**CHAROTAR UNIVERSITY OF SCIENCE AND TECHNOLOGY**
**Chandubhai S. Patel Institute of Technology**
**CHARUSAT Campus, Highway 139,**
**Off, Nadiad - Petlad Road,**
**Changa, Gujarat 388421**

**B.TEC. IT/CE SEM-III** **YEAR 2023**

**SUBJECT & SUBJECT CODE: DISCRETE MATHEMATICS & ALGEBRA (MA253)**

# CHAPTER: PREDICATE CALCULUS

**TOPICS INCLUDED ARE**

- **PROPOSITION, TYPES OF PROPOSITION, TAUTOLOGY, CONTRADICTIONS.**
- **CONTRADICATIONS, CONNECTIVES, TYPES OF CONNECTIVES AND ITS PROPERTIES.**
- **LOGICAL EQUIVQLENCE, VERIFICATION USING TRUTH TABLE.**
- **CONVERSE, INVERSE AND CONTRAPOSITIVE.**
- **MINIMAL FUNCTIONALLY COMPLETE SET OF CONNECTIVES.**
- **NORMAL FORMS: DNF, CNF, PDNF, PCNF.**
- **LOGIC IN PROOF**
- **RULE OF INFERENCES**

**SENTENCE:** A collection of words making a complete grammatical structure with meaning and sense is known as sentence.

**PROPOSITION:** A declarative sentence which can be either true or false is known a Proposition. It is also known as statement.

Consider, for example, the following sentence in English

1. New Dehli is a capital of India.
2. The square of 4 is 16.
3. Bring me coffee.
4. This statement is false.

The statement 1-2 is proposition as the sentence 1 and 2 are true. To sentence 3, we cannot decide whether it is true or false as it is not a declarative sentence. Sentence 4 has both true as well as false value so 4 is not a proposition.

**TRUTH VALUE:** The truth or falsity of a statement is known as its truth value. Propositions are denoted by *p,q,r*..... and also known as propositional variables.

**ATOMIC STATEMENT:** A proposition consisting of only a single propositional variable or a single propositional constant is known as atomic statement. It is also known as primary or primitive proposition.

**COMPOUND STATEMENT:** A proposition, which is combination of two or more propositional variables, is known as compound statement. It is also known as molecular or composite statement.

**TRUTH TABLE:** A truth table is a table that shows the truth value of a compound proposition for all possible value.

**CONNECTIVES:** The words or phrases which are used to form a proposition are known as connectives. There are five basic connectives: Negation, Conjunction, Disjunction, Conditional and Bi- conditional.

**NEGATION:** If **p** is any proposition, the negation of p is denoted by **~p** , is a proposition which is false when p is true and true when p is false. It is also known as **unary operator.**

| p | ~p |
|---|---|
| T | F |
| F | T |

**CONJUNCTION:** If p and q are two statements then conjunction of p and q is the compound statement of the form " p and q" and it is denoted by $p \wedge q$, which is true when both p and q are true.

| p | q | $p \wedge q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

**DISJUNCTION:** If p and q are two statements then disjunction of p and q is the compound statement of the form " p or q" and it is denoted by $p \vee q$, which is false when both p and q are false.

| p | q | $p \vee q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

**CONDITIONAL:** If p and q are two statement then conditional statement of p and q is the compound statement of the form " if p then q" and it is denoted by $p \to q$, which is false when p is true and q is false.

| p | q | $p \to q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| **F** | **F** | **T** |

**Remark**:

1. In $p \to q$**,**the proposition p is called **antecedent or hypothesis** and proposition q is called **consequent or conclusion**.
2. The connective **if........then** can be also read as follows:

**a. p implies q.**
**b. p is sufficient for q.**
**c. p only if q.**
**d. q is necessary for p.**
**e. q if p.**
**f. q follows from p.**
**g. q is consequence of p.**

3. $p \to q \equiv \sim p \vee q$

**BICONDITIONAL:** If p and q are two statement then bi-conditional statement of p and q is the compound statement of the form "if p only if q" and it is denoted by $p \leftrightarrow q$, which true when either p and q both are true or false simultaneously.

| p | q | $p \leftrightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |

**Example: If P represents** *This book is good* **and Q represent** *This book is cheap***, then write the following sentences in symbolic form.**

**Solution: Here**

| **Sentences** | **Symbol** |
|---|---|
| **This book is good and cheap.** | $P \wedge Q$ |
| **This book is not good and cheap.** | $\sim P \wedge Q$ |

**This book is costly but good.** $\sim Q \wedge P$
**This book is neither good nor cheap.** $\sim P \wedge \sim Q$
**This book is either good or cheap.** $P \vee Q$

**Example: If P: It is raining, Q: I have the time, R: I will go to a movie. Write the sentences in English corresponding to the following propositional form**

1. $(\sim P \wedge Q) \leftrightarrow R$
2. $(Q \to R) \wedge (R \to Q)$
3. $\sim Q \vee R$
4. $R \to (\sim P \wedge Q)$

**Solution:**

| | | |
|---|---|---|
| 1. | $(\sim P \wedge Q) \leftrightarrow R$ | I will go to a movie if and only if it is not raining and I have the time. |
| 2. | $(Q \to R) \wedge (R \to Q)$ | I will go to movie if and only if I have the time. |
| 3. | $\sim Q \vee R$ | It is not the case that I have the time or I will go to a movie. |
| 4. | $R \to (\sim P \wedge Q)$ | I will go to a movie only if it is not raining and I have the time. |

## <u>CONVERSE, CONTRAPOSITIVE AND INVERSE OF AN IMPLICATION</u>

There are some related implication that can be formed from $p \to q$.They are as follows:

- **When** $p \to q$ **an implication, then converse of is** $p \to q$ **is an implication** $q \to p$ **.**
- **When** $p \to q$ **an implication, then inverse of is** $p \to q$ **is an implication** $\sim p \to \sim q$ **.**
- **When** $p \to q$ **an implication, then contra positive of is** $p \to q$ **is an implication** $\sim q \to \sim p$ **.**

**Remark:** A conditional proposition and its converse or inverse are not logically equivalent. On other hand, a conditional proposition and its contra positive are logically equivalent **(Can be check using the truth table). The importance of the contra positive derives from the fact that mathematical theorems in the form** $p \to q$ **can sometimes be proved easily when restarted in the form** $\sim q \to \sim p$ **. For example**

**Example: Prove that if $x^2$ is divisible by 4, then x is even.**

**Solution:** Let ***p*:** $x^2$ is divisible by 4.
***q*:** x is even.

The implication is of the form $p \to q$.The contra positive is $\sim q \to \sim p$, which state in words:
If x is odd, then $x^2$ is not divisible by 4.

**The proof of contra positive is easy.**

Since x is odd, x=2k+1 , for some integer k. Hence

$x^2=(2k+1)^2=4k^2+4k+1=4(k^2+k+1/4)$

Since $k^2+k$ is an integer, $k^2+k+1/4$ is not integer therefore $x^2$ is not divisible by 4.

**EXAMPLE: Prove that if $n^2$ is odd, then n is odd.**

**Solution:** Let **p**: $n^2$ is odd

**q**: n is odd

Proof by contra positive method $\sim q \to \sim p$

~**q**: n is even.

~**p**: $n^2$ is odd.

Since ~q: n is even therefore n=2x, where x is any integer

$n^2=(2x)^2=4x^2=2(2x^2)=2a$ (assume $a=2x^2$) which is an even number

Therefore, $n^2$ is an even number.

Now we know that $\sim q \to \sim p \equiv p \to q$

Therefore, by proof of contra positive we can say that $n^2$ is odd then n is odd.

**PRATICE EXAMPLE**

1. Prove that if $x, y \in Z$ ( set of integers) such that, xy is odd then both x and y are odd.
2. If 3n+2 is odd, then n is odd.
3. Prove that if $m+n \geq 73$, then $m \geq 37$ *or* $n \geq 37$ .m and n being the positive integer.

