HW3 Repeat the above analysis

1) LINEARIZATION
2) Polar Coord & examine

for (a) $\begin{cases} \dot{x} = -y - x^3 - xy^2 \\ \dot{y} = x - y^3 - x^2 y \end{cases}$

and for (b) $\begin{cases} \dot{x} = -y + x^3 + xy^2 \\ \dot{y} = x + y^3 + x^2 y \end{cases}$

Linearization $\Rightarrow \underbrace{f(x) = f(0) + Df(0)\,x}\, + O(x^2)$

D matrix? $\Rightarrow f(x) = \begin{cases} \dot{x} \\ \dot{y} \end{cases} \Rightarrow Df(\vec{x}) = \begin{pmatrix} \dot{x}_x & \dot{x}_y \\ \dot{y}_x & \dot{y}_y \end{pmatrix}$

Polar Coords | $\dot{r} = \dfrac{x\dot{x} + y\dot{y}}{r}$ & $\dot{\theta} = \dfrac{x\dot{y} - y\dot{x}}{r^2}$

1a) Let $\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}$, then $Df(\vec{x}) = \begin{pmatrix} (\dot{x}_1)_{x_1} & (\dot{x}_1)_{x_2} \\ (\dot{x}_2)_{x_1} & (\dot{x}_2)_{x_2} \end{pmatrix}$

$$= \begin{pmatrix} -3x_1^2 - x_2^2 & -1 - 3x_2^2 - x_1^2 \\ 1 - 2x_1x_2 & -3x_2^2 - x_1^2 \end{pmatrix}$$

$$\Rightarrow Df(\vec{0}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Linearization of $\dot{x} = f(0) + Df(0)\,x$

$$= 0 + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -x_2 - x_1^3 - x_1x_2^2 \\ x_1 - x_2^3 - x_1^2 x_2 \end{pmatrix}$$

$$\Rightarrow \dot{x} = \overbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}^{A} x$$

$\Rightarrow \lambda = \pm i$

$\Rightarrow (A - \lambda I)\eta = 0 \Rightarrow (A \mp iI)\eta = 0$

$\Rightarrow \mp i\eta_1 - \eta_2 \Rightarrow \mp i\eta_1 = \eta_2$

$(\eta_1 \mp i\eta_2 = 0)$ ← redundant solution

$\Rightarrow \eta = c_k \begin{pmatrix} \mp i \\ 1 \end{pmatrix}$, $c_k \in \mathbb{C} - \{0\}$ for $k \in \{1,2\}$

$\Rightarrow$ Generalized Sol|

$$x = c_1 \begin{pmatrix} -i \\ 1 \end{pmatrix} e^{it} + c_2 \begin{pmatrix} i \\ 1 \end{pmatrix} e^{-it}$$

$$x = C_1 \begin{pmatrix} -i \\ 1 \end{pmatrix} e^{it} + C_2 \begin{pmatrix} i \\ 1 \end{pmatrix} e^{-it}$$

$$= C_1 \left(-i \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) (\cos(t) + i\sin(t))$$

$$+ C_2 \left(i \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) (\cos(t) - i\sin(t))$$

$$= C_1(-i)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos(t) + C_1\cancel{(-i)}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cancel{i}\sin(t)$$

$$+ C_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos(t) + C_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} i\sin(t)$$

$$+ C_2(i)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos(t) + C_2\cancel{(i)}\begin{pmatrix} 1 \\ 0 \end{pmatrix}(\cancel{-i}\sin(t))$$

$$+ C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos(t) + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}(-i\sin(t))$$

$$= C_1(-i)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos(t) + C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin(t)$$

$$+ C_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos(t) + C_1(i)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin(t)$$

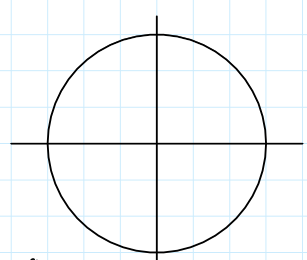
$$+ C_2(i)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos(t) + C_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin(t)$$

$$+ C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos(t) + C_2(-i)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin(t)$$

$$= (C_1 + C_2)\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos(t) \right)$$

$$+ (i)(C_1 - C_2)\left(\begin{pmatrix} -1 \\ 0 \end{pmatrix}\cos(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin(t) \right)$$

$$\Rightarrow \boxed{\begin{aligned} Re(x) &= (c_1 + c_2)\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos(t) \right) \\ Im(x) &= i(c_1 - c_2)\left(\begin{pmatrix} -1 \\ 0 \end{pmatrix}\cos(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin(t) \right) \end{aligned}}$$



