HW2

Tuesday, August 31, 2021 9:11 AM

HW2 1) $x' = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix} x$

2) $x' = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} x$

Solve for α, β being zero or non-zero

$\alpha = \{ +, 0, -$ $\beta = \{ +, 0, -$

Do ANALYSIS as this example: traj & Flow &

Decompose $\mathbb{R}^2$ into W^s, W^u, W^c direct sum.

W^c = 'center' space

(find λ, and decouple ode & solve)

(at least 2-dim)

$\alpha, \beta \in \mathbb{R}$

(1) Let $x' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$ and let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

Then $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}}_{m} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

eigenvalues of m

$$0 = \det(m - \lambda I) = \det \begin{pmatrix} \alpha-\lambda & \beta \\ \beta & \alpha-\lambda \end{pmatrix}$$

$$= (\alpha-\lambda)^2 - \beta^2$$

$$\Rightarrow (\alpha-\lambda)^2 = \beta^2$$

$$\Rightarrow \alpha - \lambda = \pm\beta$$

$$\Rightarrow \lambda = \alpha \pm \beta$$

$\lambda_1 = \alpha + \beta$

$\lambda_2 = \alpha - \beta$

Cases

✓(0) $\alpha = \beta = 0 \Rightarrow$ no eigenvalues

✓(1) $\alpha \neq 0, \beta \neq 0 \Rightarrow \lambda_1 = \alpha + \beta, \lambda_2 = \alpha - \beta$

✓(i) $\alpha > 0$ ✓(ii) $\alpha < 0$

✓(a) $|\alpha| > |\beta|$ ✓(a) $|\alpha| > |\beta|$

✓(b) $|\alpha| < |\beta|$ ✓(b) $|\alpha| < |\beta|$

✓(2) $\alpha = 0, \beta \neq 0 \Rightarrow \lambda_1 = +\beta, \lambda_2 = -\beta$

✓(3) $\alpha = \beta$

$\lambda_1 = \alpha + \beta$ $(m - \lambda I)x = \begin{pmatrix} \alpha - (\alpha+\beta) & \beta \\ \beta & \alpha - (\alpha+\beta) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$$\Rightarrow \begin{pmatrix} -\beta & \beta \\ \beta & -\beta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} -\beta & \beta \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Rightarrow -\beta x_1 + \beta x_2 = 0$$

$$\Rightarrow \beta x_1 = \beta x_2$$

Beta = 0 => x1, x2 free => R^2 is fixed (every point in R^2 is stable)

for $\beta \neq 0$ $x_1 = x_2$ (let $x_2 = 1$)

1st eigenvector $x = c \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $c \in \mathbb{R} - \{0\}$

$\lambda_2 = \alpha - \beta$ $(m - \lambda I)x = \begin{pmatrix} \alpha - (\alpha-\beta) & \beta \\ \beta & \alpha - (\alpha-\beta) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$$\Rightarrow \begin{pmatrix} \beta & \beta \\ \beta & \beta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \beta x_1 = -\beta x_2$$

for $\beta \neq 0$ $x_1 = -x_2$ (let $x_2 = 1$)

2nd eigenvector $x = k \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $k \in \mathbb{R} - \{0\}$

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \alpha+\beta & 0 \\ 0 & \alpha-\beta \end{pmatrix}$$

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \Rightarrow P^{-1} = P^T$$

Inverse[$\frac{1}{\text{Sqrt[2]}}$ $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$]

trixForm=

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\Rightarrow x' = m x$$

Change of basis to eigenvector basis

$$y' = Dy \rightarrow \begin{cases} y_1' = (\alpha+\beta) y \\ y_2' = (\alpha-\beta) y \end{cases}$$

Trajectory of (1)

$$\boxed{\begin{aligned} y_1 &= c_1 e^{(\alpha+\beta)t}, \ c_1 \neq 0 \\ y_2 &= c_2 e^{(\alpha-\beta)t}, \ c_2 \neq 0 \end{aligned}}$$

Flow of (1)

$$y_1 = c_1 (e^t)^{(\alpha+\beta)}$$

$$\Rightarrow e^t = c_1' y_1^{\frac{1}{(\alpha+\beta)}}$$

$$y_2 = c_2 (e^t)^{(\alpha-\beta)}$$

$$\Rightarrow \boxed{y_2 = c_2' \, y_1^{\frac{(\alpha-\beta)}{(\alpha+\beta)}}}$$

$y_2 = C_2 (e^t)^{(\alpha-\beta)}$

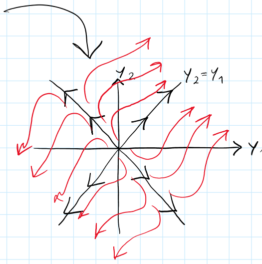
$\Rightarrow \boxed{y_2 = C_2' \, y_1^{\frac{(\alpha-\beta)}{(\alpha+\beta)}}}$

