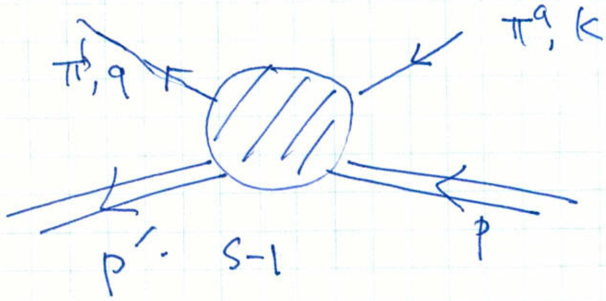
⑤ The Weinberg-Tomozawa formula

S-wave elastic scattering of a pion off any hadron target.

General Method, same as Adler's Rule., except now, we will take two pions off the mass-shell.

Consider a process of the type

$$\pi^a + i \longrightarrow \pi^b + f.$$



Need to study

$$I = \int d^4x\, d^4y\, e^{iq\cdot x}\, e^{-ik\cdot y}\, T \langle f |\, \partial^\mu A_\mu^b(x)\ \partial^\nu A_\nu^a(y)\, | i \rangle$$

→ off-mass shell

$$= -\frac{(2\pi)^4 \delta^4(p+k-p'-q)\ F_\pi^2 m_\pi^4}{(q^2-m_\pi^2)(k^2-m_\pi^2)(2\pi)^3(4EE')^{1/2}} \cdot \widetilde{\mathcal{M}}.$$

→ dep. on def. com pt anywhere

Related to what we are interested in as

$$\langle f\, b |\, S-1\, | i\, a \rangle = (k.F.)\, \mathcal{M}\ (2\pi)^4 \delta^4(p+k-p'-q)$$

$$\widetilde{\mathcal{M}} \text{ at } q^2 = m_\pi^2 = k^2.$$

then

$$\cancel{d\sigma = \frac{1}{|v_i - v_f|}\frac{1}{4\omega_i\omega_f}|\mathcal{M}|^2 \frac{d^3q}{(2\pi)^{3/2} 2q_0}\frac{d^3}{(2\pi)^{3/2}}}$$

Now.

$$\partial_x^\mu \partial_y^\nu T[A_\mu^b(x)\, A_\nu^b(y)] = T\{\partial^\mu A_\mu^b(x)\, \partial^\nu A_\nu^a(y)]$$

$$+ \delta(x^0-y^0)\,[A_0^b(x), \partial^\nu A_\nu^a(y)] \to I_1$$

$$- \partial^\mu(x)\,\delta(x^0-y^0)\,[A_\mu^b(x), A_0^a(y)]$$

Commutator.

∴ Integrating

~~$\int T[\partial^\mu A_\mu^a(x), \partial^\nu A_\nu^b(y)]$~~

$$I = I_1 + I_2 + I_3$$

$$I_1 = -\int d^4x\, d^4y\, e^{+iqx} e^{-ik\cdot y} \delta(x_0-y_0) \langle f|[A_0^b(x), \partial^\nu A_\nu^a(y)]|i\rangle$$

$$I_2 = \int d^4x\, d^4y\, e^{+iqx} e^{-ik\cdot y} \partial_x^\mu \partial_y^\nu \langle f| T[A_\mu^b(x)\, A_\nu^a(y)]|i\rangle$$

$$I_3 = -\int d^4x\, d^4y\, e^{+iqx} e^{-ik\cdot y} \partial_x^\mu \delta(x_0-y_0) \langle f|[A_0^a(y), A_\mu^b(x)]|i]$$

$$= i\int d^4x\, d^4y\, e^{+iqx} e^{-ik\cdot y} q^\mu \delta(x_0-y_0) \langle f|[A_0^a(y), A_\mu^b(x)]|i\rangle$$

i) I_3 terms

we will chose $\vec{q} = \vec{k} = 0$.

$$I_3 = -2 g_v \varepsilon_{abc} \int d^4x\, d^4y\, e^{-ik_0 y_0} e^{iq_0 y_0} \delta^4(x-y)\, q^0 \langle f| V_0^c(x)|i\rangle$$

$$\int d^3x \langle f| V_0^c(x)|i\rangle = 2 g_v \langle f| I^a|i\rangle = 2 g_v I_t^c \delta^3(p-p')$$

from norm. of state

I : total Isospin

I_t : isospin of target

$$\therefore \; I_3 = -4 g_v^2 \varepsilon_{abc} I_t^c \, 2\pi \, \delta^4(p+k-p'-q') q_0 .$$

$$\therefore \; M_3 = \frac{8 g_v^2}{F_\pi^2} \varepsilon_{abc} I_t^c \, p \cdot q$$

also note

$$\langle b | I^c | a \rangle = i \varepsilon_{abc} .$$

$$\therefore \; M_3 = -i \frac{8 g_v^2}{F_\pi^2} (I_\pi \cdot I_t) \, p \cdot q .$$