2a) Polar Coords | $\dot{r} = \frac{x_1\dot{x}_1 + x_2\dot{x}_2}{r}$ & $\dot{\theta} = \frac{x_1\dot{x}_2 - x_2\dot{x}_1}{r^2}$

$$\dot{r} = \underline{x_1(-x_2 - x_1^3 - x_1x_2^2) + x_2(x_1 - x_2^3 - x_1^2x_2)}$$

r

$$= \frac{-\cancel{x_1x_2} - x_1^4 - x_1^2x_2^2 + \cancel{x_1x_2} - x_2^4 - x_1^2x_2^2}{r}$$

$$= \frac{-\left(x_1^4 + x_2^4 + 2x_1^2x_2^2\right)}{r}$$

$$\dot{\theta} = \frac{x_1\left(x_1 - x_2^3 - x_1^2x_2\right) - x_2\left(-x_2 - x_1^3 - x_1x_2^2\right)}{r^2}$$

$$= \frac{x_1^2 - \cancel{x_1x_2^3} - \cancel{x_1^3x_2} + x_2^2 + \cancel{x_1^3x_2} + \cancel{x_1x_2^3}}{r^2}$$

$$= 1$$

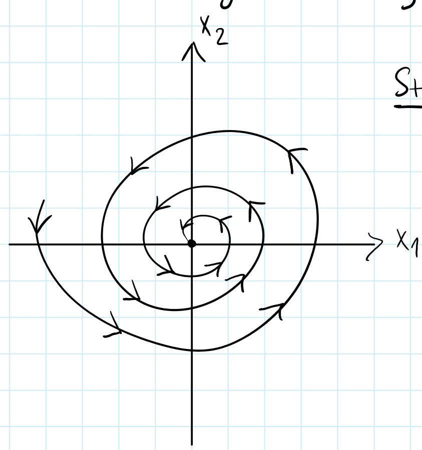
Part 2a

```
x1dot = -x2 - x1^3 - x1 x2^2;
x2dot = x1 - x2^3 - x1^2 x2;
r = Sqrt[x1^2 + x2^2];
rdot = (x1 x1dot + x2 x2dot)/r // FullSimplify
thetadot = (x1 x2dot - x2 x1dot)/r^2 // FullSimplify
```

$-\left(x1^2 + x2^2\right)^{3/2}$

1

Since $\dot{r} \leq 0 \ \forall x_1, x_2$, radius is decreasing w/ time.
Sine $\dot{\theta} = 1 > 0$, angle is increasing w/ time.



Stable

1b) Let $\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}$. Then $Df(x) = \begin{pmatrix} (\dot{x}_1)_{x_1} & (\dot{x}_1)_{x_2} \\ (\dot{x}_2)_{x_1} & (\dot{x}_2)_{x_2} \end{pmatrix}$

$$= \begin{pmatrix} 3x_1^2 + x_2^2 & -1 + 2x_1x_2 \\ 1 + 2x_1x_2 & 3x_2^2 + x_1^2 \end{pmatrix}$$

$$\Rightarrow Df(0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$\Rightarrow$ Linearization of $\dot{x} = f(0) + Df(0)x$

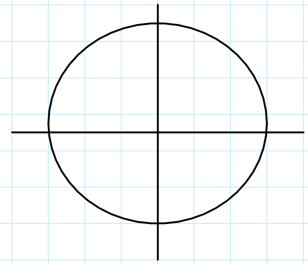
$$= 0 + \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{A} x$$

$\Rightarrow \; \dot{x} = Ax$

$\Rightarrow \; \lambda = \pm i$

$\Rightarrow$ Same resultas in part 1a.

$$\Rightarrow \boxed{\begin{array}{l} Re(x) = (c_1+c_2)\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(t) \right) \\[2ex] Im(x) = i(c_1-c_2)\left(\begin{pmatrix} -1 \\ 0 \end{pmatrix} \cos(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(t) \right) \end{array}}$$



2b) Polar Coords $\quad \dot{r} = \dfrac{x_1\dot{x}_1 + x_2\dot{x}_2}{r}$ & $\dot{\theta} = \dfrac{x_1\dot{x}_2 - x_2\dot{x}_1}{r^2}$

$$\dot{r} = \frac{x_1(-x_2 + x_1^3 + x_1x_2^2) + x_2(x_1 + x_2^3 + x_1^2x_2)}{r}$$

$$= \frac{\cancel{-x_1x_2} + x_1^4 + x_1^2x_2^2 + \cancel{x_1x_2} + x_2^4 + x_1^2x_2^2}{r}$$

$$= \frac{x_1^4 + x_2^4 + 2x_1^2x_2^2}{r}$$

$$\dot{\theta} = \frac{x_1(x_1 + x_2^3 + x_1^2x_2) - x_2(-x_2 + x_1^3 + x_1x_2^2)}{r^2}$$

$$= \frac{x_1^2 + \cancel{x_1x_2^3} + \cancel{x_1^3x_2} + x_2^2 - \cancel{x_1^3x_2} - \cancel{x_1x_2^3}}{r^2}$$

$$= 1$$

Part 2b

```
x1dot = -x2 + x1^3 + x1 x2^2;
x2dot = x1 + x2^3 + x1^2 x2;
r = Sqrt[x1^2 + x2^2];
rdot = (x1 x1dot + x2 x2dot)/r // FullSimplify
thetadot = (x1 x2dot - x2 x1dot)/r^2 // FullSimplify
```

$(x1^2 + x2^2)^{3/2}$

1

Since $\dot{r} \geq 0 \; \forall x_1, x_2$, radius is increasing w/ time.
Since $\dot{\theta} = 1 > 0$, angle is increasing w/ time.

Unstable