In[1]:=

```
Clear["Global`*"]
rp = 0.1; (* used for radius in polar *)
xp = rp * Cos[t]; (* used for x-coord in polar *)
yp = rp * Sin[t]; (* used for y-coord in polar *)
Index[p_, q_] := 1/(2 π) N[Integrate[Re[(p D[q, t] - q D[p, t])/(p^2 + q^2)], {t, 0, 2 π}]];
a = 2;
```

Use Poincare-Bendixson Theorem to find and classify limit cycles for Problem 1 and Problem 2.

Problem 1

In[7]:=

```
(* x_dot = x-y-x^3 *)
(* y_dot = x+y-y^3 *)
(xp - yp - xp^3) D[xp + yp - yp^3, t] - (xp + yp - yp^3) D[xp - yp - xp^3, t] // FullSimplify
(xp - yp - xp^3)^2 + (xp + yp - yp^3)^2 // FullSimplify
1/(2 π) Integrate[((0.019850375000000003`) + 0.00004962500000000002` Cos[4 t] -
    0.00005000000000000002` Sin[4 t]) / (0.019850625` -
    0.000049625000000000014` Cos[4 t] + 0.000050000000000000016` Sin[4 t]), {t, 0, 2 π}]
 (* I had to get the intgrand in this form in order to integrate
  since the computer did not know how to handle numerical
  0 as seen in the immidiate outputs 1 and 2 below *)
```

Out[7]= $(0.0198504 + 1.73472 \times 10^{-18}\, i) + 0.000049625\, \text{Cos}[4 t] - 0.00005\, \text{Sin}[4 t]$

Out[8]= $0.0198506 + 7.86047 \times 10^{-19}\, \text{Cos}[2 t] - 0.000049625\, \text{Cos}[4 t] + 0.00005\, \text{Sin}[4 t]$

Out[9]= 1.

Having an index of 1 indicates (x_dot,y_dot) is stable, or unstable, has a node or focus at the origin.

In[103]:=

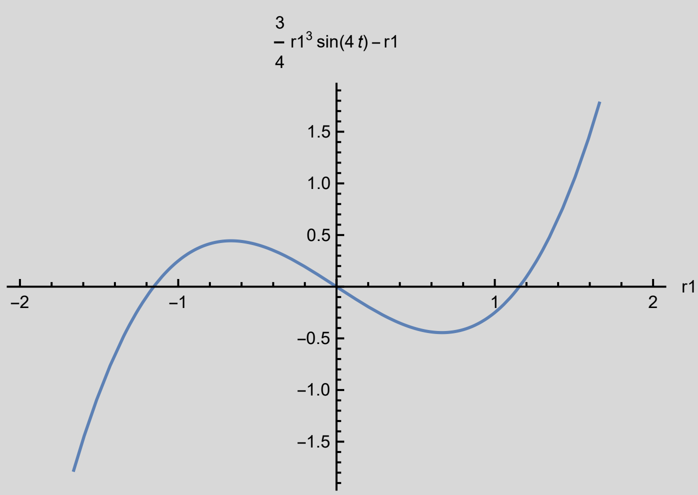
```
x1 = r1 Cos[t];
y1 = r1 Sin[t];
x1dot = x1 - y1 - x1^3;
y1dot = x1 + y1 - y1^3;
r1dot = (x1 D[x1dot, t] + y1 D[y1dot, t])/r1 // FullSimplify
theta1dot = (x1 D[y1dot, t] - y1 D[x1dot, t])/r1^2 // FullSimplify
N[Roots[r1dot == 0 /. t -> π/8, r1]]
Plot[r1dot /. t -> π/8, {r1, -a, a}, AxesLabel -> {r1, r1dot}]
N[Roots[theta1dot == 0 /. t -> π/8, r1]]
Plot[theta1dot /. t -> π/8, {r1, 0, a}, AxesLabel -> {r1, theta1dot}]
```

Out[107]= $-r1 + \frac{3}{4} r1^3 \operatorname{Sin}[4t]$

Out[108]= $\frac{1}{4} \times \left(4 - 3 r1^2 + 3 r1^2 \operatorname{Cos}[4t]\right)$

