Support Vector Machine (SVM)

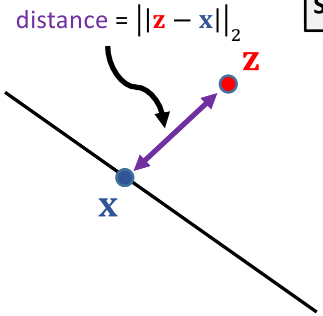
Shusen Wang

Project a Point onto a Hyperplane

Project a Point onto a Hyperplane

Question: how to project $\mathbf{z}$ to the hyperplane?

Solution: find $\mathbf{x}$ on the hyperplane such that $||\mathbf{z} - \mathbf{x}||_2^2$ is minimized.



Hyperplane $\mathbf{w}^T\mathbf{x} + b = 0$

- $\min_{\mathbf{x}} ||\mathbf{z} - \mathbf{x}||_2^2; \quad$ s.t. $\mathbf{w}^T\mathbf{x} + b = 0$

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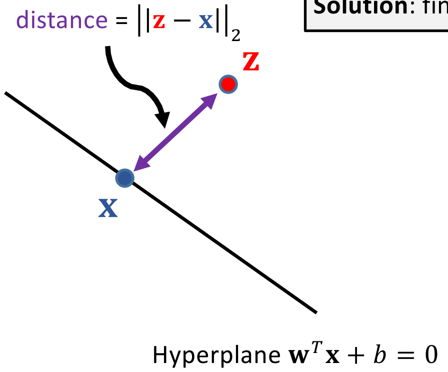


- $\min_{\mathbf{x}} ||\mathbf{z} - \mathbf{x}||_2^2;$ s.t. $\mathbf{w}^T\mathbf{x} + b = 0$
- Solve the problem using the KKT condition:

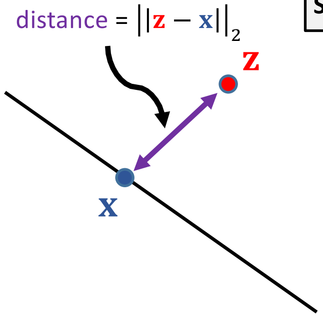
$$\begin{cases} \dfrac{\partial ||\mathbf{z} - \mathbf{x}||_2^2}{\partial \mathbf{x}} + \lambda \dfrac{\partial \left(\mathbf{w}^T\mathbf{x} + b\right)}{\partial \mathbf{x}} = 0; \\ \mathbf{w}^T\mathbf{x} + b = 0. \end{cases}$$

- Solution: $\mathbf{x} = \mathbf{z} - \frac{\mathbf{w}^T\mathbf{z} + b}{||\mathbf{w}||_2^2}\mathbf{w}$

Project a Point onto a Hyperplane

Question: how to project $\mathbf{z}$ to the hyperplane?

Solution: find $\mathbf{x}$ on the hyperplane such that $\|\mathbf{z}-\mathbf{x}\|_2^2$ is minimized.



Hyperplane $\mathbf{w}^T\mathbf{x}+b=0$

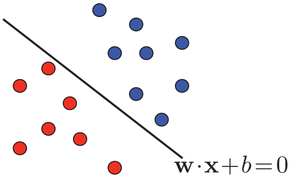
- Solution: $\mathbf{x}=\mathbf{z}-\frac{\mathbf{w}^T\mathbf{z}+b}{\|\mathbf{w}\|_2^2}\mathbf{w}$
- The ℓ_2 distance between $\mathbf{z}$ and the hyperplane is

$$\|\mathbf{z}-\mathbf{x}\|_2=\frac{|\mathbf{w}^T\mathbf{z}+b|}{\|\mathbf{w}\|_2}.$$

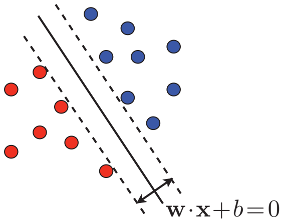
Support Vector Machine

Support Vector Machine (SVM)

Separate data by a hyperplane (assume the data are separable)



An arbitrary hyperplane.

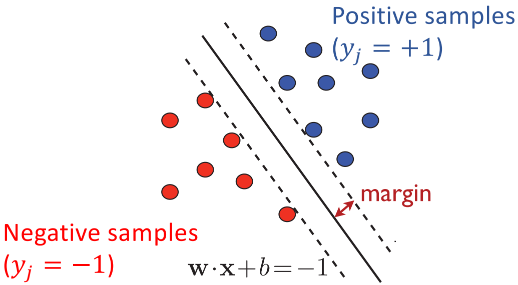


The hyperplane that maximizes the margin.

