19th April, 2023

UNIT-3 Laplace Transforms

Definition: Let $f(t)$ be a given function defined for all $t > 0$. The Laplace transform of $f(t)$ is denoted by $L[f(t)]$ and it is defined to be

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

provided that the integral exists, where s is a parameter real / complex. Here L is the Laplace Transform Operator

* $f(t)$ is the inverse Laplace Transform of $F(s)$ and is denoted as $L^{-1}[f(s)]$

$t \rightarrow$ time $s \rightarrow$ space

Linearity Property Of Laplace Transform

1. If $L[f(t)] = F(s)$ and $L[g(t)] = G(s)$ and if c_1 and c_2 are any two arbitrary constants then

$$L[c_1 f(t) \pm c_2 g(t)] = c_1 L[f(t)] \pm c_2 L[f(t)]$$
$$= c_1 F(s) \pm c_2 G(s)$$

2. If $L^{-1}[f(t)] = f(t)$ and $L^{-1}[g(s)] = g(t)$ then

$$L^{-1}[c_1 F(s) \pm c_2 G(s)] = c_1 L^{-1}[F(s)] \pm c_2 L^{-1}[G(s)]$$
$$= c_1 f(t) \pm c_2 g(t)$$

Date
Page

Q: Find the Laplace Transform of

$$f(t) = \begin{cases} e^t & 0 < t < 1 \\ 0 & t > 1 \end{cases}$$

Sol. $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$= \int_0^1 e^{-st} f(t)\, dt + \int_1^\infty e^{-st} f(t)\, dt$$

$$= \int_0^1 e^{-st} e^t\, dt + \int_1^\infty e^{-st} (0)\, dt$$

$$= \int_0^1 e^{-t(s-1)}\, dt$$

$$= \left[\frac{e^{-(s-1)t}}{-(s-1)} \right]_0^1$$

$$= \frac{e^{-(s-1)}}{-(s-1)} - \frac{1}{-(s-1)}$$

$$= \frac{1}{s-1} - \frac{e^{-(s-1)}}{s-1}$$

Laplace Transform of Standard Functions.

1. $L[k] = \frac{k}{s}$, where $k \to$ constant and $t > 0$

PROOF: By definition: $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$L[k] = \int_0^\infty e^{-st}\, k\, dt$$

$$= k \left[\frac{e^{-st}}{-s} \right]_0^\infty \qquad e^{-\infty} = 0$$

$$= \frac{k}{s}$$

* Laplace transform doesn't exist for $s < 0$
* $L[0] = 0$
* $L[1] = \frac{1}{s}$

2. $L[e^{at}] = \frac{1}{s-a}$

PROOF: By definition: $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$L[e^{at}] = \int_0^\infty e^{-st} e^{at}\, dt$$

$$= \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^\infty$$

$$= 0 - \left(\frac{-1}{(s-a)} \right) = \frac{1}{s-a}$$

* $s - a > 0 \Rightarrow s > a$

3. $L[e^{-at}] = \frac{1}{s+a}$, $s > -a$

4. $L[\sin at] = \frac{a}{s^2 + a^2}$

5. $L[\cos at] = \frac{s}{s^2 + a^2}$

Date
Page

PROOF: wkt, $e^{iat} = \cos at + i \sin at$

$$L[e^{iat}] = L[\cos at] + iL[\sin at]$$

$$\frac{1}{s-ia} = L[\cos at] + iL[\sin at]$$

$$\frac{s+ia}{s^2+a^2} = L[\cos at] + iL[\sin at]$$

By equating the real and imaginary parts

$$L[\cos at] = \frac{s}{s^2+a^2}, \quad L[\sin at] = \frac{a}{s^2+a^2}$$

6. $L[\sinh at] = \dfrac{a}{s^2-a^2}$

PROOF: $L[\sinh at] = L\left[\dfrac{e^{at}-e^{-at}}{2}\right]$

$$= \frac{1}{2}\left\{L[e^{at}] - L[e^{-at}]\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-a} - \frac{1}{s+a}\right\}$$

$$= \frac{1}{\cancel{2}}\left[\frac{\cancel{2}a}{s^2-a^2}\right]$$

$$= \frac{a}{s^2-a^2}$$

7. $L[\cosh at] = \dfrac{s}{s^2-a^2}$

8. $L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$ where n is a non-ve real no. or negative fraction

PROOF : By definition, $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$L[t^n] = \int_0^\infty e^{-st}\, t^n \, dt$$

$$st = x$$

$$s\,dt = dx$$

$$dt = \frac{dx}{s}$$

$t = 0;\ x = 0$

$t = \infty;\ x = \infty$

$$= \int_0^\infty e^{-x} \left(\frac{x^n}{s^{n+1}}\right) dx$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x}\, x^n \, dx$$

$\because \Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx$

$$= \frac{\Gamma(n+1)}{s^{n+1}}$$

9. $L[t^n] = \frac{n!}{s^{n+1}}$ where n is non-ve integer

PROOF: $L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}$

Note: $L[t^n] = \frac{n!}{s^{n+1}}$ for $n = 0, 1, 2, 3 \ldots$ & $s > 0$

$L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$ for $n > -1$ & $s > 0$

Summary

1. $L[k] = \frac{k}{s}$; $L^{-1}\left[\frac{k}{s}\right] = 1$

2. $L[e^{at}] = \frac{1}{s-a}$; $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

3. $L[e^{-at}] = \frac{1}{s+a}$; $L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$

** 4. $L[\sin at] = \frac{a}{s^2+a^2}$; $L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{\sin at}{a}$

5. $L[\cos at] = \frac{s}{s^2+a^2}$; $L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$

** 6. $L[\sin h at] = \frac{a}{s^2-a^2}$; $L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{\sin h at}{a}$

7. $L[\cos h at] = \frac{s}{s^2-a^2}$; $L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cos h at$

8. $L[t^n] = \frac{n!}{s^{n+1}}$; $L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!}$; $n = 0, 1, 2 \ldots$

(or)

$L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$; $L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{\Gamma(n+1)}$; $n > -1$

q 1. $L[2] = \frac{2}{s}$

2. $L[3e^{-3t}] = \frac{3}{s+3}$

3. $L[5e^{4t}] = 5L[e^{4t}] = \frac{5}{s-4}$

4. $L[\sin 3t] = \frac{3}{s^2+9}$

5. $L[2\cos 5t] = \frac{2 \cdot s}{s^2+25}$

6. $L[7\sin h3t] = 7 \cdot \frac{3}{s^2-9} = \frac{21}{s^2-9}$

7. $L[2\cosh 3t] = 2 \cdot \frac{s}{s^2-9} = \frac{2s}{s^2-9}$

q. 1. Find $L^{-1}\left[\frac{s}{s^2+25}\right] = \cos 5t$

2. Find $L^{-1}\left[\frac{6}{s^2+4}\right] = \frac{6\sin 2t}{2}$

3 Find $L^{-1}\left[\frac{1}{2s-5}\right] = L^{-1}\left[\frac{1}{2(s-\frac{5}{2})}\right] = \frac{1}{2}e^{\frac{5t}{2}}$

4. Find $L^{-1}\left[\frac{1}{s^4}\right] = \frac{t^3}{3!}$

5. Find $L^{-1}\left[\frac{1}{s^{7/2}}\right] = \frac{t^{5/2}}{\Gamma(7/2)} = \frac{t^{5/2}}{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}} = \frac{8t^{5/2}}{15\sqrt{\pi}}$

q. Find $L[t^3 + 2 + 2e^{-4t} + 5\sin 6t]$

Sol. $L\{t^3+2+2e^{-4t}+5\sin 6t\} = L\{t^3\} + L\{2\}$
$+ 2L\{e^{-4t}\} + 5L\{\sin 6t\}$

$= \frac{3!}{s^4} + \frac{2}{s} + 2\cdot\frac{1}{s+4} + 5\cdot\frac{6}{s^2+36}$

$= \frac{6}{s^4} + \frac{2}{s} + \frac{2}{s+4} + \frac{30}{s^2+36}$

q. Find $L\left[\sin 2t \cos 2t\right]$

$= L\left[\frac{\sin 4t}{2}\right]$ (or) $\sin A \cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B))$

$= \frac{1}{2} \cdot \frac{4}{s^2+16}$

q. Find $L\left[\cos^2 at\right]$

$L\left[\cos^2 at\right] = L\left[\frac{1+\cos 2at}{2}\right]$

$= \frac{1}{2}\left\{L[1] + L[\cos 2at]\right\}$

$= \frac{1}{2}\left\{\frac{1}{s} + \frac{s}{s^2+(2a)^2}\right\}$

$= \frac{1}{2}\left\{\frac{1}{s} + \frac{s}{s^2+4a^2}\right\}$

q. $L\left[\cos(at+b)\right]$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

Sol. $L\left[\cos(at+b)\right] =$

$L\left[\cos at \cos b\right] - L\left[\sin at \sin b\right]$

$= \cos b \, L[\cos at] - \sin b \, L[\sin at]$

$= \cos b \cdot \frac{s}{s^2+a^2} - \sin b \cdot \frac{a}{s^2+a^2}$

q. $L\left[\sinh^2 2t + 5\cosh 2t\right]$

$= \frac{1}{4} L[e^{4t} + e^{-4t} - 2] + 5L[\cos 2t]$

$\sinh^2 2t = (\sinh 2t)^2$

$= \left(\frac{e^{2t} - e^{-2t}}{2}\right)^2$

$= \frac{1}{4}[e^{4t} + e^{-4t} - 2]$

$= \frac{1}{4}\left[\frac{1}{s-4} + \frac{1}{s+4} - \frac{2}{s}\right] + 5 \cdot \frac{s}{s^2-4}$

→ Inverse Laplace

q. $L^{-1}\left[\frac{3s-8}{s^2+4} - \frac{4s-24}{s^2-16}\right]$

$= L^{-1}\left\{\frac{3s-8}{s^2+4} - \frac{4s-24}{s^2-16}\right\} = 3L^{-1}\left\{\frac{s}{s^2+4}\right\} - 8L^{-1}\left\{\frac{1}{s^2+4}\right\}$

$- 4L^{-1}\left\{\frac{s}{s^2-16}\right\} + 24L^{-1}\left\{\frac{1}{s^2-16}\right\}$

$= 3\cos 2t - 8\frac{\sin 2t}{2} - 4\cosh 4t + 24\frac{\sinh 4t}{4}$

$= 3\cos 2t - 4\sin 2t - 4\cosh 4t + 6\sinh 4t$

q. $L^{-1}\left[\frac{3s-8}{4s^2+5}\right]$

$= 3L^{-1}\left[\frac{s}{4s^2+5}\right] - 8L^{-1}\left[\frac{1}{4s^2+5}\right]$

$= 3L^{-1}\left[\frac{s}{4(s^2+\frac{5}{4})}\right] - 8L^{-1}\left[\frac{1}{4(s^2+\frac{5}{4})}\right]$

$= \frac{3}{4}L^{-1}\left[\frac{s}{s^2+\left(\frac{\sqrt{5}}{2}\right)^2}\right] - \frac{8}{4}L^{-1}\left[\frac{1}{s^2+\left(\frac{\sqrt{5}}{2}\right)^2}\right]$

$= \frac{3}{4}\cos\frac{\sqrt{5}}{2}t - 2L^{-1}\frac{\sin\frac{\sqrt{5}}{2}t}{\sqrt{5}/2} = \frac{3}{4}\cos\frac{\sqrt{5}}{2}t - 4\frac{\sin\frac{\sqrt{5}}{2}t}{\sqrt{5}}$

24th April, 2024

q. $L^{-1}\left[\frac{5s-4}{s^2+8}\right]$

$= 5L^{-1}\left[\frac{s}{s^2+8}\right] - 4L^{-1}\left[\frac{1}{s^2+8}\right]$

$= 5L^{-1}\left[\frac{s}{s^2+(2\sqrt{2})^2}\right] - 4L^{-1}\left[\frac{1}{s^2+(2\sqrt{2})^2}\right]$

