## 3 The Shell Model

Liquid drop model and Fermi gas model describes the gross properties of nuclei. No specifics. In the 1930s Barlett and Elsaser pointed out Magic numbers.

If A or N= 2, 8, 20, 28, 50, 82, 126⋯

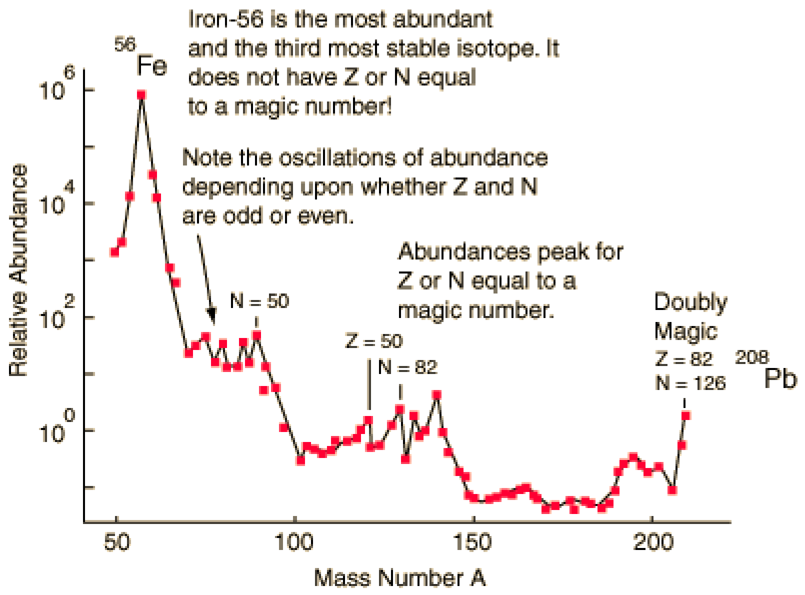
The Nuclei are very stable. Tried to understand them in terms of N, P moving in V(x) central potential, but was not accepted at first because nuclear force is short ranged. Although they could explain 2, 8, 20.

→ In 1949, Goeppert Mayer and J. H. D. Jensen could explain all using spin orbit effects.

**I) What is a magic number**.

1) Relative abundance

Nuclei with Z or N=magic number are abundant in Nature.




b) Separation Energy (of last nucleon)

Atomic example: the peak in separation energy or ionization energy comes from the shell structure.

Nuclear Analogy: for fixed Z,

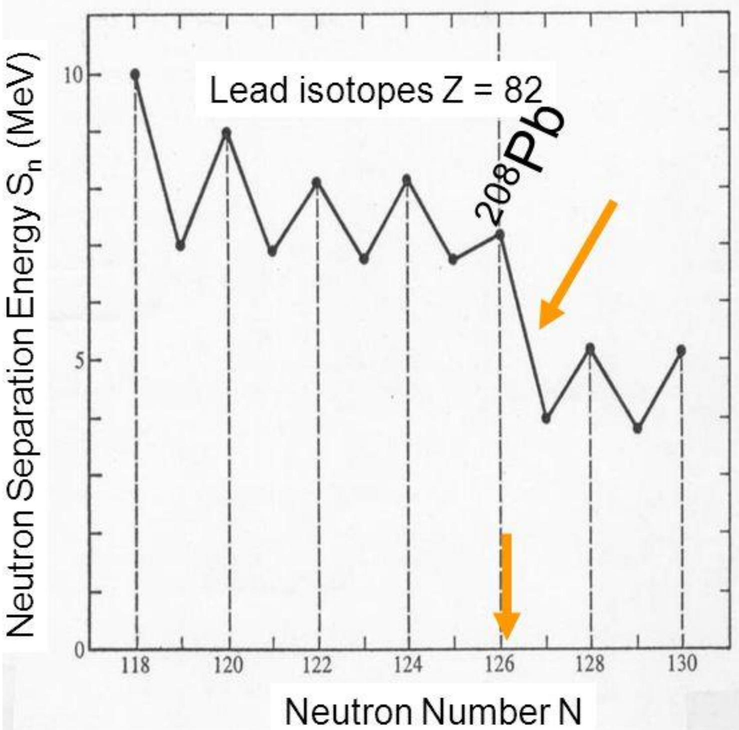
$$S_N(Z,N) = B(Z,N) - B(Z,N-1)$$

# Nuclear Binding Energy

For constant Z
$S_n$ (even N) > $S_n$ (odd N)
For constant N
$S_p$ (even Z) > $S_p$ (odd Z)

Remember HW 14 (Krane 3.14).

