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The Sampling Error in Estimates of Mean-Variance Efficient Portfolio Weights

MARK BRITTEN-JONES*

ABSTRACT

This paper presents an exact finite-sample statistical procedure for testing hypotheses about the weights of mean-variance efficient portfolios. The estimation and inference procedures on efficient portfolio weights are performed in the same way as for the coefficients in an OLS regression. OLS t- and F-statistics can be used for tests on efficient weights, and when returns are multivariate normal, these statistics have exact t and F distributions in a finite sample. Using 20 years of data on 11 country stock indexes, we find that the sampling error in estimates of the weights of a global efficient portfolio is large.

MEAN-VARIANCE ANALYSIS IS IMPORTANT for both practitioners and researchers in finance. For practitioners, theory suggests that mean-variance efficient[1] portfolios can play an important role in portfolio management applications. For researchers in finance, mean-variance analysis is central to many asset pricing theories as well as to empirical tests of those theories; however, practitioners have reported difficulties in implementing mean-variance analysis.[2] For example, Black and Litterman (1992) note that, "when investors have tried to use quantitative models to help optimize the critical allocation decision, the unreasonable nature of the results has often thwarted their efforts" (p. 28).

The main difficulty concerns the extreme weights that often arise when sample efficient portfolios are constructed. This phenomenon is studied by Best and Grauer (1991), who show that sample efficient portfolios are extremely sensitive to changes in asset means. The sensitive nature of portfolio weights implies that sampling error in estimates of asset means and covariances feeds through to the estimates of efficient portfolio weights. This has prompted a variety of 'fixes' in order to reduce the influence of sampling error, ranging from direct and extensive restrictions on portfolio weights (see, e.g., Haugen (1997)) to Bayesian shrinkage methods (see Jorion (1985), Frost and Savarino (1986, 1988), and Black and Litterman (1992)).

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[1] Hereafter, the term *efficient* refers to mean-variance efficient.

[2] Academics have also reported such difficulties. See Kroll, Levy, and Markowitz (1984), Pulley (1981), and, in a slightly different context involving conditional moments, Gallant, Hansen, and Tauchen (1990).

Despite the important (and possibly damaging) role of sampling error in the construction of sample efficient portfolios, little formal analysis of sampling error in the context of efficient portfolio weights has been published. Jobson and Korkie (1980) derive an asymptotic distribution for estimates of efficient weights; however, Ledoit (1995) and Gibbons, Ross, and Shanken (1989) show that asymptotic results can be misleading when the number of assets is not small. Ledoit (1995) develops a general asymptotic theory that bypasses some problems with existing asymptotic theory when the number of assets is large, but his focus is the covariance matrix of returns, not portfolio weights. A variety of Bayesian approaches have been developed for purposes of testing the efficiency of a given portfolio, but these works have not focused on the sampling error in estimates of efficient portfolio weights.[3]

For researchers in finance, mean-variance analysis can play an important role in tests of asset pricing theories. For example, tests of the CAPM have traditionally focused on the expected return-beta relation. But recent research by Roll and Ross (1994) and Kandel and Stambaugh (1995) shows that the linear relation between expected return and beta is not robust to slight inefficiency in the market portfolio. These studies suggest that CAPM tests should be based on the CAPM's central prediction—the mean-variance efficiency of the market portfolio.[4] Fama (1996) and Grinblatt and Titman (1983) show that multifactor asset pricing models can also be tested by examining the mean-variance efficiency of linear combinations of certain key portfolios.

Traditionally, researchers have thought about portfolio efficiency in terms of returns. Such an approach is complicated by the fact that the location (in mean-variance space) of both the test portfolio and the efficient frontier are subject to sampling error.[5] For some purposes, a simpler way of thinking about portfolio efficiency is in terms of portfolio weights. We present procedures and test statistics enabling formal statistical inference on efficient portfolio weights. We show that the Gibbons, Ross, and Shanken (1989) (GRS) F-test of portfolio efficiency can be thought of (and derived) as a test of the restriction that the weights of the 'tangency' portfolio equal the weights of the test portfolio. Rejection of this restriction implies rejection of the efficiency of the test portfolio.

The tests we present can be implemented in a simple and intuitive manner that is formally identical to the standard inference procedures used for OLS regression coefficients. One result is a simpler implementation method for the GRS F-test. For practitioners this procedure could be used to assess

[3] See Kandel, McCulloch, and Stambaugh (1995), McCulloch and Rossi (1990, 1991), Harvey and Zhou (1990), and Shanken (1987).

[4] These studies also support the use of a generalized least squares (GLS) regression to analyze the linear relation between expected return and beta. However, the goodness-of-fit of this regression is related in a simple manner to the degree of mean-variance efficiency of the market portfolio. In other words, the GLS approach boils down to testing the efficiency of the market portfolio.

[5] See Roll (1985) and Gibbons, Ross, and Shanken (1989).

the precision of the weights of a sample efficient portfolio. For researchers in finance, this procedure provides a simple and flexible way of testing a given portfolio's efficiency, and of testing hypotheses concerning the weights of efficient portfolios.

