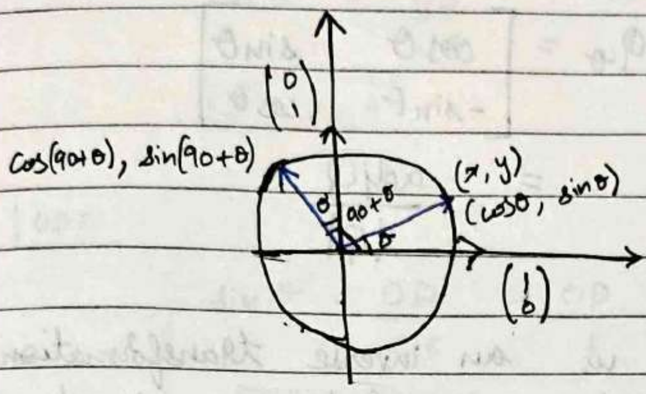
4th March, 2023

Unit-3 Rotations, Reflection and projection in R^2

Rotation matrix in R^2 (Q)

$x = r\cos\theta$; $y = r\sin\theta$



$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Let $Q: R^2 \longrightarrow R^2$ be the linear transformation representing in anticlockwise direction through on θ

Let the basis vectors : $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ in R^2

Rotating $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ by an angle θ

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$||^{ly}$ rotating $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ by an angle θ

$$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos(90+\theta) \\ \sin(90+\theta) \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

$$\therefore Q_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \quad -(1)$$

* If we change $\theta \rightarrow -\theta$ then we get

$$Q_{-\theta} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

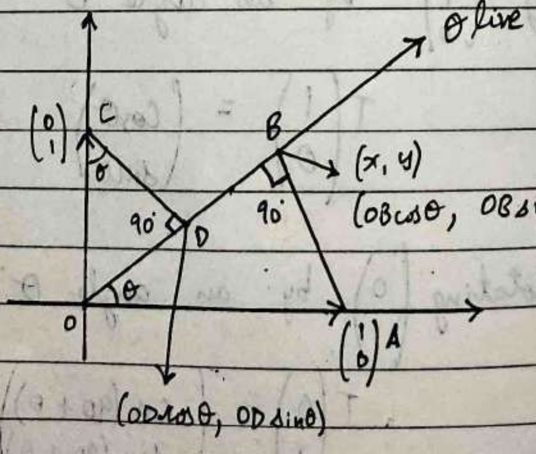
$$= \frac{\operatorname{adj} Q}{|Q|}$$

* This gives us an inverse transformation, i.e rotation in clockwise direction.

Note 1. If the rotation is 1st by angle θ and then by an angle ϕ, then the product of the transformation is the same as the product of individual matrices representing the transformation i.e

$$\begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

Projection matrix in R^2 (P):



$\underline{|OBA}$

$$\sin\theta = \frac{BA}{OA}$$

$\underline{|OBA}$

$$\cos\theta = \frac{OB}{OA} = \frac{OB}{1}$$

$$\boxed{OB = \cos\theta}$$

$\underline{|ODC}$

$$\sin\theta = \frac{OD}{OC} = \frac{OD}{1}$$

$$\boxed{OD = \sin\theta}$$

$$\therefore D = D(OD\cos\theta, OD\sin\theta)$$
$$= D(\sin\theta\cos\theta, \sin^2\theta)$$

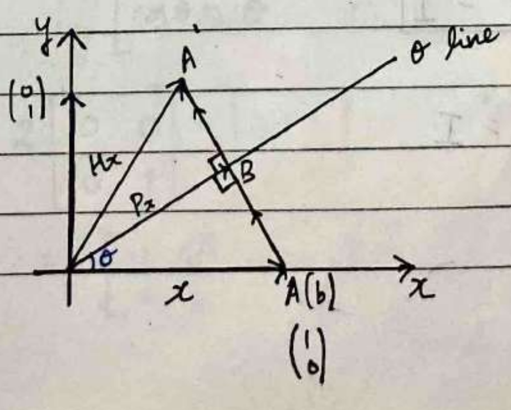
$$B = B(\cos^2\theta, \sin\theta\cos\theta)$$

$$\therefore P = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

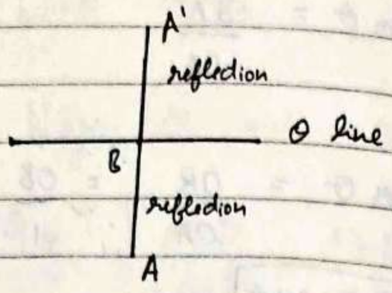
Note : 1. P will not have the inverse $(|P| \neq 0)$

2. $P^2 = P$ (Projecting twice is same as projecting once)

Reflection matrix (H)



Let A' be reflected to A'



$A'B = AB$

from the direction

$$\vec{OA} = \vec{OB} - \vec{OA}$$
$$\vec{OA'} = \vec{OB} + \vec{A'B} \quad -① \quad -②$$

we have

$$A'B = AB$$

solving ① & ②

$$OA + OA' = 2\,OB$$
$$x + Hx = 2Px$$
$$x(I + H) = 2Px$$

$$2P = I + H$$

$$\boxed{H = 2P - I}$$

Note : $H^2 = (2P - I)^2$
$$= 4P^2 + I^2 - 4PI$$
$$= 4P + I - 4P$$

$$\boxed{H^2 = I}$$

$$H \cdot H = I$$

2. Twice the reflection will give back original

3. The reflection has its own inverse.

CWP 1. Find the image of these points applying the transformation given

i) Rotate $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ counter clockwise through 30° line

ii) Reflect $(2, 3)$ across 90° line and then project on x-axis

Sol. i) $\theta = 30^\circ$ (anti-clockwise)

$$Q_{30^\circ} = \begin{bmatrix} \cos 30^\circ & -\sin 30 \\ \sin 30 & \cos 30 \end{bmatrix}$$

Let $x = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ then

$$Q_{30^\circ}\, x = \begin{bmatrix} \cos 30 & -\sin 30 \\ \sin 30 & \cos 30 \end{bmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} =$$

ii) Reflection across $\theta = 90^\circ$

$$H_{90^\circ} = 2 \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= 2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ take $\theta = 90^\circ$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Projection on x-axis

$\theta = 0^\circ$

$$P_{\theta=0^\circ} = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Let $x = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$$H_{90^\circ} x = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{matrix} -2 \\ 3 \end{matrix} \; y$$

Q. Determine the new point after applying the transformation to the given point.

a) Project $x = (-2, 1)$ on the y-axis and then rotate by 45° counter-clockwise

b) Rotate $x = (-2, 1)$ 60° counter-clockwise and then project on the x-axis.

Sol. Projection matrix to project any vector of $\mathbb{R}^2$ on the y-axis is given by

$$P_{90^\circ} = \begin{bmatrix} \cos^2(90^\circ) & \cos(90^\circ)\sin(90^\circ) \\ \cos(90^\circ)\sin(90^\circ) & \sin^2(90^\circ) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

6th March, 2024

Orthogonality

The olthogonal basis is one of the foundations of linear algebra

① Length of a vector

Let $x_1, x_2, x_3 \ldots x_n$ be the vector in R^n, then the length of 'x' which is called ndm

Inner Product

Let $x = (x_1, x_2, \ldots x_n)$ & $y = (y_1, y_2, \ldots y_n)$ then the inner product of x & y in R^n is denoted by $\langle x, y \rangle$ or $x^T y$, called the dot product / scalar product / inner product.

