☞Units

$$\hbar = 1.0545718\times10^{-34}\ \mathrm{m}^2\ \mathrm{kg/s}$$

$$c = 2.99792458\times10^{8}\ \mathrm{m/s}$$

$$\hbar c = 3.16152649\times10^{-26}\ \mathrm{m\ kg\ m^2/s^2} = 197.32697\ \mathrm{MeV\ fm}$$

where, $\mathrm{eV} = 1.6021766208\times10^{-19}\mathrm{kg\ m^2/s^2},\ \ \mathrm{fm}=1\times10^{-15}\ \mathrm{m}$

We will take the limit where, $\hbar = c = 1$ and only keep one unit, MeV or fm

Basic units are : m, kg, s

Therefore, can always reconstruct the full units and multiplicative factor because for a given physical unit, the combination of c and hbar is unique

That is , for given, a,b,c $\quad \mathrm{MeV}^{\gamma} c^{\alpha} \hbar^{\beta} = \mathrm{m}^{a}\ \mathrm{kg}^{b}\ \mathrm{s}^{c}$

there exits a unique α,β,γ

Units of hadronic world is order 1fm or 200 MeV

☞Energy, momentum, mass

$$E = mc^2 = pc$$

$$E \to \mathrm{MeV}, \quad \mathrm{mass} \to \mathrm{MeV}/\,c^2, \quad \mathrm{momentum} \to \mathrm{MeV}/c$$

☞Time, distance

$$\mathrm{fm{=}1fm} \times \frac{1}{197\mathrm{MeV\ fm}} = \frac{1}{197\mathrm{MeV}}$$

$$\mathrm{t{=}}\frac{x}{c} = \mathrm{fm}\,/\,c$$

Isospin symmetry in strong interaction

☞Proton and Neutron:

	Mass (MeV)	difference	Spin	Lifetime	Mag Mom
Proton	938.272		1	Stable	2.793
Neutron	939.507	+1.294	1	881.5 s	-1.913

Strong interaction part is similar

Weak+ E&M part

☞Pion:

	Mass (MeV)	difference	Spin	Lifetime	Charge
π^+	139.59	+4.59	0	2.6×10^{-8} s	+e
π^0	135.0		0	8.4×10^{-17} s	0
π^-	139.59	+4.59	0	2.6×10^{-8} s	-e

Strong interaction part is similar

Weak+ E&M part

☞Proton and Neutron can be written as fundamental representation of SU(2)

- Introduce two dimensional space to represent the proton and neutron

$$|p\rangle = \psi(x,t,s)\begin{pmatrix}1\\0\end{pmatrix}, \quad |n\rangle = \psi(x,t,s)\begin{pmatrix}0\\1\end{pmatrix}$$

- Introduce operators to identify and interpolate between p and n

$$\tau_3 = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \qquad \tau_+ = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, \quad \tau_- = \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}$$

$$\tau_3|p\rangle = +1|p\rangle, \quad \tau_3|n\rangle = -1|n\rangle \quad \tau_+|p\rangle = 0, \quad \tau_+|n\rangle = |p\rangle, \quad \tau_-|p\rangle = |n\rangle, \quad \tau_-|n\rangle = 0$$

- Re-express in terms of Hermitian operators (Pauli Matrix)

$$\tau_3 = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \qquad \tau_1 = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \quad \tau_2 = \begin{pmatrix}0 & -i\\i & 0\end{pmatrix}$$

SU(2) group

$$U\left(\varepsilon_1,\varepsilon_2,\varepsilon_3\right)=\exp\left(-i\varepsilon_\nu T_\nu\right)=\exp\left(-\frac{i}{2}\varepsilon_\nu\tau_\nu\right)=\sum_{n=0}^{\infty}\left(-\frac{i}{2}\varepsilon_\nu\tau_\nu\right)^n$$

note $$\left(\varepsilon\cdot\tau\right)^2=\varepsilon^\alpha\varepsilon^\beta\tau_a\tau_b=\varepsilon^\alpha\varepsilon^\beta\left(i\varepsilon_{\alpha\beta\gamma}\tau_a+\delta_{\alpha\beta}\right)=\varepsilon^2$$

$$U\left(\varepsilon\right)=1_{2\times2}\cos\left(\varepsilon/2\right)-in_\nu\tau_\nu\sin\left(\varepsilon/2\right)$$ where $$n_\nu=\hat{\varepsilon}_\nu=\frac{1}{\sqrt{\varepsilon_1^2+\varepsilon_2^2+\varepsilon_3^2}}\varepsilon_\nu$$

☞ Group multiplication $U\left(\alpha\right)=\exp\left(-i\alpha_\nu T_\nu\right)\equiv\exp\left(-i\alpha\right)$ simplified notation

$$U\left(\alpha\right)U\left(\beta\right)=\exp\left(-i\alpha\right)\exp\left(-i\beta\right)\equiv\exp\left(-i\gamma\right)$$

where $$\gamma=\alpha+\beta+\frac{1}{2}[\alpha,\beta]+\text{ higher commutations}$$

