



MODULE

7

Linear Programming: The Simplex Method

LEARNING OBJECTIVES

After completing this chapter, students will be able to:

1. Convert LP constraints to equalities with slack, surplus, and artificial variables.
2. Set up and solve LP problems with simplex tableaus.
3. Interpret the meaning of every number in a simplex tableau.
4. Recognize special cases such as infeasibility, unboundedness and degeneracy.
5. Use the simplex tables to conduct sensitivity analysis.
6. Construct the dual problem from the primal problem.

CHAPTER OUTLINE

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M7.1 Introduction

Recall that the theory of LP states the optimal solution will lie at a corner point of the feasible region. In large LP problems, the feasible region cannot be graphed because it has many dimensions, but the concept is the same.

The simplex method systematically examines corner points, using algebraic steps, until an optimal solution is found.

In Chapter 7 we looked at examples of linear programming (LP) problems that contained two decision variables. With only two variables it is possible to use a graphical approach. We plotted the feasible region and then searched for the optimal corner point and corresponding profit or cost. This approach provides a good way to understand the basic concepts of LP. Most real-life LP problems, however, have more than two variables and are thus too large for the simple graphical solution procedure. Problems faced in business and government can have dozens, hundreds, or even thousands of variables. We need a more powerful method than graphing, so in this chapter we turn to a procedure called the *simplex method*.

How does the simplex method work? The concept is simple, and it is similar to graphical LP in one important respect. In graphical LP we examine each of the corner points; LP theory tells us that the optimal solution lies at one of them. In LP problems containing several variables, we may not be able to graph the feasible region, but the optimal solution will still lie at a corner point of the many-sided, many-dimensional figure (called an n -dimensional polyhedron) that represents the area of feasible solutions. The simplex method examines the corner points in a systematic fashion, using basic algebraic concepts. It does so in an *iterative* manner, that is, repeating the same set of procedures time after time until an optimal solution is reached. Each iteration brings a higher value for the objective function so that we are always moving closer to the optimal solution.

Why should we study the simplex method? It is important to understand the ideas used to produce solutions. The simplex approach yields not only the optimal solution to the decision variables and the maximum profit (or minimum cost), but valuable economic information as well. To be able to use computers successfully and to interpret LP computer printouts, we need to know what the simplex method is doing and why.

We begin by solving a maximization problem using the simplex method. We then tackle a minimization problem and look at a few technical issues that are faced when employing the simplex procedure. From there we examine how to conduct sensitivity analysis using the simplex tables. The chapter concludes with a discussion of the dual, which is an alternative way of looking at any LP problem.

M7.2 How to Set Up the Initial Simplex Solution

Let us consider the case of the Flair Furniture Company from Chapter 7. Instead of the graphical solution we used in that chapter, we now demonstrate the simplex method. You may recall that we let

T = number of tables produced

C = number of chairs produced

and that the problem was formulated as

$$\begin{array}{ll}
 \text{Maximize profit} & = \$70T + \$50C & \text{(objective function)} \\
 \text{subject to} & 2T + 1C \leq 100 & \text{(painting hours constraint)} \\
 & 4T + 3C \leq 240 & \text{(carpentry hours constraint)} \\
 & T, C \geq 0 & \text{(nonnegativity constraints)}
 \end{array}$$

Slack variables are added to each less-than-or-equal-to constraint. Each slack variable represents an unused resource.

Converting the Constraints to Equations

The first step of the simplex method requires that we convert each inequality constraint (except nonnegativity constraints) in an LP formulation into an equation.¹ Less-than-or-equal-to constraints (\leq) such as in the Flair problem are converted to equations by adding a *slack variable* to each constraint. Slack variables represent unused resources; these may be in the form of time on a machine, labor hours, money, warehouse space, or any number of such resources in various business problems.

In our case at hand, we can let

S_1 = slack variable representing unused hours in the painting department

S_2 = slack variable representing unused hours in the carpentry department

The constraints to the problem may now be written as

$$2T + 1C + S_1 = 100$$

and

$$4T + 3C + S_2 = 240$$

Thus, if the production of tables (T) and chairs (C) uses less than 100 hours of painting time available, the unused time is the value of the slack variable, S_1 . For example, if $T = 0$ and $C = 0$ (in other words, if nothing is produced), we have $S_1 = 100$ hours of slack time in the painting department. If Flair produces $T = 40$ tables and $C = 10$ chairs, then

$$2T + 1C + S_1 = 100$$

$$2(40) + 1(10) + S_1 = 100$$

$$S_1 = 10$$

and there will be 10 hours of slack, or unused, painting time available.

To include all variables in each equation, which is a requirement of the next simplex step, slack variables not appearing in an equation are added with a coefficient of 0. This means, in effect, that they have no influence on the equations in which they are inserted; but it does allow us to keep tabs on all variables at all times. The equations now appear as follows:

$$2T + 1C + 1S_1 + 0S_2 = 100$$

$$4T + 3C + 0S_1 + 1S_2 = 240$$

$$T, C, S_1, S_2 \geq 0$$

Because slack variables yield no profit, they are added to the original objective function with 0 profit coefficients. The objective function becomes

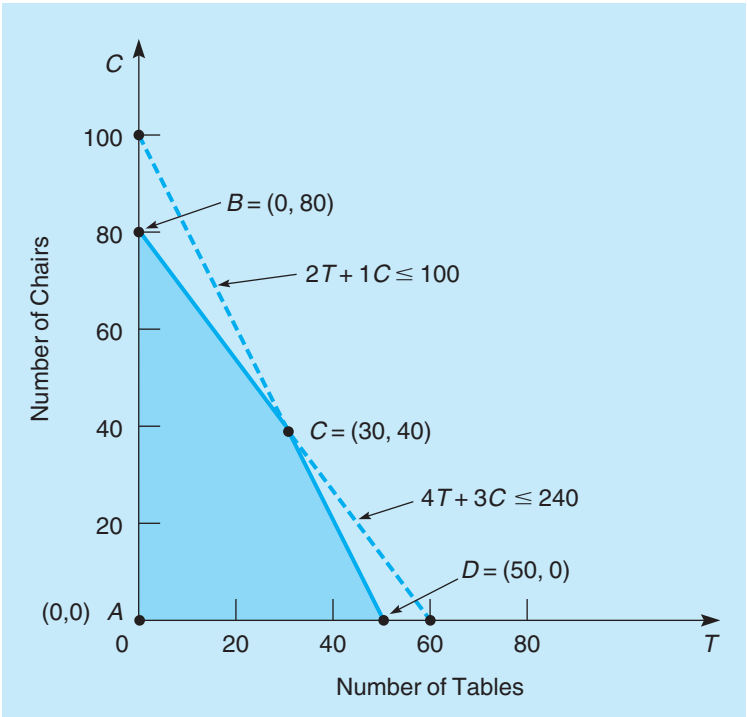
$$\text{Maximize profit} = \$70T + \$50C + \$0S_1 + \$0S_2$$

Finding an Initial Solution Algebraically

Let's take another look at the new constraint equations. We see that there are two equations and four variables. Think back to your last algebra course. When you have the same number of unknown variables as you have equations, it is possible to solve for unique values of the variables. But when there are four unknowns (T , C , S_1 , and S_2 , in this case) and only two equations, you can

¹This is because the simplex is a matrix algebra method that requires all mathematical relationships to be equations, with each equation containing all of the variables.

FIGURE M7.1
Corner Points of the Flair Furniture Company Problem



A basic feasible solution to a system of n equations is found by setting all but n variables equal to 0 and solving for the other variables.

Simplex considers only corner points as it seeks the best solution.

let two of the variables equal 0 and then solve for the other two. For example, if $T = C = 0$, then $S_1 = 100$ and $S_2 = 240$. A solution found in this manner is called a *basic feasible solution*.

The simplex method begins with an initial feasible solution in which all real variables (such as T and C) are set equal to 0. This trivial solution always produces a profit of \$0, as well as slack variables equal to the constant (right-hand-side) terms in the constraint equations. It's not a very exciting solution in terms of economic returns, but it is one of the original corner point solutions (see Figure M7.1). As mentioned, the simplex method will start at this corner point (A) and then move up or over to the corner point that yields the most improved profit (B or D). Finally, the technique will move to a new corner point (C), which happens to be the optimal solution to the Flair Furniture problem. The simplex method considers only feasible solutions and hence will touch no possible combinations other than the corner points of the shaded region in Figure M7.1.

The First Simplex Tableau

To simplify handling the equations and objective function in an LP problem, we place all of the coefficients into tabular form. The first *simplex tableau* is shown in Table M7.1. An explanation of its parts and how the tableau is derived follows.

CONSTRAINT EQUATIONS We see that Flair Furniture's two constraint equations can be expressed as follows:

SOLUTION MIX	T	C	S_1	S_2	QUANTITY (RIGHT-HAND SIDE)
S_1	2	1	1	0	100
S_2	4	3	0	1	240

Here are the constraints in tabular form.

TABLE M7.1 Flair Furniture's Initial Simplex Tableau

C_j	SOLUTION MIX	\$70 T	\$50 C	\$0 S_1	\$0 S_2	QUANTITY	
\$0	S_1	2	1	1	0	100	
\$0	S_2	4	3	0	1	240	
	Z_j	\$0	\$0	\$0	\$0	\$0	
	$C_j - Z_j$	\$70	\$50	\$0	\$0	\$0	

The numbers (2, 1, 1, 0) in the first row represent the coefficients of the first equation, namely, $2T + 1C + 1S_1 + 0S_2$. The numbers (4, 3, 0, 1) in the second row are the algebraic equivalent of the constraint $4T + 3C + 0S_1 + 1S_2$.

The initial solution mix begins with real, or decision, variables set equal to zero.

As suggested earlier, we begin the initial solution procedure at the origin, where $T = 0$ and $C = 0$. The values of the other two variables must then be nonzero, so $S_1 = 100$ and $S_2 = 240$. These two slack variables constitute the *initial solution mix*; their values are found in the *quantity* (or right-hand-side [RHS]) *column*. Because T and C are not in the solution mix, their initial values are automatically equal to 0.

This initial solution is a *basic feasible solution* and is described in vector, or column, form as

Here is the basic feasible solution in column form.

$$\begin{bmatrix} T \\ C \\ S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 100 \\ 240 \end{bmatrix}$$

Variables in the solution mix are called *basic*. Those not in the solution are called *nonbasic*.

Variables in the solution mix, which is called the *basis* in LP terminology, are referred to as basic variables. In this example, the basic variables are S_1 and S_2 . Variables not in the solution mix or basis and that have been set equal to zero (T and C in this case), are called *nonbasic variables*. Of course, if the optimal solution to this LP problem turned out to be $T = 30$, $C = 40$, $S_1 = 0$, and $S_2 = 0$, or

$$\begin{bmatrix} T \\ C \\ S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} 30 \\ 40 \\ 0 \\ 0 \end{bmatrix} \quad (\text{in vector form})$$

then T and C would be the final basic variables, and S_1 and S_2 would be the *nonbasic variables*. Notice that for any corner point, exactly two of the four variables will equal zero.

SUBSTITUTION RATES Many students are unsure as to the actual meaning of the numbers in the columns under each variable. We know that the entries are the coefficients for that variable.

Under T are the coefficients $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$, under C are $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, under S_1 are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and under S_2 are $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Substitution rates are numbers in the body of the table. This paragraph explains how to interpret their meaning.

But what is their interpretation? The numbers in the body of the simplex tableau (see Table M7.1) can be thought of as *substitution rates*. For example, suppose we now wish to make T larger than 0, that is, produce some tables. For every unit of the T product introduced into the current solution, 2 units of S_1 and 4 units of S_2 must be removed from the solution. This is so because each table requires 2 hours of the currently unused painting department slack time, S_1 . It also takes 4 hours of carpentry time; hence 4 units of variable S_2 must be removed from the solution for every unit of T that enters. Similarly, the substitution rates for each unit of C that enters the current solution are 1 unit of S_1 and 3 units of S_2 .

Another point that you are reminded of throughout this chapter is that for any variable ever to appear in the solution mix column, it must have the number 1 someplace in its column and 0s in every other place in that column. We see that column S_1 contains $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, so variable S_1 is in the solution. Similarly, the S_2 column is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, so S_2 is also in the solution.²

ADDING THE OBJECTIVE FUNCTION We now continue to the next step in establishing the first simplex tableau. We add a row to reflect the objective function values for each variable. These contribution rates, called C_j , appear just above each respective variable, as shown in the following table:

C_j	SOLUTION MIX	$\begin{matrix} \rightarrow & \$70 & \$50 & \$0 & \$0 \\ \downarrow & T & C & S_1 & S_2 \end{matrix}$				QUANTITY
		T	C	S_1	S_2	
\$0	S_1	2	1	1	0	100
\$0	S_2	4	3	0	1	240

The unit profit rates are not just found in the top C_j row: in the leftmost column, C_j indicates the unit profit for each variable *currently* in the solution mix. If S_1 were removed from the solution and replaced, for example, by C , \$50 would appear in the C_j column just to the left of the term C .

THE Z_j AND $C_j - Z_j$ ROWS We can complete the initial Flair Furniture simplex tableau by adding two final rows. These last two rows provide us with important economic information, including the total profit and the answer as to whether the current solution is optimal.

We compute the Z_j value for each column of the initial solution in Table M7.1 by multiplying the 0 contribution value of each number in the C_j column by each number in that row and

²If there had been *three* less-than-or-equal-to constraints in the Flair Furniture problem, there would be three slack variables, S_1 , S_2 , and S_3 . The 1s and 0s would appear like this:

SOLUTION MIX	S_1	S_2	S_3
S_1	1	0	0
S_2	0	1	0
S_3	0	0	1



IN ACTION

Resource Allocation at Pantex

Companies often use optimization techniques such as LP to allocate limited resources to maximize profits or minimize costs. One of the most important resource allocation problems faced by the United States is dismantling old nuclear weapons and maintaining the safety, security, and reliability of the remaining systems. This is the problem faced by Pantex, a \$300 million corporation.

Pantex is responsible for disarming, evaluating, and maintaining the U.S. nuclear stockpile. The company is also responsible for storing critical weapon components that relate to U.S.–Russian nonproliferation agreements. Pantex constantly makes trade-offs in meeting the requirements of disarming some nuclear weapons versus maintaining existing nuclear weapon systems, while effectively allocating limited resources. Like many manufacturers, Pan-

tex must allocate scarce resources among competing demands, all of which are important.

The team charged with solving the resource allocation problem at Pantex developed the Pantex Process Model (PPM). PPM is a sophisticated optimization system capable of analyzing nuclear needs over different time horizons. Since its development, PPM has become the primary tool for analyzing, planning, and scheduling issues at Pantex. PPM also helps to determine future resources. For example, it was used to gain government support for \$17 million to modify an existing plant with new buildings and \$70 million to construct a new plant.

Source: Based on Edwin Kjeldgaard, et al. "Swords into Plowshares: Nuclear Weapon Dismantlement, Evaluation, and Maintenance at Pantex," *Interfaces* 30, 1 (January–February 2000): 57–82.

The Z-row entry in the quantity column provides the gross profit.

the j th column, and summing. The Z_j value for the quantity column provides the total contribution (gross profit in this case) of the given solution:

$$\begin{aligned} Z_j \text{ (for gross profit)} &= (\text{Profit per unit of } S_1) \times (\text{Number of units of } S_1) \\ &\quad + (\text{Profit per unit of } S_2) \times (\text{Number of units of } S_2) \\ &= \$0 \times 100 \text{ units} + \$0 \times 240 \text{ units} \\ &= \$0 \text{ profit} \end{aligned}$$

The Z_j values for the other columns (under the variables T , C , S_1 , and S_2) represent the gross profit *given up* by adding one unit of this variable into the current solution. Their calculations are as follows:

$$\begin{aligned} Z_j &= (\text{Profit per unit of } S_1) \times (\text{Substitution rate in row 1}) \\ &\quad + (\text{Profit per unit of } S_2) \times (\text{Substitution rate in row 2}) \end{aligned}$$

Thus,

$$\begin{aligned} Z_j \text{ (for column } T) &= (\$0)(2) + (\$0)(4) = \$0 \\ Z_j \text{ (for column } C) &= (\$0)(1) + (\$0)(3) = \$0 \\ Z_j \text{ (for column } S_1) &= (\$0)(1) + (\$0)(0) = \$0 \\ Z_j \text{ (for column } S_2) &= (\$0)(0) + (\$0)(1) = \$0 \end{aligned}$$

We see that there is no profit *lost* by adding one unit of either T (tables), C (chairs), S_1 , or S_2 .

The $C_j - Z_j$ number in each column represents the net profit, that is, the profit gained minus the profit given up, that will result from introducing 1 unit of each product or variable into the solution. It is not calculated for the quantity column. To compute these numbers, simply

The $C_j - Z_j$ row gives the net profit from introducing one unit of each variable into the solution.

subtract the Z_j total for each column from the C_j value at the very top of that variable's column. The calculations for the net profit per unit (the $C_j - Z_j$ row) in this example follow:

	COLUMN			
	T	C	S_1	S_2
C_j for column	\$70	\$50	\$0	\$0
Z_j for column	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>
$C_j - Z_j$ for column	\$70	\$50	\$0	\$0

We reach an optimal solution when the $C_j - Z_j$ row has no positive numbers in it.

It is obvious to us when we compute a profit of \$0 that the initial solution is not optimal. By examining the numbers in the $C_j - Z_j$ row of Table M7.1, we see that the total profit can be increased by \$70 for each unit of T (tables) and by \$50 for each unit of C (chairs) added to the solution mix. A negative number in the $C_j - Z_j$ row would tell us that profits would *decrease* if the corresponding variable were added to the solution mix. An optimal solution is reached in the simplex method when the $C_j - Z_j$ row contains no positive numbers. Such is not the case in our initial tableau.

M7.3 Simplex Solution Procedures

Here are the five simplex steps.

After an initial tableau has been completed, we proceed through a series of five steps to compute all the numbers needed in the next tableau. The calculations are not difficult, but they are complex enough that even the smallest arithmetic error can produce a wrong answer.

We first list the five steps and then carefully explain and apply them in completing the second and third tableaus for the Flair Furniture Company data.

Five Steps of the Simplex Method for Maximization Problems

1. Variable entering the solution has the largest positive $C_j - Z_j$.

1. Determine which variable to enter into the solution mix next. One way of doing this is by identifying the column, and hence the variable, with the largest positive number in the $C_j - Z_j$ row of the preceding tableau. This means that we will now be producing some of the product contributing the greatest additional profit per unit. The column identified in this step is called the *pivot column*.

2. Variable leaving the solution is determined by a ratio we must compute.

2. Determine which variable to replace. Because we have just chosen a new variable to enter the solution mix, we must decide which basic variable currently in the solution will have to leave to make room for it. Step 2 is accomplished by dividing each amount in the *quantity* column by the corresponding number in the column selected in step 1. The row with the *smallest nonnegative number* calculated in this fashion will be replaced in the next tableau. (This smallest number, by the way, gives the maximum number of units of the variable that may be placed in the solution.) This row is often referred to as the *pivot row*. The number at the intersection of the pivot row and pivot column is referred to as the *pivot number*.