In Section I we present the 'regression' procedure for inference on portfolio weights. In Section II we demonstrate the procedure by using international stock return data to analyze the weights of an international efficient portfolio. Section III concludes. Technical proofs are contained in the Appendix.

I. The Regression Approach to Portfolio Analysis

In this paper a riskless asset is assumed available for both borrowing and lending in each period. *Excess returns* are calculated by subtracting the return of this riskless asset from the total return.[6] There are K risky assets indexed by $k = 1,\ldots,K$. The excess returns on the K assets in some period t in $(1,\ldots,T)$ are denoted by the K elements of the vector $\mathbf{x}_t$:

$$\mathbf{x}_t' = [x_{1t},\ldots x_{kt},\ldots,x_{Kt}]. \tag{1}$$

The T observations of excess returns are contained in the $T \times K$ matrix $\mathbf{X}$:

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1' \\ \vdots \\ \mathbf{x}_T' \end{bmatrix}. \tag{2}$$

Note that a portfolio of risky assets and a riskless asset has an excess return that is determined solely by the weights and excess returns of the risky assets. Thus, given a K-vector of risky asset weights $\mathbf{b}$, the excess return of this portfolio in period t is simply $\mathbf{x}_t'\mathbf{b}$, where the weights in $\mathbf{b}$ need not sum to one.

Let $\mathbf{l}$ represent a vector of ones with length conforming to the rules of matrix algebra. Viewed as a portfolio excess return, the T-vector of ones $\mathbf{l}$ is highly desirable as it has positive excess return with zero sample standard deviation. The regression approach to portfolio selection[7] is based on minimizing the squared deviations between the excess returns on a constructed portfolio and the excess returns in $\mathbf{l}$. This minimization problem can be per-

[6] Hereafter, the term *return* refers to excess return unless otherwise stated.

[7] Cochrane (1997), building on the work of Hansen and Richards (1987), shows that the identification of stochastic discount factors can be analyzed in terms of projections onto asset payoffs without an intercept term. Cochrane's results are in terms of population moments, whereas the focus here is on statistical inference when sample moments are used. Future research could attempt to analyze and identify stochastic discount factors using the statistical inference procedures developed here.

formed using an artificial ordinary least squares (OLS) regression and the following proposition states that such a regression recovers the weights of a sample efficient portfolio.

THEOREM 1: *OLS regression of a constant* $\mathbf{1}$ *onto a set of asset's excess returns* $\mathbf{X}$, *without an intercept term,*

$$\underset{(T\times 1)}{\mathbf{1}} = \underset{(T\times k)\,(k\times 1)}{\mathbf{X}\mathbf{b}} + \underset{(T\times 1)}{\mathbf{u}}, \tag{3}$$

results in an estimated coefficient vector

$$\hat{\mathbf{b}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{1}, \tag{4}$$

that is a set of risky-asset-only portfolio weights for a sample efficient portfolio. The scaled (so that weights sum to one) coefficient vector $\hat{\mathbf{b}}/\mathbf{1}'\hat{\mathbf{b}}$ *is thus the familiar tangency portfolio*

$$\frac{\bar{\mathbf{\Sigma}}^{-1}\bar{\mathbf{x}}}{\mathbf{1}'\bar{\mathbf{\Sigma}}^{-1}\bar{\mathbf{x}}}, \tag{5}$$

derived from quadratic programming, where the sample mean $\bar{\mathbf{x}} = \mathbf{X}'\mathbf{1}/T$, *and the (maximum likelihood) sample covariance* $\bar{\mathbf{\Sigma}} = (\mathbf{X} - \mathbf{1}\bar{\mathbf{x}}')'(\mathbf{X} - \mathbf{1}\bar{\mathbf{x}}')/T$, *are used as parameters.*

Proof: Using the *updating formula* for an inverse matrix,[8] express the coefficient vector $\hat{\mathbf{b}}$ from the regression in equation (3) in terms of the sample mean $\bar{\mathbf{x}}$ and sample covariance $\bar{\mathbf{\Sigma}}$:

$$\begin{aligned}\hat{\mathbf{b}} &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{1} \\ &= (\bar{\mathbf{\Sigma}} + \bar{\mathbf{x}}\bar{\mathbf{x}}')^{-1}\bar{\mathbf{x}} \\ &= \left(\bar{\mathbf{\Sigma}}^{-1} - \frac{\bar{\mathbf{\Sigma}}^{-1}\bar{\mathbf{x}}\bar{\mathbf{x}}'\bar{\mathbf{\Sigma}}^{-1}}{1 + \bar{\mathbf{x}}'\bar{\mathbf{\Sigma}}^{-1}\bar{\mathbf{x}}}\right)\bar{\mathbf{x}} \\ &= \frac{\bar{\mathbf{\Sigma}}^{-1}\bar{\mathbf{x}}}{1 + \bar{\mathbf{x}}'\bar{\mathbf{\Sigma}}^{-1}\bar{\mathbf{x}}}.\end{aligned} \tag{6}$$

Scaling $\hat{\mathbf{b}}$ so that the coefficients sum to one results in the tangency portfolio

$$\frac{\hat{\mathbf{b}}}{\mathbf{1}'\hat{\mathbf{b}}} = \frac{\bar{\mathbf{\Sigma}}^{-1}\bar{\mathbf{x}}}{\mathbf{1}'\bar{\mathbf{\Sigma}}^{-1}\bar{\mathbf{x}}} \tag{7}$$

when sample means and covariances are used as parameters. Q.E.D.

