1. Goal

The project work requires to study the approach proposed by Michaud (resampling strategy).

1.1 Section Required

- literature overview on the topic
- provide some known models
    - solution methods with pros/cons.
- Provide a problem statement

1.2 Aim

The aim of this case is to discuss a few relevant problems caused by the exposure to estimation risk when investors deal with portfolio selection and illustrate the resampling technique as originally proposed by Michaud (1998).

Following the procedure in **Herold and Maurer (2002)**, the resampling approach is empirically tested and insample and out-of-sample performances are compared with plug-in approach results.

2. Literature Review

2.1 Plug-In Approach in MVO

Markowitz mean-variance theory (MV) provides a classic solution to the portfolio selection problem. The risk of a portfolio (as measured by the variance of its return) can be reduced by combining assets whose returns are imperfectly correlated.

Diversification, however, is not boundless. Implementation of this approach nonetheless requires knowledge of both the **expected returns** on all assets comprised in a portfolio and their **covariances**; an information set which by definition is not available.

A common way to circumvent this problem is thus to **use sample estimates of such measures in the optimization procedure** (the so-called **plug-in approach**). It follows that a prototypical investor is not only exposed to market risk, but also to **estimation risk**. The latter can, therefore, be defined as the loss of utility which arises from forming portfolios on the basis of sample estimates rather than true values.

Clearly, even if the true moments of the asset return distributions of a portfolio were known with certainty, MV optimized portfolios would not beat other portfolios in every future investment period, since return realizations usually differ from their expected values. However, over an appropriately large investment period, MV would provide on average the optimal portfolio composition.

2.2 Errors in Means and Covariance

1. Intuition

Inputs into the efficient frontier algorithm are measured with error, and the optimiser tends to pick those assets with very attractive features (high return and low risk and/or correlation) and tends to short or deselect those with the worst features.

These are exactly the cases where estimation error is likely to be highest, hence maximising the impact of estimation error on portfolio weights. If, for example, assets have high correlations, they appear similar to the quadratic programming algorithm. An algorithm that takes point estimates as inputs and treats them as if they were known with certainty (which they are not) will react to tiny differences in returns that are well within measurement error.

The problem gets worse as the number of assets rises because this increases the chance of outliers.

2. Poor OOS Forecast

$$W = \sum_{t=1}^{T}\left(\frac{\sigma_{rollingEstimation}}{\sigma_{Realized}}\right)^2 \tag{1}$$

3. Model Review

3.1 Classic MVO

3.1.1 Model

In modern finance, Markowitz's MV portfolio selection technique provides a solution to the problem of optimally allocating assets.

In each period, an investor chooses a portfolio by such that portfolio variance is minimized given a predetermined level expected return, which means, in each period, the investor trades off portfolio expected return with portfolio variance.

$$\begin{gathered} min_w : w\Sigma w^T \\ s.t : w^T r = \mu \\ w^T 1 = 1 \end{gathered} \tag{2}$$

3.1.2 Plug-In Method

The inputs to the classical portfolio selection model are the expected return vector and the variance–covariance matrix . However, since these parameters are not known with certainty, they need to be estimated from sample data, thereby exposing the asset allocation choice to an estimation risk. The optimization problem then becomes

$$
\begin{aligned}
min_w : w\hat{\Sigma}w^T \\
s.t : w^T\hat{r} = \mu \\
w^T 1 = 1
\end{aligned} \tag{3}
$$

We define **estimation risk** as the difference between the optimal solution to (1) and the optimal solution to (2).

3.1.3 Consequences of Estimation Risk

1. **Low degree of diversification**

MV portfolios often involve very extreme positions. In particular, as the number of assets grows, the weight on each single asset does not tend to zero.

2. Sudden shifts in the allocation of the optimal weights along the efficient frontier

The composition of the optimal portfolio is very different for individuals that differ slightly in their attitude towards risk.

3. **High sensitivity of portfolio weights to small variations in expected returns**

Since little changes in expected returns can completely alter the composition of MV optimal portfolios, while modifications in the variance–covariance matrix have smaller impact, it follows that errors in the sample estimates of expected return have great bearing on allocation choices.

Chopra and Ziemba (1993) find that errors in mean estimates are about ten times as important as errors in variance estimates. Errors in variance estimates are in turn about twice as important as errors in covariance estimates.

This leads to instability of the optimal portfolio in time.