3. New pivot-row calculations are done next.

3. Compute new values for the pivot row. To do this, we simply divide every number in the row by the pivot number.

4. Other new rows are calculated with Equation M7-1.

4. Compute the new values for each remaining row. (In our Flair Furniture problem there are only two rows in the LP tableau, but most larger problems have many more rows.) All remaining row(s) are calculated as follows:

$$(\text{New row numbers}) = (\text{Numbers in old row})$$

$$- \left[\begin{pmatrix} \text{Number above} \\ \text{or below} \\ \text{pivot number} \end{pmatrix} \times \begin{pmatrix} \text{Corresponding number in} \\ \text{the new row, that is, the} \\ \text{row replaced in step 3} \end{pmatrix} \right] \quad (\text{M7-1})$$

5. Finally Z_j and $C_j - Z_j$ rows are recomputed.

5. Compute the Z_j and $C_j - Z_j$ rows, as demonstrated in the initial tableau. If all numbers in the $C_j - Z_j$ row are 0 or negative, an optimal solution has been reached. If this is not the case, return to step 1.

M7.4 The Second Simplex Tableau

Here we apply the five steps to Flair Furniture.

First, T (tables) enters the solution mix because its $C_j - Z_j$ value of \$70 is largest.

Now that we have listed the five steps needed to move from an initial solution to an improved solution, we apply them to the Flair Furniture problem. Our goal is to add a new variable to the solution mix, or basis, to raise the profit from its current tableau value of \$0.

Step 1. To decide which of the variables will enter the solution next (it must be either T or C , since they are the only two nonbasic variables at this point), we select the one with the largest positive $C_j - Z_j$ value. Variable T , tables, has a $C_j - Z_j$ value of \$70, implying that each unit of T added into the solution mix will contribute \$70 to the overall profit. Variable C , chairs, has a $C_j - Z_j$ value of only \$50. The other two variables, S_1 and S_2 , have 0 values and can add nothing more to profit. Hence, we select T as the variable to enter the solution mix and identify its column (with an arrow) as the pivot column. This is shown in Table M7.2.

Step 2. Since T is about to enter the solution mix, we must decide which variable is to be replaced. There can only be as many basic variables as there are constraints in any LP problem, so either S_1 or S_2 will have to leave to make room for the introduction of T , tables, into the basis. To identify the pivot row, each number in the quantity column is divided by the corresponding number in the T column.

For the S_1 row:

$$\frac{100 \text{ (hours of painting time available)}}{2 \text{ (hours required per table)}} = 50 \text{ tables}$$

For the S_2 row:

$$\frac{240 \text{ (hours of carpentry time available)}}{4 \text{ (hours required per table)}} = 60 \text{ tables}$$

S_1 leaves the solution mix because the smaller of the two ratios indicates that the next pivot row will be the first row.

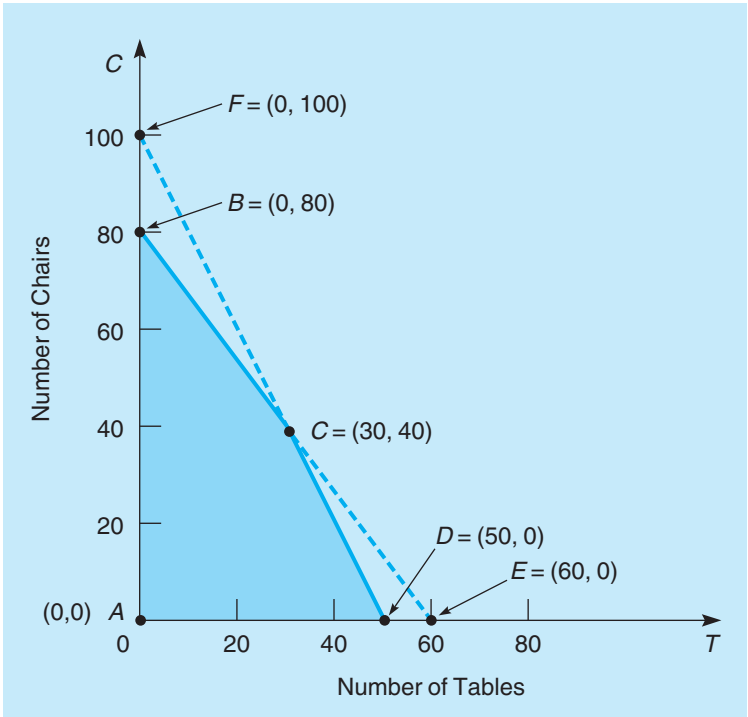
The smaller of these two ratios, 50, indicates the maximum number of units of T that can be produced without violating either of the original constraints. This corresponds to point D in Figure M7.2. The other ratio (60) corresponds to point E on this graph. Thus, the smallest ratio is chosen so that the next solution is feasible. Also, when $T = 50$, there is no slack in constraint 1, so $S_1 = 0$. This means that S_1 will be the next variable to be replaced at this iteration of the simplex method. The row with the smallest ratio (row 1) is the pivot row. The pivot row and the pivot number (the number at the intersection of the pivot row and pivot column) are identified in Table M7.3.

Step 3. Now that we have decided which variable is to enter the solution mix (T) and which is to leave (S_1), we begin to develop the second, improved simplex tableau. Step 3 involves

TABLE M7.2
Pivot Column Identified in the Initial Simplex Tableau

C_j		\$70	\$50	\$0	\$0	
	SOLUTION MIX	T	C	S_1	S_2	QUANTITY (RHS)
\$0	S_1	2	1	1	0	100
\$0	S_2	4	3	0	1	240
	Z_j	\$0	\$0	\$0	\$0	\$0
	$C_j - Z_j$	\$70	\$50	\$0	\$0	total profit
		Pivot column				

FIGURE M7.2
Graph of the Flair Furniture Company Problem



The new pivot row is computed by dividing every number in the pivot row by the pivot number.

$$\frac{2}{2} = 1 \quad \frac{1}{2} = 0.5 \quad \frac{1}{2} = 0.5 \quad \frac{0}{2} = 0 \quad \frac{100}{2} = 50$$

The new version of the entire pivot row appears in the accompanying table. Note that *T* is now in the solution mix and that 50 units of *T* are being produced. The C_j value is listed as a \$70 contribution per unit of *T* in the solution. This will definitely provide Flair Furniture with a more profitable solution than the \$0 generated in the initial tableau.

C_j	SOLUTION MIX	<i>T</i>	<i>C</i>	<i>S</i> ₁	<i>S</i> ₂	QUANTITY
\$70	<i>T</i>	1	0.5	0.5	0	50

TABLE M7.3
Pivot Row and Pivot Number Identified in the Initial Simplex Tableau

C_j		\$70	\$50	\$0	\$0	
	SOLUTION MIX	<i>T</i>	<i>C</i>	<i>S</i> ₁	<i>S</i> ₂	QUANTITY
\$0	<i>S</i> ₁	②	1	1	0	100
\$0	<i>S</i> ₂	4	3	0	1	240
	<i>Z</i> _{<i>j</i>}	\$0	\$0	\$0	\$0	\$0
	<i>C</i> _{<i>j</i>} - <i>Z</i> _{<i>j</i>}	\$70	\$50	\$0	\$0	

Step 4. This step is intended to help us compute new values for the other row in the body of the tableau, that is, the S_2 row. It is slightly more complex than replacing the pivot row and uses the formula (Equation M7.1) shown earlier. The expression on the right side of the following equation is used to calculate the left side.

We can now recompute the S_2 row.

$\left(\begin{array}{c} \text{Number in} \\ \text{New } S_2 \text{ Row} \end{array} \right) = \left(\begin{array}{c} \text{Number in} \\ \text{Old } S_2 \text{ Row} \end{array} \right) - \left[\left(\begin{array}{c} \text{Number Below} \\ \text{Pivot Number} \end{array} \right) \times \left(\begin{array}{c} \text{Corresponding Number} \\ \text{in the New } T \text{ Row} \end{array} \right) \right]$							
0	=	4	−	(4)	×		(1)
1	=	3	−	(4)	×		(0.5)
−2	=	0	−	(4)	×		(0.5)
1	=	1	−	(4)	×		(0)
40	=	240	−	(4)	×		(50)

This new S_2 row will appear in the second tableau in the following format:

C_j	SOLUTION MIX	T	C	S_1	S_2	QUANTITY
\$70	T	1	0.5	0.5	0	50
\$0	S_2	0	1	−2	1	40

We note that the T column contains $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and the S_2 column contains $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. These 0s and 1s indicate that T and S_2 are in the basis (the solution mix).

Now that T and S_2 are in the solution mix, take a look at the values of the coefficients in their respective columns. The T column contains $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, a condition necessary for that variable to be in the solution. Similarly, the S_2 column has $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, that is, it contains a 1 and a 0. Basically, the algebraic manipulations we just went through in steps 3 and 4 were simply directed at producing 0s and 1s in the appropriate positions. In step 3 we divided every number in the pivot row by the pivot number; this guaranteed that there would be a 1 in the T column's top row. To derive the new second row, we multiplied the first row (each row is really an equation) by a constant (the number 4 here) and subtracted it from the second equation. The result was the new S_2 row with a 0 in the T column.

Step 5. The final step of the second iteration is to introduce the effect of the objective function. This involves computing the Z_j and $C_j - Z_j$ rows. Recall that the Z_j entry for the quantity column gives us the gross profit for the current solution. The other Z_j values represent the gross profit given up by adding one unit of each variable into this new solution. The Z_j values are calculated as follows:

$$\begin{aligned}
 Z_j \text{ (for } T \text{ column)} &= (\$70)(1) + (\$0)(0) = \$70 \\
 Z_j \text{ (for } C \text{ column)} &= (\$70)(0.5) + (\$0)(1) = \$35 \\
 Z_j \text{ (for } S_1 \text{ column)} &= (\$70)(0.5) + (\$0)(-2) = \$35 \\
 Z_j \text{ (for } S_2 \text{ column)} &= (\$70)(0) + (\$0)(1) = \$0 \\
 Z_j \text{ (for total profit)} &= (\$70)(50) + (\$0)(40) = \$3,500
 \end{aligned}$$

Note that the current profit is \$3,500.

We find the new profit in the Z row.

TABLE M7.4
Completed Second
Simplex Tableau for
Flair Furniture

C_j		\rightarrow	\$70	\$50	\$0	\$0	
	SOLUTION MIX		T	C	S_1	S_2	QUANTITY
\$70	T		1	0.5	0.5	0	50
\$0	S_2		0	1	-2	1	40
	Z_j		\$70	\$35	\$35	\$0	\$3,500
	$C_j - Z_j$		\$0	\$15	-\$35	\$0	

The $C_j - Z_j$ row indicates the net profit, given the current solution, of one more unit of each variable. For example, C has a profit of \$15 per unit.

The $C_j - Z_j$ numbers represent the net profit that will result, given our present production mix, if we add one unit of each variable into the solution:

	COLUMN			
	T	C	S_1	S_2
C_j for column	\$70	\$50	\$0	\$0
Z_j for column	\$70	\$35	\$35	\$0
$C_j - Z_j$ for column	\$0	\$15	-\$35	\$0

The Z_j and $C_j - Z_j$ rows are inserted into the complete second tableau as shown in Table M7.4.

Interpreting the Second Tableau

Table M7.4 summarizes all of the information for the Flair Furniture Company's production mix decision as of the second iteration of the simplex method. Let's briefly look over a few important items.

We can look at the current solution as a corner point in the graphical method.

CURRENT SOLUTION At this point, the solution point of 50 tables and 0 chairs ($T = 50, C = 0$) generates a profit of \$3,500. T is a basic variable; C is a nonbasic variable. Using a graphical LP approach, this corresponds to corner point D , as shown earlier in Figure M7.2.

RESOURCE INFORMATION We also see in Table M7.4 that slack variable S_2 , representing the amount of unused time in the carpentry department, is in the basis. It has a value of 40, implying that 40 hours of carpentry time remain available. Slack variable S_1 is nonbasic and has a value of 0 hours. There is no slack time in the painting department.

SUBSTITUTION RATES We mentioned earlier that the substitution rates are the coefficients in the heart of the tableau. Look at the C column. If 1 unit of C (1 chair) is added to the current solution, 0.5 units of T and 1 unit of S_2 must be given up. This is because the solution $T = 50$ tables uses up all 100 hours of time in the painting department. (The original constraint, you may recall, was $2T + 1C + S_1 = 100$.) To capture the 1 painting hour needed to make 1 chair, 0.5 of a table *less* must be produced. This frees up 1 hour to be used in making 1 chair.

But why must 1 unit of S_2 (i.e., 1 hour of carpentry time) be given up to produce 1 chair? The original constraint was $4T + 3C + S_2 = 240$ hours of carpentry time. Doesn't this indicate that 3 hours of carpentry time are required to produce 1 unit of C ? The answer is that we are looking at *marginal* rates of substitution. Adding 1 chair replaced 0.5 table. Because 0.5 table required $(0.5 \times 4 \text{ hours per table}) = 2$ hours of carpentry time, 2 units of S_2 are freed. Thus, only 1 *more* unit of S_2 is needed to produce 1 chair.

Here is an explanation of the meaning of substitution rates.

Just to be sure you have this concept down pat, let's look at one more column, S_1 , as well. The coefficients are $\begin{pmatrix} 0.5 \\ -2 \end{pmatrix}$. These substitution rate values mean that if 1 hour of slack painting time is added to the current solution, 0.5 of a table (T) *less* will be produced. However, note that if 1 unit of S_1 is added into the solution, 2 hours of carpentry time (S_2) will no longer be used. These will be *added* to the current 40 slack hours of carpentry time. Hence, a *negative* substitution rate means that if 1 unit of a column variable is added to the solution, the value of the corresponding solution (or row) variable will be increased. A *positive* substitution rate tells us that if 1 unit of the column variable is added to the solution, the row variable will decrease by the rate.

Can you interpret the rates in the T and S_2 columns now?

NET PROFIT ROW The $C_j - Z_j$ row is important to us for two reasons. First, it indicates whether the current solution is optimal. When there are no positive numbers in the bottom row, an optimum solution to an LP maximization problem has been reached. In the case of Table M7.4, we see that $C_j - Z_j$ values for columns T , S_1 , and S_2 are 0 or negative. The value for column C (15) means that the net profit can be increased by \$15 for each chair added into the current solution.

Because the $C_j - Z_j$ value for T is 0, for every unit of T added the total profit will remain unchanged, because we are already producing as many tables as possible. A negative number, such as the -35 in the S_1 column, implies that total profit will *decrease* by \$35 if 1 unit of S_1 is added to the solution. In other words, making one slack hour available in the painting department ($S_1 = 0$ currently) means that we would have to produce one-half table less. Since each table results in a \$70 contribution, we would be losing $0.5 \times \$70 = \35 , for a net loss of \$35.

Later in this chapter we discuss in detail the subject of *shadow prices*. These relate to $C_j - Z_j$ values in the slack variable columns. Shadow prices are simply another way of interpreting negative $C_j - Z_j$ values; they may be viewed as the potential increase in profit if one more hour of the scarce resource (such as painting or carpentry time) could be made *available*.

We mentioned previously that there are two reasons to consider the $C_j - Z_j$ row carefully. The second reason, of course, is that we use the row to determine which variable will enter the solution next. Since an optimal solution has not been reached yet, let's proceed to the third simplex tableau.

The $C_j - Z_j$ row tells us (1) whether the current solution is optimal and (2) if it is not, which variable should enter the solution mix next.

M7.5 Developing the Third Tableau

Since not all numbers in the $C_j - Z_j$ row of the latest tableau are 0 or negative, the previous solution is not optimal, and we must repeat the five simplex steps.

C (chairs) will be the next solution mix variable because it has the only positive value in the $C_j - Z_j$ row.

Step 1. Variable C will enter the solution next by virtue of the fact that its $C_j - Z_j$ value of 15 is the largest (and only) positive number in the row. This means that for every unit of C (chairs) we start to produce, the objective function will increase in value by \$15. The C column is the new pivot column.

Step 2. The next step involves identifying the pivot row. The question is, which variable currently in the solution (T or S_2) will have to leave to make room for C to enter? Again, each number in the quantity column is divided by its corresponding substitution rate in the C column:

$$\text{For the } T \text{ row: } \frac{50}{0.5} = 100 \text{ chairs}$$

$$\text{For the } S_2 \text{ row: } \frac{40}{1} = 40 \text{ chairs}$$

TABLE M7.5

Pivot Row, Pivot Column, and Pivot Number Identified in the Second Simplex Tableau

C_j		\$70	\$50	\$0	\$0	
	SOLUTION MIX	T	C	S_1	S_2	QUANTITY
\$70	T	1	0.5	0.5	0	50
\$0	S_2	0	①	-2	1	40 ← Pivot row
						Pivot number
	Z_j	\$70	\$35	\$35	\$0	\$3,500
	$C_j - Z_j$	\$0	\$15	-\$35	\$0	(total profit)
						Pivot column

We replace the variable S_2 because it is in the pivot row.

These ratios correspond to the values of C (the variable entering the solution mix) at points F and C seen earlier in Figure M7.2. The S_2 row has the smallest ratio, so variable S_2 will leave the basis (and will become a nonbasic variable equal to zero) and will be replaced by C (which will have a value of 40). The new pivot row, pivot column, and pivot number are all shown in Table M7.5.

The pivot row for the third tableau is replaced here.

Step 3. The pivot row is replaced by dividing every number in it by the (circled) pivot number. Since every number is divided by 1, there is no change:

$$\frac{0}{1} = 0 \quad \frac{1}{1} = 1 \quad \frac{-2}{1} = -2 \quad \frac{1}{1} = 1 \quad \frac{40}{1} = 40$$

The entire new C row looks like this:

C_j	SOLUTION MIX	T	C	S_1	S_2	QUANTITY
\$50	C	0	1	-2	1	40

It will be placed in the new simplex tableau in the same row position that S_2 was in before (see Table M7.5).

Step 4. The new values for the T row may now be computed:

$$\begin{pmatrix} \text{Number} \\ \text{in new} \\ T \text{ row} \end{pmatrix} = \begin{pmatrix} \text{Number} \\ \text{in old} \\ T \text{ row} \end{pmatrix} - \left[\begin{pmatrix} \text{Number} \\ \text{above pivot} \\ \text{number} \end{pmatrix} \times \begin{pmatrix} \text{Corresponding} \\ \text{number in new} \\ C \text{ row} \end{pmatrix} \right]$$

$$\begin{array}{rclclcl} 1 & = & 1 & - & (0.5) & \times & (0) \\ 0 & = & 0.5 & - & (0.5) & \times & (1) \\ 1.5 & = & 0.5 & - & (0.5) & \times & (-2) \\ -0.5 & = & 0 & - & (0.5) & \times & (1) \\ 30 & = & 50 & - & (0.5) & \times & (40) \end{array}$$

The new T row is computed here.

Hence, the new T row will appear in the third tableau in the following position:

C_j	SOLUTION MIX	T	C	S_1	S_2	QUANTITY
\$70	T	1	0	1.5	-0.5	30
\$50	C	0	1	-2	1	40

The final step is again computing the Z_j and $C_j - Z_j$ values.

Step 5. Finally, the Z_j and $C_j - Z_j$ rows are calculated for the third tableau:

$$Z_j \text{ (for } T \text{ column)} = (\$70)(1) + (\$50)(0) = \$70$$

$$Z_j \text{ (for } C \text{ column)} = (\$70)(0) + (\$50)(1) = \$50$$

$$Z_j \text{ (for } S_1 \text{ column)} = (\$70)(1.5) + (\$50)(-2) = \$5$$

$$Z_j \text{ (for } S_2 \text{ column)} = (\$70)(-0.5) + (\$50)(1) = \$15$$

$$Z_j \text{ (for total profit)} = (\$70)(30) + (\$50)(40) = \$4,100$$

The net profit per unit row appears as follows:

	COLUMN			
	T	C	S_1	S_2
C_j for column	\$70	\$50	\$0	\$0
Z_j for column	\$70	\$50	\$5	\$15
$C_j - Z_j$ for column	\$0	\$0	-\$5	-\$15

An optimal solution is reached because all $C_j - Z_j$ values are zero or negative.

