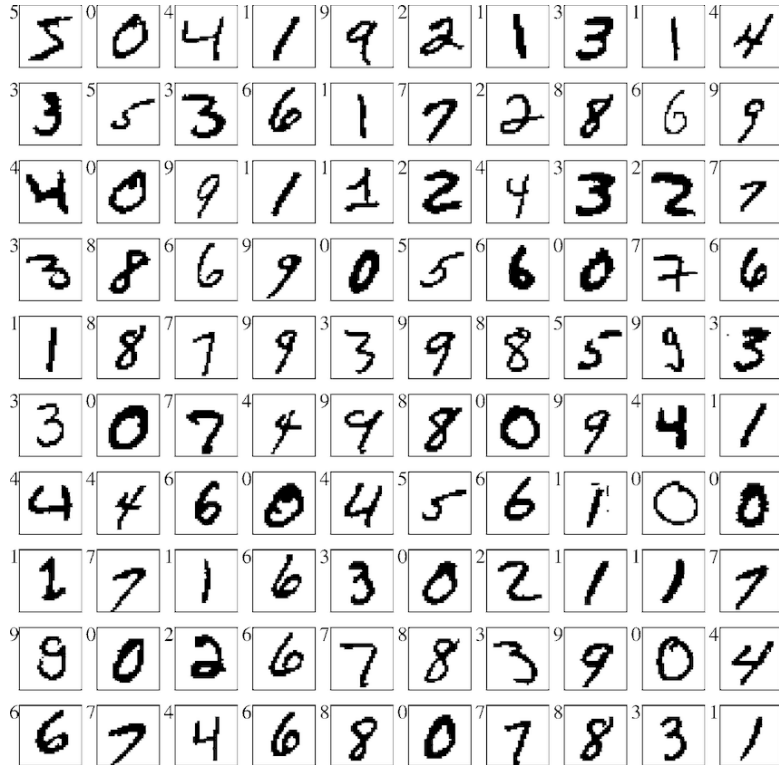


Neural Networks: Basics

Shusen Wang

Revisit Softmax Classifier

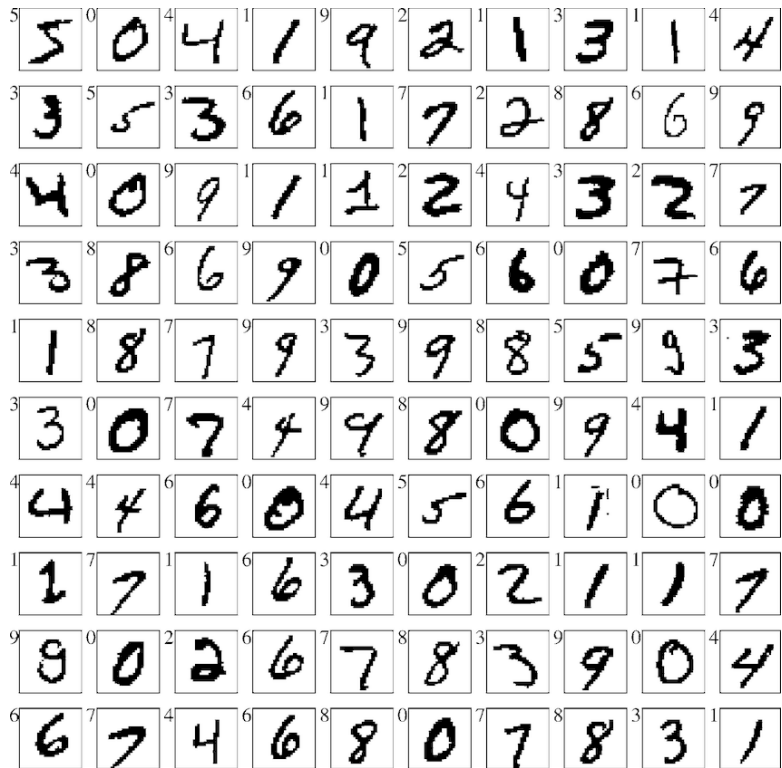
Train a Softmax Classifier



The MNIST Dataset

- $n = 60,000$ training samples $(\mathbf{x}_1, y_1), \dots, (\mathbf{x}_n, y_n)$.
- Each \mathbf{x}_j is a 28×28 image.
- Each y_j is an integer in $\{0, 1, 2, \dots, 9\}$.

Train a Softmax Classifier



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- Each \mathbf{x}_j is a 28×28 image.
- Each y_j is an integer in $\{0, 1, 2, \dots, 9\}$.

Task: multi-class classification

- Given a 28×28 image, predict the digit.
- Learn a function $\mathbf{f}: \mathbb{R}^{28 \times 28} \mapsto \mathbb{R}^{10}$.
- The i -th entry of $\mathbf{f}(\mathbf{x})$ indicates how likely the image \mathbf{x} is the digit i .

Train a Softmax Classifier



Linear model: softmax classifier

- Vectorize each 28×28 image to a 784-dim vector.
- Add a feature of all ones. (So \mathbf{x} becomes 785-dim.)

Train a Softmax Classifier



Linear model: softmax classifier

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- Add a feature of all ones. (So \mathbf{x} becomes 785-dim.)
- Let $\mathbf{W} \in \mathbb{R}^{10 \times 785}$ contain the parameters.
- Let $\mathbf{z} = \mathbf{W}\mathbf{x} \in \mathbb{R}^{10}$.
- Output a 10-dim vector:

$$\mathbf{f}(\mathbf{x}) = \text{SoftMax}(\mathbf{z}) = \text{SoftMax}(\mathbf{W}\mathbf{x}).$$


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$$\mathbf{f}(\mathbf{x}) = \text{SoftMax}(\mathbf{z}) = \text{SoftMax}(\mathbf{W}\mathbf{x}).$$

$$\text{SoftMax}(\mathbf{z}) = \frac{1}{\sum_{i=0}^9 \exp(\mathbf{z}_i)} [\exp(\mathbf{z}_0), \dots, \exp(\mathbf{z}_9)]$$


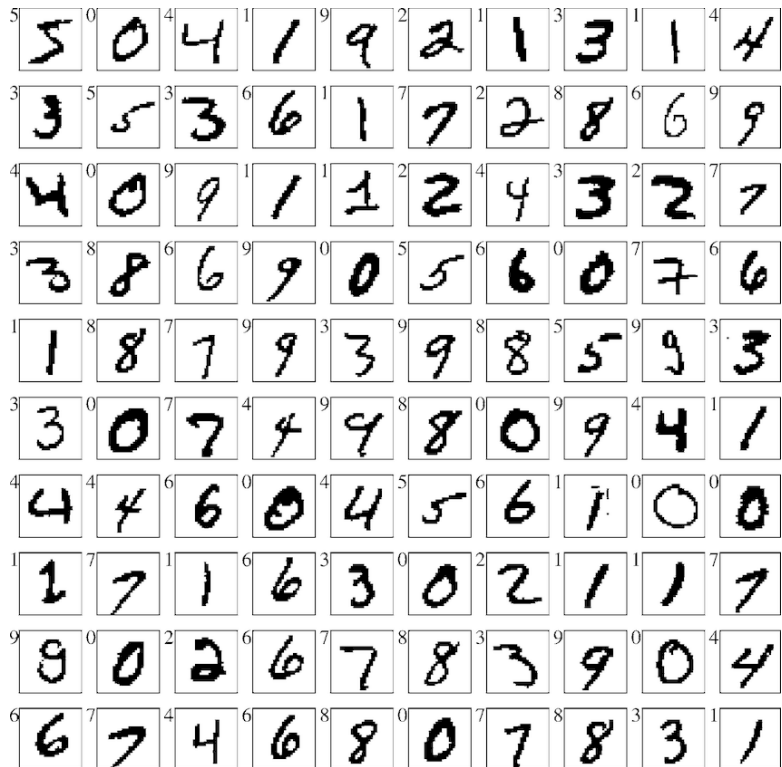
Train a Softmax Classifier



Learn $\mathbf{W} \in \mathbb{R}^{10 \times 785}$ from the training data

- One-hot encode of the labels
 - Originally, the label is a scalar in $\{0, 1, 2, \dots, 9\}$.
 - The one-hot encode \mathbf{y} is a 10-dim vector $\{0, 1\}^{10}$.
 - E.g., the one-hot encode of 2 is $[0, 0, 1, 0, 0, 0, 0, 0, 0, 0]$.

Train a Softmax Classifier



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- Cross-entropy loss:

$$\text{CrossEntropy}(\mathbf{f}, \mathbf{y}) = -\sum_{i=0}^9 y_i \cdot \log(f_i).$$

Train a Softmax Classifier



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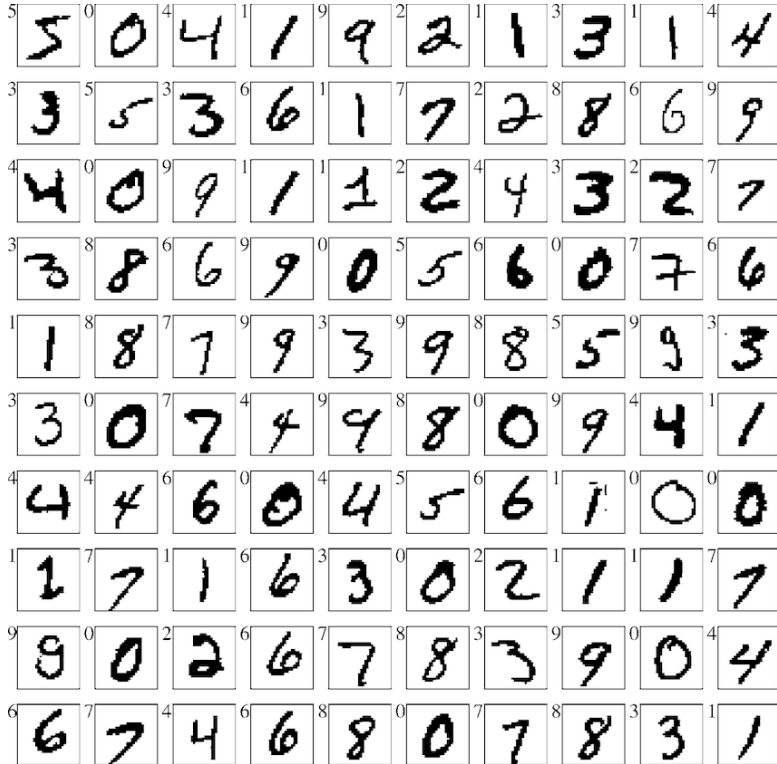
$$\text{CrossEntropy}(\mathbf{f}, \mathbf{y}) = - \sum_{i=0}^9 y_i \cdot \log(f_i).$$

- Solve the optimization model:

$$\mathbf{W}^* = \underset{\mathbf{W}}{\operatorname{argmin}} \left\{ \frac{1}{n} \sum_{j=1}^n \text{CrossEntropy}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j) \right\}.$$

\mathbf{W} is the parameter of \mathbf{f}

Train a Softmax Classifier



Make prediction for a test sample \mathbf{x}'

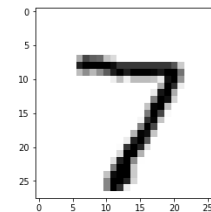
- Now we have $\mathbf{W}^* \in \mathbb{R}^{10 \times 785}$.
- For a test sample \mathbf{x}' , compute $\mathbf{z} = \mathbf{W}^* \mathbf{x}' \in \mathbb{R}^{10}$.
- Make prediction by $\text{argmax } \mathbf{z}$.
 - If the 7-th entry of \mathbf{z} is the largest, then the model thinks the image is digit "7".