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

Note: $x^T y = y^T x$

* $\|x\| = 0$, x is a zero vector

Orthogonal vectors

Two vectors x & y are said to be orthogonal in R^n iff $x^T y = 0$

(or)

Two vectors x & y are said to be orthogonal (if they form the right angle and satisfy)

$$\|x\|^2 + \|y\|^2 = \|x + y\|^2$$

Note: 1. $x^T x = 0$, x is a zero vector

2. $x^T y = 0$, $\theta = 90°$

3. $x^T y < 0$, θ is obtuse

4. $x^T y > 0$, θ is acute

5. If a non-zero vector $V_1, V_2 \ldots V_n$ are orthogonal then those vectors are linearly independent but the converse is not true.

eg: $x = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ $y = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

$x^T y = (1 \ \ -2) \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 0$

$\begin{bmatrix} 1 & 4 \\ -2 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 4 \\ 0 & 10 \end{bmatrix}$ L·I

eg: $A = \begin{bmatrix} -1 & -4 \\ 2 & 2 \end{bmatrix}$

$= \begin{bmatrix} -1 & 4 \\ 0 & 10 \end{bmatrix}$

L I

$x^T y \neq 0$

converse not true

* $\dim(U) + \dim(V) \leq \dim(U+V) \rightarrow$ orthogonal subspace
$\dim(U) + \dim(V) = \dim(U+V) \rightarrow$ orthogonal complement

Theorem: If a set of non-zero vectors $v_1, v_2 \ldots v_n$ are mutually orthogonal then the set is Linearly independent

Orthonormal Basis :

Vectors which are orthogonal to each other and has a norm one, are called orthonormal vectors. If the orthonormal vectors are rotated they will form orthonormal basis

Orthogonal Subspaces :

Every vector in one subspace should be orthogonal to every vector in other subspace

A line is orthogonal to the line or can be orthogonal to plane
Plane cannot be orthogonal to plane
[$\because$ dim = 4 but it should be 3]

In general 2-subspaces, V and W of the same plane R^n are orthogonal If every vector v in V is orthogonal to every vector w in W

i.e $v^T w = 0$

Fundamental Theorem of Orthogonality :

1. The row space is orthogonal to the null space in R^n, column space is orthogonal to null space in R^m

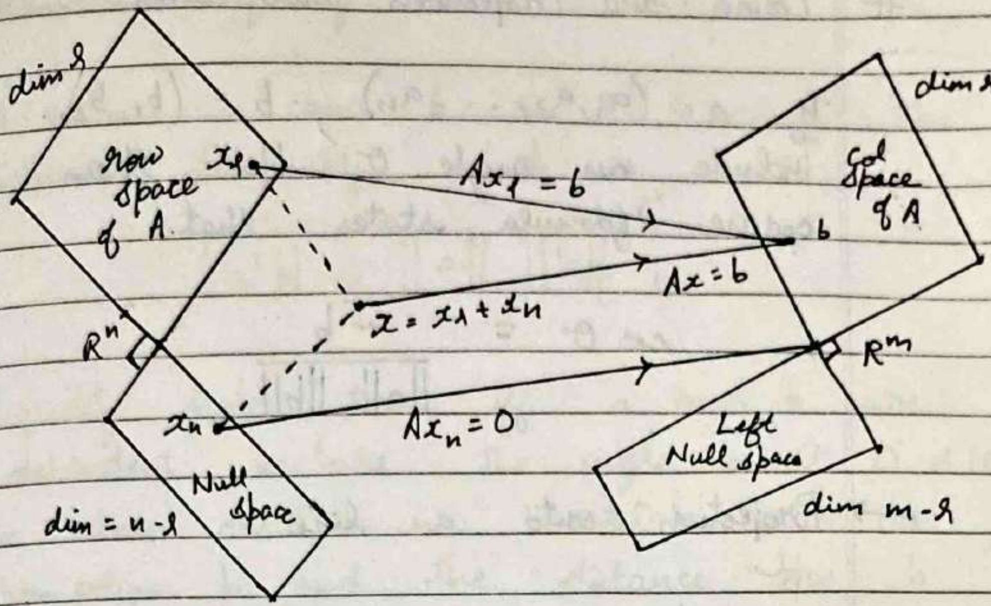
Orthogonal Complements

Given a subspace of V, the space of all vectors orthogonal to V is called orthogonal complements of V written as $V^{\perp}$

Note: 1. The orthogonal complement of a subspace V is unique.

2. If S and T are orthogonal complements in R^n then its always true that

$$\dim S + \dim T = n$$



* Splitting R^n into orthogonal parts V and W will split every vector into $x = v + w$.

* The vector v is the projection onto the subspace V and the orthogonal component w is the projection of x onto W.

* The true effect of matrix multiplication is that every Ax is in $C(A)$. The null space goes to 0. The row space component goes to $C(A)$. Nothing is carried to the left Null space.

Note: * $Ax = b$ is solvable iff $y^T b = 0$ whenever $y^T A = 0$.

* From the row space to column space A is actually invertible, every vector 'b' in the column space comes from exactly one vector in the row space.

* Every vector transforms its row space onto its column space.

Cosines and Projections onto Lines

If $a = (a_1, a_2, \dots a_n)$, $b = (b_1, b_2, \dots b_n)$ include an angle θ b/w them then cosine formula states that

$$\cos\theta = \frac{a^T b}{\|a\| \, \|b\|}$$

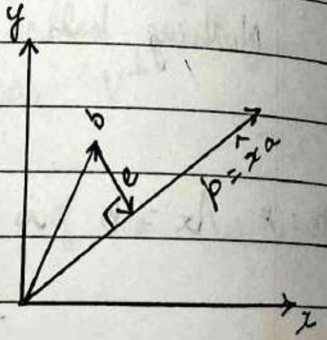
→ Projection onto a line

To find the projection of b onto the line through a given vector 'a', we find the point p on the line that is closest to b.

This point must be some multiple of 'a' say $p = \hat{x} a$

Now, the line from b to the closest point p is $\perp$ to the vector a & hence

$e = b - p$ since $a \perp e$



$\Rightarrow a^T e = 0 \Rightarrow a^T(b-p) = 0$

$$a^T b - a^T(\hat{x} a) = 0$$

$$\hat{x} = \frac{a^T b}{a^T a}$$

$$p = \hat{x} a$$

→ Schwarz Inequality

All vectors a and b in R^n satisfy the Schwarz Inequality which is

$$|a^T b| \leq ||a|| \, ||b||$$

Note : 1. InEquality holds true iff a and b are dependent vectors. The angle is $\theta = 0°$ or $180°$. In this case, b is identical with its projection p and the distance b/w b and p is 0.

2. Schwarz inequality is also stated as $|\cos\theta| \leq 1$

→ Projection Matrix of Rank 1

Projections onto a line through a given vector 'a' is carried out by a Projection matrix given by

$$P = \frac{aa^T}{a^T a}$$

This matrix multiplies b and produces p

i.e $\quad Pb = \dfrac{aa^T b}{a^T a} = a \dfrac{a^T b}{a^T a} = a\hat{x} = p$

Projections and Least Squares

The failure of Gaussian Elimination is almost certain when we have several equations and 1 unknown

$$a_1 x = b_1$$
$$a_2 x = b_2$$
$$\vdots$$
$$a_m x = b_m$$

The system is solvable if $b = (b_1, \dots b_m)$ is a multiple of $a = (a_1, \dots a_m)$

If the system is inconsistent, then we choose that value of a that minimizes an avg. error E in the m eqns. The most convinient avg. comes from the sum of squares

squared Error $\quad E^2 = \sum_{i=1}^{m} (a_i x - b_i)^2$

If there is an exact soln. the min. error is $E = 0$. If not min. error occurs when $\frac{dE^2}{dx} = 0$

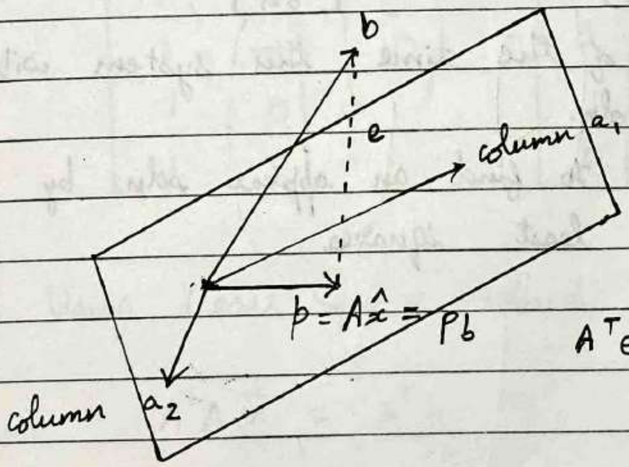
Solving for x, the least squares soln. is $\Big\}$ $\hat{x} = \frac{a^T b}{a^T a}$

→ Least Squares Problem with Several Variables

Consider system of eqns. $Ax = b$ is inconsistent

The vector b lies outside $C(A)$ and we need to project it onto $C(A)$ to get the point p in $C(A)$ that is closest to b. The problem here is the same as to minimize the error $E = \|Ax - b\|$ and this is exactly the distance from b to the point Ax in $C(A)$.

Searching for the least squares soln. $\hat{x}$ is the same as locating the point p that is closest to b.



$a_1^T e = 0$
$a_2^T e = 0$
combines into

$$A^T e = A^T(b - A\hat{x}) = 0$$

The error vector e must be $\perp$ $(e = b - A\hat{x})$ to $C(A)$ & hence can be found in left null space of A

$$\Rightarrow \quad A^T(b - A\hat{x}) = 0 \quad \text{or} \quad A^T A\hat{x} = A^T b$$

These are called Normal equations
Solving them we get the optimal soln. $\hat{x}$

Note: If b is orthogonal to $C(A)$ then its projection is the zero vector.