All results for the third iteration of the simplex method are summarized in Table M7.6. Note that since every number in the tableau's $C_j - Z_j$ row is 0 or negative, an optimal solution has been reached.

That solution is

$$T = 30 \text{ tables}$$

$$C = 40 \text{ chairs}$$

$$S_1 = 0 \text{ slack hours in the painting department}$$

$$S_2 = 0 \text{ slack hours in the carpentry department}$$

$$\text{profit} = \$4,100 \text{ for the optimal solution}$$

The final solution is to make 30 tables and 40 chairs at a profit of \$4,100. This is the same as the graphical solution presented earlier.

TABLE M7.6

Final Simplex Tableau for the Flair Furniture Problem

C_j		\$70	\$50	\$0	\$0	
	SOLUTION MIX	T	C	S_1	S_2	QUANTITY
\$70	T	1	0	1.5	-0.5	30
\$50	C	0	1	-2	1	40
	Z_j	\$70	\$50	\$5	\$15	\$4,100
	$C_j - Z_j$	\$0	\$0	-\$5	-\$15	

T and C are the final basic variables, and S_1 and S_2 are nonbasic (and thus automatically equal to 0). This solution corresponds to corner point C in Figure M7.2.

It's always possible to make an arithmetic error when you are going through the numerous simplex steps and iterations, so it is a good idea to verify your final solution. This can be done in part by looking at the original Flair Furniture Company constraints and objective function:

First constraint: $2T + 1C \leq 100$ painting department hours

$$2(30) + 1(40) \leq 100$$

$$100 \leq 100 \checkmark$$

Second constraint: $4T + 3C \leq 240$ carpentry department hours

$$4(30) + 3(40) \leq 240$$

$$240 \leq 240 \checkmark$$

Objective function: profit = $\$70T + \$50C$

$$= \$70(30) + \$50(40)$$

$$= \$4,100$$

Verifying that the solution does not violate any of the original constraints is a good way to check that no mathematical errors were made.

M7.6 Review of Procedures for Solving LP Maximization Problems

Before moving on to other issues concerning the simplex method, let's review briefly what we've learned so far for LP maximization problems.

- I. Formulate the LP problem's objective function and constraints.
- II. Add slack variables to each less-than-or-equal-to constraint and to the problem's objective function.
- III. Develop an initial simplex tableau with slack variables in the basis and the decision variables set equal to 0. Compute the Z_j and $C_j - Z_j$ values for this tableau.
- IV. Follow these five steps until an optimal solution has been reached:
 1. Choose the variable with the greatest positive $C_j - Z_j$ to enter the solution. This is the pivot column.
 2. Determine the solution mix variable to be replaced and the pivot row by selecting the row with the smallest (nonnegative) ratio of the quantity-to-pivot column substitution rate. This row is the pivot row.
 3. Calculate the new values for the pivot row.
 4. Calculate the new values for the other row(s).
 5. Calculate the Z_j and $C_j - Z_j$ values for this tableau. If there are any $C_j - Z_j$ numbers greater than 0, return to step 1. If there are no $C_j - Z_j$ numbers that are greater than 0, an optimal solution has been reached.

Here is a review of the five simplex steps.

M7.7 Surplus and Artificial Variables

To handle \geq and $=$ constraints, the simplex method makes a conversion like it made to \leq constraints.

Up to this point in the chapter, all of the LP constraints you have seen were of the less-than-or-equal-to (\leq) variety. Just as common in real-life problems—especially in LP minimization problems—are greater-than-or-equal-to (\geq) constraints and equalities. To use the simplex method, each of these must be converted to a special form also. If they are not, the simplex technique is unable to set up an initial solution in the first tableau.

Before moving on to the next section of this chapter, which deals with solving LP minimization problems with the simplex method, we take a look at how to convert a few typical constraints:

$$\text{Constraint 1: } 5X_1 + 10X_2 + 8X_3 \geq 210$$

$$\text{Constraint 2: } 25X_1 + 30X_2 = 900$$

Surplus Variables

We subtract a surplus variable to form an equality when dealing with a \geq constraint.

Greater-than-or-equal-to (\geq) constraints, such as constraint 1 as just described, require a different approach than do the less-than-or-equal-to (\leq) constraints we saw in the Flair Furniture problem. They involve the subtraction of a *surplus variable* rather than the addition of a slack variable. The surplus variable tells us how much the solution exceeds the constraint amount. Because of its analogy to a slack variable, surplus is sometimes simply called *negative slack*. To convert the first constraint, we begin by subtracting a surplus variable, S_1 , to create an equality:

$$\text{Constraint 1 rewritten: } 5X_1 + 10X_2 + 8X_3 - S_1 = 210$$

If, for example, a solution to an LP problem involving this constraint is $X_1 = 20$, $X_2 = 8$, $X_3 = 5$, the amount of surplus could be computed as follows:

$$\begin{aligned} 5X_1 + 10X_2 + 8X_3 - S_1 &= 210 \\ 5(20) + 10(8) + 8(5) - S_1 &= 210 \\ 100 + 80 + 40 - S_1 &= 210 \\ -S_1 &= 210 - 220 \\ S_1 &= 10 \text{ surplus units} \end{aligned}$$

There is one more step, however, in preparing a \geq constraint for the simplex method.

Artificial Variables

There is one small problem in trying to use the first constraint (as it has just been rewritten) in setting up an initial simplex solution. Since all “real” variables such as X_1 , X_2 , and X_3 are set to 0 in the initial tableau, S_1 takes on a negative value:

$$\begin{aligned} 5(0) + 10(0) + 8(0) - S_1 &= 210 \\ 0 - S_1 &= 210 \\ S_1 &= -210 \end{aligned}$$

All variables in LP problems, be they real, slack, or surplus, *must* be nonnegative at all times. If $S_1 = -210$, this important condition is violated.

Artificial variables are needed in \geq and $=$ constraints.

To resolve the situation, we introduce one last kind of variable, called an *artificial variable*. We simply add the artificial variable, A_1 , to the constraint as follows:

$$\text{Constraint 1 completed: } 5X_1 + 10X_2 + 8X_3 - S_1 + A_1 = 210$$

Now, not only the X_1 , X_2 , and X_3 variables may be set to 0 in the initial simplex solution, but the S_1 surplus variable as well. This leaves us with $A_1 = 210$.

Let’s turn our attention to constraint 2 for a moment. This constraint is already an equality, so why worry about it? To be included in the initial simplex solution, it turns out, even an equality must have an artificial variable added to it:

$$\text{Constraint 2 rewritten: } 25X_1 + 30X_2 + A_2 = 900$$

The reason for inserting an artificial variable into an equality constraint deals with the usual problem of finding an initial LP solution. In a simple constraint such as number 2, it's easy to guess that $X_1 = 0$, $X_2 = 30$ would yield an initial feasible solution. But what if our problem had 10 equality constraints, each containing seven variables? It would be *extremely* difficult to sit down and “eyeball” a set of initial solutions. By adding artificial variables, such as A_2 , we can provide an automatic initial solution. In this case, when X_1 and X_2 are set equal to 0, $A_2 = 900$.

Artificial variables have no meaning in a physical sense and are nothing more than computational tools for generating initial LP solutions. If an artificial variable has a positive (nonzero) value, then the original constraint where this artificial variable was added has not been satisfied. A feasible solution has been found when all artificial variables are equal to zero, indicating all constraints have been met. Before the final simplex solution has been reached, all artificial variables must be gone from the solution mix. This matter is handled through the problem's objective function.

Surplus and Artificial Variables in the Objective Function

Whenever an artificial or surplus variable is added to one of the constraints, it must also be included in the other equations and in the problem's objective function, just as was done for slack variables. Since artificial variables must be forced out of the solution, we can assign a very high C_j cost to each. In minimization problems, variables with *low* costs are the most desirable ones and the first to enter the solution. Variables with *high* costs leave the solution quickly, or never enter it at all. Rather than set an actual dollar figure of \$10,000 or \$1 million for each artificial variable, however, we simply use the letter M to represent a very large number.³ Surplus variables, like slack variables, carry a zero cost. In maximization problems, we use negative M .

If a problem had an objective function that read

$$\text{Minimize cost} = \$5X_1 + \$9X_2 + \$7X_3$$

and constraints such as the two mentioned previously, the completed objective function and constraints would appear as follows:

$$\begin{array}{ll} \text{Minimize cost} = & \$5X_1 + \$9X_2 + \$7X_3 + \$0S_1 + \$MA_1 + \$MA_2 \\ \text{subject to} & 5X_1 + 10X_2 + 8X_3 - 1S_1 + 1A_1 + 0A_2 = 210 \\ & 25X_1 + 30X_2 + 0X_3 + 0S_1 + 0A_1 + 1A_2 = 900 \end{array}$$

M7.8 Solving Minimization Problems

Now that we have discussed how to deal with objective functions and constraints associated with minimization problems, let's see how to use the simplex method to solve a typical problem.

The Muddy River Chemical Company Example

The Muddy River Chemical Corporation must produce exactly 1,000 pounds of a special mixture of phosphate and potassium for a customer. Phosphate costs \$5 per pound and potassium costs \$6 per pound. No more than 300 pounds of phosphate can be used, and at least 150 pounds of potassium must be used. The problem is to determine the least-cost blend of the two ingredients.

³A technical point: If an artificial variable is ever used in a *maximization* problem (an occasional event), it is assigned an objective function value of $-M$ to force it from the basis.

Artificial variables have no physical meaning and drop out of the solution mix before the final tableau.

To make sure that an artificial variable is forced out before the final solution is reached, it is assigned a very high cost (M).



IN ACTION

Linear Programming Modeling in the Forests of Chile

Faced with a series of challenges in making short-term harvesting decisions, Forestal Arauco, a Chilean forestry firm, turned to professors at the University of Chile for LP modeling help. One of the problems in short-term harvesting of trees is to match demand of products—defined by length and diameter—with the supply of standing timber.

The manual system used at the time by foresters led to a significant amount of waste of timber, where higher diameter logs, suited for export or sawmills, ended up being used for pulp, with a considerable loss in value. An LP model, labeled OPTICORT by the professors, was the logical way to get better schedules.

“The system not only optimized the operational decisions in harvesting, but also changed the way managers looked at the

problem,” says Professor Andres Weintraub. “The model and its concepts became the natural language to discuss the operations. They had to negotiate the parameters, and the model would do the dirty work. The system had to run in a few minutes to allow discussion and negotiation; that was a critical feature for the success of this tool,” he adds.

The LP program took about two years to develop, and the researchers were careful to observe two cardinal rules: (1) The solution approach had to be comfortable and clear to the user, and (2) the system had to provide answers to the user in a fast development, so the user could see quick improvements.

Source: Based on J. Summerour. “Chilean Forestry Firm a ‘Model’ of Success,” *OR/MS Today* (April 1999): 22–23.

Here is the mathematical formulation of the minimization problem for Muddy River Chemical Corp.

This problem may be restated mathematically as

$$\begin{aligned} \text{Minimize cost} &= \$5X_1 + \$6X_2 \\ \text{subject to } X_1 + X_2 &= 1,000 \text{ lb} \\ X_1 &\leq 300 \text{ lb} \\ X_2 &\geq 150 \text{ lb} \\ X_1, X_2 &\geq 0 \end{aligned}$$

where

X_1 = number of pounds of phosphate

X_2 = number of pounds of potassium

Note that there are three constraints, not counting the nonnegativity constraints; the first is an equality, the second a less-than-or-equal-to, and the third a greater-than-or-equal-to constraint.

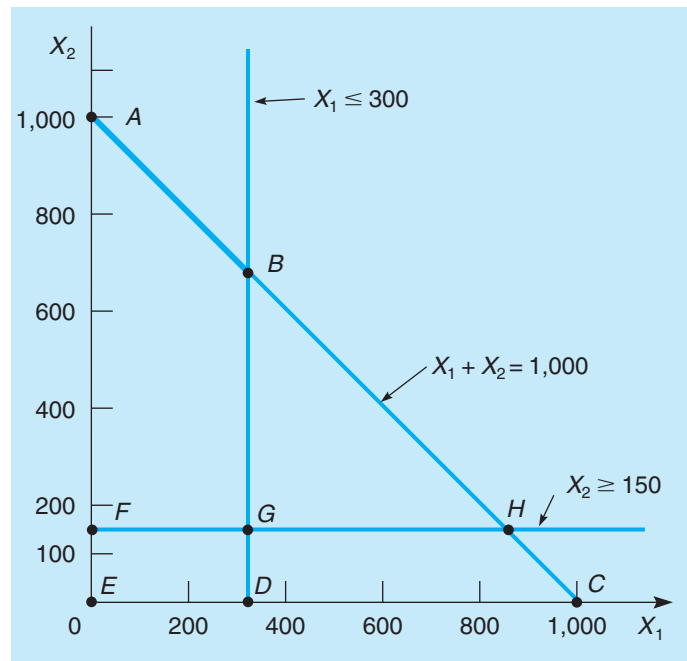
Graphical Analysis

To have a better understanding of the problem, a brief graphical analysis may prove useful. There are only two decision variables, X_1 and X_2 , so we are able to plot the constraints and feasible region. Because the first constraint, $X_1 + X_2 = 1,000$, is an equality, the solution must lie somewhere on the line ABC (see Figure M7.3). It must also lie between points A and B because of the constraint $X_1 \leq 300$. The third constraint, $X_2 \geq 150$, is actually redundant and nonbinding since X_2 will automatically be greater than 150 pounds if the first two constraints are observed. Hence, the feasible region consists of all points on the line segment AB . As you recall from Chapter 7, however, an optimal solution will always lie at a corner point of the feasible region (even if the region is only a straight line). The solution must therefore be either at point A or point B . A quick analysis reveals that the least-cost solution lies at corner B , namely $X_1 = 300$ pounds of phosphate, $X_2 = 700$ pounds of potassium. The total cost is \$5,700.

Looking at a graphical solution first will help us understand the steps in the simplex method.

FIGURE M7.3

Muddy River Chemical Corporation's Feasible Region Graph



You don't need the simplex method to solve the Muddy River Chemical problem, of course. But we can guarantee you that few problems will be this simple. In general, you can expect to see several variables and many constraints. The purpose of this section is to illustrate the straightforward application of the simplex method to minimization problems. When the simplex procedure is used to solve this, it will methodically move from corner point to corner point until the optimal solution is reached. In Figure M7.3, the simplex method will begin at point *E*, then move to point *F*, then to point *G*, and finally to point *B*, which is the optimal solution.

Converting the Constraints and Objective Function

The first step is to apply what we learned in the preceding section to convert the constraints and objective function into the proper form for the simplex method. The equality constraint, $X_1 + X_2 = 1,000$, just involves adding an artificial variable, A_1 :

$$X_1 + X_2 + A_1 = 1,000$$

The second constraint, $X_1 \leq 300$, requires the insertion of a slack variable—let's call it S_1 :

$$X_1 + S_1 = 300$$

The last constraint is $X_2 \geq 150$, which is converted to an equality by subtracting a surplus variable, S_2 , and adding an artificial variable, A_2 :

$$X_2 - S_2 + A_2 = 150$$

Finally, the objective function, $\text{cost} = \$5X_1 + \$6X_2$, is rewritten as

$$\text{Minimize cost} = \$5X_1 + \$6X_2 + \$0S_1 + \$0S_2 + \$MA_1 + \$MA_2$$

First, insert slack, surplus, and artificial variables. This makes it easier to set up the initial simplex tableau in Table M7.7.

The complete set of constraints can now be expressed as follows:

$$1X_1 + 1X_2 + 0S_1 + 0S_2 + 1A_1 + 0A_2 = 1,000$$

$$1X_1 + 0X_2 + 1S_1 + 0S_2 + 0A_1 + 0A_2 = 300$$

$$0X_1 + 1X_2 + 0S_1 - 1S_2 + 0A_1 + 1A_2 = 150$$

$$X_1, X_2, S_1, S_2, A_1, A_2 \geq 0$$

Rules of the Simplex Method for Minimization Problems

The minimization simplex rules are slightly different. Now, the new variable to enter the solution mix will be in the column with the negative $C_j - Z_j$ value indicating the greatest improvement.

Minimization problems are quite similar to the maximization problems tackled earlier in this chapter. The significant difference involves the $C_j - Z_j$ row. Our objective is to minimize cost, and a negative $C_j - Z_j$ value indicates that the total cost will decrease if that variable is selected to enter the solution. Thus, the new variable to enter the solution in each tableau (the pivot column variable) will be the one with a negative $C_j - Z_j$ that gives the largest improvement. We choose the variable that decreases costs the most. In minimization problems, an optimal solution is reached when all the numbers in the $C_j - Z_j$ row are 0 or *positive*—just the opposite from the maximization case.⁴ All other simplex steps, as seen in the following, remain the same.

Steps for Simplex Minimization Problems

1. Choose the variable with a negative $C_j - Z_j$ that indicates the largest decrease in cost to enter the solution. The corresponding column is the pivot column.
2. Determine the row to be replaced by selecting the one with the smallest (nonnegative) quantity-to-pivot column substitution rate ratio. This is the pivot row.
3. Calculate new values for the pivot row.
4. Calculate new values for the other rows.
5. Calculate the Z_j and $C_j - Z_j$ values for this tableau. If there are any $C_j - Z_j$ numbers less than 0, return to step 1.

First Simplex Tableau for the Muddy River Chemical Corporation Problem

Now we solve Muddy River Chemical Corporation's LP formulation using the simplex method. The initial tableau is set up just as in the earlier maximization example. Its first three rows are shown in the accompanying table. We note the presence of the \$*M* costs associated with artificial variables A_1 and A_2 , but we treat them as if they were any large number. As noted earlier, they have the effect of forcing the artificial variables out of the solution quickly because of their large costs.

⁴We should note that there is a second way to solve minimization problems with the simplex method: It involves a simple mathematical trick. It happens that minimizing the cost objective is the same as *maximizing* the negative of the cost objective function. This means that instead of writing the Muddy River objective function as

$$\text{Minimize cost} = 5X_1 + 6X_2$$

we can instead write

$$\text{Maximize } (-\text{Cost}) = -5X_1 - 6X_2$$

The solution that maximizes $(-\text{Cost})$ also minimizes cost. It also means that the same simplex procedure shown earlier for maximization problems can be used if this trick is employed. The only change is that the objective function must be multiplied by (-1) .