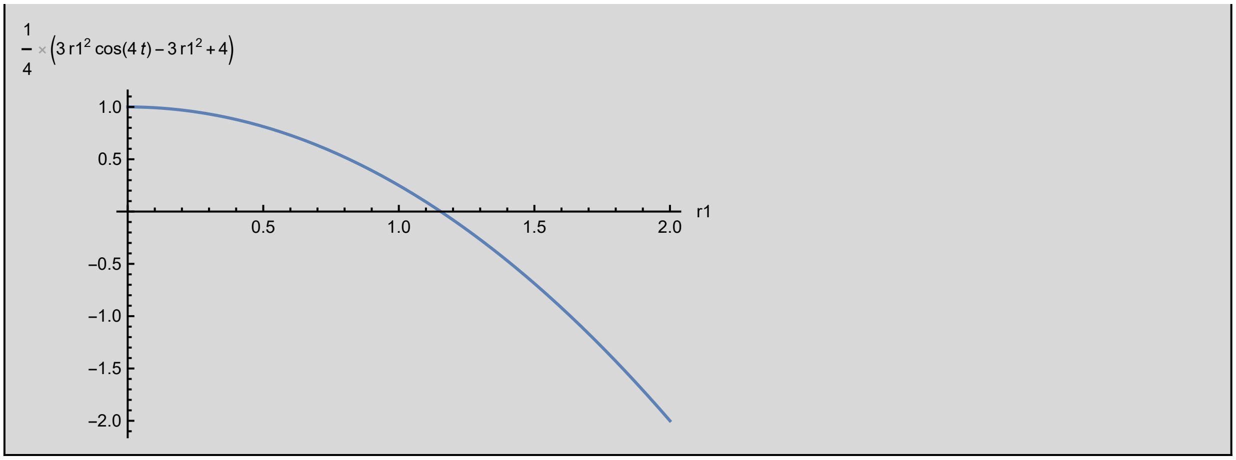
Out[109]= r1 == 0. || r1 == 1.1547 || r1 == -1.1547

Out[110]=



Out[111]= r1 == 1.1547 || r1 == -1.1547

Out[112]=



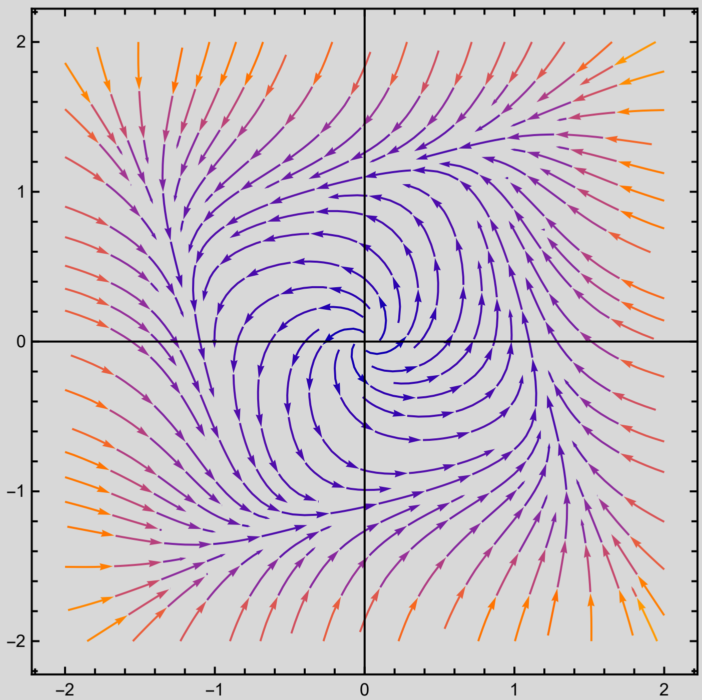
Note that both phase plane plots above are for $t = \frac{\pi}{8}$ and change in time, but all have the same general behavior.

