Recurrence Relations

Recurrence Relations

A **recurrence relation** for the sequence $\{a_n\}$ is an equation that expresses a_n is terms of one or more of the previous terms of the sequence, namely, a_0, $a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

A sequence is called a **solution** of a recurrence relation if its terms satisfy the recurrence relation.

Consider a sequence 1, 1, 2, 3, 5, 8, 13, 21, ...

This sequence of numbers is called the **Fibonacci sequence**.

Here

$$a_n = a_{n-1} + a_{n-2}, \qquad n \geq 2$$

It can be also written as

$$a_n = a_{n+2} - a_{n+1}, \qquad n \geq 0$$

Consider a general expression

$$a_n = 3^n, n \geq 0.$$

Suppose that, we have the relation

$$a_n = 3a_{n-1} \; with \; a_0 = 1$$

Then for $n = 1,2,3, \ldots,$ we get

$$a_1 = 3a_0 = 3$$

$$a_2 = 3a_1 = 9$$

$$a_3 = 3a_2 = 27$$

$$a_4 = 3a_3 = 81$$

Hence, 3^n is a solution of the recurrence relation

$a_n = 3a_{n-1}$.

Example:

Consider the recurrence relation

$a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$

Is the sequence $\{a_n\}$ with $a_n=3n$ a solution of this recurrence relation?

Solution:

For $n \geq 2$ we see that

$2a_{n-1} - a_{n-2} = 2(3(n - 1)) - 3(n - 2) = 3n = a_n$.

Therefore, $\{a_n\}$ with $a_n=3n$ is a solution of the recurrence relation.

Example:

Is the sequence $\{a_n\}$ with $a_n=5$ a solution of the same recurrence relation?

Solution:

For $n \geq 2$ we see that

$2a_{n-1} - a_{n-2} = 2 \cdot 5 - 5 = 5 = a_n$.

Therefore, $\{a_n\}$ with $a_n=5$ is also a solution of the recurrence relation.

Remarks:

In other words, a recurrence relation is like a recursively defined sequence, but **without specifying any initial values (initial conditions)**.

Therefore, the same recurrence relation can have (and usually has) **multiple solutions**.

If **both** the initial conditions and the recurrence relation are specified, then the sequence is **uniquely** determined.

Solution of Recurrence Relations

In general, we would prefer to have an **explicit formula** to compute the value of a_n rather than conducting n iterations.

For one class of recurrence relations, we can obtain such formulas in a systematic way.

Those are the recurrence relations that express the terms of a sequence as **linear combinations** of previous terms.

Linear recurrences

A recurrence relation of the form

$$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = F(n),$$

Where $c_0, c_1, \ldots, c_k$ are constants is called a ***linear recurrence relation with constant coefficients***.

It is of k^{th} order (or of degree k), provided that both c_0 and c_k are non-zero.

For example,

$$2a_n + 2a_{n-1} = 2^n \text{ (First order)}$$

$$3a_n - 5a_{n-1} + 2a_{n-2} = n^2 + 5 \text{ (Second order)}$$

There are two types of linear recurrence relations with constant coefficients:

1. Linear homogeneous recurrences
2. Linear non-homogeneous recurrences

There are two methods of solving Recurrence Relation

1. Characteristic roots and
2. Generating Functions

Definition: A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form:

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$,

Where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

A sequence satisfying such a recurrence relation is uniquely determined by the recurrence relation and the k initial conditions

$a_0 = C_0, a_1 = C_1, a_2 = C_2, \ldots, a_{k-1} = C_{k-1}$.

Examples of Linear Homogeneous Recurrence Relations

The recurrence relation $P_n = (1.05)P_{n-1}$
is a linear homogeneous recurrence relation of **degree one**.

The recurrence relation $f_n = f_{n-1} + f_{n-2}$
is a linear homogeneous recurrence relation of **degree two**.

The recurrence relation $a_n = a_{n-5}$
is a linear homogeneous recurrence relation of **degree five**.

Solution of a linear homogeneous recurrence relation using the method of Characteristics roots

we try to find solutions of the form **$a_n = r^n$**, where r is a constant.

$a_n = r^n$ is a solution of the recurrence relation

$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$

if and only if

$r^n = c_1r^{n-1} + c_2r^{n-2} + \ldots + c_kr^{n-k}$.

Divide this equation by r^{n-k} and subtract the right-hand side from the left:

$r^k - c_1r^{k-1} - c_2r^{k-2} - \ldots - c_{k-1}r - c_k = 0$

This is called the **characteristic equation** of degree k of the recurrence relation.