C_j	SOLUTION MIX	X_1	X_2	S_1	S_2	A_1	A_2	QUANTITY
\$M	A_1	1	1	0	0	1	0	1,000
\$0	S_1	1	0	1	0	0	0	300
\$M	A_2	0	1	0	-1	0	1	150

The numbers in the Z_j row are computed by multiplying the C_j column on the far left of the tableau times the corresponding numbers in each other column. They are then entered in Table M7.7:

$$\begin{aligned}
 Z_j \text{ (for } X_1 \text{ column)} &= \$M(1) + \$0(1) + M(0) = \$M \\
 Z_j \text{ (for } X_2 \text{ column)} &= \$M(1) + \$0(0) + \$M(1) = \$2M \\
 Z_j \text{ (for } S_1 \text{ column)} &= \$M(0) + \$0(1) + \$M(0) = \$0 \\
 Z_j \text{ (for } S_2 \text{ column)} &= \$M(0) + \$0(0) + \$M(-1) = -\$M \\
 Z_j \text{ (for } A_1 \text{ column)} &= \$M(1) + \$0(0) + \$M(0) = \$M \\
 Z_j \text{ (for } A_2 \text{ column)} &= \$M(0) + \$0(0) + \$M(1) = \$M \\
 Z_j \text{ (for total cost)} &= \$M(1,000) + \$0(300) + \$M(150) = \$1,150M
 \end{aligned}$$

The $C_j - Z_j$ entries are determined as follows:

	COLUMN					
	X_1	X_2	S_1	S_2	A_1	A_2
C_j for column	\$5	\$6	\$0	\$0	\$M	\$M
Z_j for column	$\underline{\$M}$	$\underline{\$2M}$	$\underline{\$0}$	$\underline{-\$M}$	$\underline{\$M}$	$\underline{\$M}$
$C_j - Z_j$ for column	$-\$M + \5	$-\$2M + \6	$\$0$	$\$M$	$\$0$	$\$0$

TABLE M7.7
Initial Simplex
Tableau for the
Muddy River
Chemical Corporation
Problem

C_j		\$5	\$6	\$0	\$0	\$M	\$M	
	SOLUTION MIX	X_1	X_2	S_1	S_2	A_1	A_2	QUANTITY
\$M	A_1	1	1	0	0	1	0	1,000
\$0	S_1	1	0	1	0	0	0	300
\$M	A_2	0	①	0	-1	0	1	150 ← Pivot row
			Pivot number					
Z_j		\$M	\$2M	0	-\$M	\$M	\$M	\$1,150M
$C_j - Z_j$		$-\$M + 5$	$-\$2M + 6$	\$0	\$M	\$0	0	(total cost)
			Pivot column					

Here is the initial simplex solution.

This initial solution was obtained by letting each of the variables X_1 , X_2 , and S_2 assume a value of 0. The current basic variables are $A_1 = 1,000$, $S_1 = 300$, and $A_2 = 150$. This complete solution could be expressed in vector, or column, form as

$$\begin{bmatrix} X_1 \\ X_2 \\ S_1 \\ S_2 \\ A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 300 \\ 0 \\ 1,000 \\ 150 \end{bmatrix}$$

An extremely high cost, \$1,150M from Table M7.7, is associated with this answer. We know that this can be reduced significantly and now move on to the solution procedures.

Developing a Second Tableau

We examine whether the current solution is optimal by looking at the $C_j - Z_j$ row.

In the $C_j - Z_j$ row of Table M7.7, we see that there are two entries with negative values, X_1 and X_2 . In the simplex rules for minimization problems, this means that an optimal solution does not yet exist. The pivot column is the one with the *negative* entry in the $C_j - Z_j$ row that indicates the largest improvement—shown in Table M7.7 as the X_2 column, which means that X_2 will enter the solution next.

Which variable will leave the solution to make room for the new variable, X_2 ? To find out, we divide the elements of the quantity column by the respective pivot column substitution rates:

$$\text{For the } A_1 \text{ row} = \frac{1,000}{1} = 1,000$$

$$\text{For the } S_1 \text{ row} = \frac{300}{0} \quad (\text{this is an undefined ratio, so we ignore it})$$

$$\text{For the } A_2 \text{ row} = \frac{150}{1} = 150 \quad (\text{smallest quotient, indicating pivot row})$$

A_2 is the pivot row because 150 is the smallest quotient.

Hence, the pivot row is the A_2 row, and the pivot number (circled) is at the intersection of the X_2 column and the A_2 row.

The entering row for the next simplex tableau is found by dividing each element in the pivot row by the pivot number, 1. This leaves the old pivot row unchanged, except that it now represents the solution variable X_2 . The other two rows are altered one at a time by again applying the formula shown earlier in step 4:

$$(\text{New row numbers}) = (\text{Numbers in old row})$$

$$- \left[\left(\frac{\text{Number above or below}}{\text{pivot number}} \right) \times \left(\frac{\text{Corresponding number}}{\text{in newly replaced row}} \right) \right]$$

$A_1 \text{ Row}$	$S_1 \text{ Row}$
$1 = 1 - (1)(0)$	$1 = 1 - (0)(0)$
$0 = 1 - (1)(1)$	$0 = 0 - (0)(1)$
$0 = 0 - (1)(0)$	$1 = 1 - (0)(0)$
$1 = 0 - (1)(-1)$	$0 = 0 - (0)(-1)$
$1 = 1 - (1)(0)$	$0 = 0 - (0)(0)$
$-1 = 0 - (1)(1)$	$0 = 0 - (0)(1)$
$850 = 1,000 - (1)(150)$	$300 = 300 - (0)(150)$

TABLE M7.8
Second Simplex
Tableau for the
Muddy River
Chemical Corporation
Problem

$C_j \rightarrow$		\$5	\$6	\$0	\$0	\$M	\$M	
\downarrow	SOLUTION MIX	X_1	X_2	S_1	S_2	A_1	A_2	QUANTITY
\$M	A_1	1	0	0	1	1	-1	850
\$0	S_1	①	0	1	0	0	0	300 ← Pivot row
								Pivot number
\$6	X_2	0	1	0	-1	0	1	150
	Z_j	\$M	\$6	\$0	$M - 6$	\$M	$-M + 6$	$\$850M + \900
	$C_j - Z_j$	$-\$M + 5$	\$0	\$0	$-\$M + 6$	\$0	$\$2M - 6$	
								Pivot column

The Z_j and $C_j - Z_j$ rows are computed next:

$$\begin{aligned}
 Z_j \text{ (for } X_1) &= \$M(1) + \$0(1) + \$6(0) = \$M \\
 Z_j \text{ (for } X_2) &= \$M(0) + \$0(0) + \$6(1) = \$6 \\
 Z_j \text{ (for } S_1) &= \$M(0) + \$0(1) + \$6(0) = \$0 \\
 Z_j \text{ (for } S_2) &= \$M(1) + \$0(0) + \$6(-1) = \$M - 6 \\
 Z_j \text{ (for } A_1) &= \$M(1) + \$0(0) + \$6(0) = \$M \\
 Z_j \text{ (for } A_2) &= \$M(-1) + \$0(0) + \$6(1) = -\$M + 6 \\
 Z_j \text{ (for total cost)} &= \$M(850) + \$0(300) + \$6(150) = -\$850M + 900
 \end{aligned}$$

	COLUMN					
	X_1	X_2	S_1	S_2	A_1	A_2
C_j for column	\$5	\$6	\$0	\$0	\$M	\$M
Z_j for column	<u>\$M</u>	<u>\$6</u>	<u>\$0</u>	<u>$M - 6$</u>	<u>\$M</u>	<u>$-\\$M + 6$</u>
$C_j - Z_j$ for column	$-\$M + 5$	\$0	\$0	$-\$M + 6$	\$0	$\$2M - 6$

All of these computational results are presented in Table M7.8.

The solution at the end of the second tableau (point F in Figure M7.3) is $A_1 = 850$, $S_1 = 300$, $X_2 = 150$. X_1 , S_2 , and A_2 are currently the nonbasic variables and have zero value. The cost at this point is still quite high, $\$850M + \900 . This answer is not optimal because not every number in the $C_j - Z_j$ row is zero or positive.

Developing a Third Tableau

The new pivot column is the X_1 column. To determine which variable will leave the basis to make room for X_1 , we check the *quantity column-to-pivot column* ratios again:

$$\text{For the } A_1 \text{ row} = \frac{850}{1} = 850$$

$$\text{For the } S_1 \text{ row} = \frac{300}{1} = 300 \quad (\text{smallest ratio})$$

$$\text{For the } X_2 \text{ row} = \frac{150}{0} = \text{undefined}$$

The solution after the second tableau is still not optimal.

The third tableau is developed in this section.

TABLE M7.9
Third Simplex
Tableau for the
Muddy River
Chemical Corporation
Problem

$C_j \rightarrow$		\$5	\$6	\$0	\$0	\$M	\$M	
\downarrow	SOLUTION MIX	X_1	X_2	S_1	S_2	A_1	A_2	QUANTITY
\$M	A_1	0	0	-1	1	1	-1	550
\$5	X_1	1	0	1	0	0	0	300
\$6	X_2	0	1	0	-1	0	1	150
	Z_j	\$5	\$6	$-\$M + 5$	$\$M - 6$	$\$M$	$-\$M + 6$	$\$550M + 2,400$
	$C_j - Z_j$	\$0	\$0	$\$M - 5$	$-\$M + 6$	\$0	$\$2M - 6$	

Hence, variable S_1 will be replaced by X_1 .⁵ The pivot number, row, and column are labeled in Table M7.8.

To replace the pivot row, we divide each number in the S_1 row by 1 (the circled pivot number), leaving the row unchanged. The new X_1 row is shown in Table M7.9. The other computations for this third simplex tableau follow:

Here are the computations for the fourth solution.

A_1 Row	S_1 Row
$0 = 1 - (1)(1)$	$0 = 0 - (0)(1)$
$0 = 0 - (1)(0)$	$1 = 1 - (0)(0)$
$-1 = 0 - (1)(1)$	$0 = 0 - (0)(1)$
$1 = 1 - (1)(0)$	$-1 = -1 - (0)(0)$
$1 = 1 - (1)(0)$	$0 = 0 - (0)(0)$
$-1 = -1 - (1)(0)$	$1 = 1 - (0)(0)$
$550 = 850 - (1)(300)$	$150 = 150 - (0)(300)$

The Z_j and $C_j - Z_j$ rows are computed next:

$$\begin{aligned}
 Z_j \text{ (for } X_1) &= \$M(0) + \$5(1) + \$6(0) = \$5 \\
 Z_j \text{ (for } X_2) &= \$M(0) + \$5(0) + \$6(1) = \$6 \\
 Z_j \text{ (for } S_1) &= \$M(-1) + \$5(1) + \$6(0) = -\$M + 5 \\
 Z_j \text{ (for } S_2) &= \$M(1) + \$5(0) + \$6(-1) = \$M - 6 \\
 Z_j \text{ (for } A_1) &= \$M(1) + \$5(0) + \$6(0) = \$M \\
 Z_j \text{ (for } A_2) &= \$M(-1) + \$5(0) + \$6(1) = -\$M + 6 \\
 Z_j \text{ (for total cost)} &= \$M(550) + \$5(300) + \$6(150) = \$550M + 2,400
 \end{aligned}$$

	COLUMN					
	X_1	X_2	S_1	S_2	A_1	A_2
C_j for column	\$5	\$6	\$0	\$0	$\$M$	$\$M$
Z_j for column	\$5	\$6	$-\$M + 5$	$\$M - 6$	$\$M$	$-\$M + 6$
$C_j - Z_j$ for column	\$0	\$0	$\$M - 5$	$-\$M + 6$	\$0	$\$2M - 6$

⁵ At this point, it might appear to be more cost-effective to replace the A_1 row instead of the S_1 row. This would remove the last artificial variable, and its large $\$M$ cost, from the basis. The simplex method, however, does not always pick the most direct route to reaching the final solution. You may be assured, though, that it *will* lead us to the correct answer. In Figure M7.3, this would involve moving to point H instead of point G .

The third solution is still not optimal.

The solution at the end of the three iterations (point *G* in Figure M7.3) is still not optimal because the S_2 column contains a $C_j - Z_j$ value that is negative. Note that the current total cost is nonetheless lower than at the end of the second tableau, which in turn is lower than the initial solution cost. We are headed in the right direction but have one more tableau to go!

Fourth Tableau for the Muddy River Chemical Corporation Problem

The pivot column is now the S_2 column. The ratios that determine the row and variable to be replaced are computed as follows:

For the A_1 row: $\frac{550}{1} = 550$ (row to be replaced)

For the X_1 row: $\frac{300}{0}$ (undefined)

For the X_2 row: $\frac{150}{-1}$ (not considered because it is negative)

Here are the computations for the fourth solution.

Each number in the pivot row is divided by the pivot number (again 1, by coincidence). The other two rows are computed as follows and are shown in Table M7.10:

$$\begin{aligned} 1 &= 1 - (0)(0) \\ 0 &= 0 - (0)(0) \\ 1 &= 1 - (0)(-1) \\ 0 &= 0 - (0)(1) \\ 0 &= 0 - (0)(1) \\ 0 &= 0 - (0)(-1) \\ 300 &= 300 - (0)(550) \end{aligned}$$

$$\begin{aligned} 0 &= 0 - (-1)(0) \\ 1 &= 1 - (-1)(0) \\ -1 &= 0 - (-1)(-1) \\ 0 &= -1 - (-1)(1) \\ 1 &= 0 - (-1)(1) \\ 0 &= 1 - (-1)(-1) \\ 700 &= 150 - (-1)(550) \end{aligned}$$

$$\begin{aligned} Z_j \text{ (for } X_1) &= \$0(0) + \$5(1) + \$6(0) = \$5 \\ Z_j \text{ (for } X_2) &= \$0(0) + \$5(0) + \$6(1) = \$6 \\ Z_j \text{ (for } S_1) &= \$0(-1) + \$5(1) + \$6(-1) = -\$1 \\ Z_j \text{ (for } S_2) &= \$0(1) + \$5(0) + \$6(0) = \$0 \\ Z_j \text{ (for } A_1) &= \$0(1) + \$5(0) + \$6(1) = \$6 \\ Z_j \text{ (for } A_2) &= \$0(-1) + \$5(0) + \$6(0) = \$0 \\ Z_j \text{ (for total cost)} &= \$0(550) + \$5(300) + \$6(700) = \$5,700 \end{aligned}$$

	COLUMN					
	X_1	X_2	S_1	S_2	A_1	A_2
C_j for column	\$5	\$6	\$0	\$0	$\$M$	$\$M$
Z_j for column	<u>\$5</u>	<u>\$6</u>	<u>-\$1</u>	<u>\$0</u>	<u>\$6</u>	<u>\$0</u>
$C_j - Z_j$ for column	\$0	\$0	\$1	\$0	$\$M - 6$	$\$M$

TABLE M7.10
Fourth and Optimal
Solution to the
Muddy River
Chemical Corporation
Problem

C_j	SOLUTION MIX	\rightarrow \$5	\$6	\$0	\$0	\$M	\$M	QUANTITY
		X_1	X_2	S_1	S_2	A_1	A_2	
\$0	S_2	0	0	-1	1	1	-1	550
\$5	X_1	1	0	1	0	0	0	300
\$6	X_2	0	1	-1	0	1	0	700
	Z_j	\$5	\$6	-\$1	\$0	\$6	\$0	\$5,700
	$C_j - Z_j$	\$0	\$0	\$1	\$0	$M - 6$	M	

The optimal solution has been reached because only positive or zero values appear in the $C_j - Z_j$ row.

On examining the $C_j - Z_j$ row in Table M7.10, only positive or 0 values are found. The fourth tableau therefore contains the optimum solution. That solution is $X_1 = 300$, $X_2 = 700$, $S_2 = 550$. The artificial variables are both equal to 0, as is S_1 . Translated into management terms, the chemical company's decision should be to blend 300 pounds of phosphate (X_1) with 700 pounds of potassium (X_2). This provides a surplus (S_2) of 550 pounds of potassium more than required by the constraint $X_2 \geq 150$. The cost of this solution is \$5,700. If you look back to Figure M7.3, you can see that this is identical to the answer found by the graphical approach.

Although small problems such as this can be solved graphically, more realistic product blending problems demand use of the simplex method, usually in computerized form.

M7.9 Review of Procedures for Solving LP Minimization Problems

Just as we summarized the steps for solving LP maximization problems with the simplex method in Section M7.6, let us do so for minimization problems here:

- I. Formulate the LP problem's objective function and constraints.
- II. Include slack variables in each less-than-or-equal-to constraint, artificial variables in each equality constraint, and both surplus and artificial variables in each greater-than-or-equal-to constraint. Then add all of these variables to the problem's objective function.
- III. Develop an initial simplex tableau with artificial and slack variables in the basis and the other variables set equal to 0. Compute the Z_j and $C_j - Z_j$ values for this tableau.
- IV. Follow these five steps until an optimal solution has been reached:
 1. Choose the variable with the negative $C_j - Z_j$ indicating the greatest improvement to enter the solution. This is the pivot column.
 2. Determine the row to be replaced by selecting the one with the smallest (nonnegative) quantity-to-pivot column substitution rate ratio. This is the pivot row.
 3. Calculate the new values for the pivot row.
 4. Calculate the new values for the other row(s).
 5. Calculate the Z_j and $C_j - Z_j$ values for the tableau. If there are any $C_j - Z_j$ numbers less than 0, return to step 1. If there are no $C_j - Z_j$ numbers that are less than 0, an optimal solution has been reached.

M7.10 Special Cases

A situation with no feasible solution may exist if the problem was formulated improperly.

No finite solution may exist in problems that are not bounded. This means that a variable can be infinitely large without violating a constraint.

In Chapter 7 we addressed some special cases that may arise when solving LP problems graphically (see Section 7.7). Here we describe these cases again, this time as they refer to the simplex method.

Infeasibility

Infeasibility, you may recall, comes about when there is no solution that satisfies all of the problem's constraints. In the simplex method, an infeasible solution is indicated by looking at the final tableau. In it, all $C_j - Z_j$ row entries will be of the proper sign to imply optimality, but an artificial variable (A_1) will still be in the solution mix.

Table M7.11 illustrates the final simplex tableau for a hypothetical minimization type of LP problem. The table provides an example of an improperly formulated problem, probably containing conflicting constraints. No feasible solution is possible because an artificial variable, A_2 , remains in the solution mix, even though all $C_j - Z_j$ are positive or 0 (the criterion for an optimal solution in a minimization case).

Unbounded Solutions

Unboundedness describes linear programs that do not have finite solutions. It occurs in maximization problems, for example, when a solution variable can be made infinitely large without violating a constraint. In the simplex method, the condition of unboundedness will be discovered prior to reaching the final tableau. We will note the problem when trying to decide which variable to remove from the solution mix. As seen earlier in this chapter, the procedure is to divide each quantity column number by the corresponding pivot column number. The row with the smallest positive ratio is replaced. But if all the ratios turn out to be negative or undefined, it indicates that the problem is unbounded.

Table M7.12 illustrates the second tableau calculated for a particular LP maximization problem by the simplex method. It also points to the condition of unboundedness. The solution is not optimal because not all $C_j - Z_j$ entries are 0 or negative, as required in a maximization problem. The next variable to enter the solution should be X_1 . To determine which variable will leave the solution, we examine the ratios of the quantity column numbers to their corresponding numbers in the X_1 , or pivot, column:

$$\begin{array}{l} \text{Ratio for the } X_2 \text{ row: } \frac{30}{-1} \\ \text{Ratio for the } S_2 \text{ row: } \frac{10}{-2} \end{array} \quad \begin{array}{l} \swarrow \\ \searrow \end{array} \quad \text{Negative ratios unacceptable}$$

Since both pivot column numbers are negative, an unbounded solution is indicated.

