Basics of Convex Optimization

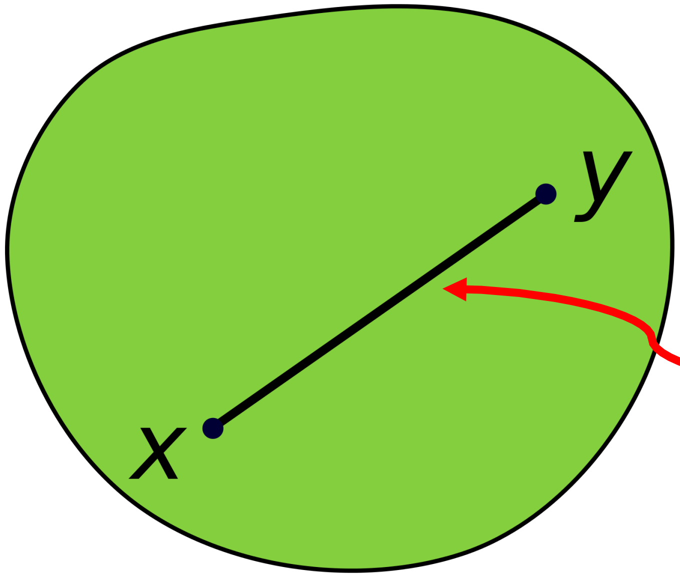
Shusen Wang

Convex Sets

Convex Set

Definition (Convex Set).

A set $\mathcal{C}$ is convex if and only if for any $\mathbf{x}, \mathbf{y} \in \mathcal{C}$ and any $\eta \in (0, 1)$, the point $\eta\mathbf{x} + (1 - \eta)\mathbf{y}$ is also in $\mathcal{C}$.



By definition, the line segment between $\mathbf{x}$ and $\mathbf{y}$ is in $\mathcal{C}$.

A convex set $\mathcal{C}$.

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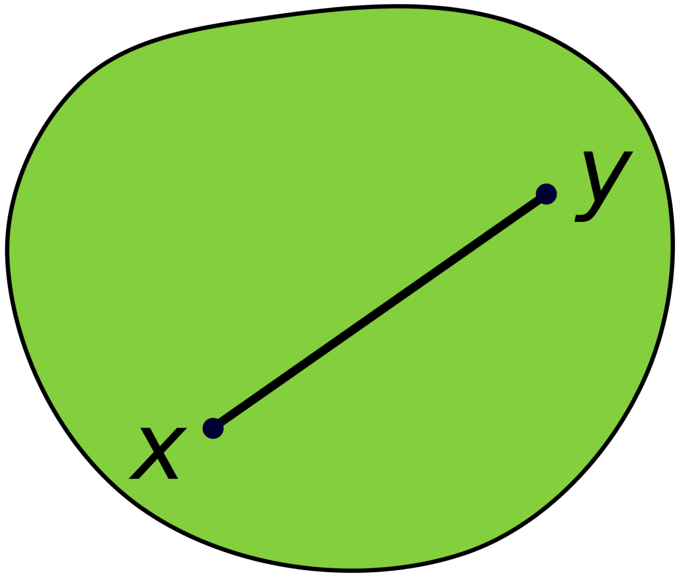
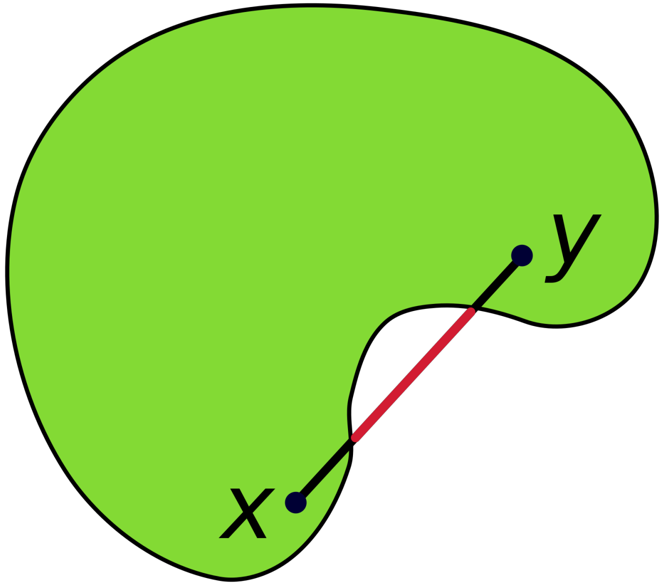


A convex set $\mathcal{C}$.



A non-convex set.

