Physics of quark-gluon plasma and high-energy heavy-ion collisions

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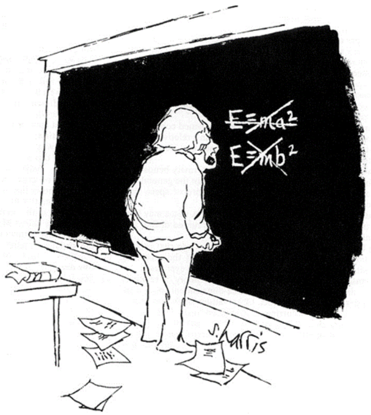
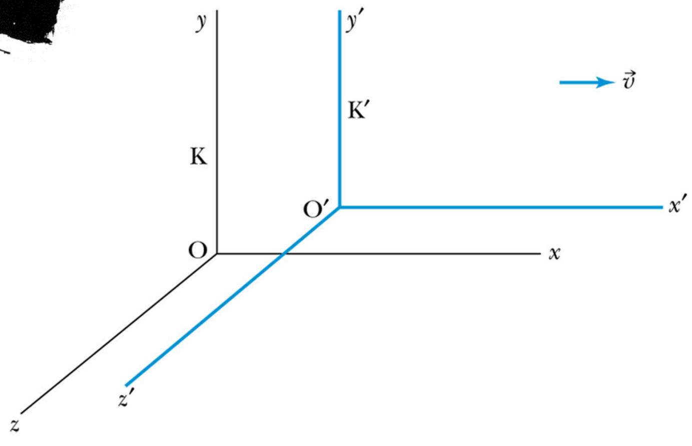
Lecture 2: Relativistic quantum mechanics and field theory

Einstein’s special relativity





Galileo’s relativity (t unchanged)

$$x' = x - \vec{v}t$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

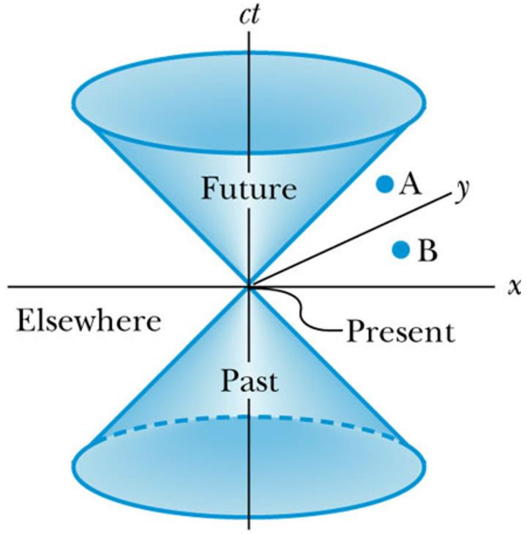
Einstein’s relativity (c unchanged)

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$$
$$y' = y$$
$$z' = z$$
$$t' = \frac{t - (vx/c^2)}{\sqrt{1 - v^2/c^2}}$$

- **Time Dilation**:
  Clocks in K’ run slow with respect to stationary clocks in K.

- **Length Contraction**:
  Lengths in K’ are contracted with respect to the same lengths stationary in K.

Light cone



$$s^2 = x^2 - c^2t^2 = (x')^2 - c^2(t')^2 = (s')^2$$

Einstein's special relativity

Covariant and contravariant four-vectors

- x^μ stands for

  x^0, x^1, x^2, x^3 for $\mu=0,1,2,3$

  $ct, x, y, z = (ct, \mathbf{r})$

- x_μ stands for

  x_0, x_1, x_2, x_3 for $\mu=0,1,2,3$

  $ct, -x, -y, -z = (ct, -\mathbf{r})$

Lorentz invariant quantities:

$x_\mu x^\mu = \sum_{\mu=0,1,2,3} x_\mu x^\mu$

(repeated index → summation implied)

