- 4-Vector

Convention 1 (Bjorken & Drell)

$$x^{\mu} = (x^0, x^k) = (t, \vec{x}) = (t, x, y, z).$$

$$x^{\mu} x_{\mu} = x^{\mu} g_{\mu\nu} x^{\nu} = t^2 - \vec{x}^2$$

Convention 2 (Sakurai)

$$b_{\mu} = b^{\mu} = (b_1, b_2, b_3, b_4) = (\vec{b}, i b_0)$$

$$b^2 = b_1^2 + b_2^2 + b_3^2 + b_4^2 = \vec{b}^2 - b_0^2$$

Either way, we are interested in transformation that leaves

① $\begin{cases} x^{\mu} x_{\mu} = x^{\mu\prime} x_{\mu}^{\prime} \\ x^{\mu\prime} = \Lambda^{\mu}{}_{\nu} x^{\nu} \end{cases}$ or or ② $\begin{cases} b^2 = b^{\prime 2} \\ b_{\mu}^{\prime} = a_{\mu\nu} b_{\nu}. \end{cases}$

• Lorentz group : rotation + boost

i) Rotation, $\mu, \nu = 1, 2, 3$

$$x_i' x_i' = x_j a_{ij} a_{il} x_l = x_l x_l$$

$$\Rightarrow a_{ij} a_{il} = \delta_{jl}$$

$O(3)$ group $\rightarrow$ 3 independent generators

if $O(n)$ " $\rightarrow \frac{n(n-1)}{2}$

ex) $x_1' = \cos\theta\, x_1 - \sin\theta\, x_2$
$x_2' = \sin\theta\, x_1 + \cos\theta\, x_2$.

2) Boost, μ, ν involves space + time.

$$t' = \gamma(t - \tfrac{\beta}{c} z) \qquad \gamma = (1-\beta^2)^{-1/2} \quad \beta = \frac{v}{c}$$
$$z' = \gamma(z - \beta c t)$$

or $x'^3 = x^3 \cosh\omega - x^0 \sinh\omega$ $\qquad \cosh\omega = \gamma$
$x^{0\prime} = -x^3 \sinh\omega + x^0 \cosh\omega$ $\qquad \sinh\omega = \gamma\beta$ $\Big\} \overset{\text{or}}{\Rightarrow} \omega = \frac{1}{2} \ln \frac{1+\beta}{1-\beta}$

☞ in matrix form

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g \qquad x^{\mu} = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = x$$

$$x^2 = x^T g x \quad \text{constant}$$

☞ Transformation matrix $\Lambda = \Lambda^{\mu}{}_{\nu}$

$$x^2 = x'^T g x' = x^T \Lambda^T g \Lambda x = x^T g x$$

$$\to \Lambda^T g \Lambda = g \qquad \to \det\left(\Lambda^T g \Lambda\right) = \left(\det \Lambda\right)^2 \det g = \det g \to \det \Lambda = \pm 1$$

$$\det g = +1 \quad \to \quad \text{proper Lorentz transformation}$$

$$\det g = -1 \quad \to \quad \text{improper Lorentz transformation}$$

This group is called $\mathrm{O}(3,1)$

if $\det \Lambda = 1 \quad \to \quad \mathrm{SL}(2, \mathrm{C})$

- Generators of Lorentz group

$$\Lambda \cong \exp\left(-i\theta\cdot G\right) \sim 1 - i\theta\cdot G$$

☞ Rotation

$$x'_1 = x_1 - \theta x_2$$
$$x'_2 = \theta x_1 + x_2$$

$$\rightarrow \quad L_3 = -i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

for other directions

$$\rightarrow \quad L_1 = -i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \qquad \rightarrow \quad L_2 = -i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

☞ Boost

$$t' = t - \beta x_3$$
$$x'_3 = z - \beta t$$

$$\rightarrow \quad K_3 = -i\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

similarly

$$\rightarrow \quad K_1 = -i\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \rightarrow \quad K_2 = -i\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- Generators of Lorentz group

☞ Can Prove

$$\left[L_i, L_j\right] = i\varepsilon_{ijk} L_k$$ → SU(2)

$$\left[L_i, K_j\right] = i\varepsilon_{ijk} K_k$$

$$\left[K_i, K_j\right] = -i\varepsilon_{ijk} L_k$$

☞ This can be Recast into

$$X_i^{\pm} = \frac{1}{2}\left(L_i \pm iK_i\right)$$

$$\left[X_i^+, X_j^+\right] = i\varepsilon_{ijk} X_k^+$$

$$\left[X_i^-, X_j^-\right] = i\varepsilon_{ijk} X_k^-$$

$$\left[X_i^+, X_j^-\right] = 0$$

☞ Independent sum of SU(2) algebra (direct sum)

A representation can be characterized by (m, n)

- Particle should be a scalar, vector, tensor Field in Lorentz group

Classical Mechanics

$q_i(t) \quad \dot{q}_i(t) \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$

Scalar field

$\phi(x) \qquad \partial_\mu \phi(x) \qquad \rightarrow \ ?$

Lorentz covariance.