The solutions of this equation are called the **characteristic roots** of the recurrence relation.

There are three types of characteristics roots:

(1) Distinct root

(2) Multiple roots

(3) Mixed roots

Distinct Roots

Let us consider linear homogeneous recurrence relations of **degree two**.

Theorem: Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2.

Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

Example: What is the solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$?

Solution:

Comparing given recurrence relation with

$a_n = c_1a_{n-1} + c_2a_{n-2}$, we get $c_1 = 1$ and $c_2 = 2$.

The characteristic equation of the recurrence relation is

$r^2 – r – 2 = 0$.

Its roots are $r = 2$ and $r = -1$.

Hence, the solution is given by

$a_n = \alpha_1 2^n + \alpha_2(-1)^n$ for some constants α_1 and α_2.

Given the equation $a_n = \alpha_1 2^n + \alpha_2(-1)^n$ and the initial conditions $a_0 = 2$ and $a_1 = 7$, it follows that

$a_0 = 2 = \alpha_1 + \alpha_2$

$a_1 = 7 = \alpha_1 \cdot 2 + \alpha_2 \cdot (-1)$

Solving these two equations gives
$\alpha_1 = 3$ and $\alpha_2 = -1$.

Therefore, the solution to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with
$a_n = 3 \cdot 2^n - (-1)^n$.

Example: **Find an explicit formula for the Fibonacci numbers.**

Solution: The Fibonacci numbers satisfy the recurrence relation $a_n = a_{n-1} + a_{n-2}$, n ≥ 2 with the initial conditions $a_0 = 0$ and $a_1 = 1$.

Comparing it with $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, we get $c_1 = 1$ and $c_2 = 1$.

So, the characteristic equation is $r^2 - r - 1 = 0$.

Its roots are

$$r = \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}.$$

Therefore, the Fibonacci numbers are given by

$$a_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n,$$

for some constants α_1 and α_2.

We can determine values for these constants so that the sequence meets the conditions $a_0 = 0$ and $a_1 = 1$.

$$a_0 = \alpha_1 + \alpha_2 = 0$$

and

$$a_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right) = 1$$

On solving these equations, we get

$$\alpha_1 = \frac{1}{\sqrt{5}}, \quad \alpha_2 = -\frac{1}{\sqrt{5}}$$

Therefore, the solution of the given recurrence relation is

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Multiple Roots

Theorem:

Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0 which is repeated two times.

A sequence $\{a_n\}$ is a solution of the recurrence relation

$a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if

$a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$, for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

Example: What is the solution of the recurrence relation

$a_n = 6a_{n-1} – 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$?

Solution: The only root of the characteristic equation

$r^2 – 6r + 9 = 0$ is $r_0 = 3$.

Hence, the solution to the recurrence relation is

$a_n = \alpha_1 3^n + \alpha_2 n3^n$ for some constants α_1 and α_2.

To match the initial condition, we need

$a_0 = 1 = \alpha_1$

$a_1 = 6 = \alpha_1\cdot 3 + \alpha_2\cdot 3$

Solving these equations yields $\alpha_1 = 1$ and $\alpha_2 = 1$.

Consequently, the overall solution is given by

$a_n = 3^n + n3^n$.

Example: **Find the solution to the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with initial conditions $a_0 = 1$, $a_1 = -2$ and $a_2 = -1$.**

Solution: The characteristic equation is

$r^3 + 3r^2 + 3r + 1 = 0$, which has a single root $r_0 = -1$ of multiplicity three.

$\therefore a_n = (\alpha_1+\alpha_2 n+\alpha_3 n^2)\, r_0^n = (\alpha_1+\alpha_2 n+\alpha_3 n^2)(-1)^n$

initial conditions are given $a_0 = 1$, $a_1 = -2$ and $a_2 = -1$.

$a_0 = \alpha_1 = 1$

$a_1 = (\alpha_1+\alpha_2+\alpha_3) \cdot (-1) = -2$

$a_2 = \alpha_1+2\alpha_2+4\alpha_3 = -1$

$\therefore \alpha_1 = 1, \alpha_2 = 3, \alpha_3 = -2 \Rightarrow \quad a_n = (1+3n-2n^2) \cdot (-1)^n$

Mixed Roots

Example: Suppose that the roots of the characteristic equation of a linear homogeneous recurrence relation are 2, 2, 2, 5, 5, and 9 . What is the form of the general solution?