$= (ct)^2 - x^2 - y^2 - z^2$

$x_\mu x^\mu$	$p_\mu x^\mu$
$p_\mu p^\mu$	$\partial x_\mu \partial x^\mu$
$p_\mu \partial x^\mu$	$\partial x_\mu A^\mu$

Example: Maxwell equations (c=1)

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}$$

Maxwell's Equations in terms of $F^{\mu\nu}$

Source equations:

$$\frac{\partial F^{\mu\nu}}{\partial x_\nu} = J^\mu$$

M1: $\nabla.\mathbf{E} = \rho/\epsilon_0$ $\quad \mu = 0,\ \nu = (1,2,3)$

M4: $\nabla \times \mathbf{B} = \mu_0(\mathbf{J} + \epsilon_0 \partial\mathbf{E}/\partial t)$ $\quad \mu = 1,\ \nu = (2,3,0)$

(similarly for $\mu = 2, 3$)

No-source equations:

$$\frac{\partial F^{\mu\nu}}{\partial x_\sigma} + \frac{\partial F^{\sigma\mu}}{\partial x_\nu} + \frac{\partial F^{\nu\sigma}}{\partial x_\mu} = 0$$

M2: $\nabla.\mathbf{B} = 0$ $\quad (\mu,\nu,\sigma) = (1,2,3)$

M3: $\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$ $\quad (\mu,\nu,\sigma) = (0,1,2)(3,0,1)(2,3,0)$

Relativistic quantum mechanics

How to “derive” the free Schroedinger equation

Free non-relativistic particle $\frac{\mathbf{p}^2}{2m} = E$ Apply QM prescription $\mathbf{p} \to -\hbar i \nabla$ $E \to \hbar i \frac{\partial}{\partial t}$

Get Schroedinger Equation $$-\frac{\hbar^2}{2m}\nabla^2 \Psi = i\hbar \frac{\partial \psi}{dt}$$

Missing phenomena: Anti-particles, pair production, spin

For relativistic particle $E^2 - \mathbf{p}^2 c^2 = m^2 c^4$ Applying QM prescription again gives:

Klein-Gordon Equation $$\boxed{-\frac{1}{c^2}\frac{\partial^2 \psi}{dt^2} + \nabla^2 \psi = \left(\frac{mc}{\hbar}\right)^2 \psi}$$

KG is 2nd order in time, need more initial information to specify the time evolution. Not a good feature.

Relativistic quantum mechanics

Library of Congress

The Dirac equation. (Square root of the KG equation)

$$E = \pm\sqrt{m^2 + |\vec{p}|^2} \longrightarrow ???$$

Dirac's idea (hbar=c=1):

$$E = \vec{\alpha}\cdot\vec{p} + \beta m \longrightarrow i\frac{\partial\psi}{\partial t} = \left(-i\vec{\alpha}\cdot\vec{\nabla} + \beta m\right)\psi$$

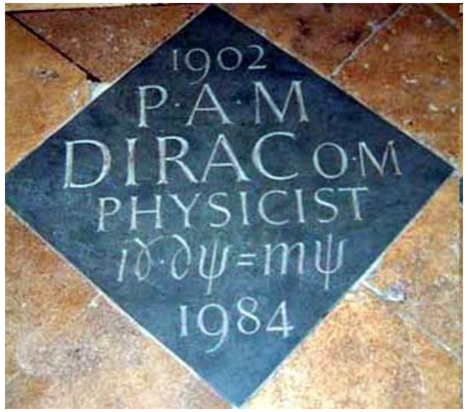
All we need to do is work out $\vec{\alpha}$ and β

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

2×2 matrices

where σ_i are the usual Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



Since these act on the field ψ, ψ itself must now be a 4 component vector, known as a **spinor**.

$$\left(-i\vec{\alpha}\cdot\vec{\nabla} + \beta m\right)\psi \sim \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}\begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

[Strictly speaking this is also just a representation.]

Relativistic quantum mechanics

The Dirac equation.

