CSC412 Notes Week 3

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1 Graphical Models

1.1 Chain Rule

The joint distribution of (N) random variables can be evaluated with the chain rule

$$P(x_{1,\dots,N}) = P(x_1)P(x_2|x_1)P(x_3|x_2,x_1)\dots P(x_n|x_{n-1,\dots,1})$$

When we have a joint distribution of discrete random variables with full dependence between variables.

More formally, in probability the chain rule for two random variables is

$$P(x,y) = P(x|y)P(y)$$

1.2 Conditional Independence

To represent large joint distributions we can assume conditional independence

$$X \perp Y|Z \Leftrightarrow P(X,Y|Z) = P(X|Z)P(Y|Z) \Leftrightarrow P(X|Y,Z) = P(X|Z)$$

This is very useful as now we can represent a large chain of N variables as a product of independent variables.

$$P(x_{1:n}) = P(x_1)\prod_{t=1}^{n} P(x_t|x_{t-1})$$

this is the (first order) Markov Assumption. Where "the future is independent of the past given the present"

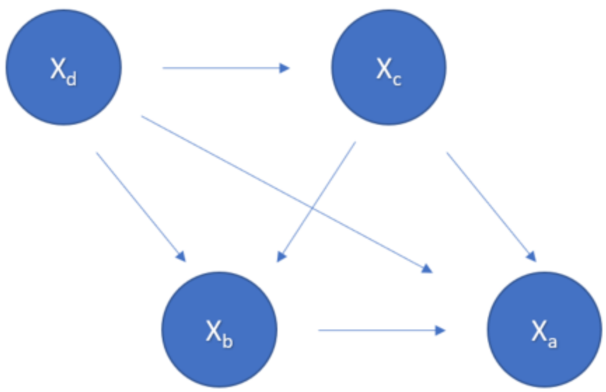
1.3 Probabilistic Graphical Models

If you don't know what a graph is, Wikipedia

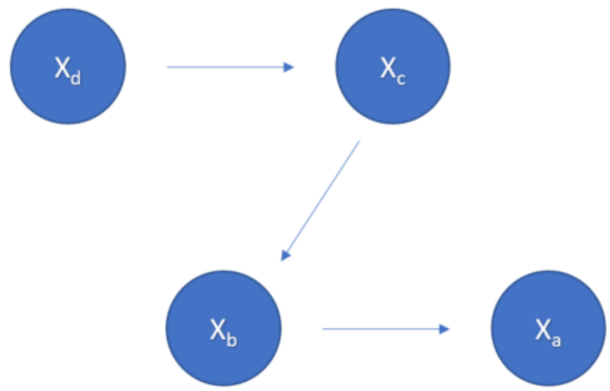
A probabilistic graphical model can modeled can be used to represent joint distributions when we assume Conditional Independence. In this model, nodes are variables and edges show conditional dependence.

1.4 Directed Acyclical Graphical Model

Using the Markov Assumption we can very easily represent complicated graphs. A 4 node starting graph can be represented with half the number of edges!



can now be represented as



its easy to see that X_a is much easier to evaluate in the second graph than the first.

$$P(x_a) = P(x_a|x_b, x_c, x_d)P(x_b|x_c, x_d)P(x_c|x_d)P(x_d)$$

can be simplified to this with the Markov assumption.

$$P(x_a) = P(x_a|x_b)P(x_b|x_c)P(x_c|x_d)P(x_d)$$

2 Conditional Independence and Directed-Separation

Directed-separation is where two variables in a DAGM may or may not be connected given a third variable.
D-connection implies conditional dependence.
D-separation implies conditional independence.

This also extrapolates into groups/sets of variables for X Y Z

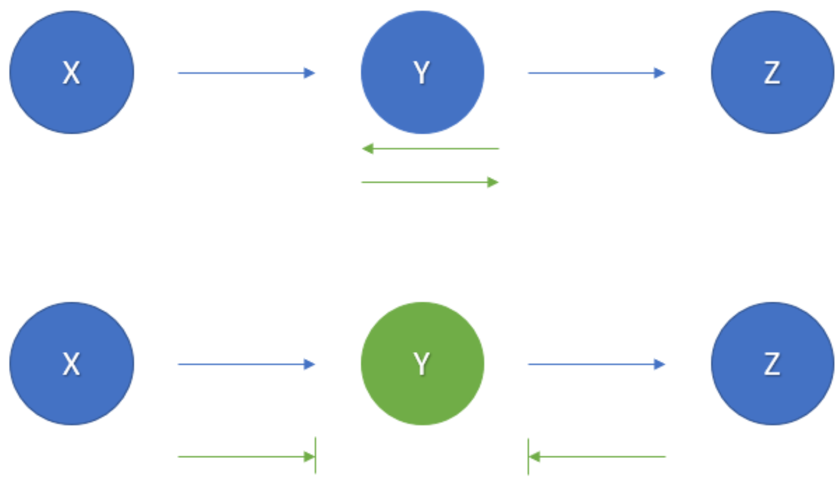
$$X = \{X_1, ...X_n\}$$
$$Y = \{Y_1, ...Y_n\}$$
$$Z = \{Z_1, ...Z_n\}$$
$$X \perp Z|Y$$

If every variable in X is d-separated from every variable in Z conditioned on all the variables in Y.