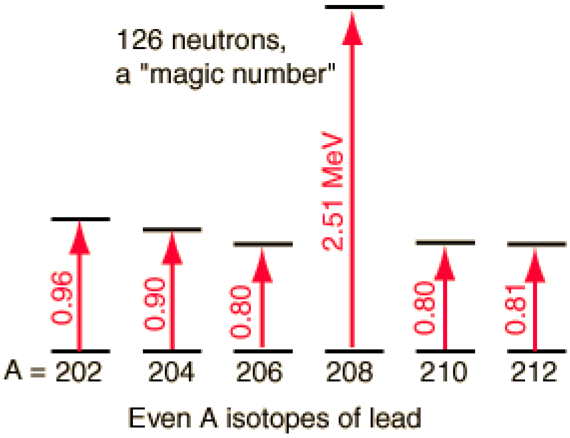
$^{208}$Pb (doubly magic) ► can then easily remove the "extra" neutron in $^{209}$Pb.






→ Two effects: Pairing and shell

2) Magic nuclei also have large excitation energy



**II) The closed shell**

1) Nuclear central potential: Start from two-body force

$V_{12} = V_0 f(x_1 - x_2)$

Average potential of 1 due to other nucleons

$V(1) = V_0 \int d^3x_2 f(x_1 - x_2)\rho(x_2)$ where $\rho$ is the nuclear density distribution ,

If force is short ranges, $f(x_1 - x_2) = C\delta(x_1 - x_2)$ where $\int dx_1^3 f(x_1 - x_2) = C = \int dx^3 f(x)$

$V(1) = CV_0 \rho(x_1)$

**III) One dimensional simple harmonic oscillator**

Schrödiner equation

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}kx^2\right)\psi = E\psi, \qquad \text{where} \quad k = m\omega^2$$

Eigenfuntions and eigenvalues

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) \quad \text{where} \ n = 0,1,2,.... \quad ,$$

Where $H_n(z) = (-1)^n e^{z^2}\frac{d^n}{dz^n}\left(e^{-z^2}\right)$.

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right)$$

**IV) Three dimensional simple harmonic oscillator in coordinate space**

1) Schrödiner equation

$$\left(-\frac{\hbar^2}{2m}\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right) + \frac{1}{2}k\left(x^2 + y^2 + z^2\right)\right)\Psi = \left(E_{n_x} + E_{n_y} + E_{n_y}\right)\psi(x)\psi(y)\psi(y), \qquad \text{where} \quad k = m\omega^2$$

,

Where $E = E_{n_x} + E_{n_y} + E_{n_z} = \hbar\omega\left(n_x + n_y + n_z + \frac{3}{2}\right) = \hbar\omega\left(N + \frac{3}{2}\right)$

2) Degeneracy for given $N$

$$d = {}_{N+2}C_2 = \frac{(N+2)(N+1)}{2}$$

**V) Three dimensional simple harmonic oscillator in spherical coordinates**

with r

# 3-Dim Simple Harmonic Oscillator

$$\nabla^2\psi + \frac{2m}{\hbar^2}\left(E - \frac{1}{2}k r'^2\right)\psi = 0$$

Change of variable $k = m\omega^2$, $r' = \left(\frac{\hbar}{m\omega}\right)^{1/2} r$, $E = \frac{1}{2}\hbar\omega\lambda$.

$$\nabla^2\psi + (\lambda - r^2)\psi = 0$$

$$\hookrightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{1}{r^2\hbar}L^2$$

$\hookrightarrow$ Angular momentum

$$\psi = R(r)\,Y_\ell^m(\theta,\varphi)$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}R\right) + \left(\lambda - r^2 - \frac{\ell(\ell+1)}{r^2}\right)R = 0$$

$$\psi_{N\ell m} = \left(\frac{2}{r}\right)^{1/2}\Lambda_k^{\ell+\frac{1}{2}}(r^2)\,Y_\ell^m(\theta,\varphi) \qquad k = \frac{1}{2}(N-\ell)$$

$\rightarrow N = 2k+\ell$.

$$\Lambda_k^{\alpha}(r^2) = \left[\Gamma(\alpha+1)\binom{k+\alpha}{k}\right]^{-1/2}\exp\left(-\frac{r^2}{2}\right)r^\alpha L_k^\alpha(r^2)$$

$$E = \hbar\omega\left(N + \frac{3}{2}\right)$$

$\hookrightarrow$ degeneracy for given $N$ ; $\frac{1}{2}(N+1)(N+2)\times 2$

spin.

degeneracy is $(N+1)(N+2)$

$N = n_1 + n_2 + n_3$ $\rightarrow$ $(N+2)$ numbers and pick up 2.