Convex Set: Examples

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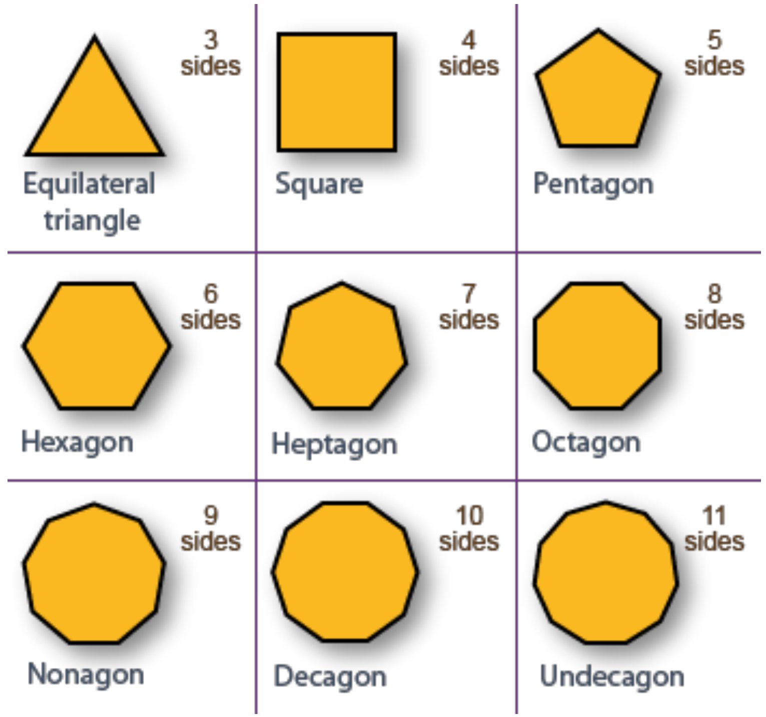
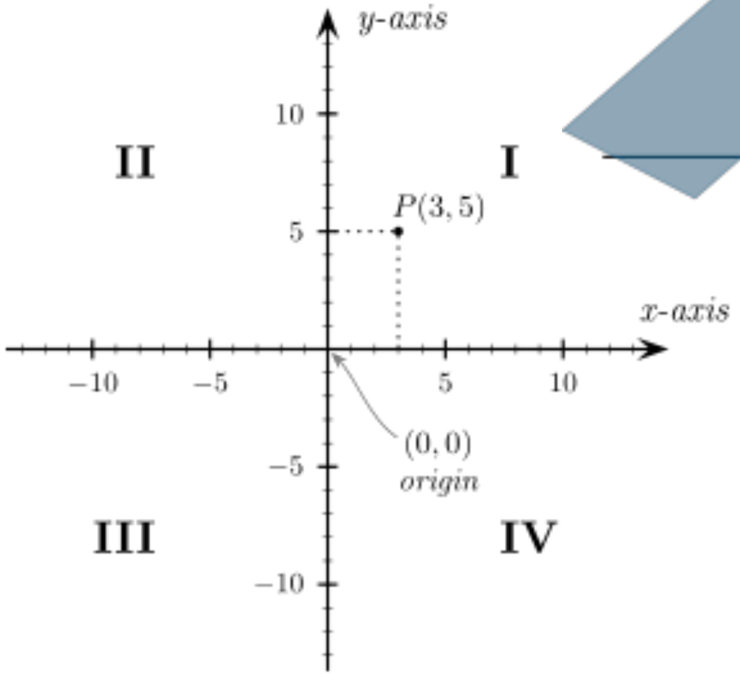
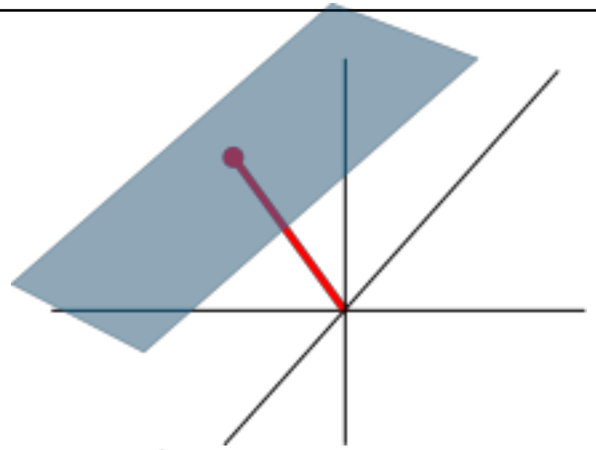
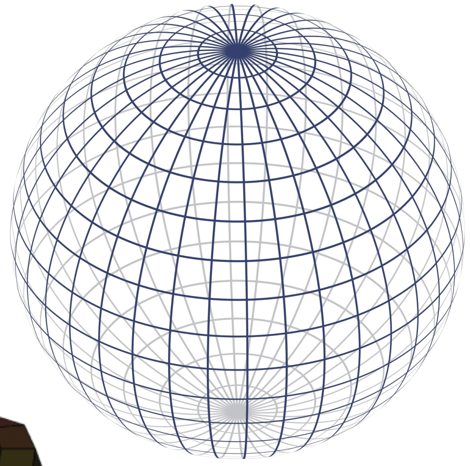
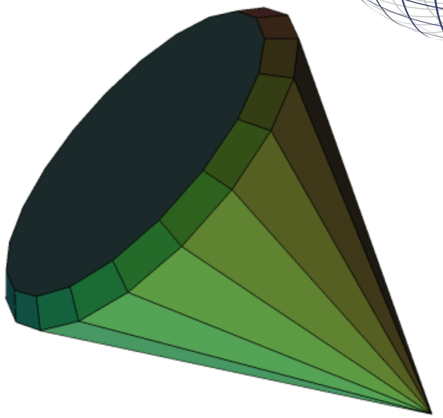










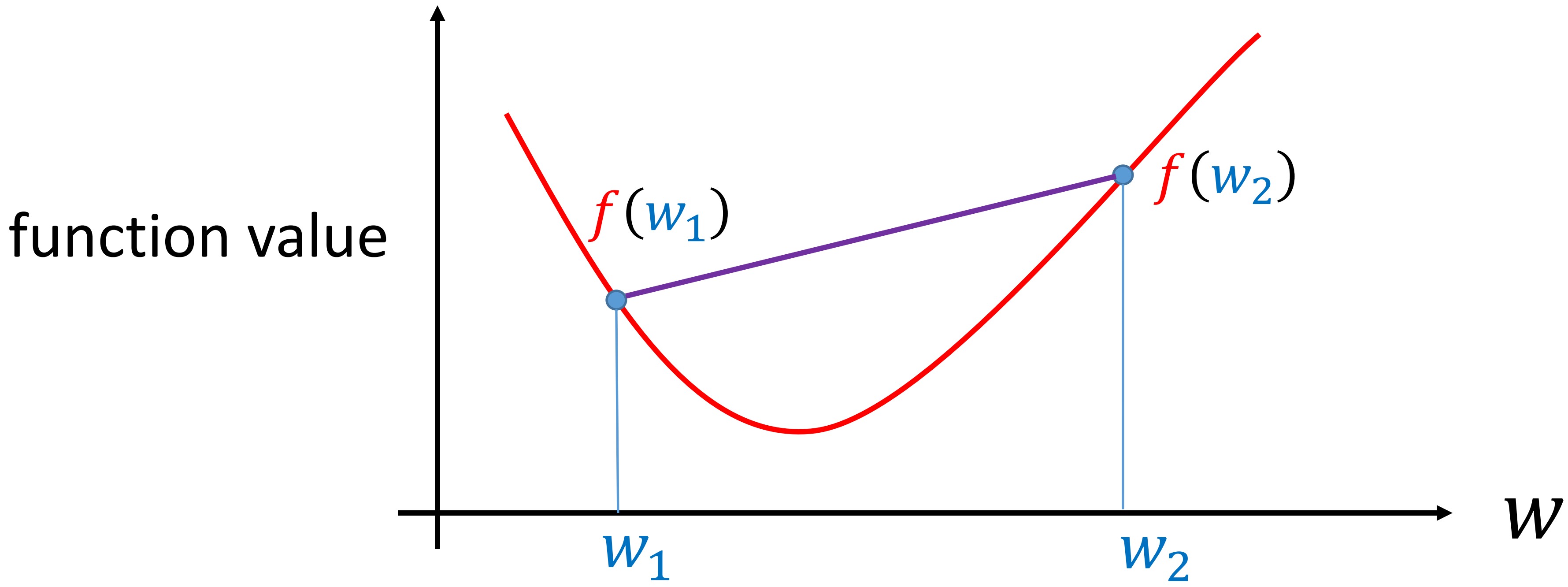
Convex Functions

Convex Function

Definition (Convex Function).