$= 5\cos 2\sqrt{2}t - 4\frac{\sin 2\sqrt{2}t}{2\sqrt{2}}$

$= 5\cos 2\sqrt{2}t - \sqrt{2}\sin 2\sqrt{2}t$

q. $L^{-1}\left[\frac{3(s^2-2)^2}{2s^5}\right]$

$= \frac{3}{2}L^{-1}\left\{\frac{s^4-4s^2+4}{s^5}\right\}$

$= \frac{3}{2}L^{-1}\left\{\frac{1}{s}-\frac{4}{s^3}+\frac{4}{s^5}\right\}$

$= \frac{3}{2}\left[1-4\cdot\frac{t^2}{2!}+4\cdot\frac{t^4}{4!}\right]$

$= \frac{3}{2}\left[1-2t^2+\frac{t^4}{6}\right]$

H.W q. Find $L[f(t)]$ if $f(t) = \left(\sqrt{t}+\frac{1}{\sqrt{t}}\right)^3$

H.W q. Find $L[f(t)]$ if $f(t) = \sin\sqrt{t}$

Existence of Laplace Transforms

1. Piecewise Continuous Function :

A func. f is a piecewise continuous on an interval $[a, b]$ if this interval can be partitioned by a finite no. of points.

$$a = t_0 < t_1 < \ldots\ldots t_n = b \text{ such that}$$

i) f is continuous in every interval (t_k, t_{k+1})
ii) f has finite limits at endpoints of each sub intervals.

q. consider the foll piecewise-defined func.

$$f(t) = \begin{cases} t^2 & 0 \le t \le 1 \\ 3-t & 1 \le t \le 2 \\ t+1 & 2 \le t \le 3 \end{cases}$$

$[0, 3]$ is divided

Theorem: Sufficient conditions for the existence of LT

1. f is piecewise continuous on $[0, b]$ for all $b > 0$.

2. $f(t)$ is of exponential order of γ

i.e $|f(t)| \le k e^{\gamma t}$, where $k > 0$ and $\gamma > 0$ are constants. Then LT of f exists for $s > \gamma$.

Note: The previous theorem gives a sufficient condition for the existence of LT. It is not a necessary condition.

Properties of Laplace Transforms

Property 1: If $L[f(t)] = F(s)$ then $L[e^{at} f(t)] = F(s-a)$

If $L^{-1}[F(s)] = f(t)$ then, $L^{-1}\{F(s-a)\} = e^{at} f(t)$
$= e^{at} L^{-1}[F(s)]$

(Inverse by completing the sq.)

Note: If $L[f(t)] = F(s)$ then $L[e^{-at} f(t)] = F(s+a)$

If $L^{-1}[F(s)] = f(t)$ then, $L^{-1}[F(s+a)] = e^{-at} f(t) = e^{-at} L^{-1}[F(s)]$

$$* \; L[e^{at} f(t)] = L[f(t)]_{s \to s-a}$$
$$= [F(s)]_{s \to s-a}$$
$$= F(s-a)$$

Q: Find $L[e^{-2t} \cosh 4t]$

Sol. $L[e^{-2t} \cosh 4t] = \{L[\cosh 4t]\}_{s \to s+2}$

$$= \left[\frac{s}{s^2-16}\right]_{s \to s+2}$$

$$= \frac{s+2}{(s+2)^2-16} = \frac{s+2}{s^2+4s-12}$$

q: Find $L\{e^{-3t}\sin 4t\}$

Sol. $L\{e^{-3t}\sin 4t\} = \left[\frac{4}{s^2+16}\right]_{s\to s+3}$

$$= \frac{4}{(s+3)^2+16} = \frac{4}{s^2+6s+25}$$

q. $L\{e^{at}\cos bt\}$

$$= \left\{\frac{s}{s^2+b^2}\right\}_{s\to s-a} = \frac{s-a}{(s-a)^2+b^2}$$

q. $L\{e^{at}\sin bt\}$

$$= \left\{\frac{b}{s^2+b^2}\right\}_{s\to s-a} = \frac{b}{(s-a)^2+b^2}$$

q. Find $L\{\cosh at \cos at\}$

$$= L\left\{\frac{e^{at}+e^{-at}}{2}\cdot \cos at\right\}$$

$$= \frac{1}{2}\left[L[e^{at}\cos at] + L[e^{-at}\cos at]\right]$$

$$= \frac{1}{2}\left[\left\{\frac{s}{s^2+a^2}\right\}_{s\to s-a} + \left\{\frac{s}{s^2+a^2}\right\}_{s\to s+a}\right]$$

$$= \frac{1}{2}\left[\frac{s-a}{(s-a)^2+a^2} + \frac{s+a}{(s+a)^2+a^2}\right]$$

q: $L^{-1}\left[\frac{1}{(s-1)^2}\right]$

$= e^t L^{-1}\left\{\frac{1}{s^2}\right\}$

$= e^t \cdot \frac{t^1}{1!}$

$L[e^t t] = \left(\frac{1}{s^2}\right)_{s \to s-1}$

$= \frac{1}{(s-1)^2}$

25th April, 2023

RESULT: Let $f(t)$ satisfy the condition of the existence theorem and $L[f(t)] = F(s)$ then,

i) $\lim_{x \to \infty} F(s) = 0$

ii) $\lim_{s \to \infty} sF(s)$ is bounded

These two results indicate that not all functions of s are Laplace transform of some function $f(t)$

q. $L[e^{3t}(3\sin 4t - 4\cos 4t)]$

$= 3L[e^{2t}\sin 4t] - 4L[e^{2t}\cos 4t]$

$= 3\left\{\frac{4}{s^2+16}\right\}_{s \to s-2} - 4\left\{\frac{s}{s^2+16}\right\}_{s \to s-2}$

$= \frac{12}{(s-2)^2+16} - \frac{4(s-2)}{(s-2)^2+16}$

q. $L^{-1}\left[\dfrac{s+2}{(s+1)^4}\right]$

$L^{-1}\left[\dfrac{s+1+1}{(s+1)^4}\right]$

$L^{-1}\left[\dfrac{s+1}{(s+1)^4}\right] + L^{-1}\left[\dfrac{1}{(s+1)^4}\right]$

$L^{-1}\left[\dfrac{1}{(s+1)^3}\right] + L^{-1}\left[\dfrac{1}{(s+1)^4}\right]$

$e^{-t} L^{-1}\left[\dfrac{1}{s^3}\right] + e^{-t} L^{-1}\left[\dfrac{1}{s^4}\right]$

$e^{-t}\dfrac{t^2}{2!} + e^{-t}\dfrac{t^3}{3!}$

q. $L^{-1}\left[\dfrac{1}{s^2-4s+8}\right]$

$2(2)(s)$

$\dfrac{s^2-2\cdot 2s+4+4}{(s-2)^2+2^2}$

$= L^{-1}\left[\dfrac{1}{(s-2)^2+2^2}\right]$

$= e^{2t} \cdot L^{-1}\left[\dfrac{1}{s^2+2^2}\right] = e^{2t} \cdot \dfrac{\sin 2t}{2}$

q. $L^{-1}\left[\dfrac{4}{s^2-s+2}\right]$

$s^2-s+2 = s^2 - 2\cdot\dfrac{1}{2}\cdot s + \dfrac{1}{4} - \dfrac{1}{4} + 2$

$= \left(s-\dfrac{1}{2}\right)^2 + \dfrac{7}{4}$

Date
Page

$$L^{-1}\left[\frac{4}{s^2-s+2}\right] = 4L^{-1}\left[\frac{1}{\left(s-\frac{1}{2}\right)^2+\frac{7}{4}}\right]$$

$$= 4e^{t/2} L^{-1}\left\{\frac{1}{s^2+\left(\frac{\sqrt{7}}{2}\right)^2}\right\}$$

$$= 4e^{t/2} \frac{\sin\left(\sqrt{7}/2\right)t}{\left(\sqrt{7}/2\right)}$$

$$= \frac{8e^{t/2}\sin\sqrt{7}/2\,t}{\sqrt{7}}$$

q. $L^{-1}\left[\frac{s+6}{s^2+6s+13}\right]$

$$-9+13$$

$$s^2+6s+13 = s^2+2\cdot(3)\cdot s+3^2 \underline{-3^2+13}$$

$$(s+3)^2+4$$

$$L^{-1}\left[\frac{s+3+3}{(s+3)^2+4}\right]$$

$$L^{-1}\left[\frac{s+3}{(s+3)^2+2^2}\right] + 3L^{-1}\left[\frac{1}{(s+3)^2+2^2}\right]$$

$$e^{-3t}\cos 2t + \frac{3e^{-3t}\sin 2t}{2}$$

→ Finding Inverse Laplace Transforms by method of Partial Fractions.

Factors in denominator	Corresponding P.F
1. non-repeated linear factors	$F(s) =$
2. repeated linear factor	
3. non-repeated quadratic factor	$F(s) = \frac{As+B}{(\quad)} + \frac{Cs+D}{(\quad)}$
4. repeated quadratic factor	$F(s) = \frac{As+B}{(\quad)} + \frac{Cs+D}{(\quad)^2}$

Q: Find inverse laplace transforms

1. $$\frac{5s^2+3s-16}{(s-1)(s-2)(s+3)}$$

'non repeated linear

$$= \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+3}$$

$$5s^2+3s-16 = A(s-2)(s+3) + B(s-1)(s+3) + C(s-1)(s+3)$$

put $s=1 \Rightarrow -8 = -4A$

$A = \cancel{4} \; 2$

put $s=2 \Rightarrow 10 = 5B$

$B = 2$

put $s=-3 \Rightarrow 20 = 20C$

$C = 1$

Date
Page

$$L^{-1}\left[\frac{5s^2+3s-16}{(s-1)(s-2)(s+3)}\right] = L^{-1}\left[\frac{2}{s-1}\right] + L^{-1}\left[\frac{2}{s-2}\right] + L^{-1}\left[\frac{1}{s+3}\right]$$

$$= 2e^{t} + 2e^{2t} + e^{-3t}$$

3. $\dfrac{s}{(s+1)^2(s^2+1)}$

$$\frac{s}{(s+1)^2(s^2+1)} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{Cs+D}{(s^2+1)}$$

$$s = A(s+1)(s^2+1) + B(s^2+1) + (Cs+D)(s+1)^2$$

put $s=-1 \Rightarrow -1 = 2B$

$$B = -1/2$$

put $s=0 \Rightarrow 0 = A+B+D$ ①

$$A+D = \frac{1}{2}$$

equating co-efficient of $s^3 \Rightarrow 0 = A+C$

of $s^2 \Rightarrow 0 = A + B + 2C + D$

$$0 = \frac{1}{2} - \frac{1}{2} + 2c$$

$$C = 0$$

$$\therefore A = 0$$

$$\therefore D = 1/2$$

$$L^{-1}\left\{\frac{s}{(s+1)^2(s^2+1)}\right\} = L^{-1}\left[0 - \frac{1}{2(s+1)^2} + \frac{1}{2(s^2+1)}\right]$$

$$= -L^{-1}\left[\frac{1}{2}\cdot\frac{1}{(s+1)^2}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= -\frac{1}{2}e^{-t}L^{-1}\left[\frac{1}{s^2}\right] + \frac{1}{2}\frac{\sin t}{1}$$

$$= \frac{-te^{-t}}{2} + \frac{1}{2}\sin t$$

26th April, 2023

PROPERTY-2 Differentiation of Transforms (multiplication by t)