[8] See Greene (1993), p. 25.

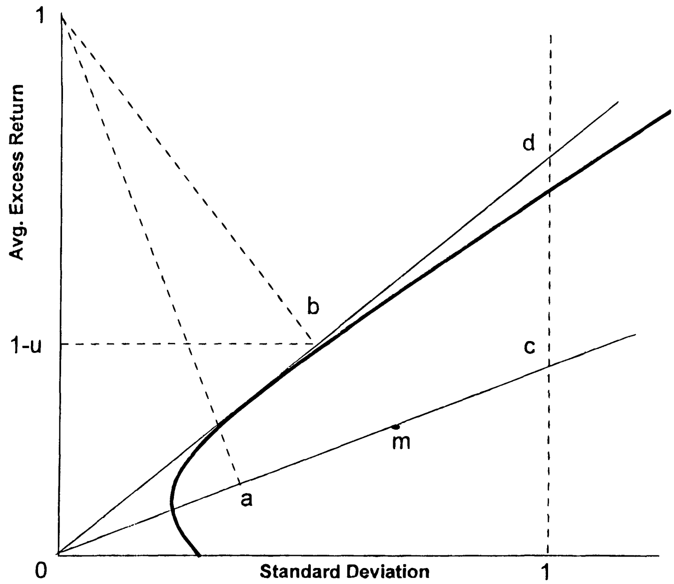

Figure 1. Sample mean standard deviation diagram. The point b is the point on the line 0d that is closest to the point (0,1) and the point a is the point on the line 0m that is closest to (0,1).

The regression in equation (3) is unusual. There is no intercept, the dependent variable is nonstochastic, and the residual vector $\mathbf{u}$ is correlated with the regressors, which are stochastic. However, the regression has a simple interpretation: The dependent variable $\mathbf{l}$ is a sample counterpart to arbitrage profits—positive excess return with zero standard deviation; the coefficients $\mathbf{b}$ represent the weights on risky assets in the portfolio; $\mathbf{X}b$ represents excess returns on this portfolio; and the residual vector $\mathbf{u}$ shows deviations in this portfolio's return from $\mathbf{1}$.

The estimated portfolio weights $\hat{\mathbf{b}}$ produce a portfolio return vector that is closest in terms of least squares distance to the arbitrage return vector $\mathbf{l}$. This least squares distance can be illustrated using the familiar mean-standard deviation diagram. The feasible set, constructed from the sample mean and (maximum-likelihood) sample covariance, has an efficient boundary shown by the line 0d from the origin passing through the tangency portfolio (Figure 1). The arbitrage return vector $\mathbf{l}$ is located at the point (0,1).

The residuals $\mathbf{u}$ can be split into a mean component[9] $\bar{u}\mathbf{l}$ and a deviation component $\mathbf{e}$:

$$\mathbf{u} = \bar{u}\mathbf{l} + \mathbf{e}, \tag{8}$$

[9] The lack of an intercept in the regression implies residuals need not sum to zero.

where $\bar{u} = (1/T)\mathbf{u}'\mathbf{1}$. Similarly, the mean squared residual (MSR) can be split into a squared mean and a squared standard deviation (SD):

$$MSR = \text{Mean}^2 + \text{SD}^2$$

$$\mathbf{u}'\mathbf{u}/T = \bar{u}^2 + \mathbf{e}'\mathbf{e}/T. \tag{9}$$

The residual's mean $\bar{u}$ equals the difference between the constructed portfolio's average return and 1; thus the constructed portfolio's mean return equals $1 - \bar{u}$:

$$\bar{\mathbf{x}}'\hat{\mathbf{b}} = 1 - \bar{u}, \tag{10}$$

and this is measured on the vertical axis of Figure 1.

Since the residual and constructed portfolio excess returns sum to one, the residuals' standard deviation, SD, equals the constructed portfolio's standard deviation, and this is measured on the horizontal axis. By Pythagoras' theorem the root mean square residual equals the distance between the portfolio's location in mean–standard deviation space and the arbitrage portfolio's location (0,1). OLS thus finds a portfolio whose returns are *located as closely as possible in mean–standard deviation space to the point* (0,1).

Such a portfolio is shown in Figure 1 by the point b. Note that this portfolio is sample efficient.

The regression framework highlights the stochastic nature of optimal weights calculated from sample data. Our next theorem states that the standard OLS F-statistic associated with a particular linear restriction applied to the regression in (3) implements the GRS F-test.

