# $\mathcal{QFT}$ Problem Sets

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## 目录



## 1 Introduction

**习题 1.1** 计算 1 Mpc =? $\mathrm{MeV}^{-1}$。

**解答 1.1** 由 $1\ \mathrm{Mpc} = 3.1 \times 10^{22}\ \mathrm{m} = 3.1 \times 10^{37}\ \mathrm{fm}$ 和

$$1\ \mathrm{fm} = \frac{1}{197}\ \mathrm{MeV}^{-1} \tag{1}$$

得到

$$1\ \mathrm{Mpc} = \frac{3.1}{1.97} \times 10^{35}\ \mathrm{MeV}^{-1} = 1.6 \times 10^{35}\ \mathrm{MeV}^{-1} \tag{2}$$

**习题 1.2** 已知 $f(x) = \int \frac{dk}{2\pi} f(k) e^{ikx}$,求证 $\int dx \left(\frac{df(x)}{dx}\right)^2 = \int \frac{dk}{2\pi} k^2 f(k) f(-k)$。

**解答 1.2** 直接计算:

$$\begin{aligned}
\int dx \left(\frac{df(x)}{dx}\right)^2 &= \int dx \frac{dk_1 dk_2}{(2\pi)^2} \left(ik_1 f(k_1) e^{ik_1 x}\right) \left(ik_2 f(f_2) e^{ik_2 x}\right) \\
&= \int \frac{dk_1 dk_2}{(2\pi)^2} dx - k_1 k_2 f(k_1) f(k_2) e^{i(k_1+k_2)x} \\
&= \int \frac{dk_1 dk_2}{(2\pi)^2} - k_1 k_2 f(k_1) f(k_2) (2\pi) \delta(k_1 + k_2) \\
&= \int \frac{dk}{2\pi} k^2 f(k) f(-k)
\end{aligned} \tag{3}$$

**习题 1.3** 泛函

$$Z[J]=e^{-\frac{1}{2}\int dxdyJ(x)D(x-y)J(y)},\quad D(x)=D(-x) \tag{4}$$

求 $\frac{\delta^2 Z}{\delta J^2}$。

**解答 1.3** 两种角度解答该题:

1. 从格点列角度, $J(x)\to J_m$, $D(x-y)\to D_{mn}$ 为一个对称矩阵, $\int dx\to\sum_m$, 我们有

$$Z[J]\to Z(J_k)=e^{-\frac{1}{2}J_mD_{mn}J_n} \tag{5}$$

从而

$$\begin{aligned}\frac{\partial Z(J_k)}{\partial J_i}&=e^{-\frac{1}{2}J_mD_{mn}J_n}\left(-\frac{1}{2}\right)\frac{\partial J_mD_{mn}J_n}{\partial J_i}\\&=-\frac{1}{2}e^{-\frac{1}{2}J_mD_{mn}J_n}\left(\delta_{mi}D_{mn}J_n+J_mD_{mn}\delta_{ni}\right)\\&=-\frac{1}{2}e^{-\frac{1}{2}J_mD_{mn}J_n}\left(D_{in}J_n+J_mD_{mi}\right)\\&=-\frac{1}{2}e^{-\frac{1}{2}J_mD_{mn}J_n}(2D_{in}J_n)\\&=-e^{-\frac{1}{2}J_mD_{mn}J_n}D_{in}J_n\end{aligned} \tag{6}$$

$$\begin{aligned}\frac{\delta^2 Z}{\delta J(x)\delta J(y)}\to\frac{\partial^2 Z(J_k)}{\partial J_i\partial J_j}&=e^{-\frac{1}{2}J_mD_{mn}J_n}(D_{jm}J_mD_{in}J_n)-e^{-\frac{1}{2}J_mD_{mn}J_n}D_{in}\delta_{nj}\\&=e^{-\frac{1}{2}J_mD_{mn}J_n}(D_{jm}J_mD_{in}J_n-D_{ij})\\&\leftarrow Z[J]\left(\int dzdw\ D(x-z)J(z)D(y-w)J(w)-D(x-y)\right)\end{aligned} \tag{7}$$