Theorem If $L[f(t)] = F(s)$ then

$$L[t\, f(t)] = -\frac{d}{ds}\left(L[f(t)]\right) = -\frac{d}{ds}[F(s)]$$

Note: 1. $L[t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2}[F(s)]$

2. $L[t^3 f(t)] = (-1)^3 \frac{d^3}{ds^3}[F(s)]$

in general $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}(F(s))$

→ Inverse L.T of derivatives

If $L^{-1}[F(s)] = f(t)$ then for $n = 1, 2, 3 \ldots\ldots$

$$L^{-1}[F^{(n)}(s)] = (-1)^n t^n f(t)$$

q. Find $L\{t(\underbrace{3\sin 2t - 2\cos 2t}_{f(t)})\}$

Sol. wkt $L[t\, f(t)] = -\frac{d}{ds} F(s)$

$$L[3\sin 2t - 2\cos 2t] = \frac{6}{s^2+4} - \frac{2s}{s^2+4}$$

$$\therefore L[t(3\sin 2t - 2\cos 2t)] = -\frac{d}{ds}\left[\frac{6}{s^2+4} - \frac{2s}{s^2+4}\right]$$

$$= -\left[6\frac{d}{ds}\left(\frac{1}{s^2+4}\right) - 2\frac{d}{ds}\left(\frac{s}{s^2+4}\right)\right]$$

$$= -\left[6\frac{[(s^2+4)(0) - 1(2s)]}{(s^2+4)^2} - 2\,\frac{[(s^2+4)(1) - s(2s)]}{(s^2+4)^2}\right]$$

$$= -\left[6\left(\frac{-2s}{(s^2+4)^2}\right) - 2\left(\frac{-s^2+4}{(s^2+4)^2}\right)\right]$$

$$= \frac{12s}{(s^2+4)^2} + \frac{8-2s^2}{(s^2+4)^2}$$

$$= \frac{-2s^2+12s+8}{(s^2+4)^2}$$

Q. Find $L[t^2 \sin at]$

$$L[t^2 \sin at] = (-1)^2 \frac{d^2}{ds^2} L[\sin at]$$

$$= \frac{d^2}{ds^2}\left[\frac{a}{s^2+a^2}\right]$$

$$= a\frac{d}{ds}\left[\frac{d}{ds}\left(\frac{1}{s^2+a^2}\right)\right]$$

$$= a\frac{d}{ds}\left[\frac{-2s}{(s^2+a^2)^2}\right]$$

$$= -2a\left[\frac{(s^2+a^2)^2(1) - s\,2(s^2+a^2)(2s)}{(s^2+a^2)^4}\right]$$

$$= -2a\left[\frac{(s^2+a^2)^2 - 4s^2(s^2+a^2)}{(s^2+a^2)^4}\right]$$

$$= -2a\left[\frac{(s^2+a^2)[a^2-3s^2]}{(s^2+a^2)^4}\right] = -2a\left[\frac{a^3-3s^2}{(s^2+a^2)^3}\right]$$

q. Find $L\{t^2 e^t \sin 4t\}$

$$L\{t^2 e^t \sin 4t\} = (-1)^2 \frac{d^2}{ds^2} L[e^t \sin 4t] = \frac{d^2}{ds^2}\left[\frac{4}{(s-1)^2+16}\right]$$

(OR) easier $L\{e^t \; t^2 \sin 4t\} = \{L[t^2 \sin 4t]\}_{s \to s-1}$

$$= \left[\frac{2\times 4 \; (3s^2-16)}{(s^2+16)^3}\right]_{s\to s-1} \quad \text{from previous q}$$

$$= \frac{8\left(3(s-1)^2-16\right)}{\left((s-1)^2+16\right)^3}$$

q. Find $L\{t^3 \cos t\}$

$$= (-1)^3 \frac{d^3}{ds^3} L[\cos t]$$

$$= \frac{-d^3}{ds^3}\left(\frac{s}{s^2+1}\right)$$

→ Inverse L·T derivatives (cont.)

$$L^{-1}\left\{\frac{d^n F(s)}{ds^n}\right\} = (-1)^n t^n f(t)$$

Note : 1. $L^{-1}\left\{\frac{d}{ds} F(s)\right\} = -t\, f(t)$ is $L^{-1}[F(s)]$

2. $L^{-1}\left\{\frac{d^2 F(s)}{ds^2}\right\} = +t^2 f(t)$

3. $L^{-1}\left\{\frac{d^3}{ds^3} F(s)\right\} = -t^3 f(t)$

q. Find $L^{-1}\left[\frac{s}{(s^2+1)^2}\right]$

$$\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = -\frac{2s}{(s^2+1)^2}$$

$$\frac{s}{(s^2+1)^2} = \frac{-1}{2}\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$$

$$L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{-1}{2}L^{-1}\left[\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right]$$

$$= \frac{-1}{2}\left(-t\, L^{-1}\left(\frac{1}{s^2+1}\right)\right)$$

$$= \frac{t}{2}\sin t$$

(OR)

use P.F $\frac{s}{(s^2+1)^2} = \frac{As+B}{s^2+1} + \frac{Cs+D}{(s^2+1)^2}$

q. Find $L^{-1}\left[\frac{2(s+1)}{(s^2+2s+2)^2}\right]$

$$2L^{-1}\left\{\frac{s+1}{((s+1)^2+1)^2}\right\} = 2e^{-t}\underbrace{L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}}_{\text{prev. ques}}$$

$$= 2e^{-t}\cdot\frac{t}{2}\sin t$$

$$= te^{-t}\sin t$$

q. Find $L\{t^5 e^{4t}\cosh 4t\}$

$$L\{t^5e^{4t}\cosh 4t\} = L\left\{t^5e^{4t}\left(\frac{e^{4t}+e^{-4t}}{2}\right)\right\}$$

$$= \frac{1}{2} L\left[t^5 (e^{8t} + 1)\right]$$

$$= \frac{1}{2}\left[L(e^{8t} t^5) + L(t^5)\right]$$

$$= \frac{1}{2}\left[\left(\frac{5!}{s^6}\right)_{s \to s-8} + \frac{5!}{s^6}\right]$$

$$= \frac{1}{2}\left[\frac{5!}{(s-8)^6} + \frac{5!}{s^6}\right]$$

q. Find $L^{-1}\left[\frac{s}{(s^2-9)^2}\right]$

$$\frac{d}{ds}\left(\frac{1}{s^2-9}\right) = -\frac{2s}{(s^2-9)^2}$$

$$\frac{s}{(s^2-9)^2} = -\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s^2-9}\right)$$

$$L^{-1}\left[\frac{s}{(s^2-9)^2}\right] = -\frac{1}{2} L^{-1}\left[\frac{d}{ds}\left(\frac{1}{s^2-9}\right)\right]$$

$$= -\frac{1}{2}\left[-t \, L^{-1}\left\{\frac{1}{s^2-9}\right\}\right]$$

$$= \frac{t}{2} \, \frac{\sinh 3t}{3}$$

PROPERTY-3 Integration of L·T (Division by t)

Theorem If $L[f(t)] = F(s)$ then $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\, ds$

Date ______
Page ______

Note: 1. $L\left\{\frac{f(t)}{t^2}\right\} = \int_s^{\infty}\int_s^{\infty} F(s)\, ds\, ds$

2. $L\left\{\frac{f(t)}{t^3}\right\} = \int_s^{\infty}\int_s^{\infty}\int_s^{\infty} F(s)\, ds\, ds\, ds$

q. Find $L\left\{\frac{\sin t}{t}\right\}$ and hence show that $\int_0^{\infty} \frac{\sin t}{t} = \frac{\pi}{2}$

Sol. $L\left\{\frac{\sin t}{t}\right\} = \int_s^{\infty} \frac{1}{s^2+1}\, ds$

$= \left[\tan^{-1}(s)\right]_s^{\infty}$

$= \frac{\pi}{2} - \tan^{-1}(s)$

$= \cot^{-1} s$

Deduction, By definition of L.T

$\int_0^{\infty} e^{-st} f(t)\, dt = L[f(t)]$

$\int_0^{\infty} e^{-st} \frac{\sin t}{t}\, dt = L\left[\frac{\sin t}{t}\right]$

$\int_0^{\infty} e^{-st} \frac{\sin t}{t} = \frac{\pi}{2} - \tan^{-1} s$

put $s=0$

$\int_0^{\infty} \frac{\sin t}{t} = \frac{\pi}{2}$

q. Find $L\left\{\frac{1-e^{-t}}{t}\right\}$

Sol. $L\left\{\frac{1-e^{-t}}{t}\right\} = \int_s^{\infty} L[1-e^{-t}]\, ds$

$= \int_s^{\infty}\left(\frac{1}{s} - \frac{1}{s+1}\right) ds = \left[\log s - \log(s+1)\right]_s^{\infty}$

$$= \left[\log \frac{s}{s+1}\right]_s^\infty$$

$$= \left[\log \frac{s}{s(1+\frac{1}{s})}\right]_0^\infty$$

$$= \left[\log \frac{1}{1+1/s}\right]_0^\infty$$

$$= \log 1 - \log \frac{s}{s+1}$$

$$= \log \frac{s+1}{s}$$

q. Find $L\left\{\frac{\sin 3t \cos t}{t}\right\}$

$$\int_s^\infty L[\sin 3t \cos t]\, ds$$

$$\int_s^\infty \frac{L}{2}[\sin 4t + \sin 2t]\, ds$$

$$\frac{1}{2}\int_s^\infty \frac{4}{s^2+16} + \frac{2}{s^2+4}\, ds$$

$$\int_s^\infty \frac{2}{s^2+16} + \frac{1}{s^2+4}\, ds$$

$$\left[\frac{2}{4}\tan^{-1}\left(\frac{s}{4}\right) + \frac{1}{2}\tan^{-1}\left(\frac{s}{2}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\frac{\pi}{2} + \frac{\pi}{2} - \left(\tan^{-1}\left(\frac{s}{4}\right) + \tan^{-1}\left(\frac{s}{2}\right)\right]\right.$$

$$= \frac{1}{2}\left[\cot^{-1}\left(\frac{s}{4}\right) + \cot^{-1}\left(\frac{s}{2}\right)\right]$$

H.W q. Find $L\left\{\frac{e^{3t}}{t}\right\}$

→ Inverse L.T of integrals

If $L^{-1}\{F(s)\} = f(t)$ then

$$L^{-1}\left\{\int_s^\infty F(s)\,ds\right\} = \frac{f(t)}{t} = \frac{L^{-1}\{F(s)\}}{t}$$

q. $L^{-1}\left[\int_s^\infty \frac{a}{s^2+a^2}\,ds\right]$

Sol. wkt $L^{-1}\left\{\int_s^\infty F(s)\,ds\right\} = \frac{L^{-1}[F(s)]}{t} = \frac{f(t)}{t}$

$L^{-1}\left[\frac{a}{s^2+a^2}\right] = \cancel{a}\cdot\frac{\sin at}{\cancel{a}} = f(t)$

$\therefore L^{-1}\left[\int_s^\infty \frac{a}{s^2+a^2}\,ds\right] = \frac{\sin at}{t}$

q. $L^{-1}\left[\int_s^\infty \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\,ds\right]$

Sol $L^{-1}\left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right] = \cos at - \cos bt$

$\therefore L^{-1}\left[\int_s^\infty \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right] = \frac{\cos at - \cos bt}{t}$

q. $L\left\{\left(\frac{\sin 2t}{\sqrt{t}}\right)^2 + t\sin 2t\right\}$

Sol. $L\left\{\frac{\sin^2 2t}{t}\right\} + L\{t\sin 2t\}$

$$L[\sin^2 2t] = L\left[\frac{1-\cos 4t}{2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+16}\right]$$

$$\therefore L\left\{\frac{\sin^2 2t}{t}\right\} + L\{t \sin 2t\} = \frac{1}{2}\int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+16}\right) ds + \left(\frac{-d}{ds}\left(\frac{2}{s^2+4}\right)\right)$$

PROPERTY-4 L·T of a derivative

Theorem: If $L\{f(t)\} = F(s)$ then $L\{f'(t)\} = sF(s) - f(0)$

Note: 1. $L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$

2. $L\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$

q. If $L[\sin at] = \frac{a}{s^2+a^2}$, find $L[\sin^2 at]$ using L·T of derivatives.