THEOREM 2: *The GRS F-test for the efficiency of a given portfolio with weights* $\mathbf{g}$ *can be implemented by the standard OLS F-statistic associated with the linear restriction that portfolio weights in regression (3) are proportional to the weights of the given portfolio. Under multivariate normality the F(OLS)-statistic,*

$$F(\text{OLS}) = \frac{(\text{SSR}_r - \text{SSR}_u)/(K-1)}{\text{SSR}_u/(T-K)}, \tag{11}$$

is distributed in finite sample as central $F_{K-1,T-K}$ *where* SSR_u *is the sum of squared residuals from the unrestricted regression in (3),* SSR_r *is the sum of squared residuals from the regression in equation (3) estimated subject to the restriction that* $\mathbf{b} \propto \mathbf{g}$ *(*$\mathbf{g}$ *is a vector containing the weights of the given portfolio), and T is the number of observations.*

Proof: This theorem can be proved using Theorem 3 and its proof, but a diagrammatic proof is simpler and provides greater economic intuition. In the context of the regression in equation (3), consider the $K - 1$ restrictions that the weights **b** be proportional to the weights of the given portfolio **g**. The proportionality restriction can be written as $\mathbf{b} = a \times \mathbf{g}$, where a is a free scalar. The restricted regression can thus be written as a regression with a single independent variable, the given portfolio's excess returns $\mathbf{Xg}$, and a single regression coefficient a:

$$\mathbf{1} = (\mathbf{Xg})a + \mathbf{u}^*. \tag{12}$$

This restriction is shown in Figure 1 by the line Om from the origin through the given portfolio's position shown as m.

The root mean squared residual associated with the restricted and unrestricted regressions is given by the minimum distances between (0,1) and the lines Om and Od. The F(OLS)-statistic can thus be written

$$F(\text{OLS}) = \left(\left(\frac{\overline{1a}}{\overline{1b}}\right)^2 - 1\right)\left(\frac{T-K}{K-1}\right), \tag{13}$$

where $\overline{1a}$ and $\overline{1b}$ are the distances shown in the diagram between 1 and a and between 1 and b, respectively. Now GRS note that their test statistic, which they denote W, is given by

$$W = \left[\frac{\overline{Od}}{\overline{Oc}}\right]^2 - 1, \tag{14}$$

where $\overline{Od}$ and $\overline{Oc}$ are the distances shown in Figure 1. The point d shows the average return attainable from taking a position on the efficient frontier with a standard deviation of one and c shows the expected return attainable from taking a position in the given portfolio with standard deviation of one. Note that[10]

$$\frac{\overline{Od}}{\overline{Oc}} = \frac{\overline{1a}}{\overline{1b}}. \tag{15}$$

Thus the F(OLS)-statistic is proportional to GRS' W-statistic:

$$F(\text{OLS}) = W\left(\frac{T-k}{k-1}\right). \tag{16}$$

Under the assumption of multivariate normality, GRS show that $W(T-K)/(K-1)$ is distributed in a finite sample as a central F distribution with $K-1$

[10] To see this write $\overline{Od}/\overline{Oc}$ as $\cos\widehat{cO1}/\cos\widehat{dO1}$, where the angles are measured to the horizontal axis. Similarly, $\overline{1a}/\overline{1b}$ can be expressed as $\cos\widehat{O1a}/\cos\widehat{O1b}$, where the angles are measured from the vertical axis. Since the angles $\widehat{cO1}$ and $\widehat{O1a}$ are equal, and the angles $\widehat{dO1}$ and $\widehat{O1b}$ are also equal, the equality holds.

and $T - K$ degrees of freedom. It therefore follows that, with multivariate normality, the F(OLS)-statistic in finite sample has an exact central F distribution with $K - 1$ and $T - K$ degrees of freedom:

$$F(\mathrm{OLS}) \sim F_{k-1,T-k}. \tag{17}$$

Q.E.D.

It is worth pausing to examine the different interpretations of the GRS F-test. The first interpretation (which underlies the derivation and proof contained in GRS) is as a joint test of zero intercepts in a multivariate regression. The second interpretation is as a ratio formed from the squared Sharpe ratios of the sample efficient portfolio and of the test portfolio. The interpretation we offer is as a test of the restriction that the weights of the tangency portfolio are proportional to the weights of the test portfolio. This test can then be implemented using a standard OLS F-test for testing a set of linear restrictions on regression coefficients. The next section shows that we can extend this regression analogy further to examining other linear restrictions.

A. Testing the Efficiency of a Linear Combination of Assets

Jobson and Korkie (1989) and GRS show, in the context of testing the efficiency of a subset of assets relative to a full set of assets, that an analogous test statistic to the GRS F-test, based on the ratio of maximal Sharpe ratios from the subset and the complete set of assets, has an exact central F distribution under the null hypothesis of efficiency of the subset. This is an extension of the GRS F-test, and the following proposition extends this result to any hypothesis that can be expressed as a linear restriction on efficient portfolio weights.

