Unit III

3.	**Partial Differential Equations and Applications:**
3.1	Formation of Partial Differential Equation
3.2	Lagrange's Linear Differential Equations
3.3	Special types of Nonlinear First Order Partial Differential Equation

Definition:

An equation containing one or more partial derivatives of an unknown function of two or more independent variables is known as a partial differential equation (PDE).

For examples:

1. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (Laplace Equation)
2. $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ (Wave Equation)
3. $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ (Heat Equation)
4. $\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial^3 u}{\partial y^3} - 2x\left(\frac{\partial u}{\partial x}\right) = 0$

Notations:

When we consider the case of two independent variables usually assume them to be x and y and assume z to be the dependent variable. We adopt the following notations

$$p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y}, \text{ and } t = \frac{\partial^2 z}{\partial y^2}.$$

When we consider the case of three independent variables usually assume them to be x, y and z and assume u to be the dependent variable. We adopt the following notations

$$u_x = \frac{\partial u}{\partial x}, u_y = \frac{\partial u}{\partial y}, u_z = \frac{\partial u}{\partial z}, u_{xx} = \frac{\partial^2 u}{\partial x^2}, u_{xy} = \frac{\partial^2 u}{\partial x\, \partial y}, \text{ and so on.}$$

Order & Degree of a PDE:

The order of a partial differential equation is the order of the highest partial derivative occurring in the equation.

The degree of a partial differential equation is the degree of the highest order derivative which occurs in it after the equation has been rationalized, that is, made free from radicals and fractions.

For examples:

1. $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \frac{z}{y}\left(\frac{\partial u}{\partial y}\right)$ is first order and second degree PDE.
2. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u$ is second order and first degree PDE.
3. $\frac{\partial^2 z}{\partial x^2} = \left(\frac{\partial z}{\partial y}\right)^3$ is second order and first degree PDE.
4. $\frac{\partial^2 u}{\partial x^2} = \left(1 + \frac{\partial u}{\partial y}\right)^{\frac{1}{2}}$ is second order and second degree PDE.

Linear & Nonlinear PDE:

A partial differential equation is said to be linear if the dependent variable and its partial derivatives occur only in the first degree and not multiplied by the dependent variable.

A partial differential equation which is not linear is called nonlinear partial differential equation.

For examples:

1. $y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = xy$ is linear PDE.
2. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = u$ is nonlinear PDE.

3.1 Formation of Partial Differential Equation:

A partial differential equation can be formed in two ways:

1. By eliminating arbitrary constant from the given relation.
2. By eliminating arbitrary functions from the given relation.

Tutorial:

1 Derive a partial differential equation (by eliminating arbitrary constants) from the equation $z = (x^2 + a)(y^2 + b)$.

Solution. Given that $z = (x^2 + a)(y^2 + b) - - - (1)$

Differentiating equation (1) partially w.r.t. x and y respectively, we get

$$\frac{\partial z}{\partial x} = 2x\,(y^2 + b)$$

$$\Rightarrow\ p = 2x\,(y^2 + b)$$

$$\Rightarrow (y^2 + b) = \frac{p}{2x} - - - (2)$$

and

$$\frac{\partial z}{\partial y} = 2y\,(x^2 + a)$$

$$\Rightarrow q = 2y\,(x^2 + a)$$

$$\Rightarrow (x^2 + a) = \frac{q}{2y} --- (3)$$

Substituting these values of $(x^2 + a)$ and $(y^2 + b)$ from (3) and (2) in (1) , we get

$$z = \frac{q}{2y} \cdot \frac{p}{2x}$$

$$\Rightarrow 4xyz = pq.$$

2 Derive a partial differential equation (by eliminating arbitrary constants) from the equation $az + b = a^2x + y$.

Ans. $pq = 1$

3 Derive a partial differential equation (by eliminating arbitrary constants) from the equation $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$.

Ans. $2z = px + qy$

4 Eliminating the arbitrary function $'g'$ from the relation

$z = y^2 + 2g\left(\frac{1}{x} + \log y\right)$.