* Another way to understand degeneracy)

Schrödinger Eq. $\frac{1}{r^2}\frac{d}{dr}r^2\frac{dR}{dr} + (\lambda - r^2 - \frac{l(l+1)}{r^2})R = 0$

$$\xrightarrow{r\to\infty} \frac{1}{r^2}\frac{d}{dr}r^2\frac{dR}{dr} - r^2R = 0 \quad \sim R = e^{-r^2/2}\phi(r)$$

so $R = e^{-r^2/2}\phi(r)$

then

$$\frac{dR}{dr} = -re^{-r^2/2}\phi + e^{-r^2/2}\phi'$$

$$\frac{d}{dr}r^2\frac{dR}{dr} = \frac{d}{dr}(-r^3e^{-r^2/2}\phi + r^2e^{-r^2/2}\phi')$$

$$= -3r^2e^{-r^2/2}\phi + r^4e^{-r^2/2}\phi - r^3e^{-r^2/2}\phi' + 2re^{-r^2/2}\phi' - r^3e^{-r^2/2}\phi' + r^2e^{-r^2/2}\phi''$$

$$= r^2e^{-r^2/2}\phi'' + (2re^{-r^2/2} - 2r^3e^{-r^2/2})\phi' - 3r^3e^{-r^2/2}\phi + r^4e^{-r^2/2}\phi$$

$$\therefore \phi'' + (\frac{2}{r} - 2r)\phi' + (\lambda - 3 - \frac{l(l+1)}{r^2})\phi = 0.$$

try solving it using series solution.

$$\phi = \sum_{n=0}^{\infty} a_n r^{n+\alpha}, \quad \phi' = \sum_n a_n(n+\alpha)r^{n+\alpha-1}$$

$$\phi'' = \sum a_n(n+\alpha)(n+\alpha-1)r^{n+\alpha-2}$$

substitute into Schrödinger Eq.

$$\sum_n a_n (n+\alpha)(n+\alpha-1) r^{n+\alpha-2} + \sum_{n=0}^{\infty} 2a_n (n+\alpha) r^{n+\alpha-2}$$

$$+ \sum_{n=0}^{\infty} (-2) a_n (n+\alpha) r^{n+\alpha} + \sum_{n=0}^{\infty} (\lambda-3) a_n r^{n+\alpha}$$

$$+ \sum_{n=0}^{\infty} -\ell(\ell+1) a_n r^{n+\alpha-2} = 0.$$

indicial Equation, Note $a_0 \neq 0$

$n=0$

$\rightarrow \alpha(\alpha-1) + 2\alpha - \ell(\ell+1) = \alpha(\alpha+1) - \ell(\ell+1) = 0$

$\alpha = \ell$ or $-(\ell+1)$

↳ no good because diverges at origin.

∴ chose $a_0 \neq 0$, $a_1 = 0$

$$\phi = \sum_{k=0}^{\infty} a_k r^{k+\ell}$$

look at $r^{k+\ell-2}$ i.e. $n=k$

$$\{(k+\ell)(k+\ell+1) - \ell(\ell+1)\} a_k + [-2(k+\ell-2) + (\lambda-3)] a_{k-2} = 0$$

at large $k \rightarrow a_k \cong \frac{2}{k} a_{k-2}$ looks like $\phi = e^{r^2}$

∴ Solution diverges at $r \to \infty$ unless.

$\lambda - 3 = 2(k+\ell-2) \geq 0$ at some $k$.