Solution:

For some constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$ we have

$$a_n = (\alpha_1(2)^n + \alpha_2 n(2)^n + \alpha_3 n^2(2)^n) + (\alpha_4(5)^n + \alpha_5 n(5)^n) + \alpha_6(9)^n$$

Practice Examples

(1) $a_n + 5a_{n-1} + 6a_{n-2} = 0, a_0 = 1\ and\ a_1 = 2$

(2) $a_n - 7a_{n-1} + 10a_{n-2} = 0, a_0 = 0\ and\ a_1 = 3$

(3) $a_n - 13a_{n-1} + 36a_{n-2} = 0, a_0 = 2\ and\ a_1 = 1$

(4) $a_r - 4a_{r-1} + 4a_{r-2} = 0, a_0 = 1\ and\ a_1 = 6$

(5) $a_r - 10a_{r-1} + 25a_{r-2} = 0, a_0 = 2\ and\ a_1 = 3$

(6) $a_n - 8a_{n-1} + 21a_{n-2} - 18a_{n-3} = 0.$

Linear Non-homogeneous Recurrence Relations with Constant Coefficients

The General form of **Linear non-homogeneous recurrence relation with constant coefficients is**

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$$

-------- (1)

For Example, $a_n = 3a_{n-1} + 2n.$

Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

The following are linear nonhomogeneous recurrence relations with constant coefficients:

$a_n = a_{n-1} + 2^n$,

$a_n = a_{n-1} + a_{n-2} + n^2 + n + 1,$

$a_n = 3a_{n-1} + n3^n$,

$a_n = a_{n-1} + a_{n-2} + a_{n-3} + n!$

where as the following are the associated linear homogeneous recurrence relations, respectively:

$a_n = a_{n-1}$,

$a_n = a_{n-1} + a_{n-2},$

$a_n = 3a_{n-1}$,

$a_n = a_{n-1} + a_{n-2} + a_{n-3}$

Every solution of a linear non-homogeneous recurrence relation is the sum of

- a solution to the associated linear homogeneous recurrence relation and
- a particular relation

Solution of Linear non-homogeneous recurrence relation

Theorem: If $\left\{a_n^{(p)}\right\}$ is a particular solution of the non-homogeneous linear recurrence relation with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

then every solution is of the form $\left\{a_n^{(p)} + a_n^{(h)}\right\}$,

where $\left\{a_n^{(h)}\right\}$ is a solution of the associated homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}.$$

Note:

There is no general method for solving such relations.

However, we can solve them for special cases.

In particular, $\mathrm{F}(n)$ is

- a polynomial function
- exponential function or
- the product of a polynomial and exponential functions

There are total four cases to obtain $\left\{a_n^{(p)}\right\}$:

Case-I:

Suppose $F(n)$ is a polynomial of degree 'q' and 1 is not a root of the characteristic equation of homogeneous part of equation (1).

Then $\left\{a_n^{(p)}\right\}$ is of the form

$$a_n^{(p)} = A_0 + A_1 n + A_2 n^2 + \cdots + A_q n^q,$$

Where $A_0, A_1, \dots, A_q$ are constants, those can be evaluated by using $a_n = a_n^{(p)}$ satisfies equation (1).

<u>Case-II:</u>

Suppose $F(n)$ is a polynomial of degree 'q' and 1 is a root of multiplicity 'm' of the characteristic equation of homogeneous part of equation (1).

Then $\left\{a_n^{(p)}\right\}$ is of the form

$$a_n^{(p)} = n^m\left(A_0 + A_1 n + A_2 n^2 + \cdots + A_q n^q\right),$$

Where $A_0, A_1, \ldots, A_q$ are constants, those can be evaluated by using $a_n = a_n^{(p)}$ satisfies equation (1).

Case-III:

Suppose $F(n) = \alpha b^n$, where α is any constant and b is not a root of the characteristic equation of homogeneous part of equation (1). Then $\left\{a_n^{(p)}\right\}$ is of the form

$$a_n^{(p)} = A_0 b^n$$

Case-IV:

Suppose $F(n) = \alpha b^n$, where α is any constant and b is a root of multiplicity 'm' of the characteristic equation of homogeneous part of equation (1). Then $\left\{a_n^{(p)}\right\}$ is of the form

$$a_n^{(p)} = A_0\, n^m\, b^n$$

Note:

$F(n)$	$a_n^{(p)}$
Any constant	A_0
n	$A_0 + A_1 n$
n^2	$A_0 + A_1 n + A_2 n^2$
r^n	$A_0 r^n$

Example. Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1=3$?

