In[327]:=

```
Clear["Global`*"];
```

In[328]:=

```
x[τ] = x0[τ] + ϵ x1[τ] + ϵ^2 x2[τ];
ode = ω^2 D[x[τ], {τ, 2}] + ϵ (ω^3 D[x[τ], {τ, 1}]^3/3 - ω D[x[τ], {τ, 1}]) + x[τ] == 0 /.
   ω → 1 + ϵ ω1 + ϵ^2 ω2
Collect[
 ode,
 ϵ]
```

$$x[\tau] = x_0[\tau] + \epsilon\, x_1[\tau] + \epsilon^2 x_2[\tau];$$

$$\text{ode} = \omega^2 D[x[\tau], \{\tau, 2\}] + \epsilon \left(\frac{\omega^3 D[x[\tau], \{\tau, 1\}]^3}{3} - \omega D[x[\tau], \{\tau, 1\}]\right) + x[\tau] == 0 \;/.\; \omega \to 1 + \epsilon\, \omega_1 + \epsilon^2 \omega_2$$

Out[329]=

$$x_0[\tau] + \epsilon\, x_1[\tau] + \epsilon^2 x_2[\tau] + \epsilon \left(-\left(\left(1 + \epsilon\, \omega_1 + \epsilon^2 \omega_2\right) \left(x_0'[\tau] + \epsilon\, x_1'[\tau] + \epsilon^2 x_2'[\tau]\right)\right) + \frac{1}{3} \left(1 + \epsilon\, \omega_1 + \epsilon^2 \omega_2\right)^3 \left(x_0'[\tau] + \epsilon\, x_1'[\tau] + \epsilon^2 x_2'[\tau]\right)^3\right) + \left(1 + \epsilon\, \omega_1 + \epsilon^2 \omega_2\right)^2 \left(x_0''[\tau] + \epsilon\, x_1''[\tau] + \epsilon^2 x_2''[\tau]\right) == 0$$

