- The U(n) group and SU(n) group

$$|\psi'\rangle = \mathrm{U}|\psi\rangle, \quad \text{where} \quad |\psi\rangle = \begin{pmatrix} \psi_1 \\ .. \\ \psi_n \end{pmatrix} \quad \rightarrow \text{ consider } n \text{ component vector}$$

☞ Unitary operators: $\mathrm{UU}^{\dagger} = 1 \quad \rightarrow n^2$ independent U operators

Hermitian operators: $\mathrm{U} = \exp(i\mathrm{H}) \quad \rightarrow n^2$ independent H operators

→U(n) group

☞ Special Unitary operators: $\mathrm{UU}^{\dagger} = 1$ and $\mathrm{Det}(\mathrm{U}) = 1$ or $\mathrm{Tr}[\mathrm{H}] = 0$

$$\rightarrow n^2 - 1 \quad \text{independent operators}$$

→SU(n) group

- Subgroup of U(n) group and SU(n) group

$$|\psi'\rangle = \mathrm{U}|\psi\rangle, \quad \text{where} \quad |\psi\rangle = \begin{pmatrix} \psi_1 \\ .. \\ \psi_n \end{pmatrix} \qquad \mathrm{U} = \begin{pmatrix} U_{11} & .. & U_{1n} \\ .. & .. & .. \\ U_{n1} & .. & U_{nn} \end{pmatrix}$$

☞ U(m) and SU(m) are subgroups if m ≤ n

$$\mathrm{U}(n \times n) = \begin{pmatrix} U(m \times m) & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathrm{U} = \exp(i\mathrm{H})$$

$$\mathrm{H}(n \times n) = \begin{pmatrix} H(m \times m) & 0 \\ 0 & 0 \end{pmatrix}$$

→SU(2) is a subgroup of SU(3)group

• Generators of the SU(3) group

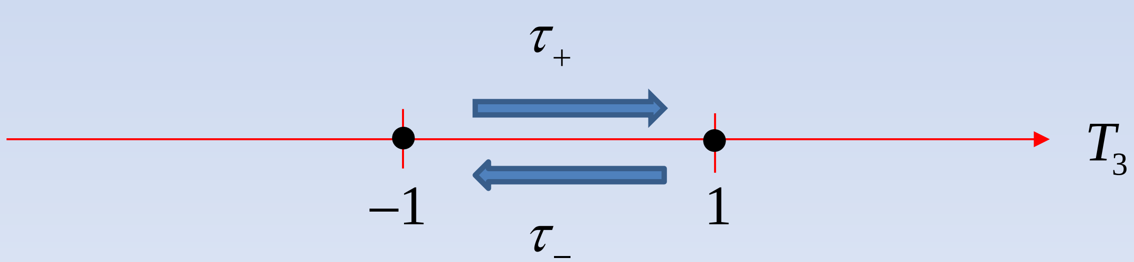
$$\mathrm{U} = \exp\left(i\mathrm{H}\right)$$

☞ Reminder of SU(2) generators

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

☞ Different view of SU(2) generators in fundamental representation

$$\tau_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \tau_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



Generators of the SU(3) group - I

☞ The Lambda matrix $2\times {}_nC_2+\left(n-1\right)=n^2-1$

$$\lambda_1=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix},\quad \lambda_2=\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix}\quad 1\leftrightarrow 2$$

$$\lambda_3=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}$$

$$\lambda_4=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix},\quad \lambda_5=\begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix}\quad 1\leftrightarrow 3$$

$$\lambda_6=\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix},\quad \lambda_7=\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix}\quad 2\leftrightarrow 3$$

$$\lambda_8=\frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}$$

Generators of the SU(3) group - II

☞ Another interpretation $2\times {}_nC_2+(n-1)=n^2-1$ $F_i=\frac{1}{2}\lambda_i$

$$\lambda_1=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix},\quad \lambda_2=\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix} \Longrightarrow T_\pm=F_1\pm iF_2 \qquad 1\leftrightarrow 2$$

$$\lambda_4=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix},\quad \lambda_5=\begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix} \Longrightarrow V_\pm=F_4\pm iF_5 \qquad 1\leftrightarrow 3$$

$$\lambda_6=\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix},\quad \lambda_7=\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix} \Longrightarrow U_\pm=F_6\pm iF_7 \qquad 2\leftrightarrow 3$$

$$\lambda_3=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}\quad \lambda_8=\frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix} \Longrightarrow \begin{array}{l}T_3=F_3\\ Y=\frac{2}{\sqrt{3}}F_8\end{array}$$