3.2 Resampling Method

3.2.1 Michaud (1998)

Michaud's resampling procedure :

- Step 1: Collecting T-period historical returns on a set of N assets
- Step 2: Computing sample means $\hat{\mu}$ and the sample variance–covariance matrix $\hat{\Sigma}$
- Step 3: Finding optimal weights for a set of mean-variance efficient portfolios (Under Z different target return)
  - Target expected returns to solve problem (1) are fixed in the following way.
    - Target Return (z) = MVP Return + $z * \delta$
    - $\delta = (Max - MVP)/(Z - 1)$
    - z = 0:(Z-1)
- Step 4: Assuming that asset returns are from a multivariate normal distribution, with mean and variance–covariance matrix equal to the sample ones and estimated in Step 2.
- Step 5: Simulating B independent draws by resampling for each asset class from the multivariate normal distribution as defined in Step 4, with each draw consisting in T*N Samples

- Step 6: For each simulation b, re-estimating a new set of optimization inputs μ_b and Σ_b and finding Z efficient portfolios in Step 3
- Step 7: For each target return z, computing the average composition across the B simulations.
  - The z portfolios with this average composition are called resampled portfolios of rank z.
  - $w_z = \frac{\sum_{b=1}^{B} w_b^*}{B}$

Mean-variance efficient frontier and statistical equivalence region

- Step 1: Calculate the expected return and the standard deviation of each of the Z MVO portfolios using the sample mean and the sample covariance matrix obtained in Step 2. (Plug-In Method)
- Step 2: Calculate the expected return and the standard deviation of each of the (B × Z) resampled optimized portfolios.
  - Plot these (B × Z) combinations on the same graph as the true MV efficient frontier.

3.2.2 In-Sample Analysis

Results in G.Fusai

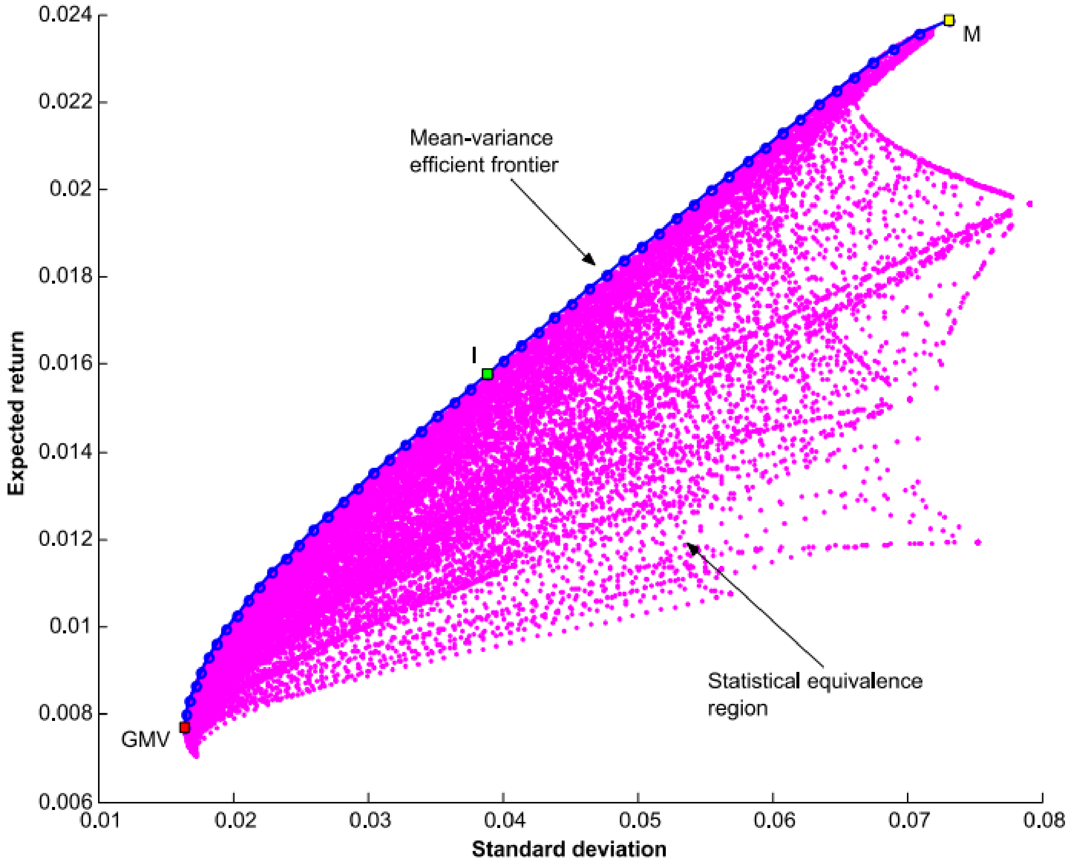


Fig. 9.1. Mean-variance efficient frontier and statistical equivalence region.

