CHAROTAR UNIVERSITY OF SCIENCE AND TECHNOLOGY
Department of Mathematical Sciences
CHARUSAT Campus, Highway 139,
Off, Nadiad - Petlad Road,
Changa, Gujarat 388421

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SUBJECT & SUBJECT CODE: DISCRETE MATHEMATICS & ALGEBRA (MA253)

CHAPTER: RELATION & LATTICE

TOPICS INCLUDED ARE

- **RELATIONS ON SETS**
- **TYPES OF RELATIONS IN A SET.**
- **PROPERTIES OF RELATIONS.**
- **REPRESENTATIONS OF RELATIONS.**
- **EQUIVALENCE RELATION.**
- **COVERING OF A SET, PARTION OF THE SET.**
- **PARTIALLY ORDERED RELATION, PARTIALLY ORDERED SETS.**
- **LATTICE, SUB LATTICES.**
- **PROPERTIES IF LATTICE.**
- **SOME SPECIAL LATTICES.**

CARTESIAN PRODUCT: CARTESIAN PRODUCT: Let A and B be sets. Cartesian product of A and B , denoted by AxB, is defined as

$$A\times B=\{(a,b):a\in A\, and\, b\in B\}$$

RELATION ON SETS: Let A and B be two sets. A relation from A to B is a subset of the Cartesian product AXB. Suppose R is a relation from A o B. Then R is a set of ordered pairs (a, b) where $a\in A, b\in B$ .Every such ordered pair is written as aRb and read as “ a is related to b by R”.

DOMAIN AND RANGE OF RELATION R:

Domain: The set $\{a\in A:(a,b)\in R \text{ for some } b\in B\}$ is called the domain of R and denoted by Dom(R).

Range: The set $\{b\in B:(a,b)\in R \text{ for some } a\in A\}$ is called the range of R and denoted by Ran(R).

Thus, the domain of relation R is the set of all first elements of the ordered pairs which belong to R and the range is the set of second element.

Example: Let A={ 2,3,5} , B={ 2,4,6,10} . A relation from A to B is given as follows: { 2R2, 2R4,2R6, 2R10,3R6,5R10}.Write R as a set of ordered pair.

Solution: R={(2,2),(2,4), (2,6),(2,10), (3,6),(5,10)}

Example:

Total number of distinct relation from a set A to a set B .

Let the number of element of A and B be m and n respectively. Then the number of elements in AXB is mn. Therefore, the number of elements of the power set of AXB is 2^{mn}.Thus AXB has 2^{mn} subsets. Now every subset of AXB is a relation from A to B. Hence, the number of different relation from A to B is 2^{mn}.

TYPES OF RELATION

Inverse Relation: Let R be the relation from set A to set B.The inverse of R is denoted by R^{-1} is the relation from B to A which consist of those ordered pairs which , when reversed, belong to R that is
$R^{-1}=\{(b, a): (a, b) \in R\}$.

Example: Let A={1,2,3} and B={a, b} and R={ (a,1), (b,1),(a,30, (b,2)} be a relation defined from A to B. Then find R^{-1}.

Solution: R^{-1}={ (a,1), (b,1),(a,3),(b,2)}

Identity Relation: A relation R in a set A is said to be identity relation I_A if $I_A=\{(x,x): x \in A\}$.For example Let A={2,4,6} .I_A={(2,2),(4,4),(6,6)} is an identity relation on A.

Complement of Relation: Let R be a relation defined on set A .Then complement of the relation is denoted by R^c or $\overline{R}$ .Then $\overline{R}$ **is defined as** $\overline{R}$ **={** $(a,b):(a,b) \in A \times B \, and \, (a,b) \notin R$ **} i.e.,** $\overline{R}$ **=(AXB)-R**

Combining Relation: Given two relations R_1 and R_2, we can combine them using basic set operations to form new relations such as $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, and$R_2 - R_1$.

For example: Let $A = \{1,2,3\}$ and $B = \{1,2,3,4\}$. The relations

$$R_1 = \{(1,1),(2,2),(3,3)\} \text{ and } R_2 = 1,1),(1,2),(1,3),(1,4)\}$$

can be combined using basic set operations to form new relations:

$\boldsymbol{R_1 \cap R_2}$ **={(1,1)}**

$\boldsymbol{R_1 \cup R_2}$ **={(1,1),(1,2),(1,3),(1,4),(2,2),(3,3)}**

$\boldsymbol{R_1 \cup R_2}$ **={(1,1),(1,2),(1,3),(1,4),(2,2),(3,3)}**

$\boldsymbol{R_2 - R_1}$ **={(1,2),(1,3),(1,4)}**

PROPERTIES OF RELATION

REFLEXIVE RELATION: Let R be a relation in set A .Then R is said to be reflexive relation if (a, a)$\in$R $\forall a \in R$.(a, a) are known as diagonal element.