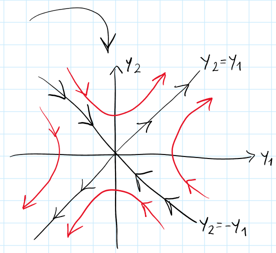
- Let $\alpha+\beta=\gamma$ and $\alpha-\beta=\delta$

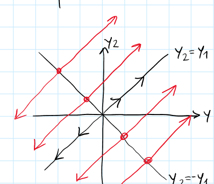
	Tradjectory	Flow
Case 1.i.a $\gamma, \delta > 0$ and $\gamma > \delta$	$y_1 = c_1 e^{\gamma t}$ $y_2 = c_2 e^{\delta t}$	$y_2 = C_2' y_1^{\frac{\delta}{\gamma}}$
Case 1.i.b $\gamma > 0, \delta < 0$ $\Rightarrow \alpha-\beta = -\lvert\delta\rvert$	$y_1 = c_1 e^{\gamma t}$ $y_2 = c_2 e^{-\lvert\delta\rvert t}$	$y_2 = C_2' y_1^{\frac{\lvert\delta\rvert}{\gamma}}$
Case 1.ii.a $\gamma, \delta < 0$ $\Rightarrow \alpha+\beta = -\lvert\gamma\rvert$ $\Rightarrow \alpha-\beta = -\lvert\delta\rvert$ and $\gamma > \delta$	$y_1 = c_1 e^{-\lvert\gamma\rvert t}$ $y_2 = c_2 e^{-\lvert\delta\rvert t}$	$y_2 = C_2' y_1^{\lvert\frac{\delta}{\gamma}\rvert}$
Case 1.ii.b $\gamma > 0, \delta < 0$ $\Rightarrow \alpha-\beta = -\lvert\delta\rvert$	$y_1 = c_1 e^{\gamma t}$ $y_2 = c_2 e^{-\lvert\delta\rvert t}$	$y_2 = C_2' y_1^{\frac{\lvert\delta\rvert}{\gamma}}$
Case 2 $\gamma > 0, \delta < 0$	$y_1 = c_1 e^{\gamma t}$ $y_2 = c_2 e^{-\lvert\delta\rvert t}$	$y_2 = C_2' y_1^{\frac{\lvert\delta\rvert}{\gamma}}$
Case 3 $\gamma =$ positive constant $\delta = 0$	$y_1 = c_1 e^{\gamma t}$ $y_2 = c_2$	Note $C_1 = \frac{C_1}{C_2} \cdot C_2 = C_1' y_2$ $y_1 = C_1' y_2 e^{\gamma t}$

Same

Same







(2) Let $x' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$ and let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

Then $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \overbrace{\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}}^{m} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

$\Rightarrow (\alpha-\lambda)^2 = -\beta^2$

$\Rightarrow \alpha - \lambda = \pm i\beta$

$\Rightarrow \lambda = \alpha \pm i\beta$

$\Rightarrow \boxed{\lambda_1 = \alpha + i\beta \quad \text{and} \quad \lambda_2 = \alpha - i\beta}$

$\lambda_1 = \alpha + i\beta$ | $(m - \lambda_1 I)\vec{x} = \begin{pmatrix} \alpha-(\alpha+i\beta) & -\beta \\ \beta & \alpha-(\alpha+i\beta) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} -i\beta & -\beta \\ \beta & -i\beta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$-iR_1 + R_2 \to R_2 \Rightarrow \begin{pmatrix} -i\beta & -\beta \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

For $\beta \neq 0 \Rightarrow -i\beta x_1 - \beta x_2 = 0$

$x_1 = i x_2$

let $x_2 = 1 \Rightarrow x_1 = i$

$\Rightarrow \boxed{\vec{x} = c \begin{pmatrix} i \\ 1 \end{pmatrix}, \; c \in \mathbb{C} - \{0\}}$

$\lambda_2 = \alpha - i\beta$ | $(m - \lambda_2 I)x = \begin{pmatrix} \alpha-(\alpha-i\beta) & -\beta \\ \beta & \alpha-(\alpha-i\beta) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} i\beta & -\beta \\ \beta & i\beta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$iR_1 + R_2 \to R_2 \Rightarrow \begin{pmatrix} i\beta & -\beta \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$i\beta x_1 - \beta x_2 = 0$$

For $\beta \neq 0 \Rightarrow \quad i x_1 = x_2$

$$x_1 = -i x_2$$

Let $x_2 = 1 \Rightarrow \quad x_1 = -i$

$$\Rightarrow \boxed{\vec{x} = k\begin{pmatrix} -i \\ 1 \end{pmatrix}, \; k \in \mathbb{C} - \{0\}}$$