Findings in Fig 9.1

Clearly, **resampled portfolios will lie below the MV efficient frontier**, as they are sub-optimal with respect to optimized portfolios based on sample estimates of the mean and the variance–covariance matrix.

Fig. 9.1 "dramatically illustrates the enormous, even startling, variability implicit in efficient frontier portfolio estimation". The size of the SER suggests that much of the effort put in the optimization procedure can be worthless or even dangerous.

Dispersion increases moving from the lower to the upper part of the efficient frontier (i.e. moving from lower expected return portfolios to the higher expected return portfolios).

More illustration from Table 9.2

At the GMV portfolio level, differences in the portfolio weights between resampling and mean-variance are practically negligible.

Moving to the intermediate return level, differences in portfolio weights become remarkable. Resampling tends to preserve diversified portfolios, whilst mean-variance optimization tends to concentrate allocation.

Table 9.2. Distribution statistics

	Resampled weights	St. dev.	5% perc.	Median	95% perc.	MV weights
Global minimum variance portfolio						
Canada	3.96	2.15	0.36	4.02	7.44	5.13
US	0.79	1.57	0	0	4.21	0
Germany	1.17	1.44	0	0.60	4.36	1.11
Japan	0.49	0.74	0	0	2.04	0.17
France	0.06	0.26	0	0	0.4	0
UK	0.33	0.78	0	0	1.96	0
Euro Bonds	3.23	2.65	0	2.95	8.06	3.53
US Bonds	89.96	3.24	84.19	90.39	94.57	90.05
Intermediate (expected) return portfolio						
Canada	1.14	4.46	0	0	6.44	0
US	19.18	19.33	0	13.64	49.54	35.65
Germany	0.31	1.56	0	0	0.41	0
Japan	1.09	4.5	0	0	7.35	0
France	8.86	14.88	0	0	43.67	0
UK	4.71	12.03	0	0	39.78	0
Euro Bonds	22.92	20.33	0	23.04	50.63	36.37
US Bonds	41.78	19.55	17.09	40.81	64.62	27.99
Maximum expected return portfolio						
Canada	0.10	5.45	0	0	0	0
US	43.20	44.62	0	0	100	100
Germany	2.00	14.07	0	0	0	0
Japan	5.00	21.90	0	0	0	0
France	33.00	47.26	0	0	100	0
UK	11.00	31.45	0	0	100	0
Euro Bonds	4.90	4.26	0	0	0	0
US Bonds	0.80	3.41	0	0	0	0

More illustration from Figure 9.4

Figure 9.4 exhibits efficient portfolio weights generated both in mean-variance and resampling settings.

Lack of diversification and tendency towards extreme positions are evident. Two main observations can be drawn. First, mean-variance and resampled portfolios are quite similar for low levels of expected return, while they tend to diverge moving to higher expected return levels.

Second, instability of portfolio weights grows moving upward along the frontier. Note, for instance, the abnormal difference in weights between rank 50 and rank 51 in the mean-variance setting: at least 15% of portfolio composition is changed.