We can write this equation in a four-vector form by defining a new quantity γ^μ

$$\gamma^0 \equiv \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \vec{\gamma} \equiv \beta\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

The anti-commutation relations become:

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}}$$

And the Dirac Equation is: (with $p^\mu \to i\partial^\mu$) Often $\gamma^\mu\partial_\mu$ is written as $\not\partial$

$$\boxed{(i\gamma^\mu\partial_\mu - m)\,\psi = 0}$$

$\left(\psi^\dagger = (\psi^*)^T\right)$

A appropriate conserved quantity is now $\rho = \psi^\dagger\psi$ with $\vec{J} = \psi^\dagger\vec{\alpha}\psi$

In four-vector notation,

$$j^\mu = \bar{\psi}\gamma^\mu\psi \qquad \text{with} \qquad \bar{\psi} \equiv \psi^\dagger\gamma^0 \qquad (\text{Note } \gamma^0\gamma^0 = \beta^2 = 1\ )$$

HW2: Show that j^μ is conserved.

Relativistic quantum mechanics

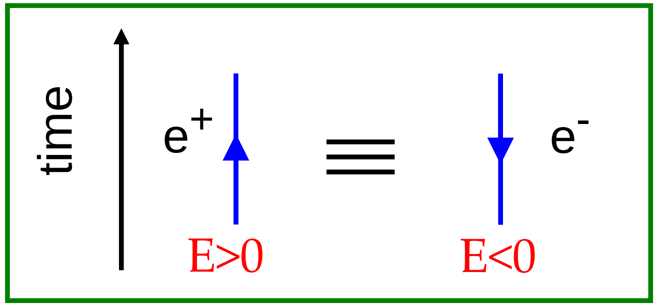
The Dirac equation.

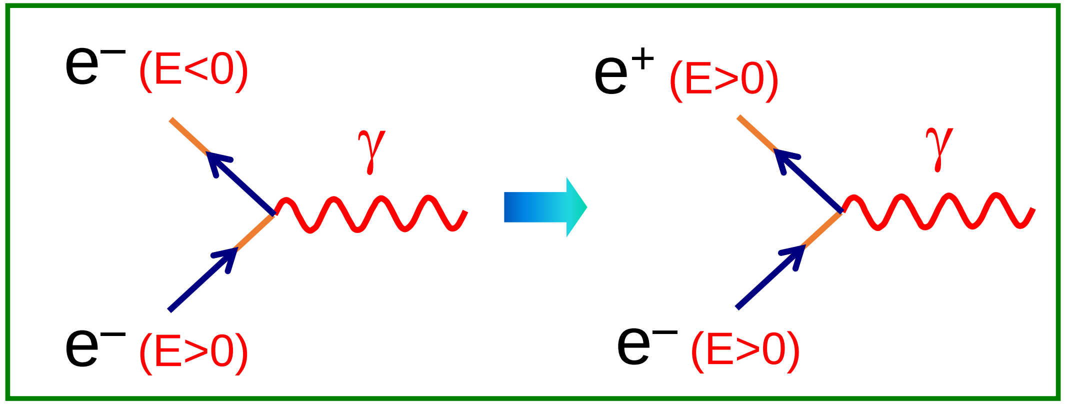
Look for a general plane wave solution: $\psi = u(E, \vec{p})e^{i(\vec{p}.\vec{r}-Et)}$

It indeed has a negative-energy state: $E = \pm\sqrt{m^2 + |\vec{p}|^2}$

<u>Feynman-Stückelberg Interpretation:</u>

Interpret a negative energy solution as a negative energy particle which propagates backwards in time or equivalently a positive energy anti-particle which propagates forwards in time





$$e^{-i(-E)(-t)} \longrightarrow e^{-iEt}$$

NOTE: in the Feynman diagram the arrow on the anti-particle remains in the backwards in time direction to label it an anti-particle solution.

Relativistic quantum mechanics

Angular momentum, spin, and helicity of Dirac particle

The angular momentum of a particle is given by $\vec{L} = \vec{r} \times \vec{p}$

$$[H, \vec{L}] = [\vec{\alpha} \cdot \vec{p}, \vec{r} \times \vec{p}] = -i\vec{\alpha} \times \vec{p}$$

This is not zero, so $\vec{L} = \vec{r} \times \vec{p}$ is **not** conserved!