We now have **four regions** to consider for the second phase plane plot above, but we only need to consider **two regions** (in quadrants 1 and 4) since the plot is symmetric.
The first region indicates that the point r1 = 0 is a stable equilibrium and r1 = 1.1547 is an unstable equilibrium.
(But this does not make sense based on further analysis of the plots.)

We now have **two regions** to consider for the second phase plane plot above.
The **first region** indicates that if a test particle were placed in the region 0 < r1 < 1.1547, it will have a **positive and decreasing angular velocity with respect to r1** until it is close enough to r1 = 1.1547 in which case the test particle would then tend towards a constant angular velocity with respect to r1.
The **second region** indicates that if a test particle were placed in the region 1.1547 < r1 < ∞, it will have a **negative and increasing angular velocity with respect to r1** until it is close enough to r1 = 1.1547 in which case the test particle would then tend towards a constant angular velocity with respect to r1.

In[20]:=
```
StreamPlot[{x - y - x^3, x + y - y^3}, {x, -a, a}, {y, -a, a}, Axes → True]
```

Out[20]=



The above plot of (x_dot,y_dot) shows that “squared-off” circle about the origin is a stable limit cycle.

Problem 2

In[21]:=
```
(* with r^2 = x^2 + y^2 *)
(* x_dot = -y+x(r^4-3r^2+1) *)
(* y_dot = x+y(r^4-3r^2+1) *)
Index[-yp + xp (rp^4 - 3 rp^2 + 1), xp + yp (rp^4 - 3 rp^2 + 1)]
```

Out[21]=
```
1.
```

Having an index of 1 indicates (x_dot,y_dot) is stable, or unstable, has a node or focus at the origin.

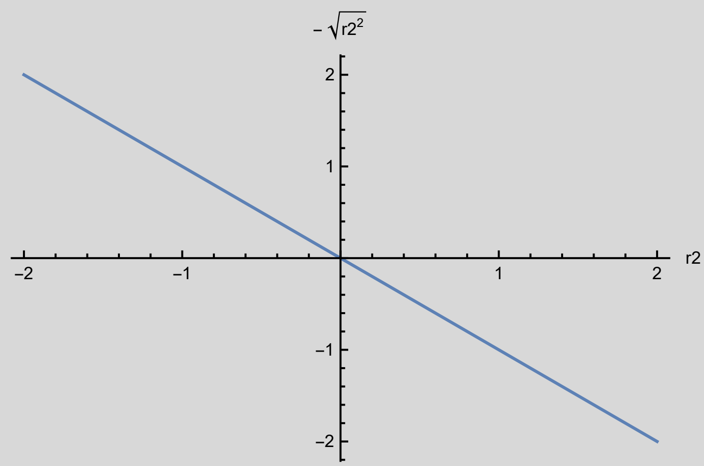
In[22]:=

```
x2 = r2 Cos[t];
y2 = r2 Sin[t];
x2dot = -y2 + x2 (r2^4 - 3 r2^2 + 1);
y2dot = x2 + y2 (r2^4 - 3 r2^2 + 1);
r2dot = (x2 D[x2dot, t] + y2 D[y2dot, t]) / Sqrt[x2^2 + y2^2] // FullSimplify
theta2dot = (x2 D[y2dot, t] - y2 D[x2dot, t]) / (x2^2 + y2^2) // FullSimplify
Plot[-r2, {r2, -a, a}, AxesLabel → {r2, r2dot}]
N[Roots[theta2dot == 0, r2]]
Plot[theta2dot, {r2, 0, a}, AxesLabel → {r2, theta2dot}]
```