TABLE M7.11
Illustration of
Infeasibility

C_j		\$5	\$8	\$0	\$0	\$M	\$M	
	SOLUTION MIX	X_1	X_2	S_1	S_2	A_1	A_2	QUANTITY
\$5	X_1	1	0	-2	3	-1	0	200
\$8	X_2	0	1	1	2	-2	0	100
\$M	A_2	0	0	0	-1	-1	1	20
	Z_j	\$5	\$8	-\$2	\$31 - M	-\$21 - M	\$M	\$1,800 + 20M
	$C_j - Z_j$	\$0	\$0	\$2	\$M - 31	\$2M + 21	\$0	

TABLE M7.12
Problem with an
Unbounded Solution

C_j	SOLUTION MIX					QUANTITY
		X_1	X_2	S_1	S_2	
\$9	X_2	-1	1	2	0	30
\$0	S_2	-2	0	-1	1	10
	Z_j	-\$9	\$9	\$18	\$0	\$270
	$C_j - Z_j$	\$15	\$0	-\$18	\$0	

↑ Pivot column

Degeneracy

Degeneracy is another situation that can occur when solving an LP problem using the simplex method. It develops when three constraints pass through a single point. For example, suppose a problem has only these three constraints $X_1 \leq 10$, $X_2 \leq 10$, and $X_1 + X_2 < 20$. All three constraint lines will pass through the point (10, 10). Degeneracy is first recognized when the ratio calculations are made. If there is a *tie* for the smallest ratio, this is a signal that degeneracy exists. As a result of this, when the next tableau is developed, one of the variables in the solution mix will have a value of zero.

Table M7.13 provides an example of a degenerate problem. At this iteration of the given maximization LP problem, the next variable to enter the solution will be X_1 , since it has the only positive $C_j - Z_j$ number. The ratios are computed as follows:

$$\text{For the } X_2 \text{ row: } \frac{10}{0.25} = 40$$

$$\text{For the } S_2 \text{ row: } \frac{20}{4} = 5 \quad \leftarrow \text{Tie for the smallest ratio indicates degeneracy}$$

$$\text{For the } S_3 \text{ row: } \frac{10}{2} = 5 \quad \leftarrow$$

Cycling may result from degeneracy.

Theoretically, degeneracy could lead to a situation known as *cycling*, in which the simplex algorithm alternates back and forth between the same nonoptimal solutions; that is, it puts a new variable in, then takes it out in the next tableau, puts it back in, and so on. One simple way of dealing with the issue is to select either row (S_2 or S_3 in this case) arbitrarily. If we are unlucky and cycling does occur, we simply go back and select the other row.

TABLE M7.13
Problem Illustrating
Degeneracy

C_j	SOLUTION MIX							QUANTITY
		X_1	X_2	X_3	S_1	S_2	S_3	
\$8	X_2	0.25	1	1	-2	0	0	10
\$0	S_2	4	0	0.33	-1	1	0	20
\$0	S_3	2	0	2	0.4	0	1	10
	Z_j	\$2	\$8	\$8	\$16	\$0	\$0	\$80
	$C_j - Z_j$	\$3	\$0	-\$6	-\$16	\$0	\$0	

↑ Pivot column

TABLE M7.14
Problem with
Alternate Optimal
Solutions

C_j	SOLUTION MIX	\$3	\$2	\$0	\$0	QUANTITY
		X_1	X_2	S_1	S_2	
\$2	X_2	1.5	1	1	0	6
\$0	S_2	1	0	0.5	1	3
	Z_j	\$3	\$2	\$2	\$0	\$12
	$C_j - Z_j$	\$0	\$0	-\$2	\$0	

Alternate optimal solutions may exist if the $C_j - Z_j$ value = 0 for a variable not in the solution mix.

More Than One Optimal Solution

Multiple, or alternate, optimal solutions can be spotted when the simplex method is being used by looking at the final tableau. If the $C_j - Z_j$ value is equal to 0 for a variable that is not in the solution mix, more than one optimal solution exists.

Let's take Table M7.14 as an example. Here is the last tableau of a maximization problem; each entry in the $C_j - Z_j$ row is 0 or negative, indicating that an optimal solution has been reached. That solution is read as $X_2 = 6$, $S_2 = 3$, profit = \$12. Note, however, that variable X_1 can be brought into the solution mix without increasing or decreasing profit. The new solution, with X_1 in the basis, would become $X_1 = 3$, $X_2 = 1.5$, with profit still at \$12. Can you modify Table M7.14 to prove this?

M7.11 Sensitivity Analysis with the Simplex Tableau

In Chapter 7 we introduce the topic of sensitivity analysis as it applies to LP problems that we have solved graphically. This valuable concept shows how the optimal solution and the value of its objective function change, given changes in various inputs to the problem. Graphical analysis is useful in understanding intuitively and visually how feasible regions and the slopes of objective functions can change as model coefficients change. Computer programs handling LP problems of all sizes provide sensitivity analysis as an important output feature. Those programs use the information provided in the final simplex tableau to compute ranges for the objective function coefficients and ranges for the RHS values. They also provide "shadow prices," a concept that we introduce in this section.

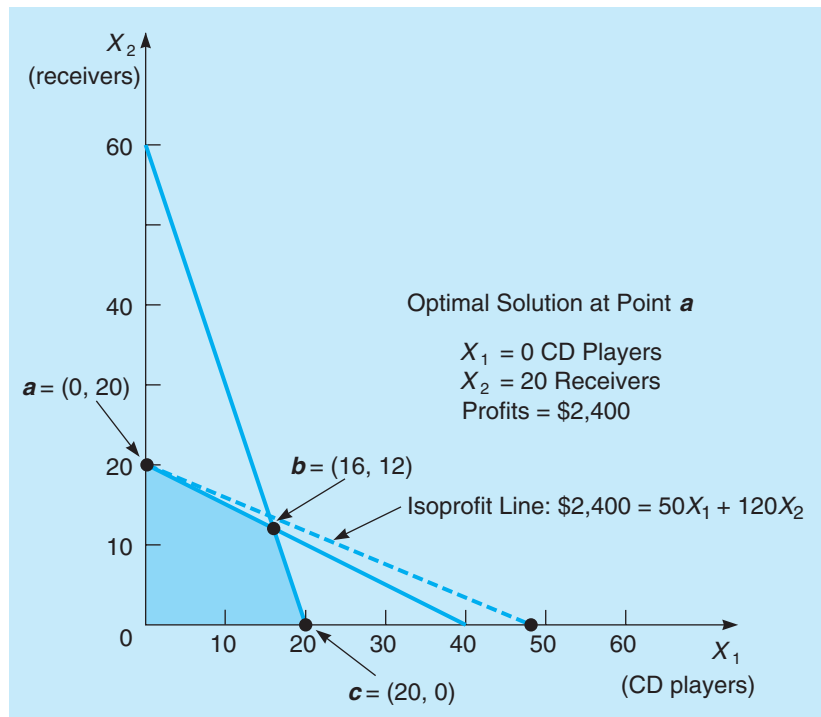
High Note Sound Company Revisited

In Chapter 7 we use the High Note Sound Company to illustrate sensitivity analysis graphically. High Note is a firm that makes compact disk (CD) players (called X_1) and stereo receivers (called X_2). Its LP formulation is repeated here:

$$\begin{aligned}
 &\text{Maximize profit} = \$50X_1 + \$120X_2 \\
 &\text{subject to} \quad 2X_1 + 4X_2 \leq 80 \quad (\text{hours of electricians' time available}) \\
 &\quad \quad \quad 3X_1 + 1X_2 \leq 60 \quad (\text{hours of audio technicians' time available})
 \end{aligned}$$

High Note's graphical solution is repeated in Figure M7.4.

FIGURE M7.4
High Note Sound
Company Graphical
Solution



Changes in the Objective Function Coefficients

In Chapter 7 we saw how to use graphical LP to examine the objective function coefficients. A second way of illustrating the sensitivity analysis of objective function coefficients is to consider the problem's final simplex tableau. For the High Note Sound Company, this tableau is shown in Table M7.15. The optimal solution is seen to be as follows:

$$\left. \begin{array}{l} X_2 = 20 \text{ stereo receivers} \\ S_2 = 40 \text{ hours of slack time of audio technicians} \end{array} \right\} \text{Basic variables}$$

$$\left. \begin{array}{l} X_1 = 0 \text{ CD players} \\ S_1 = 0 \text{ hours of slack time of electricians} \end{array} \right\} \text{Nonbasic variables}$$

Basic variables (those in the solution mix) and *nonbasic variables* (those set equal to 0) must be handled differently using sensitivity analysis. Let us first consider the case of a nonbasic variable.

TABLE M7.15
Optimal Solution by
the Simplex Method

C_j		\$50	\$120	\$0	\$0	
	SOLUTION MIX	X_1	X_2	S_1	S_2	QUANTITY
\$120	X_2	0.5	1	0.25	0	20
\$0	S_2	2.5	0	-0.25	1	40
	Z_j	\$60	\$120	\$30	\$0	\$2,400
	$C_j - Z_j$	-\$10	\$0	-\$30	\$0	

Nonbasic variables are variables that have a value of zero.

The solution is optimal as long as all $C_j - Z_j \leq 0$.

The range over which C_j rates for nonbasic variables can vary without causing a change in the optimal solution mix is called the range of insignificance.

Testing basic variables involves reworking the final simplex tableau.

NONBASIC OBJECTIVE FUNCTION COEFFICIENT Our goal here is to find out how sensitive the problem's optimal solution is to changes in the contribution rates of variables not currently in the basis (X_1 and S_1). Just how much would the objective function coefficients have to change before X_1 or S_1 would enter the solution mix and replace one of the basic variables?

The answer lies in the $C_j - Z_j$ row of the final simplex tableau (as in Table M7.15). Since this is a maximization problem, the basis will not change unless the $C_j - Z_j$ value of one of the nonbasic variables becomes positive. That is, the current solution will be optimal as long as all numbers in the bottom row are less than or equal to 0. It will not be optimal if X_1 's $C_j - Z_j$ value is positive, or if S_1 's $C_j - Z_j$ value is greater than 0. Therefore, the values of C_j for X_1 and S_1 that do not bring about any change in the optimal solution are given by

$$C_j - Z_j \leq 0$$

This is the same as writing

$$C_j \leq Z_j$$

Since X_1 's C_j value is \$50 and its Z_j value is \$60, the current solution is optimal as long as the profit per CD player does not exceed \$60, or correspondingly, does not increase by more than \$10. Similarly, the contribution rate per unit of S_1 (or per hour of electrician's time) may increase from \$0 up to \$30 without changing the current solution mix.

In both cases, when you are maximizing an objective function, you may increase the value of C_j up to the value of Z_j . You may also *decrease* the value of C_j for a nonbasic variable to negative infinity ($-\infty$) without affecting the solution. This range of C_j values is called the *range of insignificance* for nonbasic variables.

$$-\infty \leq C_j (\text{for } X_1) \leq \$60$$

$$-\infty \leq C_j (\text{for } S_1) \leq \$30$$

BASIC OBJECTIVE FUNCTION COEFFICIENT Sensitivity analysis on objective function coefficients of variables that are in the basis or solution mix is slightly more complex. We saw that a change in the objective function coefficient for a nonbasic variable affects only the $C_j - Z_j$ value for that variable. But a change in the profit or cost of a basic variable can affect the $C_j - Z_j$ values of *all* nonbasic variables because this C_j is not only in the C_j row but also in the C_j column. This then impacts the Z_j row.

Let us consider changing the profit contribution of stereo receivers in the High Note Sound Company problem. Currently, the objective function coefficient is \$120. The change in this value can be denoted by the Greek capital letter delta (Δ). We rework the final simplex tableau (first shown in Table M7.15) and see our results in Table M7.16.

Notice the new $C_j - Z_j$ values for nonbasic variables X_1 and S_1 . These were determined in exactly the same way as we did earlier in this chapter. But wherever the C_j value for X_2 of \$120 was seen in Table M7.15, a new value of $120 + \Delta$ is used in Table M7.16.

Once again, we recognize that the current optimal solution will change only if one or more of the $C_j - Z_j$ row values becomes greater than 0. The question is, how may the value of Δ vary so that all $C_j - Z_j$ entries remain negative? To find out, we solve for Δ in each column.

From the X_1 column:

$$-10 - 0.5\Delta \leq 0$$

$$-10 \leq 0.5\Delta$$

$$-20 \leq \Delta \text{ or } \Delta \geq -20$$

TABLE M7.16
Change in the Profit
Contribution of Stereo
Receivers

C_j		\$50	\$120 + Δ	\$0	\$0	
	SOLUTION MIX	X_1	X_2	S_1	S_2	QUANTITY
\$120 + Δ	X_2	0.5	1	0.25	0	20
\$0	S_2	2.5	0	-0.25	1	40
	Z_j	\$60 + 0.5 Δ	\$120 + Δ	\$30 + 0.25 Δ	\$0	\$2,400 + 20 Δ
	$C_j - Z_j$	-\$10 - 0.5 Δ	\$0	-\$30 - 0.25 Δ	\$0	

This inequality means that the optimal solution will not change unless X_2 's profit coefficient decreases by at least \$20, which is a change of $\Delta = -\$20$. Hence, variable X_1 will not enter the basis unless the profit per stereo receiver drops from \$120 to \$100 or less. This, interestingly, is exactly what we noticed graphically in Chapter 7. When the profit per stereo receiver dropped to \$80, the optimal solution changed from corner point a to corner point b .

Now we examine the S_1 column:

$$-30 - 0.25\Delta \leq 0$$

$$-30 \leq 0.25\Delta$$

$$-120 \leq \Delta \text{ or } \Delta \geq -120$$

This inequality implies that S_1 is less sensitive to change than X_1 . S_1 will not enter the basis unless the profit per unit of X_2 drops from \$120 all the way down to \$0.

Since the first inequality is more binding, we can say that the *range of optimality* for X_2 's profit coefficient is

$$\$100 \leq C_j(\text{for } X_2) \leq \infty$$

As long as the profit per stereo receiver is greater than or equal to \$100, the current production mix of $X_2 = 20$ receivers and $X_1 = 0$ CD players will be optimal.

In analyzing larger problems, we would use this procedure to test for the range of optimality of every real decision variable in the final solution mix. The procedure helps us avoid the time-consuming process of reformulating and resolving the entire LP problem each time a small change occurs. Within the bounds set, changes in profit coefficients would not force a firm to alter its product mix decision or change the number of units produced. Overall profits, of course, will change if a profit coefficient increases or decreases, but such computations are quick and easy to perform.

Changes in Resources or RHS Values

Making changes in the RHS values (the resources of electricians' and audio technicians' time) result in changes in the feasible region and often the optimal solution.

SHADOW PRICES This leads us to the important subject of *shadow prices*. Exactly how much should a firm be willing to pay to make additional resources available? Is one more hour of machine time worth \$1 or \$5 or \$20? Is it worthwhile to pay workers an overtime rate to stay one extra hour each night to increase production output? Valuable management information could be provided if the worth of additional resources were known.

Fortunately, this information is available to us by looking at the final simplex tableau of an LP problem. An important property of the $C_j - Z_j$ row is that the negatives of the numbers in its slack variable (S_i) columns provide us with what we call shadow prices. A *shadow price* is

The range of optimality is the range of values over which a basic variable's coefficient can change without causing a change in the optimal solution mix.

The shadow price is the value of one additional unit of a scarce resource. Shadow pricing provides an important piece of economic information.

The final simplex tableau for the High Note Sound Company problem is repeated as Table M7.17 (it was first shown as Table M7.15). The tableau indicates that the optimal solution is $X_1 = 0$, $X_2 = 20$, $S_1 = 0$, and $S_2 = 40$ and that profit = \$2,400. Recall that S_1 represents slack availability of the electricians' resource and S_2 the unused time in the audio technicians' department.

Should High Note also hire a part-time audio technician at a rate of \$14 per hour? The answer is no: The shadow price is \$0, implying no increase in the objective function by making more of this second resource available. Why? Because not all of the resource is currently being used—40 hours are still available. It would hardly pay to buy more of the resource.

Ranging is simple in that it resembles the simplex process we used earlier in this chapter to find the minimum ratio for a new variable. The S_1 column and quantity column from Table M7.17 are repeated in the following table; the ratios, both positive and negative, are also shown:

QUANTITY	S_1	RATIO
20	0.25	$20/0.25 = 80$
40	-0.25	$40/-0.25 = -160$

TABLE M7.17
Final Tableau for the
High Note Sound
Company

C_j		\$50	\$120	\$0	\$50	
	SOLUTION MIX	X_1	X_2	S_1	S_2	QUANTITY
\$120	X_2	0.5	1	0.25	0	20
\$0	S_2	2.5	0	-0.25	1	40
	Z_j	\$60	\$120	\$30	\$0	\$2,400
	$C_j - Z_j$	-\$10	\$0	-\$30	\$0	
		Objective function increases by \$30 if 1 additional hour of electricians' time is made available				

The smallest positive ratio (80 in this example) tells us by how many hours the electricians' time resource can be *reduced* without altering the current solution mix. Hence, we can decrease the RHS resource by as much as 80 hours—basically from the current 80 hours all the way down to 0 hours—without causing a basic variable to be pivoted out of the solution.

The smallest negative ratio (-160) tells us the number of hours that can be added to the resource before the solution mix changes. In this case, we can increase electricians' time by 160 hours, up to $240 (=80 \text{ currently} + 160 \text{ may be added})$ hours. We have now established the range of electricians' time over which the shadow price of \$30 is valid. That range is from 0 to 240 hours.

The audio technician resource is slightly different in that all 60 hours of time originally available have not been used. (Note that $S_2 = 40$ hours in Table M7.17.) If we apply the ratio test, we see that we can reduce the number of audio technicians' hours by only 40 (the smallest positive ratio $= 40/1$) before a shortage occurs. But since we are not using all the hours currently available, we can increase them indefinitely without altering the problem's solution. Note that there are no negative substitution rates in the S_2 column, so there are no negative ratios. Hence, the valid range for *this* shadow price would be from $20 (= 60 - 40)$ hours to an unbounded upper limit.

The substitution rates in the slack variable column can also be used to determine the actual values of the solution mix variables if the right-hand side of a constraint is changed. The following relationship is used to find these values:

$$\text{New quantity} = \text{Original quantity} + (\text{Substitution rate})(\text{Change in right-hand side})$$

For example, if 12 more electrician hours were made available, the new values in the quantity column of the simplex tableau are found as follows:

ORIGINAL QUANTITY	S_1	NEW QUANTITY
20	0.25	$20 + (0.25)(12) = 23$
40	-0.25	$40 + (-0.25)(12) = 37$

Thus, if 12 hours are added, $X_2 = 23$ and $S_2 = 37$. All other variables are nonbasic and remain zero. This yields a total profit of $50(0) + 120(23) = \$2,760$, which is an increase of \$360 (or the shadow price of \$30 per hour for 12 hours of electrician time). A similar analysis with the other constraint and the S_2 column would show that if any additional audio technician hours were added, only the slack for that constraint would increase.

Changes in the RHS values of constraints may change the optimal quantity values of the solution mix variables.

M7.12 The Dual

Every LP primal has a dual. The dual provides useful economic information.

Every LP problem has another LP problem associated with it, which is called its *dual*. The first way of stating a linear problem is called the *primal* of the problem; we can view all of the problems formulated thus far as primals. The second way of stating the same problem is called the *dual*. The optimal solutions for the primal and the dual are equivalent, but they are derived through alternative procedures.

The dual contains economic information useful to management, and it may also be easier to solve, in terms of less computation, than the primal problem. Generally, if the LP primal involves maximizing a profit function subject to less-than-or-equal-to resource constraints, the

dual will involve minimizing total opportunity costs subject to greater-than-or-equal-to product profit constraints. Formulating the dual problem from a given primal is not terribly complex, and once it is formulated, the solution procedure is exactly the same as for any LP problem.