THEOREM 3: *Denoting a set of efficient (with respect to population moments) portfolio weights by* $\mathbf{b}$, *we can test the linear restriction*

$$\mathbf{Rb} = 0, \tag{18}$$

where $\mathbf{R}$ *is a* $(q \times K)$ *matrix of restrictions, by the standard OLS F-statistic associated with the restriction* $\mathbf{Rb} = 0$ *applied to the regression in equation (3). Thus,*

$$\frac{(\mathrm{SSR}_r - \mathrm{SSR}_u)/q}{\mathrm{SSR}_u/(T-K)}, \tag{19}$$

has an exact central F distribution with q and $T - K$ *degrees of freedom.* SSR_u *is the sum of squared residuals from the unrestricted regression (3),* SSR_r *is the sum of squared residuals from estimation of regression (3)* *subject to the restriction* $\mathbf{Rb} = \mathbf{0}$, *q is the number of linear restrictions (the number of rows of* $\mathbf{R}$*), K is the number of assets, and T is the number of observations.*

Proof: In the Appendix. Q.E.D.

B. Exact Statistical Inference on Efficient Portfolio Weights

Theorem 3 enables simple derivations of formal inference procedures for efficient portfolio weights. The following corollary shows how a standard t-test can be used to test whether an asset's weight in an ex ante efficient portfolio is different from zero.

COROLLARY 1: *Denoting a set of ex ante efficient portfolio weights (for the case of a riskless asset) by* $\mathbf{b}$, *we can test whether the ex ante efficient weight b_i on a particular asset i is zero by testing the restriction*

$$b_i = 0, \tag{20}$$

using the standard OLS t-statistic associated with this restriction in the regression in equation (3). Thus,

$$\frac{\hat{b}_i}{s\sqrt{a_{ii}}} \tag{21}$$

has an exact t distribution with $T - K$ degrees of freedom, where s^2 is the usual estimate of residual variance:

$$s^2 = SSR_u/(T - K), \tag{22}$$

and a_{ii} is the i^{th} element on the principal diagonal of $(\mathbf{X}'\mathbf{X})^{-1}$.

Proof: In the Appendix. Q.E.D.

Theorem 3 provides a very simple way of conducting formal exact statistical inference on the weights of the efficient portfolio. The usual t- and F-statistics associated with OLS regression can be used in the context of regression equation (3), and with multivariate normality, these statistics have exact t and F distributions in finite sample.

The only pitfall concerns the nature of testable hypotheses. The single-regression procedure identifies sample efficient portfolio weights only up to a scaling factor (reflecting the fact that the set of efficient portfolios is a straight line in mean–standard deviation space). So restrictions need to be imposed in the form of proportionality constraints. For example, in order to test the hypothesis that the tangency portfolio's weight on the first asset is equal to a number, say 0.1, one cannot simply test the restriction

$$b_1 = 0.1. \tag{23}$$

A meaningful restriction is that the *scaled* weight is equal to the number; that is,

$$\frac{b_1}{\mathbf{l}'\mathbf{b}} = 0.1, \tag{24}$$

and this restriction is implemented by

$$b_1 - 0.1 \times \mathbf{l}'\mathbf{b} = 0. \tag{25}$$

This restriction is still linear and can therefore be implemented using Theorem 3. Consider the proportionality restriction in Theorem 2: $\mathbf{b} \propto \mathbf{g}$ where $\mathbf{g}' = (g_1, \ldots, g_K)$ is the vector of weights on the given portfolio. Since $\mathbf{g}'\mathbf{1} = 1$, this restriction can be expressed as

$$\frac{b_1}{\mathbf{l}'\mathbf{b}} = g_1 \tag{26}$$

$$\frac{b_2}{\mathbf{l}'\mathbf{b}} = g_2 \tag{27}$$

$$\vdots$$

$$\frac{b_{k-1}}{\mathbf{l}'\mathbf{b}} = g_{K-1}, \tag{28}$$

which is written in standard matrix notation as $\mathbf{Rb} = \mathbf{0}$, where the $K-1$ by K restriction matrix $\mathbf{R}$ is

$$\mathbf{R} = \begin{bmatrix} 1-g_1 & -g_1 & -g_1 & \cdots & -g_1 \\ -g_2 & 1-g_2 & -g_2 & \cdots & -g_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -g_{K-1} & -g_{K-1} & \cdots & 1-g_{K-1} & -g_{K-1} \end{bmatrix}. \tag{29}$$

II. An International Mean-Variance Efficient Portfolio

The composition of an international efficient portfolio is of interest, not only from the practical viewpoint of investors seeking to optimize their risk-return tradeoff, but also from a theoretical viewpoint. International asset pricing theories can have implications for the composition of a globally efficient portfolio,[11] and the composition of such a portfolio can alert us to the presence and importance of barriers to international investment.

[11] A useful survey of the portfolio implications of various international asset pricing theories is Stulz (1995).

