CHAROTAR UNIVERSITY OF SCIENCE AND TECHNOLOGY

FACULTY OF APPLIED SCIENCES

DEPARTMENT OF MATHEMATICAL SCIENCES

SEMESTER 3 B.Tech. CE, IT, CSE

DISCRETE MATHEMATICS AND ALGEBRA

MA253

UNIT 6

LINEAR ALGEBRA

Field:

A non-empty set $\boldsymbol{F}$ is said to form a field under the binary operations '+' and '$\cdot$' if

(i) $(\boldsymbol{F}, +)$ is an abelian group

(ii) $(\boldsymbol{F}, \ \cdot)$ is an abelian group

(iii)'$\cdot$' is distributive over '$+$'.

i.e., for all $a, b, c \in \boldsymbol{F}$,

$a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$

For example,

$\boldsymbol{N}, \boldsymbol{Z}$ are not field under '+' and '$\cdot$'.

$\boldsymbol{Q}, \boldsymbol{R}, \boldsymbol{C}$ are field under '+' and '$\cdot$'.

Vector Space:

Let V be a non-empty set and $\boldsymbol{F}$ be a field under the binary operations '+' and '·' . Then V is called a vector space over the field $\boldsymbol{F}$ if

for all $u, v, w \in V$ and for all $\alpha, \beta \in \boldsymbol{F}$

(1) $u + v \in V$

(2) $u + (v + w) = (u + v) + w$

(3) There exists an element $0 \in V$ such that for all $u \in V$,

$$u + \mathbf{0} = u = u + \mathbf{0}$$

(4) There exists an element $-u \in V$ such that for all $u \in V$,

$$u + (-\boldsymbol{u}) = 0 = (-\boldsymbol{u}) + u$$

(5) $u + v = v + u$

(6) $\alpha u \in V$

(7) $(\alpha\beta)u = \alpha(\beta u) = \beta(\alpha u)$

(8) $\alpha(u + v) = \alpha u + \alpha v$

(9) $(\alpha + \beta)u = \alpha u + \beta u$

(10) $1 \cdot u = u$, where '1' is the identity element of $\boldsymbol{F}$.

It is also known as linear space or linear vector space.

The vector space V over $\boldsymbol{F}$ is denoted by $\mathrm{V}(\boldsymbol{F})$.

If $\boldsymbol{F} = \boldsymbol{R}$ then $\mathrm{V}(\boldsymbol{R})$ is a vector space over the set of real numbers which is also called " *Real Vector space* ".

If $\boldsymbol{F} = \boldsymbol{C}$ then $\mathrm{V}(\boldsymbol{C})$ is called " *Complex Vector space* ".

Ex: Show that R is a vector space over R under the usual addition and scalar multiplication.

Ex: Show that C is a vector space over R under the usual addition and scalar multiplication.

Ex: Show that R^2, the set of *2*-tuples of real numbers (u_1, u_2) forms a vector space over R with the usual addition and scalar multiplication.

Ex: Determine whether the set V of all pairs of real numbers (x, y) with the operations $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2 + 1, y_1 + y_2 + 1)$ and $k(x, y) = (kx, ky)$, where $k \in R$, is a vector space over R or not.

Ex: Show that R^3, the set of *3*-tuples of real numbers (u_1, u_2, u_3) forms a vector space over R with the usual addition and scalar multiplication. (H.W)

Note: The $\boldsymbol{R^n}$, the set of *n*-tuples of real numbers $(u_1, u_2, \dots, u_n)$ with the usual addition and scalar multiplication forms a vector space over $\boldsymbol{R}$. **(H.W.)**

Ex: Prove that M_{22}, the set of all 2 $\times$2 matrices of real numbers forms a vector space over R with the matrix addition and scalar multiplication.