Least Square Fitting of Data

series of experiment whose $I/P \xrightarrow{\text{linear}} O/P$

will be of form

$$b = C + Dt$$

at $t = t_1$	$b = b_1$	$b_1 = c + dt_1$
$t = t_2$	$b = b_2$	$b_2 = c + dt_2$
$\vdots$		$\vdots$
$t = t_m$	$b = b_m$	$b_m = c + dt_m$

Matrix Form:

$$\begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ 1 & t_3 \\ \vdots & \vdots \\ 1 & t_m \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}$$

majority of the time this system will not have a soln.

$\therefore$ We try to find an approx soln. by using proj. eq least squares.

11th March, 2024

→ Problems on Least Square Fitting of Data

Q.1 Use the method of least square to fit the best line to the data $b = 4, 3, 1, 0$ at $t = -2, -1, 0, 2$ respectively. Find the projection of $b = (4, 3, 1, 0)$ onto the column space of A. Calculate the error vector 'e' and check that 'e' is orthogonal to the columns of A.

Sol: Let $b = C + Dt$ —(1) be the best straight line for the given data.

We need to find C & D Now

$$Ax = b$$

$$\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{bmatrix} 4 \\ 3 \\ 1 \\ 0 \end{bmatrix}$$

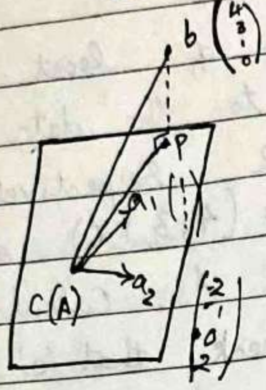
By Using Least Square method

$$A^T A \hat{x} = A^T b$$

$$\underbrace{\begin{bmatrix} 4 & -1 \\ -1 & 9 \end{bmatrix}}_{2\times 2} \begin{pmatrix} C \\ D \end{pmatrix} = \underbrace{\begin{pmatrix} 8 \\ -11 \end{pmatrix}}_{2\times 1}$$

$$\hat{x} = \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{pmatrix} 61/35 \\ -36/35 \end{pmatrix}$$

① $\Rightarrow$ $b = \frac{61}{35} - \frac{36}{35}t$



Point of Projection

(or) proj b on C(A) $\qquad P = A\hat{x}$

$$= \begin{pmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 61/35 \\ -36/35 \end{pmatrix} = \begin{pmatrix} 133/35 \\ 97/35 \\ 61/35 \\ -11/35 \end{pmatrix}$$

error $e = b - p$

$$= \begin{pmatrix} 4 \\ 3 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 133/35 \\ 97/35 \\ 61/35 \\ -11/35 \end{pmatrix}$$

$$= \begin{pmatrix} 1/5 \\ 8/35 \\ -36/35 \\ 11/35 \end{pmatrix}$$

To check $e \perp$ bar C(A)

Let $a_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ (first column in A & C(A))

Now we need to check $a_1 \perp e$

$$a_1^T e = 0 \quad (\text{or}) \quad e^T a_1 = 0$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1/5 \\ 8/35 \\ -36/35 \\ 11/35 \end{pmatrix} = 0$$

Now we need to check $e \perp a_2$

$$e^T a_2 = 0 \quad (\text{or}) \quad a_2^T e = 0$$

Let $a_2 = \begin{pmatrix} -2 \\ -1 \\ 0 \\ 2 \end{pmatrix}$ (Second column in A or $C(A)$)

$$\begin{pmatrix} -2 & -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1/5 \\ 8/5 \\ -36/35 \\ 11/35 \end{pmatrix} = 0$$

Q2. Find a vector x orthogonal to the row space of A, a vector y orthogonal to the column space of A and a vector y orthogonal to Null space of A

where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 4 & 6 \end{bmatrix}$

Sol. rowspace $\perp$ Null space
columnspace $\perp$ Left Null space
Null space $\perp$ Row space

We need to find

x i.e $N(A)$
y i.e $N(A^T)$
z i.e $C(A^T)$ of A

consider $Ax = b$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 4 & 6 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$(A|b) = \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 2 & 3 & 4 & b_2 \\ 2 & 4 & 6 & b_3 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 2R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & -1 & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - 2b_1 \end{array}\right]$$

$$\Rightarrow y = N(A^T) = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow z = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \right\}.$$

$x = N(A)$

$$Ux = 0 \quad \text{or} \quad Ax = 0$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_3 \rightarrow$ free var $= 1$

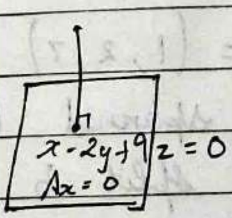
$$x_1 + 2x_2 + 3x_3 = 0$$
$$-x_2 - 2x_3 = 0$$

$$x = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

q3. Let P be the plane whose eqn. is $x - 2y + 9z = 0$. Find a vector $\perp$ to P. What matrix has the plane P as its null space and what matrix has P as its row space?

Sol. Plane eqn $\rightarrow$ form of $Ax = 0$
$\Downarrow$ Null space

now $\perp$ to Null space $\Rightarrow$ Row space



Let $P = \{(x, y, z)$ such that $x - 2y + 9z = 0\}$

$$\underbrace{(1, -2, 9)}_{\text{Row space}} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\text{Null space}} = 0$$

$\therefore$ Vector $\perp$ to P is $\begin{pmatrix} 1 \\ -2 \\ 9 \end{pmatrix}$

So matrix $A = (1, -2\ 9)$ has its plane P as its null space

To find B find the Null space of P

Let $C(B^T) = P$ (Null space of A)

Consider $A = \begin{array}{ccc} x & y & z \\ \hline \end{array}$ $\begin{bmatrix} 1 & -2 & 9 \end{bmatrix}$

y & z are free var.

$x - 2y + 9z = 0$

y	z	x
1	0	2
0	1	-9

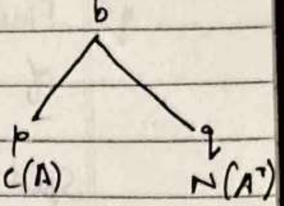
$N(A) = \left\{ \alpha \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -9 \\ 0 \\ 1 \end{pmatrix} \right\}$

$B = \begin{bmatrix} 2 & 1 & 0 \\ -9 & 0 & 1 \end{bmatrix}$

Q4. Find the projection of $b = (1, 2, 7)$ onto the column space of A spanned by $(1, 1, -2)$ and $(1, -1, 4)$, split b into $p + q$ with p in $C(A)$ and q in $N(A^T)$

Sol. Let p be the projection of b onto $C(A)$ which is spanned by $(1, 1, -2)$ & $(-1, -1, 4)$

Here we need to go for proj with several variables



split => proj.

point of proj. $P = A\hat{x}$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix} \quad \& \quad b = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$$

To find $\hat{x}$

$$A^T A \hat{x} = A^T b$$

$$\begin{bmatrix} 6 & -8 \\ -8 & 18 \end{bmatrix} \hat{x} = \begin{pmatrix} -11 \\ 27 \end{pmatrix}$$

$$\hat{x} = \begin{pmatrix} 9/22 \\ 37/22 \end{pmatrix}$$

$$P = \begin{pmatrix} 23/11 \\ -14/11 \\ 65/11 \end{pmatrix} = A\hat{x}$$

$$p + q = b$$

$$q = b - p$$

$$q = \begin{bmatrix} -12/11 \\ 36/11 \\ 12/11 \end{bmatrix}$$

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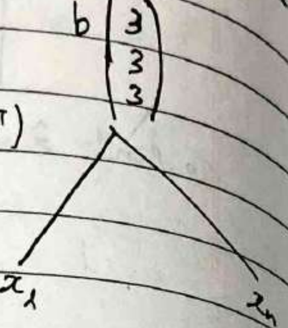
Q5. Find the basis for the orthogonal component of the row space of $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix}$

Split the vector $(3,3,3)$ into a row space component x_r and a null space component x_n



Sol: x_r and x_n are projections of $x = (3,3,3)$ onto $C(A^T)$ & $N(A^T)$

Row space $\perp$ Null space

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix} \quad N \quad \begin{array}{ccc} x & y & z \\ \end{array} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

Null space $z = 1$

$$x + 2z = 0 \quad ; \quad x = -2$$
$$y + 2z = 0 \quad ; \quad y = -2$$

(a)

$$N(A) = \left\{ \alpha \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} \middle| \alpha \in R \right\}$$

Take projection for 1D line rather than several variables

Let x_n be the projection of x (b) $(3,3,3)$ on $N(A)$

i.e $x_n = \left(\dfrac{a^T b}{a^T a} \right) a$

$$= \left(\frac{(-2 \ -2 \ 1)\begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}}{(-2 \ -2 \ 1)\begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}} \right) \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

$$x = x_l + x_n$$

$$\begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = x_l + \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

$$x_l = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$$

Projection Matrix of rank 1 (onto the line)

Note:

1. P is a symmetric matrix

2. $P^n = P$ for $n = 1, 2, 3 \ldots$

3. Rank of P is one

4. The Trace of P is one

5. If 'a' is a n-dimensional vector then P is a square matrix of order n

6. If 'a' is a unit vector then $P = aa^T$

Q6. What multiple of $a = (1,1,1)$ is closest to $b = (2,4,4)$? Also find the point on the line through 'b' that is closest to a?