**Example: State Converse, Contra positive and Inverse of the following statement**

1. **If it rain then the crop will grow.**

   ***p***: It rains.

   ***q***: The crop will grow.

   **Converse ($q \to p$)If** the crop grow then there has been rain.

   **Contra positive ($\sim q \to \sim p$):** If the crop do not grow then there has been no rain.

   **Inverse**$(\sim p \to \sim q)$: If it does not rain then crop will not grow.

2. **If a triangle is not isosceles then it is not equilateral.**

   ***p:*** A triangle is not isosceles.

   ***q:*** It is not equilateral.

   **Converse ($q \to p$) If** a triangle is not equilateral then it is not isosceles.

   **Contra positive ($\sim q \to \sim p$):** If the triangle is equilateral then it is isosceles.

   **Inverse**$(\sim p \to \sim q)$: If a triangle is isosceles then it is equilateral.

**PRATICE EXAMPLE**

**Example**: **State Converse, Contra positive and Inverse of the following statement**

1. If a triangle is isosceles, then two of its sides are equal.
2. If there is no unemployment in India, then the Indian's won't go to the USA for employment.

**Example: Construct a truth table for each compound proposition:**

**(i)** $p \wedge (\sim q \vee q)$

| p | q | ~q | $(\sim q \vee q)$ | $p \wedge (\sim q \vee q)$ |
|---|---|---|---|---|
| T | T | F | T | T |
| T | F | T | T | T |
| F | T | F | T | F |
| F | F | T | T | F |

**(ii)** $(P \vee Q) \wedge (P \rightarrow Q) \wedge (Q \rightarrow P)$

| P | Q | $(P \vee Q)$ | $P \rightarrow Q$ | $(P \vee Q) \wedge (P \rightarrow Q)$ | $Q \rightarrow P$ | $(P \vee Q) \wedge (P \rightarrow Q) \wedge (Q \rightarrow P)$ |
|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T |
| T | F | T | F | F | T | F |
| F | T | T | T | T | F | F |
| F | F | F | T | F | T | F |

**PRATICE EXAMPLE**

**Obtain the truth table for the following**

1. $(P \vee Q) \rightarrow (P \vee R) \rightarrow (Q \vee R)$
2. $(\sim P \leftrightarrow \sim Q) \leftrightarrow Q \leftrightarrow R$
3. $(P \wedge Q) \vee (\sim P \wedge Q) \vee (P \wedge \sim Q) \vee (\sim \mathrm{P} \wedge \sim \mathrm{Q})$
4. $p \wedge (q \vee r)$
5. $\sim (p \vee q) \vee (\sim p \wedge \sim q)$

## <u>SOME DERIVED CONNECTIVES</u>

1. **NAND**: NAND is the negation of conjunction of two statements. Assume p and q are any two statement then NAND of p and q is a proposition which is false when both p and q are true otherwise true. It is denoted by $p \uparrow q$.

| p | q | $p \uparrow q$ |
|---|---|---|
| T | T | F |
| T | F | T |
| F | T | T |
| F | F | T |

2. **NOR:** NOR is negation of disjunction of two statements. Assume p and q be two proposition. NOR of p and q is a proposition which is true when both p and q are false, otherwise false. It is denoted by $p \downarrow q$

| p | q | $p \downarrow q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| **F** | **F** | **F** |

**Remark:** Both NAND and NOR satisfy commutative law but they are not associative

3. **XOR (Exclusive or):** It is a proposition that is true when exactly one of p and q is true but not both and is false otherwise. It is denoted by $p \oplus q$ .

| p | q | $p \oplus q$ |
|---|---|---|
| T | T | F |
| T | F | T |
| F | T | T |
| F | F | F |

## ALGEBRA OF PROPOSITION

| **Idempotent law** | **Associative Law** | **Commutative Law** | **Distributive Law** |
|---|---|---|---|
| $p \vee p \equiv p$<br>$p \wedge p \equiv p$ | $p \vee (q \vee r) \equiv (p \vee q) \vee r$<br>$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$ | $p \vee q \equiv q \vee p$<br>$p \wedge q \equiv q \wedge p$ | $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$<br>$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ |
| **Identity Law** | **Complement Law** | **Involution Law** | **De Morgan's Law** |
| $p \vee F \equiv p$ | $p \vee \sim \mathrm{p} \equiv T$ | $\sim (\sim \mathrm{p}) \equiv p$ | $\sim (p \vee q) \equiv \sim p \wedge \sim q$<br>$\sim (p \wedge q) \equiv \sim p \vee \sim q$ |
| $p \vee T \equiv T$ | $p \vee \sim p \equiv F$ | | |
| $p \wedge T \equiv p$ | $\sim F \equiv T$ | | |
| $p \wedge F \equiv F$ | $\sim T \equiv F$ | | |

## LOGICAL EQUIVALENCE

If two propositions P and Q have the same truth values in every possible case, the propositions are **logically equivalent.** It is denoted by $P \equiv Q$ .

**To test whether two proposition P and Q are logically equivalent**

**Method 1: Using Truth Table.**

**Step 1:** Construct the truth Table for compound statement P and Q.
**Step 2:** Check each combination of truth values of the propositional variables to see whether the value of P is same as the truth value of Q. If in each row the truth value of P is the same as the truth value of Q, then P and Q are logically equivalent.

**Example: Use truth tables to prove** $p\vee(q\wedge r)\equiv(p\vee q)\wedge(p\vee r)$.

| p | q | r | $(q\wedge r)$ | $(p\vee q)$ | $(p\vee r)$ | $p\vee(q\wedge r)$ | $(p\vee q)\wedge(p\vee r)$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T |
| T | | T | F | F | T | T | T |
| T | F | T | F | T | T | T | T |
| T | F | F | F | T | T | T | T |
| F | T | T | T | T | T | T | T |
| F | T | F | F | F | T | F | F |
| F | F | T | F | F | F | T | F |
| F | F | F | F | F | F | F | F |

Since the entries in 7 and 8 column is same, therefore $p\vee(q\wedge r)\equiv(p\vee q)\wedge(p\vee r)$.

**Example: Use truth tables to prove** $\sim(p\wedge q)\equiv(\sim p\vee\sim q)$.

| p | q | $(p\wedge q)$ | $\sim(p\wedge q)$ | $\sim p$ | $\sim q$ | $(\sim p\vee\sim q)$ |
|---|---|---|---|---|---|---|
| T | T | T | F | F | F | F |
| T | F | F | T | F | T | T |
| F | T | F | T | T | F | T |
| F | F | F | T | T | T | T |

Since the entries in 4 and 7 column is same, therefore $\sim(p\wedge q)\equiv(\sim p\vee\sim q)$.

**Method 2: Using Algebra of proposition (Properties of Proposition).**
**Example: Show that** $(P\wedge Q)\vee(P\wedge\sim Q)\equiv P$

**Solution:** $(P\wedge Q)\vee(P\wedge\sim Q)\equiv P\wedge(Q\vee\sim Q)$ (Distributive Law)

$\equiv P\wedge T$ (Complement Law)

$\equiv P$ (Identity Law)

$= R.H.S$

$\therefore (P\wedge Q)\vee(P\wedge\sim Q)\equiv P$

**Example: Show that** $(P\rightarrow Q)\wedge(R\rightarrow Q)\equiv(P\vee R)\rightarrow Q$

**Solution:** $(P\Rightarrow Q)\wedge(R\Rightarrow Q)\equiv(\sim P\vee Q)\wedge(\sim R\vee Q)$(Equivalent form of $P\rightarrow Q\equiv\sim P\vee Q$)

$$\equiv (Q \vee \sim P) \wedge (Q \vee \sim R) \text{(Commutative Law)}$$

$$\equiv Q \vee (\sim P \wedge \sim R) \text{ (Distributive Law)}$$

$$\equiv Q \vee (\sim (P \vee R)) \text{(De Morgan's Law)}$$

$$\equiv (\sim (P \vee R)) \vee Q \text{(Commutative Law)}$$

$$\equiv (P \vee R) \Rightarrow Q \text{ (Equivalent form of } P \rightarrow Q \equiv \sim P \vee Q \text{)}$$

$$= R.H.S$$

$\therefore (P \rightarrow Q) \wedge (R \rightarrow Q) \equiv (P \vee R) \rightarrow Q$