Solution. Given that $z = y^2 + 2g\left(\frac{1}{x} + \log y\right) --- (1)$

Differentiating equation (1) partially w.r.t. x and y respectively, we get

$$\frac{\partial z}{\partial x} = 2g'\left(\frac{1}{x} + \log y\right)\left(-\frac{1}{x^2}\right)$$

$$\Rightarrow p = 2\left(-\frac{1}{x^2}\right) g'\left(\frac{1}{x} + \log y\right)$$

$$\Rightarrow -\frac{x^2p}{2} = g'\left(\frac{1}{x} + \log y\right) --- (2)$$

and

$$\frac{\partial z}{\partial y} = 2y + 2g'\left(\frac{1}{x} + \log y\right)\left(\frac{1}{y}\right)$$

$$\Rightarrow q = 2y + 2g'\left(\frac{1}{x} + \log y\right)\left(\frac{1}{y}\right)$$

$$\Rightarrow \frac{y}{2}(q - 2y) = g'\left(\frac{1}{x} + \log y\right) --- (3)$$

From (2) and (3), we get

$$-\frac{x^2p}{2} = \frac{y}{2}(q - 2y)$$

$$\Rightarrow x^2p = -qy + 2y^2$$

$$\Rightarrow x^2p + yq = 2y^2.$$

5 Eliminating the arbitrary functions $'f'$ and $'g'$ from the relation

$z = f(x + at) + g(x - at)$.

Ans. $\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}$

6 From the partial differential equation (by eliminating arbitrary function) from

$f(x^2 + y^2, z - xy) = 0$.

Solution. Given that $f(x^2 + y^2, z - xy) = 0$

Let $u = x^2 + y^2, v = z - xy$. Then $f(u, v\ ) = 0 ----(1)$

Also,

$$\frac{\partial u}{\partial x} = 2x, \frac{\partial u}{\partial y} = 2y, \frac{\partial u}{\partial z} = 0, \frac{\partial v}{\partial x} = -y, \frac{\partial v}{\partial y} = -x, \frac{\partial v}{\partial z} = 1\ .$$

We know that partial differential equation corresponding equation (1) is

$$Pp + Qq = R --- (2)$$

where,

$$P = \frac{\partial(u, v)}{\partial(y,\ z)} = \begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{vmatrix} = \begin{vmatrix} 2y & 0 \\ -x & 1 \end{vmatrix} = 2y,$$

$$Q = \frac{\partial(u, v)}{\partial(z,\ x)} = \begin{vmatrix} \frac{\partial u}{\partial z} & \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial z} & \frac{\partial v}{\partial x} \end{vmatrix} = \begin{vmatrix} 0 & 2x \\ 1 & -y \end{vmatrix} = -2x,$$

and

$$R = \frac{\partial(u, v)}{\partial(x,\ y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ -y & -x \end{vmatrix} = -2x^2 + 2y^2 = 2y^2 - 2x^2.$$

Substituting these values of P, Q and R in (2) , we get

$$2yp - 2xq = 2(y^2 - x^2)$$

$$\Rightarrow yp - xq = y^2 - x^2$$

$$\Rightarrow xp - yq = x^2 - y^2.$$

7 From the partial differential equation (by eliminating arbitrary function)

from $f(x^2 + y^2 + z^2, xyz) = 0$.

Ans. $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$

3.2 Lagrange's Linear Differential Equations or First Order Linear Partial Differential Equation

A differential equation involving first order partial derivatives p and q only is called a PDE of the first order.

A partial differential equation of order one is of the form $Pp + Qq = R$, where $P,\ Q$, and R are functions of x, y, z. Such type of partial differential equation is known as first order linear partial differential equation or Lagrange's equation.

Working rule to solve Lagrange's Equation

1. Consider a linear partial differential equation in the standard form
$$Pp + Qq = R. - - -(1)$$
2. Then the auxiliary equations of (1) are
$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. - - -(2)$$
3. The auxiliary equation can be solved by method of grouping (described below) or method of multipliers (described below) or both to get two independent solutions of auxiliary equation (2), denoted by $u = c_1,\ v = c_2$ where c_1 and c_2 are arbitrary constants.
4. $F(u, v) = 0$ or $u = \phi(v)$ or $v = \psi(u)$ is the general solution of the equation (1).