But, if we define $\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} = -i\alpha_1\alpha_2\alpha_3\vec{\alpha} \quad (= -i\gamma_1\gamma_2\gamma_3\vec{\gamma})$

then $[H, \vec{\Sigma}] = [\vec{\alpha} \cdot \vec{p}, -i\alpha_1\alpha_2\alpha_3\vec{\alpha}] = 2i\vec{\alpha} \times \vec{p}$

So the quantity $\vec{J} = \vec{L} + \frac{1}{2}\vec{\Sigma}$ **is conserved!** $[H, \vec{J}] = 0$

Total angular momentum, orbital AM, spin:
Dirac equation describe spin-1/2 particle (fermion)

Relativistic quantum mechanics

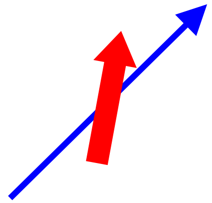
Angular momentum, spin, and helicity of Dirac particle

★ The component of a particles spin along its direction of flight is a good quantum number:

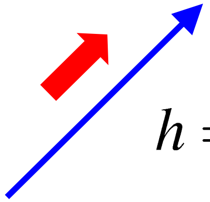
$$[\hat{H}, \hat{S}.\hat{p}] = 0$$

★ Define the component of a particles spin along its direction of flight as HELICITY:

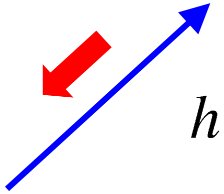
$$h \equiv \frac{\vec{S}.\vec{p}}{|\vec{S}||\vec{p}|} = \frac{2\vec{S}.\vec{p}}{|\vec{p}|} = \frac{\vec{\Sigma}.\vec{p}}{|\vec{p}|}$$



- If we make a measurement of the component of spin of a spin-half particle along any axis it can take two values $\pm 1/2$, consequently the eigenvalues of the helicity operator for a spin-half particle are: ± 1



$h = +1$



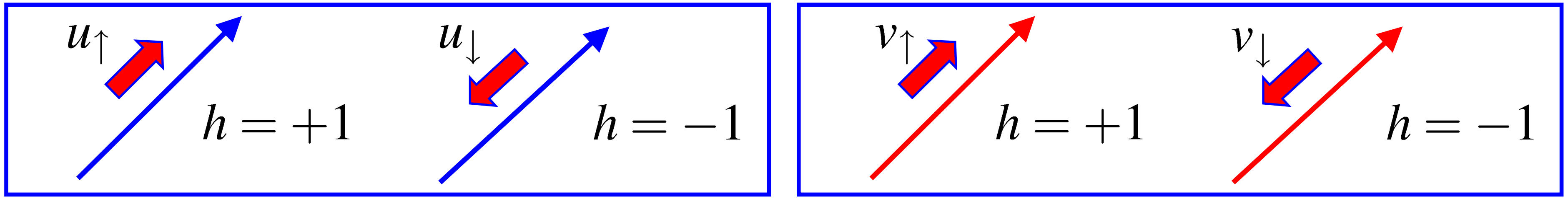
$h = -1$

Often termed: "right-handed" "left-handed"

Relativistic quantum mechanics

Angular momentum, spin, and helicity of Dirac particle

★ Using u and v to denote particle ant anti-particle states, we have the energy and helicity eigenstates:



★ For massive particle, helicity is not boost invariant:

If a particle has a mass, I can always move to a reference frame where I am going faster than it, causing the momentum to reverse direction. This causes the helicity to change sign.