Sol. $f(t) = \sin^2 at$; $f(0) = 0$

$f'(t) = 2\sin at \cos at \cdot a$

$f'(t) = a \cdot \sin 2at$

$L[f'(t)] = a\,L[\sin 2at]$

$sL[F(s)] - f(0) = a \cdot \frac{2a}{s^2+4a^2}$

$sL[\sin^2 at] - 0 = \frac{2a^2}{s^2+4a^2}$

Date
Page

$$\therefore L[\sin^2 at] = \frac{2a^2}{s(s^2+4a^2)}$$

(OR) $L[\sin^2 at] = L\left[\frac{1-\cos 2at}{2}\right]$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4a^2}\right]$$

$$= \frac{1}{2}\left[\frac{s^2+4a^2-s^2}{s(s^2+4a^2)}\right] = \frac{2a^2}{s(s^2+4a^2)}$$

27th April, 2023

q. Find L.T of $f(t) = t^3$ using differentiation formula $L[f'''(t)]$

Sol. $f(t) = t^3$; $f(0) = 0$

$f'(t) = 3t^2$; $f'(0) = 0$

$f''(t) = 6t$; $f''(0) = 0$

$f'''(t) = 6$

wkt $L[f'''(t)] = s^3 L[f(t)] - s^2 f(0) - s^2 f'(0) - f''(0)$

$$L[6] = s^3 L[t^3] - 0 - 0 - 0$$

$$\frac{6}{s} = s^3 L[t^3]$$

$$L[t^3] = \frac{6}{s^4}$$

$L[t^3] = \frac{3!}{s^4}$ (by formula)

q. Given $L[\sin\sqrt{t}] = \frac{\sqrt{\pi}}{2s^{3/2}} e^{-1/4s}$

Prove that : $L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \left(\frac{\pi}{s}\right)^{1/2} e^{-1/4s}$

Sol. $f(t) = \sin\sqrt{t}$; $f(0) = 0$

$f'(t) = \cos\sqrt{t} \cdot \frac{1}{2\sqrt{t}}$

$L[f'(t)] = sF(s) - f(0)$

$L\left[\frac{\cos\sqrt{t}}{2\sqrt{t}}\right] = sL[\sin\sqrt{t}] - 0$

$\frac{1}{2} L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = \frac{s\sqrt{\pi}}{2s^{3/2}} e^{-1/4s}$

$= \frac{\sqrt{\pi}}{s^{1/2}} e^{-1/4s}$

$= \sqrt{\frac{\pi}{s}} e^{-1/4s}$

→ Inverse L.T of a derivative

If $L^{-1}[F(s)] = f(t)$ and $f(0) = 0$ then $L^{-1}\{sF(s)\} = \frac{df(t)}{dt}$

q: $L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right]$ → this can be solved by Partial fraction / convolution theorem

Sol. $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{t\sin at}{2a}$ (from PROPERTY-2 done in class q)

↓

$L^{-1}[F(s)] = f(t)$ where $f(t) = \frac{t\sin at}{2a}$ $\therefore f(0) = 0$

wkt $L^{-1}[sF(s)] = f'(t)$ and $f(0) = 0$

Date ______
Page ______

$$L^{-1}\left[s\cdot\frac{s}{(s^2+a^2)^2}\right] = \frac{1}{2a}\left[t\cos at\cdot a + \sin at\cdot 1\right]$$

$$= \frac{at\cos at + \sin at}{2a}$$

PROPERTY-5 L·T of integral of a function

Theorem If $L[f(t)] = F(s)$ then $L\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$, $s>0$

Note: 1. If $L[f(t)] = F(s)$ then

$$L\left\{\int_0^t\int_0^t f(t)\,dt\,dt\right\} = \frac{F(s)}{s^2}$$

2. $$L\left\{\int_0^t \cdots\cdots n \text{ times} \cdots \int_0^t f(t)\,dt\right\} = \frac{F(s)}{s^n}$$

q. P.T $L\left\{\int_0^t \frac{\sin t}{t}\,dt\right\} = \frac{1}{s}\tan^{-1}\left(\frac{1}{s}\right)$

$$L\left[\frac{\sin t}{t}\right] = \int_s^\infty \frac{1}{s^2+1}\,ds$$

$$= \left(\tan^{-1}(s)\right)_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}s = \cot^{-1}s$$

$$\therefore L\left[\int_0^t \frac{\sin t}{t}\,dt\right] = \frac{\cot^{-1}s}{s}$$

q. Find L·T of $\int_0^t\int_0^t\int_0^t \cos at\,dt\,dt\,dt$

$L[\cos at] = \frac{s}{s^2+a^2}$ $\quad\therefore \int_0^t\int_0^t\int_0^t \cos at\,dt\,dt\,dt = \frac{s}{s^2+a^2}\cdot\frac{1}{s^3}$

q. Find $L\left[e^{-4t}\int_0^t \frac{\sin 3t}{t}\,dt\right]$

Sol. $L\left[\int_0^t \frac{\sin 3t}{t}\,dt\right] = \frac{L\left[\frac{\sin 3t}{t}\right]}{s}$ — ①

$$L\left[\frac{\sin 3t}{t}\right] = \int_s^{\infty} \frac{3}{s^2+9}\,ds$$

$$= \cancel{3}\cdot\frac{1}{\cancel{3}}\left[\tan^{-1}\frac{s}{3}\right]_s^{\infty}$$

$$= \frac{\pi}{2} - \tan^{-1}\frac{s}{3} = \cot^{-1}\frac{s}{3}$$

① $\Rightarrow$ $L\left[\int_0^t \frac{\sin 3t}{t}\,dt\right] = \frac{\cot^{-1}(s/3)}{s}$

$$L\left[e^{-4t}\int_0^t \frac{\sin 3t}{t}\,dt\right] = \frac{\cot^{-1}\left(\frac{s+4}{3}\right)}{s+4}$$

→ Inverse L.T of integral of a function

If $L^{-1}\{F(s)\} = f(t)$ then $L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(t)\,dt$

$$L^{-1}\left\{\frac{F(s)}{s^n}\right\} = \int_0^t\int_0^t \cdots\cdots \int_0^t f(t)\,dt^n$$

q. $L^{-1}\left[\frac{1}{s(s^2+9)}\right]$

$$= L^{-1}\left[\frac{\frac{1}{s^2+9}}{s}\right] = \int_0^t \frac{\sin 3t}{3}\,dt$$

$F(s)$

Date
Page

$$= \frac{1}{3}\left[-\frac{\cos 3t}{3}\right]_0^t$$

$$= \frac{1}{9}\left[-\cos 3t + 1\right]$$

$$= \frac{1 - \cos 3t}{9}$$

Q: $L^{-1}\left[\frac{1}{s^2(s^2+9)}\right]$

$$= \int_0^t \int_0^t \frac{\sin 3t}{3}\, dt$$

$$= \frac{1}{9}\int_0^t 1 - \cos 3t \; dt$$

$$= \frac{1}{9}\left[t - \frac{\sin 3t}{3}\right]_0^t$$

$$= \frac{1}{9}\left[t - \frac{\sin 3t}{3}\right]$$

28th April, 2023

Applications of L.T

Remember:

1. $L\{y'(t)\} = sL\{y(t)\} - y(0)$

2. $L\{y''(t)\} = s^2 L\{y(t)\} - s\,y(0) - y'(0)$

3. $L\{y'''(t)\} = s^3 L\{y(t)\} - s^2 y(0) - s y'(0) - y''(0)$

WORKING RULE

1. Express the D.E using notation : $y(t), y'(t), y''(t)$

2. Take Laplace Transform on both sides of D.E

3. Substitute for $L\{y'(t)\}$, $L\{y''(t)\}$, $L\{y'''(t)\}$ ....

4. Substitute the given initial conditions to obtain $L\{y(t)\}$ as func. of s. This reduces the DE to an algebraic eqn.

5. Solve the algebraic eqns. to get y in terms of s

6. Find inverse Laplace Transform to obtain $y(t)$

Q. Solve the initial value problem:

$$y'' + 4y = 0 \text{ given that } y(0) = 1 \text{ \& } y'(0) = 6$$

Sol. $y''(t) + 4y(t) = 0$

$$L[y''(t)] + 4L[y(t)] = L[0]$$

$$s^2 L[y(t)] - s\,y(0) - y'(0) + 4L[y(t)] = 0$$

$$s^2 L[y(t)] - s - 6 + 4L[y(t)] = 0$$

$$L[y(t)](s^2+4) = s+6$$

$$L[y(t)] = \frac{s+6}{s^2+4}$$

$$y(t) = L^{-1}\left[\frac{s+6}{s^2+4}\right]$$

Date
Page

$$y(t) = L^{-1}\left[\frac{s}{s^2+4}\right] + 6L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$= \cos 2t + 6\frac{\sin 2t}{2}$$

$$y(t) = \cos 2t + 3\sin 2t$$

q: Solve the initial value problem $y'' + 4y' + 4y = 12t^2e^{-2t}$ given that $y(0) = 2$, $y'(0) = 1$.

sol $y''(t) + 4y'(t) + 4y(t) = 12t^2e^{-2t}$

$$L[y''(t)] + 4L[y'(t)] + 4L[y(t)] = 12\,L[e^{-2t}t^2]$$

$$s^2L[y(t)] - sy(0) - y'(0) + 4[sL[y(t)] - y(0)] + 4L[y(t)]$$

$$= \frac{12 \times 2!}{(s+2)^3}$$

$$s^2L[y(t)] - 2s - 1 + 4sL[y(t)] - 8 + 4L[y(t)] = \frac{24}{(s+2)^3}$$

$$L[y(t)](s^2+4s+4) = \frac{24}{(s+2)^3} + 2s + 9$$

$$L[y(t)](s+2)^2 = \frac{24}{(s+2)^3} + 2s + 9$$

$$L[y(t)] = \frac{24}{(s+2)^5} + \frac{2s+9}{(s+2)^2}$$

$$y(t) = 24L^{-1}\left\{\frac{1}{(s+2)^5}\right\} + L^{-1}\left\{\frac{2s}{(s+2)^2}\right\} + 9L^{-1}\left\{\frac{1}{(s+2)^2}\right\}$$

$$= 24e^{-2t}L^{-1}\left\{\frac{1}{s^5}\right\} + 2L^{-1}\left\{\frac{(s+2)-2}{(s+2)^2}\right\} + 9e^{-2t}L^{-1}\left\{\frac{1}{s^2}\right\}$$

$$= 24e^{-2t}\frac{t^4}{4!} + 2L^{-1}\left\{\frac{1}{s+2}\right\} - 4L^{-1}\left\{\frac{1}{(s+2)^2}\right\} + 9e^{-2t}\frac{t}{1!}$$

$$= e^{-2t}t^4 + 2e^{-2t} - 4e^{-2t}t + 9e^{-2t}t$$

$$y(t) = e^{-2t} + 2e^{-2t} + 5e^{-2t}$$

q: Consider a series RLC circuit where $R = 20\,\Omega$, $L = 0.05H$ and $C = 10^{-4}F$ and is driven by an alternating emf given by $E = 100t$. Given current i and capacitor charge q are 0 at time $t = 0$. Find an expression for $i(t)$ in region $t > 0$.