↳ Equation should be a scalar or Vector or Tensor

$A_\mu A^\mu \ , \ A_\mu \qquad A_\mu B^\mu$ etc.

$$\frac{\partial L}{\partial q_i} \rightarrow \frac{\partial \mathcal{L}(\phi, \partial_\mu \phi)}{\partial \phi(x)}$$

$$\frac{\partial L}{\partial \dot{q}_i} \rightarrow \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \ , \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \rightarrow \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}$$

Klein-Gordon Equation $\frac{\partial}{\partial x_\mu}\left[\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\right] - \frac{\partial \mathcal{L}}{\partial \phi} = 0$

$\mathcal{L} = ?$ → should be a scalar

$$\mathcal{L} = -\frac{1}{2}\left(\left(\frac{\partial \phi}{\partial x_\mu} \frac{\partial \phi}{\partial x_\mu} \right) + \mu^2 \phi^2 \right).$$

$$\Rightarrow \quad -\frac{1}{2}\frac{\partial}{\partial x_\mu}\left(2 \frac{\partial \phi}{\partial x_\mu} \right) + \mu^2 \phi = 0 \qquad \rightarrow \underbrace{\Box \phi - \mu^2 \phi = 0.}_{\text{Klein-Gordon Eq.}}$$

$$\Box \phi = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}.$$

Laplacian or D'Alembert Op

Note $\quad E \rightarrow i\hbar\frac{\partial}{\partial t}, \qquad P_k \rightarrow -i\hbar\frac{\partial}{\partial x_k}$

Klein-Gordon Equation.

$$\left(E^2 - p^2c^2 - \mu^2c^2\hbar^2\right)\phi = 0.$$

$$E^2 - p^2c^2 = m^2c^4 \qquad m^2c^4 = \mu^2c^2\hbar^2 \quad \Rightarrow \mu = \frac{mc}{\hbar}$$

☞ Basic Equations

$$\nabla \cdot E = \rho \ , \qquad \nabla \times B - \frac{1}{c}\frac{\partial E}{\partial t} = \frac{j}{c}$$ ⟸ 3-conditions

$$\nabla \cdot B = 0 \ , \qquad \nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t} = 0$$ ⟸ Relates E and B

☞ Introduce

$$F_{\mu\nu} = \begin{pmatrix} 0 & B_3 & -B_2 & -iE_1 \\ -B_3 & 0 & B_1 & -iE_2 \\ B_2 & -B_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{pmatrix} \qquad j_\mu = \left(\vec{j}, ic\rho\right)$$

First set $\rightarrow$ $$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \frac{j_\mu}{c}$$

☞ Furthermore $$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Second set $\rightarrow$ $$\varepsilon_{\alpha\lambda\mu\nu}\left(\left[\partial_\lambda, F_{\mu\nu}\right] + \left[\partial_\mu, F_{\nu\lambda}\right] + \left[\partial_\nu, F_{\lambda\mu}\right]\right) = 0$$

Jacobi Identity

Maxwell Field -Lagrangian

☞ Lagrangian density

$$L = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \frac{1}{c} j_\mu A_\mu \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

* Hamiltonian density

$$\mathcal{H}^{em} = \frac{\partial \mathcal{L}}{\partial(\partial A_\mu/\partial x_4)} \frac{\partial A_\mu}{\partial x_4} - \mathcal{L}$$

$$= -F_{4\mu}\left(F_{\mu\nu} + \frac{\partial A_4}{\partial x_\mu}\right) + \frac{1}{2}\left(|B|^2 - |E|^2\right)$$

$$= \frac{1}{2}\left(|B|^2 + |E|^2\right) - i E \cdot \nabla A_4$$

in free field case $\quad + (\nabla \cdot E) A_4 + \text{surface term}$, $(\nabla \cdot E) = 0$

☞ Lagrangian density for QED

$$L = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \bar{\psi}\left(i D_\mu \gamma^\mu - m\right)\psi \qquad D_\mu \psi = \left(\partial_\mu - i g A_\mu\right)\psi$$