Train a Softmax Classifier

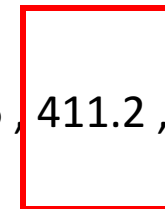


Make prediction for a test sample \mathbf{x}'

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- For a test sample \mathbf{x}' , compute $\mathbf{z} = \mathbf{W}^* \mathbf{x}' \in \mathbb{R}^{10}$.
- Make prediction by $\text{argmax } \mathbf{z}$.
 - If the 7-th entry of \mathbf{z} is the largest, then the model thinks the image is digit "7".



$$\mathbf{z} = [-55.7, -141.4, 18.1, 188.3, -91.3, -26.8, -183.6, 411.2, -142.1, 96.2]$$



Train a Softmax Classifier



Results

- The training set has 60,000 samples.
- The test set has 10,000 samples.
- The accuracy on the training set is 84.64%.
- The accuracy on the test set is 83.58%.
- Not too bad!
- The accuracy of a random guess is merely 10%.

Train a Softmax Classifier: Re-cap

Define a function $\mathbf{f}: \mathbb{R}^{785} \mapsto \mathbb{R}^{10}$:

- **Input:** vector $\mathbf{x} \in \mathbb{R}^{785}$.
- $\mathbf{z} = \mathbf{W} \mathbf{x} \in \mathbb{R}^{10}$.
- **Output:** $\mathbf{f}(\mathbf{x}) = \text{SoftMax}(\mathbf{z})$.

Trainable parameters: $\mathbf{W} \in \mathbb{R}^{10 \times 785}$

Train a Softmax Classifier: Re-cap

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- **Output:** $\mathbf{f}(\mathbf{x}) = \text{SoftMax}(\mathbf{z})$.

Trainable parameters: $\mathbf{W} \in \mathbb{R}^{10 \times 785}$

Train the function by empirical risk minimization (ERM):

- **Training set:** $(\mathbf{x}_1, \mathbf{y}_1), \dots, (\mathbf{x}_n, \mathbf{y}_n) \in \mathbb{R}^{785} \times \mathbb{R}^{10}$.
- **Loss function:** $\text{CrossEntropy}(\mathbf{f}, \mathbf{y}) = -\sum_{i=1}^{10} y_i \cdot \log(\mathbf{f}(\mathbf{x})_i)$.
- **Solve ERM:** $\underset{\mathbf{W}}{\operatorname{argmin}} \left\{ \frac{1}{n} \sum_{j=1}^n \text{CrossEntropy}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j) \right\}$.

Train a Softmax Classifier: Re-cap

- **How to solve** $\underset{\mathbf{W}}{\operatorname{argmin}} \left\{ \frac{1}{n} \sum_{j=1}^n \text{CrossEntropy}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j) \right\} ?$
- **Stochastic gradient descent (SGD) with momentum** repeats:
 1. Randomly pick j from $\{1, 2, \dots, n\}$.
 2. Evaluate the gradient $\mathbf{G}_j = \frac{\partial \text{CrossEntropy}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j)}{\partial \mathbf{W}} \Big|_{\mathbf{W}=\mathbf{W}_{\text{old}}}$.
 3. Update the momentum: $\mathbf{V}_{\text{new}} = \beta \mathbf{V}_{\text{old}} + \mathbf{G}_j$.
 4. Update \mathbf{W} by $\mathbf{W}_{\text{new}} \leftarrow \mathbf{W}_{\text{old}} - \alpha \mathbf{V}_{\text{new}}$.

Fully-Connected Neural Network (Multi-layer Perceptron)

From Linear Model to Neural Network

Define a function $\mathbf{f}: \mathbb{R}^{785} \mapsto \mathbb{R}^{10}$:

- **Input:** vector $\mathbf{x}^{(0)} \in \mathbb{R}^{785}$.
- $\mathbf{z}^{(1)} = \mathbf{W}^{(0)} \mathbf{x}^{(0)} \in \mathbb{R}^{10}$.
- **Output:** $\mathbf{f}(\mathbf{x}^{(0)}) = \text{SoftMax}(\mathbf{z}^{(1)})$.

Trainable parameter:

- $\mathbf{W}^{(0)} \in \mathbb{R}^{10 \times 785}$.

From Linear Model to Neural Network

Define a function $\mathbf{f}: \mathbb{R}^{785} \mapsto \mathbb{R}^{10}$:

- **Input:** vector $\mathbf{x}^{(0)} \in \mathbb{R}^{785}$.
- $\mathbf{z}^{(1)} = \mathbf{W}^{(0)} \mathbf{x}^{(0)} \in \mathbb{R}^{d_1}$.
- $\mathbf{x}^{(1)} = \max\{\mathbf{0}, \mathbf{z}^{(1)}\} \in \mathbb{R}^{d_1}$.

Trainable parameters:

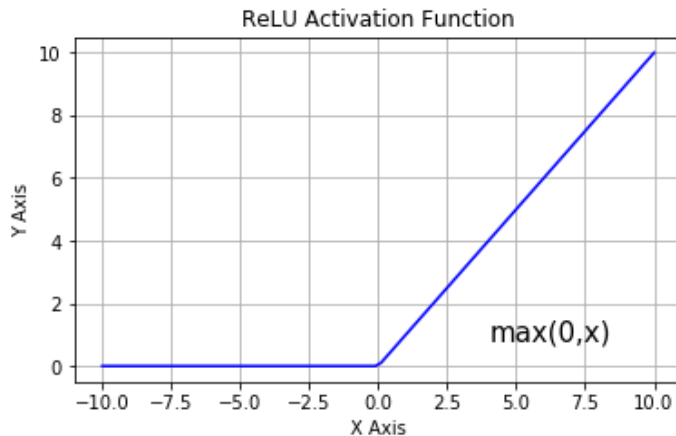
- $\mathbf{W}^{(0)} \in \mathbb{R}^{d_1 \times 785}$,

From Linear Model to Neural Network

Define a function $\mathbf{f}: \mathbb{R}^{785} \mapsto \mathbb{R}^{10}$:

- Input: vector $\mathbf{x}^{(0)} \in \mathbb{R}^{785}$.
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- $\mathbf{x}^{(1)} = \max\{0, \mathbf{z}^{(1)}\} \in \mathbb{R}^{d_1}$.

ReLU (activation function)



Trainable parameters:

- $\mathbf{W}^{(0)} \in \mathbb{R}^{d_1 \times 785}$,

From Linear Model to Neural Network

Define a function $\mathbf{f}: \mathbb{R}^{785} \mapsto \mathbb{R}^{10}$:

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Hidden Layer 1

Trainable parameters:

- $\mathbf{W}^{(0)} \in \mathbb{R}^{d_1 \times 785}$,

From Linear Model to Neural Network

Define a function $\mathbf{f}: \mathbb{R}^{785} \mapsto \mathbb{R}^{10}$:

- **Input:** vector $\mathbf{x}^{(0)} \in \mathbb{R}^{785}$.

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- $\mathbf{x}^{(1)} = \max\{\mathbf{0}, \mathbf{z}^{(1)}\} \in \mathbb{R}^{d_1}$.

Hidden Layer 1

- $\mathbf{z}^{(2)} = \mathbf{W}^{(1)} \mathbf{x}^{(1)} \in \mathbb{R}^{d_2}$.

Trainable parameters:

- $\mathbf{W}^{(0)} \in \mathbb{R}^{d_1 \times 785}$,
- $\mathbf{W}^{(1)} \in \mathbb{R}^{d_2 \times d_1}$,

It should be $\mathbf{z}^{(2)} = \mathbf{W}^{(1)} \mathbf{x}^{(1)} + \mathbf{b}^{(1)}$ in practice.
I leave out $\mathbf{b} \in \mathbb{R}^{d_2}$ in the slides for simplicity.

From Linear Model to Neural Network

Define a function $\mathbf{f}: \mathbb{R}^{785} \mapsto \mathbb{R}^{10}$:

- **Input:** vector $\mathbf{x}^{(0)} \in \mathbb{R}^{785}$.

- $\mathbf{z}^{(1)} = \mathbf{W}^{(0)} \mathbf{x}^{(0)} \in \mathbb{R}^{d_1}$.
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Hidden Layer 1

- $\mathbf{z}^{(2)} = \mathbf{W}^{(1)} \mathbf{x}^{(1)} \in \mathbb{R}^{d_2}$.
- $\mathbf{x}^{(2)} = \max\{\mathbf{0}, \mathbf{z}^{(2)}\} \in \mathbb{R}^{d_2}$.

Hidden Layer 2

Trainable parameters:

- $\mathbf{W}^{(0)} \in \mathbb{R}^{d_1 \times 785}$,
- $\mathbf{W}^{(1)} \in \mathbb{R}^{d_2 \times d_1}$,

From Linear Model to Neural Network

Define a function $\mathbf{f}: \mathbb{R}^{785} \mapsto \mathbb{R}^{10}$:

• **Input:** vector $\mathbf{x}^{(0)} \in \mathbb{R}^{785}$.