$$\left[Ri + L\frac{di}{dt} + \frac{1}{c}\int i\,dt = E\right]$$

Sol $R\,i(t) + L\,i'(t) + \frac{1}{c}\int i(t)\,dt = E$

$$20\,i(t) + 0.05\,i'(t) + \frac{1}{10^{-4}}\int i(t)\,dt = 100t$$

$$400\,i(t) + i'(t) + 20 \times 10^4 \int i(t)\,dt = 2000 \qquad \times 20$$

$$400\,L[i(t)] + L[i'(t)] + 200000\,L\left[\int i(t)\,dt\right] = 2000\,L[t]$$

$$400\,L[i(t)] + s[i(t)] - i(0) + 200000\,\frac{L[i(t)]}{s} = 2000 \cdot \frac{1}{s^2}$$

$$L[i(t)]\left\{400 + s + \frac{200000}{s}\right\} = \frac{2000}{s^2}$$

$$L[i(t)]\left\{\frac{400s + s^2 + 200000}{\cancel{s}}\right\} = \frac{2000}{s^{\cancel{2}}}$$

$$L[i(t)] = \frac{2000}{s(s^2 + 400s + 200000)}$$

$$i(t) = 2000\,L^{-1}\left\{\frac{1}{s(s^2 + 400s + 200000)}\right\} \quad -①$$

$$\frac{1}{s(s^2 + 400s + 200000)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 400s + 200000}$$

$$1 = A(s^2 + 400s + 200000) + (Bs + C)s$$

equating co-efficients of s^2 : $0 = A + B$

Date
Page

equating co-efficients of s : $0 = 400A + C$

equating co-efficients of constant : $1 = 200000A$

$$\boxed{A = \frac{1}{200000}}$$

$$C = -400\left(\frac{1}{200000}\right)$$

$$\boxed{C = \frac{-1}{500}} , \boxed{B = \frac{-1}{200000}}$$

$$\therefore ① \rightarrow 2000\left[L^{-1}\left[\frac{1}{200000}\cdot\frac{1}{s}\right] - \frac{1}{200000}L^{-1}\left[\frac{s}{s^2+400s+200000}\right] - \frac{1}{500}L^{-1}\left[\frac{1}{s^2+400s+200000}\right]\right]$$

$$= \frac{1}{100}\times 1 - \frac{1}{100}L^{-1}\left\{\frac{s+200-200}{(s+200)^2+400^2}\right\} - 4L^{-1}\left\{\frac{1}{(s+200)^2+400^2}\right\}$$

$$= \frac{1}{100} - \frac{1}{100}\left[L^{-1}\left\{\frac{s+200}{(s+200)^2+400^2}\right\} - 200L^{-1}\left\{\frac{1}{(s+200)^2+400^2}\right\}\right]$$

$$- 4e^{-200t}L^{-1}\left\{\frac{1}{s^2+400^2}\right\}$$

$$= \frac{1}{100} - \frac{1}{100}\left[e^{-200t}L^{-1}\left\{\frac{s}{s^2+400^2}\right\} - 200e^{-200t}L^{-1}\left\{\frac{1}{s^2+400^2}\right\}\right] - 4e^{-200t}\frac{\sin 400t}{400}$$

$$= \frac{1}{100} - \frac{1}{100}\left[e^{-200t}\cos 400t - 200e^{-200t}\frac{\sin 400t}{400}\right] - 4e^{-200t}\frac{\sin 400t}{400}$$

$$= \frac{1}{100} - \frac{1}{100}e^{-200t}\cos 400t + \frac{1}{200}e^{-200t}\sin 400t - \frac{1}{100}e^{-200t}\sin 400t$$

$$= \frac{1}{100} - \frac{1}{100}e^{-200t}\cos 400t - \frac{e^{-200t}\sin 400t}{200}$$

2nd May, 2023

q. Solve $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t$ given $x(0) = 2$ & $x'(0) = -1$

Sol. $x''(t) - 2x'(t) + x(t) = e^t$

$$L[x''(t)] - 2L[x'(t)] + L[x(t)] = L[e^t]$$

$$s^2L[x(t)] - sx(0) - x'(0) - 2[sL[x(t)] - x(0)] + L[x(t)] = \frac{1}{s-1}$$

$$s^2L[x(t)] - 2s - (-1) - 2sL[x(t)] + 4 + L[x(t)] = \frac{1}{s-1}$$

$$L[x(t)](s^2 - 2s + 1) - 2s + 5 = \frac{1}{s-1}$$

$$L[x(t)](s^2 - 2s + 1) = \frac{1}{s-1} + 2s - 5$$

$$L[x(t)](s-1)^2 = \frac{1}{s-1} + 2s - 5$$

$$L[x(t)] = \frac{1}{(s-1)^3} + \frac{2s}{(s-1)^2} - \frac{5}{(s-1)^2}$$

$$x(t) = L^{-1}\left[\frac{1}{(s-1)^3}\right] + 2L^{-1}\left[\frac{s}{(s-1)^2}\right] - 5L^{-1}\left[\frac{1}{(s-1)^2}\right]$$

$$= e^t L^{-1}\left[\frac{1}{s^3}\right] + 2L^{-1}\left[\frac{s-1+1}{(s-1)^2}\right] - 5e^t L^{-1}\left[\frac{1}{s^2}\right]$$

$$= e^t \cdot \frac{t^2}{2!} + 2L^{-1}\left[\frac{\cancel{(s-1)}}{(s-1)^{\cancel{2}}}\right] + 2L^{-1}\left[\frac{1}{(s-1)^2}\right] - 5e^t\frac{t}{1!}$$

$$= e^t\frac{t^2}{2} + 2e^t + 2e^t \cdot t - 5e^t \cdot t$$

$$= \frac{e^t t^2}{2} + 2e^t - 3e^t t$$

2nd May, 2023

Date ____
Page ____

UNIT-3 Laplace Transforms & Inverse Laplace Transforms cont.

q: Evaluate $\int_0^{\infty} t^3 e^{-t} \sin t \, dt$

By def. of LT

$$\int_0^{\infty} e^{-3t} f(t)\, dt = L[f(t)]$$

$$\int_0^{\infty} e^{-3t} t^3 \sin t \, dt = L[t^3 \sin t]$$

$$= -\frac{d^3}{ds^3}\left(\frac{1}{s^2+1}\right)$$

$$= -\frac{d^2}{ds^3}\left(\frac{-1}{(s^2+1)^2} \cdot 2s\right)$$

$$= 2\frac{d}{ds}\left[\frac{d}{ds}\frac{s}{(s^2+1)^2}\right]$$

$$= 2\frac{d}{ds}\left[\frac{(s^2+1)^2 1 - s \cdot 2(s^2+1)(2s)}{(s^2+1)^4}\right]$$

$$= 2\frac{d}{ds}\left[\frac{(s^2+1)(s^2+1-4s^2)}{(s^2+1)^4}\right]$$

$$= 2\frac{d}{ds}\left[\frac{1-3s^2}{(s^2+1)^3}\right]$$

$$= 2\left[\frac{(s^2+1)^3(-6s) - (1-3s^2)3(s^2+1)^2 2s}{(s^2+1)^6}\right]$$

$$= 2\left[\frac{(s^2+1)^2[(-6s^2-6s) - 6s(1-3s^2)]}{(s^2+1)^6}\right]$$

$$= 2\left[\frac{-6s^3-6s-6s+18s^3}{(s^2+1)^4}\right]$$

$$\int_0^\infty e^{-st}\,t^3 \sin t \, dt = 2\left[\frac{12s^3-12s}{(s^2+1)^4}\right]$$

put $s=1$

$$\int_0^\infty e^{-t}\,t^3 \sin t\, dt = 0$$

H/w q. Evaluate $\int_0^\infty \frac{\cos 6t - \cos 4t}{t}\, dt$

Unit Step Function (Heavyside unit step Function)

Definition: The unit step func. denoted by $u(t-a)$ or $H(t-a)$ is defined by

$$H(t-a) = u(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases}$$

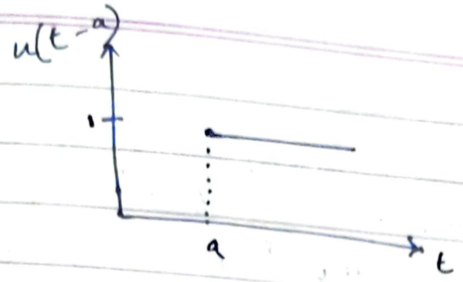
where a is a non-negative constant.

It is used to find the Laplace Transform of discontinuous function.

In particular where $a=0$

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

Date
Page



* L.T of unit step func : $L[u(t-a)] = \frac{e^{-as}}{s}$

Note: If $a = 0$ $\quad L[u(t)] = \frac{1}{s}$

→ Second Translation (or Second Shifting) Theorem

$$L[f(t-a)u(t-a)] = e^{-as} L[f(t)] = e^{-as} F(s)$$

— Points to remember

1. $L[u(t-a)] = \frac{e^{-as}}{s}$

2. $L[f(t-a)u(t-a)] = e^{-as} L[f(t)]$

3. $L[f(t)u(t-a)] = e^{-as} L[f(t+a)]$

q. Find $e^{t-1}u(t-1)$

Sol. $L[e^{t-1}u(t-1)] = e^{-1s} L[f(t)]$

here $f(t-1) = e^{t-1}$

$t \to t+1 \quad f(t) = e^{t}$

$L[f(t)] = \frac{1}{s-1}$

$\therefore$ ① $\to L[e^{t-1}u(t-1)] = e^{-s} \cdot \frac{1}{s-1}$

q. Find L.T of $4\sin(t-3)u(t-3)$

Sol. $L[4\sin(t-3)\,u(t-3)] = 4e^{-3s}L[f(t)]$

$= 4e^{-3s}\dfrac{1}{s^2+1}$

$f(t-3) = \sin(t-3)$

$f(t) = \sin t$

$L[f(t)] = \dfrac{1}{s^2+1}$

q. Evaluate $L[t^2 u(t-3)]$

Sol. $L[f(t)\,u(t-a)] = e^{-3s}L[f(t+3)]$

$= e^{-3s}\left[\dfrac{2}{s^3} + \dfrac{6}{s^2} + \dfrac{9}{s}\right]$

$f(t) = t^2$

$f(t+3) = (t+3)^2$

$= t^2 + 9 + 6t$

$L[f(t+3)] = L[t^2] + L[9] + L[6t]$

$= \dfrac{2}{s^3} + \dfrac{6}{s^2} + \dfrac{9}{s}$

q. Find L.T of $(4t - t^2)u(t-2)$

$L[f(t)\,u(t-a)] = e^{-as}L[f(t+a)]$

$L[(4t - t^2)u(t-2)] = e^{-2s}L[4(t+2) - (t+2)^2]$

here $f(t) = 4t - t^2$

$f(t+2) = 4(t+2) - (t+2)^2$

$= e^{-2s}L[4t + 8 - t^2 - 4 - 4t]$

$= e^{-2s}L[4 - t^2]$

$= e^{-2s}\left(\dfrac{4}{s} - \dfrac{2}{s^3}\right)$

Date
Page

RESULTS

1. If $f(t) = \begin{cases} f_1(t) & t < a \\ f_2(t) & t \geq a \end{cases}$

Then $f(t) = f_1(t) + \{f_2(t) - f_1(t)\} u(t-a)$

2. If $f(t) = \begin{cases} f_1(t) & t \leq a \\ f_2(t) & a < t \leq b \\ f_3(t) & t > b \end{cases}$

Then $f(t) = f_1(t) + \{f_2(t) - f_1(t)\} u(t-a) + \{f_3(t) - f_2(t)\} u(t-b)$

assuming $f \in [0, \infty]$

Q: Express the func. $f(t) = \begin{cases} \overset{f_1(t)}{t^2} & 0 < t < 2 \\ 4t \; {-f_2(t)} & t > 2 \end{cases}$ interms of unit step func. and hence find its L.T

Sol. $f(t) = t^2 + (4t - t^2) u(t-2)$

$L[f(t)] = L[t^2] + L[(4t - t^2) u(t-2)]$

$= \frac{2}{s^3} + e^{-2s}\left(\frac{4}{s} - \frac{2}{s^3}\right)$ from prev. ques

