HW4

Saturday, September 25, 2021 8:48 PM

Use $V(x) = x_1^2 + x_2^2$ on

1.

$$f(x) = \begin{cases} \dot{x_1} = -x_2 - x_1 x_2 \\ \dot{x_2} = x_1 + x_1^2 \end{cases}$$

what is the conclusion? On what set does the flow lie?

2.

$$g(x) = \begin{cases} \dot{x_1} = -x_2 - x_1 x_2^2 - x_1^3 \\ \dot{x_2} = x_1 - x_2 x_1^2 - x_2^3 \end{cases}$$

What is the conclusion?

3.

$$h(x) = \begin{cases} \dot{x_1} = -x_2 + x_1 x_2^2 + x_1^3 \\ \dot{x_2} = x_1 + x_2 x_1^2 + x_2^3 \end{cases}$$

What is the conclusion?

4. What does the linearization say for all three cases?

#1 $\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -x_2 - x_1 x_2 \\ x_1 + x_1^2 \end{pmatrix}$ and $x(0) = c$

$$\boxed{V(x) = x_1^2 + x_2^2}$$

$$\Rightarrow \dot{V} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$$

$$= 2x_1(-x_2 - x_1x_2) + 2x_2(x_1 + x_1^2)$$

$$= \cancel{-2x_1x_2} - \cancel{2x_1^2x_2} + \cancel{2x_2x_1} + \cancel{2x_2x_1^2} = 0 \quad \forall x.$$

$$\left.\begin{aligned} V = \int \dot{V}\,dt &= 2\int x_1\dot{x}_1\,dt + 2\int x_2\dot{x}_2\,dt \\ &= 2\left(\tfrac{1}{2}x_1^2\right) + 2\left(\tfrac{1}{2}x_2^2\right) \end{aligned}\right\} (*)$$

$$= x_1^2 + x_2^2 = \text{constant on } \varphi_t(c) \quad \text{and } V(0) = 0.$$

Let $x = \varphi_t(c)$, then $x_1 = c_2$ and $x_2 = c_2$, so $V(\varphi_t(c)) = c_1^2 + c_2^2$

Conclusion $\left.\begin{aligned} &V(x) \geq 0 \ \forall x \text{ and } V(0) = 0. \\ &\dot{V}(x) \leq 0 \ \forall x \text{ and } \dot{V}(0) = 0. \end{aligned}\right\}$ Thus our system is stable.

Flow lies on a circle of radius k centered at the origin.

Flow lies on a circle of radius K centered at the origin.

$$\varphi_t(c) = \{ c \mid c_1^2 + c_2^2 = K^2 \text{ for some } K \in \mathbb{R} \}.$$

#2 $\dot{V} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$

$$= 2x_1(-x_2 - x_1x_2^2 - x_1^3) + 2x_2(x_1 - x_2x_1^2 - x_2^3)$$

$$= \cancel{-2x_1x_2} - 2x_1^2x_2^2 - 2x_1^4 + \cancel{2x_2x_1} - 2x_2^2x_1^2 - 2x_2^4$$

$$= -4x_1^2x_2^2 - 2x_1^4 - 2x_2^4 < 0 \quad \forall x.$$

$V = \int \dot{V}\,dt$ by $(*)$

Let $x = \varphi_t(c)$, then $V(\varphi_t(c)) = c_1^2 + c_2^2$.

Conclusion Flow spirals around the origin with decreasing radius eventually reaching $\vec{0}$.

#3 $\dot{V} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$

$$= 2x_1(-x_2 + x_1x_2^2 + x_1^3) + 2x_2(x_1 + x_2x_1^2 + x_2^3)$$

$$= \cancel{-2x_1x_2} + 2x_1^2x_2^2 + 2x_1^4 + \cancel{2x_2x_1} + 2x_2^2x_1^2 + 2x_2^4$$

$$= 4x_1^2x_2^2 + 2x_1^4 + 2x_2^4 \geq 0 \quad \forall x.$$

$V = \int \dot{V}\,dt$ by $(*)$ ← [illegible]

$V = \int \dot{V}\,dt$ by $(*)$ ← What is the purpose of this step?

Let $x = \varphi_t(c)$, then $V(\varphi_t(c)) = c_1^2 + c_2^2$

<u>Conclusion</u> Flow spirals around the origin with increasing radius never reaching equilibrium.