• Gauge transformation

☞ Physics is determined by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$A_\mu^{new} = A_\mu^{old} + \frac{\partial \chi}{\partial x_\mu} \quad \text{will give same } F_{\mu\nu}$$

$$F_{\mu\nu}^{new} = \partial_\mu A_\nu^{new} - \partial_\nu A_\mu^{new} = F_{\mu\nu}^{old} + \left(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu\right)\chi$$

☞ Chose a non homogenous part of the gauge

$$\partial_\mu A_\mu^{new} = \partial_\mu A_\mu^{old} + \partial_\mu \partial_\mu \chi = 0 \rightarrow \text{ chose } \square\chi = -\partial_\mu A_\mu^{old}$$

Field equation becomes $\partial_\mu F_{\mu\nu} = \partial_\mu \partial_\mu A_\nu^{new} - \partial_\mu \partial_\nu A_\mu^{new} = -\frac{j_\nu}{c} \rightarrow \square A_\mu = -\frac{j_\mu}{c}$

☞ Further homogenous gauge freedom

$$A_\mu^{'} = A_\mu + \frac{\partial \Lambda}{\partial x_\mu} \quad \text{with } \square\Lambda = 0$$

preserves Field equation and gauge condition $\square A_\mu = -\frac{j_\mu}{c}$ and $\partial_\mu A_\mu = 0$

- Gauge Invariance in the Lagrangian density

☞ Lagrangian density for QED

$$L = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \bar{\psi}\left(iD_\mu \gamma^\mu - m\right)\psi \qquad D_\mu \psi = \left(\partial_\mu - igA_\mu\right)\psi$$

☞ Gauge transformation: Lagrangian is invariant under

$$\psi \to U\psi \quad , \qquad A_\mu \to U\left(A_\mu - \frac{i}{g}\partial_\mu\right)U^\dagger$$

$$D_\mu \psi \to U D_\mu \psi \qquad D_\mu \to U D_\mu U^\dagger$$

$$F_{\mu\nu} = \frac{i}{g}\left[D_\mu , D_\nu\right] \to U F_{\mu\nu} U^\dagger$$

where $U = \exp\left(-ig\chi\right)$ or $U = \exp\left(-ig\Lambda\right)$

A : Elektrodynamics in Radiation (Coulomb) Gauge.

We can always chose A s.t.

$$\nabla \cdot A(x,t) = 0$$

if $\nabla \cdot A^{old} \neq 0$

chose $\vec{A}^{new} = A^{old} + \nabla \chi(x,t)$

$$i A_0^{new} = i A_0^{old} + \frac{\partial}{\partial \underbrace{x_4}_{ict}} \chi \quad \rightarrow \quad A_0^{new} = A_0^{old} - \frac{\partial}{\partial ct} \chi$$

$$-\nabla \cdot A^{new} = \underbrace{\nabla \cdot A^{old}}_{\text{source}} + \nabla^2 \chi = 0$$

$$\chi(x,t) = \frac{1}{4\pi} \int \frac{d^3x'}{|x-x'|} \nabla' \cdot A^{old}(x',t)$$

because.

$$\nabla^2 \chi = \frac{1}{4\pi} \nabla^2 \int \frac{d^3x'}{|x-x'|} \nabla' \cdot A^{old}(x',t)$$

$$\nabla^2 \frac{1}{4\pi} \frac{1}{|x-x'|} = -\delta^3(x-x')$$

$$= -\nabla \cdot A^{old}(x,t)$$

what about A^0 → look at field Eq.

$$\Box A_\mu^{new} - \frac{\partial}{\partial x_\mu}\left(\frac{\partial A_\nu^{new}}{\partial x_\nu}\right) = -\frac{j_\mu}{c}$$

⇓ for 4 → 0 component

$$\nabla^2 A_0^{new} - \frac{1}{c^2}\frac{\partial^2 A_0^{new}}{\partial t^2} + \frac{1}{c}\frac{\partial}{\partial t}\left(\nabla\cdot A^{new} + \frac{1}{c}\frac{\partial A_0^{new}}{\partial t}\right) = -\rho$$

$$\therefore \nabla^2 A_0^{new}(x,t) = -\rho(x,t).$$

no time derivative

$$\therefore A_0^{new}(x,t) = \frac{1}{4\pi}\int \frac{\rho(x',t)\, d^3x'}{|x-x'|}$$