Therefore SU(2) group is defined through commutation relations of generators

$$\left[T_i,T_j\right]=i\varepsilon_{ijk}T_k,\ \text{ where } i,j,k=1,2,3$$

☞SU(2) representations:

$$\left[T_i, T_j\right] = i\varepsilon_{ijk} T_k, \quad \text{where } i, j, k = 1,2,3$$

- Maximum commuting operator is 1:

  defines the states that the group operators acts $\quad T_3 \left| m \right\rangle = \mathrm{m} \left| T_3 \right\rangle$

- The remaining operators are used to introduce Raising and lowering operator

$$T_{\pm} = T_1 \pm i T_2$$

$$\left[T_3, T_{\pm}\right] = \pm T_{\pm}$$

$$\left(T_3 T_{\pm} - T_{\pm} T_3\right)\left| m \right\rangle = \left(T_3 T_{\pm} - T_{\pm} m\right)\left| m \right\rangle = \pm T_{\pm}\left| m \right\rangle \qquad \rightarrow T_{\pm}\left| m \right\rangle \propto \left| m \pm 1 \right\rangle$$

- Casimir operator

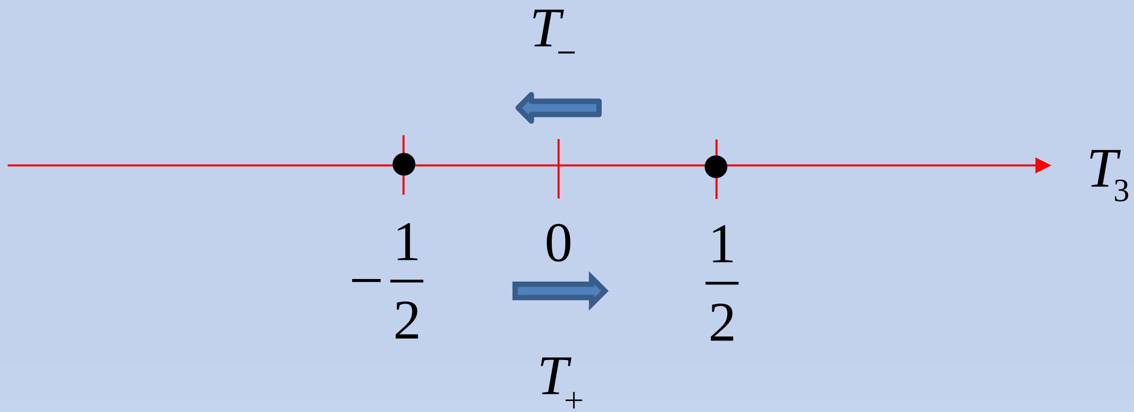
$$\left[T_3, T^2\right] = 0, \qquad T^2 = T_1^2 + T_2^2 + T_3^2 = T_- T_+ + T_3 + T_3^2$$

maxium $m = j$, such that $T_+ \left| j, j \right\rangle = 0$ and $T^2 \left| j, j \right\rangle = j\left(j+1\right)\left| j, j \right\rangle$

A state in can be labeled $\left| j, m \right\rangle$, where j, m are integers

☞ SU(2) fundamental representation:

- Highest and lowest weight states are $p = \left| \frac{1}{2}, \frac{1}{2} \right\rangle, \quad n = \left| \frac{1}{2}, -\frac{1}{2} \right\rangle$



- Matrix form

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad n = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

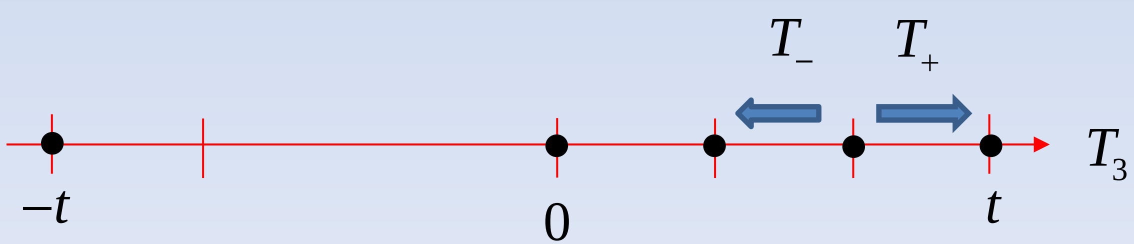
$$T_i = \frac{1}{2}\tau_i \qquad \left|\psi'\right\rangle = \mathrm{U}\left|\psi\right\rangle, \quad \text{where} \quad \mathrm{U} = \exp\left(-i\alpha_i T_i\right)$$