Ex: Prove that M_{33}, the set of all 3 $\times$ 3 matrices of real numbers forms a vector space over R with the matrix addition and scalar multiplication. (H.W)

<u>Notes:</u>

(1) The M_{nn}, the set of all n $\times$ n matrices of real numbers forms a vector space over $\boldsymbol{R}$ with the matrix addition and scalar multiplication. **(H.W)**

(2) The M_{mn}, the set of all m $\times$ n matrices of real numbers forms a vector space over $\boldsymbol{R}$ with the matrix addition and scalar multiplication. **(H.W)**

Ex: Check whether the set R^+ with the binary operations $x + y = xy$ and $kx = x^k$, where k is any scalar, forms a vector space over R or not.

Ex: Check whether R^+ with usual vector addition and scalar multiplication forms a vector space over R or not.

Ex: Show that P_2, the set of all polynomials of degree ≤ 2 with addition and scalar multiplication of polynomials forms a vector space over R.

Ex: Show that P_3, the set of all polynomials of degree ≤ 3 with addition and scalar multiplication of polynomials forms a vector space over R. (H.W.)

<u>Note:</u>

The set P_n of all polynomials of degree $\leq n$ with addition and scalar multiplication of polynomials forms a vector space.

Subspace:

A non-empty subset S of a vector space V is said to be a subspace if S itself a vector space under the operations defined on V.

Note:

(1) Every vector space has at least two subspaces, vector space it self V and $\{0\}$.

(2) The subspace $\{0\}$ is called the zero subspace containing the zero element or vector.

Condition to check whether a non-empty subset of a vector space is subspace or not:

Theorem:

A non-empty subset S of a vector space V is a subspace iff

(i) $u + v \in S,$ for all $u, v \in S$

(ii) $\alpha u \in S,$ for all $u \in S$ and $\alpha \in \boldsymbol{R}$

Ex: Check whether the following sets are subspace over the respective vector spaces or not:

(1) $\boldsymbol{S = \{(x, y)\ /x = 3y\},\ V = R^2}$

(2) $\boldsymbol{S = \{a_0 + a_1x + a_2x^2 + a_3x^3\ /\ a_0 = 0\},\ V = P_3}$

(3) $\boldsymbol{S = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a + b + c + d = 0\right\},\ V = M_{22}}$

(4) $\boldsymbol{S = \{(x, y) :\ y = x^2\},\ V = R^2}$ **(H.W.)**

(5) $\boldsymbol{S = \{(x, y)\ /x^2 = y^2\},\ V = R^2}$

(6) $\boldsymbol{S = \{(x, y, z) : y = x + z + 1\},\ V = R^3}$ **(H.W.)**

(7) $\boldsymbol{S = \{A_{nn}\ : AB = BA \text{ for fixed } B_{nn}\},\ V = M_{nn}}$

(8) $\boldsymbol{S = \{(x, kx) : k, x \in R\},\ V = R^2}$

Linear Combination:

A vector $v \in V$ is said to be a linear combination of vectors $v_1, v_2, \dots, v_n$ if it can be expressed as

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n,$$

where $\alpha_1, \alpha_2, \dots, \alpha_n$ are scalars.

<u>Method:</u>

(1) Express v as a linear combination of $v_1, v_2, \dots, v_n$ and form a system of linear equations.
i.e., $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$

(2) If the system of equations is consistent then v is a linear combination of $v_1, v_2, \dots, v_n$ and
if it is inconsistent then v is not a linear combination of $v_1, v_2, \dots, v_n$.

Ex: Which of the following vectors is/are linear combinations of $\boldsymbol{v_1} = (\mathbf{0}, -\mathbf{2}, \mathbf{2})$ and $\boldsymbol{v_2} = (\mathbf{1}, \mathbf{3}, -\mathbf{1})$?
(i) $(\mathbf{3}, \mathbf{1}, \mathbf{5})$ (ii) $(\mathbf{0}, \mathbf{4}, \mathbf{5})$.

Ex: Express the vector $(\mathbf{2}, -\mathbf{2}, \mathbf{3})$ as a linear combination of the set of vectors $\{(\mathbf{0}, \mathbf{1}, -\mathbf{1}), (\mathbf{2}, \mathbf{0}, \mathbf{1}), (-\mathbf{3}, \mathbf{2}, \mathbf{5})\}$ of $\boldsymbol{R}^{\mathbf{3}}$.