**Example: Show that** $(\sim P \wedge (\sim \mathrm{Q} \wedge \mathrm{R})) \vee (Q \wedge R) \vee (P \wedge R) \equiv R$

**Solution:** $(\sim P \wedge (\sim \mathrm{Q} \wedge \mathrm{R})) \vee (Q \wedge R) \vee (P \wedge R) \equiv ((\sim P \wedge \sim \mathrm{Q}) \wedge \mathrm{R}) \vee (Q \wedge R) \vee (P \wedge R)$

(Distributive Law)

$$\equiv (\sim (P \vee \mathrm{Q}) \wedge \mathrm{R}) \vee (Q \wedge R) \vee (P \wedge R)$$

(De Morgan’s Law)

$$\equiv (\sim (P \vee \mathrm{Q}) \wedge \mathrm{R}) \vee ((Q \vee P) \wedge R) \text{(Distributive Law)}$$

$$\equiv (\sim (P \vee \mathrm{Q}) \wedge \mathrm{R}) \vee ((P \vee Q) \wedge R) \text{ (Commutative Law)}$$

$$\equiv (\sim (P \vee \mathrm{Q}) \wedge (P \vee Q)) \wedge (R) \text{ (Distributive Law)}$$

$$\equiv T \wedge R \text{(Complement Law)}$$

$$\equiv R \text{(Identity Law)}$$

$$= R.H.S$$

$\therefore (\sim P \wedge (\sim \mathrm{Q} \wedge \mathrm{R})) \vee (Q \wedge R) \vee (P \wedge R) \equiv R$.

**PRATICE EXAMPLE**

**Example: Show that the following statements are logically equivalent using truth table.**

1. $p \leftrightarrow q \equiv ((p \rightarrow q) \wedge (q \rightarrow p))$
2. $p \leftrightarrow q \equiv ((p \vee q) \rightarrow (p \wedge q))$
3. $(p \vee q) \rightarrow r \equiv ((p \rightarrow r) \wedge (q \rightarrow r))$

**Example: Show that the following statements are logically equivalent without using truth table.**

4. $p \leftrightarrow q \equiv (p \vee q) \rightarrow (p \wedge q)$

**TAUTOLOGY:** A statement which is always true is known **as Tautology**. It is denoted by ***t***. It is also known as logical truth. There are two methods to check whether the given proposition is a Tautology

1. Using truth table method.
2. without using truth table. (using Identities or algebra of proposition)

**Example: Show** $(P \to (Q \to R)) \to ((P \to Q) \to (P \to R))$ **is tautology using truth table.**

**Solution: Let** $\alpha = (P \to (Q \to R)) \to ((P \to Q) \to (P \to R))$

| P | Q | R | $Q \to R$ | $(P \to (Q \to R))$ | $(P \to Q)$ | $(P \to R)$ | $((P \to Q) \to (P \to R))$ | $\alpha$ |
|---|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T | T |
| T | T | F | F | F | T | F | F | T |
| T | F | T | T | T | F | T | T | T |
| T | F | F | T | T | F | F | T | T |
| F | T | T | T | T | T | T | T | T |
| F | T | F | F | T | T | T | T | T |
| F | F | T | T | T | T | T | T | T |
| F | F | F | T | T | T | T | T | T |

Since the truth value is true for all possible values of the propositional variables which can be seen in the last column of the table, the given proposition is a tautology.

**Example: Using identities prove that:** $Q \vee (P \wedge \sim Q) \vee (\sim P \wedge \sim Q)$

**Solution:** $Q \vee (P \wedge \sim Q) \vee (\sim P \wedge \sim Q) \Leftrightarrow [(Q \vee P) \wedge (Q \vee \sim Q)] \vee (\sim P \wedge \sim Q)$ (Distributive Law)

$\Leftrightarrow [(Q \vee P) \wedge T] \vee (\sim P \wedge \sim Q)$ (Complement Law)

$\Leftrightarrow [(Q \vee P)] \vee (\sim P \wedge \sim Q)$ (Identity Law)

$\Leftrightarrow [(Q \vee P)] \vee \sim (P \vee Q)$ (De Morgan's Law)

$\Leftrightarrow [(Q \vee P)] \vee \sim (Q \vee P)$ (Commutative Law)

$\Leftrightarrow T$

Hence the given compound proposition is a tautology.

**CONTRADICATION:** A statement which is always false is known **as contradiction** or **Fallacy**. It is denoted by ***c***. It is also known as Contradiction.

**Example: Verify that the proposition $p \wedge (q \wedge \sim p)$ is a contradiction.**

**Solution:**

| p | q | ~p | $(q \wedge \sim p)$ | $p \wedge (q \wedge \sim p)$ |
|---|---|---|---|---|
| T | T | F | F | F |
| T | F | F | F | F |
| F | T | T | T | F |
| F | F | T | F | F |

It is clear from the truth table that for any possible value of p and q, the given proposition is false which establish that given proposition is a contradiction.

**PRACTICE EXAMPLE**

1. Show that $(p \vee q) \wedge (\sim p \wedge \sim q)$ is a contradiction.
2. Show that $p \leftrightarrow q \equiv (p \vee q) \rightarrow (p \wedge q)$, using truth table as well properties/algebra of proposition.
3. Which of the following proposition is a tautology?

   (a) $(p \vee q) \rightarrow q$

   (b) $p \vee (q \rightarrow p)$

   (c) $p \vee (p \rightarrow q)$

   (d) $p \rightarrow (p \rightarrow q)$

**CONTINGENCY:** A statement which is neither tautology nor fallacy is known **as Contingency**.

## FUNCTIONALLY COMPLETE SET OF CONNECTIVES

**Functionally Complete Set of Connectives:** Any set of connective in which every formula can be expressed in terms of an equivalence formula containing the connectives from the set is called functionally complete set of connectives.

**Remark:** A minimal functionally complete **set of connectives does not contain a connective** which can be expressed in terms of other connectives.

**Result: Prove that $\{\wedge, \sim\}$ is Minimal functionally complete set of connectives.**

**Proof:** We five basic connectives $\{\wedge, \vee, \rightarrow, \leftrightarrow, \sim\}$.

To eliminate the conditional, one uses the following logical equivalence:

$p \rightarrow q \equiv \sim p \vee q$

To eliminate the bi conditional, one uses the following logical equivalence:

$p \to q \equiv (\sim p \vee q) \wedge (p \vee \sim q)$

Thus Conditional and Bi conditional connective can be replaced by conjunction, disjunction and negation only using above equivalence. Also disjunction can be replaced by negation and conjunction using De Morgan's Law.

Hence any statement formula can be expressed using conjunction and negation only.

Hence, it is prove that $\{\wedge, \sim\}$ **is Minimal functionally complete set of connectives.**

**Result: Prove that $\{\vee, \sim\}$ is Minimal functionally complete set of connectives.**

**Proof:** To eliminate the conditional, one uses the following logical equivalence:

$p \to q \equiv \sim p \vee q$

To eliminate the bi conditional, one uses the following logical equivalence:

$p \leftrightarrow q \equiv (\sim p \vee q) \wedge (p \vee \sim q)$

Thus Conditional and Bi conditional connective can be replaced by conjunction, disjunction and negation only using above equivalence. Also disjunction can be replaced by negation and conjunction using De Morgan's Law.

Hence any statement formula can be expressed using conjunction and negation only.

