All measurements—and therefore the calculations using those measurements—will have some inherent uncertainty that needs to be reported. There are two methods we will use in this class to report uncertainty. The method you use will depend on how the data is collected.

If you make only single measurements to be entered into a calculation, you will analyze propagation of error. When you are working with large data sets, you will do statistics to find the mean and standard deviation.

1 Significant figures

For either method you use, the uncertainty is rounded to one significant figure, and is of the same order of magnitude (decimal place) as the last significant figure of your calculated quantity. Similarly, the uncertainty on any given measurement is of the same order of magnitude as the last significant figure in the measurement.

2 Propagation of error

In general, propagation of uncertainty depends on the rate at which a function changes with respect to change in experimental quantities. Say you have a function $F = f(x_1, x_2, \ldots, x_n)$. If the errors in measuring each variable x_i are small, random, and independent of each other, then the uncertainty in F is

$$(\delta F)^2 = \sum_{i=1}^{n} \left[\left(\frac{\partial F}{\partial x_i} \right)^2 (\delta x_i)^2 \right] \tag{1}$$

where δ denotes the experimental uncertainty.

As an example, say you want to calculate the volume of a cylinder by measuring the height and radius:

$$V = \pi r^2 h \tag{2}$$

Each uncertainty in a measured value leads to more uncertainty in a calculated quantity. We have two measured values, so we will have two terms in our uncertainty for the volume of the cylinder. If the uncertainty in the measurement of the radius is indicated by δr and the uncertainty in the height measurement is δh, then the uncertainty in the volume δV is given by

$$\delta V = \sqrt{\left(\frac{\partial V}{\partial r} \right)^2 (\delta r)^2 + \left(\frac{\partial V}{\partial h} \right)^2 (\delta h)^2} \tag{3}$$

Suppose we have the following measurements and associated uncertainties

$$\begin{aligned} r &= 0.73 & \delta r &= 0.01\,\mathrm{cm} \\ h &= 1.26\,\mathrm{cm} & \delta h &= 0.01\,\mathrm{cm} \end{aligned}$$

The volume of our cylinder would be, using significant figures rules,

$$V = \pi(0.73\,\mathrm{cm})^2(1.26\,\mathrm{cm}) = 2.1\,\mathrm{cm}^3$$

The uncertainty in the measurement would be, from equation (3),

$$\begin{aligned}\delta V &= \sqrt{(2\pi rh)^2(\delta r)^2 + (\pi r^2)^2(\delta h)^2} \\ &= 6.0 \times 10^{-2}\,\mathrm{cm}^3\end{aligned} \tag{4}$$

So, your final answer would be

$$V = 2.1 \pm 0.1\,\mathrm{cm}^3$$

3 Standard deviation

One way of "beating down" the uncertainty of an overall experiment is to take the results from many experiments and use statistics. Two common metrics are *mean* and *standard deviation.* A mean is simply an arithmetic average; add up each measurement and divide by the total number of measurements. In symbols:

$$\overline{x} = \frac{\sum_{i=1}^{N} x_i}{N}$$

The standard deviation gives information about how far each datum lies from the average:

$$\sigma_x = \sqrt{\frac{\sum_{i=1}^{N}(x_i - \overline{x})^2}{N-1}}$$

Squaring the difference between a data point and the average removes any plus or minus sign, and taking the square root gets us back to the appropriate units. When taking many measurements and using statistical analysis, we can report values as $x = \overline{x} \pm \sigma_x$.

For example, say you perform a projectile motion experiment where you measure the range of a projectile. You do ten trials and make the measurements:

Trial	x (cm)	Trial	x (cm)
1	50.5	6	48.0
2	49.5	7	49.0
3	51.0	8	50.0
4	55.0	9	50.5
5	48.5	10	49.5

The average, to three significant figures, is $\overline{x} = 50.2\,\mathrm{cm}$ and the standard deviation is $\sigma_x = 1.94\,\mathrm{cm}$. Our final result would be reported as

$$x = 50 \pm 2 \text{ cm}$$