Homework of Week 5

Deadline: 9:00am, November 26 (Thursday), 2020

1. Consider the probability that every bin receives exactly one ball when n balls are thrown randomly into n bins.

   - Give an upper bound on this probability using the condition-free Poisson approximation.
   - Determine the exact probability of this event.

2. Let $X_1, ..., X_n$ be independent and identically distributed Poisson random variables and $X = \sum_{i=1}^{n} X_i$. Let $\mathcal{E}$ be the event that all X_i's are nonzero. Prove that $\Pr(\mathcal{E}|X = k)$ increases with k.

3. Let $X_1, ..., X_n$ be independent and identically distributed Poisson random variables and $X = \sum_{i=1}^{n} X_i$. Let $\mathcal{E}$ be the event that all X_i's are nonzero. We know that $\Pr(\mathcal{E}|X = k)$ increases with k. Prove that for any $0 < k_1 < k_2$, $\Pr(\mathcal{E}|X = k_1) \leq \Pr(\mathcal{E}|X \in [k_1, k_2])$.

4. Let $X_1, ..., X_n$ be independent and identically distributed Poisson random variables and $X = \sum_{i=1}^{n} X_i$. Let $\mathcal{E}$ be the event that all X_i's are nonzero. Prove that $\lim_{n\to\infty} \Pr(\mathcal{E}|X = m + \sqrt{2m \ln m}) - \Pr(\mathcal{E}|X = m - \sqrt{2m \ln m}) = 0$ where $m = n \ln n$.

5. Do Bernoulli experiment for 20 trials, using a new 1-Yuan coin. Record the result in a string $s_1 s_2 ... s_i ... s_{20}$, where s_i is 1 if the i^{th} trial gets Head, and otherwise is 0.