- Let $\mathcal{C}$ be a convex set and $f: \mathcal{C} \mapsto \mathbb{R}$ be a function.
- f is convex if for any $\mathbf{w}_1, \mathbf{w}_2 \in \mathcal{C}$ and any $\eta \in (0,1)$,

$$f(\eta \mathbf{w}_1 + (1-\eta)\mathbf{w}_2) \;\leq\; \eta f(\mathbf{w}_1) + (1-\eta) f(\mathbf{w}_2).$$



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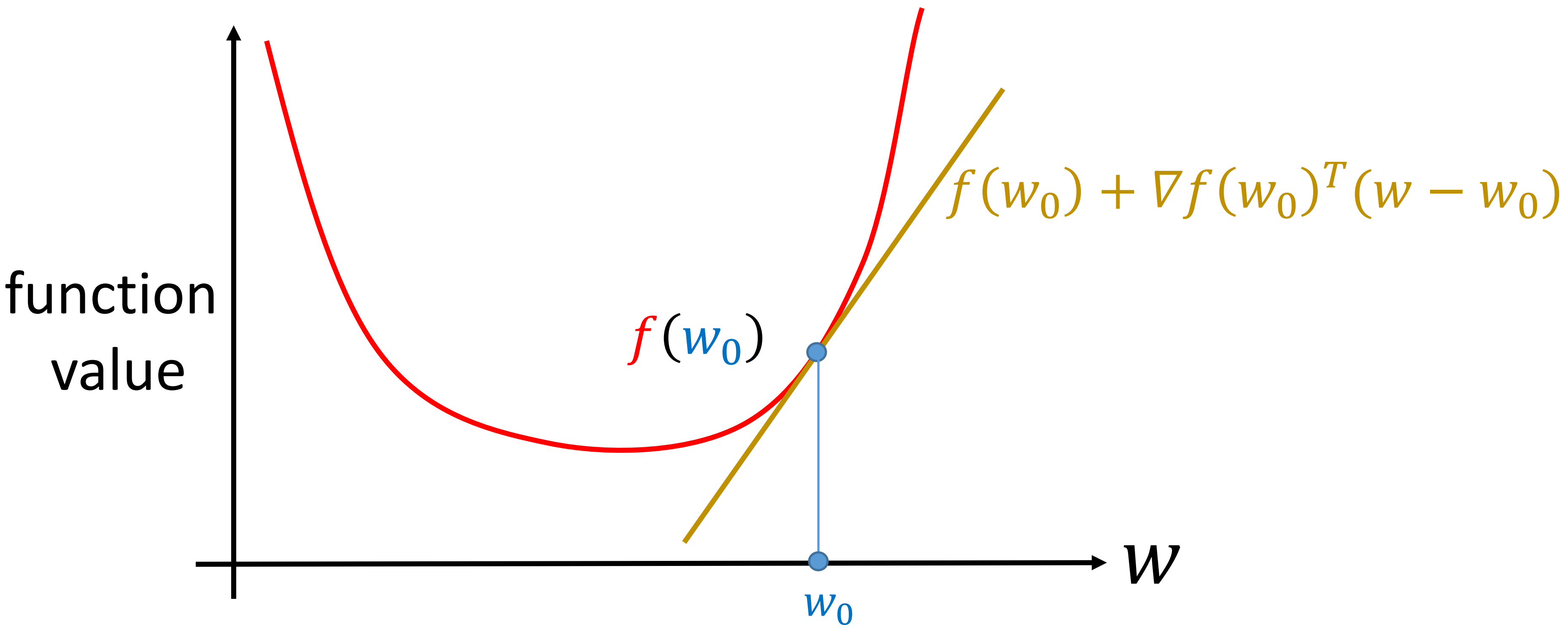
- Examples of convex functions:
  - Linear function $f(\mathbf{w}) = \mathbf{a}^T\mathbf{w} + b$. (Here $\mathbf{w} \in \mathbb{R}^d$, $\mathbf{a} \in \mathbb{R}^d$, and $b \in \mathbb{R}$.)
  - Vector norm $f(\mathbf{w}) = ||\mathbf{w}||_p$ for all $p \geq 1$. (Defined on $\mathbb{R}^d$)
  - Vector norm $f(\mathbf{w}) = ||\mathbf{w}||_p^p$ for all $p \geq 1$. (Defined on $\mathbb{R}^d$)
  - Exponential $f(w) = e^w$. (Defined on $\mathbb{R}$)
  - Sum of convex functions $f(\mathbf{w}) = f_1(\mathbf{w}) + \cdots + f_k(\mathbf{w})$.

Convex Function: Properties

Properties of convex function:

1. $f(\mathbf{w}_0) + \nabla f(\mathbf{w}_0)^T(\mathbf{w} - \mathbf{w}_0) \leq f(\mathbf{w})$. (Assume f is differentiable).



Convex Function: Properties

Properties of convex function:

1. $f(\mathbf{w}_0) + \nabla f(\mathbf{w}_0)^T(\mathbf{w} - \mathbf{w}_0) \leq f(\mathbf{w})$. (Assume f is differentiable).
2. The Hessian matrix is everywhere positive semi-definite: $\nabla^2 f(\mathbf{w}) \succcurlyeq \mathbf{0}$.

- Assume f is twice differentiable.
- $\mathbf{H} \in \mathbb{R}^{d \times d}$ is positive semi-definite $\Longleftrightarrow$ for all $\mathbf{x} \in \mathbb{R}^d$, $\mathbf{x}^T\mathbf{H}\mathbf{x} \geq 0$.

Convex Optimization

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Definition (Convex Optimization).

- Convex optimization: $\min_{\mathbf{w}} f(\mathbf{w})\,; \quad \text{s.t.}\, \mathbf{w} \in \mathcal{C}$.
  1. $\mathcal{C}$ (feasible set) is convex set,
  2. f (objective function) is convex function.