• Generators of the SU(3) group - III

☞ Commutation relation

$$\left[F_i, F_j\right] = i f_{ijk} F_k$$

☞ Maximum number of commuting operators=Rank →2

$$\left[T_3, Y\right] = 0 \quad \rightarrow \quad \left[F_3, F_8\right] = 0$$

☞ Rank→2 = 2 Casimir operators

- Generators of the SU(3) group - IV

☞ Subalgebra

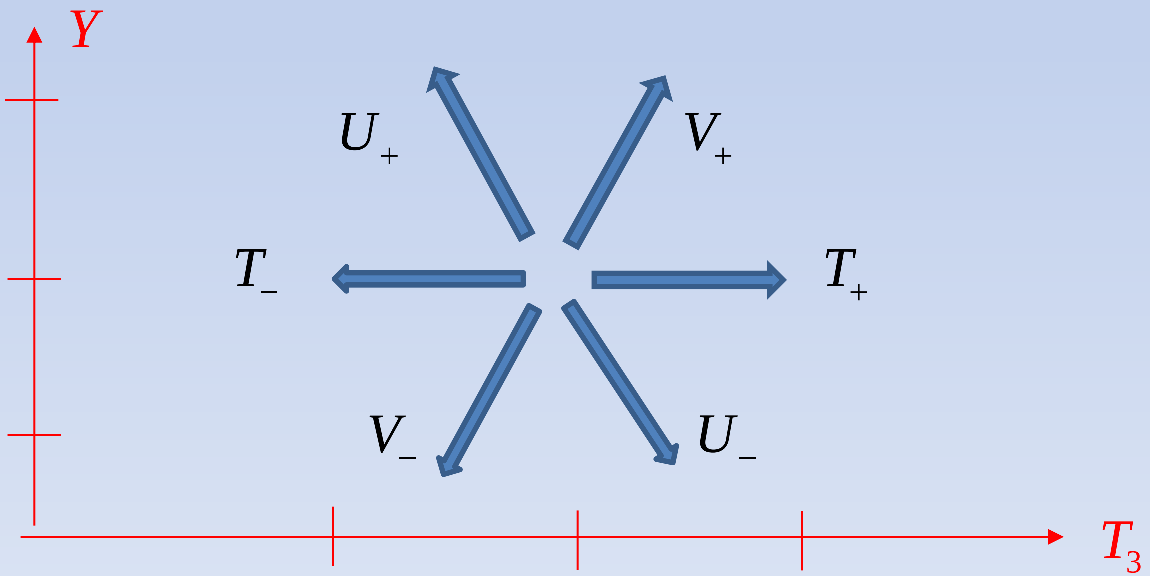
$$[T_3, T_\pm] = \pm T_\pm \qquad [T_+, T_-] = 2T_3$$

$$[U_3, U_\pm] = \pm U_\pm \qquad [U_+, U_-] = 2U_3 \equiv \tfrac{3}{2}Y - T_3$$

$$[V_3, V_\pm] = \pm V_\pm \qquad [V_+, V_-] = 2V_3 \equiv \tfrac{3}{2}Y + T_3$$

☞ Raising operators in T3 and Y space

$$[T_3, U_\pm] = \mp\frac{1}{2}U_\pm \qquad [T_3, V_\pm] = \pm\frac{1}{2}V_\pm \qquad [Y, U_\pm] = \pm U_\pm \qquad [Y, V_\pm] = \pm V_\pm$$



