习题

我没有答案

高寒
gaoh26@mail2.sysu.edu.cn

1.0

What is the partition function?

- 玻尔兹曼分布的归一化因子;
- 均匀分布的矩母函数;
- 自由能的指数$e^{-\beta F}$;
- 能量态密度$g(E)$的拉普拉斯变换;

1.1.1

Compute the partition function of a quantum harmonic oscillator with frequency ω and energy levels

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) \quad n \in \mathbf{Z}$$

Find the average energy U and entropy S as a function of temperature T.

1.1.2

Einstein constructed a simple model of a solid as N atoms, each of which vibrates with the same frequency ω. Treating these vibrations as a harmonic oscillator, show that at high temperatures, $k_B T \gg \hbar\omega$, the Einstein model correctly predicts the Dulong-Petit law for the heat capacity of a solid,

$$C_V = 3Nk_B$$

At low temperatures, the heat capacity of many solids is experimentally observed to tend to zero as $c_V \sim T^3$. Was Einstein right about this?

1.2

A particle moving in one dimension has Hamiltonian

$$H = \frac{p^2}{2m} + \lambda q^4$$

Show that the heat capacity for a gas of N such particles is $C_V = 3Nk_B/4$. Explainwhy the heat capacity is the same regardless of whether the particles are distinguishable or indistinguishable.

1.3

A non-relativistic particle is confined in a spherical container with radii R with potential

$$V(r) = \begin{cases} -h & , \ r < a \\ 0 & , \ a \le r < R \\ +\infty & , \ r > R \end{cases}$$

Firstly predict the rough behavior of the pressure for a gas of N such particles as a function of T, then determine $P(T)$ by statistics.

2.1

Show that

1. $$C_P - C_V = T\left(\frac{\partial V}{\partial T}\right)_P\left(\frac{\partial P}{\partial T}\right)_V = -T\left(\frac{\partial V}{\partial T}\right)_P^2\left(\frac{\partial P}{\partial V}\right)_T$$

2. $$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P$$

3. $$\left(\frac{\partial U}{\partial P}\right)_T = -T\left(\frac{\partial V}{\partial T}\right)_P - P\left(\frac{\partial V}{\partial P}\right)_T$$

2.1

4. $$\left(\frac{\partial C_V}{\partial V}\right)_T = T\left(\frac{\partial^2 P}{\partial T^2}\right)_V$$

5. $$\left(\frac{\partial C_P}{\partial P}\right)_T = -T\left(\frac{\partial^2 V}{\partial T^2}\right)_P$$

From equations above, show that for any non-ideal gas with constant C_V and C_P, the equation of state can be written as

$$(C_P - C_V)T = (P + a)(V + b)$$

where a and b are constants.

2.2

Consider the neutral gas of electrons, protons and Hydrogen. They undergo a reaction as

$$\mathrm{e}^- + \mathrm{p}^+ \rightleftharpoons \mathrm{H}$$

You know from Quantum Mechanics that the Hydrogen atom has binding energy $E = -I$ (where $I = 13.6\ \mathrm{eV}$). Let the number of Hydrogen atoms be $N_\mathrm{H} = (1-x)N$ and the e number of electrons and protons be $N_\mathrm{e} = N_\mathrm{p} = xN$ with $x \in [0, 1]$.

1. What is the equilibrium condition for the gas?
2. From the condition above, show that

$$\frac{x^2}{1-x} = \frac{V}{N}\left(\frac{m_\mathrm{e} m_\mathrm{p}}{2\pi\hbar^2 m_\mathrm{H}}\right)^{3/2} (k_B T)^{3/2}\, e^{-I/k_B T}$$

3.1

A Wigner crystal is a triangular lattice of electrons in a two dimensional plane. The longitudinal vibration modes of this crystal are bosons with dispersion relation $\omega = \sigma\sqrt{k}$. Show that, at low temperatures, these modes provide a contribution to the heat capacity that scales as $C \sim T^4$.

3.2

Consider a gas of non-interacting ultra-relativistic electrons, whose mass may be neglected. Show that $U = 3PV$. Show that at zero temperature $PV^{4/3} = \mathrm{const}$. Show that at high temperatures $E = 3Nk_B T$ and the equation of state coincides with that of a classical ultra-relativistic gas.
What are the results if we count the existance of positrons ($\mu_{\mathrm{e}^+} = -\mu_{\mathrm{e}^-}$)?

3.3

Consider the free energy $F = a(T)m^2 + b(T)m^4 + c(T)m^6$ where $b(T) < 0$ and, for stability, $c(T) > 0$ for all T. Sketch the possible behaviours of the free energy as $a(T)$ varies and, in each case, identify the ground state and metastable states. Show that the system undergoes a first order phase transition at some temperature T_c. Determine the value $a(T_c)$ and the discontinuity in m at the transition.