Ex: Express the matrix $\begin{bmatrix} \mathbf{5} & \mathbf{1} \\ -\mathbf{1} & \mathbf{9} \end{bmatrix}$ as a linear combination of $\begin{bmatrix} \mathbf{1} & -\mathbf{1} \\ \mathbf{0} & \mathbf{3} \end{bmatrix}$, $\begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{2} \end{bmatrix}$ and $\begin{bmatrix} \mathbf{2} & \mathbf{2} \\ -\mathbf{1} & \mathbf{1} \end{bmatrix}$.

Span of a set:

The span of a non-empty subset *S* of a vector space *V* is the set of all linear combinations of finite number of elements of *S*.

It is denoted by *span (S)* or *L(S)*.

i.e., $span(S) = \{\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n : v_i \in S, \alpha_i \in \boldsymbol{R},$
$1 \leq i \leq n\}$

For example, the span of $(4,2)$ in R^2 can be determined as

Let $v_1 = (4,2)$.

Then for $\alpha_1 \in \boldsymbol{R}$, we have

$$\begin{aligned} v &= \alpha_1 v_1 \\ &= \alpha_1(4,2) = (4\alpha_1, 2\alpha_1) \end{aligned}$$

Hence,

$span(v_1) = \{v \in \boldsymbol{R}^2 : v = (4\alpha_1, 2\alpha_1), \alpha_1 \in \boldsymbol{R}\}.$

Ex: Determine the span of $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ in R^3.

Ex: Let $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ in M_{22}. Then determine the $span(A_1, A_2, A_3)$.

Ex: Let $p_1(x) = 1 + 3x, p_2(x) = x + x^2$. Then find the $span(p_1, p_2)$.

Ex: Determine whether $v_1 = (2, 2, 2), v_2 = (0, 0, 3), v_3 = (0, 1, 1)$ span the vectors of R^3 or not.

Linear Dependence and Independence of a set:

Linearly Dependent set:

A finite set $\{v_1, v_2, \dots, v_n\}$ of a vector space V over a field F is said to be a linearly dependent (LD) set if there exists scalars $\alpha_1, \alpha_2, \dots, \alpha_n$, not all zero (i.e., at least one of them is non zero) such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0.$$

Linearly Independent set:

A finite set $\{v_1, v_2, \dots, v_n\}$ of a vector space V over a field F is said to be a linearly independent (LI) set if there exists scalars $\alpha_1, \alpha_2, \dots, \alpha_n$ such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0 \Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0.$$

Notes:

(1) Method to check a set to be LI or LD:
If the system of equations $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$ has trivial solution then the set of vectors is LI otherwise LD (i.e., for non trivial solutions)

(2) In a vector space *V*, any set of vectors containing zero vector is LD set.

(3) An infinite set of vectors in *V(**F**)* is said to be LI if every finite subset of it is LI.

Ex: Check whether the following set of vectors are LI or LD:

(1) $\{(\mathbf{1},\mathbf{2},\mathbf{3}),(\mathbf{0},\mathbf{2},\mathbf{1}),(\mathbf{0},\mathbf{1},\mathbf{3})\}$, $\boldsymbol{V}=\boldsymbol{R}^{3}$

(2) $\{\mathbf{2}+\boldsymbol{x}+\boldsymbol{x}^{2},\boldsymbol{x}+\mathbf{2}\boldsymbol{x}^{2},\mathbf{2}+\mathbf{2}\boldsymbol{x}+\mathbf{3}\boldsymbol{x}^{2}\}$, $\boldsymbol{V}=\boldsymbol{P}_{2}$

(3) $\left\{\begin{bmatrix}\mathbf{1} & -\mathbf{1}\\ \mathbf{2} & \mathbf{3}\end{bmatrix},\begin{bmatrix}\mathbf{1} & -\mathbf{4}\\ \mathbf{0} & \mathbf{1}\end{bmatrix},\begin{bmatrix}\mathbf{1} & \mathbf{3}\\ \mathbf{4} & \mathbf{5}\end{bmatrix}\right\}$, $\boldsymbol{V}=\boldsymbol{M}_{22}$

Basis:

A subset S of vectors of a vector space V is said to be a basis for V if

(i) S is LI.