To determine d-separation we will use the Bayes ball algorithm

For Bayes Ball there are 3 structures that you must know.

2.1 Bayes Ball - Chain



X and Z are conditionally dependent when y is unknown and conditionally independent when y is known.
From the chain's graph, we can encode the structure as.

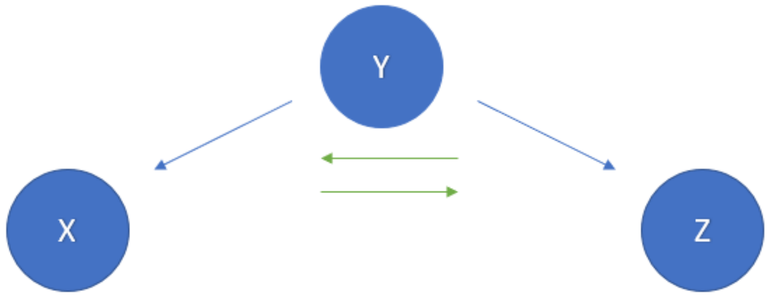
$$P(x, y, z) = P(x)P(y|x)P(z|y)$$

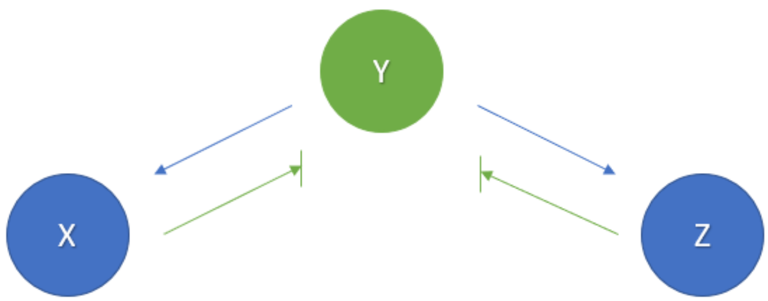
once we condition on y we get.

$$P(x, z|y) = \frac{P(x)p(y|x)p(z|y))}{P(y)} = \frac{P(x, y)P(z|y)}{P(y)} = P(x|y)P(z|y)$$

$$\therefore x \perp z | y$$

2.2 Bayes Ball - Fork





X and Z are conditionally dependent when y is unknown and conditionally independent when y is known.
from the fork's graph we get the equation

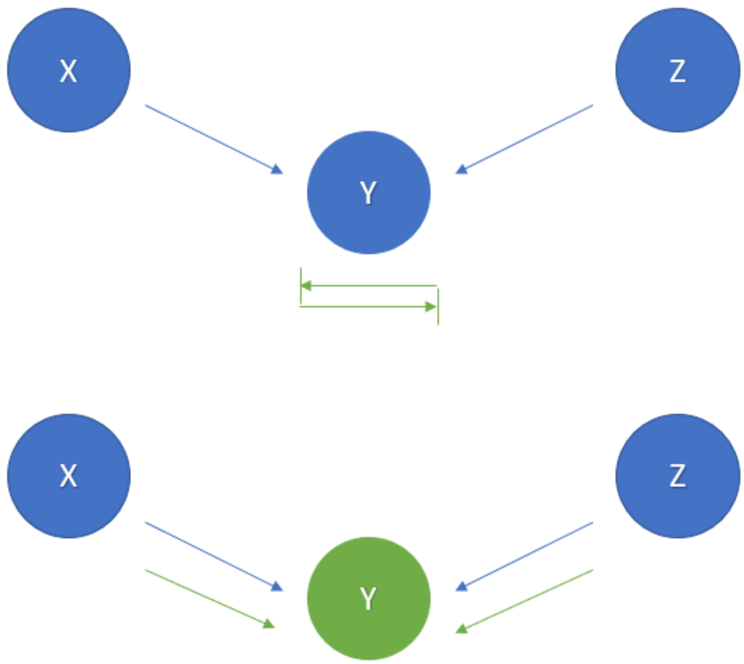
$$P(x, y, z) = P(y)P(x|y)P(z|y)$$

conditioning on y we get.

$$P(x, z|y) = \frac{P(x, y, z)}{P(y)} = \frac{P(y)P(x|y)P(z|y))}{P(y)} = P(x|y)P(z|y)$$

$$\therefore x \perp z|y$$

2.3 Bayes Ball - Collider



X and Z are conditionally independent when y is unknown and conditionally dependent when y is known.
From the collider's graph we get the equation

$$p(x, y, z) = p(x)p(z)p(y|x, z)$$

conditioning on y we get.

$$P(x, z|y) = \frac{P(x, y, z)}{P(y)} = \frac{p(x)p(z)p(y|x, z))}{P(y)} = P(x)P(z)$$

$$\therefore x \not\perp z|y$$

however if we do not condition on y it's easy to see that...