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- $\mathbf{x}^{(2)} = \max\{\mathbf{0}, \mathbf{z}^{(2)}\} \in \mathbb{R}^{d_2}$.

Hidden Layer 2

- $\mathbf{z}^{(3)} = \mathbf{W}^{(2)} \mathbf{x}^{(2)} \in \mathbb{R}^{10}$.
- $\mathbf{x}^{(3)} = \text{SoftMax}(\mathbf{z}^{(3)}) \in \mathbb{R}^{10}$.

Output Layer

Trainable parameters:

- $\mathbf{W}^{(0)} \in \mathbb{R}^{d_1 \times 785}$,
- $\mathbf{W}^{(1)} \in \mathbb{R}^{d_2 \times d_1}$,
- $\mathbf{W}^{(2)} \in \mathbb{R}^{10 \times d_2}$.

From Linear Model to Neural Network

Define a function $\mathbf{f}: \mathbb{R}^{785} \mapsto \mathbb{R}^{10}$:

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Hidden Layer 2

- $\mathbf{z}^{(3)} = \mathbf{W}^{(2)} \mathbf{x}^{(2)} \in \mathbb{R}^{10}$.
- $\mathbf{x}^{(3)} = \text{SoftMax}(\mathbf{z}^{(3)}) \in \mathbb{R}^{10}$.

Output Layer

- **Output:** $\mathbf{f}(\mathbf{x}^{(0)}) = \mathbf{x}^{(3)}$.

Trainable parameters:

- $\mathbf{W}^{(0)} \in \mathbb{R}^{d_1 \times 785}$,
- $\mathbf{W}^{(1)} \in \mathbb{R}^{d_2 \times d_1}$,
- $\mathbf{W}^{(2)} \in \mathbb{R}^{10 \times d_2}$.

Fully-Connected Layer

Define a function $\mathbf{f}: \mathbb{R}^{785} \mapsto \mathbb{R}^{10}$:

- Input: vector $\mathbf{x}^{(0)} \in \mathbb{R}^{785}$.

- $\mathbf{z}^{(1)} = \mathbf{W}^{(0)} \mathbf{x}^{(0)} \in \mathbb{R}^{d_1}$.

- $\mathbf{x}^{(1)} = \max\{\mathbf{0}, \mathbf{z}^{(1)}\} \in \mathbb{R}^{d_1}$.

- $\mathbf{z}^{(2)} = \mathbf{W}^{(1)} \mathbf{x}^{(1)} \in \mathbb{R}^{d_2}$.

- $\mathbf{x}^{(2)} = \max\{\mathbf{0}, \mathbf{z}^{(2)}\} \in \mathbb{R}^{d_2}$.

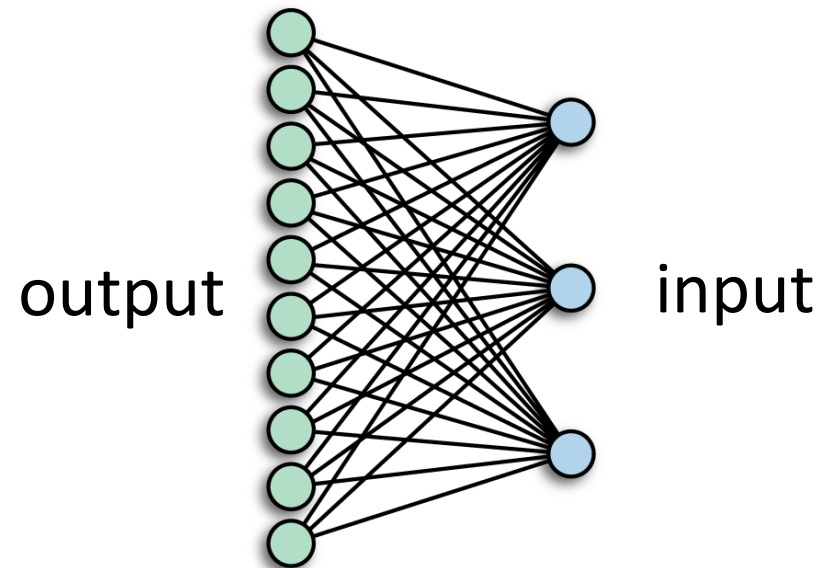
- $\mathbf{z}^{(3)} = \mathbf{W}^{(2)} \mathbf{x}^{(2)} \in \mathbb{R}^{10}$.

- $\mathbf{x}^{(3)} = \text{SoftMax}(\mathbf{z}^{(3)}) \in \mathbb{R}^{10}$.

- Output: $\mathbf{f}(\mathbf{x}^{(0)}) = \mathbf{x}^{(3)}$.

“Fully-Connected” or “Dense” Layer

Each entry of $\mathbf{z}^{(1)}$ depends on (i.e., connected to) all the entries of $\mathbf{x}^{(0)}$.



Activation Functions

Define a function $\mathbf{f}: \mathbb{R}^{785} \mapsto \mathbb{R}^{10}$:

- Input: vector $\mathbf{x}^{(0)} \in \mathbb{R}^{785}$.
- $\mathbf{z}^{(1)} = \mathbf{W}^{(0)} \mathbf{x}^{(0)} \in \mathbb{R}^{d_1}$.
- $\mathbf{x}^{(1)} = \max\{\mathbf{0}, \mathbf{z}^{(1)}\} \in \mathbb{R}^{d_1}$. ReLU
- $\mathbf{z}^{(2)} = \mathbf{W}^{(1)} \mathbf{x}^{(1)} \in \mathbb{R}^{d_2}$.
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- $\mathbf{z}^{(3)} = \mathbf{W}^{(2)} \mathbf{x}^{(2)} \in \mathbb{R}^{10}$.
- $\mathbf{x}^{(3)} = \text{SoftMax}(\mathbf{z}^{(3)}) \in \mathbb{R}^{10}$. SoftMax
- Output: $\mathbf{f}(\mathbf{x}^{(0)}) = \mathbf{x}^{(3)}$.

Activation Functions

Define a function $f: \mathbb{R}^{785} \mapsto \mathbb{R}^{10}$:

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- $\mathbf{x}^{(2)} = \max\{\mathbf{0}, \mathbf{z}^{(2)}\} \in \mathbb{R}^{d_2}$.
- $\mathbf{z}^{(3)} = \mathbf{W}^{(2)} \mathbf{x}^{(2)} \in \mathbb{R}^{10}$.

- $\mathbf{x}^{(3)} = \text{SoftMax}(\mathbf{z}^{(3)}) \in \mathbb{R}^{10}$.

SoftMax

- Use **SoftMax** because this is a **multi-class classification** problem.

- Output: $f(\mathbf{x}^{(0)}) = \mathbf{x}^{(3)}$.

Activation Functions

Define a function $f: \mathbb{R}^{785} \mapsto \mathbb{R}^{10}$:

- Input: vector $\mathbf{x}^{(0)} \in \mathbb{R}^{785}$.
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- $\mathbf{z}^{(2)} = \mathbf{W}^{(1)} \mathbf{x}^{(1)} \in \mathbb{R}^{d_2}$.
- $\mathbf{x}^{(2)} = \max\{\mathbf{0}, \mathbf{z}^{(2)}\} \in \mathbb{R}^{d_2}$.
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- $\mathbf{x}^{(3)} = \text{SoftMax}(\mathbf{z}^{(3)}) \in \mathbb{R}^{10}$.

SoftMax

- Output: $f(\mathbf{x}^{(0)}) = \mathbf{x}^{(3)}$.

- Use **Logistic function** for **binary classification problem**.

- For regression:
 - **No activation function**, if the labels are in \mathbb{R} .
 - Use **ReLU** if the labels are positive.

- Use **SoftMax** because this is a **multi-class classification problem**.

Activation Functions

Define a function $f: \mathbb{R}^{785} \mapsto \mathbb{R}^{10}$:

- Input: vector $\mathbf{x}^{(0)} \in \mathbb{R}^{785}$.
- $\mathbf{z}^{(1)} = \mathbf{W}^{(0)} \mathbf{x}^{(0)} \in \mathbb{R}^{d_1}$.
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- $\mathbf{z}^{(3)} = \mathbf{W}^{(2)} \mathbf{x}^{(2)} \in \mathbb{R}^{10}$.
- $\mathbf{x}^{(3)} = \text{SoftMax}(\mathbf{z}^{(3)}) \in \mathbb{R}^{10}$.
- Output: $f(\mathbf{x}^{(0)}) = \mathbf{x}^{(3)}$.

Question: Why bothering using ReLU?

- Without the two activation functions, $\mathbf{z}^{(3)}$ would be a linear function of $\mathbf{x}^{(0)}$.
- A linear function can be represented by $\mathbf{z}^{(3)} = \mathbf{W}\mathbf{x}^{(0)}$.
- The neural network would be equally expressive as a linear model!!!

Gradient and Backpropagation

Train the Neural Network

Define a function $\mathbf{f}: \mathbb{R}^{785} \mapsto \mathbb{R}^{10}$:

- **Input:** vector $\mathbf{x}^{(0)} \in \mathbb{R}^{785}$.
- $\mathbf{z}^{(1)} = \mathbf{W}^{(0)} \mathbf{x}^{(0)} \in \mathbb{R}^{d_1}$.
- $\mathbf{x}^{(1)} = \max\{\mathbf{0}, \mathbf{z}^{(1)}\} \in \mathbb{R}^{d_1}$.
- $\mathbf{z}^{(2)} = \mathbf{W}^{(1)} \mathbf{x}^{(1)} \in \mathbb{R}^{d_2}$.
- $\mathbf{x}^{(2)} = \max\{\mathbf{0}, \mathbf{z}^{(2)}\} \in \mathbb{R}^{d_2}$.
- $\mathbf{z}^{(3)} = \mathbf{W}^{(2)} \mathbf{x}^{(2)} \in \mathbb{R}^{10}$.
- $\mathbf{x}^{(3)} = \text{SoftMax}(\mathbf{z}^{(3)}) \in \mathbb{R}^{10}$.
- **Output:** $\mathbf{f}(\mathbf{x}^{(0)}) = \mathbf{x}^{(3)}$.