Sol i) Proj of 'b' onto 'a'

$$p = \hat{x}\, a$$

$$\hat{x} = \frac{a^T b}{a^T a}$$

where $a = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $b = \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix}$

$\hat{x} = \frac{10}{3}$ $\quad \therefore p = \frac{10}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

ii) proj of 'a' onto 'b'

$$p = \left(\frac{b^T a}{b^T b}\right) b = \frac{(2\ 4\ 4)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}{(2\ 4\ 4)\begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix}} \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix}$$

$$= \frac{10}{36}\begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix}$$

Q7. Find the matrix the projects every point in R^3 onto the line of intersection of the planes $x+y+z=0$ and $x-z=0$. What are the column space and row space of this matrix?

Sol 1. We need to find the line of intersection b/w $x+y+z=0$ & $x-z=0$

gaussian elimin. $(A|b) = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right]$

$R_2 \to R_2 - R_1$

$$\simeq \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right]$$

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rank (A) = rank (A|b) < 3 $\Rightarrow \infty$ solns.

$$x + y + z = 0$$
$$-y - 2z = 0$$
$$z = 1 \quad \text{(free var.)}$$
$$y = -2$$
$$x = 1$$

The line of intersection is $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = a$

projection matrix $= \dfrac{aa^T}{a^Ta}$

$$= \frac{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} (1\ -2\ 1)}{(1\ -2\ 1)\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}$$

$$P = \frac{1}{6}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

here rank = 1 $\Rightarrow$ 1 pivot

column space = $\left\{ \alpha \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \middle| \alpha \in R \right\}$

raw space = $\left\{ \alpha \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \middle| \alpha \in R \right\}$

CWP. 2. i) Find the lengths, inner product, cosine of the angle b/w $u = (1, 2, -1, 3)$, $v = (0, 0, -1, -2)$

ii) Produce all vectors in R^3 that are orthogonal to both $u = (1, 2, 1)$ $v = (1, -1, 1)$. Produce an orthogonal basis from these mutually orthogonal vectors

Sol. i) $u = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \end{pmatrix}$ $v = \begin{pmatrix} 0 \\ 0 \\ -1 \\ -2 \end{pmatrix}$

length:

$$\|u\| = \sqrt{1^2 + 2^2 + (-1)^2 + (3)^2} = \sqrt{15}$$

$$\|v\| = \sqrt{0^2 + 0^2 + (-1)^2 + (-2)^2} = \sqrt{5}$$

inner product:

$$\langle u, v \rangle = u^T v = \begin{pmatrix} 1 & 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -1 \\ -2 \end{pmatrix} = -5$$

$$\cos\theta = \frac{u^T v}{\|u\| \, \|v\|} = \frac{-5}{\sqrt{15}\sqrt{5}} = \frac{-1}{\sqrt{3}}$$

ii) given $u = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ $v = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

We observe that u & v are orthogonal $\left(\because u^T v = \begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0 \right)$

$\Rightarrow$ u & v are linearly independent vectors

In R^3 we need 3 L.I vectors (basis)

$\Rightarrow$ we need to find a vector w that is orthogonal to u & v.

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Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$ why row space?
(Row space)

$R_2 \to R_2 - R_1$

$$\sim \begin{array}{c} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & 0 \end{bmatrix} \\ \begin{matrix} x & \ y & \ z \end{matrix} \end{array} = U$$

To find Nullspace (rowspace $\perp$ Null space)

$$Ux = 0$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\boxed{z = 1}$ (free var.)

$\boxed{y = 0}$ $\boxed{x = -1}$

Now $\{u, v, w\}$ will form the orthogonal basis

$$\therefore w = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{Let } B = \begin{array}{c} \begin{matrix} u & \quad v & \quad w \end{matrix} \\ \begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

orthogonal vectors

$$\begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 2/\sqrt{6} & -1/\sqrt{3} & 0 \\ 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}$$

3. If S is the subspace of R^4 containing only the 0 vector, what is $S^\perp$? If S is spanned by $(1,1,1,1)$, what is $S^\perp$? If S is spanned by $(2,0,0,0)$, $(0,4,0,0)$ and $(0,0,0,3)$ What is $S^\perp$?

Sol. wkt $\dim(S) + \dim(S^\perp) = 4$

ⓐ $S =$ zero vector

$\dim(S) = 0$

$\dim(S^\perp) = 4$

we should have 4 L.I vectors in R^4

ⓑ $S = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ $\dim(S) = 1$

$\dim(S^\perp) = 3$

we should have 3 L.I vectors that should be orthogonal to S

Let $A = \begin{array}{cccc} x & y & z & t \\ \end{array}$
$A = [1 \ 1 \ 1 \ 1]$ $\quad x + y + z + t = 0$

x	y	z	t
-1	0	1	0
-1	0	0	1

$$S^\perp = \left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

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ⓒ $S = \left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 4 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 3 \end{pmatrix} \right\}$

$$\dim(S) = 3$$
$$\dim(S^{\perp}) = 1$$

we need to find 1 L.I vector orthogonal to S

Let A = row space
$$\begin{array}{cccc} & & & \\ \left[\begin{matrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{matrix}\right] \\ x \quad y \quad z \quad t \end{array} = U$$

$$Ux = 0$$

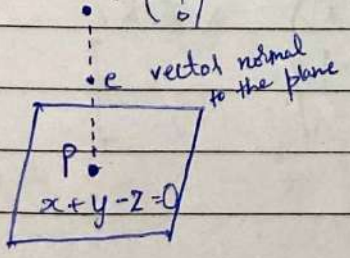
$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$z = 1 \qquad x = 0, y = 0, t = 0$

$$\therefore S^{\perp} = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

5. What point on the plane $x + y - z = 0$ is closest to $b = (2, 1, 0)$?

Sol.



$$p = b - e$$
$$= \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

7. Find the matrix P that projects every point in R^3 onto the line of intersection of the planes $x-2y+3z=0$ & $y-z=0$. What are the column space and row space of this matrix?

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \end{array}$$

$x-2y+3z=0$

$y-z=0$

$\boxed{z=1}$ $\boxed{y=1}$ $\boxed{x=-1}$

line of intersection $= \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = a$

Proj matrix $= \dfrac{aa^T}{a^Ta} = \dfrac{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\begin{pmatrix} -1 & 1 & 1 \end{pmatrix}}{\begin{pmatrix} -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}}$

$$= \frac{1}{3}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

$R_2 \rightarrow R_2 + R_1$

$R_3 \rightarrow R_3 + R_1$

$$= \frac{1}{3}\begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

rank = 1

column space $= \left\{ \alpha\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \alpha \in R \right\}$

row space $= \left\{ \alpha\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \alpha \in R \right\}$

8. Let $A = \begin{bmatrix} 3 & 1 & -1 \end{bmatrix}$ and let V be the nullspace of A. Find:

i) a basis for V and basis for $V^\perp$

ii) a projection matrix P_1 onto $V^\perp$

iii) the projection matrix P_2 onto V

Sol. Let $A = \begin{bmatrix} 3 & 1 & -1 \end{bmatrix} = U$

row space - why?