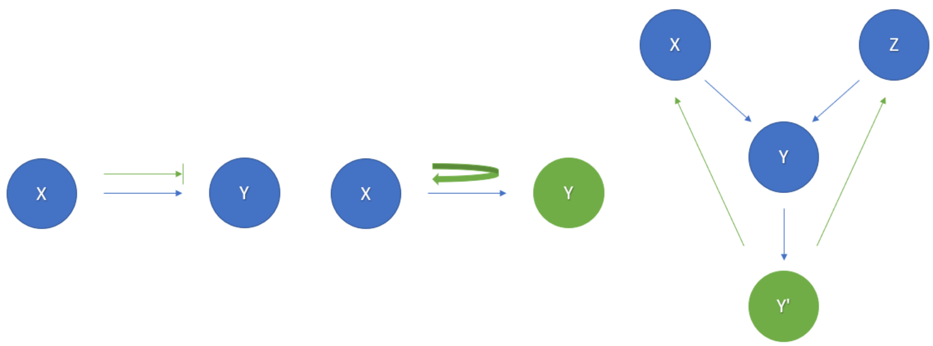
$$P(x, z) = P(x)P(z)$$

$$\therefore x \perp z$$

So we see that conditioning on a common child at the bottom of a collider/v-structure makes its parents become dependent.
This important effect is called explaining away, inter-causal reasoning, or Berkson's paradox.

As an example,
X is the event of a Toronto Raptors parade $P(X) = 0.01$
Z is the event of a car accident $P(Z) = 0.1$
Y is the event of a traffic jam downtown
Lets say that these are the only 2 sources of traffic. So if we know a traffic jam has occurred, then at least one of the two events has happened.

2.4 Boundary Conditions

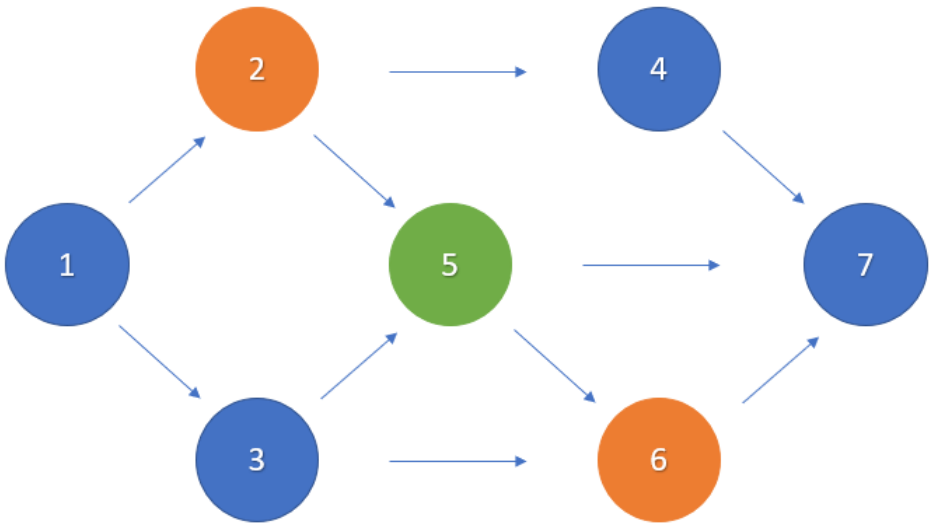


(1) (2) (3)

In example 3, if any child of Y is observed then Y is effectively observed, so the information 'bounces back'.
This is shown again in examples 1 and 2. Where if Y is known then the information goes up the chain.

2.5 Putting It Together

Now that we understand Conditional Independence with Bayes Ball on simple graphs we can apply this to complex graphs.



Say, we want to determine the conditional dependence of 2 and 6 given 5.
There are 3 paths from 2 to 6.
$2 \to 5 \to 6$ cannot be traversed $2 \perp 6|5$ (known chain)
$2 \to 4 \to 7 \to 6$ cannot be traversed $4 \perp 6|7$ (unknown collider)
$2 \to 1 \to 3 \to 6$ cannot be traversed $2 \perp 3|1$ (unknown fork)
so we can say $2 \perp 6|5$.
This would change if we knew 1 or 6 or didn't know 5.