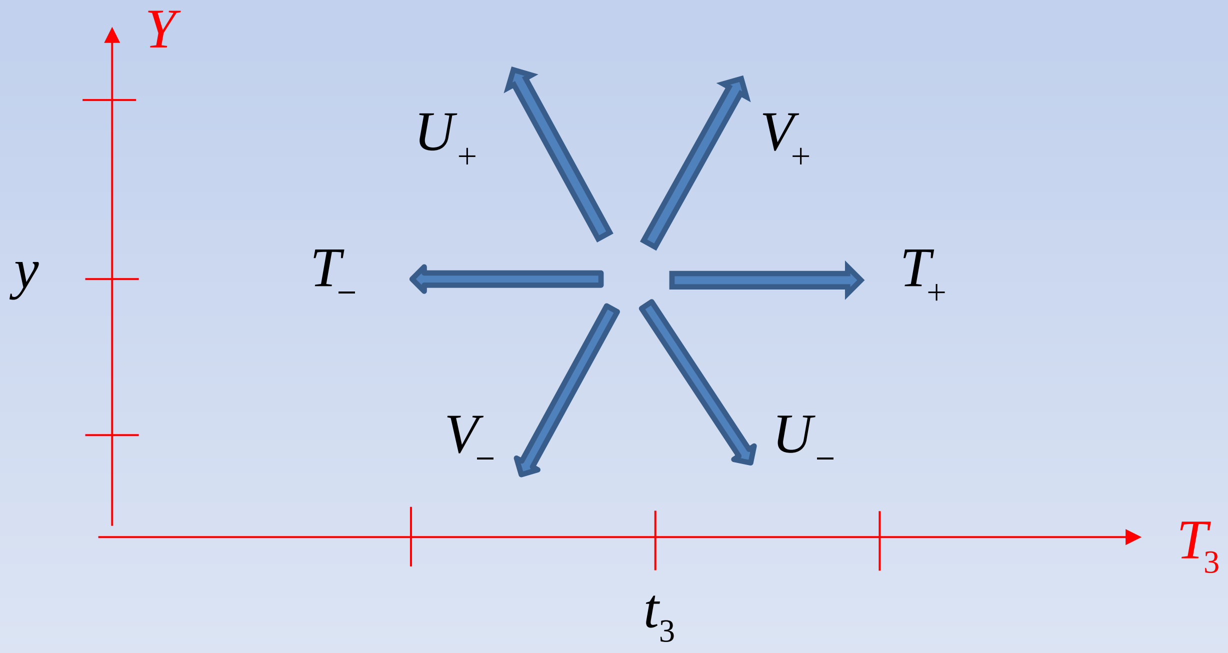
A representation of SU(3)

☞ Identify a state by T3 and Y $\left|t_3, y\right\rangle$

$$T_3\left|t_3, y\right\rangle = t_3\left|t_3, y\right\rangle \qquad Y\left|t_3, y\right\rangle = y\left|t_3, y\right\rangle$$