A number of papers have constructed ex post global efficient portfolios,[12] but these studies have been unable to formally assess the statistical reliability of their results. The problem of statistical inference for this type of analysis is well summarized by Adler and Dumas (1983), p. 945, who write (concerning estimation of the weights of an international efficient portfolio they call the log-portfolio):

> ". . . the estimates are plagued by major statistical problems which undermine their significance. No statistical theory, to our knowledge, gives the sample distribution of the estimated $w_{\log}$ [weights of international efficient portfolio]. . . . We are, therefore, unable to build confidence intervals for the optimal log-portfolio composition. . . ."

In this section, we use the inference procedures developed in the first section to examine the magnitude and effect of sampling error in estimates of the composition of an efficient international equity portfolio. Table I shows the weights of an ex post international tangency portfolio for a U.S. investor. We use monthly data from Morgan Stanley Capital International (MSCI) for the 20-year period from January 1977 to December 1996, for the equity markets of 11 developed countries.[13] Weights for the full 20-year period are shown as well as weights for two 10-year subperiods. The estimates contain several extreme positions such as a short position of 45 percent of portfolio value in the Canadian market, and a short position of 18 percent in the German market. Note that many weights change dramatically between the two subperiods. For example, in the first subperiod the optimal weight in Denmark is a short position of 29.6 percent, but in the second subperiod the optimal weight in Denmark is a long position of 68.8 percent.

Using the statistical inference procedures developed in the first part of this paper, we construct t-statistics for testing the hypothesis that a country's weight is zero. As pointed out earlier, hypotheses must be expressed in terms of a proportional relation, and the zero restriction is very simply implemented. The standard errors we show are derived imposing the null hypothesis. Thus they are constructed by dividing the estimated weight (scaled to sum to one) by the associated t-statistic. This provides a useful indication of the sampling error, but a formal confidence interval cannot be constructed for one weight alone due to the required proportional nature of testable hypotheses.

Over the full 20-year period, the standard errors of the estimates are large. The smallest standard error is for Japan at slightly more than 20 percent of portfolio value. The largest standard error is for the United States, which at 47.0 percent is almost half of total portfolio value. Not surprisingly, the standard errors are generally larger in the two 10-year subperiods. The result of such statistical imprecision is that none of the zero-weight restrictions can be rejected at the standard significance level of 0.05.

[12] See, for example, Adler and Dumas (1983) and Solnik (1982).

[13] We do not examine the issue of currency hedging, but assume all foreign positions are unhedged with regard to currency exposure.

Table I

Estimates of a Global Tangency Portfolio

This table contains estimates of the weights of a global tangency portfolio, from the viewpoint of a U.S. investor. The monthly country returns are the Morgan Stanley Capital International (MSCI) country stock index returns converted into U.S. dollar returns using the MSCI exchange rate series. Excess returns are calculated by subtracting the one-month T-bill return obtained from Ibbotson Associates. Weights are in percentage form (i.e., they sum to 100). The t-statistics test the hypothesis that the country weight is zero. Standard errors (SE) are calculated under the null hypothesis of a zero weight, as the estimated weight divided by the t-statistic.

	1977–1996		1977–1986		1987–1996	
	Weight (t-statistic)	SE	Weight (t-statistic)	SE	Weight (t-statistic)	SE
Australia	12.8	23.6	6.8	33.6	21.6	32.6
	(0.54)		(0.20)		(0.66)	
Austria	3.0	25.5	−9.7	44.1	22.5	30.3
	(0.12)		(−0.22)		(0.74)	
Belgium	29.0	35.1	7.1	46.8	66.0	54.5
	(0.83)		(0.15)		(1.21)	
Canada	−45.2	38.9	−32.7	51.0	−68.9	62.7
	(−1.16)		(−0.64)		(−1.10)	
Denmark	14.2	30.2	−29.6	45.3	68.8	38.7
	(0.47)		(−0.65)		(1.78)	
France	1.2	28.7	−0.7	37.3	−22.8	47.3
	(0.04)		(−0.02)		(−0.48)	
Germany	−18.2	35.4	9.4	49.4	−58.6	52.1
	(−0.51)		(0.19)		(−1.13)	
Italy	5.9	20.2	22.2	27.9	−15.3	29.5
	(0.29)		(0.79)		(−0.52)	
Japan	5.6	23.4	57.7	40.4	−24.5	28.1
	(0.24)		(1.43)		(−0.87)	
U.K.	32.5	32.1	42.5	42.9	3.5	49.8
	(1.01)		(0.99)		(0.07)	
U.S.	59.3	47.0	27.0	65.2	107.9	70.6
	(1.26)		(0.41)		(1.53)	

In order to examine the question of the benefits of global diversification for a U.S. investor, we construct F-statistics for the restriction that weights on all foreign (i.e., non-U.S.) countries' equity markets are zero. For the full 20-year period the F-statistic is 0.59 with an associated p-value of 0.82. For the 1977 to 1986 subperiod the F-statistic is 0.81 with a p-value of 0.62, and for the 1987 to 1996 subperiod the F-statistic is 0.69 with a p-value of 0.73. Based on the data, we are unable to reject the hypothesis that the tangency portfolio for a U.S. investor has no exposure to foreign equity markets.

