Probabilistic Method and Random Graphs

Lecture 9. De-randomization and Second Moment Method

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[1]The slides are mainly based on Chapter 6 of Probability and Computing.

Comments, questions, or suggestions?

A Review of Lecture 8

- Principle of probabilistic method

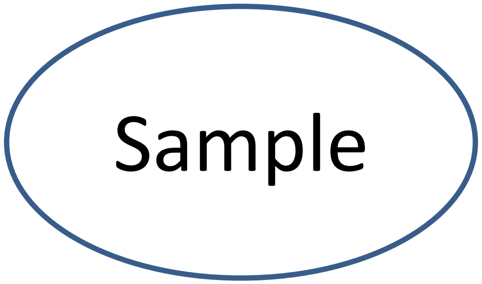
Probability space of all objects $\Rightarrow$ Nonzero probability of satisfying the property $\Rightarrow$ Existence

- Counting: Tournament, Ramsey number
- First moment method: Max-3SAT, MIS
  - Expectation argument: $\Pr(X \geq \mathbb{E}[X]) > 0, \Pr(X \leq \mathbb{E}[X]) > 0$
  - Markov's inequality: $\Pr(X \geq a) \leq \frac{\mathbb{E}[X]}{a}$

$$\Pr(X \neq 0) = \Pr(X > 0) = \Pr(X \geq 1) \leq \mathbb{E}[X]$$

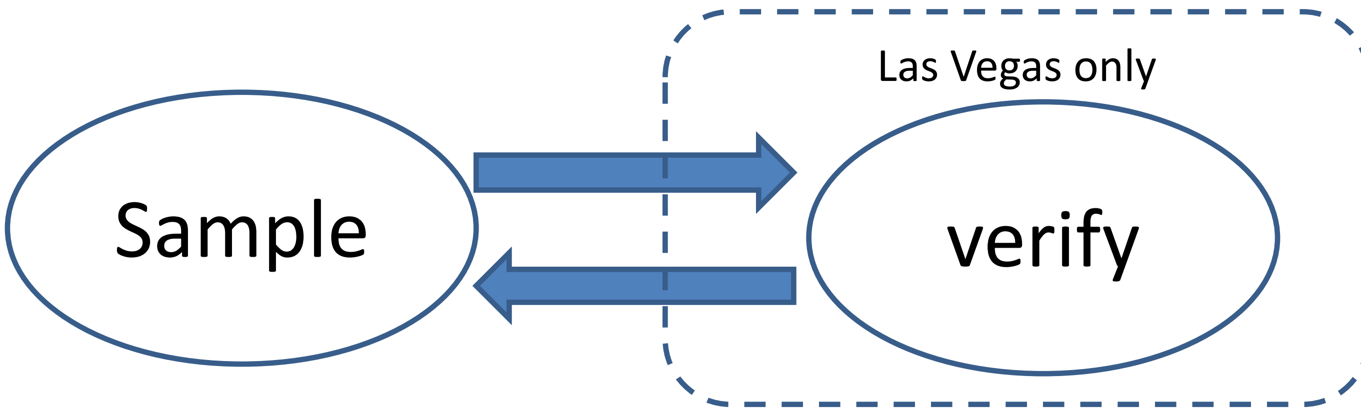
A Review of Lecture 8

- How to find a desirable object? By sampling!
- Algorithmic paradigm



A Review of Lecture 8

- How to find a desirable object? By sampling!
- Algorithmic paradigm



- First moment method guarantees efficiency

- Cool to get an efficient randomized algorithm

- Can we derive a deterministic one?

- Yes, if expectation argument is used

De-randomization: an example

- **MAX-3SAT**: Given a 3-CNF Boolean formula, find a truth assignment satisfying the maximum number of clauses
  - E.g.: $(x_1 \vee x_2 \vee x_3) \wedge \ldots \wedge (\overline{x_1} \vee \overline{x_3} \vee x_4)$
- Known: at least $\frac{7}{8}n$ clauses can be satisfied
- Randomized algo. to find a good assignment
  - Independently, randomly assign values
  - Succeed if lucky
    - Can we make good choice, rather than pray for luck?