$= 2(n+\ell) = 2(N).$

Note then

$$\phi = r^{l}\left(a_0 + \cdots \quad a_{k-2} r^{k-2}\right)$$

$$= r^{l} a_0 + \cdots \quad + \underbrace{a_{k-2} r^{l+k-2}}_{\Downarrow}$$

$$a_n r^{l+n} \Rightarrow a_n r^{N}$$

$E_n = (N + \frac{3}{2})\hbar\omega.$

Note $k-2 = n \Rightarrow$ even

$N = \underset{\text{even} := 2(n_p - 1)}{\underbrace{n}} + l$

the Reason there is a degeneracy is

given $N$, $l = 0, 1 \cdots N$.

for each $l \Longrightarrow 2l+1$ degeneracy

if $N =$ even.

$$\sum_{\substack{l=0 \\ l:\text{even}}}^{N} (2l+1) = \sum_{i=0}^{[N/2]} \left(2(2i)+1\right)$$

$$= \frac{2 \cdot 2 \times \frac{N}{2}\left(\frac{N}{2}+1\right)}{2} + \left(\frac{N}{2}+1\right)$$

$$= \frac{(N+1)(N+2)}{2}$$

if $N =$ odd

$$\sum_{\substack{l=1 \\ l:\text{odd}}}^{N} (2l+1) = \sum_{i=1}^{\frac{N+1}{2}} \left(2(2i-1)+1\right) = \frac{(N+1)(N+2)}{2}.$$

∴ Oscillator shell for 3-D HO.

| N. | deg. ~~[illegible]~~ $(N+1)(N+2)$ | Orbitals. $n_p L$. | parity | Total # of leve |
|---|---|---|---|---|
| 0 | 2 | ~~1~~s | + | 2 |
| 1 | 6 | 1p | — | 8 |
| 2 | 12 | 2s, 1d | + | 20 |
| 3 | ~~[illegible]~~ 20 | 2p, 1f | − | 40 |
| 4 | | | | 70 |
| 5 | | ⋮ | | 112 |
| ⋮ | | | | |

O.

does not work from there on

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Why does it break down after $N=2$ ?

o.

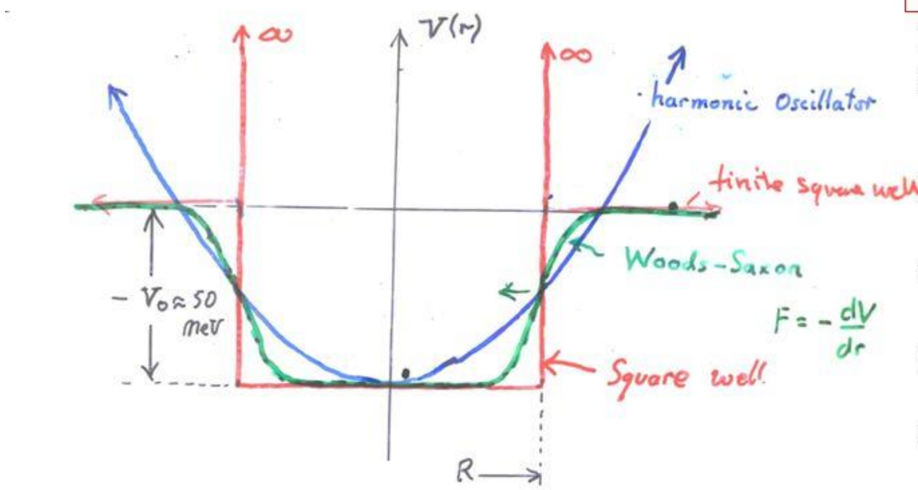
$$\psi \sim r^N \sim r^{2(n_p-1)+\ell}$$

Hydrgen atom $\psi_H \sim r^{n-1} = r^{n-\ell-1} \times r^{\ell}$

$n-\ell \geq 0$

# Solutions to the Infinite Square Well

$$-\frac{\hbar^2}{2m}\left[\frac{d^2R_{nl}}{dr^2}+\frac{2}{r}\frac{dR_{nl}}{dr}\right]+\left[V(r)+\frac{l(l+1)\hbar^2}{2mr^2}\right]R_{nl}=E_{nl}R_{nl}$$




