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HW 15: 3.8-3.10
M328K
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3.8 Theorem. *Suppose* $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$ *is a polynomial of degree* $n > 0$ *with integer coefficients. Let* a*,* b*, and* m *be integers with* $m > 0$*. If* $a \equiv b \pmod{m}$*, then* $f(a) \equiv f(b) \pmod{m}$*.*

Proof. Suppose $f(x) = c_n x^n + c_{n-1}x^{n-1} + \ldots + c_0$ is a polynomial of degree $n > 0$ with integer coefficients. Let a, b, and m be integers with $m > 0$ and $a \equiv b \pmod{m}$. We will show $f(a) \equiv f(b) \pmod{m}$. First, since $a \equiv b \pmod{m}$, we know that $a^k \equiv b^k \pmod{m}$ for any integer k by theorem 1.18. Because $c_k \equiv c_k \pmod{m}$ for all k, we can see that for each term in f, $c_{n-j}a^{n-j} \equiv c_{n-j}b^{n-j} \pmod{m}$, where j is some integer such that $0 \leq j \leq n$, by theorem 1.14. Now, by theorem 1.12, since $c_n a^n \equiv c_n b^n \pmod{m}$ and $c_{n-1}a^{n-1} \equiv c_{n-1}b^{n-1} \pmod{m}$, we have that $c_n a^n + c_{n-1}a^{n-1} \equiv c_n b^n + c_{n-1}b^{n-1} \pmod{m}$. By theorem 1.12 again, $c_n a^n + c_{n-1}a^{n-1} + c_{n-2}a^{n-2} \equiv c_n b^n + c_{n-1}b^{n-1} + c_{n-2}b^{n-2} \pmod{m}$. We can apply this same logic for each term in $f(x)$ to show that $f(a) \equiv f(b) \pmod{m}$. $\square$

3.9 Corollary. *Let the natural number n be expressed in base 10 as*

$$n = a_k a_{k-1} \dots a_1 a_0.$$

Let $m = a_k + a_{k-1} + \dots + a_1 + a_0$. Then $9|n$ if and only if $9|m$.

Proof. Let the natural number n be expressed in base 10 as

$$n = a_k a_{k-1} \dots a_1 a_0.$$

Let $m = a_k + a_{k-1} + \dots + a_1 + a_0$. We will show that $9 \mid n$ if and only if $9 \mid m$.

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 x^0$. We can now express n as $f(10)$ and m as $f(1)$. Since $10 - 1 = 9$, by definition of congruence mod n we have that $10 \equiv 1 \pmod 9$. So by theorem 3.8, $f(10) \equiv f(1) \pmod 9$, or equivalently, $n \equiv m \pmod 9$. Therefore by definition of congruence mod n, $9 \mid n - m$.

First, assume that $9 \mid m$. In this case, $9 \mid m$ and $9 \mid n - m$, so by theorem 1.1 $9 \mid m + (n - m)$, that is $9 \mid n$.

Now assume that $9 \mid n$. In this case, $9 \mid n$ and $9 \mid n - m$, so by theorem 1.2 $9 \mid n - (n - m)$, that is $9 \mid m$.

Therefore $9|n$ if and only if $9|m$. □

3.10 Corollary. *Let the natural number n be expressed in base 10 as*

$$n = a_k a_{k-1} \dots a_1 a_0.$$

If $m = a_k + a_{k-1} + \dots + a_1 + a_0$. Then $3|n$ if and only if $3|m$.

Proof. Let the natural number n be expressed in base 10 as

$$n = a_k a_{k-1} \dots a_1 a_0.$$

If $m = a_k + a_{k-1} + \dots + a_1 + a_0$. We will show that $3|n$ if and only if $3|m$.

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 x^0$. We can now express n as $f(10)$ and m as $f(1)$. Since $10 - 1 = 9$ and $3 \mid 9$, by definition of congruence mod n we have that $10 \equiv 1 \pmod 3$. So by theorem 3.8, $f(10) \equiv f(1) \pmod 3$, or equivalently, $n \equiv m \pmod 3$. Therefore by definition of congruence mod n, $3 \mid n - m$.

First, assume that $3 \mid m$. In this case, $3 \mid m$ and $3 \mid n - m$, so by theorem 1.1 $3 \mid m + (n - m)$, that is $3 \mid n$.

Now assume that $3 \mid n$. In this case, $3 \mid n$ and $3 \mid n - m$, so by theorem 1.2 $3 \mid n - (n - m)$, that is $3 \mid m$.

Therefore $3|n$ if and only if $3|m$. $\square$