Out[26]= $-\sqrt{r2^2}$

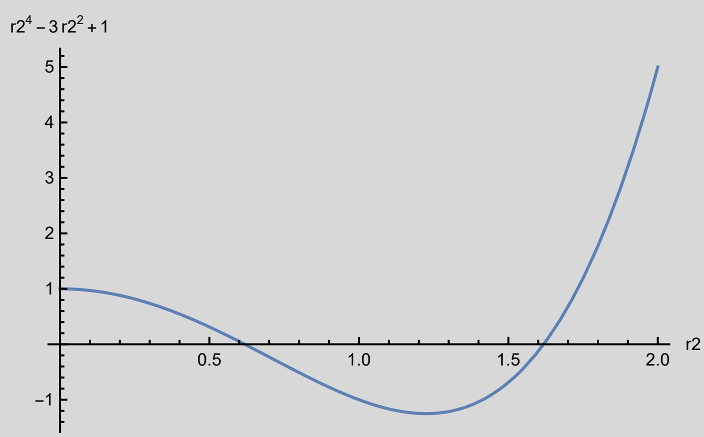
Out[27]= $1 - 3\,r2^2 + r2^4$

Out[28]=



Out[29]= r2 == -1.61803 || r2 == 0.618034 || r2 == -0.618034 || r2 == 1.61803

Out[30]=



The first phase plane plot above shows that r2 = 0 is a stable equilibrium.
(But this does not make sense based on further analysis of the plots.)

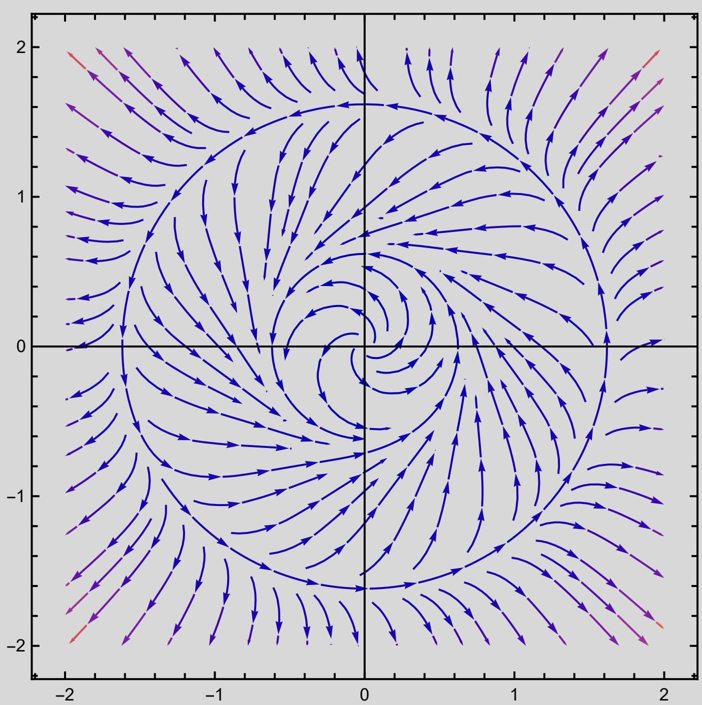
We now have **three regions** to consider for the second phase plane plot above.
The **first region** indicates that if a test particle were placed in the region 0 < r2 < 0.618034, it will have a

The first phase plane plot above shows that r2 = 0 is a stable equilibrium.
(But this does not make sense based on further analysis of the plots.)

We now have **three regions** to consider for the second phase plane plot above.
The **first region** indicates that if a test particle were placed in the region 0 < r2 < 0.618034, it will have a **positive and decreasing angular velocity with respect to r2** until it reaches r2 = 0.618034; upon reaching r2, the test particle would then tend towards a constant angular velocity with respect to r2.
The **second region** indicates that if a test particle were placed in the region 0.618034 < r2 < 1.61803, it will have a **negative and increasing angular velocity with respect to r2** until it reaches r2 = 0.618034; upon reaching r2, the test particle would then tend towards a constant angular velocity with respect to r2.
The **third region** indicates that if a test particle were placed in the region 1.61803 < r2 < ∞, it will have a **positive and increasing angular velocity with respect to r2** never reaching a limit cycle.

In[31]:=
```
r = Sqrt[x^2 + y^2];
StreamPlot[{-y + x (r^4 - 3 r^2 + 1), x + y (r^4 - 3 r^2 + 1)}, {x, -a, a}, {y, -a, a}, Axes → True]
```

Out[32]=



The above plot of (x_dot,y_dot) shows that there are two limit cycles about the origin, the first being the circle of smaller radius is a stable limit cycle and the second being the circle of larger radius is an unstable limit cycle.