These results provide no *statistical* support for the proposition that there are benefits to global diversification for a U.S. investor. This is in line with a recent study by Sinquefield (1996). However the magnitude of sampling error tells us that the data (by themselves) actually provide little information for international portfolio construction. A possible remedy is to allow prior information, perhaps derived from theory, to influence the estimates of an efficient portfolio. Such an approach is pursued, within a Bayesian framework, by Black and Litterman (1992) who combine an international CAPM with historical data to generate portfolio weights.

III. Conclusions and Further Research

In summary, the regression approach to portfolio analysis provides a new and simple tool for the empirical analysis of mean-variance problems. Using this approach, we derive exact formal inference procedures for hypotheses about the weights of efficient portfolios. Furthermore, we show how to implement the GRS F-test for portfolio efficiency using linear restrictions on a single linear OLS regression. Finally, we use these inference procedures to show the importance and magnitude of sampling error in estimates of the weights of an international mean-variance efficient portfolio.

Further research could attempt to apply some of the regression estimation techniques and inference procedures that are designed to deal with nonnormal and nonindependent data. Another line of research could seek to extend the approach developed here to the analysis of conditional mean-variance efficiency[14] by expanding the asset universe to include 'managed' portfolios (excess returns scaled by instrumental variables).

Appendix

Proof of Theorem 3: The sum of squared residuals (SSR) from the unrestricted regression

$$\mathbf{l} = \mathbf{X}\mathbf{b} + \mathbf{u} \tag{A1}$$

is given by

$$\text{SSR} = \mathbf{l}'\mathbf{M}\mathbf{l}, \tag{A2}$$

where

$$\mathbf{M} = \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'. \tag{A3}$$

The restriction $\mathbf{R}\mathbf{b} = 0$ can be written as

$$\mathbf{R}_1\mathbf{b}_1 + \mathbf{R}_2\mathbf{b}_2 = 0, \tag{A4}$$

[14] See Cochrane (1996) and Hansen and Richard (1987).

where $\mathbf{R} = (\mathbf{R}_1, \mathbf{R}_2), \mathbf{b}' = (\mathbf{b}_1', \mathbf{b}_2')$, $\mathbf{R}_1$ is a $(q \times q)$ nonsingular matrix, and $\mathbf{R}_2$ is a $q \times (k - q)$ matrix. Solving for $\mathbf{b}_1$ gives

$$\mathbf{b}_1 = -\mathbf{R}_1^{-1}\mathbf{R}_2\mathbf{b}_2. \tag{A5}$$

The restricted regression

$$\mathbf{l} = \mathbf{X}\mathbf{b} + \mathbf{u}, \tag{A6}$$

$$\mathbf{R}\mathbf{b} = 0, \tag{A7}$$

can be rewritten with the restriction directly imposed as

$$\begin{aligned} \mathbf{l} &= \mathbf{X}_1\mathbf{b}_1 + \mathbf{X}_2\mathbf{b}_2 + \mathbf{u} \\ &= -\mathbf{X}_1\mathbf{R}_1^{-1}\mathbf{R}_2\mathbf{b}_2 + \mathbf{X}_2\mathbf{b}_2 + \mathbf{u} \\ &= (\mathbf{X}_2 - \mathbf{X}_1\mathbf{R}_1^{-1}\mathbf{R}_2)\mathbf{b}_2 + \mathbf{u}. \end{aligned} \tag{A8}$$

If we define $\mathbf{X}_*$ as $\mathbf{X}_2 - \mathbf{X}_1\mathbf{R}_1^{-1}\mathbf{R}_2$ we can write the restricted regression more compactly as

$$\mathbf{l} = \mathbf{X}_*\mathbf{b}_2 + \mathbf{u}. \tag{A9}$$

The sum of squared residuals from the restricted regression is given by

$$\mathrm{SSR}_r = \mathbf{l}'\mathbf{M}_*\mathbf{l}, \tag{A10}$$

where

$$\mathbf{M}_* = \mathbf{I} - \mathbf{X}_*(\mathbf{X}_*'\mathbf{X}_*)^{-1}\mathbf{X}_*'. \tag{A11}$$

Now $\mathbf{X}_*$ is the set of excess returns on a set of hypothetical basis assets which span the space of portfolio returns generated by the original assets subject to the restrictions on portfolio weights. The restriction $\mathbf{R}\mathbf{b} = 0$ implies that the maximal ex ante Sharpe ratio from the full set of assets is equal to the maximal ex ante Sharpe ratio achievable from the restricted set. Now the ex post Sharpe ratio for the complete set of assets is $(\bar{\mathbf{x}}'\bar{\boldsymbol{\Sigma}}^{-1}\bar{\mathbf{x}})^{1/2}$. Similarly, for the restricted case the maximal ex post Sharpe ratio is given by $(\bar{\mathbf{x}}_*'\bar{\boldsymbol{\Sigma}}_*^{-1}\bar{\mathbf{x}}_*)^{1/2}$, where $\bar{\mathbf{x}}_* = \mathbf{X}_*'\mathbf{l}/T$ and $\bar{\boldsymbol{\Sigma}}_* = (\mathbf{X}_* - \mathbf{l}\bar{\mathbf{x}}_*')'(\mathbf{X}_* - \mathbf{l}\bar{\mathbf{x}}_*')/T$. Some matrix algebra shows that the sum of squared residuals from the unrestricted regression is