Look closer at the randomized algorithm

- In equivalence, choose values sequentially
- Good choices lead to a good final result
  - Which choice is good?
    - Easy to know with hindsight, but how to predict
  - A tentative approach: always make the choice which allows a good final result
    - Fact: a $\frac{7n}{8}$ expect. means the existence of a $\frac{7}{8}$-approx.
    - Make the current choice, keeping the expectation $\geq \frac{7n}{8}$
  - Nice, but does such a choice exist? How to find it?

Conditional expectation says yes!

- The first step
  - $\frac{7n}{8} = \mathbb{E}[X] = \boxed{\sum_{v_1} \Pr(x_1 = v_1)}\, \mathbb{E}[X | x_1 = v_1]$
  - There must be v_1 s.t. $\mathbb{E}[X | x_1 = v_1] \geq \frac{7n}{8}$
- Likewise, if $\mathbb{E}[X | x_1 = v_1, \ldots, x_{k-1} = v_{k-1}] \geq \frac{7n}{8}$, then $\mathbb{E}[X | x_1 = v_1, \ldots, x_k = v_k] \geq \frac{7n}{8}$ for some v_k
- Final correctness
  - $X(x_1 = v_1, \ldots, x_m = v_m) = \mathbb{E}[X | x_1 = v_1, \ldots, x_m = v_m] \geq \frac{7n}{8}$
- Given $v_1, \ldots, v_{k-1}$, what's the v_k?
  - Let v_k s.t. $\mathbb{E}[X | x_1 = v_1, \ldots, x_k = v_k]$ is maximized

Deterministic $\frac{7}{8}$-algorithm for MAX-3SAT

For $k = 1 \cdots m$ do

$$x_k = \text{argmax}_{v_k} \mathbb{E}[X | x_1 = v_1, \ldots x_{k-1} = v_{k-1}, x_k = v_k]$$

Endfor

- Cool! And this approach can be generalized

De-randomization via conditional expectation

- Expectation argument $\Longrightarrow$ deterministic algorithm
- Basic idea
  - Expectation argument guarantees existence
  - Sequentially make deterministic choices
    - Each choice maintains the expectation, given the past ones

- Only valid for expectation argument where randomness lies in a sequence of random variables
- What if the expectation is hard to compute?

Example: Turán Theorem

- Any graph $G = (V, E)$ contains an independent set of size at least $\frac{|V|}{D+1}$, where $D = \frac{2|E|}{|V|}$
- Expectation argument: the expected size of an independent set S is at least $\frac{|V|}{D+1}$
- Randomly choose vertices into S one by one

- Try the de-randomization routine

Idea of the algorithm (1)

- Choose valid vertices sequentially
- At step $t+1$, find u to maximize $\mathbb{E}\left[Q \mid S^{(t)}, u\right]$
  - $S^{(t)}$: the independent set at step t
  - Q: the size of the final independent set
- Hard to compute the expectation ☹
  - $\mathbb{E}[Q] \geq \sum \frac{1}{d(w)+1} \geq \frac{|V|}{D+1}$
- It suffices to show $\mathbb{E}\left[Q \mid S^{(t)}\right] \geq \frac{|V|}{D+1}$ for any t

Idea of the algorithm (2)

- Note that $\mathbb{E}\left[Q \middle| S^{(t)}\right] \geq \left|S^{(t)}\right| + \sum_{w \in R^{(t)}} \frac{1}{d(w)+1} \triangleq X^{(t)}$
  - $R^{(t)}$: set of vertices away from $S^{(t)}$ by distance >1
- $X^{(0)} \geq \frac{|V|}{D+1} \Rightarrow$ it's enough if $X^{(t)}$ is non-decreasing
  - Can we achieve this?
- If at step $t+1$, $u \in R^{(t)}$ is chosen,
  $X^{(t+1)} - X^{(t)} = 1 - \sum_{w \in \Gamma^{+}(u)} \frac{1}{d(w)+1}$

Can it be non-negative?