Let A={ 1,2,3,4}

RELATION	REFLEXIVE	REASON
R_1={(1,1),(2,4),(3,3),(4,1),(4,4)}	Not Reflexive	(2,2),(3,3)$\notin$ R
R_2={(1,1),(2,2),(2,3),(3,3),(4,4)}	Reflexive	(1,1),(2,2),(3,3),(4,4) $\in$R
R_3={(1,1),(2,2),(3,3),(4,4)}	Reflexive	It contain all diagonal element.
R_4={(1,2),(2,3),(2,4),(3,4),(4,2)}	Not Reflexive	(1,1),(2,2),(3,3),(4,4) $\in$R i.e. all diagonal element belong to relation R_4.
$R_5=\phi$	Not Reflexive	(1,1) $\notin$ R ,(2,2) $\notin$ R,(3,3)$\notin$ R, (4,4) $\notin$ R_5. No diagonal element belongs to relation R_3.
R_6=AXA	Reflexive	(1,1),(2,2),(3,3),(4,4) $\in R_6$ i.e. all diagonal element belong to relation R_3.

Remark: Total number of element in A =n. Therefore no. of element in Cartesian product =nXn

IRREFLEXIVE RELATION: A relation R on a set A is said to be irreflexive if a A, (a,a) A. It is also called as antireflexive

RELATION	REFLEXIVE	REASON
R_1={(1,1),(2,4),(3,3),(4,1),(4,4)}	Not Irreflexive	((1,1), (2,2) belong to the set R_1
R_2={(1,1),(2,2),(2,3),(3,3),(4,4)}	Not Irreflexive	(1,1),(2,2),(3,3),(4,4) $\in$ R
R_3={(1,1),(2,2),(3,3),(4,4)}	Not Irreflexive	It contains all diagonal elements.
R_4={(1,2),(2,3),(2,4),(3,4),(4,2)}	Irreflexive	Diagonal element does not belong to set R_4.
$R_5=\phi$	Irreflexive	Diagonal element does not belong to set R_5.
R_6=AXA	Not Irreflexive	all diagonal element belong to relation R_3.

Remark: If the relation is not reflexive then it is not compulsory that it is irreflexive. For example, relation R_1 is neither reflexive nor irreflexive.

SYMMETRIC RELATION: A relation R on a set A is said to symmetric if for $\forall$ a,b $\in$ A (a,b) $\in$ R then (b,a) $\in$ R.

Let A={ a, b, c}

RELATION	SYMMERTIC	REASON
$R_1=\phi$	Symmetric	((1,1), (2,2) belong to the set R_1
R_2= AXA	Symmetric	(1,1),(2,2),(3,3),(4,4) $\in$ R
R_3={(a,b),(b,a)}	Symmetric	It contains all diagonal elements.
R_4={(b,c),(c,b),(b,b),(c,c)}	Symmetric	Diagonal element does not belong to set R_4.
R_5={(a,a),(b,b),(c,c)}	Symmetric	Diagonal element does not belong to set R_5.
R_6={(a,b),(b,c),(a,c)}	Not symmetric	all diagonal element belong to relation R_3.
R_7={(a,b),(b,a),(a,c)}	Not symmetric	

ANTISYMMETRIC RELATION: A relation R on the set A is antisymmetric relation if for all a,b $\in$ A,(a,b) $\in$ R and (b,a) $\in$ R then a=b.

RELATION	ANTISYMMETRIC	REASON
R_1={(a,b),(b,c),(a,c)}	Antisymmetric	No symmetric pair exist for any element
R_2={(a,b),(a,a),(b,b)}	Antisymmetric	No symmetric pair exist for any element
R_3={(a,a),(b,b),(c,c)}	Antisymmetric	Diagonal elements are allowed in symmetric and no symmetric element exist.
R_4={(a,b),(b,a),(b,c),(c,c)}	Not Antisymmetric	Symmetric element of (a,b) exist in R_4.
$R_5=\phi$	Antisymmetric	Symmetric pair does not exist
R_6= AXA	Not Antisymmetric	Symmetric pair exist

Remark: Antisymmetric is not the same as not symmetric. A relation may be symmetric as well as antisymmetric at the same time. For example R_3={(a,a),(b,b) ,(c,c)} is both symmetric and antisymmetric on A ={a,b,c}

ASYMMETRIC RELATION: A relation R on set A is asymmetric if (a,b) $\in$R then (b,a) $\notin$ R fora$\neq$b. This means that the presence of (a, b) in R excludes the possibility of presence of (b,a) in R.