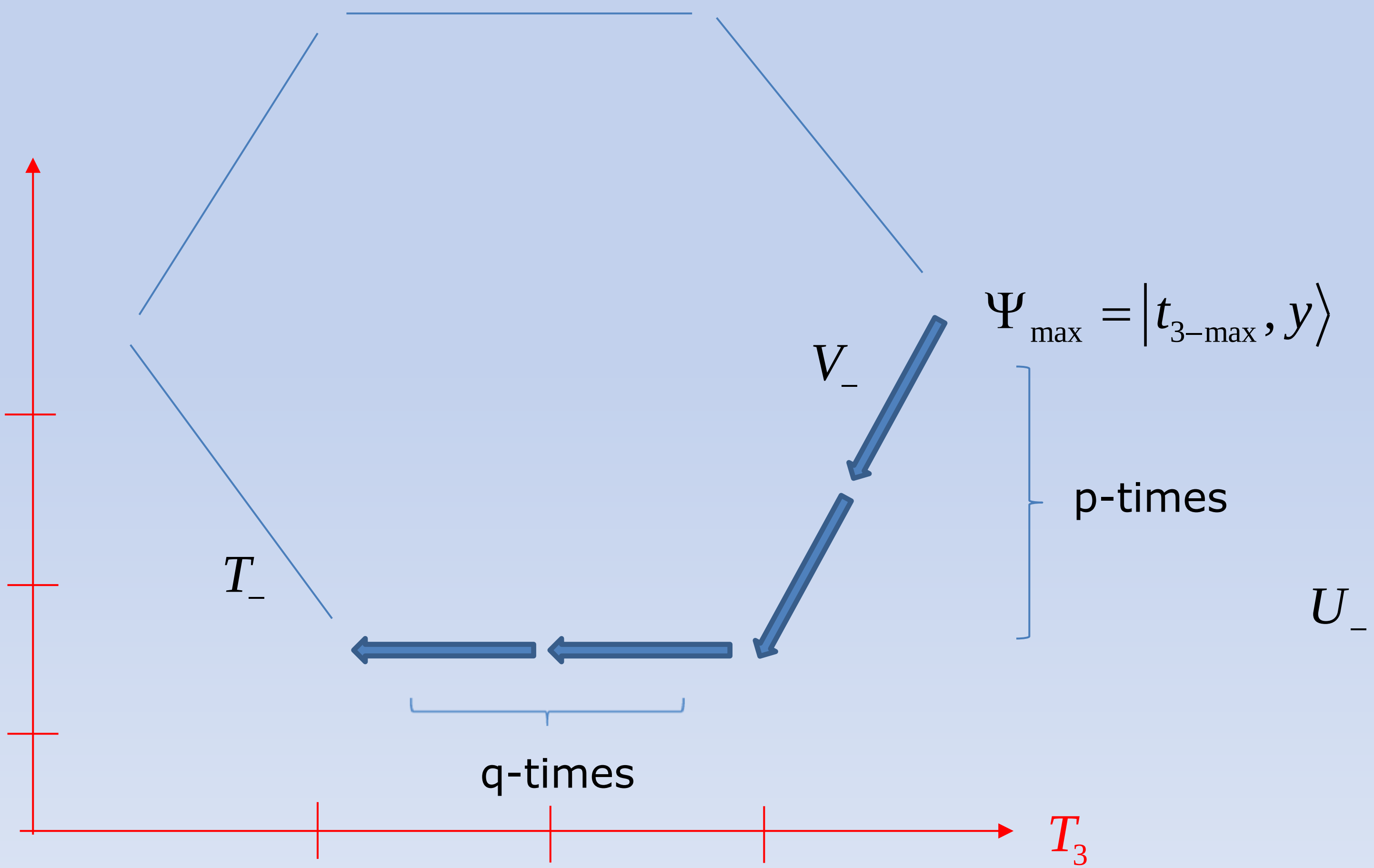
☞ $$T_3\left(V_\pm\left|t_3, y\right\rangle\right) = \left(t_3 \pm \frac{1}{2}\right)\left(V_\pm\left|t_3, y\right\rangle\right) \qquad T_3\left(U_\pm\left|t_3, y\right\rangle\right) = \left(t_3 \mp \frac{1}{2}\right)\left(U_\pm\left|t_3, y\right\rangle\right)$$

$$Y\left(V_\pm\left|t_3, y\right\rangle\right) = \left(y \pm 1\right)\left(V_\pm\left|t_3, y\right\rangle\right) \qquad Y\left(U_\pm\left|t_3, y\right\rangle\right) = \left(y \pm 1\right)\left(U_\pm\left|t_3, y\right\rangle\right)$$