The figure is from the book "*Foundations of Machine Learning*"

Support Vector Machine (SVM)

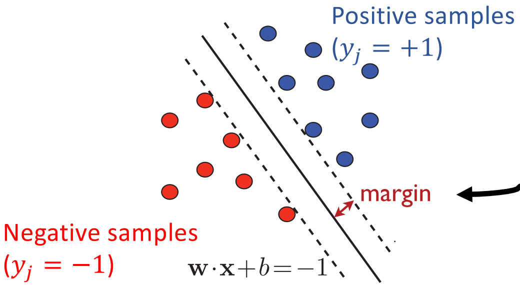
Separate data by a hyperplane (assume the data are separable)



The figure is from the book "*Foundations of Machine Learning*"

Support Vector Machine (SVM)

Separate data by a hyperplane (assume the data are separable)



- The distance between any $\mathbf{x}$ and the hyperplane is
$$\frac{|\mathbf{w}^T\mathbf{x}+b|}{||\mathbf{w}||_2}.$$
- The margin is
$$\min_j \frac{|\mathbf{w}^T\mathbf{x}_j+b|}{||\mathbf{w}||_2} = \min_j \frac{y_j(\mathbf{w}^T\mathbf{x}_j+b)}{||\mathbf{w}||_2}.$$

The figure is from the book "*Foundations of Machine Learning*"

Support Vector Machine (SVM)

Margin = $\min_j \frac{y_j(\mathbf{w}^T\mathbf{x}_j + b)}{||\mathbf{w}||_2}$; we want to maximize the margin.

Support Vector Machine (SVM): $\max_{\mathbf{w},b} \min_j \frac{y_j(\mathbf{w}^T\mathbf{x}_j + b)}{||\mathbf{w}||_2}$

Support Vector Machine (SVM)

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$$\max_{\mathbf{w},b} \min_j \frac{y_j\left(\mathbf{w}^T\mathbf{x}_j + b\right)}{||\mathbf{w}||_2} = \max_{\mathbf{w},b} \frac{\min_j y_j\left(\mathbf{w}^T\mathbf{x}_j + b\right)}{||\mathbf{w}||_2}$$

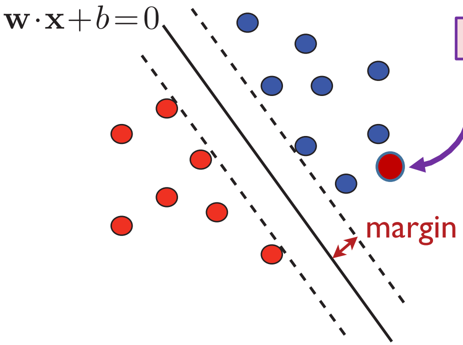
$$= \max_{\mathbf{w},b} \frac{1}{||\mathbf{w}||_2}, \qquad \text{s.t.} \quad \min_j \; y_j\left(\mathbf{w}^T\mathbf{x}_j + b\right) = 1$$

$$= \min_{\mathbf{w},b} ||\mathbf{w}||_2^2, \qquad \text{s.t.} \quad \min_j \; y_j\left(\mathbf{w}^T\mathbf{x}_j + b\right) = 1$$

$$= \min_{\mathbf{w},b} ||\mathbf{w}||_2^2, \qquad \text{s.t.} \quad y_j\left(\mathbf{w}^T\mathbf{x}_j + b\right) \geq 1 \text{ for all } j$$

Support Vector Machine (SVM)

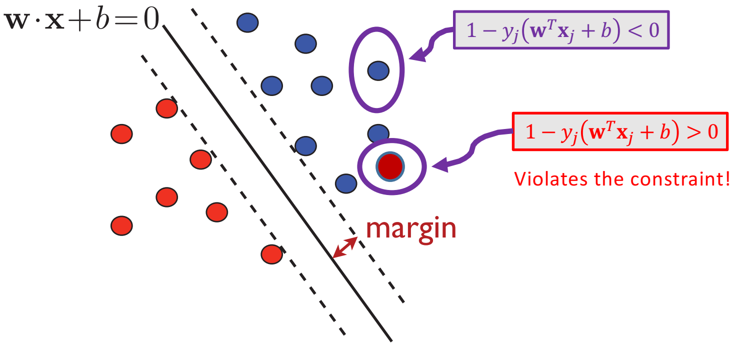
$$\min_{\mathbf{w},b} ||\mathbf{w}||_2^2, \quad \text{s.t.} \quad 1 - y_j(\mathbf{w}^T\mathbf{x}_j + b) \leq 0 \text{ for all } j \in \{1, \cdots, n\}.$$



What if the data is inseparable?

Support Vector Machine (SVM)

$$\min_{\mathbf{w},b} \|\mathbf{w}\|_2^2, \quad \text{s.t.} \quad 1 - y_j\left(\mathbf{w}^T\mathbf{x}_j + b\right) \leq 0 \ \text{ for all } j \in \{1, \cdots, n\}.$$



Support Vector Machine (SVM)

$$\min_{\mathbf{w},b} \|\mathbf{w}\|_2^2, \quad \text{s.t.} \quad 1 - y_j\left(\mathbf{w}^T\mathbf{x}_j + b\right) \leq 0 \ \text{ for all } j \in \{1, \cdots, n\}.$$

$\Big\Downarrow$ **Relax**

$$\min_{\mathbf{w},b,\xi_j} \|\mathbf{w}\|_2^2 + \lambda \sum_j [\xi_j]_+, \quad \text{s.t.} \quad 1 - y_j\left(\mathbf{w}^T\mathbf{x}_j + b\right) = \xi_j \ \text{ for all } j \in \{1, \cdots, n\}.$$

- $[\xi_j]_+ = \max\{\xi_j, 0\}$
- $\xi_j \leq 0$ means the constraint $1 - y_j\left(\mathbf{w}^T\mathbf{x}_j + b\right) \leq 0$ is satisfied
  ➔ no penalty!
- $\xi_j > 0$ means the constraint is violated (because the data is inseparable)
  ➔ penalize the violation ξ_j.