•**Define the matrix**

$$\boxed{\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}}$$

•**<u>In the massless limit,</u> the helicity states are also eigenstates of** γ^5

$$\gamma^5 u_\uparrow = +u_\uparrow;\ \ \gamma^5 u_\downarrow = -u_\downarrow;\ \ \gamma^5 v_\uparrow = -v_\uparrow;\ \ \gamma^5 v_\downarrow = +v_\downarrow$$

Relativistic quantum mechanics

Angular momentum, spin, and helicity of Dirac particle

★ In general, define the eigenstates of γ^5 as LEFT and RIGHT HANDED CHIRAL states u_R; u_L; v_R; v_L

i.e. $$\gamma^5 u_R = +u_R; \;\; \gamma^5 u_L = -u_L; \;\; \gamma^5 v_R = -v_R; \;\; \gamma^5 v_L = +v_L$$

• In the LIMIT $E \gg m$ (and ONLY IN THIS LIMIT):

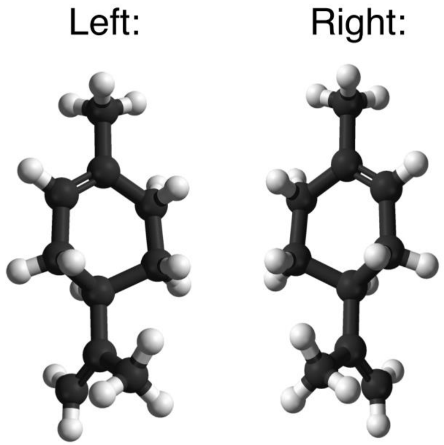
$$u_R \equiv u_\uparrow; \quad u_L \equiv u_\downarrow; \quad v_R \equiv v_\uparrow; \quad v_L \equiv v_\downarrow$$

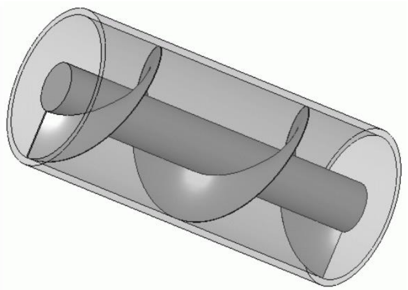
★ This is a subtle but important point: in general the HELICITY and CHIRAL eigenstates are not the same. It is only in the ultra-relativistic limit that the chiral eigenstates correspond to the helicity eigenstates.

★ **The concept of chirality is very important in many subjects.**

Greek word: χει ρ (cheir) - hand

Limonene





The Archimedes screw

Field theory

KG and Dirac equations from Lagrangians.

The action:

$$S = \int \mathcal{L}(\phi, \partial_\mu \phi)\, d^4x \qquad\qquad \mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi + m^2 \phi^* \phi$$

Least action principle. Varying S gives the Euler-Lagrange equation:

KG equation: $$\left(\partial^2 - m^2\right) \phi = 0$$

The Dirac Lagrangian $$\mathcal{L} = \bar{\psi} \left(\gamma^\mu \partial_\mu - m\right) \psi$$

Field theory: re-interpret the wave function as field.

Field theory

KG and Dirac equations from Lagrangians.

Consider phase shifting the electron field by $\psi \longrightarrow e^{i\theta}\psi$.

The adjoint field transforms as $\bar{\psi} \longrightarrow \bar{\psi}\, e^{-i\theta}$ and the Lagrangian transforms as

$$\mathcal{L} = \bar{\psi}\,(i\gamma^\mu\partial_\mu - m)\,\psi \longrightarrow \bar{\psi}\,e^{-i\theta}\,(i\gamma^\mu\partial_\mu - m)\,e^{i\theta}\psi = \mathcal{L}$$

The Lagrangian doesn't change so the physics stays the same.

This is known as a **global U(1) symmetry**

(since $e^{i\theta}$ doesn't vary with space-time coordinate)

(since $e^{i\theta}$ is a unitary 1×1 matrix)

Field theory

What happens if we make our transformation **local**, i.e. depend on space-time point?

$$\psi \longrightarrow e^{i\theta(x)}\psi \qquad\qquad \bar{\psi} \longrightarrow \bar{\psi}\, e^{-i\theta(x)}$$

$$\mathcal{L} \longrightarrow \bar{\psi}\, e^{-i\theta(x)}\left(i\gamma^\mu\partial_\mu - m\right) e^{i\theta(x)}\psi \;= \mathcal{L} - \bar{\psi}\gamma^\mu\left(\partial_\mu\theta(x)\right)\psi$$

The free Dirac Lagrangian is no longer invariant. If we really want this to be a symmetry of the theory, we will have to add in something new.

