**“One type of combinatorial problem”**

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**I.Introduction:**

Beginning from problem“How many ways to devide m candies for n children?”.This problem seems to be easy but it is too hard for students. Now I introduce this problem and some application.

**II.Standard knowledge:**

***Beginning Problem:*** m identical candies are distributed among n children. In how many ways can this be done?

This is also this problem: Find the number of all nonnegative integer solutions of the equation: x1+ x2+…+ xn=m (n, m ).

Solution:

We had n tuple (x1, x2,…, xn) is a solution of the given equation x1+ x2+…+ xn=m. We encode this solution with a sequence of m-ones and n zeros . The number of solution is equal to the number of ways to choose n-1 positions among m+n-1 positions for zeros then the number of solution is: d = .

***Standard Problem 1:*** Find the number of all positive integer solutions of the equation: x1+ x2+…+ xn=m (n, m ) (1).

Solution:

The positive integers x1, x2,…, xn are solution of the given equation if and only if the nonnegative numbers yi=xi-1 (i=) are solution of equation: y1+ y2 +…+ yn= m-n

+) If m<n equation has no solution.

+) If m n use beginning problem the number of solution of the equation (1) is : d= .

***(\*) Generalization Problem:*** Given n natural numbers a1,a2,…,an. Find the number of natural solution of the equation: x1+ x2+…+ xn=m satisfy xi  ai (i=).

Solution:

Let yi=xi- ai (i=) and A= a1+a2+…+an we have:

y1+ y2 +…+ yn= x1+ x2+…+ xn- a1+a2+…+an = m-A.

+) If m<A equation has no solution.

+) If m=A equation has 1 solution.

+) m>A equation has  solutions.

***Standard Problem 2:*** Find the number of all nonnegative integer solutions of the equation: x+y+z=n satisfy x y z (2).

Solution:

First we have lemma (\*):

“Find the number of all nonnegative integer solutions of the equation x+y=n under the condition x y.”

Because x y => x  

+) If n is divided by 2 , x has  + 1= +1 values 0,1,2,…, and one y for each x ( y=n-x) => We have  +1 solutions (x,y) satisfy x+y=n and x y.

+) If n is divided by 2 , x   and x has  + 1= +1 values and we have  + 1 solutions (x,y) satisfy the equation.

Return the main problem:

+) Let r1xyz is the number of solution of the equation (2) satisfy x<y<z.

Because x,y,z equalize on the equation : x+y+x=n.

We have: r1xyz = r1xzy = r1yzx = r1yxz = r1zxy = r1zyx=r1.

+) Let r2xyz is the number of solution of the equation (2) satisfy x=y<z.

We have: r2xyz = r2yzx = r2zxy = r2.

+) Let r3xyz is the number of solution of the equation (2) satisfy x<y=z.

We have: r3xyz = r3yzx = r3zxy = r3.

+) Let r4 is the number of solution of the equation (2) satisfy x=y=z.

The number of 3-tuples (x,y,z) satisfy x+y+z=n: = each tuple belongs to one of those cases. Thus:

6r1 + 3r2 + 3r3 + r4 =  (3)

We have : r2 + r3 + r4  is the number of all nonnegative integer solutions of the equation 2u+v=n (because x=y or y=z or z=x or x=y=z.)

Suppose that x=y, it has cases x=y>z, x=y<z, x=y=z.

The equation 2u+v=n <=> u+(u+v)=n. Symbolize t=u+v  u, u+t=n with u  t .

Use lemma (\*) the number of solutions of this equation is  +1

or r2 + r3 + r4  =  +1.

+) If n is divided by 3 => r4 = 1.

+) If n is not divided by 3 => r4 = 0.

=> r4 =.

Because x y z has cases x<y<z or x=y<z or x<y=z or x=y=z.

So the number of solutions is: d=r1 + r2 + r3 + r4.

Use equation (3) we have:

d = 

=

=

=

***Problem expand for 4 numbers:*** Find the number of all nonnegative integer solutions of the equation: x1+ x2+x3+ x4 = n and x1  x2  x3  x4 .

Solution:

Burnside’s lemma:  where v() denotes for the number of those elements a X that are mapped onto themselves

by , that is:

v() = 

Let X={(x1,x2,x3,x4); x1,x2,x3,x4N; x1+ x2+x3+ x4 = n}

Let  be the set of bijection on x induced by the permutation of the index set: A = {1;2;3;4}.

+) If = id ,have one , then v() is the number of solutions of the equation: x1+ x2+x3+ x4= n => v()=.

+) If  is induced by the interchange of one pair of indices (there are  = 6 ()), v() is the number of solutions of the equation: 2x+y+z=n

The unknown x can assume values from the set {0;1;…;} and equation has n-2x+1 solutions each x the total number of solutions is:

v() =

= 

+) If  is induced by the interchange of two pair of the indices (there are  = 3 ()), v() is the number of solutions of the equation: 2x+2y=n.

-) If n is odd equation has no solution.

-) If n is even equation has  solutions.

=> v() = .

+) If  is induced by a permutation that leaves exactly one index unchanged (there are 2.4 = 8 ()), v() is the number of solutions of the equation: 3x+y=n.

=> v() = .

+) If  is induced by a cyclic permutation (there are = 6 ()) v() is the number of solutions of the equation: 4x=n

=> v() = .

The number of  is: 1+6+3+8+6=24.

Use Burnside’s lemma the number of solution of the equation is:



***Generalization problem:*** Find the number of solutions of the equation: x1+ x2+…+ xn=m (4) under the condition: x1  x2  …  xn..