Train the Neural Network

Define a function $\mathbf{f}: \mathbb{R}^{785} \mapsto \mathbb{R}^{10}$:

- **Input:** vector $\mathbf{x}^{(0)} \in \mathbb{R}^{785}$.
- $\mathbf{z}^{(1)} = \mathbf{W}^{(0)} \mathbf{x}^{(0)} \in \mathbb{R}^{d_1}$.
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- $\mathbf{z}^{(3)} = \mathbf{W}^{(2)} \mathbf{x}^{(2)} \in \mathbb{R}^{10}$.
- $\mathbf{x}^{(3)} = \text{SoftMax}(\mathbf{z}^{(3)}) \in \mathbb{R}^{10}$.
- **Output:** $\mathbf{f}(\mathbf{x}^{(0)}) = \mathbf{x}^{(3)}$.

Build an optimization model:

$$\underset{\mathbf{W}^{(0)}, \mathbf{W}^{(1)}, \mathbf{W}^{(2)}}{\operatorname{argmin}} \left\{ \frac{1}{n} \sum_{j=1}^n \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j) \right\}$$

E.g., the cross-entropy loss

Train the Neural Network

Define a function $\mathbf{f}: \mathbb{R}^{785} \mapsto \mathbb{R}^{10}$:

- **Input:** vector $\mathbf{x}^{(0)} \in \mathbb{R}^{785}$.
- $\mathbf{z}^{(1)} = \mathbf{W}^{(0)} \mathbf{x}^{(0)} \in \mathbb{R}^{d_1}$.
- $\mathbf{x}^{(1)} = \max\{\mathbf{0}, \mathbf{z}^{(1)}\} \in \mathbb{R}^{d_1}$.
- $\mathbf{z}^{(2)} = \mathbf{W}^{(1)} \mathbf{x}^{(1)} \in \mathbb{R}^{d_2}$.
- $\mathbf{x}^{(2)} = \max\{\mathbf{0}, \mathbf{z}^{(2)}\} \in \mathbb{R}^{d_2}$.
- $\mathbf{z}^{(3)} = \mathbf{W}^{(2)} \mathbf{x}^{(2)} \in \mathbb{R}^{10}$.
- $\mathbf{x}^{(3)} = \text{SoftMax}(\mathbf{z}^{(3)}) \in \mathbb{R}^{10}$.
- **Output:** $\mathbf{f}(\mathbf{x}^{(0)}) = \mathbf{x}^{(3)}$.

How to solve

$$\underset{\mathbf{W}^{(0)}, \mathbf{W}^{(1)}, \mathbf{W}^{(2)}}{\operatorname{argmin}} \left\{ \frac{1}{n} \sum_{j=1}^n \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j) \right\} ?$$

Stochastic gradient descent (SGD)

Train the Neural Network

Define a function $\mathbf{f}: \mathbb{R}^{785} \mapsto \mathbb{R}^{10}$:

- Input: vector $\mathbf{x}^{(0)} \in \mathbb{R}^{785}$.
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- $\mathbf{x}^{(2)} = \max\{\mathbf{0}, \mathbf{z}^{(2)}\} \in \mathbb{R}^{d_2}$.
- $\mathbf{z}^{(3)} = \mathbf{W}^{(2)} \mathbf{x}^{(2)} \in \mathbb{R}^{10}$.
- $\mathbf{x}^{(3)} = \text{SoftMax}(\mathbf{z}^{(3)}) \in \mathbb{R}^{10}$.
- Output: $\mathbf{f}(\mathbf{x}^{(0)}) = \mathbf{x}^{(3)}$.

How to solve

$$\underset{\mathbf{W}^{(0)}, \mathbf{W}^{(1)}, \mathbf{W}^{(2)}}{\operatorname{argmin}} \left\{ \frac{1}{n} \sum_{j=1}^n \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j) \right\} ?$$

Stochastic gradient descent (SGD):

- Randomly pick j from $\{1, 2, \dots, n\}$.
- Compute the stochastic gradient w.r.t. $\mathbf{W}^{(0)}$ at the current iteration $\mathbf{W}_{\text{old}}^{(0)}$:

$$\mathbf{g}_j^{(0)} = \left. \frac{\partial \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j)}{\partial \mathbf{W}^{(0)}} \right|_{\mathbf{W}^{(0)} = \mathbf{W}_{\text{old}}^{(0)}}$$

- Update $\mathbf{W}^{(0)}$: $\mathbf{W}_{\text{new}}^{(0)} = \mathbf{W}_{\text{old}}^{(0)} - \alpha \mathbf{g}_j^{(0)}$.
- Do the same for $\mathbf{W}^{(1)}$ and $\mathbf{W}^{(2)}$.

Backpropagation

Define a function $\mathbf{f}: \mathbb{R}^{785} \mapsto \mathbb{R}^{10}$:

- Input: vector $\mathbf{x}^{(0)} \in \mathbb{R}^{785}$.
- $\mathbf{z}^{(1)} = \mathbf{W}^{(0)} \mathbf{x}^{(0)} \in \mathbb{R}^{d_1}$.
- $\mathbf{x}^{(1)} = \max\{\mathbf{0}, \mathbf{z}^{(1)}\} \in \mathbb{R}^{d_1}$.
- $\mathbf{z}^{(2)} = \mathbf{W}^{(1)} \mathbf{x}^{(1)} \in \mathbb{R}^{d_2}$.
- $\mathbf{x}^{(2)} = \max\{\mathbf{0}, \mathbf{z}^{(2)}\} \in \mathbb{R}^{d_2}$.
- $\mathbf{z}^{(3)} = \mathbf{W}^{(2)} \mathbf{x}^{(2)} \in \mathbb{R}^{10}$.
- $\mathbf{x}^{(3)} = \text{SoftMax}(\mathbf{z}^{(3)}) \in \mathbb{R}^{10}$.
- Output: $\mathbf{f}(\mathbf{x}^{(0)}) = \mathbf{x}^{(3)}$.

How to compute $\frac{\partial \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j)}{\partial \mathbf{W}^{(k)}}$?

Backpropagation

Define a function $\mathbf{f}: \mathbb{R}^{785} \mapsto \mathbb{R}^{10}$:

- Input: vector $\mathbf{x}^{(0)} \in \mathbb{R}^{785}$.
- $\mathbf{z}^{(1)} = \mathbf{W}^{(0)} \mathbf{x}^{(0)} \in \mathbb{R}^{d_1}$.
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How to compute $\frac{\partial \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j)}{\partial \mathbf{W}^{(k)}}$?

Backpropagation:

- Denote $L = \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j)$.

Backpropagation

Define a function $\mathbf{f}: \mathbb{R}^{785} \mapsto \mathbb{R}^{10}$:

- Input: vector $\mathbf{x}^{(0)} \in \mathbb{R}^{785}$.
- $\mathbf{z}^{(1)} = \mathbf{W}^{(0)} \mathbf{x}^{(0)} \in \mathbb{R}^{d_1}$.
- $\mathbf{x}^{(1)} = \max\{\mathbf{0}, \mathbf{z}^{(1)}\} \in \mathbb{R}^{d_1}$.
- $\mathbf{z}^{(2)} = \mathbf{W}^{(1)} \mathbf{x}^{(1)} \in \mathbb{R}^{d_2}$.
- $\mathbf{x}^{(2)} = \max\{\mathbf{0}, \mathbf{z}^{(2)}\} \in \mathbb{R}^{d_2}$.
- $\mathbf{z}^{(3)} = \mathbf{W}^{(2)} \mathbf{x}^{(2)} \in \mathbb{R}^{10}$.
- $\mathbf{x}^{(3)} = \text{SoftMax}(\mathbf{z}^{(3)}) \in \mathbb{R}^{10}$.
- **Output:** $\mathbf{f}(\mathbf{x}^{(0)}) = \mathbf{x}^{(3)}$.

How to compute $\frac{\partial \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j)}{\partial \mathbf{W}^{(k)}}$?

Backpropagation:

- Denote $L = \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j)$.
- Compute $\frac{\partial L}{\partial \mathbf{z}^{(3)}}$.

10-dim vector

Backpropagation

Define a function $\mathbf{f}: \mathbb{R}^{785} \mapsto \mathbb{R}^{10}$:

- Input: vector $\mathbf{x}^{(0)} \in \mathbb{R}^{785}$.
- $\mathbf{z}^{(1)} = \mathbf{W}^{(0)} \mathbf{x}^{(0)} \in \mathbb{R}^{d_1}$.
- $\mathbf{x}^{(1)} = \max\{\mathbf{0}, \mathbf{z}^{(1)}\} \in \mathbb{R}^{d_1}$.
- $\mathbf{z}^{(2)} = \mathbf{W}^{(1)} \mathbf{x}^{(1)} \in \mathbb{R}^{d_2}$.
- $\mathbf{x}^{(2)} = \max\{\mathbf{0}, \mathbf{z}^{(2)}\} \in \mathbb{R}^{d_2}$.
- $\mathbf{z}^{(3)} = \mathbf{W}^{(2)} \mathbf{x}^{(2)} \in \mathbb{R}^{10}$.
- $\mathbf{x}^{(3)} = \text{SoftMax}(\mathbf{z}^{(3)}) \in \mathbb{R}^{10}$.
- Output: $\mathbf{f}(\mathbf{x}^{(0)}) = \mathbf{x}^{(3)}$.

How to compute $\frac{\partial \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j)}{\partial \mathbf{W}^{(k)}}$?

Backpropagation:

- Denote $L = \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j)$.
- Compute $\frac{\partial L}{\partial \mathbf{z}^{(3)}}$.

$\mathbf{z}^{(3)}$ is a function of $\mathbf{z}^{(2)}$ and $\mathbf{W}^{(2)}$.

Apply the chain rule.

Backpropagation

Define a function $\mathbf{f}: \mathbb{R}^{785} \mapsto \mathbb{R}^{10}$:

- Input: vector $\mathbf{x}^{(0)} \in \mathbb{R}^{785}$.
- $\mathbf{z}^{(1)} = \mathbf{W}^{(0)} \mathbf{x}^{(0)} \in \mathbb{R}^{d_1}$.
- $\mathbf{x}^{(1)} = \max\{\mathbf{0}, \mathbf{z}^{(1)}\} \in \mathbb{R}^{d_1}$.
- $\mathbf{z}^{(2)} = \mathbf{W}^{(1)} \mathbf{x}^{(1)} \in \mathbb{R}^{d_2}$.
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- $\mathbf{x}^{(3)} = \text{SoftMax}(\mathbf{z}^{(3)}) \in \mathbb{R}^{10}$.
- Output: $\mathbf{f}(\mathbf{x}^{(0)}) = \mathbf{x}^{(3)}$.