Out[330]=

$$
\begin{aligned}
& x_0[\tau] + \frac{1}{3}\epsilon^{13}\omega_2^3 x_2'[\tau]^3 + \\
& \epsilon^7 \Big(\frac{1}{3}\omega_2^3 x_0'[\tau]^3 + 3\omega_1\omega_2^2 x_0'[\tau]^2 x_1'[\tau] + 3\omega_1^2\omega_2 x_0'[\tau]\, x_1'[\tau]^2 + 3\omega_2^2 x_0'[\tau]\, x_1'[\tau]^2 + \\
& \quad \frac{1}{3}\omega_1^3 x_1'[\tau]^3 + 2\omega_1\omega_2 x_1'[\tau]^3 + 3\omega_1^2\omega_2 x_0'[\tau]^2 x_2'[\tau] + 3\omega_2^2 x_0'[\tau]^2 x_2'[\tau] + 2\omega_1^3 x_0'[\tau] \\
& \quad x_1'[\tau]\, x_2'[\tau] + 12\omega_1\omega_2 x_0'[\tau]\, x_1'[\tau]\, x_2'[\tau] + 3\omega_1^2 x_1'[\tau]^2 x_2'[\tau] + 3\omega_2 x_1'[\tau]^2 x_2'[\tau] + \\
& \quad 3\omega_1^2 x_0'[\tau]\, x_2'[\tau]^2 + 3\omega_2 x_0'[\tau]\, x_2'[\tau]^2 + 3\omega_1 x_1'[\tau]\, x_2'[\tau]^2 + \frac{1}{3} x_2'[\tau]^3\Big) + \\
& \epsilon^8 \Big(\omega_2^3 x_0'[\tau]^2 x_1'[\tau] + 3\omega_1\omega_2^2 x_0'[\tau]\, x_1'[\tau]^2 + \omega_1^2\omega_2 x_1'[\tau]^3 + \omega_2^2 x_1'[\tau]^3 + \\
& \quad 3\omega_1\omega_2^2 x_0'[\tau]^2 x_2'[\tau] + 6\omega_1^2\omega_2 x_0'[\tau]\, x_1'[\tau]\, x_2'[\tau] + 6\omega_2^2 x_0'[\tau]\, x_1'[\tau]\, x_2'[\tau] + \\
& \quad \omega_1^3 x_1'[\tau]^2 x_2'[\tau] + 6\omega_1\omega_2 x_1'[\tau]^2 x_2'[\tau] + \omega_1^3 x_0'[\tau]\, x_2'[\tau]^2 + \\
& \quad 6\omega_1\omega_2 x_0'[\tau]\, x_2'[\tau]^2 + 3\omega_1^2 x_1'[\tau]\, x_2'[\tau]^2 + 3\omega_2 x_1'[\tau]\, x_2'[\tau]^2 + \omega_1 x_2'[\tau]^3\Big) + \\
& \epsilon^9 \Big(\omega_2^3 x_0'[\tau]\, x_1'[\tau]^2 + \omega_1\omega_2^2 x_1'[\tau]^3 + \omega_2^3 x_0'[\tau]^2 x_2'[\tau] + 6\omega_1\omega_2^2 x_0'[\tau]\, x_1'[\tau]\, x_2'[\tau] + \\
& \quad 3\omega_1^2\omega_2 x_1'[\tau]^2 x_2'[\tau] + 3\omega_2^2 x_1'[\tau]^2 x_2'[\tau] + 3\omega_1^2\omega_2 x_0'[\tau]\, x_2'[\tau]^2 + 3\omega_2^2 x_0'[\tau]\, x_2'[\tau]^2 + \\
& \quad \omega_1^3 x_1'[\tau]\, x_2'[\tau]^2 + 6\omega_1\omega_2 x_1'[\tau]\, x_2'[\tau]^2 + \omega_1^2 x_2'[\tau]^3 + \omega_2 x_2'[\tau]^3\Big) + \\
& \epsilon^{10} \Big(\frac{1}{3}\omega_2^3 x_1'[\tau]^3 + 2\omega_2^3 x_0'[\tau]\, x_1'[\tau]\, x_2'[\tau] + 3\omega_1\omega_2^2 x_1'[\tau]^2 x_2'[\tau] + 3\omega_1\omega_2^2 x_0'[\tau]\, x_2'[\tau]^2 + \\
& \quad 3\omega_1^2\omega_2 x_1'[\tau]\, x_2'[\tau]^2 + 3\omega_2^2 x_1'[\tau]\, x_2'[\tau]^2 + \frac{1}{3}\omega_1^3 x_2'[\tau]^3 + 2\omega_1\omega_2 x_2'[\tau]^3\Big) + \\
& \epsilon^{11} \Big(\omega_2^3 x_1'[\tau]^2 x_2'[\tau] + \omega_2^3 x_0'[\tau]\, x_2'[\tau]^2 + 3\omega_1\omega_2^2 x_1'[\tau]\, x_2'[\tau]^2 + \omega_1^2\omega_2 x_2'[\tau]^3 + \omega_2^2 x_2'[\tau]^3\Big) + \\
& \epsilon^{12} \Big(\omega_2^3 x_1'[\tau]\, x_2'[\tau]^2 + \omega_1\omega_2^2 x_2'[\tau]^3\Big) + x_0''[\tau] + \\
& \epsilon \Big(x_1[\tau] - x_0'[\tau] + \frac{1}{3} x_0'[\tau]^3 + 2\omega_1 x_0''[\tau] + x_1''[\tau]\Big) + \\
& \epsilon^2 \Big(x_2[\tau] - \omega_1 x_0'[\tau] + \omega_1 x_0'[\tau]^3 - x_1'[\tau] + \\
& \quad x_0'[\tau]^2 x_1'[\tau] + \omega_1^2 x_0''[\tau] + 2\omega_2 x_0''[\tau] + 2\omega_1 x_1''[\tau] + x_2''[\tau]\Big) + \\
& \epsilon^3 \Big(-\omega_2 x_0'[\tau] + \omega_1^2 x_0'[\tau]^3 + \omega_2 x_0'[\tau]^3 - \omega_1 x_1'[\tau] + 3\omega_1 x_0'[\tau]^2 x_1'[\tau] + x_0'[\tau]\, x_1'[\tau]^2 - \\
& \quad x_2'[\tau] + x_0'[\tau]^2 x_2'[\tau] + 2\omega_1\omega_2 x_0''[\tau] + \omega_1^2 x_1''[\tau] + 2\omega_2 x_1''[\tau] + 2\omega_1 x_2''[\tau]\Big) + \\
& \epsilon^4 \Big(\frac{1}{3}\omega_1^3 x_0'[\tau]^3 + 2\omega_1\omega_2 x_0'[\tau]^3 - \omega_2 x_1'[\tau] + 3\omega_1^2 x_0'[\tau]^2 x_1'[\tau] + 3\omega_2 x_0'[\tau]^2 x_1'[\tau] + \\
& \quad 3\omega_1 x_0'[\tau]\, x_1'[\tau]^2 + \frac{1}{3} x_1'[\tau]^3 - \omega_1 x_2'[\tau] + 3\omega_1 x_0'[\tau]^2 x_2'[\tau] + \\
& \quad 2 x_0'[\tau]\, x_1'[\tau]\, x_2'[\tau] + \omega_2^2 x_0''[\tau] + 2\omega_1\omega_2 x_1''[\tau] + \omega_1^2 x_2''[\tau] + 2\omega_2 x_2''[\tau]\Big) + \\
& \epsilon^5 \Big(\omega_1^2\omega_2 x_0'[\tau]^3 + \omega_2^2 x_0'[\tau]^3 + \omega_1^3 x_0'[\tau]^2 x_1'[\tau] + 6\omega_1\omega_2 x_0'[\tau]^2 x_1'[\tau] + 3\omega_1^2 x_0'[\tau]\, x_1'[\tau]^2 + \\
& \quad 3\omega_2 x_0'[\tau]\, x_1'[\tau]^2 + \omega_1 x_1'[\tau]^3 - \omega_2 x_2'[\tau] + 3\omega_1^2 x_0'[\tau]^2 x_2'[\tau] + 3\omega_2 x_0'[\tau]^2 x_2'[\tau] + \\
& \quad 6\omega_1 x_0'[\tau]\, x_1'[\tau]\, x_2'[\tau] + x_1'[\tau]^2 x_2'[\tau] + x_0'[\tau]\, x_2'[\tau]^2 + \omega_2^2 x_1''[\tau] + 2\omega_1\omega_2 x_2''[\tau]\Big) + \\
& \epsilon^6 \Big(\omega_1\omega_2^2 x_0'[\tau]^3 + 3\omega_1^2\omega_2 x_0'[\tau]^2 x_1'[\tau] + 3\omega_2^2 x_0'[\tau]^2 x_1'[\tau] + \omega_1^3 x_0'[\tau]\, x_1'[\tau]^2 + \\
& \quad 6\omega_1\omega_2 x_0'[\tau]\, x_1'[\tau]^2 + \omega_1^2 x_1'[\tau]^3 + \omega_2 x_1'[\tau]^3 + \omega_1^3 x_0'[\tau]^2 x_2'[\tau] + \\
& \quad 6\omega_1\omega_2 x_0'[\tau]^2 x_2'[\tau] + 6\omega_1^2 x_0'[\tau]\, x_1'[\tau]\, x_2'[\tau] + 6\omega_2 x_0'[\tau]\, x_1'[\tau]\, x_2'[\tau] + \\
& \quad 3\omega_1 x_1'[\tau]^2 x_2'[\tau] + 3\omega_1 x_0'[\tau]\, x_2'[\tau]^2 + x_1'[\tau]\, x_2'[\tau]^2 + \omega_2^2 x_2''[\tau]\Big) == 0
\end{aligned}
$$

