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HW 12: 2.42-2.44
M328K
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2.42 Theorem. *If n is a natural number and $2^n - 1$ is prime, then n must be prime.*

Proof. Let n be a natural number with $2^n - 1$ being prime. Assume by way of contradiction that n is not prime. Then by the definition of not prime there will be two natural numbers x and y such that $n = xy$ and $x > 1$. So $2^n - 1 = 2^{xy} - 1 = (2^x)^y - 1$. We can now use polynomial long division, similar to exercise 2.41, to find that:

$$(2^x)^y - 1 = (2^x - 1)((2^x)^{y-1} + (2^x)^{y-2} + \cdots + (2^x) + 1)$$

Since $x > 1$, $2^x - 1 > 1$. Now we have shown that $2^n - 1$ is not prime, which is a contradiction. Therefore n is prime. $\square$

2.43 Theorem. *If n is a natural number and $2^n + 1$ is prime, then n must be a power of 2.*

Proof. Let n be a natural number with $2^n + 1$ being prime. Assume by way of contradiction that n is not a power of 2. This means that the prime factorization of n contains some term other than 2. Let x be the product of all the terms in the prime factorization of n that are not 2. Let y be 2^k for some integer k such that $n = xy$. Now $2^n + 1 = (2^x)^y + 1$. Preforming polynomial division gives us:

$$(2^x)^y + 1 = (2^x + 1)((2^x)^{y-1} - (2^x)^{y-2} - \cdots - 2^x - 1$$

Since $x > 1$, we have found a factorization of $2^n + 1$. Since $2^n + 1$ is prime, this is a contradiction. Therefore n is a power of 2. $\square$

2.44 Exercise. *Find the first few Mersenne primes and Fermat primes.*

Solution. Mersenne: 3, 7, 31, 127, 8191, 131071, 524287.
Fermat: 3, 5, 17, 257, 65537. $\square$