**Hence, it is prove that $\{\vee, \sim\}$ is Minimal functionally complete set of connectives.**

**Result: Prove that $\{\vee, \wedge\}$ is not a Minimal functionally complete set of connectives.**

**PRACTICE EXAMPLE**

1. Write an equivalent formula for $p \wedge (Q \leftrightarrow R)$ which contains neither the biconditional nor the conditional.
2. Write an equivalent formula for $p \wedge (Q \leftrightarrow R) \vee (R \leftrightarrow P)$ which contains neither the biconditional nor the conditional.

## NORMAL FORMS

**NORMAL FORM:** By comparing truth tables, one determine whether two logical expression P and Q are equivalent. But the process is very tedious when the number of variables increases. A better method is to transform the expression P and Q to some standard forms of expression P' and Q' such that a simple comparison of P' and Q' shows whether $P \equiv Q$. **The standard forms are called normal forms or canonical forms.**

**ELEMENTARY PRODUCT:** In logical expression, a product of variables and their negation is called an elementary product. Example: $p \wedge q, p \wedge q, \sim \mathrm{p} \wedge \mathrm{q}, \mathrm{p} \wedge \sim \mathrm{q}, \mathrm{p} \wedge \mathrm{q} \wedge \sim \mathrm{r}$ are elementary products in to variables.

**ELEMENTARY SUM**: In logical expression, a sum of the variables and their negation is called elementary sum. Example $p \vee q, p \vee q, \sim p \vee q, p \vee \sim q$.

## DISJUNCTIVE NORMAL FORM (DNF)

**Disjunctive Normal Form (DNF):** A logical expression is said to be in disjunctive normal form if it is the **sum of elementary products**.

### PROCEDURE TO OBTAIN DNF OF A GIVEN LOGICAL EXPRESSION

1. Remove Conditional and Bi conditional using an equivalent expression which contains negation, disjunction and conjunction only.
2. Eliminate negation before the sum and products using De Morgan's Law.
3. Apply distributive law until a sum of elementary product is obtained.

**EXAMPLE: Obtain the disjunctive normal form of the following**

1. $p \wedge (p \to q)$

   **Solution: Step1: Remove $p \to q$ by logically equivalent statement**

   $p \wedge (p \to q) \equiv p \wedge (\sim p \vee q)$

   **Step 3: Apply distributive law**

   $p \wedge (p \to q) \equiv (p \wedge \sim p) \vee (p \wedge q)$

   which is the required disjunctive normal form (DNF)

2. $p \vee \left(\sim p \to \left(q \vee (q \to \sim r)\right)\right)$

   **Solution:** $p \vee \left(\sim p \to \left(q \vee (q \to \sim r)\right)\right) \equiv p \vee \left(\sim p \to \left(q \vee (\sim q \vee \sim r)\right)\right)$

   $\equiv p \vee \left(p \vee \left(q \vee (\sim q \vee \sim r)\right)\right)$

   $\equiv p \vee p \vee q \vee \sim q \vee \sim r$

   $\equiv p \vee q \vee \sim q \vee \sim r$

   which is the required disjunctive normal form.

3. $\left(p \wedge \sim (q \wedge r)\right) \vee (p \to q)$

**Solution:** $\left(p \wedge \sim (q \wedge r)\right) \vee (p \Rightarrow q) \equiv \left(p \wedge \sim (q \wedge r)\right) \vee (\sim p \vee q)$

(Remove $\to$ by logically equivalent statement**)**

$\equiv \left(p \wedge (\sim q \vee \sim r)\right) \vee (\sim p \vee q)$ (De Morgan's Law)

$\equiv \left((p \wedge \sim q) \vee (p \wedge \sim r)\right) \vee (\sim p \vee q)$ (Distributive law)

is the required DNF

4. $p \to \left((p \to q) \wedge \sim (\sim q \vee \sim p)\right)$

**Solution:** $p\to\left(\left(p\to q\right)\wedge\sim\left(\sim q\vee\sim p\right)\right)\equiv\sim p\vee\left(\left(p\to q\right)\wedge\sim\left(\sim q\vee\sim p\right)\right)$

$\equiv\sim p\vee\left(\left(\sim p\vee q\right)\wedge\sim\left(\sim q\vee\sim p\right)\right)$

(Remove $\to$by logically equivalent statement**)**

$\equiv\sim p\vee\left(\left(\sim p\vee q\right)\wedge\left(q\wedge p\right)\right)$( De Morgan's Law)

$\equiv\sim p\vee\left[\left(\left(\sim p\wedge\left(q\wedge p\right)\vee\left(q\wedge\left(q\wedge p\right)\right)\right)\right)\right]$

(Distributive law)

$\equiv\sim p\vee\left[\left(\left(\sim p\wedge\left(q\wedge p\right)\vee\left(q\wedge\left(q\wedge p\right)\right)\right)\right)\right]$

$\equiv\sim p\vee\left[\left(\left(\left[\left(\sim p\wedge q\right)\wedge p\right]\vee\left(\left(q\wedge q\right)\wedge p\right)\right)\right)\right]$

$\equiv\sim p\vee\left[\left(\left(\left[\left(q\wedge\sim p\right)\wedge p\right]\vee\left(q\wedge p\right)\right)\right)\right]$

$\equiv\sim p\vee\left[\left(\left(\left[q\wedge\left(\sim p\wedge p\right)\right]\vee\left(q\wedge p\right)\right)\right)\right]$

$\equiv\sim p\vee\left[\left(\left(\left[q\wedge F\right]\vee\left(q\wedge p\right)\right)\right)\right]$

$\equiv\sim p\vee\left[\left(\left(F\vee\left(q\wedge p\right)\right)\right)\right]$

$\equiv\sim p\vee\left(q\wedge p\right)$

$\equiv\left(\sim p\vee q\right)\wedge\left(\sim p\vee p\right)$

$\equiv\left(\sim p\vee q\right)\wedge T$

$\equiv\left(\sim p\wedge T\right)\vee\left(q\wedge T\right)$

$\equiv\left(\sim p\wedge\left(\sim q\vee q\right)\right)\vee\left(q\wedge\left(\sim \mathrm{p}\vee p\right)\right)$

$\equiv\left(\left(\sim p\wedge\sim q\right)\vee\left(\sim p\wedge q\right)\right)\vee\left(\left(q\wedge\sim \mathrm{p}\right)\vee\left(q\wedge \mathrm{p}\right)\right)$

$\equiv\left(\sim p\wedge\sim q\right)\vee\left(\sim p\wedge q\right)\vee\left(q\wedge \mathrm{p}\right)$

**Remark:** Disjunctive normal form (DNF)of logical expression is not unique.

## CONJUNCTIVE NORMAL FORM (CNF)

A logical expression is said to be in disjunctive normal form if it is the **product of elementary sum**.

## PROCEDURE TO OBTAIN CNF OF A GIVEN LOGICAL EXPRESSION

1. Remove Conditional and Bi conditional using an equivalent expression which contains negation, disjunction and conjunction only.
2. Eliminate negation before the sum and products using De Morgan's Law.
3. Apply distributive law until a sum of elementary product is obtained.