Method of grouping

Consider the auxiliary equation is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$. Taking $\frac{dx}{P} = \frac{dy}{Q}$ from which z is absent. Integrating it we get first solution $u(x, y) = c_1$. Similarly, taking $\frac{dx}{P} = \frac{dz}{R}$ or $\frac{dy}{Q} = \frac{dz}{R}$ from which y or x is absent or can be removed with the help of $u = c_1$. Integrating it we get second solution $v(x, z) = c_2$ or $v(y, z) = c_2$. Hence the required solution is $F(u, v) = 0$.

Method of multipliers

Consider the auxiliary equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$. Let l, m, n may be constants or functions of x, y, z. Then we have $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$. Chosen l, m, n are in such way that $lP + mQ + nR = 0$. Thus $l\,dx + m\,dy + n\,dz = 0$. Integrating it we get the solution in the form $u = c_1$. Similarly, choosing another set of multipliers l', m', n' in such way that $l'P + m'Q + n'R = 0$. Thus $l'\,dx + m'\,dy + n'\,dz = 0$. Integrating it we get the solution in the form $v = c_2$. Hence the required solution is $F(u, v) = 0$.

Tutorial:

1 Find the general solution of the partial differential equation $xp + yq = z$.

Solution. Given that $xp + yq = z - - - (1)$

Comparing (1) with $Pp + Qq = R$, we have

$$P = x, Q = y, R = z$$

The auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \Rightarrow \frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

Taking first two fractions, we have

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrating, we get

$$\log x = \log y + \log c_1$$

$$\Rightarrow \log\left(\frac{x}{y}\right) = \log c_1$$

$$\Rightarrow \frac{x}{y} = c_1 = u$$

Taking last two fractions, we have

$$\frac{dy}{y} = \frac{dz}{z}$$

Integrating, we get

$$\log y = \log z + \log c_2$$

$$\Rightarrow \log\left(\frac{y}{z}\right) = \log c_2$$

$$\Rightarrow \frac{y}{z} = c_2 = v$$

The general solution of given equation is

$$f(u, v) = 0 \Rightarrow f\left(\frac{x}{y}, \frac{y}{z}\right) = 0\,.$$

2 Solve $yz\frac{\partial z}{\partial x} + zx\,\frac{\partial z}{\partial y} = xy$.

Ans. $\boldsymbol{f(x^2 - y^2, x^2 - z^2) = 0}$**.**

3 Solve $(z^2 - 2yz - y^2)\,p + x(y+z)q = x(y-z)$.

Solution. Given that

$$(z^2 - 2yz - y^2)\,p + x(y+z)q = x(y-z) - - - (1)$$

Comparing (1) with $Pp + Qq = R$, we have

$$P = z^2 - 2yz - y^2, Q = x(y+z), R = x(y-z)$$

The auxiliary equations are

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}\Rightarrow\frac{dx}{z^2-2yz-y^2}=\frac{dy}{x(y+z)}=\frac{dz}{x(y-z)}$$

Taking last two fractions, we have

$$\frac{dy}{x(y+z)}=\frac{dz}{x(y-z)}$$

$$\Rightarrow\frac{dy}{y+z}=\frac{dz}{y-z}$$

$$\Rightarrow ydy-zdy=ydz+zdz$$

$$\Rightarrow ydy-zdz-(ydz+zdz)=0$$

$$\Rightarrow ydy-zdz-d(yz)=0$$

Integrating, we get

$$\frac{y^2}{2}-\frac{z^2}{2}-yz=c_1'$$

$$\Rightarrow y^2-z^2-2yz=2c_1'=c_1$$

$$\Rightarrow y^2-z^2-2yz=c_1=u$$

Now, choose the multiplies x , y , z, we have

$$\text{Each fraction}=\frac{x\,dx+y\,dy+z\,dz}{xz^2-2xyz-xy^2+xy^2+xyz+xyz-xz^2}$$