Solution:

We are given $a_n = 3a_{n-1} + 2n$ ----------(1),

which is non-homogeneous recurrence relation.

To find $a_n^{(h)}$, we consider the homogeneous part of (1).

$$a_n = 3a_{n-1}$$

Therefore, the characteristic equation is

$$r - 3 = 0 \Rightarrow r = 3 \Rightarrow a_n^{(h)} = \alpha \cdot 3^n.$$

Here, $F(n) = 2n$.

Then $a_n^{(p)} = A_0 + A_1 n$.

Now, from (1), we have

$$A_0 + A_1 n = 3\big(A_0 + A_1(n-1)\big) + 2n$$

$$\therefore A_0 + A_1 n = 3A_0 + 3A_1 n - 3A_1 + 2n$$

$$\therefore A_0 + A_1 n - 3A_0 + 3A_1 - 3A_1 n - 2n = 0$$

$$-2A_0 + 3A_1 - 2A_1 n - 2n = 0$$

Now, by comparing coefficients of *1* and *n* on both the sides, we get

$$\therefore -2A_0 + 3A_1 = 0\,; -2A_1 - 2 = 0$$

Therefore, $A_1 = -1, A_0 = -\frac{3}{2}$.

So, $a_n^{(p)} = -\frac{3}{2} - n$.

Hence, $a_n = a_n^{(h)} + a_n^{(p)} = \alpha \cdot 3^n - \frac{3}{2} - n$.

If $a_1 = 3$ then $\alpha = \frac{11}{6}$ and hence the required solution is

$$a_n = \frac{11}{6} 3^n - \frac{3}{2} - n.$$

Example. Solve the recurrence relation

$$\boldsymbol{a_n = 5a_{n-1} - 6a_{n-2} + 7^n} \qquad \text{-----------}\mathit{(1)}$$

Solution:

The homogeneous part of given recurrence relation is $a_n = 5a_{n-1} - 6a_{n-2}$

The characteristic equation is $r^2 - 5r + 6 = 0$

$\Rightarrow r_1 = 3, r_2 = 2$

$\Rightarrow a_n^{(h)} = a_1 3^n + a_2 2^n.$

Here $F(n) = 7^n$

Then $a_n^{(p)} = A_0 7^n$

Now, from (1), we have

$$A_0 7^n = 5 A_0 7^{n-1} - 6 A_0 7^{n-2} + 7^n$$

$$\therefore A_0 7^2 = 5 A_0 7^1 - 6 A_0 + 7^2$$

$$\therefore 49 A_0 = 35 A_0 - 6 A_0 + 49$$

Therefore,

$$A_0 = \frac{49}{20}.$$

So,

$$a_n^{(p)} = \frac{49}{20} 7^n.$$

Hence,

$$a_n = a_n^{(h)} + a_n^{(p)}$$
$$= \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n + \frac{49}{20} 7^n,$$

Where α_1 and α_2 are real constants.

Example. What form does a particular solution of the linear non-homogeneous recurrence relation

$a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ have when

$F(n) = 3^n, F(n) = n3^n, F(n) = n^2 2^n$ and

$F(n) = (n^2 + 1)3^n$.

Solution :

The associated linear homogeneous recurrence relation is

$a_n = 6a_{n-1} - 9a_{n-2}$.

The characteristic equation is $r^2 - 6r + 9 = 0 \Rightarrow r = 3$, which is multiple root.

If $F(n) = 3^n$ and 3 is a root, then the particular solution is

$$a_n^{(p)} = A_0 n^2 3^n$$

for some constants A_0.

If $F(n) = n3^n$ and 3 is a root, then the particular solution is

$$a_n^{(p)} = A_0 n^2 3^n$$

for some constants A_0.

If $F(n) = n^2 2^n$ and 2 is not a root, then the particular solution is

$$a_n^{(p)} = (A_0 + A_1 n + A_2 n^2) 2^n$$

for some constants A_0, A_1, A_2.

If $F(n) = (n^2 + 1)3^n$ and 3 is a root, then the particular solution is

$$a_n^{(p)} = n^2 (A_0 + A_1 n + A_2 n^2) 3^n$$

for some constants A_0, A_1, A_2.