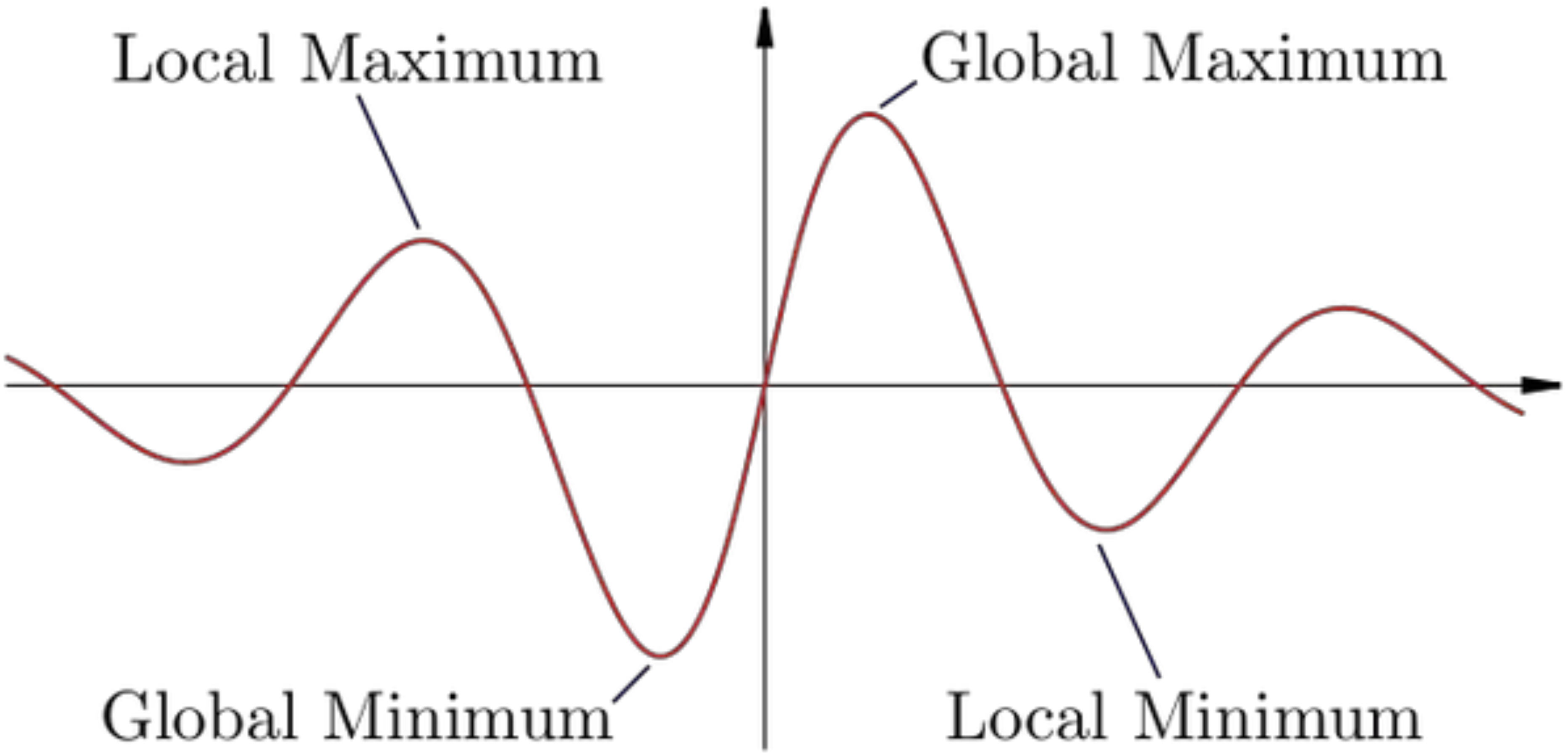
Convex Optimization: Examples

Definition (Convex Optimization).

- Convex optimization: $\min_{\mathbf{w}} f(\mathbf{w})$; $\quad$ s. t. $\mathbf{w} \in \mathcal{C}$.

- Examples:
  - Least squares regression: $\min_{\mathbf{w}} \|\mathbf{X}\mathbf{w} - \mathbf{y}\|_2^2$.
  - Logistic regression: $\min_{\mathbf{w}} \sum_j \log\left(1 + \exp\left(-y_j \mathbf{w}^T \mathbf{x}_j\right)\right)$.
  - SVM: $\min_{\mathbf{w},b} \|\mathbf{w}\|_2^2 + \lambda \sum_j \left[1 - y_j\left(\mathbf{w}^T \mathbf{x}_j + b\right)\right]_+$.
  - LASSO: $\min_{\mathbf{w}} \|\mathbf{X}\mathbf{w} - \mathbf{y}\|_2^2$; $\quad s.t. \ \|\mathbf{w}\|_1 \leq t$.

