3.1 Probability conservation in Rel. Quan Mech

- Early Attempts

$H = p^2/2m$

$H = i\hbar \frac{\partial}{\partial t} \qquad p = -i\hbar \nabla$

$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi \quad \rightarrow \quad \frac{\partial \psi}{\partial t} = +\frac{i\hbar}{2m} \nabla^2 \psi$

$\frac{\partial \psi^*}{\partial t} = -\frac{i\hbar}{2m} \nabla^2 \psi^*$

$\rho = |\psi|^2 > 0$ probability density

$S = -(i\hbar/2m)(\psi^* \nabla \psi - \psi \nabla \psi^*)$: flux density

Continuity eq

$\frac{\partial \rho}{\partial t}$

$= \left(\frac{\partial}{\partial t}\psi^*\right)\psi + \psi^* \frac{\partial}{\partial t}\psi$

$= -\frac{i\hbar}{2m}(\nabla^2 \psi^*)\psi + \psi^* \frac{i\hbar}{2m} \nabla^2 \psi$

$= -\frac{i\hbar}{2m}(\nabla \cdot (\nabla \psi^* \psi - \psi^* \nabla \psi))$

$\therefore \frac{\partial \rho}{\partial t} + \nabla \cdot S = 0.$

Relativistic form ?

$$H = \sqrt{p^2c^2 + m^2c^4}$$

$$i\hbar \frac{\partial \psi}{\partial t} = \sqrt{-\hbar^2 c^2 \nabla^2 + m^2c^4}\,\psi \qquad ?$$

So use

$$H^2 = p^2c^2 + m^2c^4$$

$$-\hbar^2 \frac{\partial^2}{\partial t^2}\psi = (-\hbar^2\nabla^2 c^2 + m^2c^4)\psi$$

$$\left[\Box\psi - \left(\frac{mc}{\hbar}\right)^2\right]\psi = 0$$

H has two solutions $\pm E$.

Can introduce

$$S_\mu = A\left(\psi^* \frac{\partial\psi}{\partial x_\mu} - \frac{\partial\psi^*}{\partial x_\mu}\psi\right)$$

$$\frac{\partial S_\mu}{\partial x_\mu} = A\left(\cancel{\frac{\partial\psi^*}{\partial x_\mu}\frac{\partial\psi}{\partial x_\mu}} + \psi^*\Box\psi - \Box\psi^*\,\psi - \cancel{\frac{\partial\psi^*}{\partial x_\mu}\frac{\partial\psi}{\partial x_\mu}}\right)$$

$$= 0.$$

But

$$S_4 = \frac{A}{ic}\left(\psi^*\frac{\partial\psi}{\partial t} - \frac{\partial\psi^*}{\partial t}\psi\right)$$

is not positive definite.

→ later it turns out to be OK. later

3-2. The Dirac Equation.

Spin이 있는 입자는

$$H = -\frac{e\hbar}{2mc}\,\sigma\cdot B \quad \text{가 나와야 한다.}$$

But why does it not come from minimal sub.

$$H^{KE} = p^2/2m$$

$$H^{KE} = (\sigma\cdot p)(\sigma\cdot p)/2m. \quad \leftarrow \text{assume}$$

Minimal sub.

$$\frac{1}{2m}\sigma\cdot(p-\frac{eA}{c})\,\sigma\cdot(p-\frac{eA}{c})$$

$$= \frac{1}{2m}\underbrace{\sigma^i\sigma^j}_{\delta^{ij}+i\epsilon^{ijk}\sigma^k}(p-\frac{eA}{c})^i(p-\frac{eA}{c})^j$$

$$= \frac{1}{2m}(p-\frac{eA}{c})^2 + \frac{i}{2m}\sigma\cdot[\underbrace{(p-\frac{eA}{c})\times(p-\frac{eA}{c})}]$$