**EXAMPLE: Obtain the conjunctive normal form of the following**

1. $p \wedge (p \rightarrow q)$

   **Solution:** $p \wedge (p \rightarrow q) \equiv p \wedge (\sim p \vee q)$ ( Remove conditional by logical connective)

   which is required conjunctive normal form

2. $\left[q \vee (p \wedge q)\right] \wedge \sim \left[(p \vee r) \wedge q\right]$

   **Solution:** $\left[q \vee (p \wedge r)\right] \wedge \sim \left[(p \vee r) \wedge q\right] \equiv \left[q \vee (p \wedge r)\right] \wedge \left[\sim (p \vee r) \vee \sim q\right]$

   ( De Morgan's Law)

   $\equiv \left[q \vee (p \wedge r)\right] \wedge \left[(\sim p \wedge \sim r) \vee \sim q\right]$ (De Morgan's Law)

   $\equiv \left[(q \vee p) \wedge (q \vee r)\right] \wedge \left[(\sim p \vee \sim q) \wedge (\sim r \vee \sim q)\right]$

   (Distributive)

   $\equiv (q \vee p) \wedge (q \vee r) \wedge (\sim p \vee \sim q) \wedge (\sim r \vee \sim q)$

   which is required CNF

**Remark:** Conjunctive normal form (CNF) of logical expression is not unique.

**Point to remember for DNF and CNF**

1. $p \rightarrow q \equiv \sim p \vee q$.
2. $p \leftrightarrow q \equiv (\sim p \vee q) \wedge (p \vee \sim q)$ *OR* $p \leftrightarrow q \equiv (p \wedge q) \vee (p \wedge q)$.
3. **De Morgan's Law**

   $\sim (p \vee q) \equiv \sim p \wedge \sim q$

   $\sim (p \wedge q) \equiv \sim p \vee \sim q$

4. Distributive Law

   $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

   $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

## PRINCIPAL DISJUNCTIVE NORMAL FORM (PDNF)

**MINTERMS:** Let p and q be two statement variables then $p \wedge q, \sim p \wedge q, p \wedge \sim q, \sim p \wedge \sim q$ are **called minterms of p and q.** The number of minterms in **n** variables **is $2^n$.**

For example the minterms for three variables p,q and r are

$p \wedge q \wedge r, \sim p \wedge q \wedge r, p \wedge \sim q \wedge r, p \wedge q \wedge \sim r, \sim p \wedge \sim q \wedge r, p \wedge \sim q \wedge \sim r, \sim p \wedge q \wedge \sim r, \sim p \wedge \sim q \wedge \sim r.$

The truth table for the minterms of p and q are given below:

**TABLE: Minterms**

| p | q | $p \wedge q$ | $\sim p \wedge q$ | $p \wedge \sim q$ | $\sim p \wedge \sim q$ |
|---|---|---|---|---|---|
| T | T | T | F | F | F |
| T | F | F | F | T | F |
| F | T | F | T | F | F |
| F | F | F | F | F | T |

**REMARKS:**

1. From the truth table it is clear that no minterms are equivalent.
2. Each minterms has truth value T for exactly one combination of the truth values of the variables p and q.

**Principal Disjunctive Norm Form (PDNF): PDNF** of a given formula can be defined as an equivalent formula consisting of disjunctives of minterms (elementary product which contains the entire variables) only. This is also known as sum of products canonical form. There are two ways to obtain the principal disjunctive normal form.

Method 1: Using Truth Table.

Method 2: Without Using Truth Table.

**Method 1: Using Truth Table.**

**Step 1:** Construct a truth table for the given compound statement.

**Step 2:** For every truth value T of the given proposition, select the minterms which has also true value T for the same combination of truth value of the statement variable.

**Step 3:** The disjunctive of minterms selected in step 2 is the required principal disjunctive normal form.

**Remark:** In PDNF, the repeated minterms are mention only once.

**Example: Obtain the principal disjunctive normal form using the truth table of $p \Rightarrow q$.**

**Solution: The truth table for $p \to q$ is given by**

| p | q | $p \to q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

The column containing $p \to q$ has truth value T for three combination of the truth values of p and q. Now

T in the first row of $p \to q$ corresponds to the minterms $p \wedge q$, **( refer TABLE: Minterms**)

T in the third row of $p \to q$ corresponds to the minterms $\sim p \wedge q$, **( refer TABLE: Minterms**)

T in the fourth row of $p \to q$ corresponds to the minterms $\sim p \wedge \sim q$, **( refer TABLE: Minterms**)

Thus , PDNF of $p \to q$ **is** $(p \wedge q) \wedge (\sim p \wedge q) \wedge (\sim p \wedge \sim q)$.

**Method 2: Without Using Truth Table.**

Step 1: Obtain the disjunctive normal form.

Step 2: Drop elementary products which are contradictions.

Step 3: If $p_i$ and $\sim p_i$ are missing in an elementary product $\alpha$, replace $\alpha$ by $(\alpha \wedge p_i) \vee (\alpha \wedge \sim p_i)$.

Step 4: Repeat step 3 until all elementary products are reduced to sum of minterms. Identical minterms appearing in the disjunction are deleted.

| |
|---|
| **POINTS TO BE REMEMBER FOR PDNF WITHOUT USING TRUTH TABLE**<br>1. After finding DNF, we apply identity law, complement law and Distributive Law.<br>2. **Identity law** for DNF is $q \equiv (q \wedge T)$<br>3. **Complement Law** : $\sim q \vee q \equiv T$ . |

**Example: Obtain the Principal Disjunctive Normal Form of the following without using the truth table.**

**1.** $p \to q$ **.**

**Solution:**

$p \to q \equiv \sim p \vee q$ **(DNF form)**

| | |
|---|---|
| **Remark:** ~p and q are individual variable and minterms consist of entire variable of compound statement. So we introduce the missing variable in 1st term ~p and 2nd term q .In 1st term ~p , q variable is missing which is introduce using identity and complement law as shown below and similarly in 2nd term q , p variable is introduce. | |
| $\sim p \equiv (\sim p \wedge T)$ (Identity Law)<br>$\equiv (\sim p \wedge (\sim q \vee q))$ ( Complement Law)<br>$\equiv ((\sim p \wedge \sim q) \vee (\sim p \wedge q))$ (Distributive Law) | $q \equiv (q \wedge T)$ ( Identity Law)<br>$\equiv (q \wedge (\sim p \vee p))$ ( Complement Law)<br>$\equiv ((q \wedge \sim p) \vee (q \wedge p))$ ( Distributive Law) |

$p \to q \equiv\equiv ((\sim p \wedge \sim q) \vee (\sim p \wedge q)) \vee ((q \wedge \sim p) \vee (q \wedge p))$

**is required PDNF.**

**2.** $q \vee (p \vee \sim q)$

**Solution:** $q \vee (p \vee \sim q) \equiv (q \wedge T) \vee [(p \wedge T) \vee (\sim q \wedge T)]$ ( Identity Law)

$\equiv (q \wedge (\sim p \vee p)) \vee [(p \wedge (\sim q \vee q)) \vee (\sim q \wedge (\sim p \vee p))]$ (Complement Law)

$\equiv [(q \wedge \sim p) \vee (q \wedge p)] \vee [(p \wedge \sim q) \vee (p \wedge q) \vee (\sim q \wedge \sim p) \vee (\sim q \wedge p)]$

(Distributive Law)

$\equiv (q \wedge \sim p) \vee [(p \wedge q) \vee (\sim q \wedge \sim p) \vee (\sim q \wedge p)]$ ( In PDNF each minterms appear once so we mention the repeated minterms only once.)