Local and Global Optima

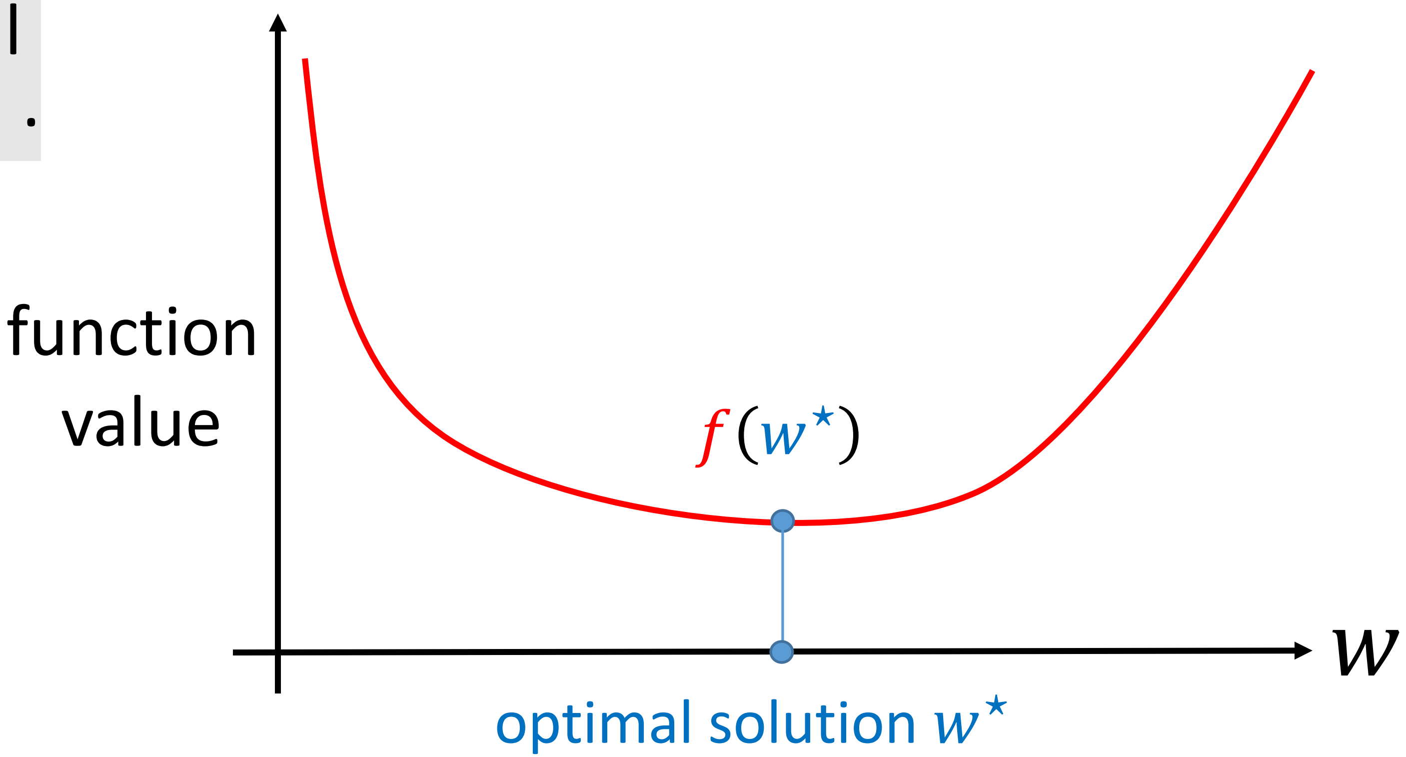


Convex Optimization: Property

Definition (Convex Optimization).

- Convex optimization: $\min_{\mathbf{w}} f(\mathbf{w})$; s.t. $\mathbf{w} \in \mathcal{C}$.

- **Property**: Every local minimum is global minimum .