$$\cancel{p\times p}^{\to 0} - \frac{e}{c}(p\times A + A\times p) + (\frac{e}{c})^2\cancel{A\times A}^{\to 0}$$

$$-i\hbar\nabla\times A = -i\hbar(\nabla\times A) + i\hbar A\times\nabla$$

$$= -\frac{e}{c}(-i\hbar)(\nabla\times A).$$

$$= \frac{1}{2m}(p-\frac{eA}{c})^2 - \frac{e\hbar}{2mc}\,\sigma\cdot B.$$

Now what is the ~~Rela~~ Relativistic form

$$(E^2/c^2) - p^2 = (mc)^2$$

$$\left(\frac{E}{c}\right)^2 - (\sigma\cdot p)^2 = \left(\frac{E}{c} - \sigma\cdot p\right)\left(\frac{E}{c} + \sigma\cdot p\right) = (mc)^2$$

$\Downarrow$

$i\cancel{\hbar\partial}$ $$E = i\hbar\frac{\partial}{\partial t} = i\hbar c\frac{\partial}{\partial x_0}.$$

$$\left(i\hbar\frac{\partial}{\partial x_0} + \sigma\cdot i\hbar\nabla\right)\left(i\hbar\frac{\partial}{\partial x_0} - \sigma\cdot i\hbar\nabla\right)\phi = (mc)^2\phi. \quad (1)$$

two component.

$\downarrow$

this is still Quadratic in time.

introduce a layer space. ϕ^R, ϕ^L

$$\phi^R = \frac{1}{mc}\left(i\hbar\frac{\partial}{\partial x_0} - i\hbar\sigma\cdot\nabla\right)\phi, \quad \phi^L = \phi.$$

we can get (1) if

$$\left[i\hbar\sigma\cdot\nabla - i\hbar(\partial/\partial x_0)\right]\phi^L = -mc\,\phi^R$$

$$\left[-i\hbar\sigma\cdot\nabla - i\hbar(\partial/\partial x_0)\right]\phi^R = -mc\,\phi^L$$

sum, and differences give

$$-i\hbar(\sigma\cdot\nabla)(\phi^R-\phi^L) - i\hbar(\partial/\partial x_0)(\phi^L+\phi^R) = -mc(\phi^L+\phi^R)$$

$$i\hbar(\sigma\cdot\nabla)(\phi^L+\phi^R) + i\hbar(\partial/\partial x_0)(\phi^R-\phi^L) = -mc(\phi^R-\phi^L)$$

$\cancel{\phi^R+\phi^L=}$

denoting $\psi = \begin{pmatrix}\psi_A \\ \psi_B\end{pmatrix} = \begin{pmatrix}\phi^R+\phi^L \\ \phi^R-\phi^L\end{pmatrix}$

$$\begin{pmatrix} -i\hbar(\partial/\partial x_0) & -i\hbar\sigma\cdot\nabla \\ i\hbar\sigma\cdot\nabla & i\hbar(\partial/\partial x_0)\end{pmatrix}\begin{pmatrix}\psi_A \\ \psi_B\end{pmatrix} = -mc\begin{pmatrix}\psi_A \\ \psi_B\end{pmatrix}$$

$$\Rightarrow \left(\gamma\cdot\nabla + \gamma_4\frac{\partial}{\partial \underbrace{ix_0}_{x_4}}\right)\psi + \frac{mc}{\hbar}\psi = 0 \quad \text{— Dirac Eq.}$$

$$\gamma_k = \begin{pmatrix} 0 & -i\sigma_k \\ i\sigma_k & 0\end{pmatrix} \qquad \gamma_4 = \begin{pmatrix} I & 0 \\ 0 & -I\end{pmatrix}$$

$\rightarrow \left(\gamma_\mu\frac{\partial}{\partial x_\mu} + \frac{mc}{\hbar}\right)\psi = 0 \rightarrow$ Dirac Eq.