#4 Linearization of $f(x)$: $f(x) = f(0) + Df(0) + O(x^2)$

<u>Linearization of #1</u>

$$Df(x) = \begin{pmatrix} (\dot{x}_1)_{x_1} & (\dot{x}_1)_{x_2} \\ (\dot{x}_2)_{x_1} & (\dot{x}_2)_{x_2} \end{pmatrix}$$

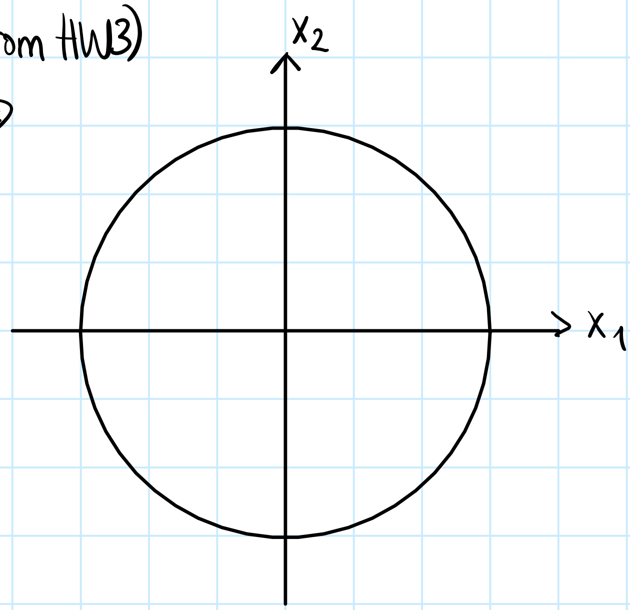
$$= \begin{pmatrix} -x_2 & -1-x_1 \\ 1+2x_1 & 0 \end{pmatrix}$$

$$Df(0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$f(x) = 0 + Df(0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(from HW3)

$\Longrightarrow$



Linearization of #2

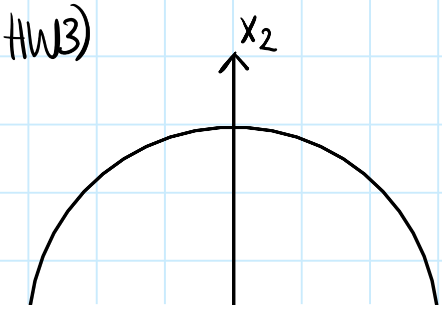
just look at coeff's of linear terms for each $\dot{x}_i$

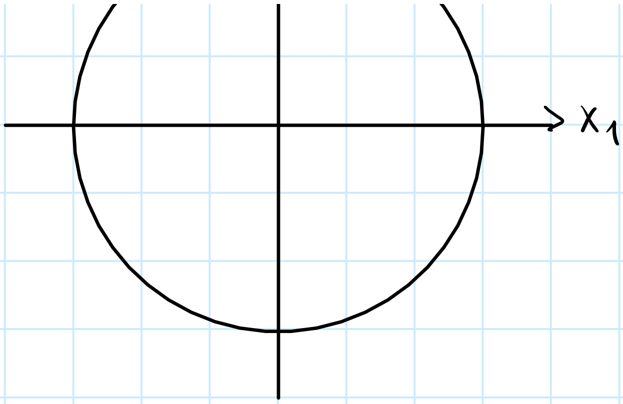
$$f(x) = f(0) + Df(0)$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(from HW3)

$\Longrightarrow$





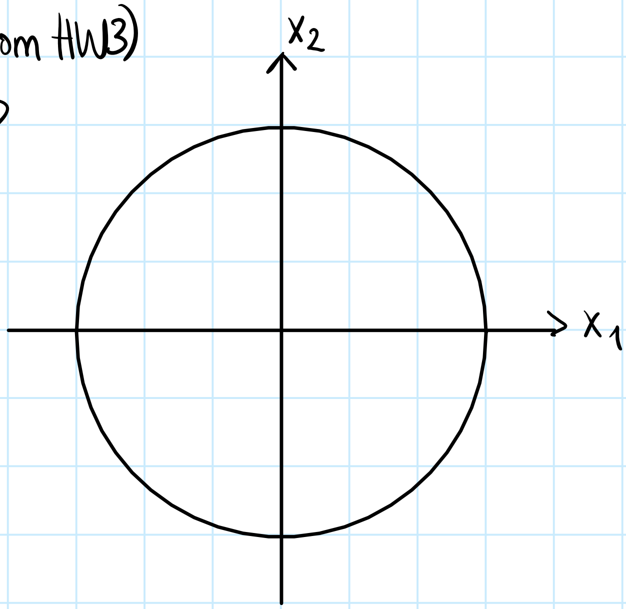
Linearization of #2 |

$f(x) = f(0) + Df(0)$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(from HW3)

$\Longrightarrow$



Conclusion | All 3 have the same linearization, but

this is just a first order approximation, so we would expect different behavior with higher order terms included as each f, g, h are different w/ different $\dot{V}$ for each $f, g,$ and h.