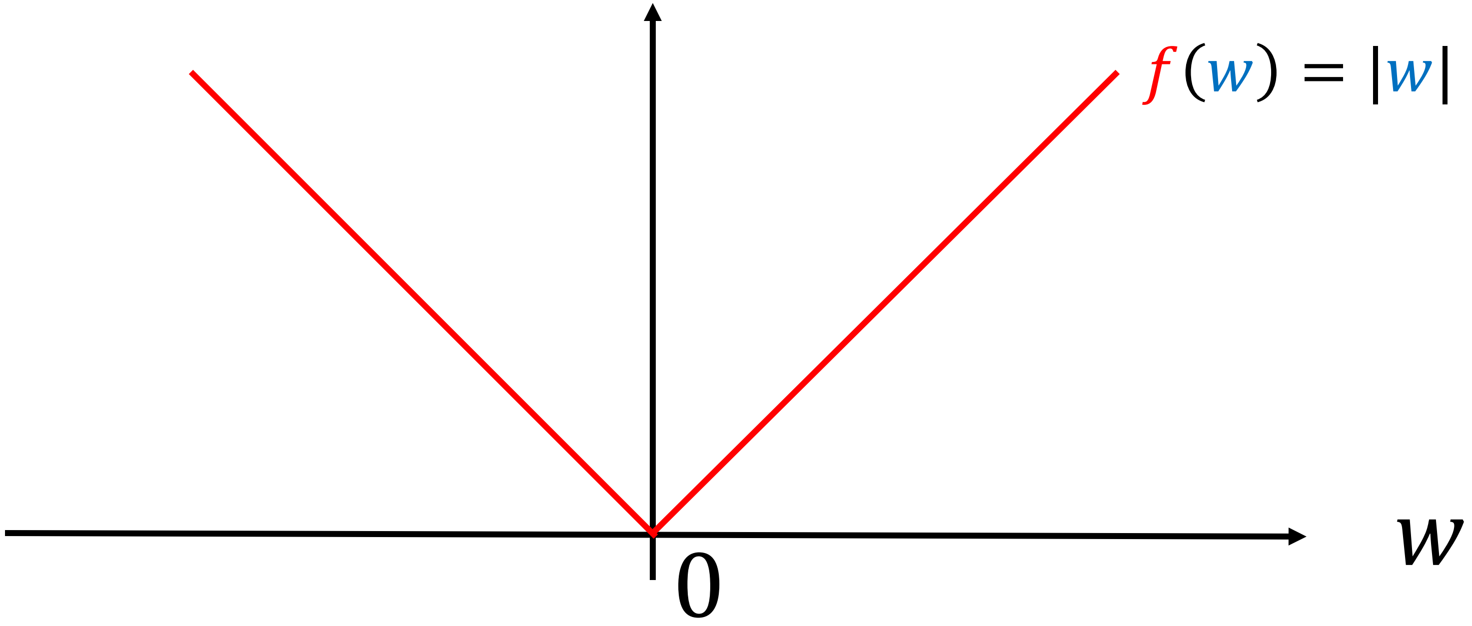


Subgradient and Subdifferential

Non-Differentiable Functions

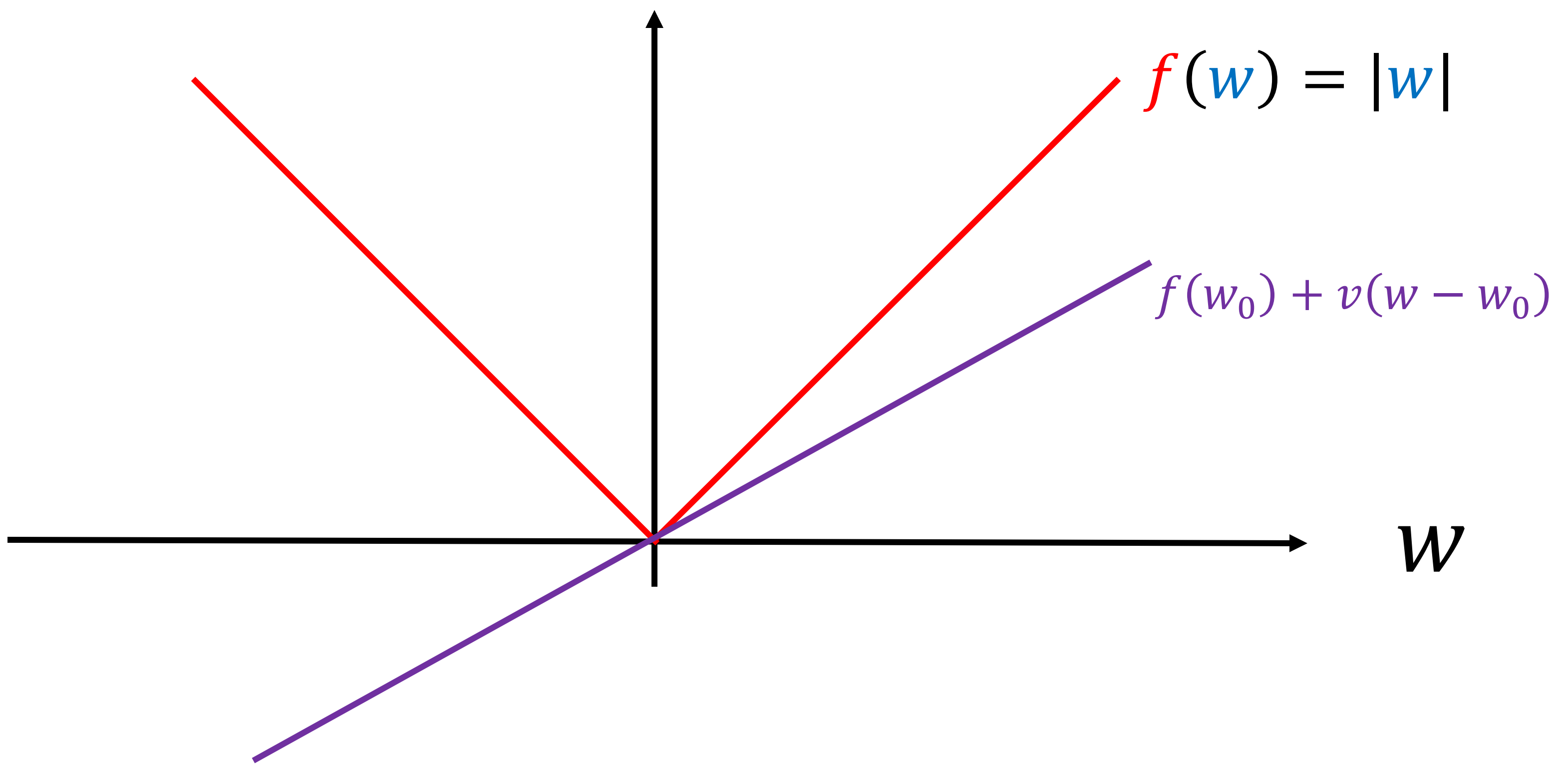
- Example of non-differentiable functions: $f(w) = |w|$

$$\frac{\partial f}{\partial w} = \begin{cases} +1, & \text{if } w > 0; \\ \text{undefined}, & \text{if } w = 0; \\ -1, & \text{if } w < 0. \end{cases}$$



Subgradient of Convex Function

Definition (Subgradient). A vector $\mathbf{v}$ is called a subgradient of f at $\mathbf{w}_0$ if for any $\mathbf{w}$, $f(\mathbf{w}) \geq f(\mathbf{w}_0) + \mathbf{v}^T(\mathbf{w} - \mathbf{w}_0)$.



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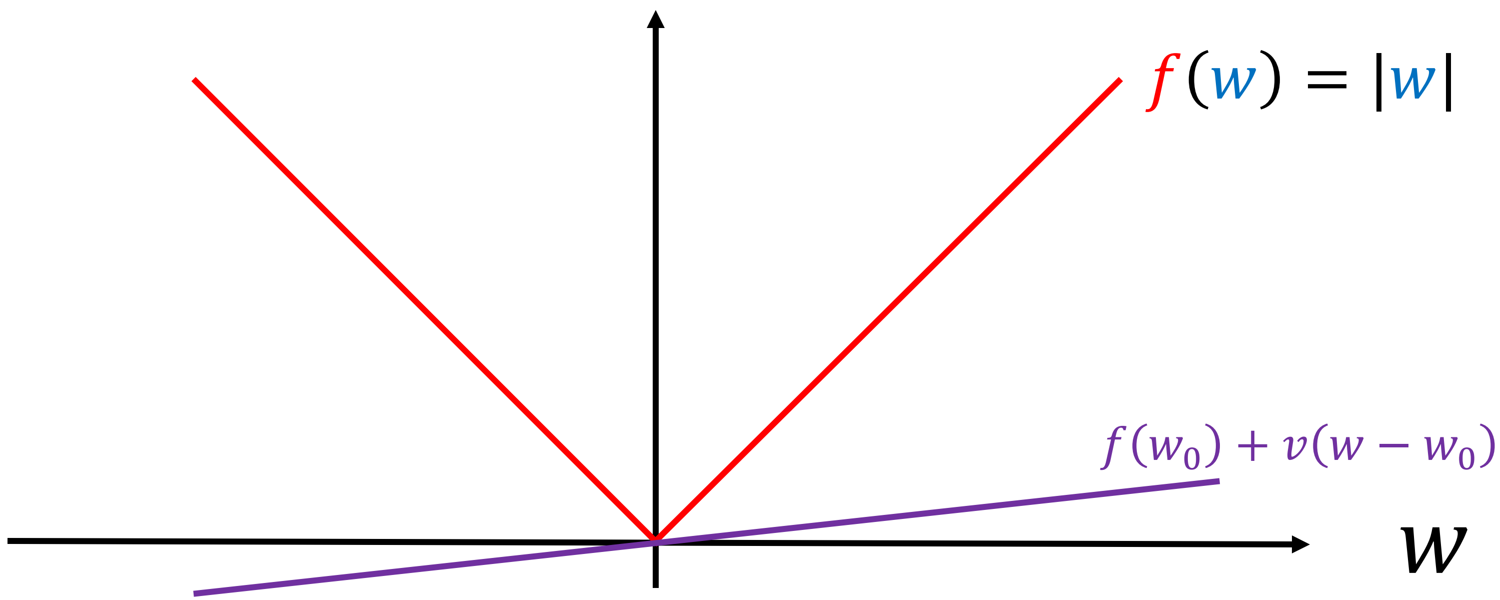