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}$ $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}$ $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}.$$

Note $\psi = \begin{pmatrix}\psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4\end{pmatrix}$ $\left(\gamma_\mu\frac{\partial}{\partial x_\mu} + \frac{mc}{\hbar}\right)\psi = 0$

Note.

$$\{\gamma_\mu \, \gamma_\nu\} = \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}.$$

$$\gamma_4^2 = 1$$

$\gamma_\mu^\dagger = \gamma_\mu$: in this Representation
and traceless.

$$\left(\gamma_4 \frac{\partial}{\partial x_4} + \gamma^i \frac{\partial}{\partial x_i} + \frac{mc}{\hbar}\psi\right) = 0 \qquad x_4 = ict$$

Multiplying Dirac Eq by $\gamma_4 \times c$

$$\left(c\underbrace{\frac{\partial}{\partial x_4}}_{\frac{\partial}{\partial ict}} + c\gamma_4\gamma^i\frac{\partial}{\partial x_i} + \gamma_4\frac{mc^2}{\hbar}\right)\psi = 0$$

$$H\psi = i\hbar\left(\frac{\partial\psi}{\partial t}\right)$$

$$H = -ic\hbar\,\alpha\cdot\nabla + \beta mc^2$$

when $\beta = \gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad \alpha_k = i\gamma_4\gamma_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}$

Note

$$\{\alpha_k, \beta\} = 0, \quad \beta^2 = 1 \qquad \{\alpha_k, \alpha_\ell\} = 2\delta_{k\ell}$$

Conserved Current

define $\bar{\psi} = \psi^{\dagger}\gamma_4$

$$= (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*)$$

Dirac Eq

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + \frac{mc}{\hbar}\right)\psi = 0 \quad -①$$

Hermitian Conjugate

$$\frac{\partial}{\partial x_k}\psi^{\dagger}\gamma_k + \frac{\partial}{\partial x_4^*}\psi^{\dagger}\gamma_4 + \frac{mc}{\hbar}\psi^{\dagger} = 0 \quad -②$$

$$\hookrightarrow -\frac{\partial}{\partial x_4} \qquad \leftarrow \times \gamma_4$$

$$-\frac{\partial}{\partial x_\mu}\bar{\psi}\gamma_\mu + \frac{mc}{\hbar}\bar{\psi} = 0$$

$$\bar{\psi} \times ① - ② \times \psi$$

$$\bar{\psi}\gamma_\mu \frac{\partial}{\partial x_\mu}\psi + \left(\frac{\partial}{\partial x_\mu}\bar{\psi}\right)\psi = 0$$

$$\therefore \frac{\partial}{\partial x_\mu}(\bar{\psi}\gamma_\mu\psi) = 0$$

conserved current

$$s_\mu = ic\bar{\psi}\gamma_\mu\psi = (\underbrace{c\psi^{\dagger}\alpha\psi}_{\text{flux density}}, \underbrace{ic\psi^{\dagger}\psi}_{\text{prob density}})$$

$$\int \psi^{\dagger}\psi \, d^3x = \text{const.}$$

$\psi^{\dagger}\psi = \psi_\alpha^*\psi_\alpha \geq 0$ positive definite $\rightarrow$ Probability

Representation independence

next time.

3-3. Simple Solutions; NonRelativistic Approximations; Plane waves.

$$p \to p - \frac{e}{c}A$$

$$-i\hbar\nabla \quad -i\hbar\nabla - \frac{e}{c}A = -i\hbar\left(\nabla - i\frac{e}{c\hbar}\right)$$

$\therefore$ with EM, Dirac Eq becomes.

$$\left(\frac{\partial}{\partial x_\mu} - \frac{ie}{\hbar c}A_\mu\right)\gamma_\mu\psi + \frac{mc}{\hbar}\psi = 0.$$