$$\mathrm{SSR} = \mathbf{l}'\mathbf{M}\mathbf{l} = \frac{T}{1 + \bar{\mathbf{x}}'\bar{\boldsymbol{\Sigma}}^{-1}\bar{\mathbf{x}}}, \tag{A12}$$

and the sum of squared residuals from the restricted regression is

$$\mathrm{SSR}_r = \mathbf{l}'\mathbf{M}_*\mathbf{l} = \frac{T}{1 + \bar{\mathbf{x}}_*'\bar{\boldsymbol{\Sigma}}_*^{-1}\bar{\mathbf{x}}_*}. \tag{A13}$$

The standard OLS F-statistic can thus be written in terms of squared Sharpe ratios:

$$\begin{aligned} F(\mathrm{OLS}) &= \frac{(\mathrm{SSR}_r - \mathrm{SSR})/q}{\mathrm{SSR}/(T-k)} \\ &= \frac{(\bar{\mathbf{x}}'\bar{\boldsymbol{\Sigma}}^{-1}\bar{\mathbf{x}} - \bar{\mathbf{x}}_*'\bar{\boldsymbol{\Sigma}}_*^{-1}\bar{\mathbf{x}}_*)/q}{(1 + \bar{\mathbf{x}}_*'\bar{\boldsymbol{\Sigma}}_*^{-1}\bar{\mathbf{x}}_*)/(T-k)}. \end{aligned} \tag{A14}$$

By replacing $\mathbf{x}^1$ with $\bar{\mathbf{x}}_*$ in Seber (1982), Theorem 2.11, p. 52, it can be seen that this statistic is distributed as an exact central F distribution with q and $T-k$ degrees of freedom under the null hypothesis that the ex ante maximal Sharpe ratios are identical in the restricted and unrestricted cases. Mardia, Kent, and Bibby (1989), p. 78, view this statistic as a 'decomposition of Mahalanobis distance' and also provide a proof that the distribution is central F under the null hypothesis.

Proof of Corollary 1: Define $\mathbf{X}_{-i}$ as the matrix $\mathbf{X}$ with the ith column removed, and define its associated orthogonal projection matrix as

$$\mathbf{M}_{-i} \equiv \mathbf{I} - \mathbf{X}_{-i}(\mathbf{X}_{-i}'\mathbf{X}_{-i})^{-1}\mathbf{X}_{-i}'. \tag{A15}$$

From standard least squares algebra[15] we can express $\hat{b}_i$ as

$$\hat{b}_i = (x^{i\prime}\mathbf{M}_{-i}x^i)^{-1}x^{i\prime}\mathbf{M}_{-i}\mathbf{l}, \tag{A16}$$

and we can express the estimate of $\hat{b}_i$'s standard error as

$$s\sqrt{a_{ii}} = s(x^{i\prime}\mathbf{M}_{-i}x^i)^{1/2}. \tag{A17}$$

Thus the t-statistic for $b_i = 0$ can be expressed as

$$\frac{x^{i\prime}\mathbf{M}_{-i}\mathbf{l}}{s(x^{i\prime}\mathbf{M}_{-i}x^i)^{1/2}}. \tag{A18}$$

[15] The necessary algebra is contained in Davidson and MacKinnon (1993), pp. 19–24, 81–86.

The square of the t-statistic is thus

$$\frac{(x^{i\prime}\mathbf{M}_{-i}\mathbf{l})^2}{s^2 x^{i\prime}\mathbf{M}_{-i}x^i}. \tag{A19}$$

From standard least squares algebra, the difference between the restricted and unrestricted sum of squares can be expressed as

$$\begin{aligned}\mathrm{SSR}_r - \mathrm{SSR}_u &= \mathbf{l}'\mathbf{M}\mathbf{l} - \mathbf{l}'\mathbf{M}_{-i}\mathbf{l} \\ &= \frac{(x^{i\prime}\mathbf{M}_{-i}\mathbf{l})^2}{x^{i\prime}\mathbf{M}_{-i}x^i},\end{aligned} \tag{A20}$$

where $\mathbf{M}$ is the orthogonal matrix associated with the unrestricted regression:

$$\mathbf{M} \equiv \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'. \tag{A21}$$

Thus the squared t-statistic can be expressed as

$$\frac{(x^{i\prime}\mathbf{M}_{-i}\mathbf{l})^2}{s^2 x^{i\prime}\mathbf{M}_{-i}x^i} = \frac{\mathrm{SSR}_r - \mathrm{SSR}_u}{\mathrm{SSR}_u/(T-K)}. \tag{A22}$$

But this is the F-statistic from Theorem 3 for the special case of $q = 1$. As the square root of a random variable distributed as $F_{1,T-k}$ is distributed as t with $T - K$ degrees of freedom, we have proved the corollary.

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