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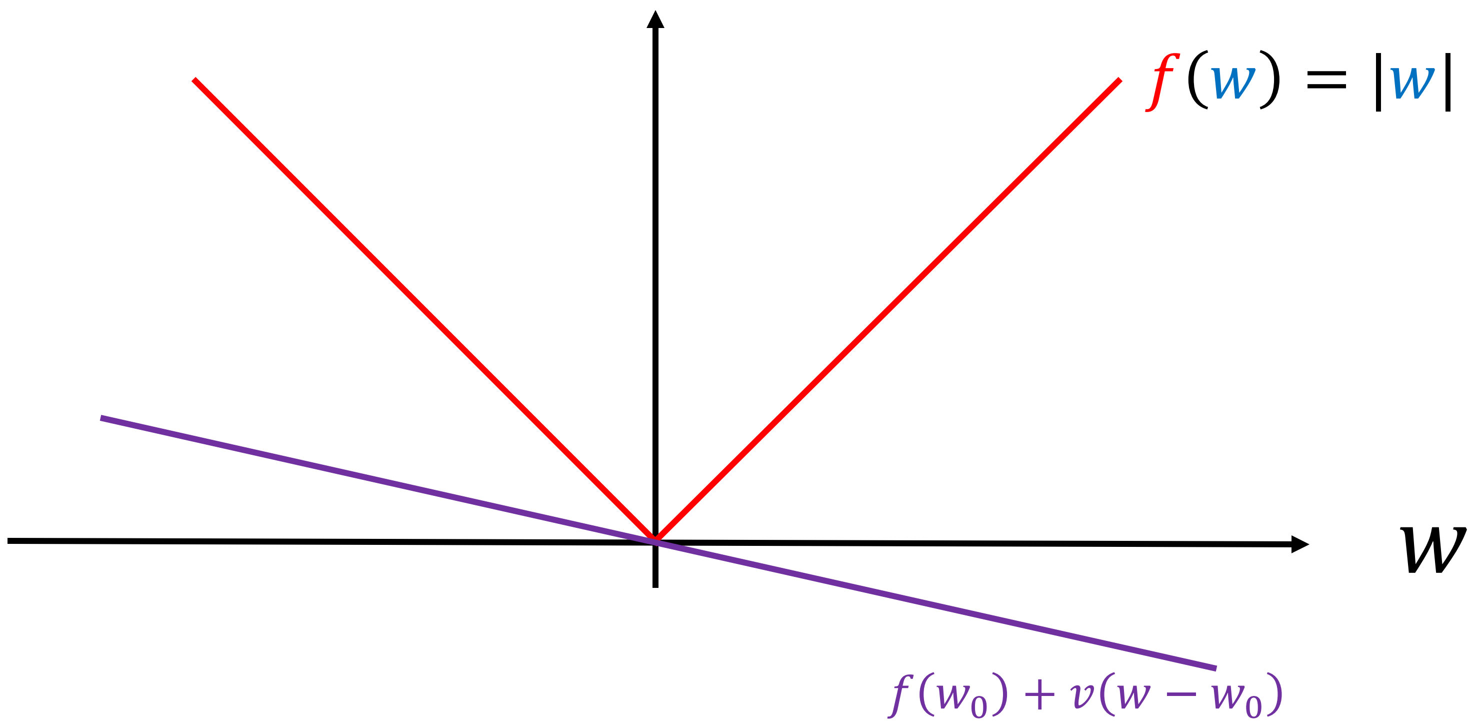

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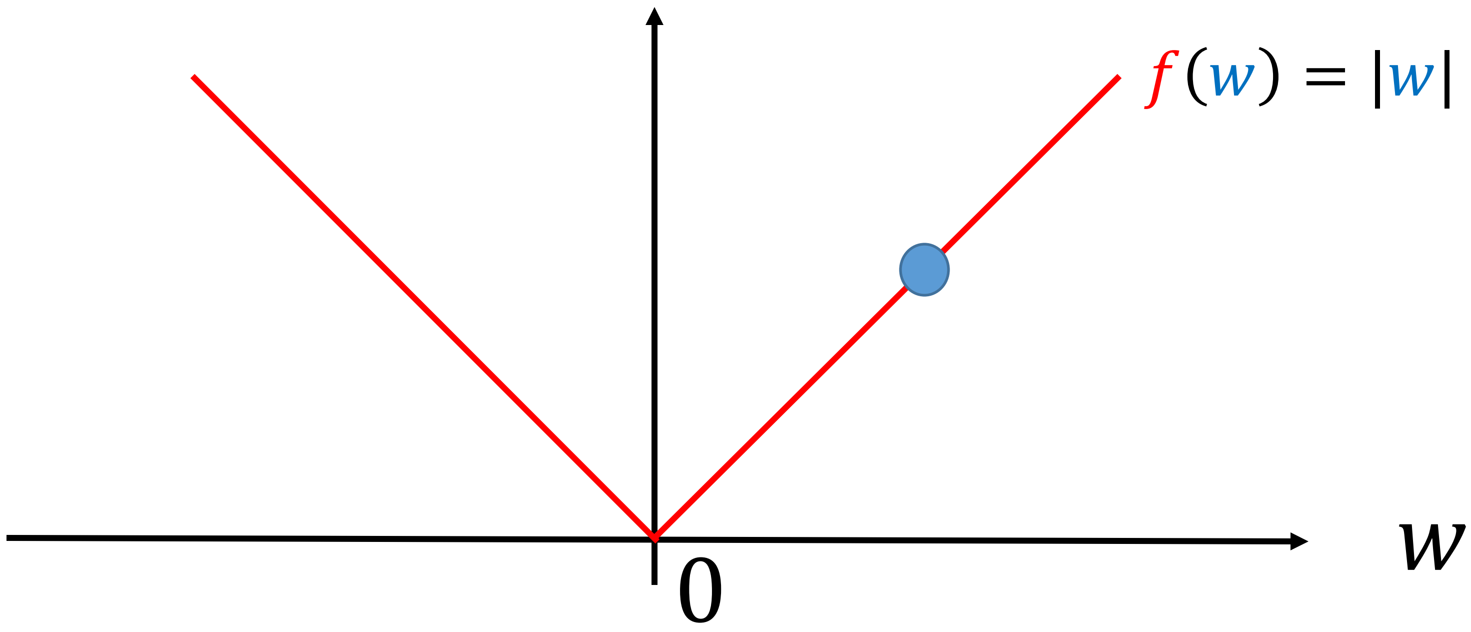
Example: $f(w) = |w|$

- $\partial f(3) = \{1\}$.