$$=\frac{x\,dx+y\,dy+z\,dz}{0}$$

$$\therefore\ x\,dx+y\,dy+z\,dz=0$$

Integrating, we get

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=c_2'$$

$$\Rightarrow x^2+y^2+z^2=2c_2'=c_2$$

$$\Rightarrow x^2+y^2+z^2=c_2=v$$

The general solution of given equation is

$$f(u,v)=0\Rightarrow f(y^2-z^2-2yz,x^2+y^2+z^2)=0\,.$$

4 Solve $x^2(y-z)p+y^2(z-x)q=z^2(x-y)$.

Ans. $f\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z},xyz\right)=0$

3.3 Special types of Nonlinear First Order Partial Differential Equation:

Case-1: Equations involving only p and q, that is, $f(p,q)=0$

Assume that $z=ax+by+c$ is a solution of $f(p,q)=0$, provided $f(a,b)=0$. Solving $f(a,b)=0$ for b or a, we have $b=\phi(a)$ or $a=\psi(b)$. Substituting this value of $b=\phi(a)$ or $a=\psi(b)$ in $z=ax+by+c$, we have the complete integral of $f(p,q)=0$ is $z=ax+\phi(a)y+c$ or $z=\psi(b)x+by+c$, where a and c are arbitrary constants.

Case 2: Equations not involving the Independent variables, that is, $f(z,p,q)=0$

Assume that $u=x+ay$ where, a is an arbitrary constant.

Then $p=\frac{\partial z}{\partial x}=\frac{dz}{du}\cdot\frac{\partial u}{\partial x}=\frac{dz}{du}\cdot 1=\frac{dz}{du}$ and $q=\frac{\partial z}{\partial y}=\frac{dz}{du}\cdot\frac{\partial u}{\partial y}=a\,\frac{dz}{du}$. Substituting the value of p and q in $f(z,p,q)=0$, we obtain ordinary differential equation of first order in variable z and u. Solve the resulting ordinary differential equation by usual methods and replace u by $x+ay$.

Case 3: Separable Equations, that is, $f(x,p)=g(y,q)$

Assume that $f(x,p)=g(y,p)=$ contant (say a). From these we can find the value of p and q as $p=F(a,x)$ and $q=G(a,y)$. Substituting the value of p and q in $dz=p\,dx+q\,dy$, we have $dz\ =F(a,x)\,dx+G(a,y)\,dy$. Integrating, we get the complete integral $z=\int F(a,x)\,dx+\int G(a,y)\,dy+b$.

Case-4: Clairaut's Equation, that is, $z=px+qy+f(p,q)$

Assume that $p=a$and $q=b$. Substituting the value of p and q in $z=px+qy+f(p,q)$, we get the complete integral is $z=ax+by+f(a,b)$.

Tutorial:

1 Solve $pq=p+q$.

Solution. Given that $pq=p+q\Rightarrow pq-p-q=0---(1)$

The equation (1) contains only p and q, i.e. it is of the form $f(p,q)=0$.

Assume that $z=ax+by+c$ is a solution of $f(p,q)=0$, provided $f(a,b)=0$.

$$\therefore ab-a-b=0\Rightarrow b=\frac{a}{a-1}$$

Thus, the complete solution is $z=ax+\frac{a}{a-1}y+c$, where a and c are arbitrary constants.

2 Solve $3p^2 - 2q^2 = 4pq$.

Ans. $z = ax + \left(\pm\sqrt{\frac{5}{2}} - 1\right)ay + c$, where a and c are arbitrary constants.

3 Solve $p^2 + q^2 = 4$.

Ans. $z = ax \pm \sqrt{4 - a^2}y + c$, where a and c are arbitrary constants.

Tutorial:

1 Solve $q^2 = z^2p^2(1 - p^2)$.

Solution. Given that $q^2 = z^2p^2(1 - p^2) \Rightarrow z^2p^2(1 - p^2) - q^2 = 0 - - - (1)$

The equation (1) contains only z, p and q, i.e. it is of the form $f(z, p, q) = 0$.