**3.** $(\sim p \wedge q \wedge \sim r) \vee (q \wedge r)$

**In the given formula** $(\sim p \wedge q \wedge \sim r)$ **is already a minterms so we convert** $q \wedge r$ **into minterm.**

**Now** $(q \wedge r) \equiv (q \wedge r) \wedge T$ (Identity Law)

$\equiv (q \wedge r) \wedge (\sim p \vee p)$ (Complement Law)

$(q \wedge r) \equiv [(q \wedge r) \wedge \sim p] \vee [(q \wedge r) \wedge p]$ (Distributive Law) (1)

Using (1) the required PDNF form is $(\sim p \wedge q \wedge \sim r) \vee [(q \wedge r) \wedge \sim p] \vee [(q \wedge r) \wedge p]$

**4.** $p \vee (\sim p \to (\sim q \to r))$

**Solution:**

$p \vee (\sim p \Rightarrow (\sim q \to r)) \equiv p \vee (\sim p \to (q \vee r))$ (Remove conditional using equivalent statement)

$\equiv p \vee (p \vee (q \vee r))$ (Remove conditional using equivalent statement)

$\equiv (p \vee p) \vee q \vee r$ (Associative Law)

$\equiv p \vee q \vee r$ (Idempotent Law)

DNF FORM (Now we convert it into PDNF form)

**(Remark: PDNF contain minterms and minterms include entire variable of compound statement so we introduce the missing terms using identity and complement law in 1st term p , 2nd term q** and3rd r)

$\equiv [(p \wedge T) \wedge T] \vee [(q \wedge T) \wedge T] \vee [(r \wedge T) \wedge T]$ ( Identity Law)

$\equiv [(p \wedge (\sim q \vee q)) \wedge (\sim r \vee r)] \vee [(q \wedge (\sim p \vee p)) \wedge (\sim r \vee r)] \vee [(r \wedge (\sim p \vee p)) \wedge (\sim q \vee q)]$

(Complement Law)

Now we simplify 1st bracket $[(p \wedge (\sim q \vee q)) \wedge (\sim r \vee r)]$ using distributive law we get

$$[(p \wedge (\sim q \vee q)) \wedge (\sim r \vee r)] \equiv [(p \wedge \sim q) \vee (p \wedge q)] \wedge (\sim r \vee r)$$

$$\equiv [(p \wedge \sim q) \wedge (\sim r \vee r)] \vee [(p \wedge q) \wedge (\sim r \vee r)]$$

$$[(p \wedge (\sim q \vee q)) \wedge (\sim r \vee r)] \equiv [(p \wedge \sim q \wedge \sim r) \vee (p \wedge \sim q \wedge r)] \vee [(p \wedge q \wedge \sim r) \vee (p \wedge q \wedge r)]$$

Similarly we can simplify 2nd and 3rd bracket $[(q \wedge (\sim p \vee p)) \wedge (\sim r \vee r)]$ and $[(r \wedge (\sim p \vee p)) \wedge (\sim q \vee q)]$

$$[(q \wedge (\sim p \vee p)) \wedge (\sim r \vee r)] \equiv (\sim p \wedge q \wedge \sim r) \vee (\sim p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (p \wedge q \wedge r)$$

$$[(r \wedge (\sim p \vee p)) \wedge (\sim q \vee q)] \equiv (\sim p \wedge q \wedge r) \vee (\sim p \wedge \sim q \wedge r) \vee (p \wedge \sim q \wedge r) \vee (p \wedge q \wedge r)$$

**Therefore required PDNF is** $(\sim p \wedge q \wedge r) \vee (\sim p \wedge \sim q \wedge r) \vee (p \wedge \sim q \wedge r) \vee (p \wedge q \wedge r) \vee (\sim p \wedge q \wedge \sim r) \vee (p \wedge q \wedge \sim r) \vee (p \wedge \sim q \wedge \sim r)$

**5.** $p \leftrightarrow q$

**Solution:** $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$ (Logical equivalent statement of biconditional)

$\equiv (\sim p \vee q) \wedge (\sim q \vee p)$ (Logical equivalent statement of conditional statement)

$\equiv [(\sim p \vee q) \wedge \sim q] \vee [(\sim p \vee q) \wedge p]$ (Distributive Law)

$\equiv (\sim p \wedge \sim q) \vee (q \wedge \sim q) \vee (\sim p \wedge p) \vee (q \wedge p)$ (Distributive Law)

**is required PDNF**

**6.** $(p \wedge q) \vee (\sim p \wedge r) \vee (q \wedge r)$

**Solution:** $(p \wedge q) \vee (\sim p \wedge r) \vee (q \wedge r)$ **DNF form .Now we convert it into PDNF form**

$(p \wedge q) \vee (\sim p \wedge r) \vee (q \wedge r)$

In order to find the PDNF we first find the minterm for 1st term $(p \wedge q)$ **( since minterm contain entire variable so we introduce the missing variable r in the 1st term using identity, complement and distributive law)**

$(p \wedge q) \equiv (p \wedge q) \wedge T$ ( Identity Law)

$\equiv (p \wedge q) \wedge (\sim r \vee r)$ ( Complement Law)

$(p \wedge q) \equiv [(p \wedge q) \wedge \sim r] \vee [(p \wedge q) \wedge r]$ ( Distributive Law)

$(p \wedge q) \equiv (p \wedge q \wedge \sim r) \vee (p \wedge q \wedge r)$

Similarly $(\sim p \wedge r)$ and $(q \wedge r)$ can be converted into minterms.

$(\sim p \wedge r) \equiv (\sim p \wedge r \wedge q) \vee (p \wedge r \wedge \sim q)$

$(q \wedge r) \equiv (q \wedge r \wedge p) \vee (q \wedge r \wedge \sim p)$

**Therefore PDNF form of**

$(p \wedge q) \vee (\sim p \wedge r) \vee (q \wedge r)$ **is** $(p \wedge q \wedge \sim r) \vee (\sim p \wedge r \wedge q) \vee (q \wedge r \wedge p) \vee (q \wedge r \wedge \sim p)$

**Advantages of obtaining principal disjunctive normal form are**

1. The principal disjunctive normal of a given formula is unique.
2. Two formulas are equivalent if and only if their principal disjunctive normal forms coincide.
3. If the given compound proposition is a tautology, then its principal disjunctive normal form will contain all possible minterms of its components.

## PRINCIPAL CONJUNCTIVE NORMAL FORM

**MAXTERMS:** Let p and q be two statement variables then $p \vee q, \sim p \vee q, p \vee \sim q, \sim p \vee \sim q$ are called maxterms of p and q. The number of maxterms in n variables is $2^n$.

For example the maxterms for three variables p, q and r are

$p \vee q \vee r, \sim p \vee q \vee r, p \vee \sim q \vee r, p \vee q \vee \sim r, \sim p \vee \sim q \vee r, p \vee \sim q \vee \sim r, \sim p \vee q \vee \sim r, \sim p \vee \sim q \vee \sim r.$

The truth table for the minterms of p and q are given below:

**Table: Maxterms**

| p | q | $p \vee q$ | $\sim p \vee q$ | $p \vee \sim q$ | $\sim p \vee \sim q$ |
|---|---|---|---|---|---|
| T | T | T | T | T | F |
| T | F | T | F | T | T |
| F | T | T | T | F | T |
| F | F | F | T | T | T |

**REMARKS:**

1. From the truth table it is clear that no max terms are equivalent.
2. Each max term has truth value F for exactly one combination of the truth values of the variables p and q.

**PRINCIPAL CONJUNCTIVE NORMAL FORM:** It is defined as an equivalent formula consists of conjunctive of max terms only. It is also called the **product of sums canonical form.** There are two ways to obtain the principal disjunctive normal form.

Method 1: Using Truth Table.

Method 2: Without using Truth Table.

**Remark:** In PCNF, the repeated maxterms are mention only once.

**Method 1: Using Truth Table**

**Step 1:** Construct a truth table for the given compound statement.

**Step 2:** For every truth value F of the given proposition, select the maxterms which has also true value F for the same combination of truth value of the statement variable.

**Step 3:** The conjunctive of maxterms selected in step 2 is the required principal conjunctive normal form.

**Example: Obtain the Principal Conjunctive Normal Form of the following t using truth table.**

1. $p \wedge q$.

**The truth table for $p \wedge q$ is given by**

| p | q | $p \wedge q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

The column containing $p \wedge q$ has truth value **F** for three combination of the truth values of p and q.

Now

**F** in the second row of $p \wedge q$ corresponds to the maxterms $\sim p \vee q$ **, (refer TABLE: Maxterms)**

**F** in the third row of $p \wedge q$ corresponds to the maxterms $p \vee \sim q$ **,( refer TABLE: Maxterms)**

**F** in the fourth row of $p \wedge q$ corresponds to the maxterms $\sim p \vee \sim q$, **(refer TABLE: Maxterms)**

Thus, PDNF of $p \wedge q$ **is** $(\sim p \vee \sim q) \wedge (\sim p \vee q) \wedge (p \vee \sim q)$.