Let's illustrate the *primal–dual relationship* with the High Note Sound Company data. As you recall, the primal problem is to determine the best production mix of CD players (X_1) and stereo receivers (X_2) to maximize profit:

$$\begin{aligned} \text{Maximize profit} &= \$50X_1 + \$120X_2 \\ \text{subject to} \quad 2X_1 + 4X_2 &\leq 80 && \text{(hours of available electrician time)} \\ 3X_1 + 1X_2 &\leq 60 && \text{(hours of audio technician time available)} \end{aligned}$$

The dual variables represent the potential value of resources.

The dual of this problem has the objective of minimizing the opportunity cost of not using the resources in an optimal manner. Let's call the variables that it will attempt to solve for U_1 and U_2 . U_1 represents the potential hourly contribution or worth of electrician time; in other words, the dual value of 1 hour of the electricians' resource. U_2 stands for the imputed worth of the audio technicians' time, or the dual technician resource. Thus, each constraint in the primal problem will have a corresponding variable in the dual problem. Also, each decision variable in the primal problem will have a corresponding constraint in the dual problem.

The RHS quantities of the primal *constraints* become the dual's *objective function* coefficients. The total opportunity cost that is to be minimized will be represented by the function $80U_1 + 60U_2$, namely,

$$\text{Minimize opportunity cost} = 80U_1 + 60U_2$$

The corresponding dual constraints are formed from the transpose⁶ of the primal constraints coefficients. Note that if the primal constraints are \leq , the dual constraints are \geq :

$$\begin{array}{l} \left(\begin{array}{l} 2U_1 + 3U_2 \geq 50 \\ 4U_1 + 1U_2 \geq 120 \end{array} \right) \begin{array}{l} \rightarrow \text{Primal profit coefficients} \\ \rightarrow \text{Coefficients from the second primal constraint} \\ \rightarrow \text{Coefficients from the first primal constraint} \end{array} \end{array}$$

Let's look at the meaning of these dual constraints. In the first inequality, the RHS constant (\$50) is the income from one CD player. The coefficients of U_1 and U_2 are the amounts of each scarce resource (electrician time and audio technician time) that are required to produce a CD player. That is, 2 hours of electricians' time and 3 hours of audio technicians' time are used up in making one CD player. Each CD player produced yields \$50 of revenue to High Note Sound Company. This inequality states that the total imputed value or potential worth of the scarce resources needed to produce a CD player must be at least equal to the profit derived from the product. The second constraint makes an analogous statement for the stereo receiver product.

⁶For example, the transpose of the set of numbers $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$. In the case of the transpose of the primal coefficients $\begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}$, the result is $\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$. Refer to Module 5, which deals with matrices and determinants, for a review of the transpose concept.

Dual Formulation Procedures

The mechanics of formulating a dual from the primal problem are summarized in the following list.

These are the five steps for formulating a dual.

Steps to Form a Dual

1. If the primal is a maximization, the dual is a minimization, and vice versa.
2. The RHS values of the primal constraints become the dual's objective function coefficients.
3. The primal objective function coefficients become the RHS values of the dual constraints.
4. The transpose of the primal constraint coefficients become the dual constraint coefficients.
5. Constraint inequality signs are reversed.⁷

Solving the Dual of the High Note Sound Company Problem

The simplex algorithm is applied to solve the preceding dual problem. With appropriate surplus and artificial variables, it can be restated as follows:

$$\begin{aligned} \text{Minimize opportunity cost} &= 80U_1 + 60U_2 + 0S_1 + 0S_2 + MA_1 + MA_2 \\ \text{subject to} \quad 2U_1 + 3U_2 - 1S_1 + 1A_1 &= 50 \\ 4U_1 + 1U_2 - 1S_2 + 1A_2 &= 120 \end{aligned}$$

The first and second tableaus are shown in Table M7.18. The third tableau, containing the optimal solution of $U_1 = 30$, $U_2 = 0$, $S_1 = 10$, $S_2 = 0$, opportunity cost = \$2,400, appears in Figure M7.5 along with the final tableau of the primal problem.

TABLE 9.18 First and Second Tableaus of the High Note Dual Problem

		C_j							
			80	60	0	0	M	M	
		SOLUTION MIX	U_1	U_2	S_1	S_2	A_1	A_2	QUANTITY
First tableau	\$M	A_1	2	3	-1	0	1	0	50
	\$M	A_2	4	1	0	-1	0	1	120
		Z_j	\$6M	\$4M	-\$M	-\$M	\$M	\$M	\$170M
		$C_j - Z_j$	$80 - 6M$	$60 - 4M$	M	M	0	0	
Second tableau	\$80	U_1	1	1.5	-0.5	0	0.5	0	25
	\$M	A_2	0	-5	2	-1	-2	1	20
		Z_j	\$80	\$120 - 5M	-\$40 + 2M	-\$M	\$40 - 2M	\$M	\$2,000 + 20M
		$C_j - Z_j$	0	$5M - 60$	$-2M + 40$	M	$3M - 40$	0	

⁷ If the j th primal constraint should be an equality, the i th dual variable is unrestricted in sign. This technical issue is discussed in L. Cooper and D. Steinberg, *Methods and Applications of Linear Programming*. Philadelphia: W. B. Saunders, 1974, p. 170.

FIGURE M7.5
Comparison of the
Primal and Dual
Optimal Tableaus

Primal's Optimal Solution						
$C_j \rightarrow$ \downarrow	Solution Mix	\$50 X_1	\$120 X_2	\$0 S_1	\$0 S_2	Quantity
\$120	X_2	0.5	1	0.25	0	20
\$0	S_2	2.5	0	-0.25	1	40
	Z_j	60	120	30	0	\$2,400
	$C_j - Z_j$	-10	0	-30	0	

Dual's Optimal Solution								
$C_j \rightarrow$ \downarrow	Solution Mix	80 U_1	60 U_2	0 S_1	0 S_2	M A_1	M A_2	Quantity
80	U_1	1	0.25	0	-0.25	0	0.25	30
0	S_1	0	-2.5	1	-0.5	-1	0.5	10
	Z_j	80	20	0	-20	0	20	\$2,400
	$C_j - Z_j$	0	40	0	20	M	M - 20	

The solution to the dual yields shadow prices.

We mentioned earlier that the primal and dual lead to the same solution even though they are formulated differently. How can this be?

It turns out that in the final simplex tableau of a primal problem, the absolute values of the numbers in the $C_j - Z_j$ row under the slack variables represent the solutions to the dual problem, that is, the optimal U_i s (see Figure M7.5). In the earlier section on sensitivity analysis we termed these numbers in the columns of the slack variables *shadow prices*. Thus, the solution to the dual problem presents the marginal profits of each additional unit of resource.

It also happens that the absolute value of the $C_j - Z_j$ values of the slack variables in the optimal *dual* solution represent the optimal values of the *primal* X_1 and X_2 variables. The minimum opportunity cost derived in the dual must always equal the maximum profit derived in the primal.

Also note the other relationships between the primal and the dual that are indicated in Figure M7.5 by arrows. Columns A_1 and A_2 in the optimal dual tableau may be ignored because, as you recall, artificial variables have no physical meaning.

M7.13 Karmarkar's Algorithm

The biggest change to take place in the field of LP solution techniques in four decades was the 1984 arrival of an alternative to the simplex algorithm. Developed by Narendra Karmarkar, the new method, called Karmarkar's algorithm, often takes significantly less computer time to solve very large-scale LP problems.⁸

As we saw, the simplex algorithm finds a solution by moving from one adjacent corner point to the next, following the outside edges of the feasible region. In contrast, Karmarkar's method follows a path of points on the *inside* of the feasible region. Karmarkar's method is also unique in its ability to handle an *extremely* large number of constraints and variables, thereby giving LP users the capacity to solve previously unsolvable problems.

Although it is likely that the simplex method will continue to be used for many LP problems, a new generation of LP software built around Karmarkar's algorithm is already becoming popular. Delta Air Lines became the first commercial airline to use the Karmarkar program, called KORBX, which was developed and is sold by AT&T. Delta found that the program streamlined the monthly scheduling of 7,000 pilots who fly more than 400 airplanes to 166 cities worldwide. With increased efficiency in allocating limited resources, Delta saves millions of dollars in crew time and related costs.

Summary

In Chapter 7 we examined the use of graphical methods to solve LP problems that contained only two decision variables. This chapter moves us one giant step further by introducing the simplex method. The simplex method is an iterative procedure for reaching the optimal solution to LP problems of any dimension. It consists of a series of rules that, in effect, algebraically examine corner points in a systematic way. Each step moves us closer to the optimal solution by increasing profit or decreasing cost, while maintaining feasibility.

This chapter explains the procedure for converting less-than-or-equal-to, greater-than-or-equal-to, and equality constraints into the simplex format. These conversions employed the inclusion of slack, surplus, and artificial variables. An initial simplex tableau is developed that portrays the problem's original data formulations. It also contains a row providing profit or cost information and a net evaluation row. The latter, identified as the $C_j - Z_j$ row, is examined in determining whether an optimal solution had yet been reached. It also points out which variable would next enter the solution mix, or basis, if the current solution was nonoptimal.

The simplex method consists of five steps: (1) identifying the pivot column, (2) identifying the pivot row and number,

(3) replacing the pivot row, (4) computing new values for each remaining row, and (5) computing the Z_j and $C_j - Z_j$ rows and examining for optimality. Each tableau of this iterative procedure is displayed and explained for a sample maximization and minimization problem.

A few special issues in LP that arise in using the simplex method are also discussed in this chapter. Examples of infeasibility, unbounded solutions, degeneracy, and multiple optimal solutions are presented.

Although large LP problems are seldom, if ever, solved by hand, the purpose of this chapter is to help you gain an understanding of how the simplex method works. Understanding the underlying principles help you to interpret and analyze computerized LP solutions.

This module also provides a foundation for another issue: answering questions about the problem after an optimal solution has been found, which is called postoptimality analysis, or sensitivity analysis. Included in this discussion is the analysis of the value of additional resources, called shadow pricing. Finally, the relationship between a primal LP problem and its dual is explored. We illustrate how to derive the dual from a primal and how the solutions to the dual variables are actually the shadow prices.

Glossary

Artificial Variable A variable that has no meaning in a physical sense but acts as a tool to help generate an initial LP solution.

Basic Feasible Solution A solution to an LP problem that corresponds to a corner point of the feasible region.

⁸For details, see Narendra Karmarkar. "A New Polynomial Time Algorithm for Linear Programming," *Combinatorica* 4, 4 (1984): 373–395, or J. N. Hooker. "Karmarkar's Linear Programming Algorithm," *Interfaces* 16, 4 (July–August 1986): 75–90.

Basis The set of variables that are in the solution, have positive, nonzero values, and are listed in the solution mix column. They are also called **basic variables**.

$C_j - Z_j$ Row The row containing the net profit or loss that will result from introducing one unit of the variable indicated in that column into the solution.

Current Solution The basic feasible solution that is the set of variables presently in the solution. It corresponds to a corner point of the feasible region.

Degeneracy A condition that arises when there is a tie in the values used to determine which variable will enter the solution next. It can lead to cycling back and forth between two nonoptimal solutions.

Infeasibility The situation in which there is no solution that satisfies all of a problem's constraints.

Iterative Procedure A process (algorithm) that repeats the same steps over and over.

Nonbasic Variables Variables not in the solution mix or basis. Nonbasic variables are equal to zero.

Pivot Column The column with the largest positive number in the $C_j - Z_j$ row of a maximization problem, or the largest negative $C_j - Z_j$ improvement value in a minimization problem. It indicates which variable will enter the solution next.

Pivot Number The number at the intersection of the pivot row and pivot column.

Pivot Row The row corresponding to the variable that will leave the basis in order to make room for the variable entering (as indicated by the new pivot column). This is the smallest positive ratio found by dividing the quantity column values by the pivot column values for each row.

Primal-Dual Relationship Alternative ways of stating an LP problem.

Quantity Column A column in the simplex tableau that gives the numeric value of each variable in the solution mix column.

Range of Insignificance The range of values over which a nonbasic variable's coefficient can vary without causing a change in the optimal solution mix.

Range of Optimality The range of values over which a basic variable's coefficient can change without causing a change in the optimal solution mix.

Right-Hand-Side Ranging A method used to find the range over which shadow prices remain valid.

Shadow Prices The coefficients of slack variables in the $C_j - Z_j$ row. They represent the value of one additional unit of a resource.

Simplex Method A matrix algebra method for solving LP problems.

Simplex Tableau A table for keeping track of calculations at each iteration of the simplex method.

Slack Variable A variable added to less-than-or-equal-to constraints in order to create an equality for a simplex method. It represents a quantity of unused resource.

Solution Mix A column in the simplex tableau that contains all the basic variables in the solution.

Substitution Rates The coefficients in the central body of each simplex table. They indicate the number of units of each basic variable that must be removed from the solution if a new variable (as represented at any column head) is entered.

Surplus Variable A variable inserted in a greater-than-or-equal-to constraint to create an equality. It represents the amount of resource usage above the minimum required usage.

Unboundedness A condition describing LP maximization problems having solutions that can become infinitely large without violating any stated constraints.

Z_j Row The row containing the figures for gross profit or loss given up by adding one unit of a variable into the solution.

Key Equation

(M7-1) (New row numbers) = (Numbers in old row)

$$-\left[\left(\begin{array}{c} \text{Number above or below} \\ \text{pivot number} \end{array} \right) \times \left(\begin{array}{c} \text{Corresponding number} \\ \text{in newly replaced row} \end{array} \right) \right]$$

Formula for computing new values for nonpivot rows in the simplex tableau (step 4 of the simplex procedure).

Solved Problems

Solved Problem M7-1

Convert the following constraints and objective function into the proper form for use in the simplex method:

$$\begin{aligned} \text{Minimize cost} &= 4X_1 + 1X_2 \\ \text{subject to} \quad &3X_1 + X_2 = 3 \\ &4X_1 + 3X_2 \geq 6 \\ &X_1 + 2X_2 \leq 3 \end{aligned}$$

Solution

$$\begin{aligned}
 \text{Minimize cost} &= 4X_1 + 1X_2 + 0S_1 + 0S_2 + MA_1 + MA_2 \\
 \text{subject to} \quad &3X_1 + 1X_2 + 1A_1 = 3 \\
 &4X_1 + 3X_2 - 1S_1 + 1A_2 = 6 \\
 &1X_1 + 2X_2 + 1S_2 = 3
 \end{aligned}$$

Solved Problem M7-2

Solve the following LP problem:

$$\begin{aligned}
 \text{Maximize profit} &= \$9X_1 + \$7X_2 \\
 \text{subject to} \quad &2X_1 + 1X_2 \leq 40 \\
 &X_1 + 3X_2 \leq 30
 \end{aligned}$$

Solution

We begin by adding slack variables and converting inequalities into equalities.

$$\begin{aligned}
 \text{Maximize profit} &= 9X_1 + 7X_2 + 0S_1 + 0S_2 \\
 \text{subject to} \quad &2X_1 + 1X_2 + 1S_1 + 0S_2 = 40 \\
 &1X_1 + 3X_2 + 0S_1 + 1S_2 = 30
 \end{aligned}$$

The initial tableau is then as follows:

$C_j \rightarrow$		\$9	\$7	\$0	\$0	
\downarrow	SOLUTION MIX	X_1	X_2	S_1	S_2	QUANTITY
\$0	S_1	2	1	1	0	40
\$0	S_2	1	3	0	1	30
	Z_j	\$0	\$0	\$0	\$0	\$0
	$C_j - Z_j$	9	7	0	0	

The correct second tableau and third tableau and some of their calculations follow. The optimal solutions, given in the third tableau, are $X_1 = 18$, $X_2 = 4$, $S_1 = 0$, $S_2 = 0$, and profit = \$190.

Step 1 and 2 To go from the first to the second tableau, we note that the pivot column (in the first tableau) is X_1 , which has the highest $C_j - Z_j$ value, \$9. The pivot row is S_1 since $40/2$ is less than $30/1$, and the pivot number is 2.

Step 3 The new X_1 row is found by dividing each number in the old S_1 row by the pivot number, namely, $2/2 = 1$, $1/2 = 0.5$, $1/2 = 0.5$, $0/2 = 0$, and $40/2 = 20$.

Step 4 The new values for the S_2 row are computed as follows:

$$\begin{aligned}
 \left(\begin{array}{c} \text{Number in} \\ \text{new } S_2 \text{ row} \end{array} \right) &= \left(\begin{array}{c} \text{Number in} \\ \text{old } S_2 \text{ row} \end{array} \right) - \left[\left(\begin{array}{c} \text{Number below} \\ \text{pivot number} \end{array} \right) \times \left(\begin{array}{c} \text{Corresponding} \\ \text{number in} \\ \text{new } X_1 \text{ row} \end{array} \right) \right] \\
 0 &= 1 - [(1) \times (1)] \\
 2.5 &= 3 - [(1) \times (0.5)] \\
 -0.5 &= 0 - [(1) \times (0.5)] \\
 1 &= 1 - [(1) \times (0)] \\
 10 &= 30 - [(1) \times (20)]
 \end{aligned}$$

Step 5 The following new Z_j and $C_j - Z_j$ rows are formed:

$$\begin{aligned} Z_j(\text{for } X_1) &= \$9(1) + 0(0) = \$9 & C_j - Z_j &= \$9 - \$9 = 0 \\ Z_j(\text{for } X_2) &= \$9(0.5) + 0(2.5) = \$4.5 & C_j - Z_j &= \$7 - 4.5 = \$2.5 \\ Z_j(\text{for } S_1) &= \$9(0.5) + 0(-0.5) = \$4.5 & C_j - Z_j &= 0 - 4.5 = -\$4.5 \\ Z_j(\text{for } S_2) &= \$9(0) + 0(1) = \$0 & C_j - Z_j &= 0 - 0 = 0 \\ Z_j(\text{profit}) &= \$9(20) + 0(10) = \$180 \end{aligned}$$

C_j		\$9	\$7	\$0	\$0	
	SOLUTION MIX	X_1	X_2	S_1	S_2	QUANTITY
\$9	X_1	1	0.5	0.5	0	20
0	S_2	0	2.5	-0.5	1	10 ← Pivot row
	Z_j	\$9	\$4.5	\$4.5	\$0	\$180
	$C_j - Z_j$	0	2.5	-4.5	0	

↑ Pivot column

This solution is not optimal, and you must perform steps 1 to 5 again. The new pivot column is X_2 , the new pivot row is S_2 , and 2.5 (circled in the second tableau) is the new pivot number.

C_j		\$9	\$7	\$0	\$0	
	SOLUTION MIX	X_1	X_2	S_1	S_2	QUANTITY
\$9	X_1	1	0	0.6	-0.2	18
7	X_2	0	1	-0.2	0.4	4
	Z_j	\$9	\$7	\$4	\$1	\$190
	$C_j - Z_j$	0	0	-4	-1	

The final solution is $X_1 = 18$, $X_2 = 4$, profit = \$190.

Solved Problem M7-3

Use the final simplex tableau in Solved Problem M7-2 to answer the following questions.

- What are the shadow prices for the two constraints?
- Perform RHS ranging for constraint 1.
- If the right-hand side of constraint 1 were increased by 10, what would the maximum possible profit be? Give the values for all the variables.
- Find the range of optimality for the profit on X_1 .

Solution

- Shadow price = $-(C_j - Z_j)$
 For constraint 1, shadow price = $-(-4) = 4$.
 For constraint 2, shadow price = $-(-1) = 1$.
- For constraint 1, we use the S_1 column.