How to compute $\frac{\partial \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j)}{\partial \mathbf{W}^{(k)}}$?

Backpropagation:

- Denote $L = \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j)$.

- Compute $\frac{\partial L}{\partial \mathbf{z}^{(3)}}$.

$$\frac{\partial L}{\partial \mathbf{z}^{(2)}} = \frac{\partial \mathbf{z}^{(3)}}{\partial \mathbf{z}^{(2)}} \frac{\partial L}{\partial \mathbf{z}^{(3)}} \quad \frac{\partial L}{\partial \mathbf{W}^{(2)}} = \frac{\partial \mathbf{z}^{(3)}}{\partial \mathbf{W}^{(2)}} \frac{\partial L}{\partial \mathbf{z}^{(3)}}$$

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Apply the chain rule.

Backpropagation

Define a function $\mathbf{f}: \mathbb{R}^{785} \mapsto \mathbb{R}^{10}$:

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How to compute $\frac{\partial \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j)}{\partial \mathbf{W}^{(k)}}$?

Backpropagation:

- Denote $L = \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j)$.
- Compute $\frac{\partial L}{\partial \mathbf{z}^{(3)}}$.
- $\frac{\partial L}{\partial \mathbf{z}^{(2)}} = \frac{\partial \mathbf{z}^{(3)}}{\partial \mathbf{z}^{(2)}} \frac{\partial L}{\partial \mathbf{z}^{(3)}}, \quad \frac{\partial L}{\partial \mathbf{W}^{(2)}} = \frac{\partial \mathbf{z}^{(3)}}{\partial \mathbf{W}^{(2)}} \frac{\partial L}{\partial \mathbf{z}^{(3)}}.$

Then free $\frac{\partial L}{\partial \mathbf{z}^{(3)}}$ from memory.

Backpropagation

Define a function $\mathbf{f}: \mathbb{R}^{785} \mapsto \mathbb{R}^{10}$:

- Input: vector $\mathbf{x}^{(0)} \in \mathbb{R}^{785}$.
- $\mathbf{z}^{(1)} = \mathbf{W}^{(0)} \mathbf{x}^{(0)} \in \mathbb{R}^{d_1}$.
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How to compute $\frac{\partial \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j)}{\partial \mathbf{W}^{(k)}}$?

Backpropagation:

- Denote $L = \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j)$.
- Compute $\frac{\partial L}{\partial \mathbf{z}^{(3)}}$.
- $\frac{\partial L}{\partial \mathbf{z}^{(2)}} = \frac{\partial \mathbf{z}^{(3)}}{\partial \mathbf{z}^{(2)}} \frac{\partial L}{\partial \mathbf{z}^{(3)}}$, $\frac{\partial L}{\partial \mathbf{W}^{(2)}}$ $= \frac{\partial \mathbf{z}^{(3)}}{\partial \mathbf{W}^{(2)}} \frac{\partial L}{\partial \mathbf{z}^{(3)}}$.

Use it to update $\mathbf{W}^{(2)}$ (e.g., by SGD).

Backpropagation

Define a function $\mathbf{f}: \mathbb{R}^{785} \mapsto \mathbb{R}^{10}$:

- Input: vector $\mathbf{x}^{(0)} \in \mathbb{R}^{785}$.
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- $\mathbf{x}^{(3)} = \text{SoftMax}(\mathbf{z}^{(3)}) \in \mathbb{R}^{10}$.
- Output: $\mathbf{f}(\mathbf{x}^{(0)}) = \mathbf{x}^{(3)}$.

How to compute $\frac{\partial \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j)}{\partial \mathbf{W}^{(k)}}$?

Backpropagation:

- Denote $L = \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j)$.
- Compute $\frac{\partial L}{\partial \mathbf{z}^{(3)}}$.
- $\frac{\partial L}{\partial \mathbf{z}^{(2)}} = \frac{\partial \mathbf{z}^{(3)}}{\partial \mathbf{z}^{(2)}} \frac{\partial L}{\partial \mathbf{z}^{(3)}}$, $\frac{\partial L}{\partial \mathbf{W}^{(2)}}$ $= \frac{\partial \mathbf{z}^{(3)}}{\partial \mathbf{W}^{(2)}} \frac{\partial L}{\partial \mathbf{z}^{(3)}}$.

Then free $\frac{\partial L}{\partial \mathbf{W}^{(2)}}$ from memory.

Backpropagation

Define a function $\mathbf{f}: \mathbb{R}^{785} \mapsto \mathbb{R}^{10}$:

- Input: vector $\mathbf{x}^{(0)} \in \mathbb{R}^{785}$.
- $\mathbf{z}^{(1)} = \mathbf{W}^{(0)} \mathbf{x}^{(0)} \in \mathbb{R}^{d_1}$.
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- $\mathbf{x}^{(3)} = \text{SoftMax}(\mathbf{z}^{(3)}) \in \mathbb{R}^{10}$.
- Output: $\mathbf{f}(\mathbf{x}^{(0)}) = \mathbf{x}^{(3)}$.

How to compute $\frac{\partial \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j)}{\partial \mathbf{W}^{(k)}}$?

Backpropagation:

- Denote $L = \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j)$.

- Compute $\frac{\partial L}{\partial \mathbf{z}^{(3)}}$.

$$\bullet \quad \boxed{\frac{\partial L}{\partial \mathbf{z}^{(2)}}} = \frac{\partial \mathbf{z}^{(3)}}{\partial \mathbf{z}^{(2)}} \frac{\partial L}{\partial \mathbf{z}^{(3)}}, \quad \frac{\partial L}{\partial \mathbf{W}^{(2)}} = \frac{\partial \mathbf{z}^{(3)}}{\partial \mathbf{W}^{(2)}} \frac{\partial L}{\partial \mathbf{z}^{(3)}}.$$

$$\bullet \quad \frac{\partial L}{\partial \mathbf{z}^{(1)}} = \frac{\partial \mathbf{z}^{(2)}}{\partial \mathbf{z}^{(1)}} \boxed{\frac{\partial L}{\partial \mathbf{z}^{(2)}}}, \quad \frac{\partial L}{\partial \mathbf{W}^{(1)}} = \frac{\partial \mathbf{z}^{(2)}}{\partial \mathbf{W}^{(1)}} \boxed{\frac{\partial L}{\partial \mathbf{z}^{(2)}}}.$$

Apply the chain rule again.

Backpropagation

Define a function $\mathbf{f}: \mathbb{R}^{785} \mapsto \mathbb{R}^{10}$:

- Input: vector $\mathbf{x}^{(0)} \in \mathbb{R}^{785}$.
- $\mathbf{z}^{(1)} = \mathbf{W}^{(0)} \mathbf{x}^{(0)} \in \mathbb{R}^{d_1}$.
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How to compute $\frac{\partial \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j)}{\partial \mathbf{W}^{(k)}}$?

Backpropagation:

- Denote $L = \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j)$.
- Compute $\frac{\partial L}{\partial \mathbf{z}^{(3)}}$.
- $\frac{\partial L}{\partial \mathbf{z}^{(2)}} = \frac{\partial \mathbf{z}^{(3)}}{\partial \mathbf{z}^{(2)}} \frac{\partial L}{\partial \mathbf{z}^{(3)}}, \quad \frac{\partial L}{\partial \mathbf{W}^{(2)}} = \frac{\partial \mathbf{z}^{(3)}}{\partial \mathbf{W}^{(2)}} \frac{\partial L}{\partial \mathbf{z}^{(3)}}$.
- $\frac{\partial L}{\partial \mathbf{z}^{(1)}} = \frac{\partial \mathbf{z}^{(2)}}{\partial \mathbf{z}^{(1)}} \frac{\partial L}{\partial \mathbf{z}^{(2)}}, \quad \frac{\partial L}{\partial \mathbf{W}^{(1)}} = \frac{\partial \mathbf{z}^{(2)}}{\partial \mathbf{W}^{(1)}} \frac{\partial L}{\partial \mathbf{z}^{(2)}}$.

Free $\frac{\partial L}{\partial \mathbf{z}^{(2)}}$ from memory.

Backpropagation

Define a function $\mathbf{f}: \mathbb{R}^{785} \mapsto \mathbb{R}^{10}$:

- Input: vector $\mathbf{x}^{(0)} \in \mathbb{R}^{785}$.
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- $\mathbf{x}^{(1)} = \max\{\mathbf{0}, \mathbf{z}^{(1)}\} \in \mathbb{R}^{d_1}$.
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- $\mathbf{x}^{(3)} = \text{SoftMax}(\mathbf{z}^{(3)}) \in \mathbb{R}^{10}$.
- Output: $\mathbf{f}(\mathbf{x}^{(0)}) = \mathbf{x}^{(3)}$.

How to compute $\frac{\partial \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j)}{\partial \mathbf{W}^{(k)}}$?

Backpropagation:

- Denote $L = \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j)$.
- Compute $\frac{\partial L}{\partial \mathbf{z}^{(3)}}$.
- $\frac{\partial L}{\partial \mathbf{z}^{(2)}} = \frac{\partial \mathbf{z}^{(3)}}{\partial \mathbf{z}^{(2)}} \frac{\partial L}{\partial \mathbf{z}^{(3)}}$, $\frac{\partial L}{\partial \mathbf{W}^{(2)}} = \frac{\partial \mathbf{z}^{(3)}}{\partial \mathbf{W}^{(2)}} \frac{\partial L}{\partial \mathbf{z}^{(3)}}$.
- $\frac{\partial L}{\partial \mathbf{z}^{(1)}} = \frac{\partial \mathbf{z}^{(2)}}{\partial \mathbf{z}^{(1)}} \frac{\partial L}{\partial \mathbf{z}^{(2)}}$, $\frac{\partial L}{\partial \mathbf{W}^{(1)}}$ $= \frac{\partial \mathbf{z}^{(2)}}{\partial \mathbf{W}^{(1)}} \frac{\partial L}{\partial \mathbf{z}^{(2)}}$.