RELATION	ASYMMETRIC	REASON
$R_1=\phi$	Asymmetric	No symmetric pair exists.
R_2= AXA	Not Asymmetric	Symmetric pairs exist for example (a,b) and (b,a)
R_3={(a,a),(b,b),(c,c)}	Not Asymmetric	Diagonal element are symmetric to its self.
R_4={(a,b),(b,c),(a,c)}	Asymmetric	No symmetric pair exits.
R_5={(a,b),(b,a),(b,c)}	Not Asymmetric	Symmetric pair (a,b) and (b,a) exist
R_6={(a,a),(b,b),(a,b),(b,a)}	Not Asymmetric	Diagonal element as well as symmetric pair exists.

Remarks:

1. A relation can be symmetric, antisymmetric and asymmetric for example: $R_1=\phi$

2.It is also possible that relation can be neither symmetric, antisymmetric nor asymmetric.
R={(a,a),(a,b),(b,a),(b,c)}

TRANSITIVE RELATION: A relation R on set A is said to be transitive if (a,b) $\in$R, (b,c) $\in$R then (a,c) $\in$R.

RELATION	TRANSITIVE	REASON
R_1={(a,b),(b,a),(a,a),(b,b)}	Transitive	According to definition
R_2={(a,a),(b,b),(c,c)}	Transitive	According to definition
R_3={(a,b)}	Transitive	According to definition
R_4={(a,b),(a,c)}	Transitive	According to definition
R_5={(a,b),(c,b)}	Transitive	According to definition
R_1={(a,b),(b,c),(a,c),(a,a)}	Transitive	According to definition
R_2{(c,a),(b,c}	Not Transitive	(b,a) does not exist
R_4=AXA	Transitive	According to definition

EQUIVALENCE RELATION: A relation R on set S is called a partial order if it is reflexive, symmetric and transitive (RST).That is ,R is an equivalence relation on A it it has following properties:

1. $(a,a)\in R \text{ for all } a\in A$
2. $(a,\text{b})\in R \text{ implies } (b,a)\in R$
3. $(a,\text{b}) \text{ and} (b,c)\in R \text{ implies } (\text{a},\text{c})\in R$

Remark: If R is an equivalence relation then R^{-1} is also an equivalence relation.

Example: If R be a relation in the set of integers Z defined by R={ (x,y): $x\in Z, y\in Z$ **,x-y is divisible by 6} Then prove that R is an equivalence relation.**

Solution: Reflexive : Let $x\in Z$ .Then x-x=0 and 0 is divisible by 6.

Therefore $(x,x)\in Z$ for all $x\in Z$.

Hence, R is reflexive.

Symmetric : Let $(x,\text{y})\in Z \Rightarrow$ (x-y) is divisible by 6.

$\Rightarrow$-(y-x) is divisible by 6.
$\Rightarrow$(y-x) is divisible by 6.
$\Rightarrow (\mathrm{y},\mathrm{x}) \in Z$

Hence, R is symmetric

Transitive : Let $(x,\mathrm{y}) \in Z$ *and* $(y,z) \in Z \Rightarrow$ (x-y) is divisible by 6 and (y-z) is divisible by 6.

$\Rightarrow$ (x-y) + (y-z) is divisible by 6.
$\Rightarrow$ (x-z) is divisible by 6.
$\Rightarrow (\mathrm{x},\mathrm{z}) \in Z$

Hence R is transitive.

Thus R is an equivalence relation.

Example: Consider the following relation on {1,2,3,4,5,6}.R={ (i,j): $|i-j|=2$ }.Is R reflexive , symmetric and transitive?

Solution: Let A= {1,2,3,4,5,6}. Then R= {(1,3), (2,4), (3,1), (4,20) ,(3,5) ,(5,3),(4,6),(6,4)}

R is not reflexive as (1,1)$\notin$R.

R is symmetric since $(\mathrm{i},\mathrm{j}) \in R \Rightarrow (j,i) \in R, \forall i, j \in R$

R is not Transitive as $(2,4)$ *and* $(4,2) \in R\, but\, (2,2) \notin R$

Example: Let R be a binary relation on the set of all integers of 0's and 1's such that R={(a, b): a and b are strings, that they have the same number's of 0's } .Is R reflexive? symmetric? Antisymmetric? Transitive? An equivalence relation?

Solution: R is reflexive since $(\mathrm{a},\mathrm{a}) \in R \;\; \forall a \in R$.