Infinite Square Well

$$V(r) = -V_0 \qquad \text{for } r < R$$
$$= +\infty \qquad \text{for } r = R$$

Finite Square Well

$$V(r) = -V_0 \qquad \text{for } r < R$$
$$= 0 \qquad \text{for } r \geq R$$

Harmonic Oscillator

$$V(r) = -V_0\left[1-\frac{1}{2}\left(\frac{r}{R}\right)^2\right]$$

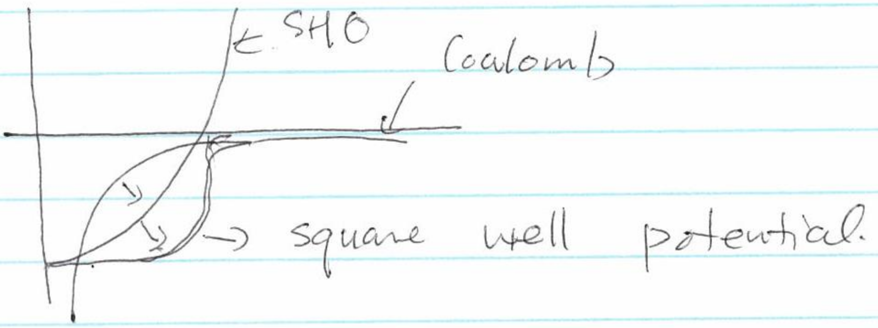
Saxon Woods

$$V(r) = \frac{-V_0}{\left[1+\exp[(r-R)/a]\right]}$$

Note Laguerre polynomial

$$L_n^k(x) = \frac{e^x x^k}{n!}\frac{d^n}{dx^n}\left(e^{-x}x^{n+k}\right)$$

What happens when a realistic potential is used




Coulomb → SHO → square well.

↓ Well know. ↓ $(N+1)(N+2)$ degeneracy ↓ Realistic case

C Real.

↓

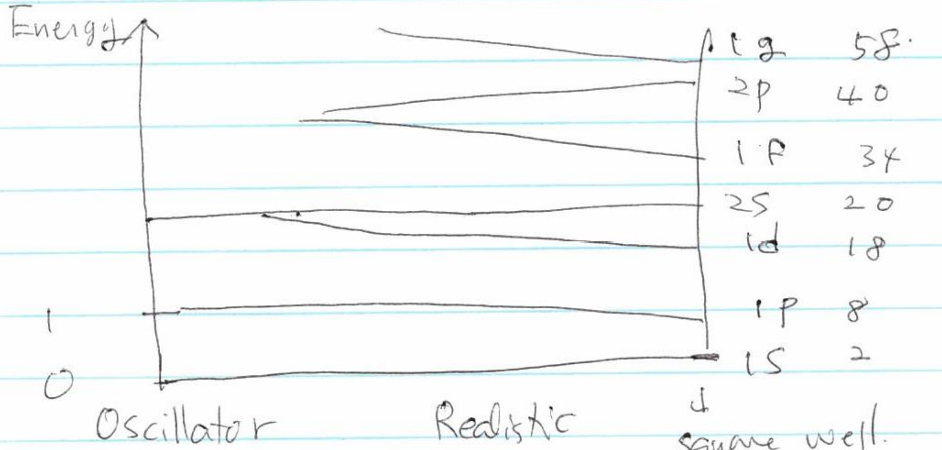
1s ------→ 1s

2s 2p ------→ 1p

2s 1d → 1d, 2s.

3s 3p 3d

when Realistic lager $l$ will have lower energy



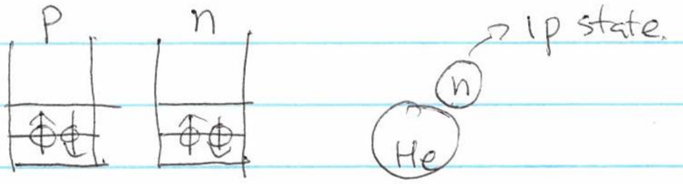
# * Nucleon Spin

a nucleon in 1p state can have

$J = L + S = \frac{3}{2}$ or $\frac{1}{2}$.

are these state degenerate?

a) Experimental Facts: look at $^5_3Li$ $^5_2He$

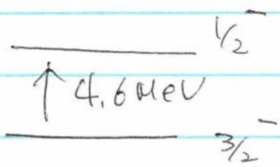



$^4_2He$ $^5_2He$

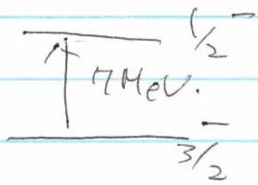
ground state excited state

$3/2^-$ $1/2^-$

$^5_2He$




similarly $^5_3Li$




therefore ground state is $^1P_{3/2}$, excited state $^1P_{1/2}$.

and splitting due to spin-Orbit is large

* Explanation of Magic number.