☞ SU(2) higher representations:

- Highest weight state $$T_3|t,t\rangle = t|t,t\rangle, \quad T^2|t,t\rangle = t(t+1)|t,t\rangle$$
- Lowest weight state $$T_3|t,-t\rangle = -t|t,-t\rangle, \quad T^2|t,-t\rangle = t(t+1)|t,-t\rangle$$
- Other states $$T_3|t,m\rangle = m|t,m\rangle, \quad T^2|t,m\rangle = t(t+1)|t,m\rangle$$
- Matrix form

$$|t,t\rangle = \begin{pmatrix} 1 \\ 0 \\ .. \\ 0 \end{pmatrix}, \quad \ldots \quad |t,m\rangle = \begin{array}{cl} \begin{pmatrix} 0 \\ 1 \\ .. \\ 0 \end{pmatrix} & \begin{array}{l} \Downarrow (t-m+1)'th \\ \\ \\ \\ \end{array} \end{array}, \qquad \text{total } 2t+1 \text{ states}$$

- Weight diagram



o Proton and Neutron are two states in the isospin space. with $\hat{T}_3 \psi_p = \frac{1}{2}\psi_p$, $\hat{T}_3 \psi_n = \frac{-1}{2}\psi_n$

$$\hat{T}_3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}.$$

also $Q = e(\hat{T}_3 + \frac{1}{2}) = \frac{1}{2}e\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = e\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$Q\psi_p = e\psi_p \qquad Q\psi_n = 0\,\psi_n.$

5.1. Isospin Operator for a Multi-Nucleon system

$T(n) = \frac{1}{2}\tau(n) \quad n = 1 \cdots A$: acts only on n'th nucleon

If total A is given, different charge states can be distinguished by T_3 or Q.

Where $$T_i = \sum_{n=1}^{A} T_i(n) = \frac{1}{2}\sum_{n=1}^{A}\tau(n)$$

$$Q = \sum_n \hat{Q}(n) = e \sum_n \frac{1}{2}\left(\tau_3(n) + 1\right) = e\left(T_3 + \frac{1}{2}A\right)$$

Any sum of nucleon states are characterized by

$$\underset{\substack{\uparrow \\ \text{total } T,}}{|T}, T_3\rangle$$

$$T^2 |T\, T_3\rangle = T(T+1)|T, T_3\rangle \qquad T = 0, \tfrac{1}{2}, \cdots$$

$$T_3 |T\, T_3\rangle = T_3 |T\, T_3\rangle \qquad T \geq T_3 \geq -T.$$

Example 5.2. Deuteron

$\psi(r_1) \quad \psi(r_2) \quad \rightarrow \quad R_{n\ell}(r)\, Y_{\ell m_\ell}(\theta,\varphi)$

$1 \qquad 2.$

$\Uparrow \qquad \Uparrow$

$\chi_{\frac{1}{2}m_1} \quad \chi_{\frac{1}{2}m_2} \quad \rightarrow \quad \chi_{s m_s}$

$$\Psi_{\text{deuteron}} = R_{n\ell}\left[Y_{\ell m} \times \chi_{s m_s}\right]^{J} |T, T_3\rangle$$

$$\left[Y_{\ell m_\ell} \times \chi_{s m_s}\right]^{(j)} = \sum_{m_\ell m_s} (\ell s j \mid m_\ell m_s m)\, Y_{\ell m_\ell}\, \chi_{s m_s}(1,2)$$

Clebsch-Gordan coefficients.

$$\chi_{s m_s}(1,2) = \sum_{m_1 m_2} \left(\tfrac{1}{2}\tfrac{1}{2} s \mid m_1 m_2 m\right) \chi_{\frac{1}{2}m_1}(1)\, \chi_{\frac{1}{2}m_2}(2)$$

— Isospin Part.

$$|T\, T_3\rangle = \sum_{t_1 t_2} \left(\tfrac{1}{2}\tfrac{1}{2} T \mid t_1 t_2 T_3\right) \tilde{\chi}_{\frac{1}{2}t_1}(1)\, \tilde{\chi}_{\frac{1}{2}t_2}(2)$$

*note $SO(3) \quad [J_i\, J_j] = i\varepsilon_{ijk} J_k$ are same group.

$SU(2) \quad [T_i\, T_j] = i\varepsilon_{ijk} T_k$

Deuteron has $A = 2$ 1 proton 1 neutron.

$T_3 = T_3(1) + T_3(2) = 0 \quad \rightarrow \quad T = 0$ or 1.

$Q = e(T_3 + \frac{1}{2}A) = e$.