To find Null space $Ux = 0$

(or) $Ax = 0$

$$\begin{bmatrix} 3 & 1 & -1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$3x + y - z = 0$$

y & z are free vars.

y	z	x
3	0	-1
0	3	1

$$\therefore V = \begin{bmatrix} -1 & 1 \\ 3 & 0 \\ 0 & 3 \end{bmatrix} = \text{Null space Matrix}$$

i) Basic Vector for V $= \left\{ \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \right\}$

$A = V^\perp$ ($\because$ A = rowspace & V = nullspace $V^\perp = A$)

ii) $(V^\perp = a)$ $\quad P_1 = \dfrac{a a^T}{a^T a}$

iii) $(A = V)$

$$P_2 = A(A^T A)^{-1} A^T$$

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9. Find $||E|| = ||Ax - b||^2$ and solve the normal eqn. $A^T A\hat{x} = A^T b$. Find the solution $\hat{x}$ and the projection p

Sol. $$||E|| = ||Ax - b||^2$$

wkt $||x||^2 = x_1^2 + x_2^2 + \dots + x_n^2$

$\hookrightarrow \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{matrix}$

Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$Ax - b = \begin{pmatrix} 2 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 2 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2x_1 + x_2 - 3 \\ x_1 - 1 \\ -x_2 - 2 \\ -x_1 + x_2 + 1 \end{pmatrix}$$

$$\Rightarrow ||Ax - b||^2 = \left[(2x_1 + x_2 - 3)^2 + (x_1 - 1)^2 + (-x_2 - 2)^2 + (-x_1 + x_2 + 1)^2\right]$$

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Solve normal eqn.

$$A^T A \hat{x} = A^T b$$

$$\begin{pmatrix} 2 & 1 & 0 & -1 \\ 1 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 & -1 \\ 1 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 2 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 4+1+0+1 & 2+0+0-1 \\ 2+0+0-1 & 1+1+1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} 6 & 1 \\ 1 & 3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$$

$$6x_1 + x_2 = 8$$
$$x_1 + 3x_2 = 0$$

$$x_1 = \frac{24}{17} \qquad x_2 = \frac{-8}{17}$$

$$\hat{x} = \begin{pmatrix} 24/17 \\ -8/17 \end{pmatrix}$$

$$\hat{x} = \frac{1}{17} \begin{pmatrix} 24 \\ -8 \end{pmatrix}$$

$$p = A\hat{x} = \frac{1}{17} \begin{pmatrix} 2 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 24 \\ -8 \end{pmatrix} = \frac{1}{17} \begin{pmatrix} 40 \\ 24 \\ 8 \\ -32 \end{pmatrix}$$

16. A person's normal orientation test

Time in room	1	2	3	4	5	6
Time to find way out of maze	0.8	2.1	2.6	2.0	3.1	3.3

a) Find least square line relating x and y

b) Hence estimate the time it will take the subjects to find his way out of ~~the~~ maze after 10 hours in room.

Sol. Let $y = C + Dx$ be the line relating x and y.

Consider the normal eqn.

$$A^T A \hat{x} = A^T b$$

$$\hat{x} = \begin{pmatrix} C \\ D \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \end{pmatrix} \qquad b = \begin{pmatrix} 0.8 \\ 2.1 \\ 2.6 \\ 2.0 \\ 3.1 \\ 3.3 \end{pmatrix}$$

input x output y

$$A^T A \hat{x} = A^T b$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 0.8 \\ 2.1 \\ 2.6 \\ 2.0 \\ 3.1 \\ 3.3 \end{pmatrix}$$

$$\begin{bmatrix} 6 & 21 \\ 21 & 91 \end{bmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 13.9 \\ 56.1 \end{pmatrix}$$

$$6C + 21D = 13.9$$
$$21C + 91D = 56.1$$

$$C = \frac{62}{75} \qquad D = \frac{149}{350}$$
$$= 0.826 \qquad\quad = 0.425$$

$$\hat{x} = \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 0.826 \\ 0.425 \end{pmatrix}$$

$$y = 0.426x + 0.826$$

b) $y = 0.426(10) + 0.826$
$= 5.087$

Orthogonal basis and Orthogonal vectors

$$\{v_1, v_2, v_3\} \text{ in } R^3$$

$$a_i^T a_j = \begin{cases} 0, & \text{whenever } i \neq j \quad \text{(orthogonality)} \\ 1, & \text{whenever } i = j \quad \text{(normalization)} \end{cases}$$

$Q \rightarrow$ Orthogonal matrix having orthonormal columns

→ Properties of matrix Q

1. If Q has orthonormal vectors then

$$Q^T Q = I$$

Left inv

2. If Q (square) with orthonormal columns then $Q^{-1} = Q^T$

3. Matrix Q preserves length of a vector

$$\|Q\|^2 = (Qx)^T Qx = x^T Q^T Q x = x^T x$$

$$\|Qx\|^2 = \|x\|^2$$

$$\|Qx\| = \|x\|$$

4. Matrix Q preserves inner product and angles

$$\because (Qx)^T Qy = x^T Q^T Qy = x^T y$$

5. $b = x_1 a_1 + x_2 a_2 + x_3 a_3 + \dots + x_n a_n$

$$q_1^T b = x_1 q_1^T q_1$$

$$x = q_1^T b$$

If $q_1, q_2 \dots q_n$ are orthonormal basis of R^n then any vector 'b' can be expressed as sum of 1-D projections onto q_i's

$\| ^ly$ $b = (q_1^T b) q_1 + (q_2^T b) q_2 + \dots (q_n^T b) q_n$

⇒ proj of b onto plane q_1 & q_2 is sum of individual 1-D projections

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6. For $Ax = b$ where $x = A^{-1}b$ is easier when $A = Q$

then $Q^T = Q^{-1}$

$$x = A^T b$$

7. The rows are orthonormal whenever columns are orthonormal

8. Least square soln. with matrix Q is easier

$$Ax \neq b$$
$$A\hat{x} = b$$
$$A^T A \hat{x} = A^T b$$
$$\hat{x} = (A^T A)^{-1} A^T b$$

If Q has orthonormal columns

$$Qx \neq b$$
$$Q^T Q \hat{x} = Q^T b$$
$$\hat{x} = (Q^T Q)^{-1} Q^T b$$

$$\hat{x} = Q^T b$$

9. Projection matrix wrt Q is $P = QQ^T$

Note : $P = I$ on column of Q

$P = 0$ on orthogonal complement

i.e Nullspace of Q^T

18th March, 2024

→ Special cases

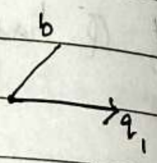
1. If Q is a square matrix, $\hat{x} = Q^T b$
$\hat{x} = Q^{-1} b \quad (Q^T = Q^{-1})$

2. If Q is a square matrix, $P = Q\hat{x}$
$= QQ^{-1} b$
$P = b$

Problems: 1. Project $b = (0,3,0)$ onto each of the orthonormal vectors $a_1 = \left(\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}\right)$, $a_2 = \left(\frac{-1}{3}, \frac{2}{3}, \frac{2}{3}\right)$ and find the projection P onto plane a_1 & a_2.

Sol. i) The proj. of 'b' onto q_1 (line)

$$P_1 = \left(\frac{q_1^T b}{q_1^T q_1}\right) q_1$$



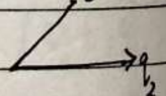
here $q_1^T q_1 = 1$ (orthonormal)

$$P_1 = (q_1^T b) q_1$$

$$= \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{-1}{3} \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} \begin{pmatrix} 2/3 \\ 2/3 \\ -1/3 \end{pmatrix}$$

$$= 2 \begin{pmatrix} 2/3 \\ 2/3 \\ -1/3 \end{pmatrix}$$

ii) The proj of b onto q_2 (line)



$$P_2 = (q_2^T b) q_2$$

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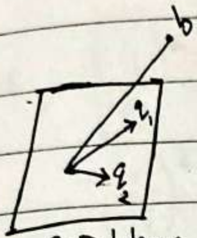
$$= \begin{pmatrix} \frac{-1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} \begin{pmatrix} -1/3 \\ 2/3 \\ 2/3 \end{pmatrix}$$

$$= 2 \begin{pmatrix} -1/3 \\ 2/3 \\ 2/3 \end{pmatrix}$$

iii) 

2-D plane

The proj 'b' onto plane containing q_1 & q_2

$$p_3 = p_1 + p_2$$

only for orthonormal vectors (proj wrt several variables = sum of 1D projections)

$$\therefore p_3 = 2 \begin{pmatrix} 2/3 \\ 2/3 \\ -1/3 \end{pmatrix} + 2 \begin{pmatrix} -1/3 \\ 2/3 \\ 2/3 \end{pmatrix}$$

$$= \begin{pmatrix} 2/3 \\ 8/3 \\ 2/3 \end{pmatrix}$$

2. $\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} & - \\ 1/\sqrt{3} & 2/\sqrt{14} & - \\ 1/\sqrt{3} & -3/\sqrt{14} & - \end{bmatrix}$ Find a 3rd column so that the matrix is orthogonal. Verify that the rows automatically become orthonormal at the same time.

Sol. Let $b = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be the vector which is orthogonal to first column q_1 & second column q_2

$b \perp q_1 \Rightarrow b^T q_1 = 0$

$$\Rightarrow (x \;\; y \;\; z) \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} = 0$$

$$\frac{x}{\sqrt{3}} + \frac{y}{\sqrt{3}} + \frac{z}{\sqrt{3}} = 0$$

(or) $x + y + z = 0$ — ①

$b \perp q_2 \Rightarrow b^T q_2 = 0$

$$(x \; y \; z)\begin{pmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ -3/\sqrt{14} \end{pmatrix} = 0$$

$$\frac{x}{\sqrt{14}} + \frac{2y}{\sqrt{14}} - \frac{3z}{\sqrt{14}} = 0$$

(or)

$x + 2y - 3z = 0$ — ②

Solving ① & ②

$x + y + z = 0$
$x + 2y - 3z = 0$

$$(A|b) = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & -3 & 0 \end{array}\right]$$

$R_2 \to R_2 - R_1$

$$\begin{array}{cccc} x & y & z & \\ \end{array}$$
$$\simeq \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -4 & 0 \end{array}\right]$$

$z \to$ free var.