QUANTITY	S_1	RATIO
18	0.6	$18/(0.6) = 30$
4	-0.2	$4/(-0.2) = -20$

The smallest positive ratio is 30, so we may reduce the right-hand side of constraint 1 by 30 units (for a lower bound of $40 - 30 = 10$). Similarly, the negative ratio of -20 tells us that we may increase the right-hand side of constraint 1 by 20 units (for an upper bound of $40 + 20 = 60$).

- c. The maximum possible profit = Original profit + 10(shadow price)
 $= 190 + 10(4) = 230$

The values for the basic variables are found using the original quantities and the substitution rates:

ORIGINAL QUANTITY	S_1	NEW QUANTITY
18	0.6	$18 + (0.6)(10) = 24$
4	-0.2	$4 + (-0.2)(10) = 2$

$X_1 = 24$, $X_2 = 2$, $S_1 = 0$, $S_2 = 0$ (both slack variables remain nonbasic variables)

profit = $9(24) + 7(2) = 230$ (which was also found using the shadow price)

- d. Let Δ = change in profit for X_1 .

C_j		$9 + \Delta$	7	0	0	
	SOLUTION MIX	X_1	X_2	S_1	S_2	QUANTITY
$9 + \Delta$	X_1	1	0	0.6	-0.2	18
7	X_2	0	1	-0.2	0.4	4
	Z_j	$9 + \Delta$	7	$4 + (0.6)\Delta$	$1 - (0.2)\Delta$	$190 + 18\Delta$
	$C_j - Z_j$	0	0	$-4 - (0.6)\Delta$	$-1 + (0.2)\Delta$	

For this solution to remain optimal, the $C_j - Z_j$ values must remain negative or zero.

$$-4 - (0.6)\Delta \leq 0$$

$$-4 \leq (0.6)\Delta$$

$$-20/3 \leq \Delta$$

and

$$-1 + (0.2)\Delta \leq 0$$

$$(0.2)\Delta \leq 1$$

$$\Delta \leq 5$$

So the change in profit (Δ) must be between $-20/3$ and 5. The original profit was 9, so this solution remains optimal as long as the profit on X_1 is between $2.33 = 9 - 20/3$ and $14 = 9 + 5$.

Self-Test

- Before taking the self-test, refer to the learning objectives at the beginning of the chapter, the notes in the margins, and the glossary at the end of the chapter.
 - Use the key at the back of the book to correct your answers.
 - Restudy pages that correspond to any questions that you answered incorrectly or material you feel uncertain about.
1. A basic feasible solution is a solution to an LP problem that corresponds to a corner point of the feasible region.
 - a. True
 - b. False
 2. In preparing a \geq constraint for an initial simplex tableau, you would
 - a. add a slack variable.
 - b. add a surplus variable.
 - c. subtract an artificial variable.
 - d. subtract a surplus variable and add an artificial variable.
 3. In the initial simplex tableau, the solution mix variables can be
 - a. only slack variables.
 - b. slack and surplus variables.
 - c. artificial and surplus variables.
 - d. slack and artificial variables.
 4. Even if an LP problem involves many variables, an optimal solution will always be found at a corner point of the n -dimensional polyhedron forming the feasible region.
 - a. True
 - b. False
 5. Which of the following in a simplex tableau indicates that an optimal solution for a maximization problem has been found?
 - a. All the $C_j - Z_j$ values are negative or zero.
 - b. All the $C_j - Z_j$ values are positive or zero.
 - c. All the substitution rates in the pivot column are negative or zero.
 - d. There are no more slack variables in the solution mix.
 6. To formulate a problem for solution by the simplex method, we must add slack variables to
 - a. all inequality constraints.
 - b. only equality constraints.
 - c. only “greater than” constraints.
 - d. only “less than” constraints.
 7. If in the optimal tableau of an LP problem an artificial variable is present in the solution mix, this implies
 - a. infeasibility.
 - b. unboundedness.
 - c. degeneracy.
 - d. alternate optimal solutions.
 8. If in the final optimal simplex tableau the $C_j - Z_j$ value for a nonbasic variable is zero, this implies
 - a. feasibility.
 - b. unboundedness.
 - c. degeneracy.
 - d. alternate optimal solutions.
 9. In a simplex tableau, all of the substitution rates in the pivot column are negative. This indicates that
 - a. there is no feasible solution to this problem.
 - b. the solution is unbounded.
 - c. there is more than one optimal solution.
 - d. the solution is degenerate.
 10. The pivot column in a maximization problem is the column with
 - a. the greatest positive $C_j - Z_j$.
 - b. the greatest negative $C_j - Z_j$.
 - c. the greatest positive Z_j .
 - d. the greatest negative Z_j .
 11. A change in the objective function coefficient (C_j) for a basic variable can affect
 - a. the $C_j - Z_j$ values of all the nonbasic variables.
 - b. the $C_j - Z_j$ values of all the basic variables.
 - c. only the $C_j - Z_j$ value of that variable.
 - d. the C_j values of other basic variables.
 12. Linear programming has few applications in the real world due to the assumption of certainty in the data and relationships of a problem.
 - a. True
 - b. False
 13. In a simplex tableau, one variable will leave the basis and be replaced by another variable. The leaving variable is
 - a. the basic variable with the largest C_j .
 - b. the basic variable with the smallest C_j .
 - c. the basic variable in the pivot row.
 - d. the basic variable in the pivot column.
 14. Which of the following must equal 0?
 - a. basic variables
 - b. solution mix variables
 - c. nonbasic variables
 - d. objective function coefficients for artificial variables
 15. The shadow price for a constraint
 - a. is the value of an additional unit of that resource.
 - b. is always equal to zero if there is positive slack for that constraint.
 - c. is found from the $C_j - Z_j$ value in the slack variable column.
 - d. all of the above.
 16. The solution to the dual LP problem
 - a. presents the marginal profits of each additional unit of resource.
 - b. can always be derived by examining the Z_j row of the primal’s optimal simplex tableau.
 - c. is better than the solution to the primal.
 - d. all of the above.
 17. The number of constraints in a dual problem will equal the number of
 - a. constraints in the primal problem.
 - b. variables in the primal problem.
 - c. variables plus the number of constraints in the primal problem.
 - d. variables in the dual problem.

Discussion Questions and Problems

Discussion Questions

- M7-1 Explain the purpose and procedures of the simplex method.
- M7-2 How do the graphical and simplex methods of solving LP problems differ? In what ways are they the same? Under what circumstances would you prefer to use the graphical approach?
- M7-3 What are slack, surplus, and artificial variables? When is each used, and why? What value does each carry in the objective function?
- M7-4 You have just formulated an LP problem with 12 decision variables and eight constraints. How many basic variables will there always be? What is the difference between a basic and a nonbasic variable?
- M7-5 What are the simplex rules for selecting the pivot column? The pivot row? The pivot number?
- M7-6 How do maximization and minimization problems differ when applying the simplex method?
- M7-7 Explain what the Z_j value indicates in the simplex tableau.
- M7-8 Explain what the $C_j - Z_j$ value indicates in the simplex tableau.
- M7-9 What is the reason behind the use of the minimum ratio test in selecting the pivot row? What might happen without it?
- M7-10 A particular LP problem has the following objective function:
- $$\text{Maximize profit} = \$8X_1 + \$6X_2 + \$12X_3 - \$2X_4$$
- Which variable should enter at the second simplex tableau? If the objective function were
- $$\text{Minimize cost} = \$2.5X_1 + \$2.9X_2 + \$4.0X_3 + \$7.9X_4$$
- which variable would be the best candidate to enter the second tableau?
- M7-11 What happens if an artificial variable is in the final optimal solution? What should the manager who formulated the LP problem do?
- M7-12 The great Romanian operations researcher Dr. Ima Student proposes that instead of selecting the variable with the largest positive $C_j - Z_j$ value (in a maximization LP problem) to enter the solution mix next, a different approach be used. She suggests that any variable with a positive $C_j - Z_j$ can be chosen, even if it isn't the largest. What will happen if we adopt this new rule for the simplex procedure? Will an optimal solution still be reached?
- M7-13 What is a shadow price? How does the concept relate to the dual of an LP problem? How does it relate to the primal?

M7-14 If a primal problem has 12 constraints and eight variables, how many constraints and variables will its corresponding dual have?

M7-15 Explain the relationship between each number in a primal and corresponding numbers in the dual.

M7-16 Create your own original LP maximization problem with two variables and three less-than-or-equal-to constraints. Now form the dual for this primal problem.

Problems*

- M7-17 The first constraint in the High Note example in this chapter is




$$2X_1 + 4X_2 \leq 80 \text{ (hours of electrician time available)}$$

Table M7.17 gives the final simplex tableau for this example on page M7-34. From the tableau, it was determined that the maximum increase in electrician hours was 160 (for a total of 240 hours).


- Change the right-hand side of that constraint to 240 and graph the new feasible region.
 - Find the new optimal corner point. How much did the profit increase as a result of this?
 - What is the shadow price?
 - Increase the electrician hours available by one unit more (to 241) and find the optimal solution. How much did the profit increase as a result of this one extra hour? Explain why the shadow price from the simplex tableau is no longer relevant.
- M7-18 The Dreskin Development Company is building two apartment complexes. It must decide how many units to construct in each complex subject to labor and material constraints. The profit generated for each apartment in the first complex is estimated at \$900, for each apartment in the second complex, \$1,500. A partial initial simplex tableau for Dreskin is given in the following table:

C_j	\$900	\$1,500	\$0	\$0	
SOLUTION MIX	X_1	X_2	S_1	S_2	QUANTITY
	14	4	1	0	3,360
	10	12	0	1	9,600
Z_j					
$C_j - Z_j$					

- Complete the initial tableau.
- Reconstruct the problem's original constraints (excluding slack variables).

*Note:  means the problem may be solved with QM for Windows;  means the problem may be solved with Excel; and  means the problem may be solved with QM for Windows and/or Excel.


- (c) Write the problem's original objective function.
- (d) What is the basis for the initial solution?
- (e) Which variable should enter the solution at the next iteration?
- (f) Which variable will leave the solution at the next iteration?
- (g) How many units of the variable entering the solution next will be in the basis in the second tableau?
- (h) How much will profit increase in the next solution?

 M7-19 Consider the following LP problem:

$$\begin{aligned} \text{Maximize earnings} &= \$0.80X_1 + \$0.40X_2 + \$1.20X_3 \\ &\quad - \$0.10X_4 \end{aligned}$$

$$\begin{aligned} \text{subject to } X_1 + 2X_2 + X_3 + 5X_4 &\leq 150 \\ X_2 - 4X_3 + 8X_4 &= 70 \\ 6X_1 + 7X_2 + 2X_3 - X_4 &\geq 120 \\ X_1, X_2, X_3, X_4 &\geq 0 \end{aligned}$$

- (a) Convert these constraints to equalities by adding the appropriate slack, surplus, or artificial variables. Also, add the new variables into the problem's objective function.
- (b) Set up the complete initial simplex tableau for this problem. Do not attempt to solve.
- (c) Give the values for all variables in this initial solution.

 M7-20 Solve the following LP problem graphically. Then set up a simplex tableau and solve the problem using the simplex method. Indicate the corner points generated at each iteration by the simplex method on your graph.


$$\begin{aligned} \text{Maximize profit} &= \$3X_1 + \$5X_2 \\ \text{subject to } X_2 &\leq 6 \\ 3X_1 + 2X_2 &\leq 18 \\ X_1, X_2 &\geq 0 \end{aligned}$$

• M7-21 Consider the following LP problem:

$$\begin{aligned} \text{Maximize profit} &= 10X_1 + 8X_2 \\ \text{subject to } 4X_1 + 2X_2 &\leq 80 \\ X_1 + 2X_2 &\leq 50 \\ X_1, X_2 &\geq 0 \end{aligned}$$

- (a) Solve this problem graphically.


- (b) Set up the initial simplex tableau. On the graph, identify the corner point represented by this tableau.
- (c) Select the pivot column. Which variable is the entering variable?
- (d) Compute the ratio of the quantity-to-pivot column substitution rate for each row. Identify the points on the graph related to these ratios.
- (e) How many units of the entering variable will be brought into the solution in the second tableau? What would happen if the largest ratio rather than the smallest ratio were selected to determine this (see the graph)?
- (f) Which variable is the leaving variable? What will the value of this variable be in the next tableau?
- (g) Finish solving this problem using the simplex algorithm.
- (h) The solution in each simplex tableau is a corner point on the graph. Identify the corner point associated with each tableau.

 M7-22 Solve the following LP problem first graphically and then by the simplex algorithm:

$$\begin{aligned} \text{Minimize cost} &= 4X_1 + 5X_2 \\ \text{subject to } X_1 + 2X_2 &\geq 80 \\ 3X_1 + X_2 &\geq 75 \\ X_1, X_2 &\geq 0 \end{aligned}$$

What are the values of the basic variables at each iteration? Which are the nonbasic variables at each iteration?

- M7-23 The final simplex tableau for an LP maximization problem is shown in the table at the bottom of this page. Describe the situation encountered here.

 M7-24 Solve the following problem by the simplex method. What condition exists that prevents you from reaching an optimal solution?

$$\begin{aligned} \text{Maximize profit} &= 6X_1 + 3X_2 \\ \text{subject to } 2X_1 - 2X_2 &\leq 2 \\ -X_1 + X_2 &\leq 1 \\ X_1, X_2 &\geq 0 \end{aligned}$$


Tableau for Problem M7-23


$C_j \rightarrow$		3	5	0	0	-M	
\downarrow	SOLUTION MIX	X_1	X_2	S_1	S_2	A_1	QUANTITY
\$5	X_2	1	1	2	0	0	6
-M	A_1	-1	0	-2	-1	1	2
	Z_j	$\$5 + M$	$\$5$	$\$10 + 2M$	$\$M$	$-\$M$	$\$30 - 2M$
	$C_j - Z_j$	$-2 - M$	0	$-10 - 2M$	-M	0	

-  M7-25 Consider the following financial problem:

$$\begin{aligned} \text{Maximize return on investment} &= \$2X_1 + \$3X_2 \\ \text{subject to} \quad 6X_1 + 9X_2 &\leq 18 \\ 9X_1 + 3X_2 &\geq 9 \\ X_1, X_2 &\geq 0 \end{aligned}$$


- Find the optimal solution using the simplex method.
- What evidence indicates that an alternate optimal solution exists?
- Find the alternate optimal solution.
- Solve this problem graphically as well, and illustrate the alternate optimal corner points.

-  M7-26 At the third iteration of a particular LP maximization problem, the tableau at the bottom of this page is established. What special condition exists as you improve the profit and move to the next iteration? Proceed to solve the problem for the optimal solution.

-  M7-27 A pharmaceutical firm is about to begin production of three new drugs. An objective function designed to minimize ingredient costs and three production constraints are as follows:


$$\begin{aligned} \text{Minimize cost} &= 50X_1 + 10X_2 + 75X_3 \\ \text{Subject to} \quad X_1 - X_2 &= 1,000 \\ 2X_2 + 2X_3 &= 2,000 \\ X_1 &\leq 1,500 \\ X_1, X_2, X_3 &\geq 0 \end{aligned}$$

- Convert these constraints and objective function to the proper form for use in the simplex tableau.
- Solve the problem by the simplex method. What is the optimal solution and cost?


-  M7-28 The Bitz-Karan Corporation faces a blending decision in developing a new cat food called Yum-Mix. Two basic ingredients have been combined and tested, and the firm has determined that to each can of Yum-Mix at least 30 units of protein and at least 80 units of riboflavin must be added. These two nutrients are available in two competing brands of animal food supplements. The cost per kilogram of the brand A supplement is \$9, and the cost per

kilogram of brand B supplement is \$15. A kilogram of brand A added to each production batch of Yum-Mix provides a supplement of 1 unit of protein and 1 unit of riboflavin to each can. A kilogram of brand B provides 2 units of protein and 4 units of riboflavin in each can. Bitz-Karan must satisfy these minimum nutrient standards while keeping costs of supplements to a minimum.

- Formulate this problem to find the best combination of the two supplements to meet the minimum requirements at the least cost.
- Solve for the optimal solution by the simplex method.

-  M7-29 The Roniger Company produces two products: bed mattresses and box springs. A prior contract requires that the firm produce at least 30 mattresses or box springs, in any combination. In addition, union labor agreements demand that stitching machines be kept running at least 40 hours per week, which is one production period. Each box spring takes 2 hours of stitching time, and each mattress takes 1 hour on the machine. Each mattress produced costs \$20; each box spring costs \$24.

- Formulate this problem so as to minimize total production costs.
- Solve using the simplex method.

-  M7-30 Each coffee table produced by Meising Designers nets the firm a profit of \$9. Each bookcase yields a \$12 profit. Meising's firm is small, and its resources are limited. During any given production period of one week, 10 gallons of varnish and 12 lengths of high-quality redwood are available. Each coffee table requires approximately 1 gallon of varnish and 1 length of redwood. Each bookcase takes 1 gallon of varnish and 2 lengths of wood. Formulate Meising's production mix decision as an LP problem, and solve using the simplex method. How many tables and bookcases should be produced each week? What will the maximum profit be?


-  M7-31 Bagwell Distributors packages and distributes industrial supplies. A standard shipment can be packaged in a class A container, a class K container, or a class


Tableau for Problem M7-26

$C_j \rightarrow$		\$6	\$3	\$5	0	0	0	
\downarrow	SOLUTION MIX	X_1	X_2	X_3	S_1	S_2	S_3	QUANTITY
\$5	X_3	0	1	1	1	0	3	5
\$6	X_1	1	-3	0	0	0	1	12
\$0	S_2	0	2	0	1	1	-1	10
	Z_j	\$6	-\$13	\$5	\$5	\$0	\$21	\$97
	$C_j - Z_j$	\$0	\$16	\$0	-\$5	\$0	-\$21	

T container. A single class A container yields a profit of \$8; a class K container, a profit of \$6; and a class T container, a profit of \$14. Each shipment prepared requires a certain amount of packing material and a certain amount of time, as seen in the following table:

CLASS OF CONTAINER	PACKING MATERIAL (LB)	PACKING TIME (HOURS)
A	2	2
K	1	6
T	3	4
Total amount of resource available each week	120 lb	240 hours


Bill Bagwell, head of the firm, must decide the optimal number of each class of container to pack each week. He is bound by the previously mentioned resource restrictions, but he also decides that he must keep his six full-time packers employed all 240 hours (6 workers, 40 hours) each week. Formulate and solve this problem using the simplex method.

 **M7-32** The Foggy Bottom Development Corporation has just purchased a small hotel for conversion to condominium apartments. The building, in a popular area of Washington, D.C., near the U.S. State Department, will be highly marketable, and each condominium sale is expected to yield a good profit. The conversion process, however, includes several options. Basically, four types of condominiums can be designed out of the former hotel rooms. They are deluxe one-bedroom apartments, regular one-bedroom apartments, deluxe studios, and efficiency apartments. Each will yield a different profit, but each type also requires a different level of investment in carpeting, painting, appliances, and carpentry work. Bank loans dictate a limited budget that may be allocated to each of these needs. Profit and cost data, and cost of conversion requirements, for each apartment are shown in the accompanying table.

Thus, we see that the cost of carpeting a deluxe one-bedroom unit is \$1,100, the cost of carpeting a regular one-bedroom unit is \$1,000, and so on. A total of \$35,000 is budgeted for all new carpet in the building.

Zoning regulations dictate that the building contain no more than 50 condominiums when the conversion is completed—and no fewer than 25 units. The development company also decides that to have a good blend of owners, at least 40% but no more than 70% of the units should be one-bedroom apartments. Not all money budgeted in each category need be spent, although profit is not affected by cost savings. But since the money represents a bank loan, under no circumstances may it be exceeded or even shifted from one area, such as carpeting, to another, such as painting.