Two states with $T_3 = 0$

$$|T=0, T_3=0\rangle = \frac{1}{\sqrt{2}}\left(\bar{\chi}_{\frac{1}{2}\frac{1}{2}}(1)\,\bar{\chi}_{\frac{1}{2}-\frac{1}{2}}(2) - \bar{\chi}_{\frac{1}{2}-\frac{1}{2}}(1)\,\bar{\chi}_{\frac{1}{2}\frac{1}{2}}(2)\right)$$

$$|T=1, T_3=0\rangle = \frac{1}{\sqrt{2}}\left(\quad '' \quad + \quad '' \quad \right)$$

— Three states with $T=1$

$$|T=1, T_3=1\rangle = \bar{\chi}_{\frac{1}{2}\frac{1}{2}}(1)\,\tilde{\chi}_{\frac{1}{2}\frac{1}{2}}(2) \qquad \text{(two proton)}$$

$$|T=1\ T_3=-1\rangle = \bar{\chi}_{\frac{1}{2}-\frac{1}{2}}(1)\,\bar{\chi}_{\frac{1}{2}-\frac{1}{2}}(2)$$

— Pauli - Principle.

Two ~~nucl~~ nucleon → Antisym - with respect to interchange $1 \leftrightarrow 2$

$$\psi_{total} = \text{space} \times \underline{\text{spin} \times \text{Isospin}}.$$

sym → S=1 ground state — T= 0, Antisym → Deuteron

Antisym ↪ S=0 — T=1 sym → un stable

if space part is sym.

— Isospin Symmetry

$$H_{total}(\text{Strong interaction}) = f(r)\mathbb{1} + g(r)T^2 \quad \text{in isospin space.}$$

$$H_{total}(T=1) = f + g\,T(T+1) = f(r) + 2\underset{\sim}{g}(r)$$

$$H_{total}(T=0) = f(r)$$

repulsive.

Ex 5.3 Charge Independence of Nuclear Force.

Show Isospin Invariance of strong Interaction

$\rightarrow$ leads to $\rightarrow$ charge Indep.

proof)

Isospin Invariance Means $[H_{strong}, T_i] = 0$.

we will show

$$\langle n(1)\, n(2) | \hat{H}_{strong} | n(1)\, n(2) \rangle = \langle n(1)\, p(2) | \hat{H}_{strong} | n(1)\, p(2) \rangle$$

$$= \langle p(1)\, p(2) | \hat{H}_{strong} | p(1)\, p(2) \rangle$$

note

$$e^{-i\pi T_2} \underbrace{|\tfrac{1}{2}\,\tfrac{1}{2}\rangle}_{p} = \left(\cos\tfrac{1}{2}\pi - iT_2 \sin\tfrac{\pi}{2}\right) |\tfrac{1}{2}\,\tfrac{1}{2}\rangle$$

$$= -i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \underbrace{|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle}_{n}$$

$$e^{-i\pi T_2} |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle = -i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -|\tfrac{1}{2}, -\tfrac{1}{2}\rangle = -p$$

$$e^{+i\pi T_2}\, e^{-i\pi T_2} = \left(\cos\tfrac{1}{2}\pi + iT_2\, i\tfrac{\pi}{2}\right)\left(\cos\tfrac{1}{2}\pi - iT_2\, i\tfrac{\pi}{2}\right)$$

$$= 1$$

$$e^{-i\pi(T_2(1) + T_2(2))} | p(1)\, p(2) \rangle = | n(1)\, n(2) \rangle$$

independent, commute.

$$e^{+i\pi T_2(1)} H_{strong}\, e^{-i\pi T_2(1)} \underset{\substack{\uparrow \\ [H, T_i] = 0}}{=} e^{i\pi T_2(1)} e^{-i\pi T_2(1)} H_{strong}$$

and so on.

⊚ Ex 5.4 The Pion Triplet.

Pion	mass [MeV]		charge	lifetime	Spin
π^+	139.59	$\uparrow 4.59$	e	2.55×10^{-8}	0
π^0	135.0	$\downarrow 4.59$	0	0.83×10^{-16}	0
π^-	139.59		$-e$		0

→ different charge state → diff Isospin
SU(2)

$SU(2) \Leftrightarrow SO(3)$ $T=1$ state with diffent T_3.

$|T=1, T_3=1\rangle = -|\pi^+\rangle$ ← why "−"

$|T=1\ T_3=0\rangle = |\pi^0\rangle$ ⇒ Adjoint Representation

$|T=1, T_3=-1\rangle = |\pi^+\rangle$

→ look at generators

$$T_\pm |T\,T_3\rangle = [T(T+1) - T_3(T_3 \pm 1)]^{1/2} |T, T_3 \pm 1\rangle$$

$$T_0 |T\,T_3\rangle = T_3 |T\,T_3\rangle$$

where $T_\pm = T_1 \pm i T_2$, $T_0 = T_3$

(Commutation Rel w.r.t + −.)

$$[T_3, T_\pm] = \pm T_\pm \qquad [T_+, T_-] = 2T_3.$$

Let us look at how they act on states and compare with commutators.