$z = 1$
$y = 4$
$x = -5$

$$b = \begin{pmatrix} -5 \\ 4 \\ 1 \end{pmatrix} \perp (q_1, q_2)$$

let $q_3 = \left[\begin{array}{c} \frac{-5}{\sqrt{(-5)^2 + 4^2 + 1^2}} = \frac{-5}{\sqrt{42}} \\ 4/\sqrt{42} \\ 1/\sqrt{42} \end{array}\right]$ is a vector $\perp (q_1, q_2)$

$$\therefore \begin{array}{c} \begin{matrix} q_1 & q_2 & q_3 \end{matrix} \\ \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} & -5/\sqrt{42} \\ 1/\sqrt{3} & 2/\sqrt{14} & 4/\sqrt{42} \\ 1/\sqrt{3} & -3/\sqrt{14} & 1/\sqrt{42} \end{bmatrix} \end{array}$$

For basis to be orthonormal norm should be = 1

i.e $\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{14}}\right)^2 + \left(\frac{-5}{\sqrt{42}}\right)^2 = 1$

$$\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{2}{\sqrt{14}}\right)^2 + \left(\frac{4}{\sqrt{42}}\right)^2 = 1$$

$$\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{-3}{\sqrt{14}}\right)^2 + \left(\frac{1}{\sqrt{42}}\right)^2 = 1$$

Grahm's Schmidt Orthogonalization process

Given : A set of linearly independent vectors say a, b, c

Note : Necessary cond. to apply this orthogonalization is to have set of independent vectors.

Aim: To find a set of orthonormal vectors say q_1, q_2, q_3

Note : No. of orthonormal vectors = obtained

Objective: Every vector in a given vector space say R^n can always be expressed in terms of the linear combination of its 1-D projection onto the orthonormal bases vectors.

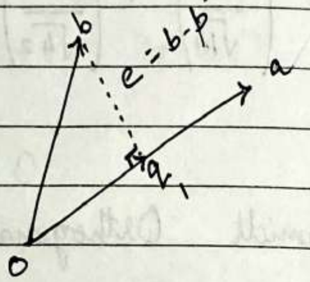
eg: vector $b \in R^n$

$$b = x_1 q_1 + x_2 q_2 + \cdots + x_n q_n$$

where $x_i \in R$ q_i - orthonormal bases set

$$\begin{matrix} a & b & c \\ q_1 & q_2 & q_3 \end{matrix}$$

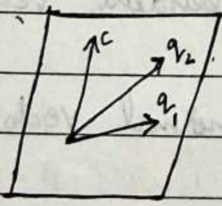
$$q_1 = \frac{a}{\|a\|}$$



$p = (q_1^T b) q_1$ since orthonormal

$$q_2 = \frac{e}{\|e\|}$$

$$e = b - p$$
$$e = b - (q_1^T b) q_1$$



proj of c onto plane containing q_1 & q_2

=> sum of individual proj

$$E = c - p$$
$$E = c - \left[(q_1^T c) q_1 + (q_2^T c) q_2 \right]$$

$$q_3 = \frac{E}{\|E\|}$$

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Problems 1. From the vectors a, b, c find the orthonormal vectors q_1, q_2 and q_3 given as $a = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $c = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Sol. By G.S process

$$q_1 = \frac{a}{\|a\|} = \frac{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}{\sqrt{1^2+1^2+0^2}} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$q_2 = \frac{e}{\|e\|}$$

$$e = b - (q_1^T b) q_1$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \left[\left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \right]$$

$$= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{2}} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix}$$

$$q_2 = \frac{\begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix}}{\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{-1}{2}\right)^2 + 1}} = \frac{1}{\frac{\sqrt{6}}{2}} \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 3/\sqrt{6} \end{bmatrix}$$

$$q_3 = \frac{e_3}{\|e_3\|}, \quad e_3 = c - \left[(q_1^T c) q_1 + (q_2^T c) q_2\right]$$

$$e_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \Bigg[\underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}_{\text{scalar}} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} + \underbrace{\begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}_{\text{scalar}} \begin{pmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix} \Bigg]$$

$$= \begin{pmatrix} -2/3 \\ 2/3 \\ 2/3 \end{pmatrix}$$

$$\|E\| = \frac{2}{\sqrt{3}} \qquad \therefore \ q_3 = \begin{pmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

same order

$A = QR$ (factorization)

Q → orthonormal vectors

R → upper Δ^{lar} sq. matrix

* Let A be a matrix whose columns are a, b, c

* Let Q be matrix whose columns are q_1, q_2, q_3 obtained by GM process.

* To get R, which connects A and Q, express a, b, c as a linear combination of q_1, q_2, q_3

$$A = Q \; R$$

$$a = (q_1^T a)\, q_1$$

$$b = (q_1^T b)\, q_1 + (q_2^T b)\, q_2$$

$$c = (q_1^T c)\, q_1 + (q_2^T c)\, q_2 + (q_3^T c)\, q_3$$

$$A = \begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \begin{bmatrix} q_1^T a & q_1^T b & q_1^T c \\ 0 & q_2^T b & q_2^T c \\ 0 & 0 & q_3^T c \end{bmatrix}$$

(OR) $Q^T A = Q R$

$$Q^T A = Q^T Q R$$

$$\boxed{R = Q^T A}$$

20th March, 2024

Note: Consider the normal eqn. for $Ax = b$

$$A^T A \hat{x} = A^T b \quad \text{but } A = QR$$

$$(QR)^T QR \hat{x} = (QR)^T b$$

$$R^T Q^T Q R \hat{x} = R^T Q^T b$$

$$R^T R \hat{x} = R^T Q^T b$$

$$R \hat{x} = Q^T b$$

When $Ax = b$ is not solvable, we consider $R\hat{x} = Q^T b$ and solve.

Problems 1. Factor $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & 0 \end{bmatrix}$ into QR

recognizing that first column is already an unit vector

Sol. Let $A = \begin{array}{cc} a & b \\ \left[\begin{array}{c} \cos\theta \\ \sin\theta \end{array}\right. & \left.\begin{array}{c} \sin\theta \\ 0 \end{array}\right] \end{array}$

To get Q

$$q_1 = \frac{a}{\|a\|} = \frac{\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}}{\sqrt{\cos^2\theta + \sin^2\theta}}$$

$$q_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$q_2 = \frac{e}{\|e\|} \qquad e = b - (q_1^T b)\, q_1$$

$$e = \begin{bmatrix} \sin\theta \\ 0 \end{bmatrix} - \left[(\cos\theta \ \sin\theta) \begin{pmatrix} \sin\theta \\ 0 \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \right]$$

$$= \begin{bmatrix} \sin\theta \\ 0 \end{bmatrix} - \left[(\cos\theta \sin\theta) \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \right]$$

$$= \begin{bmatrix} \sin\theta \\ 0 \end{bmatrix} - \begin{bmatrix} \cos^2\theta \sin\theta \\ \cos\theta \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \sin\theta - \cos^2\theta \sin\theta \\ -\cos\theta \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \sin\theta(1 - \cos^2\theta) \\ -\cos\theta \sin^2\theta \end{bmatrix}$$

$$\|e\| = \sqrt{[\sin\theta(1-\cos^2\theta)]^2 + [-\cos\theta \sin^2\theta]^2}$$

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$$= \sqrt{\sin^2\theta \sin^4\theta + \cos^2\theta \sin^4\theta}$$
$$= \sqrt{\sin^4\theta(\sin^2\theta + \cos^2\theta)}$$
$$= \sqrt{\sin^4\theta}$$
$$= \sin^2\theta$$

$$q_2 = \frac{e}{\|e\|} = \frac{1}{\sin^2\theta}\begin{bmatrix} \sin^3\theta \\ -\cos\theta\sin^2\theta \end{bmatrix} = \begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix}$$

$$R = \begin{bmatrix} q_1^T a & q_1^T b \\ 0 & q_2^T b \end{bmatrix} = \begin{bmatrix} 1 & \sin\theta\cos\theta \\ 0 & \sin^2\theta \end{bmatrix}$$

(or) $R = Q^T A$

$$Q = \begin{array}{c} \begin{array}{cc} q_1 & q_2 \end{array} \\ \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \end{array}$$

2. Find an orthonormal set q_1, q_2, q_3 for which q_1, q_2 span the column space of $A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}$. Which fundamental subspace contains q_3. What is the least squares solution of $Ax = b$ if $b = [1 \ 2 \ 7]^T$?