Support Vector Machine (SVM)

$$\min_{\mathbf{w},b} ||\mathbf{w}||_2^2, \quad \text{s.t.} \quad 1 - y_j(\mathbf{w}^T\mathbf{x}_j + b) \leq 0 \text{ for all } j \in \{1, \cdots, n\}.$$

$\Downarrow$ Relax

$$\min_{\mathbf{w},b,\xi_j} ||\mathbf{w}||_2^2 + \lambda \sum_j [\xi_j]_+, \quad \text{s.t.} \quad 1 - y_j(\mathbf{w}^T\mathbf{x}_j + b) = \xi_j \text{ for all } j \in \{1, \cdots, n\}.$$

$\Updownarrow$ Equivalent

$$\min_{\mathbf{w},b} ||\mathbf{w}||_2^2 + \lambda \sum_j [1 - y_j(\mathbf{w}^T\mathbf{x}_j + b)]_+.$$

Support Vector Machine (SVM)

$$\boxed{\min_{\mathbf{w},b} ||\mathbf{w}||_2^2 + \lambda \sum_j \left[1 - y_j\left(\mathbf{w}^T\mathbf{x}_j + b\right)\right]_+ .}$$

Support Vector Machine (SVM)

$$\min_{\mathbf{w},b} ||\mathbf{w}||_2^2 + \lambda \sum_j [1 - y_j(\mathbf{w}^T\mathbf{x}_j + b)]_+ .$$

SVM: $\min_{\mathbf{w},b} ||\mathbf{w}||_2^2 + \lambda \sum_j g\left(y_j(\mathbf{w}^T\mathbf{x}_j + b)\right).$

Hinge loss: $g(z) = [1 - z]_+ .$

Comparisons

$$\min_{\mathbf{w},b} ||\mathbf{w}||_2^2 + \lambda \sum_j [1 - y_j(\mathbf{w}^T\mathbf{x}_j + b)]_+.$$

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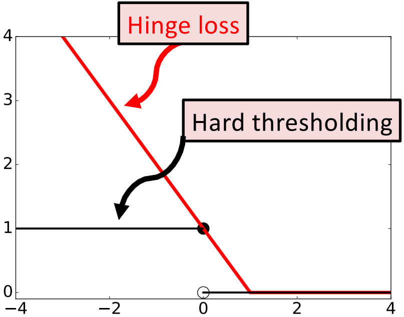

Hard thresholding: $h(z) = \begin{cases} 1, & \text{if } z < 0; \\ 0, & \text{if } z \geq 0. \end{cases}$

Comparisons

$$\min_{\mathbf{w},b} ||\mathbf{w}||_2^2 + \lambda \sum_j \left[1 - y_j\left(\mathbf{w}^T\mathbf{x}_j + b\right)\right]_+.$$

SVM: $\min_{\mathbf{w},b} ||\mathbf{w}||_2^2 + \lambda \sum_j g\left(y_j\left(\mathbf{w}^T\mathbf{x}_j + b\right)\right).$

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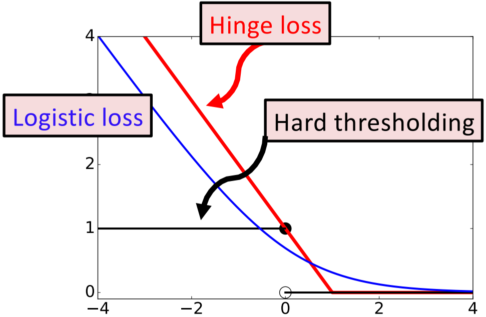
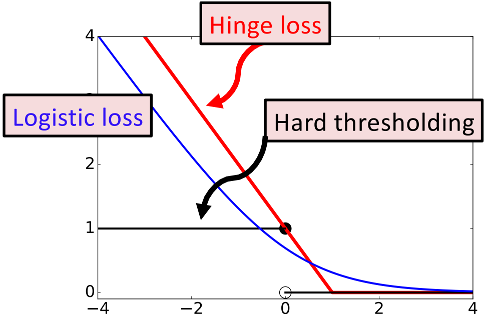


Hard thresholding: $h(z) = \begin{cases} 1, & \text{if } z < 0; \\ 0, & \text{if } z \geq 0. \end{cases}$

Logistic loss: $l(z) = \log(1 + e^{-z}).$

Comparisons



- Convexity
  - Hinge loss and logistic loss are convex.
  - Global optimal can be efficiently found.
- Smoothness
  - Hinge loss is non-smooth.
  - Logistic loss is smooth.
- Logistic regression is easier to solve than SVM.
  - GD for logistic regression has linear convergence.
  - Algorithms for SVM have sub-linear convergence.