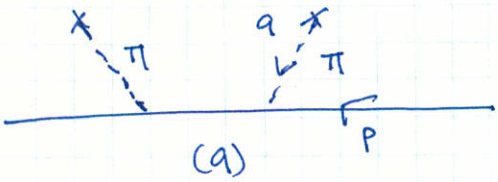
pion ↑ ↑ target

Note the factor of gv and additional factor 1/fpi

ii) I_2 part.

note

$q^\mu k^\nu \langle f | T[A_\mu^b(x) \, A_\nu^a(y)] | i \rangle$ is something we discussed before. only two possibility



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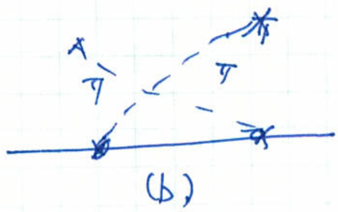

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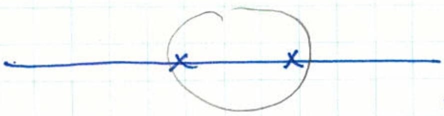


vanish when multiplied by $k_\mu q_\mu$.

These are possible additional off shell factors

$\therefore$ should only consider (a) (b)

(a) $\rightarrow$ $\bar{u}' \gamma_5 \not{k} \frac{1}{\not{p}+\not{q}-m_N} \not{q} \gamma_5 u.$

$= \bar{u}' \not{k} \frac{1}{\not{p}+\not{q}+m_t} \not{q} u$

Note these should be multiplied by factors of gA

Therefore compared to M3 we multiply by factors of gA/gV for every Axial current

Heavy target soft momentum limit.

$K_\mu = q_\mu = \frac{m_\pi}{m_t} p_\mu$

This is also the basis for why simplification occurs when derivative coupling in NNpi is used

$= \bar{u}' m_\pi \frac{1}{2m_t+m_\pi} m_\pi u. \sim O(m_\pi^2) \therefore$ neglect it.

III) I_π part

since $\partial^\nu A_\nu \rightsquigarrow m_\pi^2 \quad \therefore$ order $O(m_\pi^2) \rightsquigarrow$ neglect

called sigma term.

IV) Total

$$M = -i \frac{8 S_\nu^2}{F_\pi^2} (I_\pi \cdot I_t) p \cdot q + O\left(\frac{m_\pi^2}{m_t^2}\right)$$

$\Rightarrow$ Scattering length

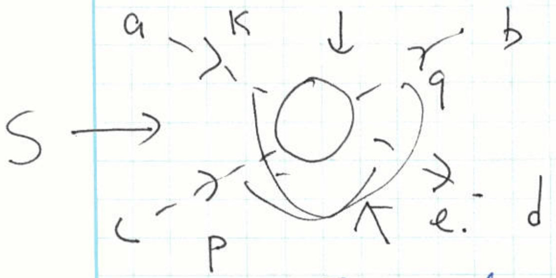
$$a = -L\left(1+\frac{m_\pi}{m_t}\right)^{-1} 2 I_\pi \cdot I_t.$$

$$= -L\left(1+\frac{m_\pi}{m_t}\right)^{-1} \times \left[I(I+1) - I_t(I_t+1) - 2 \right]$$

$a_{1/2} = 0.20 m_\pi^{-1} \quad a_{3/2} = -0.1 m_\pi^{-1} \quad a_{1/2} + 2a_{3/2} = 0$

exp $\quad 0.17 \quad -0.09. \quad \Rightarrow$ 18

⓪ π-π scattering à la Weinberg.



(previous formula does not work for π-π, because we assumed $m_\pi \ll M_{target}$)

In general k^2, q^2, p^2, l^2 + 3 are independent variables.

$$s = (k+p)^2 \quad t = (k-q)^2 \quad u = (p-q)^2.$$

but it can not depend on k^2 alone it must come as $k^2+q^2+p^2+l^2$ isospin + time reversal

note since $k+p \doteq q+l$,

$$s+t+u = p^2+k^2+q^2+l^2. \quad \therefore \text{ only two are invariant.}$$

In General. The scattering amplitude can be written as

Isospin, Bose statistic, crossing symmetry

δ_{ab} p, k. $q \to -k$.

$$iM = \delta_{ac}\delta_{bd}[A m_\pi^2 + B(u+t) + Cs] + \delta_{ab}\delta_{cd}(A m_\pi^2 + B(u+s) + Ct] + \delta_{ad}\delta_{bc}[A m_\pi^2 + B(s+t) + Cu]$$

$A, B, C = ?$

i) Adler's Rule.

if $k \to 0$, M should Vanish. (derivative coupl.)

then since all particle are on mass shell,

$s = t = u = m_\pi^2$.

$$\Rightarrow \quad A + 2B + C = 0 \quad \text{- - - -} \quad (1)$$

ii) If $k, q \to 0$, by analysis of preceeding section, all terms are zero except σ-term contribution

i.e. term prop. to $\partial_\nu A^\nu \sim m_\pi^2$

then

$S = u = m_\pi^2$, $t = 0$.