Sol. Let $A = \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix} \end{array}$ a & b are independent vectors

By G.S process

$$q_1 = \frac{a}{\|a\|} = \frac{\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}}{\sqrt{1^2 + 2^2 + (-2)^2}} = \begin{pmatrix} 1/3 \\ 2/3 \\ -2/3 \end{pmatrix}$$

$$q_2 = \frac{e}{\|e\|} \qquad e = b - (q_1^T b)q_1$$

$$e = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} - \left[\begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{-2}{3} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} \begin{pmatrix} 1/3 \\ 2/3 \\ -2/3 \end{pmatrix} \right]$$

$$= \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} - \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$$\|e\| = \sqrt{9} = 3$$

$$\therefore q_2 = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$$

$$A' = \begin{array}{c} \begin{array}{cc} q_1 & q_2 \end{array} \\ \underbrace{\begin{bmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \\ -2/3 & 2/3 \end{bmatrix}}_{\text{orthonormal vectors}} \end{array}$$

To find q_3

Let $c = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be the vector which is orthogonal to q_1 & q_2

$c^T q_1 = 0$ $\qquad$ $c^T q_2 = 0$

$x + 2y - 2z = 0$ $\qquad$ $2x + y + 2z = 0$

$$(A|b) = \left[\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 2 & 1 & 2 & 0 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

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$$\simeq \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 0 & -3 & 6 & 0 \end{array}\right] \end{array}$$

$z \to$ free variable

$$x + 2y - 2z = 0$$
$$-3y + 6z = 0$$

$\boxed{z = 1}$
$\boxed{y = 2}$
$\boxed{x = -2}$

$$C = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$$

orthogonal vector

To get orthonormal vector $q_3 = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$

$$\therefore Q = \begin{array}{c} \begin{array}{ccc} q_1 & q_2 & q_3 \end{array} \\ \begin{bmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix} \end{array}$$

q_1 & q_2 are in column space
we got $q_3 \perp$ to q_1 & q_2
$\therefore q_3 \in$ Left Null space

Since $(q_1, q_2) \in C(A)$
& $C(A) \perp N(A^T)$
$\Rightarrow q_3 \in N(A^T)$

To get Least Squares soln. Let $A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}$ & $b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$

By Least Square Method

$$R\hat{x} = Q^T b$$

$$Q = \begin{bmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \\ -2/3 & 2/3 \end{bmatrix} \qquad \underset{3\times2}{A} = \underset{3\times2}{Q}\ \underset{2\times2}{\overset{\text{upper } \Delta}{R}}$$

$$\therefore R = \begin{bmatrix} q_1^T a & q_1^T b \\ 0 & q_2^T b \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 0 & 3 \end{bmatrix}$$

$$R\hat{x} = Q^T b$$

$$\begin{bmatrix} 3 & -3 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \end{bmatrix}\begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$$

Eigen Values and Eigen Vectors

$$\underset{\text{Square}}{A}x = \underset{\text{scalar}}{\lambda} x$$

Let A be a square matrix of order n. If there exists a real / complex no. λ and a non zero vector x such that

Note: 1. The vector x is in the null space $A-\lambda I$.

2. The number λ is chosen so that $A-\lambda I$ has a null space.

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3. $A - \lambda I$ must be singular

4. Det $(A - \lambda I) = 0$ is called the characteristic eqn. of A and roots of this eqn. are called characteristic roots / Eigen values / Latent

$$Ax = \lambda x$$
$$Ax - \lambda x = 0$$
$$(A - \lambda I)\lambda = 0$$
$$Bx = 0$$

* Corresponding to 'n' distinct Eigen values we get 'n' independent Eigen vectors. But when 2 or more eigen values are equal, it may or may not be possible to get linearly independent Eigen vectors corresponding to repeated roots.

Q. Find the eigen values and corresponding eigen vectors of
$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

Sol. Consider characteristic eqn. $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & -3 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$+\lambda^3 - (\quad)\lambda^2 + (\quad)\lambda - (\quad) = 0$

$\lambda^3 - \left(\begin{array}{c}\text{sum of element of}\\ \text{diagonal of } \lambda^2\end{array}\right)\lambda^2 + \left(\begin{array}{c}\text{sum of all the}\\ \text{minors of A}\end{array}\right)\lambda$

$- (\text{determinant of A}) = 0$

minors $\begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix}$

$$4 - 8 + 4 = 0$$

$$\lambda^3 - 7\lambda^2 + 0\lambda - 36 = 0$$

$$\lambda = -2, 3, 6$$ ~~Eigen~~

(eigen values)

To get eigen vectors

consider $(A - \lambda I)x = 0$

$$\begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} x = 0$$

$$(1-\lambda)x + y + 3z = 0$$
$$x + (5-\lambda)y + z = 0$$
$$3x + y + (1-\lambda)z = 0$$

Case i) when $\lambda = -2$

$$3x + y + 3z = 0$$
$$x + 7y + z = 0$$
$$3x + y + 3z = 0$$

* Eigen values are unique
Eigen vectors are not unique

$$\frac{x}{\begin{vmatrix} b & c \\ e & f \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a & c \\ d & f \end{vmatrix}} = \frac{z}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}$$

$$\frac{x}{\begin{vmatrix} 1 & 3 \\ 7 & 1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 3 & 3 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 3 & 1 \\ 1 & 7 \end{vmatrix}}$$

$$\frac{x}{-20} = \frac{-y}{0} = \frac{z}{20} \qquad X_1 \text{ (Eigen Vector)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Case ii) when $\lambda = 3$

$$\left.\begin{aligned} -2x + y + 3z &= 0 \\ x + 2y + z &= 0 \end{aligned}\right] \text{choose diff eqns}$$

$$3x + y + -2z = 0$$

$$\frac{x}{\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -2 & 3 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -2 & 1 \\ 1 & 2 \end{vmatrix}}$$

$$\frac{x}{-5} = \frac{-y}{-5} = \frac{z}{-5}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 \\ 5 \\ -5 \end{pmatrix} \text{ or } \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Case iii) $\lambda = 6$

$$-5x + y + 3z = 0$$
$$x - y + z = 0$$
$$3x + y - 5z = 0$$

$$\frac{x}{\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -5 & 3 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -5 & 1 \\ 1 & -1 \end{vmatrix}}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

21st March, 2024

Problems

q.1 Find the Eigen value and Eigen vectors for the given vector

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Sol. characteristic eqn. is

$$|A - \lambda I| = 0$$

↳ subtract diagonal elements with λ

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$f(\lambda)\lambda^3 - \underbrace{(12)}_{\text{sum of diagonal elements}}\lambda^2 + \underbrace{(36)}_{\text{minors}}\lambda - \underbrace{(32)}_{\text{determinant}} = 0$$

$\lambda_1 c_1 \quad \lambda_2 c_2 \quad \lambda_3 c_3$

$8 + 14 + 14$

$\lambda = 8, 2$

ones of them is repeated

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sum of Eigen values = trace

$10 + x = 12$

$x = 2 \Rightarrow 2$ is repeated

To get Eigen Vectors

$$(A - \lambda I)x = 0$$

$$(6-\lambda)x - 2y + 2z = 0$$
$$-2x + (3-\lambda)y - z = 0$$
$$2x - y + (3-\lambda)z = 0$$

case i) when $\lambda = 8$

$$-2x - 2y + 2z = 0$$
$$-2x - 5y - z = 0$$

choose diff eqns.

$$2x - y - 5z = 0$$

$$\frac{x}{\begin{vmatrix} -2 & 2 \\ -5 & -1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -2 & 2 \\ -2 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -2 & -2 \\ -2 & -5 \end{vmatrix}}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

case ii) when $\lambda = 2$

$$\left.\begin{aligned} 4x - 2y + 2z &= 0 \\ -2x + y - z &= 0 \\ 2x - y + z &= 0 \end{aligned}\right\}$$

all 3 eqns. are same
cannot use prev method

consider $2x - y + z = 0$

2 free variables

y & z are free variables

$$\begin{array}{ccc} y & z & x \\ 1 & 0 & 1/2 \\ 0 & 1 & -1/2 \end{array}$$

eigen values are : $\begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix}$

q2. Find Eigen value & Eigen vector for the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix}$$