**Method 2: Without Using Truth Table**

**POINTS TO BE REMEMBER FOR PCNF WITHOUT USING TRUTH TABLE**

1.After finding CNF, we apply identity law, complement law and Distributive Law.

2.**Identity law** for DNF is $q \equiv (q \vee F)$

3.**Complement Law**: $\sim q \vee q \equiv F$.

**Example: Obtain the Principal Conjunctive Normal Form of the following without using truth table.**

2. $p \wedge q$.

**Solution:** $p \wedge q$ **which is the CNF form**

**[Remark: Since p and q are individual variable so we introduce the missing variable q in the 1st term p and missing variable p in the 2nd term q to obtain the maxterms.]**

$p \wedge q \equiv (p \vee F) \wedge (q \vee F)$ (Identity Law)

$\equiv (p \vee (\sim q \wedge q)) \wedge (q \vee (\sim \mathrm{p} \wedge p))$ (Complement Law)

$\equiv ((p \vee \sim q) \wedge (p \vee q)) \wedge ((q \vee \sim \mathrm{p}) \wedge (q \vee p))$ (Distributive Law)

$p \wedge q \equiv (p \vee \sim q) \wedge (p \vee q) \wedge (q \vee \sim \mathrm{p}) \wedge (q \vee p)$ is required PCNF form

3. $(\sim p \to r) \wedge (q \leftrightarrow p)$.

**Solution: [ First we find CNF form using logical equivalent statement of** $p \to q \equiv \sim p \vee q$ **and ]**

$(\sim p \to r) \wedge (q \leftrightarrow p) \equiv (p \vee r) \wedge [(p \to q) \wedge (q \to p)]$

$\equiv (p \vee r) \wedge [(\sim p \vee q) \wedge (\sim q \vee p)]$ **( CNF form)**

**[Remark: From CNF we find PCNF by introducing the missing variable q, r and r in 1st , 2nd , 3rd term respectively using identity , complement to obtain the maxterms and apply distributive law to find the PCNF ]**

$(\sim p \to r) \wedge (q \leftrightarrow p) \equiv (p \vee r) \wedge [(\sim p \vee q) \wedge (\sim q \vee p)]$ .................... (1)

Now $(p \vee r) \equiv (p \vee r) \vee F$

$\equiv (p \vee r) \vee (q \wedge \sim q)$

$(p \vee r) \equiv (p \vee r \vee q) \wedge (p \vee r \sim q)$ (A)

Similarly $(\sim p \vee q) \equiv (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r)$ (B)

$(\sim q \vee p) \equiv (\sim q \vee p \vee r) \wedge (\sim q \vee p \vee \sim r)$ (C)

**Putting the value from (A), (B), (C) in (1) we get**

$(\sim p \to r) \wedge (q \leftrightarrow p) \equiv (p \vee r \vee q) \wedge (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r) \wedge (\sim q \vee p \vee r) \wedge (\sim q \vee p \vee \sim r)$

**which is required PCNF form**

**Advantages of obtaining principal conjunctive normal form are**

1. The principal conjunctive normal of a given formula is unique.
2. Every compound proposition, which is not a tautology, has an equivalent principal conjunctive normal form.
3. If the given compound proposition is a contradiction, then its principal conjunctive normal form will contain all possible maxterms of its components.

**PRATICE EXAMPLE**

**Obtain the PDNF and PCNF form of the following without using truth table.**

1. $(\sim p \vee \sim q) \to (p \leftrightarrow \sim q)$
2. $q \wedge (p \vee \sim q)$
3. $p \vee \left(\sim p \to \left(q \vee (\sim q \to r)\right)\right)$
4. $\left(p \to (q \wedge r)\right) \wedge \left(\sim p \to (\sim q \wedge \sim r)\right)$
5. $\left(\neg p \longrightarrow (\neg p \longrightarrow (\neg p \wedge q))\right) \equiv p \vee q$
6. $\neg\left((\neg p \wedge q) \vee (\neg p \wedge \neg q)\right) \vee (p \wedge q) \equiv p$
7. Obtain Principal Disjunctive Normal Form of $p \to \left((p \to q) \wedge (\sim(\sim q \vee \sim p))\right)$.
8. Using truth table ,find the principal disjunctive normal form of $(q \wedge \neg r \wedge \neg s) \vee \left((r \wedge s)\right)$.
9. Using truth table, find the principal conjunctive normal form of $\left(q \wedge (p \vee \neg q)\right)$.
10. Obtain Principal Disjunctive Normal Form of $p \to \left((p \to q) \wedge (\sim(\sim q \vee \sim p))\right)$.

# LOGIC IN PROOF

**Theorem:** A theorem is a proposition that can be proved to be true.

**Premises:** It is a proposition on the basis of which we would able to draw a conclusion. You can think of premise as evidence or assumption.

Therefore, we initially assume that something is true and on the basis of that we draw the conclusion.

**Argument:** An argument is a sequence of statement. All statements except last one are known as premises or hypothesis. And the last statement is known as conclusion.

**Proof:** An argument which establishes the truth of the theorem is known as proof.

**Valid Argument**: An argument is said to be valid, if all premises are true, conclusion must be true in the same possibility. Validity of the argument can be check using two methods

1. Truth Table Technique
2. Without using Truth Table Technique / using rule of inference.

**Method 1: Truth Table Technique**

**Step 1:** Construct a truth table for **Premises** and **the Conclusion**.

**Step2:** Find the rows in which **all the premises are true**. This rows are called as **critical rows**

**Step 3:** Check conclusion of all critical rows.

**a.** If in each critical row conclusion is **true, then** the argument is **valid.**

**b.** If there is a row in which conclusion is **false, then** the argument form is **invalid.**

**Example:** Determine whether the following argument form is valid or invalid

$$p \to q \vee \sim r$$
$$q \to p \wedge r$$
$$\therefore p \to r$$

**Solution:** **Premises 1 Premises 2 Conclusion**

| p | q | r | ~r | $q\vee\sim r$ | $p\wedge r$ | $p\to q\vee\sim r$ | $q\to p\wedge r$ | $p\to r$ |
|---|---|---|---|---|---|---|---|---|
| T | T | T | F | T | T | T | T | T |
| T | T | F | T | T | F | T | F | F |
| T | F | T | F | F | T | F | T | T |
| T | F | F | T | T | F | T | T | F |
| F | T | T | F | T | F | T | F | T |
| F | T | F | T | T | F | T | F | T |
| F | F | T | F | F | F | T | T | T |
| F | F | F | T | T | F | T | T | T |

**PRATICE EXAMPLE**

**Determine whether the conclusion C follows logically from the premises $H_1$ and $H_2$ using truth table technique.**

1. $H_1 : P \Rightarrow Q$, $H_2 : \mathrm{P}$, $C : Q$
2. $H_1 :\sim P$, $H_2 : \mathrm{P} \Leftrightarrow \mathrm{Q}$, $C :\sim (P \wedge Q)$
3. $H_1 : P \Rightarrow Q$, $\mathrm{H}_2 :\sim (P \wedge Q)$ , $\mathrm{C} :\sim P$

Theoretically, it is possible to determine in a finite number of steps whether a conclusion follows from a given set of premises by constructing the appropriate truth table. However, this method become tedious when the number of atomic variables present in all the formulas representing the premises and conclusion is large. This disadvantage, coupled with the fact that the inference theory is applicable in more general situations where the truth table technique is tedious task.

## RULES OF INFERENCE

The rules of inference are criteria for determining the validity of an argument. Any conclusion which is arrived by following rules of inference is called a valid conclusion, and the argument is called a valid argument.