Use it to update $\mathbf{W}^{(1)}$ (e.g., by SGD).

Backpropagation

Define a function $\mathbf{f}: \mathbb{R}^{785} \mapsto \mathbb{R}^{10}$:

- Input: vector $\mathbf{x}^{(0)} \in \mathbb{R}^{785}$.
- $\mathbf{z}^{(1)} = \mathbf{W}^{(0)} \mathbf{x}^{(0)} \in \mathbb{R}^{d_1}$.
- $\mathbf{x}^{(1)} = \max\{\mathbf{0}, \mathbf{z}^{(1)}\} \in \mathbb{R}^{d_1}$.
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- $\mathbf{z}^{(3)} = \mathbf{W}^{(2)} \mathbf{x}^{(2)} \in \mathbb{R}^{10}$.
- $\mathbf{x}^{(3)} = \text{SoftMax}(\mathbf{z}^{(3)}) \in \mathbb{R}^{10}$.
- Output: $\mathbf{f}(\mathbf{x}^{(0)}) = \mathbf{x}^{(3)}$.

How to compute $\frac{\partial \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j)}{\partial \mathbf{W}^{(k)}}$?

Backpropagation:

- Denote $L = \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j)$.
- Compute $\frac{\partial L}{\partial \mathbf{z}^{(3)}}$.
- $\frac{\partial L}{\partial \mathbf{z}^{(2)}} = \frac{\partial \mathbf{z}^{(3)}}{\partial \mathbf{z}^{(2)}} \frac{\partial L}{\partial \mathbf{z}^{(3)}}, \quad \frac{\partial L}{\partial \mathbf{W}^{(2)}} = \frac{\partial \mathbf{z}^{(3)}}{\partial \mathbf{W}^{(2)}} \frac{\partial L}{\partial \mathbf{z}^{(3)}}$.
- $\frac{\partial L}{\partial \mathbf{z}^{(1)}} = \frac{\partial \mathbf{z}^{(2)}}{\partial \mathbf{z}^{(1)}} \frac{\partial L}{\partial \mathbf{z}^{(2)}}, \quad \boxed{\frac{\partial L}{\partial \mathbf{W}^{(1)}}} = \frac{\partial \mathbf{z}^{(2)}}{\partial \mathbf{W}^{(1)}} \frac{\partial L}{\partial \mathbf{z}^{(2)}}$.

Free $\frac{\partial L}{\partial \mathbf{W}^{(1)}}$ from memory.

Backpropagation

Define a function $\mathbf{f}: \mathbb{R}^{785} \mapsto \mathbb{R}^{10}$:

- Input: vector $\mathbf{x}^{(0)} \in \mathbb{R}^{785}$.
- $\mathbf{z}^{(1)} = \mathbf{W}^{(0)} \mathbf{x}^{(0)} \in \mathbb{R}^{d_1}$.
- $\mathbf{x}^{(1)} = \max\{\mathbf{0}, \mathbf{z}^{(1)}\} \in \mathbb{R}^{d_1}$.
- $\mathbf{z}^{(2)} = \mathbf{W}^{(1)} \mathbf{x}^{(1)} \in \mathbb{R}^{d_2}$.
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- $\mathbf{z}^{(3)} = \mathbf{W}^{(2)} \mathbf{x}^{(2)} \in \mathbb{R}^{10}$.
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- Output: $\mathbf{f}(\mathbf{x}^{(0)}) = \mathbf{x}^{(3)}$.

How to compute $\frac{\partial \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j)}{\partial \mathbf{W}^{(k)}}$?

Backpropagation:

- Denote $L = \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j)$.

- Compute $\frac{\partial L}{\partial \mathbf{z}^{(3)}}$.

$$\frac{\partial L}{\partial \mathbf{z}^{(2)}} = \frac{\partial \mathbf{z}^{(3)}}{\partial \mathbf{z}^{(2)}} \frac{\partial L}{\partial \mathbf{z}^{(3)}}, \quad \frac{\partial L}{\partial \mathbf{W}^{(2)}} = \frac{\partial \mathbf{z}^{(3)}}{\partial \mathbf{W}^{(2)}} \frac{\partial L}{\partial \mathbf{z}^{(3)}}.$$

$$\frac{\partial L}{\partial \mathbf{z}^{(1)}} = \frac{\partial \mathbf{z}^{(2)}}{\partial \mathbf{z}^{(1)}} \frac{\partial L}{\partial \mathbf{z}^{(2)}}, \quad \frac{\partial L}{\partial \mathbf{W}^{(1)}} = \frac{\partial \mathbf{z}^{(2)}}{\partial \mathbf{W}^{(1)}} \frac{\partial L}{\partial \mathbf{z}^{(2)}}.$$

$$\frac{\partial L}{\partial \mathbf{W}^{(0)}} = \frac{\partial \mathbf{z}^{(1)}}{\partial \mathbf{W}^{(0)}} \frac{\partial L}{\partial \mathbf{z}^{(1)}}.$$

Apply the chain rule again.

Backpropagation

Define a function $\mathbf{f}: \mathbb{R}^{785} \mapsto \mathbb{R}^{10}$:

- Input: vector $\mathbf{x}^{(0)} \in \mathbb{R}^{785}$.
- $\mathbf{z}^{(1)} = \mathbf{W}^{(0)} \mathbf{x}^{(0)} \in \mathbb{R}^{d_1}$.
- $\mathbf{x}^{(1)} = \max\{\mathbf{0}, \mathbf{z}^{(1)}\} \in \mathbb{R}^{d_1}$.
- $\mathbf{z}^{(2)} = \mathbf{W}^{(1)} \mathbf{x}^{(1)} \in \mathbb{R}^{d_2}$.
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- $\mathbf{z}^{(3)} = \mathbf{W}^{(2)} \mathbf{x}^{(2)} \in \mathbb{R}^{10}$.
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- Output: $\mathbf{f}(\mathbf{x}^{(0)}) = \mathbf{x}^{(3)}$.

How to compute $\frac{\partial \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j)}{\partial \mathbf{W}^{(k)}}$?

Backpropagation:

- Denote $L = \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j)$.

- Compute $\frac{\partial L}{\partial \mathbf{z}^{(3)}}$.

$$\frac{\partial L}{\partial \mathbf{z}^{(2)}} = \frac{\partial \mathbf{z}^{(3)}}{\partial \mathbf{z}^{(2)}} \frac{\partial L}{\partial \mathbf{z}^{(3)}}, \quad \frac{\partial L}{\partial \mathbf{W}^{(2)}} = \frac{\partial \mathbf{z}^{(3)}}{\partial \mathbf{W}^{(2)}} \frac{\partial L}{\partial \mathbf{z}^{(3)}}.$$

$$\frac{\partial L}{\partial \mathbf{z}^{(1)}} = \frac{\partial \mathbf{z}^{(2)}}{\partial \mathbf{z}^{(1)}} \frac{\partial L}{\partial \mathbf{z}^{(2)}},$$

Free $\frac{\partial L}{\partial \mathbf{z}^{(1)}}$ from memory.

$$\frac{\partial L}{\partial \mathbf{W}^{(0)}} = \frac{\partial \mathbf{z}^{(1)}}{\partial \mathbf{W}^{(0)}} \frac{\partial L}{\partial \mathbf{z}^{(1)}}.$$

Backpropagation

Define a function $\mathbf{f}: \mathbb{R}^{785} \mapsto \mathbb{R}^{10}$:

- Input: vector $\mathbf{x}^{(0)} \in \mathbb{R}^{785}$.
- $\mathbf{z}^{(1)} = \mathbf{W}^{(0)} \mathbf{x}^{(0)} \in \mathbb{R}^{d_1}$.
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- $\mathbf{z}^{(3)} = \mathbf{W}^{(2)} \mathbf{x}^{(2)} \in \mathbb{R}^{10}$.
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- Output: $\mathbf{f}(\mathbf{x}^{(0)}) = \mathbf{x}^{(3)}$.

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- Denote $L = \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j)$.
- Compute $\frac{\partial L}{\partial \mathbf{z}^{(3)}}$.
- $\frac{\partial L}{\partial \mathbf{z}^{(2)}} = \frac{\partial \mathbf{z}^{(3)}}{\partial \mathbf{z}^{(2)}} \frac{\partial L}{\partial \mathbf{z}^{(3)}}, \quad \frac{\partial L}{\partial \mathbf{W}^{(2)}} = \frac{\partial \mathbf{z}^{(3)}}{\partial \mathbf{W}^{(2)}} \frac{\partial L}{\partial \mathbf{z}^{(3)}}$.
- $\frac{\partial L}{\partial \mathbf{z}^{(1)}} = \frac{\partial \mathbf{z}^{(2)}}{\partial \mathbf{z}^{(1)}} \frac{\partial L}{\partial \mathbf{z}^{(2)}}, \quad \frac{\partial L}{\partial \mathbf{W}^{(1)}} = \frac{\partial \mathbf{z}^{(2)}}{\partial \mathbf{W}^{(1)}} \frac{\partial L}{\partial \mathbf{z}^{(2)}}$.
- $\frac{\partial L}{\partial \mathbf{W}^{(0)}} = \frac{\partial \mathbf{z}^{(1)}}{\partial \mathbf{W}^{(0)}} \frac{\partial L}{\partial \mathbf{z}^{(1)}}$. Use it to update $\mathbf{W}^{(0)}$.

Backpropagation: Example

1D Example

Define a function $f: \mathbb{R} \mapsto \mathbb{R}$

- **Input:** scalar $x^{(0)}$.
- $z^{(1)} = w^{(0)} x^{(0)}$.
- $x^{(1)} = \max\{0, z^{(1)}\}$.
- $z^{(2)} = w^{(1)} x^{(1)}$.
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Random sampling:

- Randomly sample j from $\{1, 2, \dots, n\}$.

Forward pass:

- Take x_j as input ($x^{(0)} = x_j$).
- Compute each layer $z^{(1)}, x^{(1)}, z^{(2)}, x^{(2)}, z^{(3)}$.

1D Example

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Backpropagation:

- Loss: $L = \frac{1}{2} (f(x_j) - y_j)^2$.