Sol. Characteristic eqn. is $|A - \lambda I| = 0$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -1 & 1 \\ 1 & -\lambda & 0 \\ -1 & 1 & -1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 + \lambda = 0$$

$$\lambda = 0, \quad \lambda^2 + 1 = 0$$

$$\lambda = 0, \pm i$$

consider $|A - \lambda I| x = 0$

$$(1-\lambda)x + y + z = 0$$

$$x - \lambda y + 0z = 0$$

$$-x + y + (1-\lambda)z = 0$$

Case i) when $\lambda = 0$

$$\left.\begin{aligned} x - y + z &= 0 \\ x &= 0 \end{aligned}\right] \quad 1x + 0y + 0z = 0$$
$$-x + y - z = 0$$

$$\frac{x}{\begin{vmatrix} -1 & 1 \\ 0 & 0 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix}}$$

eigen vector
$$X_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ +1 \\ 1 \end{pmatrix}$$

Case ii) when $\lambda = i$

$$\left.\begin{aligned} (1-i)x - y + z &= 0 \\ x - iy + 0z &= 0 \end{aligned}\right]$$
$$-x + y + (1-i)z = 0$$

$$\frac{x}{\begin{vmatrix} -1 & 1 \\ -i & 0 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1-i & 1 \\ 1 & 0 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1-i & -1 \\ 1 & -i \end{vmatrix}}$$

$$\frac{x}{i} = \frac{-y}{-1} = \frac{z}{-i}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} i \\ 1 \\ -i \end{pmatrix}$$ eigen vector X_2

Case iii) when $\lambda = -i$

$$\begin{aligned} (1+i)x - y + z &= 0 \\ x + iy + 0z &= 0 \\ -x + y + (1+i)z &= 0 \end{aligned} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \ \end{pmatrix}$$

Q3. Find the Eigen value & Eigen vector of

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

Sol. $|A - \lambda I| = 0$

$$\begin{vmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix} = 0$$

$$+\lambda^2 - \underbrace{(-3)}_{\text{trace}}\lambda + \underbrace{(2)}_{\text{determinant}} = 0$$

$$\lambda = -1, -2$$

To get Eigen Vectors

$$-\lambda x + y = 0$$
$$-2x + (-3-\lambda) y = 0$$

Case i) $\lambda = -1$

$$\left.\begin{matrix} x + y = 0 \\ -2x - 2y = 0 \end{matrix}\right] \text{Both eqns. are same}$$

$x + y = 0$

let $y = 1$ (free variable)

$x = -1$

eigen vector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

case ii) $\lambda = -2$

$$\left.\begin{matrix} +2x + y = 0 \\ -2x - y = 0 \end{matrix}\right] \text{same eqns.}$$

let $y = 1$

$x = -1/2$

eigen vector $\begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$

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→ Properties of Eigen values and Eigen vectors

* If λ is an eigen value of A with x as the corresponding eigen vector then λ^2 is an Eigen value of A^2 with same eigen vector x

* For a given Eigen vector x, there corresponds only one Eigen value λ

* For a given Eigen value there corresponds infinitely many Eigen vectors.

eg: $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

$\lambda = -1, -2$

$A^2 = \begin{bmatrix} -2 & -3 \\ 6 & 7 \end{bmatrix}$

$\lambda = 1, 4$

* $\lambda = 0$ is an Eigen value of A, iff A is singular i.e $|A| = 0$

* If λ is an Eigen value of A with x as the Eigen vector then $1/\lambda$ is an Eigen value of A^{-1} provided A^{-1} exists

* A and A^T have same Eigen Values

* The Eigen values of a diagonal matrix are just the diagonal elements of the matrix

* The Eigen values of upper ∇ and lower Δ are just the diagonal elements

* The Eigen values of an idempotent matrix are either 0 or 1

* The sum of the Eigen values of a matrix is the sum of the elements of the principal diagonal.

* The product of Eigen values of a matrix is equal to its determinant.

Cayley - Hamilton Theorem

Theorem : Every square matrix satisfies its own characteristic eqn.

$$\Rightarrow A = \begin{bmatrix} \\ \end{bmatrix}$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$\Rightarrow A^2 - 3A + 2 = 0$$

Problems

1. $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$ Evaluate A^{-1} and A^{-2} using Cayley Hamilton theorem.

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Sol. $|A - \lambda I| = 0$

$$\begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -4 & -3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$$

By Cayley Hamilton theorem $A^3 - 4A^2 - A + 4I = 0$

To get A^{-1} $\times A^{-1} \Rightarrow A^2 - 4A - I + 4A^{-1} = 0$

$$A^{-1} = \frac{1}{4}\left[-A^2 + 4A + I\right]$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 6 & 6 \\ -1 & 6 & 2 \\ 1 & -10 & -6 \end{bmatrix}$$

To get A^{-2} $\times A^{-2} \Rightarrow A - 4I - A^{-1} + 4A^{-2} = 0$

$$A^{-2} = \frac{1}{4}\left[-A + A^{-1} + 4I\right]$$

$$= \frac{1}{16}\begin{bmatrix} 1 & -18 & -18 \\ -5 & 10 & -6 \\ 5 & 6 & 22 \end{bmatrix}$$

25th March, 2024

q. Find Eigen values and Eigen vectors of $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$,

shift A to $A - 7I$ what are the eigen values and eigen vectors and how are they related to those of A ?

Sol. $|A - \lambda I| = 0$

$$\begin{vmatrix} 4-\lambda & 3 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - (6)\lambda + (5) = 0$$

$$\lambda = 1, 5$$

To find eigen vectors

$$(A - \lambda I)x = 0$$

$$(4-\lambda)x + 3y = 0$$

$$x + (2-\lambda)y = 0$$

case 1. when $\lambda = 1$

$$\left.\begin{matrix} 3x + 3y = 0 \\ x + y = 0 \end{matrix}\right\} \text{same}$$

$x + y = 0$

let $y = 1$, $x = -1$

eigen vector $X_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

case 2. when $\lambda = 5$

$$\left.\begin{matrix} -x + 3y = 0 \\ x - 3y = 0 \end{matrix}\right\} \text{same}$$

$x - 3y = 0$

$y = 1$, $x = 3$

eigen vector $X_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

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Let $A - 7I = B$

$$B = \begin{bmatrix} -3 & 3 \\ 1 & -5 \end{bmatrix}$$

$|B - \lambda I| = 0$

$$\lambda^2 - (-8\lambda) + (12) = 0$$
$$\lambda^2 + 8\lambda + 12 = 0$$
$$\lambda = -2, -6$$

To find eigen vectors

$$(A - \lambda I)x = 0$$

$$(-3-\lambda)x + 3y = 0$$
$$x + (-5-\lambda)y = 0$$

i) when $\lambda = -2$

$$\left.\begin{aligned} -x + 3y &= 0 \\ x - 3y &= 0 \end{aligned}\right\} \text{same}$$

$y = 1 \quad x = 3$

$$X_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

ii) when $\lambda = -6$

$$\left.\begin{aligned} 3x + 3y &= 0 \\ x + y &= 0 \end{aligned}\right\} \text{same}$$

$y = 1, \ x = -1$

$$X_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

* Eigen vectors are same for A and $B = A - \lambda I$ and Eigen values have been shifted from λ to $\lambda - 7$

q: Find eigen values of A, A^2, A^{-1} and $A+4I$

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Sol. $|A-\lambda I| = \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0$

$$\lambda^2 - 4\lambda + 3\lambda = 0$$

for A : $\lambda = 1, 3$

A^{-1} : $\lambda = \frac{1}{1}, \frac{1}{3}$

A^2 : $\lambda = 1^2, 3^2$

$A+4I$: $\lambda = 5, 7$

q: Write 3 diff 2x2 matrices for which the eigen values are 4, 5 and determinant = 20

can be anything

$$\begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix} \quad \begin{bmatrix} 4 & 7 \\ 0 & 5 \end{bmatrix} \quad \begin{bmatrix} 4 & 0 \\ 7 & 5 \end{bmatrix}$$

CWP q10.

Let S be the 2D subspace of R^3 with orthonormal basis

$$v_1 = \begin{pmatrix} 2/3 \\ -1/3 \\ -2/3 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \qquad \& \text{ let } v = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

Write $v = u + w$ with w in S & v in $S^\perp$

Sol. proj of v onto S (w)

split $\Rightarrow$ projection

$$\begin{array}{ccc} & v & \\ u & & w \\ S^\perp & & S \end{array}$$

b on a

w = proj of v on v_1
+ proj of v on v_2

$$= \left(\frac{v_1^T v}{\cancel{v_1^T v_1}^{\,1}}\right) v + \left(\frac{v_2^T v}{\cancel{v_2^T v_2}^{\,1}}\right) v$$

$$= \begin{pmatrix} 11/6 \\ 1/3 \\ 19/6 \end{pmatrix}$$

$v = u + w$

$$u = v - w = \begin{pmatrix} 1/6 \\ 2/3 \\ -1/6 \end{pmatrix}$$