Solution for HW-1

October 29, 2020

1

Consider a geometric random variable X with success probability p, for any $n \in Z^+$, we have

$$\Pr(X = n) = (1-p)^{n-1}p$$

So for any $n, k \in Z^+$,

$$\begin{aligned}
\Pr(X = n+k | X > k) &= \frac{\Pr(X = n+k, X > k)}{\Pr(X > k)} \\
&= \frac{\Pr(X = n+k)}{\Pr(X > k)} \\
&= \frac{\Pr(X = n+k)}{1 - \Pr(X <= k)} \\
&= (1-p)^{n-1}p \\
&= \Pr(X = n)
\end{aligned}$$

So X is memoryless.

2

Given a memoryless random variable X, for any $n, k \in Z^+$, we have

$$\Pr(X = n+k | X > k) = \Pr(X = n)$$

then,

$$\begin{aligned}
\Pr(X > n+k | X > k) &= \Pr(X > n) \\
\Pr(X > n+k) &= \Pr(X > n) \Pr(X > k)
\end{aligned}$$

So for any $n \in Z^+$,

$$\begin{aligned}
\Pr(X > n) &= \Pr(X > 1) \Pr(X > n-1) \\
&= [\Pr(X > 1)]^2 \Pr(X > n-2) \\
&\vdots \\
&= [\Pr(X > 1)]^n
\end{aligned}$$

Note $\Pr(X > 1) = 1 - p$, we have

$$\Pr(X = n) = \Pr(X > n-1) - \Pr(X > n) = (1-p)^{n-1} - (1-p)^n = (1-p)^{n-1}p$$

So X is geometric.

3

Let X_k be the result of k-th roll for $1 <= k <= n$, and $Y_k = \sum_{i=1}^{k} X_i$.

$$\begin{aligned}
\Pr(Y_n \equiv 0 \pmod 6) &= \sum_{i=1}^{6} \Pr(Y_n \equiv 0 \pmod 6 \cap X_n = i) \\
&= \sum_{i=1}^{6} \Pr(Y_{n-1} \equiv 6 - i \pmod 6) \Pr(X_n = i) \\
&= \frac{1}{6} \sum_{i=1}^{6} \Pr(Y_{n-1} \equiv 6 - i \pmod 6) \\
&= \frac{1}{6}
\end{aligned}$$