习题

我没有答案

高寒
gaoh26@mail2.sysu.edu.cn

1.0

What is the partition function?

- 玻尔兹曼分布的归一化因子;
- 均匀分布的矩母函数;
- 自由能的指数$e^{-\beta F}$;
- 能量态密度$g(E)$的拉普拉斯变换;

1.1.1

Compute the partition function of a quantum harmonic oscillator with frequency ω and energy levels

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) \quad n \in \mathbf{Z}$$

Find the average energy U and entropy S as a function of temperature T.

1.1.2

Einstein constructed a simple model of a solid as N atoms, each of which vibrates with the same frequency ω. Treating these vibrations as a harmonic oscillator, show that at high temperatures, $k_B T \gg \hbar\omega$, the Einstein model correctly predicts the Dulong-Petit law for the heat capacity of a solid,

$$C_V = 3Nk_B$$

At low temperatures, the heat capacity of many solids is experimentally observed to tend to zero as $c_V \sim T^3$. Was Einstein right about this?

1.2

A particle moving in one dimension has Hamiltonian

$$H = \frac{p^2}{2m} + \lambda q^4$$

Show that the heat capacity for a gas of N such particles is $C_V = 3Nk_B/4$. Explainwhy the heat capacity is the same regardless of whether the particles are distinguishable or indistinguishable.

1.3

A non-relativistic particle is confined in a spherical container with radii R with potential

$$V(r) = \begin{cases} -h & , \ r < a \\ 0 & , \ a \leq r < R \\ +\infty & , \ r > R \end{cases}$$

Firstly predict the rough behavior of the pressure for a gas of N such particles as a function of T, then determine $P(T)$ by statistics.

2.1

Show that

1. $$C_P - C_V = T\left(\frac{\partial V}{\partial T}\right)_P \left(\frac{\partial P}{\partial T}\right)_V = -T\left(\frac{\partial V}{\partial T}\right)_P^2 \left(\frac{\partial P}{\partial V}\right)_T$$

2. $$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P$$

3. $$\left(\frac{\partial U}{\partial P}\right)_T = -T\left(\frac{\partial V}{\partial T}\right)_P - P\left(\frac{\partial V}{\partial P}\right)_T$$

2.1

4. $$\left(\frac{\partial C_V}{\partial V}\right)_T = T\left(\frac{\partial^2 P}{\partial T^2}\right)_V$$

5. $$\left(\frac{\partial C_P}{\partial P}\right)_T = -T\left(\frac{\partial^2 V}{\partial T^2}\right)_P$$

From equations above, show that for any non-ideal gas with constant C_V and C_P, the equation of state can be written as

$$(C_P - C_V)T = (P + a)(V + b)$$

where a and b are constants.

2.2

Consider the neutral gas of electrons, protons and Hydrogen. They undergo a reaction as

$$\mathrm{e}^- + \mathrm{p}^+ \rightleftharpoons \mathrm{H}$$

You know from Quantum Mechanics that the Hydrogen atom has binding energy $E = -I$ (where $I = 13.6\ \mathrm{eV}$). Let the number of Hydrogen atoms be $N_\mathrm{H} = (1-x)N$ and the e number of electrons and protons be $N_\mathrm{e} = N_\mathrm{p} = xN$ with $x \in [0, 1]$.

1. What is the equilibrium condition for the gas?
2. From the condition above, show that

$$\frac{x^2}{1-x} = \frac{V}{N}\left(\frac{m_\mathrm{e} m_\mathrm{p}}{2\pi\hbar^2 m_\mathrm{H}}\right)^{3/2} (k_B T)^{3/2}\, e^{-I/k_B T}$$

3.1

A Wigner crystal is a triangular lattice of electrons in a **two dimensional plane**. The longitudinal vibration modes of this crystal are bosons with dispersion relation $\omega = \sigma\sqrt{k}$. Show that, at low temperatures, these modes provide a contribution to the heat capacity that scales as $C \sim T^4$.

3.2

Consider a gas of non-interacting ultra-relativistic electrons, whose mass may be neglected. Show that $U = 3PV$. Show that at zero temperature $PV^{4/3} = \mathrm{const}$. Show that at high temperatures $E = 3Nk_BT$ and the equation of state coincides with that of a classical ultra-relativistic gas.
What are the results if we count the existance of positrons $(\mu_{\mathrm{e}^+} = -\mu_{\mathrm{e}^-})$?

3.3

Consider the free energy $F = a(T)m^2 + b(T)m^4 + c(T)m^6$ where $b(T) < 0$ and, for stability, $c(T) > 0$ for all T. Sketch the possible behaviours of the free energy as $a(T)$ varies and, in each case, identify the ground state and metastable states. Show that the system undergoes a first order phase transition at some temperature T_c. Determine the value $a(T_c)$ and the discontinuity in m at the transition.

3.4

The purpose of this question is to explain why the microwave background radiation still has a black body spectrum, even though it has not been in thermal equilibrium with matter since very early in the universe's history.

Consider a region of volume V in the cosmos containing black body radiation of temperature T. Suppose the cosmos expands (slowly) by a scale factor α, so that the wavevector $\vec{k}$ and angular frequency ω of each electromagnetic radiation mode are rescaled by $1/\alpha$. Explain why you should expect the mean number of photons in each mode not to change. Show that the Planck distribution is valid after the expansion provided the temperature is also rescaled by $1/\alpha$.

Verify, from the formula for the entropy of black body radiation, that the entropy in the expanded volume is the same as the original entropy, thus confirming the adiabatic character of the expansion.

3.5

Consider a diluted plasma which can be treated as non-relastivistic and non-interacting gas. Each molecule carries a electric charge of e. Now apply a magenitic field along the z-direction $\vec{B} = B\vec{e_z}$.

1. In classical picture, the hamitonian for each molecule is $H = \left(\vec{p} + e\vec{A}\right)^2 / 2m$. Determine the heat energy of such a plasma as a function of (N, T, V).
2. In quantum picture, the energy of a molecule is determined by a integer n and the z-component of the momentum p_z as

$$E = \left(n + \frac{1}{2}\right)\hbar\omega_c + \frac{p_z^2}{2m}$$

, where $n = 0, 1, 2, \cdots$ and $\omega_c = \frac{eB}{m}$. The degeneracy for a given n is $g(n) = \frac{eBA}{h}$ where A stands for the area of the xy plane. Determine the heat energy of such a plasma as a function of (N, T, V).