Same time

1) $\nabla\cdot A^{new} = 0$ can be chosen

2) then $A_0^{new}(x,t) = \frac{1}{4\pi}\int \frac{\rho(x',t)}{|x-x'|} d^3x'$

Radiation or Coulomb Gauge

Note we still have ~~ga~~ Residual gauge freedom

$$A_\mu \to A_\mu + \partial_\mu \Lambda \quad \text{when} \quad \nabla^2 \chi = 0$$

$$\to \nabla\cdot A \to \nabla\cdot A + \nabla^2\chi = 0$$

$$\nabla^2 A_0 \to \nabla^2 A_0 + \partial_0 \underbrace{\nabla^2 \chi}_{=0}$$

can get Rid of $\vec{A}_\parallel$

— Hamiltonian density.

from chapter 1

$$\mathcal{H}^{em} = \frac{1}{2}\left(|B|^2 + |E|^2\right) - i E\cdot\nabla A_4$$

$$\nabla\cdot(E A_4) - (\nabla\cdot E) A_4 \quad A_4 \to iA_0$$

$$\nabla\cdot(A_0 E) - \rho A_0$$

surface term → 0

$$\mathcal{L}^{int} = -j_\mu A_\mu/c$$

$$= -j_4 A_4/c - j\cdot A = \rho_0 A_0$$

$$\therefore\ H_{em} + H_{int} = \int d^3x \left(\mathcal{H}^{em} + \mathcal{H}^{int}\right)$$

$$= \frac{1}{2}\int d^3x\left(|B|^2 + |E|^2\right) - \int (\vec{j}\cdot\vec{A}/c)\, d^3x$$

— different form.

Note $\quad E = -\nabla A_0 - \frac{1}{c}\frac{\partial A}{\partial t}$

$$= E_\parallel + E_\perp$$

where $\quad \nabla\times E_\parallel = 0 \qquad \nabla\cdot E_\perp = 0$

because $\nabla\cdot A = 0$

Hence $E_\perp = -\frac{1}{c}\frac{\partial A_\perp}{\partial t}$

then

$$\int |E|^2 d^3x = \int \left(E_\perp^2 + \underbrace{2E_\perp \cdot E_\parallel}_{\underbrace{-\nabla A_0 \cdot E_\perp}_{\text{or } -\frac{1}{c}\frac{\partial A_\perp}{\partial t}}} + |E_\parallel|^2 \right) d^3x$$

$$= -\nabla(E_\perp \cdot A_0) + A_0 \underbrace{\nabla \cdot E_\perp}_{0}$$

$\nabla(E_\perp \cdot A_0) = 0$ surface term

Also

$$\int |E_\parallel|^2 d^3x = \int \nabla A_0 \cdot \nabla A_0 \, d^3x$$

$$= \int \left(\underbrace{\nabla \cdot (A_0 \nabla A_0)}_{\to 0} - A_0 \nabla^2 A_0 \right) d^3x$$

$$= \int \rho A_0 \, d^3x$$

$$\hookrightarrow \int d^3x \, d^3x' \, \frac{\rho(x,t)\, \rho(x',t)}{4\pi |x-x'|}$$

Hence

$$H_{em} + H_{int} = \frac{1}{2}\int \left(|B|^2 + |E_\perp|^2 \right) d^3x$$

$$+ \frac{1}{2}\int d^3x \, d^3x' \, \frac{\rho(x,t)\, \rho(x',t)}{4\pi |x-x'|}$$

$$- \frac{1}{c}\int j \cdot A^\perp d^3x$$

Since $\nabla \cdot A = 0$ only ~~$\nabla \cdot A^\perp = 0$~~
$A^\perp$ is dynamical.

Only $A_\perp$ is dynamical, H can also be written as

$$H_{em} + H_{in} = \frac{1}{2}\int \left(|\nabla \times A^\perp|^2 + \left|\frac{1}{c}\frac{\partial A^\perp}{\partial t}\right|^2 \right) d^3x$$

$$- \frac{1}{c}\int \vec{j}\cdot A^\perp d^3x + \frac{1}{2}\int d^3x\, d^3x' \frac{\rho(x,t)\,\rho(x',t)}{4\pi|x-x'|}.$$

* Note when $j^\mu = 0$, only $A_\perp$ contribute and the field Equations become

$$\nabla^2 \vec{A}_\perp - \frac{1}{c^2}\frac{\partial^2 A_\perp}{\partial t^2} = 0$$

$$B = \nabla \times A_\perp \quad , \quad E = -\frac{1}{c}\frac{\partial A_\perp}{\partial t}.$$