In[331]:=

```
DSolve[{x0[τ] + x0''[τ] == 0, x0[0] == a0, x0'[0] == 0}, x0[τ], τ]
d0 = Cos[τ] a0;
```

Out[331]= $\{\{x_0[\tau] \to \mathrm{Cos}[\tau]\, a_0\}\}$

```
TrigExpand[
 DSolve[{x1[τ] - (D[d0, {τ, 1}]) + 1/3 (D[d0, {τ, 1}])^3 + 2 ω1 (D[d0, {τ, 2}]) + x1''[τ] == 0,
   x1[0] == 0, x1'[0] == 0}, x1[τ], τ]]
Expand[FullSimplify[1/2 τ Cos[τ] a0 - 1/2 Sin[τ] a0 - 1/8 τ Cos[τ] a0^3 + 3/32 Sin[τ] a0^3 + 1/32 Cos[τ]^2
     Sin[τ] a0^3 - 1/96 Sin[τ]^3 a0^3 + τ Sin[τ] a0 ω1]] /. {τ Cos[τ] → 0, τ Sin[τ] → 0}
d1 = Expand[1/96 × (48 (-Sin[τ]) a0 + (+9 Sin[τ] + Sin[3 τ]) a0^3)];
```

Out[333]= $\left\{\left\{x_1[\tau] \to \frac{1}{2}\tau\,\mathrm{Cos}[\tau]\, a_0 - \frac{1}{2}\mathrm{Sin}[\tau]\, a_0 - \frac{1}{8}\tau\,\mathrm{Cos}[\tau]\, a_0^3 + \frac{3}{32}\mathrm{Sin}[\tau]\, a_0^3 + \frac{1}{32}\mathrm{Cos}[\tau]^2\,\mathrm{Sin}[\tau]\, a_0^3 - \frac{1}{96}\mathrm{Sin}[\tau]^3\, a_0^3 + \tau\,\mathrm{Sin}[\tau]\, a_0\,\omega_1\right\}\right\}$

Out[334]= $-\frac{1}{2}\mathrm{Sin}[\tau]\, a_0 + \frac{3}{32}\mathrm{Sin}[\tau]\, a_0^3 + \frac{1}{96}\mathrm{Sin}[3\,\tau]\, a_0^3$