Practice Examples

(1) $a_n - 7a_{n-1} + 10a_{n-2} = 3^n, a_0 = 0 \; and \; a_1 = 1$

(2) $a_n + 6a_{n-1} - 9a_{n-2} = 3, a_0 = 0 \; and \; a_1 = 1$

(3) $a_r + 5a_{r-1} + 6a_{r-2} = 3r^2 - 2r + 1$

(4) $a_n = a_{n-1} + 3, a_0 = 1$

Generating Functions

Definition. The generating function for the sequence $\{a_k\}$, i.e., terms $a_0, a_1, a_2, \ldots$, of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + \ldots + a_n x^n + \ldots = \sum_{k=0}^{\infty} a_k x^k$$

Note:

(1) $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}, |r| < 1.$

(2) $\sum_{k=0}^{\infty} (-1)^k r^k = \frac{1}{1+r}, |r| < 1.$

Example. Find the generating functions for the sequences $\{a_k\}$ with

(1) $a_k = 3$

(2) $a_k = k+1$

(3) $a_k = 2^k$

Solution:

$$(1)\ G(x) = \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} 3x^k$$

$$(2)\ G(x) = \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} (k+1)x^k$$

$$(3)\ G(x) = \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} 2^k x^k$$

Example. Solve the recurrence relation $a_n = 3a_{n-1}$ for $n=1,2,3,\ldots$ and the initial condition $a_0 = 2$ using generating function.

Solution:

We are given that $a_n = 3\, a_{n-1}$.

Then the characteristic equation is

$$r - 3 = 0 \Rightarrow r = 3 \Rightarrow a_n = \alpha \cdot 3^n$$

$$\because a_0 = 2 = \alpha$$

$$\therefore a_n = 2 \cdot 3^n$$

Another Method to solve recurrence relation is using Generating Function.

Let $G(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{k=0}^{\infty} a_k x^k$

be the generating function for $\{a_k\}$.

Here given that $a_k = 3a_{k-1}$ for $k = 1,2,3, \ldots$

Therefore,

$$\sum_{k=1}^{\infty} a_k x^k = 3 \sum_{k=1}^{\infty} a_{k-1} x^k = 3x \sum_{k=1}^{\infty} a_{k-1} x^{k-1} = 3x \sum_{k=0}^{\infty} a_k x^k.$$

$$\therefore \sum_{k=0}^{\infty} a_k x^k - a_0 = 3x \sum_{k=0}^{\infty} a_k x^k.$$

$$\therefore G(x) - a_0 = 3x\, G(x).$$

Here $a_0 = 2$.

Then

$$G(x) - 2 = 3x\,G(x).$$

$$\therefore G(x) - 3x\,G(x) = 2.$$

$$\therefore G(x) = \frac{2}{1-3x} = 2\sum_{k=0}^{\infty}(3x)^k = 2\sum_{k=0}^{\infty} 3^k\,x^k\,.$$

Hence,

$$a_n = 2 \cdot 3^n.$$

Example. **Solve $a_n = 8a_{n-1} + 10^{n-1}$ for $n=1,2,3,\ldots$ and initial condition $a_0 = 1$.**

Solution:

Let $G(x) = a_0 + a_1 x + a_2 x^2 + \ldots = \sum_{k=0}^{\infty} a_k x^k$

be the generating function for $\{a_k\}$.

$$G(x) - 1 = \sum_{k=1}^{\infty} a_k x^k = \sum_{k=1}^{\infty} (8a_{k-1} + 10^{k-1}) x^k$$

$$= 8\sum_{k=1}^{\infty} a_{k-1} x^k + \sum_{k=1}^{\infty} 10^{k-1} x^k = 8x\sum_{k=0}^{\infty} a_k x^k + x\sum_{k=0}^{\infty} 10^k x^k$$

$$= 8xG(x) + \frac{x}{1-10x}$$

$$(1-8x)G(x)=1+\frac{x}{1-10x}=\frac{1-9x}{1-10x}$$

$$G(x)=\frac{1-9x}{(1-10x)(1-8x)}=\frac{1}{2}(\frac{1}{1-10x}+\frac{1}{1-8x})$$

$$=\frac{1}{2}(\sum_{k=0}^{\infty}10^k x^k+\sum_{k=0}^{\infty}8^k x^k)=\sum_{k=0}^{\infty}\frac{1}{2}(10^k+8^k)x^k$$

Hence,

$$a_n=\frac{1}{2}(10^n+8^n).$$

Practice Examples

(1) $a_n = 3a_{n-1} + 2, a_0 = 1$

(2) $a_{n+1} - a_n = 3^n$, for all $n \geq 0$ with $a_0 = 1$

(3) $a_n - 3a_{n-1} = n$, for all $n \geq 1$ with $a_0 = 1$