**Table: Rule of Inference**

| **Rule of Inference** | **Implication Form** | **Name** |
|---|---|---|
| $\dfrac{p}{\therefore p \vee q}$ | $p \rightarrow (p \vee q)$ | **Addition** |
| $\dfrac{p \wedge q}{\therefore p}$ | $(p \wedge q) \rightarrow p$ | **Simplification** |
| $p$<br>$\dfrac{q}{\therefore p \wedge q}$ | $((p) \wedge (q)) \rightarrow (p \wedge q)$ | **Conjunction** |
| $p \rightarrow q$<br>$\dfrac{p}{\therefore q}$ | $[(p \rightarrow q) \wedge p] \rightarrow q$ | **Modus Ponens** |
| $p \rightarrow q$<br>$\dfrac{\sim q}{\therefore \sim p}$ | $[(p \rightarrow q) \wedge \sim q] \rightarrow \sim p$ | **Modus Tollens** |
| $p \rightarrow q$<br>$\dfrac{q \rightarrow r}{\therefore p \rightarrow r}$ | $[(p \rightarrow q) \wedge (q \rightarrow r)] \Rightarrow (p \rightarrow r)$ | **Hypothetical Syllogism** |

| | | |
|---|---|---|
| $p \vee q$<br>$\underline{\sim p}$<br>$\therefore q$ | $\left[(p \vee q) \wedge \sim p\right] \rightarrow q$ | **Disjunction Syllogism** |
| $p \rightarrow q \wedge (r \rightarrow s)$<br>$p \vee r$<br>__________<br>$\therefore q \vee s$ | $(p \rightarrow q) \wedge (r \rightarrow s) \wedge (p \vee r) \rightarrow (q \vee s)$ | **Constructive Dilemma** |
| $p \rightarrow q \wedge (r \rightarrow s)$<br>$\sim q \vee \sim s$<br>__________<br>$\therefore \sim \mathrm{p} \vee \sim \mathrm{r}$ | $(p \rightarrow q) \wedge (r \rightarrow s) \wedge (\sim q \vee \sim s) \rightarrow (\sim p \vee \sim r)$ | **Destructive Dilemma** |

**Example: Can we conclude S from the following premises?**

**(i).** $P \rightarrow Q$

**(ii).** $P \rightarrow R$

**(iii).** $\sim (Q \wedge R)$

**(iv)** $S \vee P$

**Solution: [The valid argument for deducing S from the given four premises is given as sequence. On left formula is given. On the right, we indicate whether the proposition is premise or a conclusion. If it is conclusion, we indicate the premises and the rules of inference or logical identities used for deriving the conclusion.]**

| | | |
|---|---|---|
| 1. | $P \rightarrow Q$ | Premises (i) |
| 2. | $P \rightarrow R$ | Premises (ii) |
| 3. | $(P \rightarrow Q) \wedge (P \rightarrow R)$ | Line 1, 2 and Conjunction Rule |
| 4. | $\sim (Q \wedge R)$ | Premises (iii) |
| 5. | $(\sim Q \wedge \sim R)$ | Line 4 and De Morgan's Law |
| 6. | $(\sim \mathrm{P} \vee \sim \mathrm{P})$ | Line 3, 5 and Destructive Dilemma |
| 7. | $\sim \mathrm{P}$ | Idempotent Law |
| 8. | $S \vee P$ | Premises (iv) |
| 9. | S | Line 7, 8 and disjunctive Syllogism |

Thus, S is the conclusion from the given premises.

**Example: Derive S from the following premises using a valid argument?**

**(i).** $P \to Q$

**(ii).** $Q \to \sim R$

**(iii).** $(P \vee S)$

**(iv)** $R$

**Solution:**

| | |
|---|---|
| 1. $P \to Q$ | Premises (i) |
| 2. $Q \to \sim R$ | Premises (ii) |
| 3. $P \to \sim$ R | Line 1, 2 and Hypothetical Syllogism |
| 4. $R$ | Premises (iii) |
| 5. $\sim(\sim R)$ | Line 4 and Double Negation |
| 6. $\sim P$ | Line 3, 5 and Modus Tollens |
| 7. $P \vee S$ | Idempotent Law |
| 8. S | Line 6, 7 and disjunctive Syllogism |

Thus, S is the conclusion from the given premises.

**Example: Check the validity of the following argument:**

**If Ram has completed B.E. or MBA, then he is assured of a good job. If Ram is assured of a good job, he is happy. Ram is not happy. So Ram has not completed MBA.**

**Solution:** Let

p: Ram has completed B.E.

q: Ram has completed MBA.

r: Ram is assured of a good job.

s: Ram is happy.

The given premises are

(i). $(p \vee q) \to r$

(ii) $r \to s$

(iii) $\sim s$

**The conclusion is ~q**

**Solution:**

| | |
|---|---|
| 1. $(p \vee q) \to r$ | Premises (i) |

| | |
|---|---|
| 2. $r \to s$ | Premises (ii) |
| 3. $(p \vee q) \to r$ | Line 1, 2 and Hypothetical Syllogism |
| 4. $\sim s$ | Premises (iii) |
| 5. $\sim (p \vee q)$ | Line 3,4 and modus Tollens |
| 6. $\sim p \vee \sim q$ | De Morgan's law |
| 7. $\sim q$ | Line 6 and simplification |

**Example: Check the validity of the following arguments:**
**If milk is black, then every cow is white. If every cow is white, then it has four legs. If every cow has four legs, then every buffalo is white and brisk. The milk is black. Therefore, the buffalo is white**
**Solution:**
P: The milk is black.
Q: Every cow is white.
R: Every cow has four legs.
S: Every buffalo is white.
T: Every buffalo is brisk.
**The given premises are**
(i). $P \to Q$
(ii) $Q \to R$
(iii) $R \to S \wedge T$
(iv) P
**The conclusion is S.**

| | |
|---|---|
| 1. $P$ | Premises (iv) |
| 2. $P \to Q$ | Premises (i) |
| 3. $Q$ | Line 1, 2 and Modus Ponens |
| 4. $Q \to R$ | Premises (ii) |
| 5. R | Line 3,4 and Modus Ponens |
| 6. $R \to S \wedge T$ | Premises (iii) |
| 7. $S \wedge T$ | Line 5,6 and Modus Ponens |

| | |
|---|---|
| 8. S | Line 7 and Simplification |

**Therefore, the argument is valid.**

**PRATICE EXAMPLE**

1.Show that $t$ is a valid conclusion, from the premises $p \to q,\ q \to r,\ r \to s, \sim s$ and $p \vee t$.

2.**Test the validity of the following argument**: If I get the notes and study well, then I will get first class. I didn't get first class. So either I didn't get the notes or I didn't study well.

3. **Determine the validity of the following using truth table:** If I like mathematics, then I will study. Either I don't study or I pass mathematics. If I don't graduate, then I didn't pass mathematics. Therefore, If I like mathematics, then I will graduate.

4.Show that $s$ is a valid conclusion from the premises $p \longrightarrow q, p \longrightarrow r, \neg(q \wedge r)$, and $s \vee p$.

5.Show that $M$ is a valid conclusion, from the premises $T \to (M \vee E),\ S \to \neg E$, and $T \wedge S$.

6.Represent the argument symbolically

"If it rains today, then we will not have a party today.
If we do not have party today, then we will have a party tomorrow.
Therefore, if it rains today we will have a party tomorrow."

Also determine whether the argument is valid using rule of inference for predicate calculus.

## PREDICATE CALCULUS

**IMPORTANCE OF PREDICATE CALCULUS:** The propositional calculus does not allow us to represent many of the statements that we use in mathematics, computer science and in everyday life. Predicate calculus is a generalization of propositional calculus. It contains all the components of the propositional calculus, including propositional variables and constants. Predicate calculus is important for several reasons; this has application in expert system, in database and also basis for the Prolog language.

**PREDICATES:** A part of a declarative sentence describing the properties of an object or relation among objects is called a predicate. For example, 'is a student 'is a predicate.

Sentence involving predicates describing the property of objects are denoted by P(x), where P denotes the predicate and x is a variable denoting any object. For example, P(x) can denote 'x is student'. In this sentence, x is a variable and P denotes the predicate 'is student '.