3rd May, 2023

Q: Express the foll. in terms of unit step func. and hence find LT

$$f(t) = \begin{cases} \frac{k}{a} t & 0 < t < a \\ \frac{k}{a}(t-a) & a < t < 2a \\ \frac{k}{a}(t-2a) & t > 2a \end{cases}$$

Sol. $f(t) = \frac{k\cancel{t}}{a} + \frac{k}{\cancel{a}}[\cancel{t} - \cancel{a} - \cancel{t}]u(t-a) + \frac{k}{a}[\cancel{t} - 2a - \cancel{t} + a]u(t-2a)$

$f(t) = \frac{kt}{a} - ku(t-a) - ku(t-2a)$

$L[f(t)] = \frac{k}{a}L[t] - kL[u(t-a)] - kL[u(t-2a)]$

$= \frac{k}{a}\frac{1}{s^2} - \frac{ke^{-as}}{s} - \frac{ke^{-2as}}{s}$

Q. $f(t) = \begin{cases} \cos t & 0 < t < \pi \\ \sin t & t > \pi \end{cases}$

$f(t) = \cos t + (\sin t - \cos t)u(t-\pi)$

$L[f(t)] = L[\cos t] + L[\sin t\, u(t-\pi)] - L[\cos t\, u(t-\pi)]$

$= \frac{s}{s^2+1} + L[\sin t\, u(t-\pi)] - L[\cos t\, u(t-\pi)]$ —①

$L[\sin t\, u(t-\pi)] = e^{-\pi s}L[\sin(t+\pi)]$

$= \cancel{e^{-as}(-\sin t)}$

$= e^{-\pi s}L[-\sin t]$

$= \frac{-e^{-\pi s}}{s^2+1}$

$L[\cos t\, u(t-\pi)] = e^{-\pi s}L[\cos(t+\pi)]$

$= e^{-\pi s}L[-\cos t]$

$= -\frac{e^{-\pi s} \cdot s}{s^2+1}$

① becomes

$L[f(t)] = \frac{s}{s^2+1} - \frac{e^{-\pi s}}{s^2+1} - \frac{e^{-\pi s} \cdot s}{s^2+1}$

Date
Page

q: Express in unit step func. and find L·T

$$f(t) = \begin{cases} t-1 & \text{when } 1<t<2 \\ 3-t & \text{when } 2<t<3 \end{cases}$$

Sol. unit step func: $(0, \infty]$

$$\Rightarrow f(t) = \begin{cases} 0 & 0<t<1 \\ t-1 & 1<t<2 \\ 3-t & 2<t<3 \\ 0 & t>3 \end{cases}$$

$$f(t) = 0 + (t-1)u(t-1) + (3-t-t+1)u(t-2) + (0-3+t)u(t-3)$$
$$= 0 + (t-1)u(t-1) + (4-2t)u(t-2) + (-3+t)u(t-3)$$

$$L[f(t)] = L[(t-1)u(t-1)] + L[-2(t-2)u(t-2)] + L[(t-3)u(t-3)]$$

$$= e^{-s}L[t] - 2e^{-2s}L[t] + e^{-3s}L[t]$$
$$= e^{-s}\cdot\frac{1}{s^2} - 2\cdot\frac{e^{-2s}}{s^2} - e^{-3s}\cdot\frac{1}{s^2}$$

→ Computation of inverse of $e^{-as}F(s)$ or $L^{-1}[e^{-as}F(s)]$

wkt, $L[f(t-a)u(t-a)] = e^{-as}F(s)$

$L^{-1}[e^{-as}F(s)] = f(t-a)u(t-a)$

WORKING RULE Find $L^{-1}[F(s)]$

then we get $F(t)$
$t \to t-a$
followed by $u(t-a)$

q. $L^{-1}\left[\frac{5-3e^{-3s}-2e^{-7s}}{s}\right]$

Sol. $5L^{-1}\left[\frac{1}{s}\right] - 3L^{-1}\left[e^{-3s}\cdot\frac{1}{s}\right] - 2L^{-1}\left[e^{-7s}\cdot\frac{1}{s}\right]$

$= 5 - 3 \cdot 1\, u(t-3) - 2 \cdot 1 \cdot u(t-7)$

$= 5 - 3u(t-3) - 2 \cdot u(t-7)$

Q. Find $L^{-1}\left\{ \frac{3}{s} - \frac{4e^{-s}}{s^2} + \frac{4e^{-3s}}{s^2} \right\}$

$= 3L^{-1}\left[\frac{1}{s}\right] - 4L^{-1}\left[e^{-s} \cdot \frac{1}{s^2}\right] + 4L^{-1}\left[e^{-3s} \cdot \frac{1}{s^2}\right]$

$= 3 \cdot 1 - 4(t-1)u(t-1) + 4(t-3)u(t-3)$

$= 3 - 4(t-1)u(t-1) + 4(t-3)u(t-3)$

$L^{-1}\left[\frac{1}{s^2}\right] = t$

$L^{-1}\left[e^{-s} \frac{1}{s^2}\right] = (t-1)u(t-1)$

$f(t) = f_1(t) + [f_2(t) - f_1(t)]u(t-a) + [f_3(t) - f_2(t)]u(t-b)$

$f_1(t) = 3$

$f_2(t) - f_1(t) = -4(t-1) = -4t + 4$

$f_2(t) - 3 = -4t + 4$

$f_2(t) = -4t + 7$

$f_3(t) - f_2(t) = 4(t-3)$

$f_3(t) = 4t - 12 - 4t + 7$

$= -5$

$\therefore f(t) = \begin{cases} 3 & 0 < t < 1 \\ 7 - 4t & 1 < t < 3 \\ -5 & t > 3 \end{cases}$

q. The current $i(t)$ in ~~an~~ circuit is given by DE

$$\frac{d^2 i}{dt^2} + 2\frac{di}{dt} = \begin{cases} 0 \quad f_1(t) & 0 < t < 10 \\ 1 \quad f_2(t) & 10 < t < 20 \\ 0 \quad f_3(t) & t > 20 \end{cases}$$

if $i(0) = 0$, $i'(0) = 0$

Determine current as a func. of t

Sol. $i''(t) + 2i'(t) = 0 + (1-0)u(t-10) + (0-1)u(t-20)$

$i''(t) + 2i'(t) = u(t-10) - u(t-20)$

$L[i''(t)] + 2L[i'(t)] = L[u(t-10)] - L[u(t-20)]$

$$s^2 L[i(t)] - \cancel{s\,i(0)} - \cancel{i'(0)} + 2[sL[i(t)] - \cancel{i(0)}] = \frac{e^{-10s}}{s} - \frac{e^{-20s}}{s}$$

$$L[i(t)]\underbrace{(s^2 + 2s)}_{s(s+2)} = \frac{e^{-10s}}{s} - \frac{e^{-20s}}{s}$$

$$L[i(t)] = \frac{e^{-10s}}{s^2(s+2)} - \frac{e^{-20s}}{s^2(s+2)}$$

$$i(t) = L^{-1}\left[e^{-10s} \cdot \frac{1}{s^2(s+2)}\right] - L^{-1}\left[e^{-20s} \cdot \frac{1}{s^2(s+2)}\right] \quad -①$$

To find $L^{-1}\left[e^{-10s} \frac{1}{s^2(s+2)}\right]$

$$L^{-1}\left[\frac{\frac{1}{s+2}}{s^2}\right] = \int_0^t \int_0^t e^{-2t}\, dt$$

$$\int_0^t \left[\frac{e^{-2t}}{-2}\right]_0^t = -\frac{1}{2}\int_0^t e^{-2t} - 1\, dt$$

$$= \cancel{\frac{1}{4}\left[e^{-2t} - t\right]_0^t} \neq \cancel{\frac{1}{4}\left[e^{-2t} - t\right]}$$

$$= \frac{-1}{2}\left[\frac{e^{-2t}}{-2} - t\right]_0^t$$

$$= \frac{1}{2}\left[\frac{e^{-2t}}{2} + t - \frac{1}{2}\right]$$

$$= \frac{1}{2}\left[\frac{e^{-2t} + 2t - 1}{2}\right]$$

$$L^{-1}\left[\frac{1}{s^2(s+2)}\right] = \frac{1}{4}\left[-1 + 2t + e^{-2t}\right] = f(t)$$

$$\therefore L^{-1}\left[e^{-10s} \cdot \frac{1}{s^2(s+2)}\right] = \frac{1}{4}\left[e^{-2(t-10)} + 2(t-10) - 1\right] u(t-10)$$

① →

$$i(t) = \frac{1}{4}\left[e^{-2(t-10)} + 2(t-10) - 1\right] u(t-10) - \frac{1}{4}\left[e^{-2(t-20)} + 2(t-20) - 1\right] u(t-20)$$

9. Solve $\frac{di}{dt} + 2i + 5\int_0^t i \, dt = u(t)$ and $i(0) = 0$

Sol.

$$i'(t) + 2i(t) + 5\int_0^t i(t) \, dt = u(t)$$

$$L[i'(t)] + 2L[i(t)] + 5L\left[\int_0^t i(t) \, dt\right] = L[u(t)]$$

$$sL[i(t)] - i(0) + 2L[i(t)] + \frac{5L[i(t)]}{s} = \frac{1}{s}$$

$$L[i(t)]\left[s + 2 + \frac{5}{s}\right] = \frac{1}{s}$$

$$L[i(t)]\left[s^2 + 2s + 5\right] = 1$$

$$i(t) = L^{-1}\left[\frac{1}{s^2 + 2s + 5}\right] = L^{-1}\left[\frac{1}{(s+1)^2 + 4}\right]$$

$$= e^{-t} L^{-1}\left[\frac{1}{s^2+4}\right] = e^{-t} \frac{\sin 2t}{2}$$

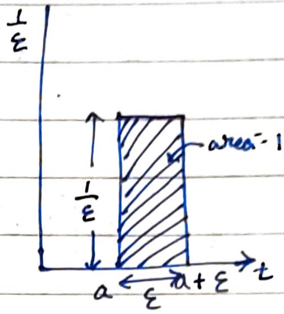
Unit impulse Function (Dirac Delta Function)

Definition: The unit impulse func. or Dirac-delta func. is denoted by $\delta(t-a)$ and is defined as:

$$\delta(t-a) = \lim_{\varepsilon \to 0} f_\varepsilon(t-a)$$

where

$$f_\varepsilon(t-a) = \begin{cases} \frac{1}{\varepsilon}, & \text{for } a \le t \le a+\varepsilon \\ 0, & \text{otherwise} \end{cases}$$



THEOREM: $L[\delta(t-a)] = e^{-as}$

PROOF: $$f_\varepsilon(t-a) = \begin{cases} \frac{1}{\varepsilon} & a < t < a+\varepsilon \\ 0 & \text{otherwise} \end{cases}$$

$$f_\varepsilon(t-a) = \begin{cases} 0 & 0 < t < a \\ \frac{1}{\varepsilon} & a < t < a+\varepsilon \\ 0 & t > a+\varepsilon \end{cases}$$