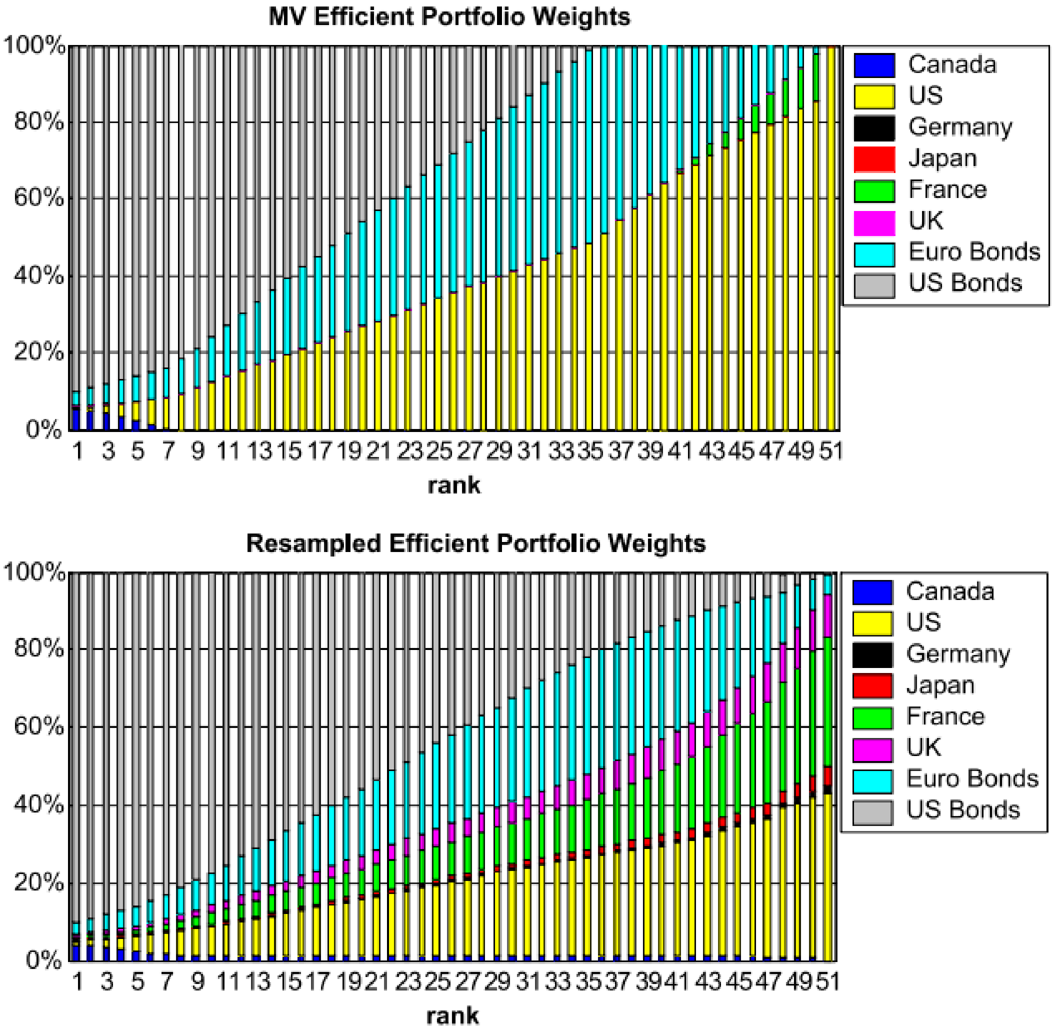


Fig. 9.4. Mean-variance and resampled portfolio weights.

3.2.3 Out-of-Sample Analysis

Implement a rolling out-of-sample analysis.

- Which asset allocation strategy performs better between the MVO and the resampled procedure?
- Design 1:
  - Rebalance MVO and Bootstrap every Month
    - Estimation Window K = 1:6
  - Given 5 Target Return (MVP:Max Return) Z=5
  - Calculate Performance Stats
- Robust Test:
  - FF-4 on Portfolio Return

Table 9.3. Sharpe ratios for different horizons

	Mean-variance			Resampling		
	Return	Risk	Sharpe ratio	Return	Risk	Sharpe ratio
$T = 30$						
GMV	0.5587	1.0828	0.5160	0.5443	1.0941	0.5249
I	0.4822	2.1829	0.2209	0.5628	1.9390	0.2903
M	0.1376	4.5968	0.0299	0.4688	3.7186	0.1261
$T = 60$						
GMV	0.5536	1.1574	0.4783	0.5597	1.1631	0.4812
I	0.6561	1.9438	0.3376	0.5682	1.9009	0.2989
M	0.7969	3.9009	0.2043	0.5352	3.4432	0.1554
$T = 90$						
GMV	0.5454	1.2255	0.4450	0.5470	1.2294	0.4449
I	0.5684	2.0567	0.2764	0.5144	1.9775	0.2601
M	0.7668	3.9053	0.1963	0.4294	3.5797	0.1200