- (a) Formulate Foggy Bottom Development Corporation’s decision as a linear program to maximize profits.
- (b) Convert your objective function and constraints to a form containing the appropriate slack, surplus, and artificial variables.

 **M7-33** The initial simplex tableau on page M7-49 was developed by Tommy Gibbs, vice president of a large cotton spinning mill. Unfortunately, Gibbs quit before completing this important LP application. Stephanie Robbins, the newly hired replacement, was immediately given the task of using LP to determine what different kinds of yarn the mill should use to minimize costs. Her first need was to be certain that Gibbs correctly formulated the objective function and constraints. She could find no statement of the problem in the files, so she decided to reconstruct the problem from the initial tableau.


- (a) What is the correct formulation, using real decision variables (that is, X_i ’s) only?
- (b) Which variable will enter this current solution mix in the second tableau? Which basic variable will leave?

Tableau for Problem M7-32

RENOVATION REQUIREMENT	TYPE OF APARTMENT				TOTAL BUDGETED (\$)
	DELUXE ONE-BEDROOM (\$)	REGULAR ONE-BEDROOM (\$)	DELUXE STUDIO (\$)	EFFICIENCY (\$)	
New carpet	1,100	1,000	600	500	35,000
Painting	700	600	400	300	28,000
New appliances	2,000	1,600	1,200	900	45,000
Carpentry work	1,000	400	900	200	19,000
Profit per unit	8,000	6,000	5,000	3,500	

Simplex Tableau for Problem M7-33


$C_j \longrightarrow$		\$12	\$18	\$10	\$20	\$7	\$8	\$0	\$0	\$0	\$0	\$0	M	M	M	M	
\downarrow SOLUTION MIX		X_1	X_2	X_3	X_4	X_5	X_6	S_1	S_2	S_3	S_4	S_5	A_1	A_2	A_3	A_4	QUANTITY
$\$M$	A_1	1	0	-3	0	0	0	0	0	0	0	0	1	0	0	0	100
0	S_1	0	25	1	2	8	0	1	0	0	0	0	0	0	0	0	900
M	A_2	2	1	0	4	0	1	0	-1	0	0	0	0	1	0	0	250
M	A_3	18	-15	-2	-1	15	0	0	0	-1	0	0	0	0	1	0	150
0	S_4	0	0	0	0	0	25	0	0	0	1	0	0	0	0	0	300
M	A_4	0	0	0	2	6	0	0	0	0	0	-1	0	0	0	1	70
Z_j		\$21M	-\$14M	-\$5M	\$5M	\$21M	\$M	\$0	\$0	-\$M	\$0	-\$M	\$M	\$M	\$M	\$M	\$570M
$C_j - Z_j$		12 - 21M	18 + 14M	10 + 5M	20 - 5M	7 - 21M	8 - M	0	0	M	0	M	0	0	0	0	

 M7-34 Consider the following optimal tableau, where S_1 and S_2 are slack variables added to the original problem:


$C_j \longrightarrow$		\$10	\$30	\$0	\$0	
	SOLUTION MIX	X_1	X_2	S_1	S_2	QUANTITY
		X_1	X_2	S_1	S_2	
\$10	X_1	1	4	2	0	160
\$ 0	S_2	0	6	-7	1	200
Z_j		\$10	\$40	\$20	\$0	\$1,600
$C_j - Z_j$		0	-10	-20	0	

- What is the range of optimality for the contribution rate of the variable X_1 ?
- What is the range of insignificance of the contribution rate of the variable X_2 ?
- How much would you be willing to pay for one more unit of the first resource, which is represented by slack variable S_1 ?
- What is the value of one more unit of the second resource? Why?
- What would the optimal solution be if the profit on X_2 were changed to \$35 instead of \$30?
- What would the optimal solution be if the profit on X_1 were changed to \$12 instead of \$10? How much would the maximum profit change?

(g) How much could the right-hand side in constraint number 2 be decreased before profit would be affected?

 M7-35 A linear program has been formulated and solved. The optimal simplex tableau for this is given at the bottom of this page.

- What are the shadow prices for the three constraints? What does a zero shadow price mean? How can this occur?
- How much could the right-hand side of the first constraint be changed without changing the solution mix (i.e., perform RHS ranging for this constraint)?
- How much could the right-hand side of the third constraint be changed without changing the solution mix?

 M7-36 Clapper Electronics produces two models of telephone-answering devices, model 102 (X_1) and model H23 (X_2). Jim Clapper, vice president for production, formulates their constraints as follows:

$$2X_1 + 1X_2 \leq 40 \text{ (hours of time available on soldering machine)}$$

$$1X_1 + 3X_2 \leq 30 \text{ (hours of time available in inspection department)}$$

Clapper's objective function is

$$\text{Maximize profit} = \$9X_1 + \$7X_2$$


Tableau for Problem M7-35

$C_j \longrightarrow$		80	120	90	0	0	0	
	SOLUTION MIX	X_1	X_2	X_3	S_1	S_2	S_3	QUANTITY
		X_1	X_2	X_3	S_1	S_2	S_3	
120	X_2	-1.5	1	0	0.125	-0.75	0	37.5
90	X_3	3.5	0	1	-0.125	1.25	0	12.5
0	S_3	-1.0	0	0	0	-0.5	1	10.0
Z_j		135	120	90	3.75	22.5	0	5,625
$C_j - Z_j$		-55	0	0	-3.75	-22.5	0	


Solving the problem using the simplex method, he produces the following final tableau:

$C_j \longrightarrow$		\$9	\$7	\$0	\$0	
\downarrow	SOLUTION Mix	X_1	X_2	S_1	S_2	QUANTITY
		X_1	X_2	S_1	S_2	QUANTITY
\$9	X_1	1	0	0.6	-0.2	18
7	X_2	0	1	-0.2	0.4	4
	Z_j	\$9	\$7	\$4	\$1	\$190
	$C_j - Z_j$	0	0	-4	-1	

- What is the optimal mix of models 102 and H23 to produce?
- What do variables S_1 and S_2 represent?
- Clapper is considering renting a second soldering machine at a cost to the firm of \$2.50 per hour. Should he do so?
- Clapper computes that he can hire a part-time inspector for only \$1.75 per hour. Should he do so?

 M7-37 Refer to Table M7.6, which is the optimal tableau for the Flair Furniture Company problem.

- What are the values of the shadow prices?
- Interpret the physical meaning of each shadow price in the context of the furniture problem.
- What is the range over which the profit per table can vary without changing the optimal basis (solution mix)?
- What is the range of optimality for C (number of chairs produced)?
- How many hours can Flair Furniture add to or remove from the first resource (painting department time) without changing the basis?
- Conduct RHS ranging on the carpentry department resource to determine the range over which the shadow price remains valid.

 M7-38 Consider the optimal solution to the Muddy River Chemical Corporation problem in Table M7.10.

- For each of the two chemical ingredients, phosphate and potassium, determine the range over which their cost may vary without affecting the basis.
- If the original constraint that “no more than 300 pounds of phosphate can be used” ($X_1 \leq 300$)

were changed to $X_1 \leq 400$, would the basis change? Would the values of X_1 , X_2 , and S_2 change?

- M7-39 Formulate the dual of this LP problem:

$$\begin{aligned} \text{Maximize profit} &= 80X_1 + 75X_2 \\ 1X_1 + 3X_2 &\leq 4 \\ 2X_1 + 5X_2 &\leq 8 \end{aligned}$$

Find the dual of the problem's dual.

- M7-40 What is the dual of the following LP problem?

$$\begin{aligned} \text{Primal: Minimize cost} &= 120X_1 + 250X_2 \\ \text{subject to} \quad 12X_1 + 20X_2 &\geq 50 \\ X_1 + 3X_2 &\geq 4 \end{aligned}$$

- M7-41 The third, and final, simplex tableau for the LP problem stated here follows:

$$\begin{aligned} \text{Maximize profit} &= 200X_1 + 200X_2 \\ \text{subject to} \quad 2X_1 + X_2 &\leq 8 \\ X_1 + 3X_2 &\leq 9 \end{aligned}$$

What are the solutions to the dual variables, U_1 and U_2 ? What is the optimal dual cost?

$C_j \longrightarrow$		\$200	\$200	\$0	\$0	
\downarrow	SOLUTION Mix	X_1	X_2	S_1	S_2	QUANTITY
		X_1	X_2	S_1	S_2	QUANTITY
\$200	X_1	1	0	0.6	-0.2	3
200	X_2	0	1	-0.2	0.4	2
	Z_j	\$200	\$200	\$80	\$40	\$1,000
	$C_j - Z_j$	0	0	-80	-40	

- M7-42 The accompanying tableau provides the optimal solution to this dual:

$$\begin{aligned} \text{Minimize cost} &= 120U_1 + 240U_2 \\ \text{subject to} \quad 2U_1 + 2U_2 &\geq 0.5 \\ U_1 + 3U_2 &\geq 0.4 \end{aligned}$$

What does the corresponding primal problem look like, and what is its optimal solution?

Tableau for Problem M7-42

$C_j \longrightarrow$		120	240	0	0	M	M	
\downarrow	SOLUTION Mix	U_1	U_2	S_1	S_2	A_1	A_2	QUANTITY
		U_1	U_2	S_1	S_2	A_1	A_2	QUANTITY
\$120	U_1	1	0	-0.75	0.5	0.75	-0.5	0.175
240	U_2	0	1	0.25	-0.5	-0.25	0.5	0.075
	Z_j	\$120	\$240	-\$30	-\$60	\$30	\$60	\$39
	$C_j - Z_j$	0	0	30	60	$M - 30$	$M - 60$	

- M7-43 Given the following dual formulation, reconstruct the original primal problem:

$$\begin{aligned}
 &\text{Minimize cost} = 28U_1 + 53U_2 + 70U_3 + 18U_4 \\
 &\text{subject to} \quad U_1 + \quad \quad \quad U_4 \geq 10 \\
 &\quad \quad \quad U_1 + 2U_2 + \quad U_3 \geq 5 \\
 &\quad \quad \quad \quad - 2U_2 + \quad \quad \quad 5U_4 \geq 31 \\
 &\quad \quad \quad \quad \quad \quad \quad 5U_3 \geq 28 \\
 &\quad 12U_1 + \quad \quad \quad 2U_3 - \quad U_4 \geq 17 \\
 &\quad \quad \quad \quad \quad \quad \quad U_1, U_2, U_3, U_4 \geq 0
 \end{aligned}$$

- M7-44 A firm that makes three products and has three machines available as resources constructs the following LP problem:

$$\begin{aligned}
 &\text{Maximize profit} = 4X_1 + 4X_2 + 7X_3 \\
 &\text{subject to} \quad 1X_1 + 7X_2 + 4X_3 \leq 100 \text{ (hours on machine 1)} \\
 &\quad \quad \quad 2X_1 + 1X_2 + 7X_3 \leq 110 \text{ (hours on machine 2)} \\
 &\quad \quad \quad 8X_1 + 4X_2 + 1X_3 \leq 110 \text{ (hours on machine 3)}
 \end{aligned}$$

Solve this problem by computer and answer these questions:

- Before the third iteration of the simplex method, which machine still has unused time available?
 - When the final solution is reached, is there any unused time available on any of the three machines?
 - What would it be worth to the firm to make an additional hour of time available on the third machine?
 - How much would the firm's profit increase if an extra 10 hours of time were made available on the second machine at not extra cost?
- M7-45 Management analysts at a Fresno laboratory have developed the following LP primal problem:

$$\begin{aligned}
 &\text{Minimize cost} = 23X_1 + 18X_2 \\
 &\text{subject cost} \quad 8X_1 + 4X_2 \geq 120 \\
 &\quad \quad \quad 4X_1 + 6X_2 \geq 115 \\
 &\quad \quad \quad 9X_1 + 4X_2 \geq 116
 \end{aligned}$$

This model represents a decision concerning number of hours spent by biochemists on certain laboratory experiments (X_1) and number of hours spent by biophysicists on the same series of experiments (X_2). A biochemist costs \$23 per hour, while a biophysicist's salary averages \$18 per hour. Both types of scientists can be used on three needed laboratory operations: test 1, test 2, and test 3. The experiments and their times are shown in the accompanying table:

LAB EXPERIMENT	SCIENTIST TYPE		MINIMUM TEST TIME NEEDED PER DAY
	BIOPHYSICIST	BIOCHEMIST	
Test 1	8	4	120
Test 2	4	6	115
Test 3	9	4	116

This means that a biophysicist can complete 8, 4, and 9 of tests 1, 2, and 3 per hour. Similarly, a biochemist can perform 4 of test 1, 6 of test 2, and 4 of test 3 per hour. The optimal solution to the lab's primal problem is

$$X_1 = 8.12 \text{ hours and } X_2 = 13.75 \text{ hours}$$

Total cost = \$434.37 per day

The optimal solution to the dual problem is

$$U_1 = 2.07, U_2 = 1.63, U_3 = 0$$

- What is the dual of the primal LP problem?
- Interpret the meaning of the dual and its solution.

- M7-46 Refer to Problem M7-45.

- If this is solved with the simplex algorithm, how many constraints and how many variables (including slack, surplus, and artificial variables) would be used?
- If the dual of this problem were formulated and solved with the simplex algorithm, how many constraints and how many variables (including slack, surplus, and artificial variables) would be used?
- If the simplex algorithm were used, would it be easier to solve the primal problem or the dual problem?

- M7-47 The Flair Furniture Company first described in Chapter 7, and again in this chapter, manufactures inexpensive tables (T) and chairs (C). The firm's daily LP formulation is given as

$$\text{Maximize profits} = 70T + 50C$$

subject to $4T + 3C \leq 240$ hours of carpentry time available

$2T + 1C \leq 100$ hours of painting time available

In addition, Flair finds that three more constraints are in order. First, each table and chair must be inspected and may need reworking. The following constraint describes the time required on the average for each:

$$0.5T + 0.6C \leq 36 \text{ hours of inspection/rework time available}$$

Second, Flair faces a resource constraint relating to the lumber needed for each table or chair and the amount available each day:

$$32T + 10C \leq 1,248 \text{ linear feet of lumber available for production}$$

Finally, the demand for tables is found to be a maximum of 40 daily. There are no similar constraints regarding chairs.

$$T \leq 40 \text{ maximize table production daily}$$

These data have been entered in the QM for Windows software that is available with this book. The inputs and results are shown in the accompanying printout. Refer to the computer output in Programs M7.1A, M7.1B, and M7.1C in answering these questions.

- How many tables and chairs should Flair Furniture produce daily? What is the profit generated by this solution?
- Will Flair use all of its resources to their limits each day? Be specific in explaining your answer.
- Explain the physical meaning of each shadow price.
- Should Flair purchase more lumber if it is available at \$0.07 per linear foot? Should it hire more carpenters at \$12.75 per hour?
- Flair's owner has been approached by a friend whose company would like to use several hours

in the painting facility every day. Should Flair sell time to the other firm? If so, how much? Explain.

- What is the range within which the carpentry hours, painting hours, and inspection/rework hours can fluctuate before the optimal solution changes?
- Within what range for the current solution can the profit contribution of tables and chairs change?

PROGRAM M7.1A
QM for Windows Input
Data for Revised Flair
Furniture Problem
(Problem M7-47)


Flair Furniture				
	T	C		RHS
Maximize	70	50		
Carpentry hours	4	3	<=	240
Painting hours	2	1	<=	100
Inspection hours	0.5	0.6	<=	36
Lumber (linear feet)	32	10	<=	1248
Demand	1	0	<=	40

PROGRAM M7.1B
Solution Results (Final
Tableau) for Flair
Furniture Problem
(Problem M7-47)

Flair Furniture Solution									
Cj	Basic Variables	70 T	50 C	0 slack 1	0 slack 2	0 slack 3	0 slack 4	0 slack 5	Quantity
Iteration 3									
0	slack 1	0	0	1	0	-3.9437	-0.0634	0	18.9296
0	slack 2	0	0	0	1	-0.8451	-0.0493	0	8.0563
50	C	0	1	0	0	2.2535	-0.0352	0	37.1831
70	T	1	0	0	0	-0.7042	0.0423	0	27.3803
0	slack 5	0	0	0	0	0.7042	-0.0423	1	12.6197
	zj	70	50	0	0	63.3803	1.1972	0	
	cj-zj	0	0	0	0	-1.1972	0	0	

PROGRAM M7.1C
Sensitivity Analysis
for Problem M7-47

Flair Furniture Solution					
Variable	Value	Reduced	Original Val	Lower Bound	Upper Bound
T	27.3803	0	70	41.6667	160
C	37.1831	0	50	21.875	84
Constraint	Dual Value	Slack/Surplus	Original Val	Lower Bound	Upper Bound
Carpentry hours	0	18.9296	240	221.0704	Infinity
Painting hours	0	8.0563	100	91.9437	Infinity
Inspection hours	63.3803	0	36	19.5	40.8
Lumber (linear feet)	1.1972	0	1248	600	1411.429
Demand	0	12.6197	40	27.3803	Infinity

 **M7-48** A Chicago manufacturer of office equipment is desperately attempting to control its profit and loss statement. The company currently manufactures 15 different products, each coded with a one-letter and three-digit designation.

- How many of each of the 15 products should be produced each month?
- Clearly explain the meaning of each shadow price.
- A number of workers interested in saving money for the holidays have offered to work overtime next month at a rate of \$12.50 per hour. What should the response of management be?
- Two tons of steel alloy are available from an overstocked supplier at a total cost of \$8,000.

Should the steel be purchased? All or part of the supply?

- The accountants have just discovered that an error was made in the contribution to profit for product N150. The correct value is actually \$8.88. What are the implications of this error?
- Management is considering the abandonment of five product lines (those beginning with the letters A through E). If no minimum monthly demand is established, what are the implications? Note that there already is no minimum for two of these products. Use the corrected value for N150.

PRODUCT	STEEL ALLOY REQUIRED (LB)	PLASTIC REQUIRED (SQ FT)	WOOD REQUIRED (BD FT)	ALUMINUM REQUIRED (LB)	FORMICA REQUIRED (BD FT)	LABOR REQUIRED (HOURS)	MINIMUM MONTHLY DEMAND (UNITS)	CONTRIBUTION TO PROFIT
A158	—	0.4	0.7	5.8	10.9	3.1	—	\$18.79
B179	4	0.5	1.8	10.3	2.0	1.0	20	6.31
C023	6	—	1.5	1.1	2.3	1.2	10	8.19
D045	10	0.4	2.0	—	—	4.8	10	45.88
E388	12	1.2	1.2	8.1	4.9	5.5	—	63.00
F422	—	1.4	1.5	7.1	10.0	0.8	20	4.10
G366	10	1.4	7.0	6.2	11.1	9.1	10	81.15
H600	5	1.0	5.0	7.3	12.4	4.8	20	50.06
I701	1	0.4	—	10.0	5.2	1.9	50	12.79
J802	1	0.3	—	11.0	6.1	1.4	20	15.88
K900	—	0.2	—	12.5	7.7	1.0	20	17.91
L901	2	1.8	1.5	13.1	5.0	5.1	10	49.99
M050	—	2.7	5.0	—	2.1	3.1	20	24.00
N150	10	1.1	5.8	—	—	7.7	10	88.88
P259	10	—	6.2	15.0	1.0	6.6	10	77.01
Availability per month	980	400	600	2,500	1,800	1,000		

Bibliography

See the Bibliography at the end of Chapter 7.