(ii) S spans V or S generates V.

Notes:

(1) Basis for a vector space is not unique.

For example, Take $S = \{1\}\ or\ \{2\}\ or\ \{u\}, where\ u \in \boldsymbol{R}$.

Then for any $\alpha_1 \in \boldsymbol{R}$**,** we have $\alpha_1 \cdot 1 = 0 \Rightarrow \alpha_1 = 0$.

Therefore, $\{1\}$ is LI.

Now, for any $\alpha_1 \in \boldsymbol{R}$ and $1 \in \boldsymbol{R}$ we have $1 = \alpha_1 \cdot 1 = \alpha_1$.

Therefore, $\{1\}$ spans $\boldsymbol{R}$.

Hence, $\{1\}$ is a basis of $\boldsymbol{R}$.

(2) Some standard Basis for various vector spaces:

(i) $\{(1,0),(0,1)\}$ is a basis for R^2.

(ii) $\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix},\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix},\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix},\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$ is a basis for M_{22}.

(iii) $\{1, x, x^2\}$ is a basis for $P_2(x)$.

Ex: Show that $\boldsymbol{S} = \{(\mathbf{1},\mathbf{0},\mathbf{0}),(\mathbf{2},\mathbf{2},\mathbf{0}),(\mathbf{3},\mathbf{3},\mathbf{3})\}$ is a basis for $\boldsymbol{R}^3$.

Ex: Determine whether the set of vectors $\{\mathbf{1}-\mathbf{3}\boldsymbol{x}+\mathbf{2}\boldsymbol{x}^2, \mathbf{1}+\boldsymbol{x}+\mathbf{4}\boldsymbol{x}^2, \mathbf{1}-\mathbf{7}\boldsymbol{x}\}$ forms a basis for $\boldsymbol{P}_2$ or not.

Ex: Determine whether the set of vectors $\left\{\begin{bmatrix}\mathbf{1} & \mathbf{2}\\\mathbf{1} & -\mathbf{2}\end{bmatrix},\begin{bmatrix}\mathbf{0} & -\mathbf{1}\\-\mathbf{1} & \mathbf{0}\end{bmatrix},\begin{bmatrix}\mathbf{0} & \mathbf{2}\\\mathbf{3} & \mathbf{1}\end{bmatrix},\begin{bmatrix}\mathbf{0} & \mathbf{0}\\-\mathbf{1} & \mathbf{2}\end{bmatrix}\right\}$ forms a basis for $\boldsymbol{M}_{22}$ or not.

Dimension:

The number of vectors in a basis of a nonzero vector space V is called the dimension of V and it is denoted by $dim(V)$.

Note:

Dimensions of some standard vector spaces can be obtained directly from their basis.

$(\boldsymbol{i})$ $dim(R^n) = n$.

$(\boldsymbol{ii})$ $dim(M_{mn}) = m \cdot n$.

$(\boldsymbol{iii})$ $dim(P_n) = n + 1$.

$(\boldsymbol{iv})$ $dim(\{0\}) = 0$ as $\{0\}$ is LD and hence vector space $\{0\}$ has no basis.

Linear Transformations:

Let V and W be two vector spaces. Then a linear transformation is a function $T: V \to W$ such that

(i) $T(u+v) = T(u) + T(v)$ for all $u, v \in V$

(ii) $T(\alpha u) = \alpha\, T(u)$ for all $\alpha \in \boldsymbol{R}$

Note:

(1) Above conditions can be also written as for all $u, v \in V$ and $\alpha, \beta \in \boldsymbol{R}$,

$$T(\alpha u + \beta v) = \alpha\, T(u) + \beta\, T(v).$$

(2) If $V = W$ then the linear transformation $T: V \to V$ is called a linear operator.