In[353]:=

```
TrigExpand[DSolve[{x2[τ] - ω1 (D[d0, {τ, 1}]) + ω1 (D[d0, {τ, 1}])^3 - (D[d1, {τ, 1}]) +
    (D[d0, {τ, 1}])^2 (D[d0, {τ, 1}]) + ω1^2 (D[d0, {τ, 2}]) + 2 ω2 (D[d0, {τ, 2}]) +
    2 ω1 (D[d1, {τ, 2}]) + x2''[τ] == 0, x2[0] == 0, x2'[0] == 0}, x2[τ], τ]]
Expand[FullSimplify[-1/4 τ Sin[τ] a0 + 1/256 Cos[τ] a0^3 - 3/8 τ Cos[τ] a0^3 -
    1/256 Cos[τ]^3 a0^3 + 9/32 Sin[τ] a0^3 + 3/64 τ Sin[τ] a0^3 + 3/32 Cos[τ]^2 Sin[τ] a0^3 +
    3/256 Cos[τ] Sin[τ]^2 a0^3 - 1/32 Sin[τ]^3 a0^3 + τ Cos[τ] a0 ω1 - Sin[τ] a0 ω1 -
    15/32 τ Cos[τ] a0^3 ω1 + 57/128 Sin[τ] a0^3 ω1 + 3/128 Cos[τ]^2 Sin[τ] a0^3 ω1 - 1/128 Sin[τ]^3 a0^3 ω1 +
    1/2 τ Sin[τ] a0 ω1^2 + τ Sin[τ] a0 ω2]] /. {τ Cos[τ] → 0, τ Sin[τ] → 0}
d2 = 1/256 Cos[τ] a0^3 - 1/256 Cos[3 τ] a0^3 + 9/32 Sin[τ] a0^3 + 1/32 Sin[3 τ] a0^3;
```

Out[353]=

$$\left\{\left\{x_2[\tau] \to -\frac{1}{4}\tau \operatorname{Sin}[\tau]\, a_0 + \frac{1}{256}\operatorname{Cos}[\tau]\, a_0^3 - \frac{3}{8}\tau\operatorname{Cos}[\tau]\, a_0^3 - \frac{1}{256}\operatorname{Cos}[\tau]^3 a_0^3 + \frac{9}{32}\operatorname{Sin}[\tau]\, a_0^3 + \frac{3}{64}\tau\operatorname{Sin}[\tau]\, a_0^3 + \frac{3}{32}\operatorname{Cos}[\tau]^2\operatorname{Sin}[\tau]\, a_0^3 + \frac{3}{256}\operatorname{Cos}[\tau]\operatorname{Sin}[\tau]^2 a_0^3 - \frac{1}{32}\operatorname{Sin}[\tau]^3 a_0^3 + \tau\operatorname{Cos}[\tau]\, a_0\,\omega_1 - \operatorname{Sin}[\tau]\, a_0\,\omega_1 - \frac{15}{32}\tau\operatorname{Cos}[\tau]\, a_0^3\,\omega_1 + \frac{57}{128}\operatorname{Sin}[\tau]\, a_0^3\,\omega_1 + \frac{3}{128}\operatorname{Cos}[\tau]^2\operatorname{Sin}[\tau]\, a_0^3\,\omega_1 - \frac{1}{128}\operatorname{Sin}[\tau]^3 a_0^3\,\omega_1 + \frac{1}{2}\tau\operatorname{Sin}[\tau]\, a_0\,\omega_1^2 + \tau\operatorname{Sin}[\tau]\, a_0\,\omega_2\right\}\right\}$$

Out[354]=

$$\frac{1}{256}\operatorname{Cos}[\tau]\, a_0^3 - \frac{1}{256}\operatorname{Cos}[3\tau]\, a_0^3 + \frac{9}{32}\operatorname{Sin}[\tau]\, a_0^3 + \frac{1}{32}\operatorname{Sin}[3\tau]\, a_0^3 - \operatorname{Sin}[\tau]\, a_0\,\omega_1 + \frac{57}{128}\operatorname{Sin}[\tau]\, a_0^3\,\omega_1 + \frac{1}{128}\operatorname{Sin}[3\tau]\, a_0^3\,\omega_1$$

In[359]:=

```
d0 + ϵ d1 + ϵ^2 d2
```

Out[359]=

$$\operatorname{Cos}[\tau]\, a_0 + \epsilon\left(-\frac{1}{2}\operatorname{Sin}[\tau]\, a_0 + \frac{3}{32}\operatorname{Sin}[\tau]\, a_0^3 + \frac{1}{96}\operatorname{Sin}[3\tau]\, a_0^3\right) + \epsilon^2\left(\frac{1}{256}\operatorname{Cos}[\tau]\, a_0^3 - \frac{1}{256}\operatorname{Cos}[3\tau]\, a_0^3 + \frac{9}{32}\operatorname{Sin}[\tau]\, a_0^3 + \frac{1}{32}\operatorname{Sin}[3\tau]\, a_0^3\right)$$