Solution:

Let Mm,n denote the set of solutions of the equation (4), and P(m,n) the desired number of its elements. We easily see that:

P(0,n)=1 for all k, P(m,n) = P(m,m) for all k  n..

Therefore, we assume further that fixed m,n we have 1< n  m. We subdivide the set Mm,n into classes Ti (i=0,1,…,n-1) such that Ti contains exactly those solutions of (4) in N0 that satisfy the condition:

0=x1=x2=…=xi<xi+1<xi+2<…<xn.

Then the assignment (x1,x2,…,xn)  (xi+1-1,xi+2-1,…,xn-1) defines a bijection from Ti to Mm-n+i,n-i since 0 xi+1-1 xi+2-1 … xn-1,

(xi+1-1)+(xi+2-1)+…+(xn-1)=m-n+i, and the inverse function is determined by the assignment

(y1,y2,…,yn-i)  (0,0,…,0, y1+1,y2+1,…,yn-i+1).

This implies that | Ti | = | Mm-n+i,n-i |,and thus

,

which can be wrinten as the recurrence relation:

P(m,n)=P(m-1,1)+P(m-2,2)+…+P(m-n,n)

(1< n  m), and together with (4) this enables us to compute, step by step, the desired number P(m,n).

**Some skills is useful on Euler problem:**

1. Numbering mẹthod.

2. Inclusion-exclusion principle.

* **II.Result and applications:**

**Problem 1:** Find the number of all nonnegative integer solutions of the equation: x+y+z+t=1000 with t  499.

Solution:

Solutions of this equation is and be divided into two cases:

Case 1: Satisfy t  499.

Cases 2: t  500 .

Because t  500 we have: x+y+z+(t-500) = 500

Let t1=t-500 the number of solution in this case is .

Thus the number of solutions satisfy is: d =  - .

**Problem 2:** Find the number of all nonnegative integer solutions of the equation: x+y+z+t  1000 .

Solution:

Let u=1000-(x+y+z+t)  0 we have the similar problem: x+y+z+t+u=1000 (x,y,z,u )

The result is: d=

**Generalization :** The number of all nonnegative integer solutions of the equation: x1+ x2+…+ xn  m is  .

**Problem 3:** Find the number of 4-tuples (x,y,z,t) nonnegative integer satisfy : 1  x  y  z  t  1000

Solution:

Let a1=x-10 , a2=y-x0, a3=z-y0, a4=t-z0, a5=1000-t0. We have:

a1 + a2 + a3 + a4 + a5 =1000.

Thus result: .

**Problem 4:** Find the number of all nonnegative integer solutions of the equation: x1+ x2+x3+ x4 = 30 satisfy 5xi10 

Solution:

Let yi=xi-5  .=> 0yi5 . We obtain:

y1+y2+y3+y4=10 (0yi5 ) (1).

Let X denote the set of solutions of the equation (1).

=> |X|=.

Let A,B,C,D denote the set of solutions satisfy

y1+y2+y3+y4=10 and 5yi.

Use problem 1:

|A|=|B|=|C|=|D|=.







Use inclusion-exclusion principle, the number of solutions is:









So the result is: - 4 = 146.

**Problem 5:** Find the number of all nonnegative integer solutions of the equation: x1+ x2+x3+ x4 = 100 satisfy x1 is divided by 3.

Solution:

Generating function of x1 is: x3+x6+…+x99 = 

Generating function of x2,x3,x4 is: 

Generating function of the equation: 

We have x100 ‘s coefficient is: 

So the number of solutions is: 115940.

**Problem 6:** Find the number of all nonnegative integer solutions of the equation: x1+ x2+x3+ x4 = 30 satisfy |xi|10 

Solution:

Because |xi|5 => -5xi5. Let yi = xi + 5

=> y1+y2+y3+y4=34 under the condition 0xi10 

Do equivalently problem 4, the result is 540

**Problem 7:** Given set A={ 1;2;3;…;18} how many ways to choose 5 number x1,x2,x3,x4,x5 satisfy |xi-xj|  2.

Solution:

On 5 number satisfy, no 2 number is consecutive, => these number devides sequence of 13 number into 6 sequence, without empty sequence.

Let a1,a2,..,a6 denote the number of integer in each sequence. Problem is equivalent to find the number of solutions of the equation a1+a2+..+a6=13. The result is 792.

**Problem 8:** How many positive integer less than 10000 satisfy the sum of digits is equal to 14 ?

Solution:

Let  denote the desired number under the condition

0a, b, c, d9. We have:

a+b+c+d=14.

Equivalent to problem 4, the number of positive integer solutions is 456.

**Problem 9:** Given 8 same balls and 12 different boxes balls. How many ways to classify 8 balls into 12 boxes satisfy the sum of balls in box 1,2,3 is even ?

Solution:

Let a1,…, a12 denote the number of balls in each box

Let a1+a2+a3=x => x is even

-) If x=0 => có a4+…+a12=8

The number of solutions (a4,…,a12) is .

The number of solutions (a1,a2,a3) is 1.

-) If x=2 => a4+…+a12=6

The number of solutions (a4,…,a12) is .

The number of solutions (a1,a2,a3) is .

-) If x=4 => a4+…+a12=4

The number of solutions (a4,…,a12) is .

The number of solutions (a1,a2,a3) is  .

-) If x=6 => a4+…+a12=2

The number of solutions (a4,…,a12) is .

The number of solutions (a1,a2,a3) is  .

-) If x=8 => a4+…+a12=0

The number of solutions (a4,…,a12) is 1.

The number of solutions (a1,a2,a3) is  .

The number of ways to classify balls into boxes is 24528 .

**References**

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