$T_+|1,1\rangle = 0, \quad T_+|1,0\rangle = \sqrt{2}|1,1\rangle \quad T_+|1,-1\rangle = \sqrt{2}|1,0\rangle$

$[T_+, T_+] = 0 \quad [T_+ T_3] = -T_+ \quad [T_+ T_-] = 2T_3$

~~$T_+(\sqrt{2}|\pi^+\rangle) = 0 \quad T_+(|\pi^0\rangle) = -(+\sqrt{2}|\pi^+\rangle)$~~

$T_+(-\sqrt{2}|1,1\rangle) = 0 \quad T_+|1,0\rangle = -(-\sqrt{2}|1,1\rangle)$ $\quad T_+|\pi$

$T_+(\sqrt{2}|1-\rangle\rangle = 2|1,0\rangle$

$T_+(\underbrace{\sqrt{2}|\pi^+\rangle}) = 0 \quad T_+(\underbrace{|\pi^0\rangle}_{T_3}) = -|\underbrace{\sqrt{2}|\pi^+\rangle}_{T_3} \quad T_+|\pi^-\rangle = 2|\pi^0\rangle$ $\quad T_3$

$T_\pm = T_1 \pm iT_2$

$\leftrightarrow \quad \sqrt{2}\pi^\pm = \pi_1 \pm i\pi_2 \quad \rightarrow \quad \pi_1 = \frac{1}{\sqrt{2}}(\pi^+ + \pi^-)$
$\pi_2 = -\frac{i}{\sqrt{2}}(\pi^+ - \pi^-)$

therefore

$|T=1, T_3=1\rangle = -|\pi^+\rangle$ and $T_+ \leftrightarrow \sqrt{2}|\pi^+\rangle$

$|T=1, T_3=0\rangle = |\pi^0\rangle$ and $T_3 \leftrightarrow |\pi^0\rangle$

$|T=1, T_3=-1\rangle = +|\pi^-\rangle$ and $T_3 \leftrightarrow \sqrt{2}|\pi^-\rangle$

action of T on $|\pi^\pm, \pi^0\rangle$ can be obtained (are the same as) commutator, $[T, \underline{\text{state}}] = [T, T_{\pm,0}]$.

$T|\pi^{\pm 0}\rangle = [T, T_{\pm,0}]$.

or $T_i|\pi_j\rangle = i\varepsilon_{ijk}|\pi_k\rangle$

5.4 Transformation Law for Isospin Vectors.

Note $\quad T_i|\pi_j\rangle = i\varepsilon_{ijk}|\pi_k\rangle$

therefore consider a Rotation

$$|\pi_j'\rangle = e^{-i\delta\varepsilon_i T_i}|\pi_j\rangle = (1 - i\delta\varepsilon_i T_i)|\pi_j\rangle$$

$$= |\pi_j\rangle + \delta\varepsilon_i\, \varepsilon_{ijk}|\pi_k\rangle$$

$$= |\pi_j\rangle - \varepsilon_{jik}\,\delta\varepsilon_i|\pi_k\rangle.$$

$\downarrow$

if i is fixed

Rotation. $\leftarrow$ involves mixing j,k

$|\pi_j'\rangle = |\pi_j\rangle - \varepsilon_{jik}\delta\varepsilon_i|\pi_k\rangle$ or $|\pi_j\rangle = |\pi_j'\rangle + \varepsilon_{jik}\,\delta\varepsilon_i|\pi_k'\rangle$

$\therefore$ can think Iso-space $\mathbb{R}^3$ generated by

$|\pi_1\rangle, |\pi_2\rangle, |\pi_3\rangle$

$\langle\pi_i|\pi_j\rangle = \delta_{ij}$

$\therefore$ can define Iso vector.

$$\vec{v} = v_k|\pi_k\rangle$$

transformation property of v_k

$$\vec{v} = v_j'|\pi_j'\rangle = v_\alpha|\pi_\alpha\rangle = v_\alpha\underbrace{|\pi_\alpha\rangle}_{|\pi_\alpha'\rangle + \varepsilon_{\alpha ik}|\pi_k'\rangle}$$

$\therefore$ $v_j' = v_j - \varepsilon_{ji\alpha}\,\delta\varepsilon_i v_\alpha.$ $\leftarrow$ passive Rotation

$v_j' = v_j + \varepsilon_{jik}\,\delta\varepsilon_i v_k.$ $\quad$ Active Rotation

$\therefore \quad \vec{V}' = \vec{V} + \delta\vec{\varepsilon} \times \vec{V}$

$\vec{V}'^2 = \vec{V}^2 + \underbrace{\delta\vec{\varepsilon} \times V \cdot V}_{=0} + O(\delta\varepsilon^2)$

length preserving.