- $\mathbb{E}_u\left[X^{(t+1)} - X^{(t)}\right] \geq 1 - \sum_{w \in R^{(t)}} \frac{1}{d(w)+1} \frac{d(w)+1}{\left|R^{(t)}\right|} = 0$
- So, there is u s.t. $X^{(t+1)} \geq X^{(t)}$

A deterministic algorithm

- Initialize S to be the empty set
- **While** there is a vertex $u \notin \Gamma(S)$
  - Add to S such a vertex u which minimizes $\sum_{w \in \Gamma^{+}(u)} \frac{1}{d(w)+1}$
- **Return** S

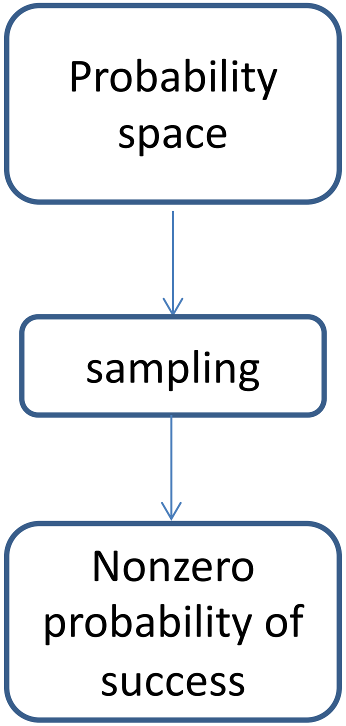
- Paul Turán (1910 –1976)
- Hungarian mathematician
- Founder of

  Probabilistic number theory

  Extremal graph theory

  (in Nazi Camp)

Sample


Probability space
sampling
Nonzero probability of success

Big Chromatic Number and Big Girth

- Chromatic number vs local structure
  - Sparse local structure → small chro. number?
  - No! (Erdős 1959)
- One of the first applications of prob. Method
- Theorem: for any integers $g, k > 0$, there is a graph with girth$\geq g$ and chro. number$\geq k$
- We just prove the special case $g = 4$, i.e. triangle-free

Basic Idea of the Proof

- Randomly pick a graph G from $G_{n,p}$
  - $\chi(G)$: the chromatic number of G
  - $\mathbb{I}(G)$: the size of a maximum independent set of G
- With high probability $\mathbb{I}(G)$ is small
  - $\mathbb{I}(G)\chi(G) \geq n$ implies that $\chi(G)$ is big
- With high probability G has few triangles
- Destroy the triangles while keeping $\mathbb{I}(G)$ small

Proof: $\mathbb{I}(G)$ is small w.h.p.

- S: a vertex set of size $\frac{n}{2k}$
- A_S: S is an independent set
- $\Pr\left(\mathbb{I}(G) \geq \frac{n}{2k}\right) = \Pr(\bigcup_S A_S)$

$$\leq \binom{n}{n/2k}(1-p)^{\binom{n/2k}{2}}$$

$$< 2^n e^{-\frac{pn(n-2k)}{8k^2}}$$

which is small if n is large and $p = \omega(n^{-1})$

Proof: triangles are few w.h.p.

- $\mathcal{T}(G)$: the number of triangles of G
- $\mathbb{E}[\mathcal{T}(G)] = \binom{n}{3}p^3 < \frac{(np)^3}{6} = \frac{n}{6}$ if $p = n^{-\frac{2}{3}}$
- By Markov ineq., $\Pr\left(\mathcal{T}(G) > \frac{n}{2}\right) \leq \frac{1}{3}$
- Recall $\Pr\left(\mathbb{I}(G) \geq \frac{n}{2k}\right) < 2^n e^{-\frac{pn(n-2k)}{8k^2}}$

$$< e^n \, e^{-\frac{pn^2}{16k^2}} = e^{n-n^{\frac{4}{3}}/16k^2} \quad \text{if } n > 4k$$