Table 9.4. Portfolio turnovers

	Mean-variance turnover	Resampling turnover	Differential turnover
$T = 30$			
GMV	3.3453	3.0274	0.3179
I	13.2883	8.2368	5.0516
M	17.1717	13.3697	3.8020
$T = 60$			
GMV	1.5135	1.5050	0.0084
I	6.7243	5.3784	1.3459
M	16.1616	10.5455	5.6162
$T = 90$			
GMV	1.1279	1.1205	0.0074
I	4.9018	4.5062	0.3956
M	5.0505	9.1131	−4.0626

- Design 2:
  - Focus on marginal improvement Given different number of assets.
    - N = 50:500
    - For each N, randomly choose N stocks in the universe and conduct design 1
      - Calculate marginal improvement
      - Repeat K times and average the marginal improvement

3.2.2 Pitfalls of Portfolio Resampling (Scherer 2002 P129)

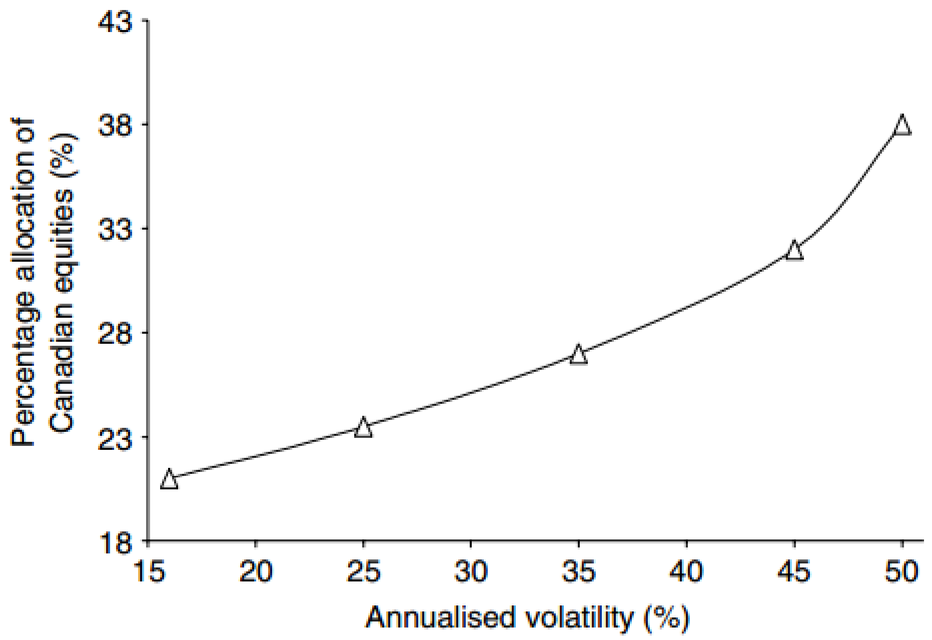
1. **Under Long-only Cases**

- Intuition: Consider two assets with the same expected return but the second one has significantly higher volatility.
  - How would resampled efficiency deal with high vol assets?
    - Repeatedly drawing from the original distribution will result in draws for the volatile asset with highly negative, as well as highly positive, returns.
    - Quadratic programming will invest heavily in this asset in the latter case and short the asset in the former case. However, as shorting is not allowed for portfolios

    with long-only constraints, this will result in positive allocation for high positive average return draws and zero allocations for high negative average return draws.
    - This is different from an unconstrained optimization, where large long positions would be offset (on average) by large negative positions.
    - Consequently, an increase in volatility will lead to an increase in the average allocation. Hence, a worsening Sharpe ratio would be accompanied by an increase in weight.
    - Again, this is not a plausible result; it arises directly from the averaging rule in combination with a long-only constraint, which creates an optionality for the allocation of the corresponding asset.

This is not a result of higher volatility leading to higher estimation error as this phenomenon does not arise in long–short portfolios – but is a direct result of averaging over long-only portfolios as the long-only constraint creates “optionality”.

Figure 4.12 Effect of volatility on allocation of Canadian equities



- **Design 3**:
    - Keep 49 same. For the left one, increase the estimated volatility

2. **Bootstrap from Sample Estimation**

The most important criticism of resampled efficiency arises from its statistical foundation, as all resamplings are derived from the same vector and covariance matrix .

Because the true distribution is unknown, all resampled portfolios suffer from the deviation of the parameters in very much the same way.

It is fair to say that all portfolios inherit the same estimation error.