— Isovector Operators,

operators that transforms as vector in isospin space

$\delta\vec{V} \to \hat{V}' - \hat{V} = \delta\vec{\varepsilon} \times \hat{V}$.

$\therefore \quad V_i' - V_i = \varepsilon_{ijk}\delta\varepsilon_j V_k \quad$ — (1)

but Note Operators transforms as

$\hat{V}_i' = U^{*} V_i U^{-1} = (1 - i\delta\varepsilon_k T_k) V_i (1 + i\delta\varepsilon_k T_k)$

$= V_i - i\delta\varepsilon_k [T_k, V_i] \quad$ — (2)

$\therefore$ (1) $\leftrightarrow$ (2) implies $[T_j, V_i] = i\varepsilon_{ijk} V_k$

definition of isovector operator.

Note $|\pi^\pm\rangle$ are charge eigenstates

but $|\pi_1\rangle = \frac{1}{\sqrt{2}}\{|\pi^+\rangle + |\pi^-\rangle\}$

$|\pi_2\rangle = \frac{-i}{\sqrt{2}}\{|\pi^+\rangle - |\pi^-\rangle\}$ $\Big\} \to$ are not.

5-6 G-Parity.

SO(3) SU(2)

parity G-Parity.

$G|\pi_j\rangle = -|\pi_j\rangle$

Can show $G = e^{-i\pi T_2}\hat{C}$

where $c|\pi^+\rangle = |\pi^-\rangle$, $c|\pi^-\rangle = |\pi^+\rangle$,

$c|\pi^0\rangle = |\pi^0\rangle$

$\pi^+ = \frac{1}{\sqrt{2}}(\pi^1 + i\pi^2)$

$\pi^- = -\frac{i}{\sqrt{2}}(\pi^1 - i\pi^2)$

$c|\pi^1 + i\pi^2\rangle = |\pi^1 - i\pi^2\rangle$

$c|\pi^1 - i\pi^2\rangle = |\pi^1 + i\pi^2\rangle$

$c|\pi^3\rangle = c|\pi^3\rangle$

$\Rightarrow$ $c|\pi_1\rangle = |\pi_1\rangle$, $c|\pi_2\rangle = -|\pi_2\rangle$

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Furthermore

$$T_2 = -i\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

~~from $T_i T_2 = i\epsilon_i$~~

$T_2|\pi_i\rangle = i\epsilon_{2ij}|\pi_j\rangle$

$$T_2^2 = (-i)^2 \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 1 \end{pmatrix}$$

$$T_2^3 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 1 \end{pmatrix} (-i) \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = T_2$$

$$\vdots$$

$$e^{-i\pi T_2} = \underset{\sim}{1} - i\pi T_2 + \frac{(-i\pi T_2)^2}{2} + \cdots$$

$$= \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix} + T_2^2 \left(1 + \frac{(-i\pi)^2}{2} + \cdots \right)$$

$$+ T_2 \left(-i\pi + \frac{(-i\pi)^3}{3!} \cdots \right)$$

$$= \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix} + T_2^2 \cos\pi + T_2 \subset \pi$$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

$$\therefore \; G = e^{-i\pi T_2} C = \begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$$

$$\therefore \; G | \pi_j \rangle = -| \pi_j \rangle.$$

$e^{-i\pi T_2}$: symmetry of strong interaction

C : charge conjugation → also sym of strong interaction

$\therefore$ G is a symmetry of strong interaction.

$$-\ \omega \to 3\pi \qquad G|\omega\rangle = -|\omega\rangle$$

$$\rho \to 2\pi \qquad G|\rho\rangle = +|\rho\rangle.$$

5-5 Experimental Test of Isospin Invariance.

— So far Isospin Invariance was used in masses.

$$[\hat{H}_{strong}, T] = 0$$ particles in same representa[tion] have same mass.

— Particle decay.

Note. Clebsch-Gordon coeff.

$$|T\,T_3\rangle = \sum_{T_3(1)+T_3(2)=T_3} (T(1)\,T(2)\,T\,|\,T_3(1)\,T_3(2)\,T_3)\,|T(1)\,T_3(1)\rangle|T(2)\,T_3(2)\rangle$$

$$T = T(1)+T(2),\ \cdots\ |T(1)-T(2)|$$

Inverse

$$|T(1)\,T_3(1)\rangle|T(2)\,T_3(2)\rangle = \sum_{T=|T(1)-T(2)|}^{T(1)+T(2)} (T(1)\,T(2)\,T'\,|\,T_3(1)\,T_3(2)\,T_3')\,|T'\,T_3'\rangle$$

when $T_3' = T_3(1) + T_3(2)$

Therefore

$$\langle T(1)\,T_3(1);\,T(2)\,T_3(2)|\,\hat{S}\,|T\,T_3\rangle$$

$$= \sum_{T} (T(1)\,T(2)\,T'\,|\,T_3(1)\,T_3(2)\,T_3'\rangle\langle T'\,T_3'|\,\hat{S}\,|T\,T_3\rangle$$

but $S = \exp(-iHt/\hbar)$

$$\downarrow$$

$$[H, T] = 0,$$

$\therefore \langle T' T_3' | \hat{S} | T T_3 \rangle = \delta_{TT'} \delta_{T_3 T_3'} \langle T \| S \| T \rangle$

Simple case of Wigner-Eckart theorem.

