Homework on nuclear models

Throughout the exam A, Z, N will respectively denote the number of nucleons, protons and neutrons of a nucleus unless otherwise stated.

1. The binding energy per nucleon is given as

$$\frac{B}{A} = a_v - a_s A^{-1/3} - a_{sym}\frac{(2Z-A)^2}{A^2} - a_c Z^2 A^{-4/3}. \quad (1)$$

The constants are

$$a_v = 15.6 \text{ MeV} \ , \ a_{sym} = 23.3 \text{ MeV}, \quad a_s = 16.8 \text{ MeV} \ , \ a_c = 0.72 \text{ MeV}. \quad (2)$$

(a) Discuss and explain the A dependence of the first term in Eq.(1).

(b) Discuss and explain the A dependence of the second term in Eq.(1).

(c) The third term is the symmetry energy. For a fixed A, derive the $(2Z-A)^2$ dependence assuming that the energy level between nucleons are constant but proportional to $1/A^2$.

(d) Given A, obtain an expression for the value of Z for the most stable nucleus.

(e) When $Z = A/2$, what would be the value of A for the most stable nucleus. Assume that A could be a non integer value.

2. The symmetry energy can also be estimated using the Fermi Gas model for the nucleus. Consider the nucleons to be non-interacting fermions that occupy a sphere of radius $R = R_0 A^{1/3}$. The maximum momentum of the nucleons is p_F (Fermi momentum).

(a) The temperature is zero. Derive the following formula for the number of states.

$$n = \frac{4A}{9\pi}\left(\frac{R_0 p_F}{\hbar}\right)^3 \quad (3)$$

(b) Apply the formula in Eq.(3) to both the proton and neutron and give an order of magnitude estimate for the Fermi momentum for proton and neutron in MeV. You can assume that $A = 2Z = 2N$.

(c) Calculate the average kinetic energy per nucleon and express the result using p_F, m.

(d) Use the previous results to show that the total energy is given as

$$\langle E(Z,N)\rangle = \frac{3}{10m}\cdot\frac{\hbar^2}{R_0^2}\left(\frac{9\pi}{4}\right)^{2/3}\frac{N^{5/3}+Z^{5/3}}{A^{2/3}}$$

(e) Make a Taylor expansion around $Z = N = A/2$ and obtain the symmetry energy in the Fermi Gas approximation.

3. Consider a sphere of radius r with uniform charge density ρ. According to Coulomb's law, the electrostatic energy needed to bring a charge q from infinity to a distance r from the center of the sphere is

$$V = \frac{1}{4\pi\epsilon_0}\frac{Qq}{r}, \quad (4)$$

where $Q = \frac{4}{3}\pi r^3\rho$. Assume that the nucleus can be thought of as a uniformly charged sphere of radius $R = R_0 A^{1/3}$ and total charge Z, where $R_0 = 1.2$ fm. Derive the Z and A dependence of the last term (Coulomb term) in Eq. (1).

4. Consider solving a three dimensional simple harmonic oscillator problem. The eigenvalue is given as

$$E = \hbar\omega\left(N + \frac{3}{2}\right), \quad (5)$$

where $N = 0, 1, 2, ...$ In terms of angular momentum l and principle quantum number n_p, one can express $N = 2(n_p - 1) + l$. In a simplified picture, the nucleons (either the proton or neutron), occupy the lowest available levels.

(a) Explain why the lowest two Magic numbers of the nucleus should be 2 and 8.

(b) According to the above picture, the next Magic numbers should be 20 and 40. However, in reality, they are 20 and then 28. To explain the the Magic number 28, one should take into account two effects. First, in a realistic potential, which is more like a square well than a simple harmonic oscillator, the higher angular momentum states should be lower in energy for a given N. Another important addition is the spin orbit coupling, which takes the following form

$$V_{ls} = C_{ls} L\cdot s. \quad (6)$$

The nucleon should be in the eigenstate of the total angular momentum $J = L + S$. Since a nucleon has spin=1/2, the total angular momentum quantum number could be either $j = l + 1/2$ or $j = l - 1/2$. Given Eq. (6), calculate V_{ls} for the two possible total angular momentum states.

(c) Now explain the Magic number 28.