Now assume commutation relation of σ-model

then $[A_0^a, \partial^\mu A^b] = i C \delta^{ab} \sigma \delta^3(\vec{x} - \vec{y})$

$\therefore A + B + C = 0 \quad \cdots$ ②

from ① & ②

$B = 0 \qquad A = -C$

iii) If $k = q \to$ small, from previous argument, linear part is given by

$$i\mathcal{M} = \frac{8 g_v^2}{F_\pi^2} i \varepsilon_{abe} (I^e)_{dc}\, p \cdot q = \frac{8 g_v^2}{F_\pi^2} (i \varepsilon_{abe})(i \varepsilon_{dce})\, p \cdot q$$

$$= \frac{8 g_v^2}{F_\pi^2} (\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})\, p \cdot q$$

Here

$s = m_\pi^2 + 2p \cdot q$. $u = m_\pi^2 - 2p \cdot q$ $t = 0$.

$$\therefore C = \frac{8 g_v^2}{F_\pi^2} = 8\pi L / m_\pi^2$$

IV) Now at ~~the~~ threshold

$s = 4m_\pi^2 \quad t = (k-q)^2 = u = 0$

$iM = m_\pi^2 c\,(3\delta_{ac}\delta_{bd} - \delta_{ab}\delta_{cd} - \delta_{ad}\delta_{bc})$

This is an matrix acting on isospin space

of $1+1 = 2 + 1 + 0 \longrightarrow$ 1 eigenvalue.

5 equal eigenvalue → Symmetric.

Zero eigenvalue. (Antisymmetric index of a, c)

Isospin 0 state

$\psi_{ac} = \delta_{ac}$

$iM_{bdac}\delta_{ac} = m_\pi^2 c\,(9-1-1)\delta_{bd}$

$\therefore iM^{(0)} = 7m_\pi^2 c \quad - (3) \rightarrow a_0 = \frac{7}{4}L = 0.20m_\pi^{-1}$

Isospin 2 state. take trace.

$i(5M^{(2)} + M^{(0)}) = m_\pi^2 c\,(9-9-3)$

$\therefore iM^{(2)} = -2m_\pi^2 c. \quad - (4) \rightarrow a_2 = \frac{1}{2}L = -0.06m_\pi^{-1}$

(3), (4) is much smaller then $1/m_\pi^4$

more like $1/m_\pi^2$.

◎ Kaon Decay.

Either leptonic

$i \to f + \pi^a + \text{leptons}.$

or non-leptonic

$i \to f + \pi^a.$

] ⇒ in all cases need $\langle f | J_W | i \rangle$
↓
weak current.

Use.

$$\int T\langle f | \partial^\mu A_\mu^a(x)\, J(y) | i \rangle = \int \partial^\mu T\langle f | A_\mu^a(x)\, J(y) | i \rangle + \langle f | [A_0(x,0)\, J(\vec{y},0)] | i \rangle \quad -①$$

∗ note from V+A Theory.

$$[A_0^a(\vec{x},0), J(\vec{y},0)] = [V_0^a(\vec{x},0), J(\vec{y},0)]$$

A) $K^+ \to$ leptons

$K^+ \to \pi^0 + \text{leptons}$

$$\langle 0 | J_\mu(0) | K^+ \rangle = -i P_\mu F_K \quad (\text{K.F})$$

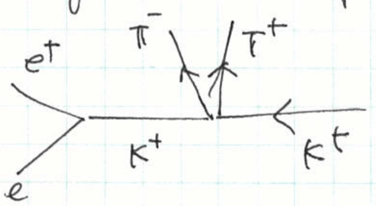
$$\langle \pi^0 | J_\mu(0) | K^+ \rangle = i(\text{K.F}) [f_+ (P_K + P_\pi)_\mu + f_- (P_K - P_\pi)_\mu]$$

using ① and Soft pion theorem

$$f_+ + f_- = g_V F_K / F_\pi \quad \to \text{Callan \& Treiman.}$$

B) $K^+ \to \pi^+ + \pi^- + \text{leptons}.$

Careful because of Pole term (Weinberg.)



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