2. 直接计算,利用泛函导数链式法则 $\frac{\delta}{\delta f(y)}g(F[f(x)])=g'(F[f(x)])\frac{\delta F[f(x)]}{\delta f(y)}$,有

$$\frac{\delta Z[J]}{\delta J(x)}=-\frac{1}{2}Z[J]\frac{\delta}{\delta J(x)}\int dzdw\ J(z)D(z-w)J(w) \tag{8}$$

后一个泛函导数按定义计算如下

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\frac{\int dzdw\ [J(z)+\epsilon\delta(x-z)]D(z-w)[J(w)+\epsilon\delta(x-w)]-\int dzdw\ J(z)D(z-w)J(w)}{\epsilon}\\
=&\int dzdw\ \delta(x-z)D(z-w)J(w)+\int dzdw\ J(z)D(z-w)\delta(x-w)\\
=&\int dwD(x-w)J(w)+\int dzJ(z)D(z-x)\\
=&2\int dwD(x-w)J(w)
\end{aligned}
\tag{9}
$$

于是

$$
\frac{\delta Z[J]}{\delta J(x)}=-Z[J]\int dwD(x-w)J(w) \tag{10}
$$

利用泛函导数的乘积法则 $\frac{\delta}{\delta f}FG=\frac{\delta F}{\delta f}G+F\frac{\delta G}{\delta f}$ 得到

$$
\begin{aligned}
\frac{\delta^2 Z[J]}{\delta J(x)\delta J(y)}=\frac{\delta}{\delta J(y)}\frac{\delta Z[J]}{\delta J(x)}&=-\frac{\delta}{\delta J(y)}Z[J]\int dwD(x-w)J(w)\\
&=Z[J]\left[\int dzD(y-z)J(z)\int dwD(x-w)J(w)-D(x-y)\right]\\
&=Z[J]\left[\int dzdwD(y-z)J(z)D(x-w)J(w)-D(x-y)\right]
\end{aligned}
\tag{11}
$$

**习题 1.4** 计算 $\langle\vec{p}|\hat{\psi}^\dagger(\vec{x})\hat{\psi}(\vec{y})|\vec{p}\rangle$。

**解答 1.4** 根据定义

$$
\hat{\psi}(\vec{x})=\int\frac{d^3\vec{k}}{(2\pi)^3}\hat{a}_{\vec{p}}e^{i\vec{p}\cdot\vec{x}} \tag{12}
$$

和对易关系

$$
[\hat{a}_{\vec{p}},\hat{a}^\dagger_{\vec{q}}]=(2\pi)^3\delta(\vec{p}-\vec{q}) \tag{13}
$$

有

$$
\begin{aligned}
\langle\vec{p}|\hat{\psi}^\dagger(\vec{x})\hat{\psi}(\vec{y})|\vec{p}\rangle&=\langle 0|\hat{a}_{\vec{p}}\hat{\psi}^\dagger(\vec{x})\hat{\psi}(\vec{y})\hat{a}^\dagger_{\vec{p}}|0\rangle\\
&=\int\frac{d^3\vec{k}_1}{(2\pi)^3}\frac{d^3\vec{k}_2}{(2\pi)^3}\langle 0|\hat{a}_{\vec{p}}\hat{a}^\dagger_{\vec{k}_1}e^{-\vec{k}_1\cdot\vec{x}}\hat{a}_{\vec{k}_2}e^{i\vec{k}_2\cdot\vec{y}}\hat{a}^\dagger_{\vec{p}}|0\rangle\\
&=\int\frac{d^3\vec{k}_1}{(2\pi)^3}\frac{d^3\vec{k}_2}{(2\pi)^3}\langle 0|\left[(2\pi)^3\delta(\vec{k}_1-\vec{p})+\hat{a}^\dagger_{\vec{k}_1}\hat{a}_{\vec{p}}\right]\left[\hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{k}_2}+(2\pi)^3\delta(\vec{k}_2-\vec{q})\right]|0\rangle e^{i\vec{k}_2\cdot\vec{y}-\vec{k}_1\cdot\vec{x}}\\
&=e^{i\vec{p}\cdot(\vec{y}-\vec{x})}
\end{aligned}
\tag{14}
$$

这一矩阵元可以诠释为在 $\vec{y}$ 处湮灭一个动量为 $\vec{p}$ 的粒子，再在 $\vec{x}$ 处产生之的振幅。