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Backpropagation:

- Loss: $L = \frac{1}{2} (f(x_j) - y_j)^2$.
- $\frac{\partial L}{\partial z^{(3)}} = z^{(3)} - y_j$.

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The value of $z^{(3)}$ is known
(after the forward pass).

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The value of $z^{(3)}$ is known
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Thus the value of $\frac{\partial L}{\partial z^{(3)}}$ is known.

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Backpropagation:

- Loss: $L = \frac{1}{2} (f(x_j) - y_j)^2$.
- $\frac{\partial L}{\partial z^{(3)}} = z^{(3)} - y_j$.
- $\frac{\partial L}{\partial z^{(2)}} = \boxed{\frac{\partial z^{(3)}}{\partial z^{(2)}}} \frac{\partial L}{\partial z^{(3)}}$

$$\frac{\partial z^{(3)}}{\partial x^{(2)}} = w^{(2)}, \quad \frac{\partial x^{(2)}}{\partial z^{(2)}} = \begin{cases} 1, & \text{if } z^{(2)} > 0; \\ 0, & \text{else.} \end{cases}$$

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Backpropagation:

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Backpropagation:

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$$\frac{\partial z^{(3)}}{\partial w^{(2)}} = x^{(2)}.$$

1D Example

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Backpropagation:

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Free $\frac{\partial L}{\partial z^{(3)}}$ from memory.

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Backpropagation:

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- $\frac{\partial L}{\partial z^{(3)}} = z^{(3)} - y_j$.
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Update $w^{(2)}$: $w^{(2)} \leftarrow w^{(2)} - \alpha \frac{\partial L}{\partial w^{(2)}}$.

1D Example

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Backpropagation:

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- $\frac{\partial L}{\partial z^{(3)}} = z^{(3)} - y_j$.
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Free $\frac{\partial z^{(3)}}{\partial w^{(2)}}$ from memory.

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- $\frac{\partial L}{\partial z^{(1)}} = \frac{\partial z^{(2)}}{\partial z^{(1)}} \frac{\partial L}{\partial z^{(2)}}$

$$\frac{\partial z^{(2)}}{\partial z^{(1)}} = \frac{\partial z^{(2)}}{\partial x^{(1)}} \frac{\partial x^{(1)}}{\partial z^{(1)}} = \begin{cases} w^{(1)}, & \text{if } z^{(1)} > 0; \\ 0, & \text{else.} \end{cases}$$

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$$\frac{\partial z^{(2)}}{\partial w^{(1)}} = x^{(1)}.$$

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Backpropagation:

- Loss: $L = \frac{1}{2} (f(x_j) - y_j)^2$.
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Free $\frac{\partial L}{\partial z^{(2)}}$ from memory.

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Update $w^{(1)}$: $w^{(1)} \leftarrow w^{(1)} - \alpha \frac{\partial L}{\partial w^{(1)}}$.

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Free $\frac{\partial L}{\partial w^{(1)}}$ from memory.

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$$\frac{\partial z^{(1)}}{\partial w^{(0)}} = x^{(0)}.$$

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- $\frac{\partial L}{\partial w^{(0)}} = \frac{\partial z^{(1)}}{\partial w^{(0)}} \frac{\partial L}{\partial z^{(1)}}$.

Update $w^{(0)}$: $w^{(0)} \leftarrow w^{(0)} - \alpha \frac{\partial L}{\partial w^{(0)}}$.

1D Example

Define a function $f: \mathbb{R} \mapsto \mathbb{R}$

Backpropagation:

One iteration:

1. Randomly sample j from $\{1, 2, \dots, n\}$.

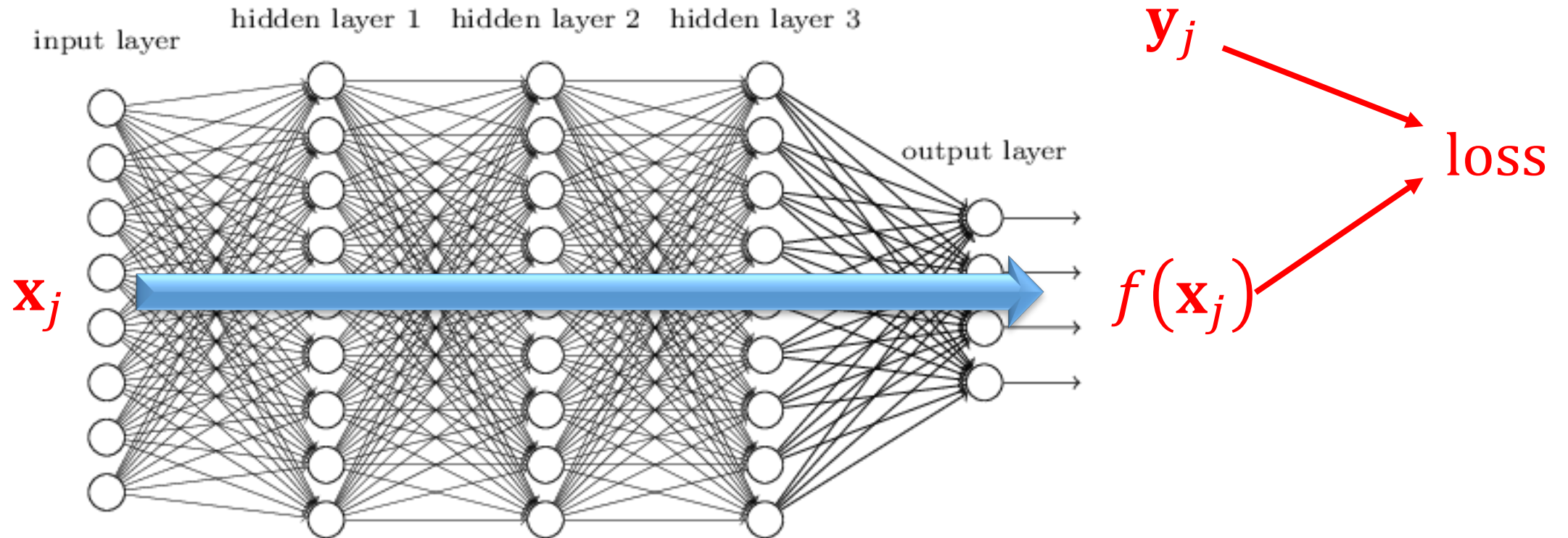
2. **Forward pass:** take x_j as input ($x^{(0)} = x_j$), compute each layer $z^{(1)}, x^{(1)}, z^{(2)}, x^{(2)}, z^{(3)}$.

3. **Backward pass:**

- Compute the derivatives $\frac{\partial L}{\partial z^{(3)}}, \frac{\partial L}{\partial w^{(2)}}, \frac{\partial L}{\partial z^{(2)}}, \frac{\partial L}{\partial w^{(1)}}, \frac{\partial L}{\partial z^{(1)}}, \frac{\partial L}{\partial w^{(0)}}$.
- Update $w^{(k)}$ using $\frac{\partial L}{\partial w^{(k)}}$.

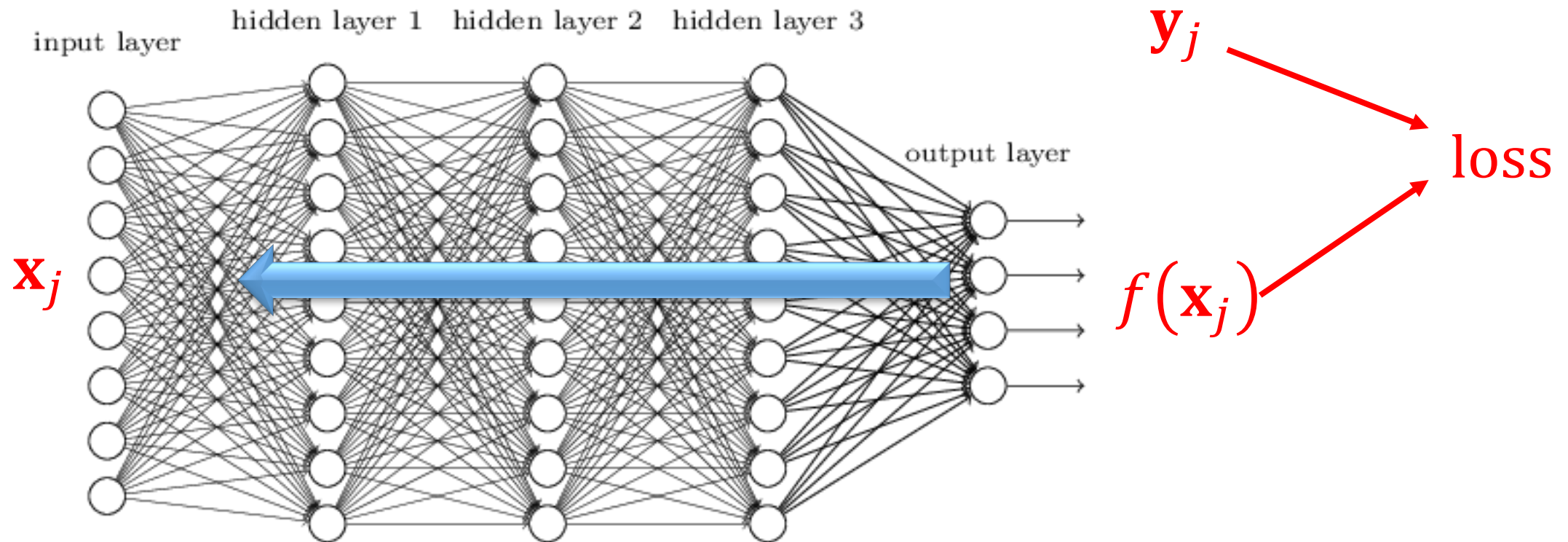
Backpropagation: Take-Home Message

1. Randomly pick a sample $(\mathbf{x}_j, \mathbf{y}_j)$.
2. Run a forward pass (from the input $\mathbf{x}^{(0)}$ to the loss function).



Backpropagation: Take-Home Message

1. Randomly pick a sample $(\mathbf{x}_j, \mathbf{y}_j)$.
2. Run a forward pass (from the input $\mathbf{x}^{(0)}$ to the loss function).
3. Run a backward pass (from the **loss** to $\mathbf{W}^{(0)}$).