Let's postulate a new field A^μ which couples to the electron according to

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu\partial_\mu - e\gamma_\mu A^\mu - m\right)\psi$$

charge of the electron = -e

Beware: conventions differ, e.g. Halzen and Martin have $+e\gamma_\mu A^\mu$ while Peskin & Schroeder have as above

Often this is written in terms of a **"covariant derivative"**

$$\boxed{D^\mu \equiv \partial^\mu + ieA^\mu}$$

Field theory

Now $\mathcal{L} = \bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi \longrightarrow \bar{\psi}\left(i\gamma^\mu e^{-i\theta(x)} D'_\mu e^{i\theta(x)} - m\right)\psi$

So, to preserve the Lagrangian, we need D_μ to transform too:

$$D_\mu \longrightarrow D'_\mu = e^{i\theta(x)} D_\mu e^{-i\theta(x)}$$

$$\Rightarrow \quad \partial_\mu + ieA'_\mu = e^{i\theta(x)}\left(\partial_\mu + ieA_\mu\right)e^{-i\theta(x)} = \partial_\mu - i\partial_\mu\theta(x) + ieA_\mu$$

therefore we need A_μ to transform as

$$A_\mu \longrightarrow A'_\mu = A_\mu - \tfrac{1}{e}\partial_\mu\theta(x)$$

This is the gauge transformation we saw for the (classical) photon earlier!

Coupling the **electron** to a **photon** makes the theory **locally U(1) symmetric**

$$\boxed{\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi}$$

$$\boxed{\left(i\gamma^\mu D_\mu - m\right)\psi = 0, \qquad \partial^2 A^\mu = j^\mu}$$

$j^\mu = e\bar{\psi}\gamma^\mu\psi$

Field theory

Symmetry and conservation law: Noether theorem

Consider a set of scalar fields $\varphi_a(x)$, and a lagrangian density

$$\mathcal{L}(x) = \mathcal{L}(\varphi_a(x), \partial_\mu \varphi_a(x))$$

let's make an infinitesimal change: $\varphi_a(x) \to \varphi_a(x) + \delta\varphi_a(x)$

$$\mathcal{L}(x) \to \mathcal{L}(x) + \delta\mathcal{L}(x)$$

$$\delta\mathcal{L}(x) = \frac{\partial \mathcal{L}}{\partial \varphi_a(x)} \delta\varphi_a(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a(x))} \partial_\mu \delta\varphi_a(x)$$

variation of the action:

$$S = \int d^4y\, \mathcal{L}(y)$$

$$\begin{aligned}
\frac{\delta S}{\delta\varphi_a(x)} &= \int d^4y\, \frac{\delta\mathcal{L}(y)}{\delta\varphi_a(x)} \\
&= \int d^4y \left[\frac{\partial\mathcal{L}(y)}{\partial\varphi_b(y)} \frac{\delta\varphi_b(y)}{\delta\varphi_a(x)} + \frac{\partial\mathcal{L}(y)}{\partial(\partial_\mu\varphi_b(y))} \frac{\delta(\partial_\mu\varphi_b(y))}{\delta\varphi_a(x)}\right] \\
&= \int d^4y \left[\frac{\partial\mathcal{L}(y)}{\partial\varphi_b(y)} \delta_{ba}\delta^4(y-x) + \frac{\partial\mathcal{L}(y)}{\partial(\partial_\mu\varphi_b(y))} \delta_{ba}\partial_\mu\delta^4(y-x)\right] \\
&= \frac{\partial\mathcal{L}(x)}{\partial\varphi_a(x)} - \partial_\mu \frac{\partial\mathcal{L}(x)}{\partial(\partial_\mu\varphi_a(x))}\,.
\end{aligned}$$

setting $\frac{\delta S}{\delta\varphi_a(x)} = 0$
we would get
equations of motion

$$\frac{\partial\mathcal{L}(x)}{\partial\varphi_a(x)} \to \partial_\mu \frac{\partial\mathcal{L}(x)}{\partial(\partial_\mu\varphi_a(x))} + \frac{\delta S}{\delta\varphi_a(x)}$$