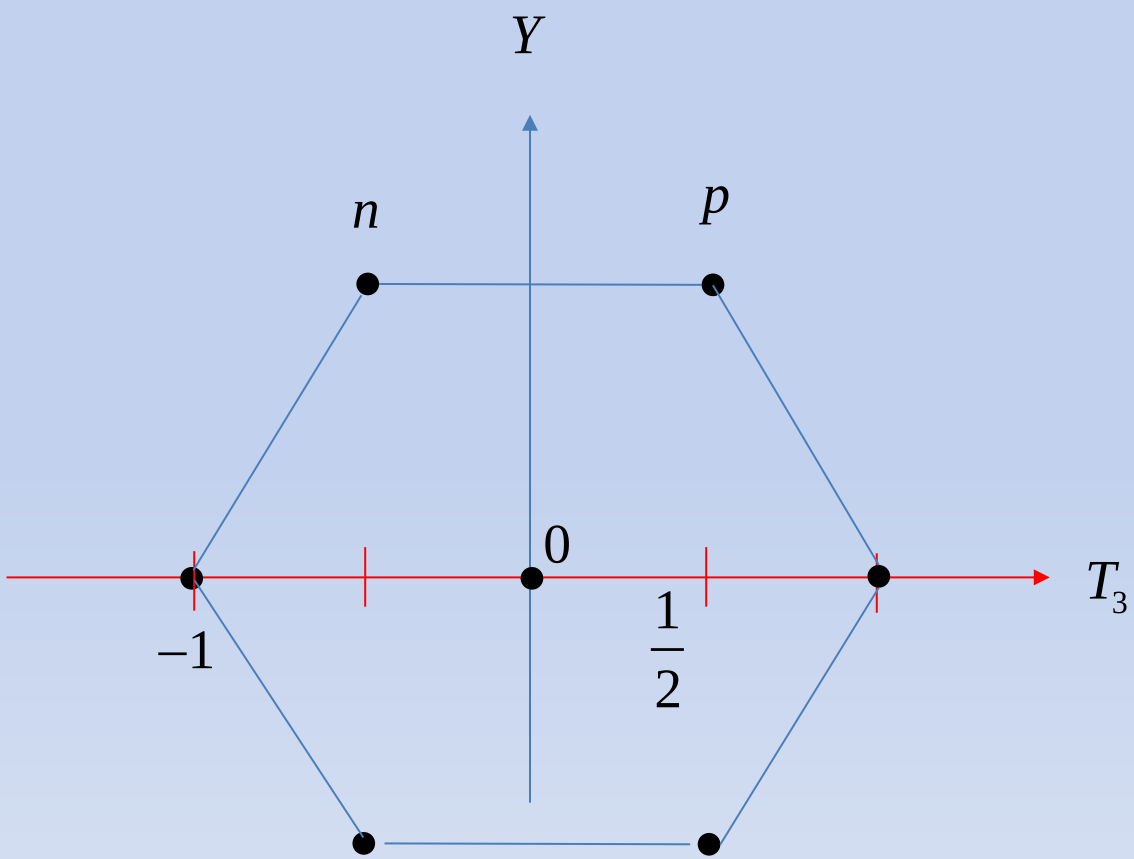


- (p,q) representation of SU(3)

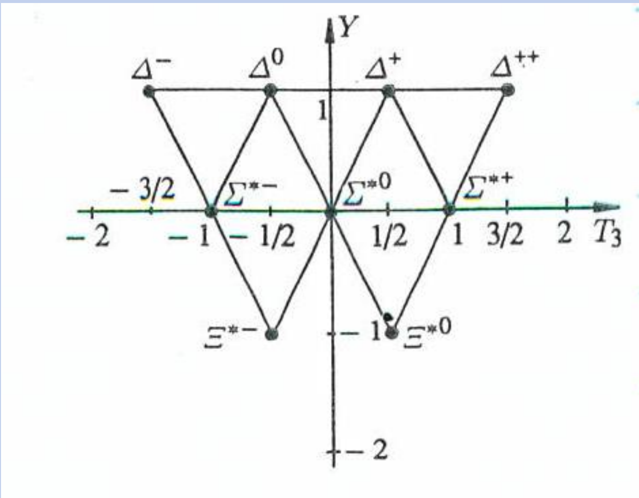
☞ Start with the largest T3 $\Psi_{\max} = \left| t_{3-\max}, y \right\rangle$

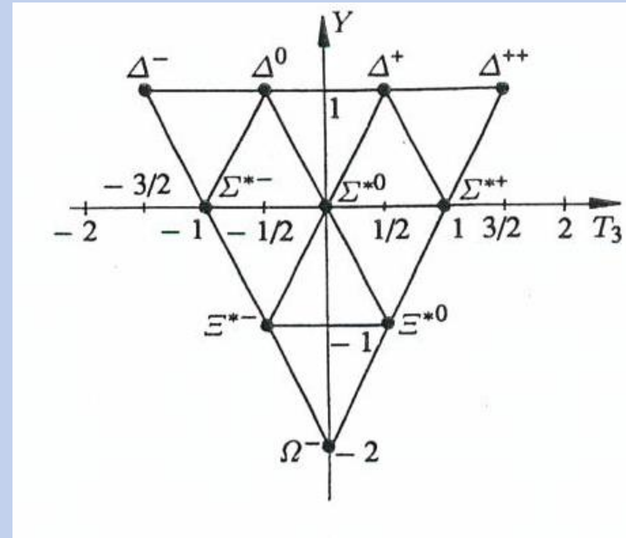


☞ (1,1) representation



☞ (3,0) representation





☞ Formula f

* Commutation Relation

$$[\lambda_i \lambda_j] = 2i f_{ijk} \lambda_k$$

① $\underbrace{[\lambda_j \lambda_i]}$ $= 2i f_{jik} \lambda_k =$ $\Rightarrow f_{ijk} = - f_j$