$$V_{\ell s} = C_{\ell s}\, \underbrace{\ell \cdot s}_{\frac{1}{2}(j^2 - \ell^2 - s^2)}$$

State is $|\alpha', j, \ell, \frac{1}{2}\rangle$ $\quad j = \ell + \frac{1}{2}$ or $\ell - \frac{1}{2}$.

$$\therefore\ \ell \cdot s\,|\alpha j \ell \tfrac{1}{2}\rangle = \tfrac{1}{2}\hbar^2\big(j(j+1) - \ell(\ell+1) - s(s+1)\big)\,|\alpha j \ell s\rangle$$

$$= \begin{cases} \frac{1}{2}\hbar^2 \ell\, |\alpha\ j = \ell + \frac{1}{2}, \ell, \frac{1}{2}\rangle \\ -\frac{1}{2}\hbar^2 (\ell+1)\, |\alpha\ j = \ell - \frac{1}{2}, \ell, \frac{1}{2}\rangle \end{cases}$$

$$\therefore\ \Delta E_{\ell s} = (\ell + \tfrac{1}{2})\hbar^2 C_{\ell s}$$

* Note

a) $\Delta E_{\ell s}$ becomes more important for Heavier Nuclei (large $\ell$).

b) for fixed $\ell$, larger $j$ gives lower Energy

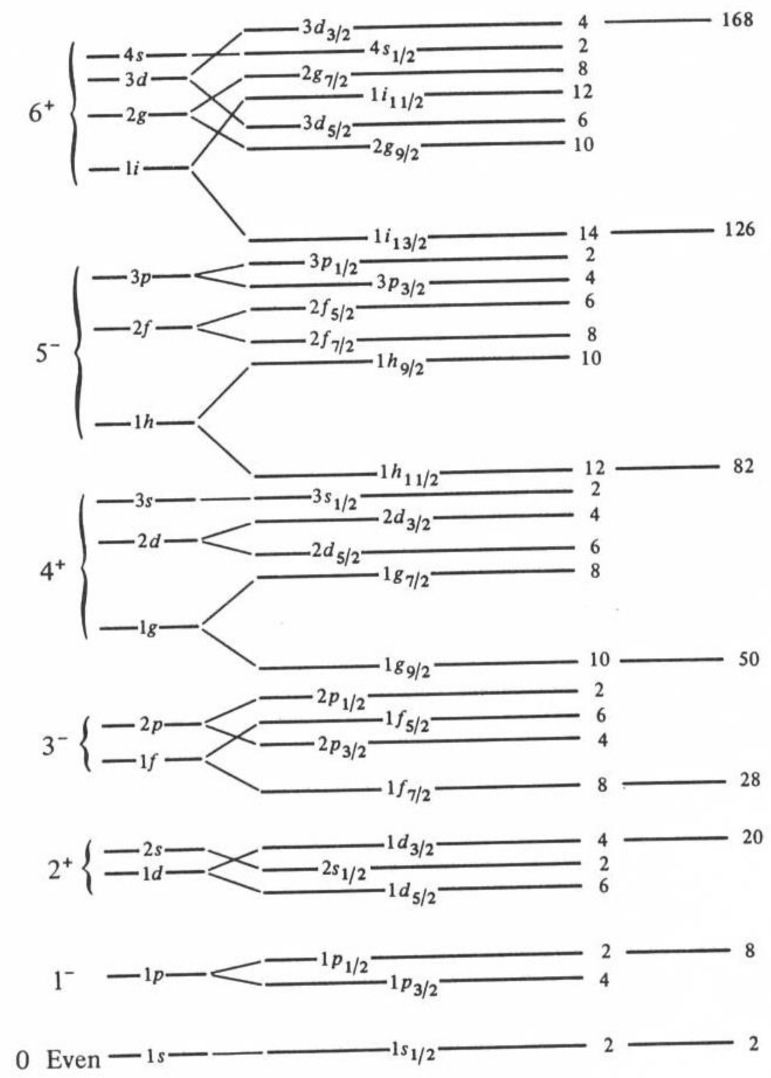



**Fig. 17.10.** Approximate level pattern for nucleons. The number of nucleons in each level and the cumulative totals are shown. The oscillator grouping is shown at the left. Neutrons and protons have essentially the same level pattern up to 50. From then on, some deviations occur. Low neutron angular momenta are more favored than low proton angular momenta.