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \alpha + i\beta & 0 \\ 0 & \alpha - i\beta \end{pmatrix}$$

$$P = \frac{1}{\sqrt{2}}\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}, \quad P^{-1} = P^{T} = \frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}$$

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Inverse[1/Sqrt[2] (I -I
                   1  1)]
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MatrixForm=

$$\begin{pmatrix} -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$x' = Mx$ Change of Basis

$y' = Dy$

Tradjectory of (2)

$$\Rightarrow \left\{ \begin{aligned} y_1' &= (\alpha + i\beta) y_1 \\ y_2' &= (\alpha - i\beta) y_2 \end{aligned} \right\} = \left\{ \begin{aligned} y_1 &= c_1 e^{(\alpha + i\beta)t}, \; c_1 \in \mathbb{C} - \{0\} \\ y_2 &= c_2 e^{(\alpha - i\beta)t}, \; c_2 \in \mathbb{C} - \{0\} \end{aligned} \right.$$

Cases

✓(0) $\alpha = \beta = 0 \Rightarrow$ no eigenvalues

(1) $\alpha \neq 0, \beta \neq 0 \Rightarrow \lambda_1 = \alpha + \beta, \lambda_2 = \alpha - \beta$

- (i) $\alpha > 0$
  - (a) $|\alpha| > |\beta|$
  - (b) $|\alpha| < |\beta|$ ← Symmetric w/ case (a)
- (ii) $\alpha < 0$ ← symmetric w/ case (i)
  - (a) $|\alpha| > |\beta|$
  - (b) $|\alpha| < |\beta|$

(2) $\alpha = 0, \beta \neq 0 \Rightarrow \lambda_1 = +i\beta, \lambda_2 = -i\beta$

	Tradjectory	Flow
Case 1.i.a	$y_1 = c_1 e^{(\alpha+i\beta)t}$ $y_2 = c_2 e^{(\alpha-i\beta)t}$	Can't get flow since we cannot get $y_1 \sim y_2$ since they cannot be functions of each other
Case 2	$y_1 = c_1 e^{i\beta t}$ $y_2 = c_2 e^{-i\beta t}$	

$$\vec{x} = c\begin{pmatrix} i \\ 1 \end{pmatrix} e^{(\alpha+i\beta)t} + k\begin{pmatrix} -i \\ 1 \end{pmatrix} e^{(\alpha-i\beta)t}$$

$$= c\,i\,e^{\alpha t}(\cos(\beta t) + i\sin(\beta t)) + k(-i)e^{\alpha t}(\cos(\beta t) - i\sin(\beta t))$$

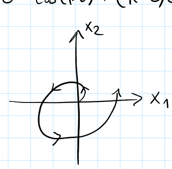
$$+ c(1)e^{\alpha t}(\cos(\beta t) + i\sin(\beta t)) + k(1)e^{\alpha t}(\cos(\beta t) - i\sin(\beta t))$$

$$= ce^{\alpha t} i\cos(\beta t) - ce^{\alpha t}\sin(\beta t) - ke^{\alpha t} i\cos(\beta t) + ke^{\alpha t}\sin(\beta t)$$

$$+ ce^{\alpha t}\cos(\beta t) + ce^{\alpha t} i\sin(\beta t) + ke^{\alpha t}\cos(\beta t) - ke^{\alpha t} i\sin(\beta t)$$

$$\mathrm{Re}(\vec{x}) = -ce^{\alpha t}\sin(\beta t) + ke^{\alpha t}\sin(\beta t) + ce^{\alpha t}\cos(\beta t) + ke^{\alpha t}\cos(\beta t)$$

$$= (k+c)e^{\alpha t}\cos(\beta t) + (k-c)e^{\alpha t}\sin(\beta t)$$



$$\vec{x} = c\begin{pmatrix} i \\ 1 \end{pmatrix} e^{i\beta t} + k\begin{pmatrix} -i \\ 1 \end{pmatrix} e^{-i\beta t}$$

$$= c(i)(\cos(\beta t) + i\sin(\beta t)) + k(-i)(\cos(\beta t) - i\sin(\beta t))$$

$$+ c(1)(\cos(\beta t) + i\sin(\beta t)) + k(1)(\cos(\beta t) + i\sin(\beta t))$$

$$= ci\cos(\beta t) - c\sin(\beta t) - ki\cos(\beta t) + k\sin(\beta t)$$

$$+ c\cos(\beta t) + ci\sin(\beta t) + k\cos(\beta t) + ki\sin(\beta t)$$

$$\mathrm{Re}(\vec{x}) = -c\sin(\beta t) + k\sin(\beta t) + c\cos(\beta t) + k\cos(\beta t)$$

$$= (k-c)\sin(\beta t) + (k+c)\cos(\beta t)$$