~~$f(t) = f_1(t) + [f_2(t) - f_1(t)]u(t-a)$~~

$$f_\varepsilon(t-a) = \frac{1}{\varepsilon}[u(t-a) - u(t-(a+\varepsilon))]$$

$$L[f_\varepsilon(t-a)] = L\left\{\frac{1}{\varepsilon}[u(t-a) - u(t-(a+\varepsilon))]\right\}$$

$$= \frac{1}{\varepsilon}\left[L[u(t-a)] - L[u(t-(a+\varepsilon))]\right]$$

$$= \frac{1}{\varepsilon}\left[\frac{e^{-as}}{s} - \frac{e^{-(a+\varepsilon)s}}{s}\right]$$

also we have,

$$L[\delta(t-a)] = \lim_{\varepsilon \to 0} L[f_\varepsilon(t-a)]$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[\frac{e^{-as}}{s} - \frac{e^{-(a+\varepsilon)s}}{s}\right]$$

$$= \lim_{\varepsilon \to 0} \frac{[e^{-as}][1 - e^{-s\varepsilon})]}{s\varepsilon}$$

$$= e^{-as}$$

RESULTS: 1: $L\{f(t)\delta(t-a)\} = e^{-as} f(a)$

2. $\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)$

q. $L[t e^{-2t} \delta(t-2)] = L[e^{-2t}\, t\, \delta(t-2)]$

$$= L[t\,\delta(t-2)]_{s \to s+2} \quad -①$$

$$L[t\,\delta(t-2)] = -\frac{d}{ds} e^{-2s} = 2e^{-2s}$$

$$① \Rightarrow L\{t e^{-2t} \delta(t-2)\} = 2e^{-2s}\Big|_{s \to s+2}$$

$$= 2e^{-2(s+2)}$$

q. $L\left[\frac{1}{t}\delta(t-a)\right]$

$$L\left\{\frac{\delta(t-a)}{t}\right\} = \int_s^\infty e^{-as}\, ds$$

$$= \left[\frac{e^{-as}}{-a}\right]_s^\infty$$

$$= 0 + \frac{e^{-as}}{a} = \frac{e^{-as}}{a}$$

Date
Page

q. $L[t\,u(t-1) + t^2\delta(t-1)]$

$$L[t\,u(t-1)] = -\frac{d}{ds}\frac{e^{-s}}{s}$$

$$= -\left[\frac{s(-e^{-s}) - e^{-s}(1)}{s^2}\right]$$

$$= \frac{se^{-s} + e^{-s}}{s^2}$$

$$L[t^2\delta(t-1)] = \frac{d^2}{ds^2}e^{-s}$$

$$= e^{-s}$$

$$\therefore L[t\,u(t-1) + t^2\delta(t-1)] = e^{-s}\left(1 + \frac{1}{s} + \frac{1}{s^2}\right)$$

q. Find the solution of initial value problem

$$y'' + 2y' + 5y = \delta(t-2)$$

given $y(0) = 0$, $y'(0) = 0$

Sol. $y''(t) + 2y'(t) + 5y(t) = \delta(t-2)$

$$L[y''(t)] + 2L[y'(t)] + 5L[y(t)] = L[\delta(t-2)]$$

$$s^2L[y(t)] - \cancel{sy(0)} - \cancel{y'(0)} + 2[sL[y(t)] - \cancel{y(0)}] + 5L[y(t)] = e^{-2s}$$

$$L[y(t)](s^2+2s+5) = e^{-2s}$$

$$L[y(t)] = \frac{e^{-2s}}{s^2+2s+5}$$

$$y(t) = L^{-1}\left[e^{-2s}\frac{1}{s^2+2s+5}\right]$$

.....

$$L^{-1}\left[\frac{1}{s^2+2s+5}\right] = \frac{e^{-t}\sin 2t}{2} = f(t)$$

$$\therefore y(t) = \frac{e^{-(t-2)}\sin 2(t-2)}{2}\, u(t-2)$$

→ USEFUL RESULTS

1. $\cos n\pi = (-1)^n$, $\cos 2n\pi = 1$, $\sin 2n\pi = 0$

2. $\sin n\pi = 0$

3. $\cos \pi = -1$

4. $e^{-\infty} = 0$

5. $$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$$

6. $$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$$

7. Bernoulli's generalised formula of integration by parts

$$\int uv\, dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\ldots$$

where u', u'', u''' denote differentiation,
v_1, v_2, v_3 denote integration

Convolution Theorem

Definition: The convolution of two funcs. $f(t)$ and $g(t)$ is denoted by $(f*g)(t)$ and defined by:

$$(f * g)(t) = \int_0^t f(u)\, g(t-u)\, du$$

$f(t) * g(t)$

$f * g$ is called convolution of f and g.

→ Convolution Theorem

Statement :

If $L^{-1}[F(s)] = f(t)$ and $L^{-1}[G(s)] = g(t)$
then $L^{-1}[F(s) \cdot G(s)] = f(t) * g(t)$

Note * $L[f(t) + g(t)] = L[f(t)] + L[g(t)]$

2. $L[f(t)\, g(t)] \neq L[f(t)] \cdot L[g(t)]$

3. $L[f(t) * g(t)] = L[f(t)] \cdot L[g(t)]$
$= F(s) \cdot G(s)$

4. $L^{-1}[F(s)\, G(s)] = f(t) * g(t)$

4th May, 2023

q. Find $L^{-1}\left[\dfrac{1}{(s+1)(s^2+1)}\right]$

Sol. * Always choose $g(s)$ to be an easier func.

$F(s) = \dfrac{1}{s^2+1}$ $\quad$ $G(s) = \dfrac{1}{s+1}$

$L^{-1}[F(s)] = \sin t = f(t)$ $\quad$ $L^{-1}[G(s)] = e^{-t}$

By C.T, $L^{-1}[F(s) \cdot G(s)] = f(t) * g(t)$

$$\therefore L^{-1}\left[\frac{1}{(s+1)(s^2+1)}\right] = \sin t * e^{-t}$$

$$= \int_0^t \sin u \, e^{-(t-u)} \, du$$

$$= e^{-t} \int_0^t e^{u} \sin u \, du$$

$a=1, b=1$

$a=1, b=1$

$$= e^{-t}\left[\frac{e^u}{1+1}(\sin u - \cos u)\right]_0^t$$

$$= e^{-t}\left[\frac{e^t(\sin t - \cos t)}{2} - \frac{1}{2}(0-1)\right]$$

$$= \frac{1}{2}(\sin t - \cos t + e^{-t})$$

q. $L^{-1}\left[\frac{s}{(s^2+1)^2}\right]$

Sol. $F(s) = \frac{s}{(s^2+1)}$ $\qquad G(s) = \frac{1}{s^2+1}$

$L^{-1}[F(s)] = \cos t = f(t)$; $L^{-1}[G(s)] = \sin t = g(t)$

By CT,

$$L^{-1}[F(s)G(s)] = f(t) * g(t)$$

$$L^{-1}\left[\frac{s}{(s^2+1)^2}\right] = \cos t * \sin t$$

$$= \int_0^t \cos u \sin(t-u) \, du$$

$$= \frac{1}{2}\int_0^t [\sin(t-u+u) + \sin(t-u-u)] \, du$$

$$= \frac{1}{2}\int_0^t [\sin t + \sin(t-2u)] \, du$$

Date
Page

$$= \frac{1}{2}\left[\sin t\,(u)_0^t + \left[\frac{-\cos(t-2u)}{-2}\right]_0^t\right]$$

$$= \frac{1}{2}\left[\sin t\,(t-0) + \frac{\cos t}{2} - \frac{\cos t}{2}\right]$$

$$= \frac{t \sin t}{2}$$

q: $L^{-1}\left[\frac{1}{(s+2)^2(s-2)}\right]$

Sol. $F(s) = \frac{1}{(s+2)^2}$ $\quad G(s) = \frac{1}{s-2}$

$L^{-1}[F(s)] = e^{-2t}\, t = f(t)$ $\quad L^{-1}[G(s)] = e^{2t} = g(t)$

By C.T

$$L^{-1}[F(s)\,G(s)] = f(t) * g(t)$$

$$L^{-1}\left[\frac{1}{(s+2)^2(s-2)}\right] = e^{-2t}\, t * e^{2t}$$

$$= \int_0^t e^{-2u} \cdot u \; e^{2(t-u)}\, du$$

$$= \int_0^t e^{-2u} \cdot u \cdot e^{2t}\, e^{-2u}\, du$$

$$= e^{2t} \int_0^t u\, e^{-4u}\, du$$

$$= e^{2t}\left[\frac{u e^{-4u}}{-4} - \frac{e^{-4u}}{16}\right]_0^t$$

$$= e^{2t}\left[\frac{-t e^{-4t}}{4} - \frac{e^{-4t}}{16} - \left(0 - \frac{1}{16}\right)\right]$$

$$= e^{2t}\left[\frac{1 - e^{-4t} - 4t}{16}\right]$$

5th May, 2023

Periodic Function

Definition: A function $f(t)$ is said to be periodic if $f(t+T) = f(t)$ for all real t and for some +ve number T $(T > 0)$.

$T \rightarrow$ period

eg: $\cos t$, $\sin t$

Theorem: If $f(t)$ is a piecewise periodic function with period T, then

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

q. Find L.T of the foll. periodic func.

$$f(t) = \begin{cases} \cos t & 0 < t \leq \pi \\ -1 & \pi < t \leq 2\pi \end{cases}$$

Sol. Here $f(t)$ is a periodic func. with period $T = 2\pi$. (i.e it will repeat itself after 2π).

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\int_0^{\pi} e^{-st} \cos t\, dt + \int_{\pi}^{2\pi} e^{-st} \cdot (-1)\, dt \right]$$

$a = -s$
$b = 1$
$x = t$

$$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]$$

Date ____
Page ____

$$= \frac{1}{1-e^{-2\pi s}}\left[\left[\frac{e^{-st}}{s^2+1}(-s\cos t + \sin t)\right]_0^{\pi} + \left[\frac{e^{-st}}{s}\right]_{\pi}^{2\pi}\right]$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\frac{e^{-\pi s}}{s^2+1}(-s(-1)+0) - \frac{1}{s^2+1}(-s+0)\right] + \frac{e^{-2\pi s}}{s} - \frac{e^{-\pi s}}{s}\Big]$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\frac{e^{-\pi s}(s)}{s^2+1} + \frac{s}{s^2+1} + \frac{e^{-2\pi s}}{s} - \frac{e^{-\pi s}}{s}\right]$$

q. Determine L.T

$$f(t) = \begin{cases} 1 \; ; & 0 \le t \le \frac{T}{2} \\ 0 \; ; & \frac{T}{2} < t < T \end{cases} \qquad f(t+T) = f(t) \quad t \geq 0$$

Sol. period $= T$

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\, dt$$

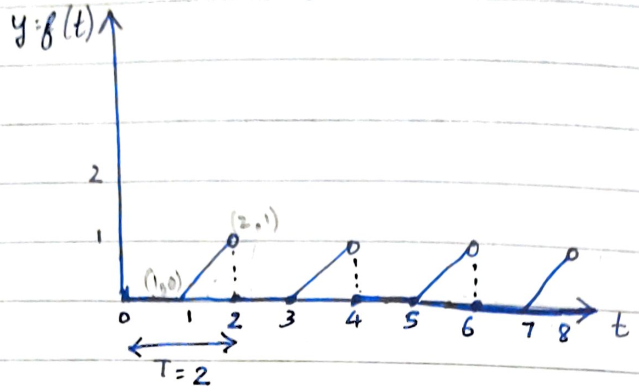
$$= \frac{1}{1-e^{-sT}}\left[\int_0^{T/2} e^{-st}\cdot 1\, dt + \cancel{\int_{T/2}^{T} e^{-st}\cdot 0\, dt}^{\,0}\right]$$

$$= \frac{1}{1-e^{-sT}}\left[\frac{e^{-st}}{-s}\right]_0^{T/2}$$

$$= \frac{1}{1-e^{-sT}}\left[\frac{e^{-Ts/2}}{-s} - \frac{1}{-s}\right]$$

$$= \frac{1}{1-e^{-sT}}\left[-\frac{e^{-Ts/2}}{s} + \frac{1}{s}\right]$$

Q. Determine L·T of periodic func. with graph:



slope $= \dfrac{1-0}{2-1} = \dfrac{1}{1}$

$y - 0 = 1(x-1)$

$y = x - 1$

$y = t - 1$

$$f(t) = \begin{cases} 0 & 0 \le t \le 1 \\ t-1 & 1 \le t < 2 \end{cases}$$

is a periodic func. with period T = 2

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2s}} \left[\int_0^1 0 + \int_1^2 e^{-st}(t-1)\,dt \right]$$