Assume $A_\mu(x,t) = A_\mu(x)$ independent of time

$$\psi = \psi(x,t)\big|_{t=0}\, e^{-iEt/\hbar}$$

$$\therefore \left(\frac{\partial}{\partial x_4} - \frac{ie}{\hbar c}\overset{\to\, iA_0}{A_4}\right)\gamma_4\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix} + \left(\frac{\partial}{\partial x_i} - \frac{ie}{\hbar c}A_i\right)\gamma_i\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix} + \frac{mc}{\hbar}\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix} = 0$$

$$\frac{\partial}{\partial ict}$$

$$\frac{1}{ic}\,\frac{-iE}{\hbar}$$

$$\frac{-1}{\hbar c}\left(E - eA_0\right)\begin{pmatrix}\psi_A\\ -\psi_B\end{pmatrix} + \frac{i}{\hbar}\left(p_i - \frac{eA_i}{c}\right)\begin{pmatrix}0 & -i\sigma_i\\ i\sigma_i & 0\end{pmatrix}\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix} =$$

$$\Rightarrow [\sigma\cdot(p-\frac{eA}{c})]\psi_B = \frac{1}{c}(E-eA_0-mc^2)\psi_A \quad -①$$

$$-[\sigma\cdot(p-\frac{eA}{c})]\psi_A = -\frac{1}{c}(E-eA_0+mc^2)\psi_B \quad -②$$

$$② \rightarrow \frac{c^2}{E-eA_0+mc^2}[\sigma\cdot(p-\frac{eA}{c})]\psi_A = c\psi_B$$

multiply by $\sigma\cdot(p-\frac{eA}{c})$

$$[\sigma\cdot(p-\frac{eA}{c})][\frac{c^2}{E-eA_0+mc^2}][\sigma\cdot(p-\frac{eA}{c})]\psi_A$$

$$= c\,\sigma\cdot(p-\frac{eA}{c})\psi_B = (\underbrace{E-eA_0-mc^2}_{E^{NR}-eA_0})\psi_A$$

note

$$\frac{c^2}{E-eA_0+mc^2} = \frac{1}{2m}\;\frac{2mc^2}{2mc^2+E^{NR}-eA_0} = \frac{1}{2m}[1-\frac{E^{NR}-eA_0}{2mc^2}+\cdots]$$

$$\frac{1}{2m}[\sigma\cdot(p-\frac{eA}{c})]^2\psi_A = (E^{NR}-eA_0)\psi_A$$

$$\rightarrow [\frac{1}{2m}(p-\frac{eA}{c})^2-\frac{e\hbar}{2mc}\sigma\cdot B+eA_0]\psi_A = E^{NR}\psi_A$$

$\frac{v^2}{c^2}$ Note

$② \rightarrow$ at order $mv\,\underset{\text{large component}}{\psi_A} = \frac{mc^2}{c}\,\underset{\text{small comp}}{\psi_B}$

* Approximate Hamiltonian for an electrostatic problem.

take $\vec{A}=0$ in Eq 3.63

$$(\sigma\cdot p)\underbrace{\left(\frac{c^2}{E-eA_0+mc^2}\right)}(\sigma\cdot p)\,\psi_A=(E-eA_0-mc^2)\psi_A$$

$$\frac{1}{2m}\left(1-\frac{E^{NR}-eA_0}{2mc^2}\right)$$

$$\therefore \underset{\sim}{H_A^{NR}}\psi_A=E^{NR}\psi_A$$

$$\hookrightarrow \sigma\cdot p\,\frac{1}{2m}\left(1-\frac{E^{NR}-eA_0}{2mc^2}\right)(\sigma\cdot p)+eA_0$$

Not an eigenvalue Eq.