Backpropagation: Take-Home Message

1. Randomly pick a sample $(\mathbf{x}_j, \mathbf{y}_j)$.
2. Run a forward pass (from the input $\mathbf{x}^{(0)}$ to the loss function).
3. Run a backward pass (from the loss to $\mathbf{W}^{(0)}$).



Get the derivatives (stochastic sub-gradients):

$$\frac{\partial \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j)}{\partial \mathbf{W}^{(2)}}, \quad \frac{\partial \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j)}{\partial \mathbf{W}^{(1)}}, \quad \frac{\partial \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j)}{\partial \mathbf{W}^{(0)}}.$$

Backpropagation: Take-Home Message

1. Randomly pick a sample $(\mathbf{x}_j, \mathbf{y}_j)$.
2. Run a forward pass (from the input $\mathbf{x}^{(0)}$ to the loss function).
3. Run a backward pass (from the loss to $\mathbf{W}^{(0)}$).



Get the derivatives (stochastic sub-gradients):

$$\frac{\partial \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j)}{\partial \mathbf{W}^{(2)}}, \frac{\partial \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j)}{\partial \mathbf{W}^{(1)}}, \frac{\partial \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j)}{\partial \mathbf{W}^{(0)}}.$$



Update $\mathbf{W}^{(0)}, \mathbf{W}^{(1)}, \mathbf{W}^{(2)}$ using the derivatives.

Mini-Batch

1. Randomly pick a sample ~~$(\mathbf{x}_j, \mathbf{y}_j)$~~ . Several random samples.
2. Run a forward pass (from the input $\mathbf{x}^{(0)}$ to the loss function).
3. Run a backward pass (from the loss to $\mathbf{W}^{(0)}$).



Get the derivatives (stochastic sub-gradients):

$$\frac{\partial \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j)}{\partial \mathbf{W}^{(2)}}, \quad \frac{\partial \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j)}{\partial \mathbf{W}^{(1)}}, \quad \frac{\partial \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j)}{\partial \mathbf{W}^{(0)}}.$$

$$\frac{1}{|\mathcal{J}|} \sum_{j \in \mathcal{J}} \frac{\partial \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j)}{\partial \mathbf{W}^{(2)}}, \quad \frac{1}{|\mathcal{J}|} \sum_{j \in \mathcal{J}} \frac{\partial \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j)}{\partial \mathbf{W}^{(1)}}, \quad \frac{1}{|\mathcal{J}|} \sum_{j \in \mathcal{J}} \frac{\partial \text{Loss}(\mathbf{f}(\mathbf{x}_j), \mathbf{y}_j)}{\partial \mathbf{W}^{(0)}}.$$

Mini-batch should always be used! Set $|\mathcal{J}|$ to 16, 32, 64, ...

Mini-Batch

SGD: $\text{BatchSize} = 1$.

- Per-iteration cost is low.
- Lots of iterations to converge.

Mini-Batch: $\text{BatchSize} > 1$.

- Better than the other two, if BatchSize is properly set.

Full Gradient: $\text{BatchSize} = n$.

- Per-iteration cost is n times higher than SGD.
- Convex problem: less number of iterations.
- Neural network: it doesn't work!

First-Order Optimization

- First-order optimization: update the parameters using gradient.
- Gradient descent algorithm (including SGD, mini-batch SGD, and full gradient descent, conjugate gradient) are typical 1st-order algorithms.
- Other 1st-order algorithms: SGD with momentum, AdaGrad, RMSprop...
- See the blogs:
 - <http://runder.io/optimizing-gradient-descent/>
 - <https://distill.pub/2017/momentum/>

Summary

Build a Fully-Connected Neural Network

- Network structure

Number of layers

Width of each layer

Activation functions

Build a Fully-Connected Neural Network

- Network structure

Number of layers

Width of each layer

Activation functions

Example:

- **Input:** vector $\mathbf{x}^{(0)} \in \mathbb{R}^{785}$. **Input layer**
- $\mathbf{z}^{(1)} = \mathbf{W}^{(0)} \mathbf{x}^{(0)} \in \mathbb{R}^{d_1}$. **Hidden Layer 1**
- $\mathbf{x}^{(1)} = \max\{\mathbf{0}, \mathbf{z}^{(1)}\} \in \mathbb{R}^{d_1}$.
- $\mathbf{z}^{(2)} = \mathbf{W}^{(1)} \mathbf{x}^{(1)} \in \mathbb{R}^{d_2}$. **Hidden Layer 2**
- $\mathbf{x}^{(2)} = \max\{\mathbf{0}, \mathbf{z}^{(2)}\} \in \mathbb{R}^{d_2}$.
- $\mathbf{z}^{(3)} = \mathbf{W}^{(2)} \mathbf{x}^{(2)} \in \mathbb{R}^{10}$. **Output layer**
- **Output:** $\mathbf{f}(\mathbf{x}^{(0)}) = \text{SoftMax}(\mathbf{z}^{(2)})$.

- Three layers (2 hidden and 1 output).
 - Input layer doesn't count (no parameter).

Build a Fully-Connected Neural Network

- Network structure

Number of layers

Width of each layer

Activation functions

Example:

- **Input:** vector $\mathbf{x}^{(0)} \in \mathbb{R}^{785}$.
- $\mathbf{z}^{(1)} = \mathbf{W}^{(0)} \mathbf{x}^{(0)} \in \mathbb{R}^{d_1}$.
- $\mathbf{x}^{(1)} = \max\{\mathbf{0}, \mathbf{z}^{(1)}\} \in \mathbb{R}^{d_1}$.
- $\mathbf{z}^{(2)} = \mathbf{W}^{(1)} \mathbf{x}^{(1)} \in \mathbb{R}^{d_2}$.
- $\mathbf{x}^{(2)} = \max\{\mathbf{0}, \mathbf{z}^{(2)}\} \in \mathbb{R}^{d_2}$.
- $\mathbf{z}^{(3)} = \mathbf{W}^{(2)} \mathbf{x}^{(2)} \in \mathbb{R}^{10}$.
- **Output:** $\mathbf{f}(\mathbf{x}^{(0)}) = \text{SoftMax}(\mathbf{z}^{(2)})$.

- Three layers (2 hidden and 1 output).
 - Input layer doesn't count (no parameter).
- Width of each layer:
 - Layer 1: d_1 ,
 - Layer 2: d_2 ,
 - Output layer: 10.

Build a Fully-Connected Neural Network

- Network structure

Number of layers

Width of each layer

Activation functions

Example:

- **Input:** vector $\mathbf{x}^{(0)} \in \mathbb{R}^{785}$.
- $\mathbf{z}^{(1)} = \mathbf{W}^{(0)} \mathbf{x}^{(0)} \in \mathbb{R}^{d_1}$.
- $\mathbf{x}^{(1)} = \max\{\mathbf{0}, \mathbf{z}^{(1)}\} \in \mathbb{R}^{d_1}$.
- $\mathbf{z}^{(2)} = \mathbf{W}^{(1)} \mathbf{x}^{(1)} \in \mathbb{R}^{d_2}$.
- $\mathbf{x}^{(2)} = \max\{\mathbf{0}, \mathbf{z}^{(2)}\} \in \mathbb{R}^{d_2}$.
- $\mathbf{z}^{(3)} = \mathbf{W}^{(2)} \mathbf{x}^{(2)} \in \mathbb{R}^{10}$.
- **Output:** $\mathbf{f}(\mathbf{x}^{(0)}) = \text{SoftMax}(\mathbf{z}^{(2)})$.

- Three layers (2 hidden and 1 output).
 - Input layer doesn't count (no parameter).
- Width of each layer:
 - Layer 1: d_1 ,
 - Layer 2: d_2 ,
 - Output layer: 10.
- Activation functions:
 - Layer 1: ReLU,
 - Layer 2: ReLU,
 - Output layer: Softmax.

Build a Fully-Connected Neural Network

- Network structure

Number of layers

Width of each layer

Activation functions

Example:

- Input: vector $\mathbf{x}^{(0)} \in \mathbb{R}^{785}$.
- $\mathbf{z}^{(1)} = \mathbf{W}^{(0)} \mathbf{x}^{(0)} \in \mathbb{R}^{d_1}$.
- $\mathbf{x}^{(1)} = \max\{\mathbf{0}, \mathbf{z}^{(1)}\} \in \mathbb{R}^{d_1}$.
- $\mathbf{z}^{(2)} = \mathbf{W}^{(1)} \mathbf{x}^{(1)} \in \mathbb{R}^{d_2}$.
- $\mathbf{x}^{(2)} = \max\{\mathbf{0}, \mathbf{z}^{(2)}\} \in \mathbb{R}^{d_2}$.
- $\mathbf{z}^{(3)} = \mathbf{W}^{(2)} \mathbf{x}^{(2)} \in \mathbb{R}^{10}$.
- Output: $\mathbf{f}(\mathbf{x}^{(0)}) = \text{SoftMax}(\mathbf{z}^{(2)})$.

- Trainable parameters:
 - $\mathbf{W}^{(0)} \in \mathbb{R}^{d_1 \times 785}$,
 - $\mathbf{W}^{(1)} \in \mathbb{R}^{d_2 \times d_1}$,
 - $\mathbf{W}^{(2)} \in \mathbb{R}^{10 \times d_2}$.
- Number of parameters:
 - $785d_1 + d_1d_2 + 10d_2$.

Build a Fully-Connected Neural Network

- Network structure

Number of layers

Width of each layer

Activation functions

- Loss functions

Cross-entropy for multi-class classification

Logistic for binary classification

L1 or squared L2 for regression (the labels are continuous)

Build a Fully-Connected Neural Network

- Network structure

Number of layers

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Activation functions

- Loss functions

Cross-entropy for multi-class classification

Logistic for binary classification

L1 or squared L2 for regression (the labels are continuous)

- Compute gradient: by a forward pass and a backward pass

Automatically performed by many deep learning systems

Batch size

Build a Fully-Connected Neural Network

- Network structure

Number of layers

Width of each layer

Activation functions

- Loss functions

Cross-entropy for multi-class classification

Logistic for binary classification

L1 or squared L2 for regression (the labels are continuous)

- Compute gradient: by a forward pass and a backward pass

Automatically performed by many deep learning systems

Batch size

- Choose an optimization algorithm (and tune its parameters)

SGD

SGD with momentum

AdaGrad

RMSprop