Ex: Determine whether the following functions are linear transformations or not:

(1) $T: R^2 \to R^2, T(x, y) = (x + 2y, 3x - y)$

(2) $T: R^3 \to R^2, T(x, y, z) = (2x - y + z, y - 4z)$

(3) $T: P_2 \to P_2, T\left(a_0 + a_1 x + a_2 x^2\right) = (a_0 + 1) + (a_1 + 1)x + (a_2 + 1)x^2.$

(4) $T: M_{nn} \to R,$ **where** $T(A) = det(A)$

Matrix Representation of a Linear Transformation:

Let $T: \boldsymbol{R^n} \to \boldsymbol{R^m}$ be a linear transformation. Then there exists a matrix A of order $m \times n$ such that $T(X) = AX$.

For example,

Let $T: \boldsymbol{R^3} \to \boldsymbol{R^3}$ be a linear transformation, where

$$T(x, y, z) = (x - y + 2z, 2x + y - z, -x - 2y).$$

Then its matrix representation is,

$$T(x, y, z) = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

i.e., $T(X) = AX$,

$$\text{Where } A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \\ -1 & -2 & 0 \end{bmatrix}$$

Range and Kernel (Null space) of a Linear Transformation:

Let V and W be two vector spaces and $T: V \to W$ be a linear transformation. Then the **range of *T*,** denoted by $R(T)$, is the set of all vectors in W which are images of at least one vector in V under T.
i.e., $R(T) = \{v \in V: T(v) = w,\ w \in W\}$

The **kernel or null space of *T***, denoted by $N(T)$, is the set of all vectors in V that maps into the zero vector.
i.e., $N(T) = \{v \in V: T(v) = 0\}$

The dimension of range of T is called rank of T and the dimension of kernel of T is called nullity of T.

Theorem:

If $T_A: R^n \to R^m$ is a transformation multiplication by $[A]_{m\times n}$, then the kernel of T_A is the null-space of the matrix A and range of T_A is the column space of A.

(**Column space:** The subspace of R^m spanned by the column vectors of A is called a column space)

i.e., Basis for $ker(T)$=Basis for the null space of A

Basis for $R(T)$=Basis for the column space of A

Rank-Nullity theorem (Dimension theorem):

If $T: V \to W$ is a linear transformation from a finite dimensional vector space V to a vectorspace W, then

$$rank(T) + nullity(T) = \dim(V)$$

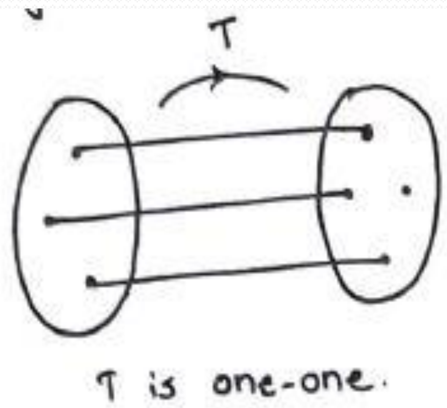
i.e., $r(T) + n(T) = \dim(Domain\ of\ a\ linear\ transformation)$

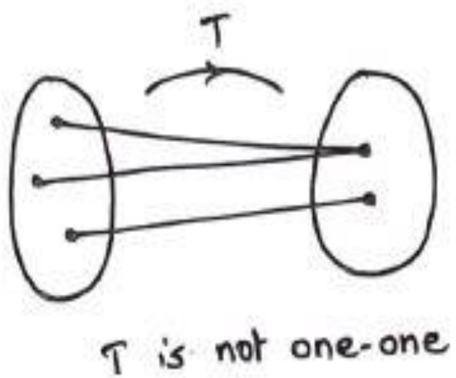
Ex: Let $T: R^2 \to R^2$ be the linear operator defined by $T(x, y) = (2x - y, -8x + 4y)$. Then find basis for $ker(T)$ and $R(T)$.