Need to correct for normalization

$$\int(\psi_A^\dagger\psi_A+\psi_B^\dagger\psi_B)\,d^3x=1.$$

from previous Eq. $\psi_B\cong-\frac{\sigma\cdot p}{2mc}\psi_A$

$$\hookrightarrow \int\psi_A^\dagger\left(1+\frac{p^2}{4m^2c^2}\right)\psi_A\,d^3x=1$$

$\therefore$ properly Normalized wave ftn is

$$\Psi=\Omega\psi_A=\left(1+p^2/8m^2c^2\right)\psi_A$$

$$\Omega = 1 + p^2/8m^2c^2$$

$$\Omega^{-1} = 1 - p^2/8m^2c^2.$$

$$\therefore \quad H_A^{NR}\psi_A = E^{NR}\psi_A$$

$$\hookrightarrow \quad \underline{\Omega^{-1} H_A^{NR}} \Omega^{-1}\psi = E^{NR}\Omega^{-2}\psi$$

$$(1 - p^2/8m^2c^2)\ \sigma\cdot p \frac{1}{2m}\left(1 - \frac{E^{NR} - eA_0}{2mc^2}\right)\sigma\cdot p\ (1 - p^2/8mc^2)$$

$$+ \ (1 - p^2/8mc^2)\ (eA_0)\ (1 - p^2/8mc^2).$$

$$\cancel{= \frac{p^2}{2m} + eA_0 - \frac{p^4}{8m^3c^2} + \frac{1}{8m^2c^2}\left\{\left\{p^2, (E^{NR} - eA_0)\right\} + - 2\sigma\cdot p\,(E^{NR} - eA_0)(\sigma\cdot p)\right\}}$$

$$\Rightarrow \left[\frac{p^2}{2m} + eA_0 - \left\{\frac{p^2}{8m^2c^2}, \left(\frac{p^2}{2m} + eA_0\right)\right\} - \frac{\sigma\cdot p}{2m}\left(\frac{E^{NR} - eA_0}{2mc^2}\right)\sigma\cdot p\right]\psi$$

$$= E^{NR}\left(1 - \frac{p^2}{4m^2c^2}\right)\psi$$

$$\Rightarrow \Big[\frac{p^2}{2m} + eA_0 - \frac{p^4}{8m^3c^2} + \frac{1}{8m^2c^2}\Big(\{p^2, (E^{NR} - eA_0)\}$$

$$-2(\sigma\cdot p)(E^{NR} - eA_0)(\sigma\cdot p)\Big)\Big]\varphi$$

$$= E^{NR}\varphi.$$

In general

$$\{A^2, B\} - 2ABA = A^2B + BA^2$$

$$= A^2B - ABA + BA^2 - ABA$$

$$= A[A,B] + [B,A]A.$$

$$= [A, [A,B]]$$

Note

$$[\sigma\cdot p, (E^{NR} - eA_0)] = [\sigma^i(-i\hbar)\partial^i, E^{NR} - eA_0]$$

$$= \sigma^i(-i\hbar)(-e)E^i$$

$$= -ie\hbar\,\sigma\cdot E$$

$$[\sigma\cdot p, -ie\hbar\,\sigma\cdot E] = -e\hbar^2\nabla\cdot E - 2e\hbar\,\sigma\cdot(E\times p)$$

$$\Big[\frac{p^2}{2m} + eA_0 - \frac{p^4}{8m^3c^2} - \frac{e\hbar\,\sigma\cdot(E\times p)}{4m^2c^2} - \frac{e\hbar^2}{8m^2c^2}\nabla\cdot E\Big]\varphi$$

$$= E^{NR}\varphi.$$

$$E = -\nabla V = -\frac{\vec{x}}{r}\frac{\partial V}{\partial r}$$

$$-\frac{e\hbar^2}{4m^2c^2}\sigma\cdot(E\times p) = -\frac{\cancel{e}\hbar}{4m^2c^2}\left(-\frac{1}{r}\frac{dv}{dr}\right)\sigma\cdot(\vec{x}\times\vec{p})$$

$$= \frac{1}{2m^2c^2}\frac{1}{r}\frac{dv}{dr}S\cdot L.$$

Spin orbit coupling.