$\therefore \langle T(1) T_3(1); T(2) T_3(2) | S | T T_3 \rangle$

$= (T(1) T(2) T | T_3(1) T_3(2) T_3) \langle T(1) T(2) \| \hat{S} \| T \rangle$

◎ Example 5.9. Pion-Production in Proton-Deuteron Scattering

$p + d \to \pi^0 + {}^3He_{(npp)} \quad |1 0\rangle |\tfrac{1}{2} \tfrac{1}{2}\rangle$

$p + d \to \pi^+ + {}^3H_{(nnp)} \quad |1 1\rangle |\tfrac{1}{2} -\tfrac{1}{2}\rangle$

$|\tfrac{1}{2} \tfrac{1}{2}\rangle |0 0\rangle \Rightarrow |\tfrac{1}{2} \tfrac{1}{2}\rangle$

$\therefore R = \dfrac{\sigma(p+d \to \pi^+ + H^3)}{\sigma(p+d \to \pi^0 + He^3)} = \dfrac{|(1 \tfrac{1}{2} \tfrac{1}{2} | 1 -\tfrac{1}{2} \tfrac{1}{2})|^2}{|(1 \tfrac{1}{2} \tfrac{1}{2} | 0 \tfrac{1}{2} \tfrac{1}{2})|^2} = \dfrac{\tfrac{2}{3}}{\tfrac{1}{3}} = 2$

34. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

		J	J	...
		M	M	...
m_1	m_2			
m_1	m_2	Coefficients		
⋮	⋮			

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

1/2×1/2

m_1	m_2	J=1, M=+1
+1/2	+1/2	1

m_1	m_2	J=1, M=0	J=0, M=0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	J=1, M=−1
−1/2	−1/2	1

1×1/2

m_1	m_2	J=3/2, M=+3/2
+1	+1/2	1

m_1	m_2	J=3/2, M=+1/2	J=1/2, M=+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

m_1	m_2	J=3/2, M=−1/2	J=1/2, M=−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

m_1	m_2	J=3/2, M=−3/2
−1	−1/2	1

2×1/2

m_1	m_2	J=5/2, M=+5/2
+2	+1/2	1

m_1	m_2	J=5/2, M=+3/2	J=3/2, M=+3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

m_1	m_2	J=5/2, M=+1/2	J=3/2, M=+1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

m_1	m_2	J=5/2, M=−1/2	J=3/2, M=−1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

m_1	m_2	J=5/2, M=−3/2	J=3/2, M=−3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

m_1	m_2	J=5/2, M=−5/2
−2	−1/2	1

3/2×1/2

m_1	m_2	J=2, M=+2
+3/2	+1/2	1

m_1	m_2	J=2, M=+1	J=1, M=+1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

m_1	m_2	J=2, M=0	J=1, M=0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	J=2, M=−1	J=1, M=−1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

m_1	m_2	J=2, M=−2
−3/2	−1/2	1

2×1

m_1	m_2	J=3, M=+3
+2	+1	1

m_1	m_2	J=3, M=+2	J=2, M=+2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

m_1	m_2	J=3, M=+1	J=2, M=+1	J=1, M=+1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

m_1	m_2	J=3, M=0	J=2, M=0	J=1, M=0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

m_1	m_2	J=3, M=−1	J=2, M=−1	J=1, M=−1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

m_1	m_2	J=3, M=−2	J=2, M=−2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

m_1	m_2	J=3, M=−3
−2	−1	1

1×1

m_1	m_2	J=2, M=+2
+1	+1	1

m_1	m_2	J=2, M=+1	J=1, M=+1
+1	0	1/2	1/2
0	+1	1/2	−1/2

m_1	m_2	J=2, M=0	J=1, M=0	J=0, M=0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

m_1	m_2	J=2, M=−1	J=1, M=−1
0	−1	1/2	1/2
−1	0	1/2	−1/2

m_1	m_2	J=2, M=−2
−1	−1	1

3/2×1

m_1	m_2	J=5/2, M=+5/2
+3/2	+1	1

m_1	m_2	J=5/2, M=+3/2	J=3/2, M=+3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

m_1	m_2	J=5/2, M=+1/2	J=3/2, M=+1/2	J=1/2, M=+1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

m_1	m_2	J=5/2, M=−1/2	J=3/2, M=−1/2	J=1/2, M=−1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

m_1	m_2	J=5/2, M=−3/2	J=3/2, M=−3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

m_1	m_2	J=5/2, M=−5/2
−3/2	−1	1

Ex 5.11 Pion - Nucleon Scattering

$$\pi + N \rightarrow \pi' + N'$$

$$|T=1\rangle \otimes |T=\tfrac{1}{2}\rangle \rightarrow \underbrace{\quad}_{\{\frac{1}{2}\} + \{\frac{3}{2}\}} \rightarrow |T=1\rangle \otimes |T=\tfrac{1}{2}\rangle$$