Ex: Verify the dimension theorem or rank-nullity theorem for a linear transformation $T: R^3 \to R^3$ defined by $T(x, y, z) = (x + 2y + 5z, 3x + 5y + 13z, -2x - y - 4z)$.

One to one transformation:

Let V and W be two vector spaces. A linear transformation $T: V \to W$ is **one-one (injective)** if T maps distinct vectors in V to distinct vectors in W.
i.e., $x = y \implies T(x) = T(y)$, for all $x, y \in V$.





Theorems:

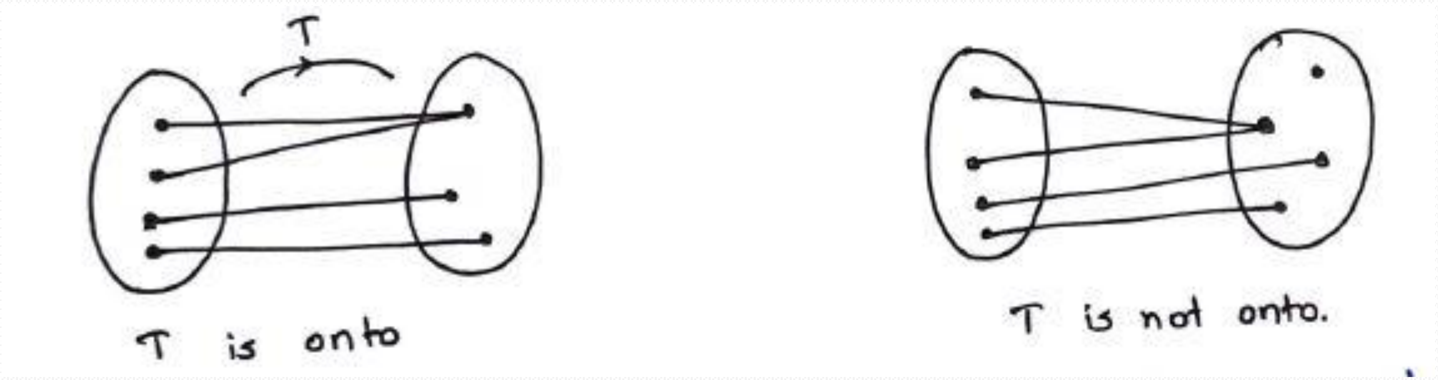
(1) A linear transformation $T: V \to W$ is **one-one** iff $ker(T) = \{0\}$.

(2) A linear transformation $T: V \to W$ is **one-one** iff $dim(ker(T)) = 0$,
i.e, $N(T) = 0$.

(3) A linear transformation $T: V \to W$ is **one-one** iff $rank(T) = \dim(V)$.

Onto transformation:

Let V and W be two vector spaces. A linear transformation $T: V \to W$ is **onto (surjective)** if the range of T is W.
i.e., for all $w \in W$ there is $a\ v \in V$ such that $T(v) = w$.



Theorem:

A linear transformation $T: V \to W$ is **onto** iff $rank(T) = \dim(W)$.

Bijective transformation:

A linear transformation $T: V \to W$ is both one-one and onto then it is called **bijective transformation.**

Isomorphism:

A bijective transformation from V to W is known as an isomorphism between V and W.

Theorems:

(1) Let V be a finite dimensional real vector space. If $\dim(V) = n$, then there is an isomorphism from V to R^n.

(2) Let V and W be finite dimensional vector spaces. If $dim(V) = \dim(W)$, then V and W are isomorphic.

Examples:

Ex: Determine that which of the following linear transformations are one-one, onto and bijective:

(1) $T: R^2 \rightarrow R^2, T(x, y) = (x + y, x - y)$

(2) $T: R^2 \rightarrow R^3, T(x, y) = (x - y, y - x, 2x - 2y)$

(3) $T: R^3 \rightarrow R^2, T(x, y, z) = (x + y + z, x - y - z)$

(4) $T: R^3 \rightarrow R^3, T(x, y, z) = (x + 3y, y, z + 2x)$