CS 341 Automata Theory
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Homework 14
Due: Tuesday, April 23

This assignment covers Sections 21.5 - 21.7

Note: We have skipped Rice's Theorem. So problems 4 - 5 are here just in case you're interested in learning about it. They are optional.

1) For each of the following languages L, state whether it is in D, SD/D or not SD. Prove your answer. Do not use Rice's Theorem. If you claim that L is not in SD, first prove that it's not in D (for practice), then prove that it's not in SD. Assume that any input of the form $\langle M \rangle$ is a description of a Turing machine.

   a) $\{\langle M \rangle$: TM M accepts exactly two strings and they are of different lengths$\}$.

   Answer. Not $\in$ SD. $\square$

   Proof. To prove that this language is not $\in$ D:
   Create a reduction R from $\langle M, w \rangle$ to $\langle M_\# \rangle$ as follows.
   Create a description $\langle M_\# \rangle$ of a machine $M_\#(x)$ that does this:

   1. Erase input
   2. Write w on tape.
   3. Run M on w.
   4. If x is `a` or `aa` accept.

   R returns $M_\#$

   Assume there is a machine $Oracle(\langle M \rangle)$ that decides this language. Then for any language-string pair $\langle M, w \rangle$ consider $C = oracle(R(\langle M, w \rangle))$.

   If $\langle M, w \rangle \in$ H: $M_\#$ accepts exactly two strings. Oracle accepts. If $\langle M, w \rangle \notin$ H: $M_\#$ accepts no strings. Oracle rejects.
   Therefore C decides H so $Oracle$ must not exist.

To prove that this language is not $\in$ SD:

Create a reduction R from $\langle M, w\rangle$ to $\langle M_{\#}\rangle$ as follows.
Create a description $\langle M_{\#}\rangle$ of a machine $M_{\#}(x)$ that does this:

1. If x is `a` or `aa` accept.
2. Erase input
3. Write w on tape.
4. Run M on w.
5. Accept

R returns $M_{\#}$
Assume there is a machine $Oracle(\langle M\rangle)$ that decides this language. Then for any language-string pair $\langle M, w\rangle$ consider $C = oracle(R(\langle M, w\rangle))$.

If $\langle M, w\rangle \in \neg$ H: $M_{\#}$ Accepts exactly two strings and loops on all others. Oracle accepts.
If $\langle M, w\rangle \notin \neg$ H: $M_{\#}$ Accepts all strings. Oracle rejects.

Therefore C semi-decides ¬H so *Oracle* must not exist. □

b) $\{\langle M, x, y\rangle : M$ accepts $xy\}$.

Answer. In SD but not in D. □

Proof.
Let R be a reduction from $\langle M, w\rangle$ to $\langle M, x, y\rangle$ as follows:

1. Create a description $\langle M_{\#}\rangle$ of a machine $M_{\#}$ that does:
   1.1 Erase the tape.
   1.2 Write w on the tape.
   1.3 Run M on w.
   1.4 Accept.
2. Return $\langle M_{\#}, \epsilon, \epsilon\rangle$

Assume by way of contradiction that there exists some machine $Oracle(\langle M, x, y\rangle)$ that decides whether M accepts xy. Then for any machine description string pair $\langle M, w\rangle$ let $C = oracle(R(\langle M, w\rangle))$. Now there are two cases:

If $\langle M, w\rangle \in$ H: $M_{\#}$ accepts all strings, so it accepts xy. Oracle accepts.
If $\langle M, w\rangle \notin$ H: $M_{\#}$ accepts no strings, so it does not accept xy. Oracle rejects.

So C decides H. Therefore *Oracle* does not exist. □

c) $\{\langle M\rangle :$ Turing machine M accepts all even length strings$\}$.

Answer. Not $\in$ SD. □

Proof. □

d) $\{\langle M\rangle : M$ rejects exactly three strings that start with `a`$\}$

Answer. □

Proof. $\square$

e) $\{\langle M_a, M_b \rangle : L(M_a) - L(M_b) = \emptyset\}$.

Answer. $\square$

Proof. $\square$

2) Prove that $\mathrm{TM}_{\mathrm{REG}}$ is not in SD.

Proof. $\square$

3) For any nonempty alphabet Σ, let L be any decidable language other than $\emptyset$ or Σ^*. Prove that $L \leq_M \neg L$.

Proof. $\square$

4) * Do the other half of the proof of Rice's Theorem, i.e., show that the theorem holds if $P(\emptyset) = True$. (Hint: use a reduction that is not a mapping reduction.)

Proof. $\square$

5) * Use Rice's Theorem to prove that $\{\langle M \rangle$: Turing machine M accepts at least two odd length strings$\}$ is not in D.

Proof. $\square$