T

Initial State

$$|1\,\mu\rangle|\tfrac{1}{2}\,\nu\rangle = (1\,\tfrac{1}{2}\,\tfrac{1}{2}\,|\,\mu\,\nu\,\mu+\nu)\,|\tfrac{1}{2},\mu+\nu\rangle + (1\,\tfrac{1}{2}\,\tfrac{3}{2}\,|\,\mu\,\nu\,\mu+\nu)\,|\tfrac{3}{2},\mu+\nu\rangle$$

↑ intermediate states

Final state.

$$|1\,\mu'\rangle|\tfrac{1}{2}\,\nu'\rangle = (1\,\tfrac{1}{2}\,\tfrac{1}{2}\,|\,\mu'\nu'\,\mu'+\nu')\,|\tfrac{1}{2},\mu'+\nu'\rangle + (1\,\tfrac{1}{2}\,\tfrac{3}{2}\,|\,\mu'\nu'\,\mu'+\nu')\,|\tfrac{3}{2},\mu'+\nu'\rangle$$

Therefore.

$$\langle 1\mu\,\tfrac{1}{2}\nu\,|\,\hat{S}\,|\,1\mu'\,\tfrac{1}{2}\nu'\rangle = (1\,\tfrac{1}{2}\,\tfrac{1}{2}\,|\,\mu\nu\,\mu+\nu)\,(1\,\tfrac{1}{2}\,\tfrac{1}{2}\,|\,\mu'\nu'\,\mu'+\nu')\,\underbrace{\langle\tfrac{1}{2}\,\mu+\nu\,|\,\hat{S}\,|\,\tfrac{1}{2}\,\mu'+\nu'\rangle}_{\langle\frac{1}{2}\|\hat{S}\|\frac{1}{2}\rangle}$$

$$+ (1\,\tfrac{1}{2}\,\tfrac{3}{2}\,|\,\mu\nu\,\mu+\nu)\,(1\,\tfrac{1}{2}\,\tfrac{3}{2}\,|\,\mu'\nu'\,\mu'+\nu')\,\langle\tfrac{3}{2}\|\hat{S}\|\tfrac{3}{2}\rangle$$

2 unknown vs. 10 physical Processes.

There are Particles called Δ, with spin $= 3/2$

$\Delta^{++}(1232)$ $\Delta^{+}(1232)$ $\Delta^{0}(1232)$ $\Delta^{-}(1232)$

these form a multiplate $T = \frac{3}{2}$.

the ratio of producing these Δ's at intermediate states

$$\sigma(\pi^+ p \to \pi^+ p) : \sigma(\pi^- p \to \pi^- p) : \sigma(\pi^- p \to \pi^0 n)$$

$$\left| \langle 1\, \tfrac{1}{2}\, \tfrac{3}{2} \,|\, 1\, \tfrac{1}{2}\, \tfrac{3}{2} \rangle \times \langle 1\, \tfrac{1}{2}\, \tfrac{3}{2} \,|\, 1\, \tfrac{1}{2}\, \tfrac{3}{2} \rangle \right|^2$$

$$1 \quad \times \quad 1$$

$$: \left| \langle 1\, \tfrac{1}{2}\, \tfrac{3}{2} \,|\, -1\, \tfrac{1}{2}\, -\tfrac{1}{2} \rangle \langle 1\, \tfrac{1}{2}\, \tfrac{3}{3} \,|\, -1\, \tfrac{1}{2}\, -\tfrac{1}{2} \rangle \right|^2 : \left| \langle 1\, \tfrac{1}{2}\, \tfrac{3}{2} \,|\, +1\, \tfrac{1}{2}\, -\tfrac{1}{2} \rangle \langle 1\, \tfrac{1}{2}\, \tfrac{3}{2} \,|\, 0\, -\tfrac{1}{2}\, -\tfrac{1}{2} \rangle \right|^2$$

$$\sqrt{\tfrac{1}{3}} \qquad \sqrt{\tfrac{1}{3}} \quad : \quad \sqrt{\tfrac{1}{3}} \qquad \sqrt{\tfrac{2}{3}}$$

$$= 1 : \frac{1}{9} : \frac{2}{9} = 9 : 1 : 2$$