°C , 37.4°C b) 8.5 min

Sol. At $t = 0$, $T = 10^\circ C$, $T_0 = 40^\circ C$

$T = T_0 + Ce^{kt}$

$10 = 40 + C$

$C = -30$

$t = 5\,min$, $T = 20$, $T_0 = 40^\circ C$

$20 = 40 + Ce^{k(5)}$

$-20 = -30e^{5k}$

$k = -0.081$

At $t = 20\,min$ $T = ?$

$T = 40 - 30e^{k(20)}$

$T = 40 - 30e^{(-0.081)(20)}$

$T = 34.07^\circ C$

$25 = 40 - 30e^{-t(0.081)}$

$t = 8.5\,min$

when $t = \frac{1}{2}\,hr = 30\,min$

$T = 40 - 30e^{(-0.081)(30)}$

$= 37.3^\circ C$