$$< e^{-n} < \frac{1}{6} \quad \text{if } n^{1/3} \geq 32k^2$$

Proof: modification

- $\Pr\left(\mathbb{I}(G) < \frac{n}{2k}, \mathcal{T}(G) \leq \frac{n}{2}\right) > \frac{1}{2}$
  - Choose G s.t. $\mathbb{I}(G) < \frac{n}{2k}, \mathcal{T}(G) \leq \frac{n}{2}$
- Remove one vertex from each triangle of G, resulting in a graph G' with $n' \geq n - \mathcal{T}(G)$
- $\mathbb{I}(G') \leq \mathbb{I}(G) < \frac{n}{2k}$
- $\chi(G') \geq \frac{n'}{\mathbb{I}(G')} \geq \frac{n-\mathcal{T}(G)}{\frac{n}{2k}} \geq k$

Algorithm for finding such a graph

- Fix $n^{1/3} \geq 32k^2$ and $p = n^{-2/3}$
- Sample G from $G_{n,p}$
- Destroy the triangles

- Success probability > ½

- Do you have any idea of de-randomizing?

Second moment argument

- Chebyshev Ineq.: $\Pr(|X - \mathbb{E}[X]| \geq a) \leq \frac{\mathrm{Var}[X]}{a^2}$
- A special case:

$$\Pr(X = 0) \leq \Pr(|X - \mathbb{E}[X]| \geq \mathbb{E}[X]) \leq \frac{\mathrm{Var}[X]}{(\mathbb{E}[X])^2}$$

  – Compare with $\Pr(X \neq 0) \leq \mathbb{E}[X]$ for integer r.v. X
- Typically works when nearly independent
  – Due to the difficulty in computing the variance

An improved version by Shepp

- $\Pr(X = 0) \leq \frac{\mathrm{Var}[X]}{\mathbb{E}[X^2]}$
- Proof: $(\mathbb{E}[X])^2 = (\mathbb{E}[1_{X\neq 0} \cdot X])^2$

$$\begin{aligned} &\leq \mathbb{E}[1_{X\neq 0}^2]\mathbb{E}[X^2] \\ &= \Pr(X \neq 0)\mathbb{E}[X^2] \\ &= \mathbb{E}[X^2] - \Pr(X = 0)\mathbb{E}[X^2] \end{aligned}$$

  – The inequality is due to $\left(\int fg\right)^2 \leq \int f^2 \int g^2$
- When $X \geq 0$, $\Pr(X > 0) > \frac{(\mathbb{E}[X])^2}{\mathbb{E}[X^2]}$

Generalizing Shepp's Theorem

- $\Pr(X > \theta\mathbb{E}[X]) \geq (1-\theta)^2 \frac{(\mathbb{E}[X])^2}{\mathbb{E}[X^2]}, \theta \in (0,1)$
- Paley&Zygmund, 1932
- Proof:

$$\mathbb{E}[X] = \mathbb{E}\left[X 1_{X \leq \theta\mathbb{E}[X]}\right] + \mathbb{E}\left[X 1_{X > \theta\mathbb{E}[X]}\right]$$

$$\leq \theta\mathbb{E}[X] + \left(\mathbb{E}[X^2]\Pr(X > \theta\mathbb{E}[X])\right)^{\frac{1}{2}}$$

- Further improvement, tight when X is constant

$$\Pr(X > \theta\mathbb{E}[X]) \geq \frac{(1-\theta)^2(\mathbb{E}[X])^2}{\mathrm{Var}[X] + (1-\theta)^2(\mathbb{E}[X])^2}$$

due to $\mathbb{E}\left[X - \theta\mathbb{E}[X]\right] \leq \mathbb{E}\left[(X - \theta\mathbb{E}[X]) 1_{X > \theta\mathbb{E}[X]}\right]$

References

- http://www.cse.buffalo.edu/~hungngo/classes/2011/Spring-694/lectures/sm.pdf

- http://www.openproblemgarden.org/

- Documentary film of Erdős: N is a Number - A Portrait of Paul Erdős

Thank you!