Field theory

Symmetry and conservation law: Noether theorem

$$\frac{\partial \mathcal{L}(x)}{\partial \varphi_a(x)} \to \partial_\mu \frac{\partial \mathcal{L}(x)}{\partial(\partial_\mu \varphi_a(x))} + \frac{\delta S}{\delta \varphi_a(x)}$$

$$\delta \mathcal{L}(x) = \frac{\partial \mathcal{L}}{\partial \varphi_a(x)} \delta \varphi_a(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a(x))} \partial_\mu \delta \varphi_a(x)$$

Emmy Noether

thus we find:

$$\delta \mathcal{L}(x) = \partial_\mu \underline{\left(\frac{\partial \mathcal{L}(x)}{\partial(\partial_\mu \varphi_a(x))} \delta \varphi_a(x) \right)} + \frac{\delta S}{\delta \varphi_a(x)} \delta \varphi_a(x)$$

$$j^\mu(x) \equiv \frac{\partial \mathcal{L}(x)}{\partial(\partial_\mu \varphi_a(x))} \delta \varphi_a(x)$$

this is called Noether current; now we have:

$$\partial_\mu j^\mu(x) = \delta \mathcal{L}(x) - \underbrace{\frac{\delta S}{\delta \varphi_a(x)}}_{= 0 \text{ if eqs. of motion are satisfied}} \delta \varphi_a(x)$$

if a set of infinitesimal transformations leaves the lagrangian unchanged, invariant, $\delta \mathcal{L} = 0$, the Noether current is conserved!

$$\partial_\mu j^\mu = 0$$

$$\frac{\partial}{\partial t} \underbrace{j^0(x)}_{\text{charge density}} + \nabla \cdot \underbrace{\mathbf{j}(x)}_{\text{current density}} = 0$$

Field theory

Symmetry and conservation law: Noether theorem

Consider a transformation of fields that change the lagrangian density by a total divergence:

$$\delta\mathcal{L}(x) = \partial_\mu K^\mu(x)$$

there is still a conserved current:

$$j^\mu(x) = \frac{\partial\mathcal{L}(x)}{\partial(\partial_\mu\varphi_a(x))}\,\delta\varphi_a(x) - K^\mu(x)$$

e.g. space-time translations:

$$\varphi_a(x) \to \varphi_a(x-a) \qquad \mathcal{L}(x) \to \mathcal{L}(x-a)$$

$$\varphi_a(x) \to \varphi_a(x) - a^\nu\partial_\nu\varphi_a(x) \qquad \delta\mathcal{L}(x) = -a^\nu\partial_\nu\mathcal{L}(x) = -\partial_\nu(a^\nu\mathcal{L}(x))$$

$$\delta\varphi_a(x) = -a^\nu\partial_\nu\varphi_a(x) \qquad K^\nu(x) = -a^\nu\mathcal{L}(x)$$

we get:

$$\begin{aligned} j^\mu(x) &= \frac{\partial\mathcal{L}(x)}{\partial(\partial_\mu\varphi_a(x))}(-a^\nu\partial_\nu\varphi_a(x)) + a^\mu\mathcal{L}(x) \\ &= a_\nu T^{\mu\nu}(x)\,, \end{aligned}$$

$$\longrightarrow \quad T^{\mu\nu}(x) \equiv -\frac{\partial\mathcal{L}(x)}{\partial(\partial_\mu\varphi_a(x))}\,\partial^\nu\varphi_a(x) + g^{\mu\nu}\mathcal{L}(x)$$

stress-energy or energy-momentum tensor