$\hookrightarrow -[\lambda_i \lambda_j] = -2i f_{ijk} \lambda_k$

② $[\lambda_i \lambda_j] \lambda_l = 2i f_{ijk} \lambda_k \times \lambda_l$

$\rightarrow Tr[\lambda_i \lambda_j \lambda_l - \lambda_j \lambda_i \lambda_l] = 4i f_{ijl}$

$[\lambda_i, \lambda_l] \times \lambda_j = 2i f_{ilk} \lambda_k \times \lambda_j$

$\rightarrow Tr[\lambda_i \lambda_l \lambda_j - \lambda_l \lambda_i \lambda_j] = 4i f_{ilj}$

$\Rightarrow \therefore f_{ijl} = - f_{ilj}$

f_{ijk} antisym for ~~all~~ any. permutation

* Anti commutation

$$[\lambda_i \lambda_j]_+ = \frac{4}{3} \delta_{ij} \mathbb{1} + 2 d_{ijk} \hat{\lambda}_k$$

$$d_{ijk} = d_{jik} = d_{ikj} \quad \cdots$$

Exercise 7.6. Relations between f, d.

A consequence of Commutation Relations. (~~Bi~~ Jacobbi Identity).

$$f_{p\ell m} f_{mkq} + f_{\ell km} f_{mpq} + f_{kpm} f_{m\ell q}$$

$p\ell k$ cyclic order. m : sum, q external

$$f_{pkm} d_{m\ell q} + f_{qkm} d_{m\ell p} + f_{\ell km} d_{mpq} = 0$$

$pq\ell$ cyclic order. m : sum, k : external

Ex 7.7 Casimir Operators of SU(3)

$$C_1 = \sum_{i=1}^{8} \overline{F}_i^{\,2}, \qquad C_2 = \sum_{ijk} d_{ijk} F_i \cdot F_j \cdot F_k$$

proof)

1) $$[C_1, F_K] = \sum_{i=1}^{8} \{ F_i [F_i, F_K] + [F_i, F_K] F_i \}$$

$$= i \sum_{i=1,m}^{8} \underbrace{f_{ikm}}_{\text{Antisym in } i,m} \underbrace{(F_i F_m + F_m F_i)}_{\text{Sym in } i,m}$$

2). $$[C_2, F_K] = \sum_{ijn} d_{ijn} \{ F_i F_j [F_n, F_K] + F_i [F_j, F_K] F_n + [F_i, F_K] F_j \cdot F_n \}$$

$$= i \sum_{ijn} d_{ijn} \{ F_i F_j f_{nkm} F_m + F_i F_m F_n f_{jkm} + F_m F_j F_n f_{ikm} \}$$

$$= i \sum_{ijn} F_i F_j F_m \{ \underbrace{d_{ijn} f_{nkm} + d_{imn} f_{nkj} + d_{mjn} f_{nki}}_{= 0} \}$$

nm

ijm cyclic, k external.

Exercise 7.8. Useful Relations

a). $\hat{C}_1(\vec{F}_i) \equiv \sum_l F_l^2 = -\frac{2i}{3} f_{ijk} F_i F_j F_k$

b) $\hat{C}_2(\vec{F}_i) \equiv \sum_{ijk} d_{ijk} F_i F_j F_k = C_1 (2 C_1 - \frac{11}{6})$.

proof).

$*$ Note $\begin{cases} [F_i\ F_j] = i f_{ijk} F_k \\ f_{ijk} f_{ijl} = 3\delta_{kl} \end{cases}$

a) $\sum_{ijk} f_{ijk} F_i F_j F_k = \sum_{ijk} f_{ijk} [\,F_j F_i + \underbrace{F_i F_j - F_j F_i}_{[F_i\ F_j]}\,] F_k$

$= \sum f_{ijk} F_j F_i F_k + \sum i f_{ijk} \cancel{F_{ijl}} F_l F_k$

$= \sum - f_{ijk} F_i F_j F_k + 3i \sum_l F_l^2.$

$\therefore C_1 = \sum_l F_l = -\frac{2i}{3} \sum_{ijk} F_i F_j F_k.$

b) Can be proven using $\{F_i\ F_j\}_+ = d_{ijk} \vec{F}_k + \frac{1}{3}\delta_{ij}$