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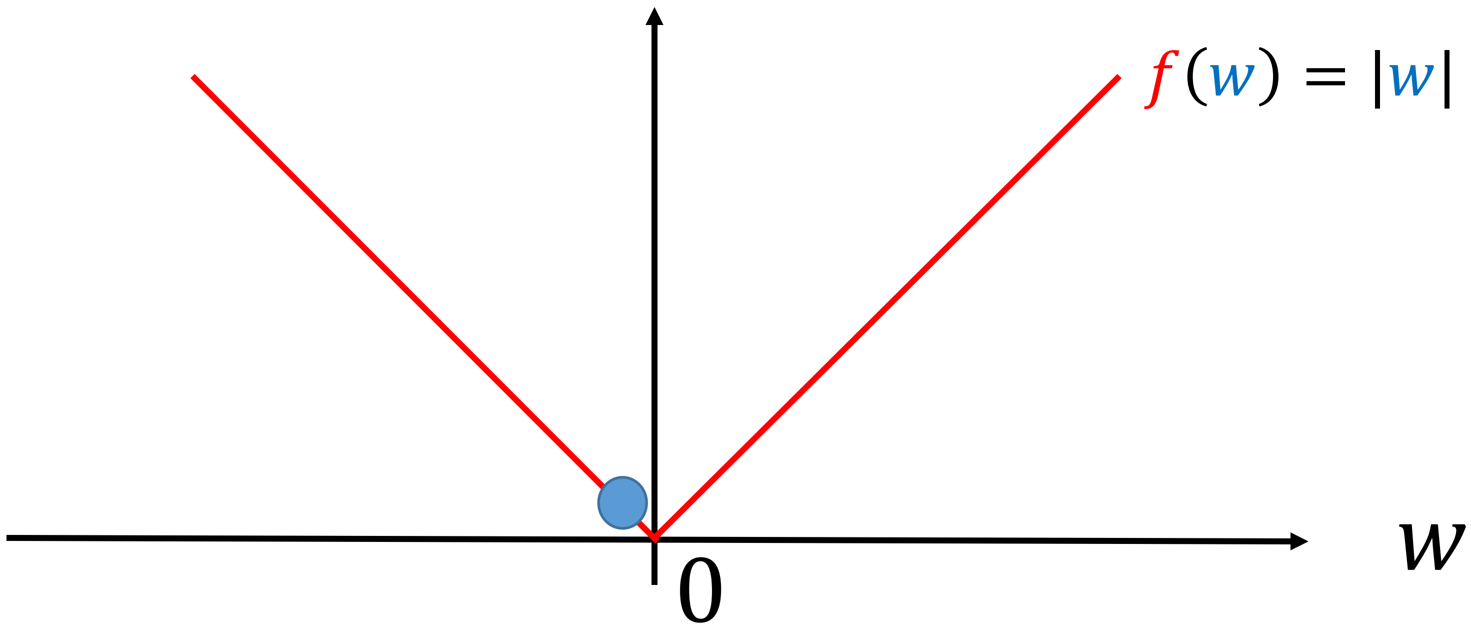
- $\partial f(-0.1) = \{-1\}$.



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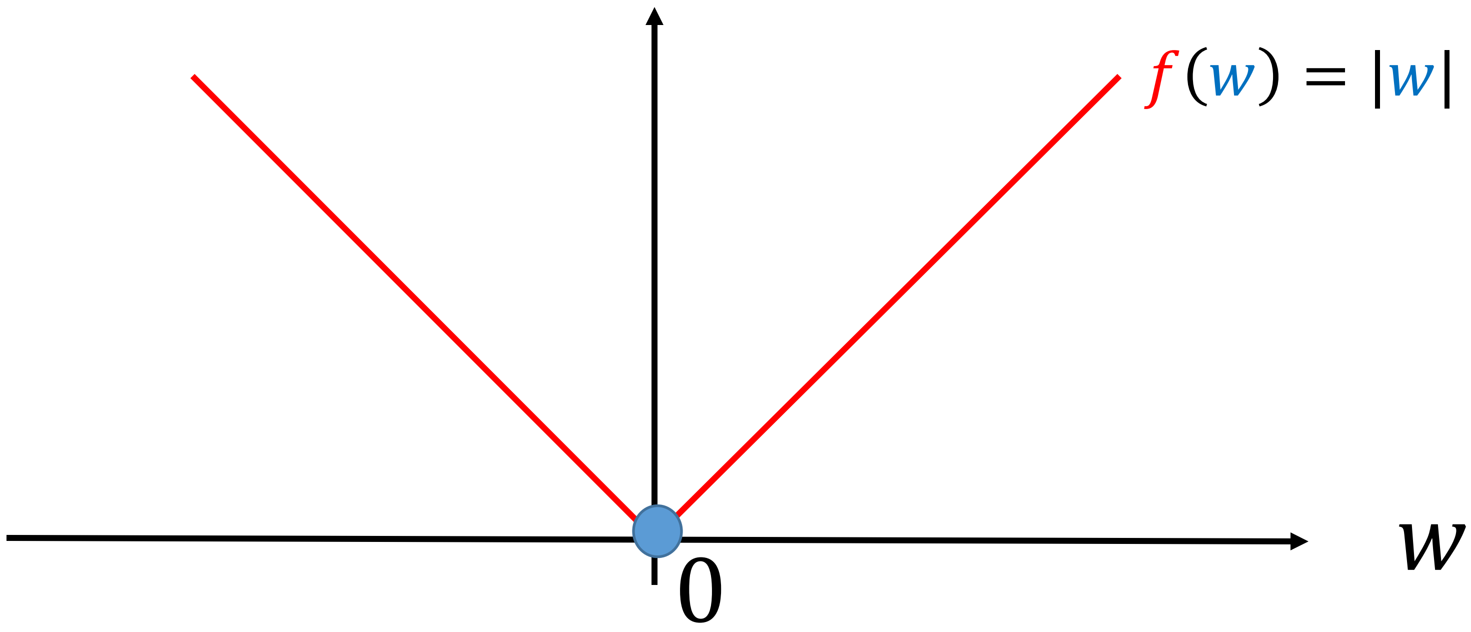

A Property of Convex Optimization

Let f be a convex function.

Property: $\mathbf{w}^\star = \min_{\mathbf{w}} f(\mathbf{w}) \iff 0 \in \partial f(\mathbf{w}^\star)$.

Example: $\min_{w} \{f(w) = |w + 5|\}$

- $\partial f(-5) = [-1, 1]$.
- Obviously $0 \in \partial f(-5)$.
- $w^\star = -5$ minimizes f.