CS 341 Automata Theory
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Homework 10
Due: Tuesday, March 27

This assignment reviews Chapter 13 and covers Chapter 14 and Sections 17.1 - 17.3.

1) For each of the following languages L, state whether L is regular, context-free but not regular, or not context-free and prove your answer.

   a) $\{w : w = uu^R \text{ or } w = ua^n \colon n = |u|, u \in \{a,b\}^*\}$.

   Answer. Context-free but not regular. □

   Proof. This language is generated by this context-free grammar so it must be context-free:

$$\begin{aligned} S &\to A \mid B \\ A &\to aAa \mid bAa \mid \epsilon \\ B &\to aBa \mid bBb \mid \epsilon \end{aligned}$$

   This language L is not regualar. Let $w = b^k a^k$ for some integer $k \geq 1$. Then y must be b^p for some integer p such that $1 \leq p \leq n$. Pumping out y generates a new string $w' = b^{k-p}a^k$ which is not in the language. Therefore by the pumping theorem this language is not regular. □

   b) $\{a^n b^{2n} c^m\} \cap \{a^n b^m c^{2m}\}$.

   Answer. Not context-free. This language is equivalent to $L = \{a^n b^{2n} c^{4n}\}$. □

   Proof. Let $w = a^k b^{2k} c^{4k}$. Let region 1 be the a's, region 2 be the b's and region 3 be the c's. In any case where vxy crosses a region boundry, pumping in will result in a new string with out of order a's b's or c's, and thus not in the language. If vxy is contained solely within a region, pumping out will disrupt the cardinality requirement, thus also generating a string not in the language. Therefore by the pumping theorem for context-free languages this language is not context-free. □

   c) L^*, where $L = \{0^*1^i0^*1^i0^* : i \geq 0\}$.

   Answer. Regular □

   Proof. L^* is equivalent to $\{w : w \in \{0,1\}^* \text{ where } \#_1(w) \text{ is even.}\}$, which we have already established to be regular. □

d) $\neg L_0$, where $L_0 = \{ww : w \in \{a,b\}^*\}$.

Answer. Context-free but not regular. □

Proof. CFG:

$$
\begin{aligned}
S &\to AB \mid BA \mid A \mid B \\
A &\to aAa \mid aAb \mid bAa \mid bAb \mid a \\
B &\to aBa \mid aBb \mid bBa \mid bBb \mid b
\end{aligned}
$$

If $\neg L_0$ were regular, then L_0 would be regular. Let $w = a^k b^k a^k b^k$, which is an element of L_0. Then y must be a^p for some integer p where $1 \le p \le k$ and y is in the first a region. So pumping out gives $w' = a^{k-p} b^k a^k b^k$, which is not an element of L_0. Therefore by the pumping theorem L_0 is not regular, so $\neg L_0$ is not regular. □

e) $\{x \in \{a,b\}^* : |x|$ is even and the first half of x has one more a than does the second half$\}$.

Answer. Not context-free. □

Proof. Let $w = ab^{2k+1}$. If vxy does not contain the a there are two cases.
If $|vxy|$ is odd, pump out once to generate an odd length string.
If $|vxy|$ is even, pump in twice to generate an odd length string.
If vxy contains the a, pump out to generate a string without an a, violating the requirement that the first half contains one more a than the second half. □

2) Give a decision procedure to answer the following question: given a context-free grammar G, does G generate any even length strings?

Solution.

1. Let L be the language generated by G.
2. Let $L' = L \cap ((a \cup b)(a \cup b))^*$.
3. Construct a PDA M' to recognize M'.
4. Determine if M' recognizes any strings.

□

3) Construct a standard, one-tape Turing machine M to decide the language $L = \{x{*}y = z : x, y, z \in 1^+$ and, when x, y, and z are viewed as unary numbers, $xy = z\}$. For example, the string $1111{*}11 = 11111111 \in L$. Describe M in the macro language described in Section 17.1.5.

Solution. See attached. □

4) Construct a standard 1-tape Turing machine M to compute the function sub_3, which is defined as follows:

$$sub_3(n) = \begin{array}{ll} n-3 & \text{if } n > 2 \\ 0 & \text{if } n \leq 2. \end{array}$$

Specifically, compute sub_3 of a natural number represented in binary. For example, on input 10111, M should output 10100. On input 11101, M should output 11010. (Hint: you may want to define a subroutine.)

Solution. See attached. □