$-\int 1 \cdot \frac{e^{-st}}{-s}$

$= -\frac{e^{-st}}{s^2}$

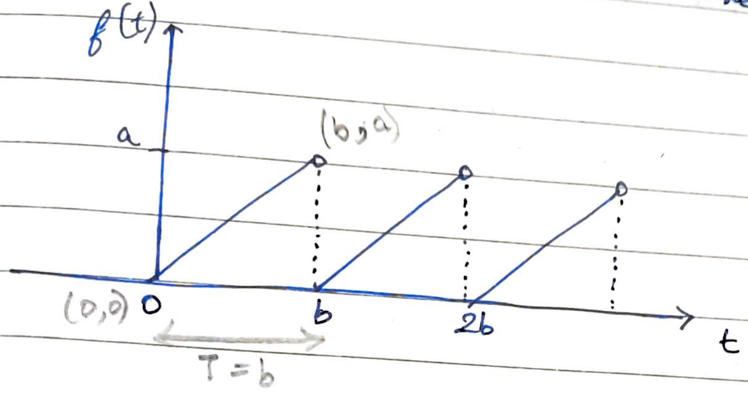
$$= \frac{1}{1-e^{-2s}} \left[(t-1)\frac{e^{-st}}{-s} - 1 \cdot \frac{e^{-st}}{s^2} \right]_1^2$$

$$= \frac{1}{1-e^{-2s}} \left[-\frac{e^{-2s}}{s} - \frac{e^{-2s}}{s^2} - \left(0 - \frac{e^{-s}}{s^2}\right) \right]$$

$$= \frac{1}{1-e^{-2s}} \left[\frac{e^{-s}}{s^2} - \frac{e^{-2s}(s+1)}{s^2} \right]$$

Graph:

Q. Determine L.T of saw tooth wave func.



slope = $\frac{a}{b}$ eqn: $y - 0 = \frac{a}{b}(x - 0)$

$y = \frac{ax}{b}$

$$f(t) = \frac{at}{b}$$

$$L[f(t)] = \frac{1}{1-e^{-2b}} \int_0^b e^{-st} \frac{at}{b}\, dt$$

$$= \frac{a}{b(1-e^{-2b})} \int_0^b t \cdot e^{-st}\, dt$$

$$= \frac{a}{b(1-e^{-2b})} \left[\frac{t \cdot e^{-st}}{-s} - \frac{1}{s^2} e^{-st} \right]_0^t$$

$$= \frac{a}{b(1-e^{-2b})} \left[\frac{-b}{s} e^{-bs} - \frac{e^{-bs}}{s^2} - 0 + \frac{1}{s^2} \right]$$

$$= \frac{a}{b(1-e^{-2b})} \left[\frac{e^{-bs}}{s^2} (-bs - 1) + \frac{1}{s^2} \right]$$

Inverse Laplace Transform of Logarithmic & trignometric functions.

$$L^{-1}[F(s)] = \frac{-1}{t} L^{-1}[F'(s)]$$

q. $L^{-1}\left[\log\left(1+\frac{1}{s^2}\right)\right]$

$$F(s) = \log\left(1+\frac{1}{s^2}\right) = \log\left(\frac{s^2+1}{s^2}\right)$$

$$F(s) = \log(s^2+1) - \log(s^2)$$

$$F'(s) = \frac{2s}{s^2+1} - \frac{2\cancel{s}}{s^{\cancel{2}}}$$

$$L^{-1}[F'(s)] = 2L^{-1}\left[\frac{s}{s^2+1}\right] - 2L^{-1}\left[\frac{1}{s}\right]$$

$$= 2\cos t - 2$$

wkt $L^{-1}[F(s)] = \frac{-1}{t} L^{-1}[F'(s)]$

$$L^{-1}\left[\log\left(1+\frac{1}{s^2}\right)\right] = \frac{-2(\cos t - 1)}{t}$$

q. $L^{-1}\left[\cot^{-1}\left(\frac{s+3}{2}\right)\right]$

$$F(s) = \cot^{-1}\left(\frac{s+3}{2}\right)$$

$$\frac{d}{dx}[\cot^{-1} x] = \frac{-1}{1+x^2}$$

$$F'(s) = \frac{-1}{1+\left(\frac{s+3}{2}\right)^2} \cdot \left(\frac{1}{2}\right)$$

$$= \frac{-1}{1+\frac{s^2+6s+9}{4}} \cdot \frac{1}{2}$$

$$= -\frac{\cancel{4}\;2}{s^2+6s+13}\cdot\frac{1}{\cancel{2}}$$

$$F'(s) = \frac{-2}{s^2+6s+13}$$

$$L^{-1}[F'(s)] = -2\,L^{-1}\left[\frac{1}{s^2+6s+13}\right]$$

$$= -2\,L^{-1}\left[\frac{1}{(s+3)^2+4}\right]$$

$$= -2\,e^{-3t}\,L^{-1}\left[\frac{1}{s^2+2^2}\right]$$

$$= -2\,e^{-3t}\,\frac{\sin 2t}{2}$$

$$= -e^{-3t}\sin 2t$$

$$\therefore\; L^{-1}\left[\cot^{-1}\left(\frac{s+3}{2}\right)\right] = \frac{e^{-3t}\sin 2t}{t}$$

q: $f(t) = \begin{cases} 1+t & \text{for } 0 \le t < 1 \\ 3-t & \text{for } 1 \le t < 2 \end{cases}$

Sol. $L[f(t)] = \dfrac{1}{-e^{-2s}+1}\left[\int_0^1 (1+t)e^{-3t}\,dt + \int_1^2 (3-t)e^{-st}\,dt\right]$

$$= \frac{1}{1-e^{-2s}}\left\{\left[(1+t)\frac{e^{-st}}{-s} - \frac{e^{-st}}{-s^2}\right]_0^1 + \left[(3-t)\frac{e^{-st}}{-s} - (-1)\frac{e^{-st}}{s^2}\right]_1^2\right\}$$

$$= \frac{1}{1-e^{-2s}}\left\{\left[\frac{2e^{-st}}{s} - \frac{e^{-s}}{s^2} - \left(\frac{-1}{s} - \frac{1}{s^2}\right)\right] + \left[\frac{e^{-2s}}{s} + \frac{e^{-2s}}{s^2} - \left(\frac{2e^{-t}}{s} + \frac{e^{-s}}{s^2}\right)\right]\right\}$$

8th May, 2023

Change of state Property

Theorem If $L\{f(t)\} = F(s)$ then $L\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$

If $L^{-1}(F(s)) = f(t)$ $L^{-1}\{F(ks)\} = \frac{1}{k} f\left(\frac{t}{k}\right)$

q. If $L\{f(t)\} = \frac{20-4s}{s^2-4s+20}$ Find $L\{f(3t)\}$

Sol. Given, $L\{f(t)\} = \frac{20-4s}{s^2-4s+20} = F(s)$

now,

$$L[f(3t)] = \frac{1}{3} F\left(\frac{s}{3}\right) = \frac{1}{3}\left[\frac{20-\frac{4s}{3}}{\left(\frac{s}{3}\right)^2 - 4\left(\frac{s}{3}\right)+20}\right]$$

$$= \frac{1}{\cancel{3}}\left[\frac{60-4s}{s^2-12s+180}\right]$$

q. If $L^{-1}\left\{\frac{s^2-1}{(s^2+1)^2}\right\} = t\cos t$, evaluate $L^{-1}\left\{\frac{9s^2-1}{(9s^2+1)}\right\}$

Sol. $L^{-1}\left\{\frac{9s^2-1}{(9s^2+1)^2}\right\} = L^{-1}\{F(3s)\}$

$$= \frac{1}{3} f\left(\frac{t}{3}\right) = \frac{1}{3}\left(\frac{t}{3}\cos\left(\frac{t}{3}\right)\right) = \frac{t}{9}\cos\frac{t}{3}$$

Date ____
Page ____

REVISION

q. $L^{-1}\left\{\dfrac{e^{4-3s}}{(s+4)^{5/2}}\right\}$

$L^{-1}\left\{e^{4}\cdot e^{-3s}\cdot\dfrac{1}{(s+4)^{5/2}}\right\}$

$= e^{4}\, L^{-1}\left\{e^{-3s}\dfrac{1}{(s+4)^{5/2}}\right\}$

$= e^{4}\, e^{-4(t-3)}\,(t-3)^{3/2}\,\dfrac{4}{3\sqrt{\pi}}\,u(t-3)$

$L^{-1}\left\{\dfrac{1}{(s+4)^{5/2}}\right\} = e^{-4t} L^{-1}\left\{\dfrac{1}{s^{5/2}}\right\}$

$= e^{-4t}\dfrac{t^{3/2}}{\Gamma(5/2)}$

$= e^{-4t}\dfrac{t^{3/2}}{\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}$

$= 4\dfrac{e^{-4t}\,t^{3/2}}{\sqrt{\pi}}$

q. $L^{-1}\left\{\dfrac{1}{\sqrt{2s+3}}\right\}$

$L^{-1}\left[\dfrac{1}{2^{1/2}\left(s+\frac{3}{2}\right)^{1/2}}\right]$

$= \dfrac{e^{-3/2t}}{\sqrt{2}} L^{-1}\left[\dfrac{1}{s^{1/2}}\right]$

$= \dfrac{e^{-3/2t}}{\sqrt{2}}\,\dfrac{t^{-1/2}}{\sqrt{\pi}}$

q. $[F(s)] = \dfrac{5s+1}{s^{2}-2s}$ find $L^{-1}[F(s)]$

$L^{-1}\left[\dfrac{5s+1}{(s-1)^{2}-1}\right] = L^{-1}\left[\dfrac{5(s-1)+5+1}{(s-1)^{2}-1}\right] = L^{-1}\left[\dfrac{5(s-1)}{(s-1)^{2}-1}\right] + 6L^{-1}\left[\dfrac{1}{(s-1)^{2}-1}\right]$

$= 5e^{t}\cosh t + 6e^{t}\sinh t$

$= 5e^{t}\left(\dfrac{e^{t}+e^{-t}}{2}\right) + 6e^{t}\left(\dfrac{e^{t}-e^{-t}}{2}\right)$

$= \dfrac{11}{2}\left(e^{2t}-1\right)$

q: $L^{-1}\left\{\frac{1}{s}\cos\left(\frac{1}{s}\right)\right\}$

$$L^{-1}\left\{\frac{\cos\frac{1}{s}}{s}\right\} = \int_0^t f(t)\,dt$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\cos\frac{1}{s} = 1 - \frac{1}{2!}\cdot\frac{1}{s^2} + \frac{1}{4!}\cdot\frac{1}{s^4} + \dots$$

$$L^{-1}\left[\cos\frac{1}{s}\right] = L^{-1}[1] - \frac{1}{2!}L^{-1}\left[\frac{1}{s^2}\right] + \frac{1}{4!}L^{-1}\left[\frac{1}{s^4}\right]$$

$$= 1 - \frac{1}{2!}t + \frac{1}{4!}\frac{t^3}{3!}$$

$$L^{-1}\left[\frac{\cos 1/s}{s}\right] = \int\left(1 - \frac{1}{2!}t + \frac{1}{4!}\frac{t^3}{3!} - \dots\right)dt$$

$$= t - \frac{t^2}{4} + \frac{1}{4!\,3!}\frac{t^4}{4} \dots$$

q: $L^{-1}\left\{\frac{1}{s(1-e^{-s})}\right\}$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$$

$$L^{-1}\left[\frac{(1-e^{-s})^{-1}}{s}\right]$$

$$L^{-1}\left[\frac{1 + e^{-s} + e^{-2s} + e^{-3s} + \dots}{s}\right]$$