Assume that $u = x + ay$ where, a is an arbitrary constant. Then $p = \frac{dz}{du}$ and $q = a\ \frac{dz}{du}$. Substituting the value of p and q in (1), we have

$$z^2\left(\frac{dz}{du}\right)^2\left[1 - \left(\frac{dz}{du}\right)^2\right] - \left(a\ \frac{dz}{du}\right)^2 = 0$$

$$\Rightarrow z^2\left[1 - \left(\frac{dz}{du}\right)^2\right] = a^2$$

$$\Rightarrow 1 - \left(\frac{dz}{du}\right)^2 = \frac{a^2}{z^2}$$

$$\Rightarrow \left(\frac{dz}{du}\right)^2 = 1 - \frac{a^2}{z^2} = \frac{z^2 - a^2}{z^2}$$

$$\Rightarrow \frac{dz}{du} = \pm\frac{(z^2 - a^2)^{\frac{1}{2}}}{z}$$

$$\Rightarrow \frac{1}{2} \cdot \frac{2z}{(z^2 - a^2)^{\frac{1}{2}}} dz = \pm du$$

$$\Rightarrow \frac{1}{2} \cdot (z^2 - a^2)^{-\frac{1}{2}}(2z)dz = \pm du$$

Integrating, we get

$$(z^2 - a^2)^{\frac{1}{2}} = \pm(u + b)$$

$$\Rightarrow z^2 = a^2 + (u + b)^2$$

$$\Rightarrow z = \pm\sqrt{a^2 + (u + b)^2}$$

Replace u by $x + ay$, we get

$z = \pm\sqrt{a^2 + (ax + b)^2}$, where a and b are arbitrary constants.

2 Solve $p(1+q^2)=q(z-1)$.

Ans. $z=\frac{1+a}{a}+\frac{1}{4a}\ (x+ay+b)^2$, where a and b are arbitrary constants.

3 Solve $zpq=q+p$.

Ans. $\frac{z^2}{2}=\left(1+\frac{1}{a}\right)(x+ay+b)$, where a and b are arbitrary constants.

Tutorial:

1 Solve $p+q=x+y$.

Solution. Given that $p+q=x+y\ \Rightarrow p-x=y-q---(1)$

The equation (1) is of the form $f(x,p)=g(y,q)$.

$$\therefore p-x=y-q=a\ (say)$$

$$\Rightarrow p=a+x\ \ \&\ q=y-a$$

Substituting these values of p and q in $dz=p\ dx+q\ dy$, we have

$$dz=(a+x)\ dx+(y-a)\ dy$$

Integrating, we get

$$z=ax+\frac{x^2}{2}+\frac{y^2}{2}-ay+b=a(x-y)+\frac{1}{2}(x^2+y^2)+b,$$

where a and b are arbitrary constants.

2 Solve $xp-y^2q^2=1$.

Ans. $z=a\log x+\sqrt{a-1}\log y+b,$ where a and b are arbitrary constants.

3 Solve $p+q=\sin x+\sin y$.

Ans. $z=ax-ay-(\cos x+\cos y)+b,$ where a and b are arbitrary constants.

Tutorial:

1 Solve $(p+q)(z-px-qy)=1$.

Solution. Given that $(p+q)(z-px-qy)=1\Rightarrow z=px+qy+\frac{1}{p+q}---(1)$

The equation (1) is of the form $z=px+qy+f(p,q)$.

Assume that $p=a$ and $q=b$. Substituting the value of p and q in (1), we get

$z=ax+by+\frac{1}{a+b}$, where a and b are arbitrary constants.

2 Solve $z=px+qy-2\sqrt{pq}$.

Ans. $z=ax+by-2\sqrt{ab}$, where a and b are arbitrary constants

3 Solve $pqz=p^2(xq+p^2)+q^2(yp+q^2)$.

Ans. $z=ax+by+\frac{a^4+b^4}{ab}$, where a and b are arbitrary constants