R is symmetric since when a and b have same numbers of 0's then b and a will also have same number of 0's. Hence $(\mathrm{a},\mathrm{b}) \in R \Rightarrow (b,a) \in R$

R is transitive since when a and b have the same number of 0's and b and c have the same number of 0's then a and c will also have same number of 0's .Hence $(\mathrm{a},\mathrm{b}) \in R, (b,c) \Rightarrow (\mathrm{a},\mathrm{c}) \in R$

Thus,R is reflexive ,symmetric, transitive and hence an equivalence relation.

R is not antisymmetric since (a,b) and (b,a) belongs to R does not imply a=b.

EQUIVALENCE CLASSES: If R is an equivalence relation on a set S and xRy, then x and y are called respect to R. The set of all elements of S that are equivalent to a given element x constitute the equivalence class of x and equivalence class is denoted by [x].

Thus [x]={y $\in$S:yRx}

PARTITIONS OF SET: A partition of a set A is a set of non –empty subsets of A denoted by { A_1,A_2,.....A_n} such that the union of A_i 's is equal to A and intersection of A_i and A_j is empty for any distinct A_i and A_j.

Example: Let A= {1,2,3,4,5} and R= {(1,2), (1,1), (2,1), (2,2), (3,3), (4,4), (4,5), (5,4), (5,5)} be equivalence relation on A. Determine the partition corresponding to R^{-1}, if it is an equivalence relation.

Solution: We know that if R is an equivalence relation on A, then R^{-1} is also an equivalence relation on A. Thus R^{-1}= {(2,1), (1,1), (1,2), (2,2), (3,3), (4,4), (5,4), (4,5), (5,5)} be an equivalence relation on A.

Then Equivalence class of

[1] = {1,2}; [2]={1,2}; [3]={3}; [4]={4,5} ; [5]={4,5}

Thus partition set corresponding to R^{-1}

P= {{1,2}, {3}, {4,5}}

Example: Let R is an equivalence relation on set A={a,b,c,d} defined by partitions of P{{a,d},{b,c}}.Determine the element of equivalence relation and also find the equivalence classes of R.

Solution: The elements of equivalence relation defined by partition P is

R={(a,a,), (d,d), (a,d),(d,a), (b,b),(c,c),(b,c),(c,b)}

The equivalence classes of R are

[a]=[d]={a,d} and [b]=[c]={b,c}

Example: Show that the relation (x,y)R(a,b) $\Leftrightarrow$ $x^2+y^2= a^2+b^2$ is an equivalence relation on plane and describe the equivalence relation classes.

Solution: Reflexive

(x,y)R(x,y) $\Leftrightarrow$ $x^2+y^2= x^2+y^2$

Therefore, R is reflexive.

Symmetric : Let (x,y)R(a,b) $\Rightarrow x^2+y^2= a^2+b^2$

$\Rightarrow a^2+b^2 =x^2+y^2$

$\Rightarrow$(a,b)R (x,y)

R is symmetric

Transitive: Let (x,y)R(a,b) and (a,b) R (c,d) $\Rightarrow x^2+y^2= a^2+b^2$ and $a^2+b^2=c^2+d^2$

$\Rightarrow x^2+y^2= c^2+d^2$

$\Rightarrow$(x,y)R (c,d)

Therefore, R is transitive.

Since R is reflexive, symmetric and transitive therefore R is an equivalence relation.

PARTIAL ORDERED RELATION: A relation R on set S is called a partial order if it is reflexive, antisymmetric and transitive (RAT).That is

1. $(a,a) \in R \ for\ all\ a \in A$
2. $(a,\mathrm{b}) \in R \ and \ (b,a) \in R \Rightarrow a = b$
3. $(a,\mathrm{b})\ \mathrm{and} (b,c) \in R \ implies \ (\mathrm{a,c}) \in R$

A set S together with a partial order relation is called partial order set or POSET and is denoted by (S, R).

Example: For example, the greater or equal relation is a partial ordering on Z, the set of integers.

Solution: Reflexive : Since $a \geq a$ for every integer a,$\geq$ is reflexive.

Antisymmetric: Since $a \geq b$ and $b \geq a$ imply a=b, $\geq$ is antisymmetric.

Transitive: Since $a \geq b$ and $b \geq c$ imply a$\geq$c, $\geq$ is transitive.

Hence , $\geq$ is partial ordering on Z and (Z, $\geq$) is a POSET.

Example: Consider P(S) as the power set i.e., the set of all subsets of a given set S. Show that the inclusion relation $\subseteq$ is a partial ordering on power set P(S).

Solution: Reflexive: A$\subseteq$A for all A$\subseteq$S, $\subseteq$ is reflexive

Antisymmetric: $A \subseteq B$ and $B \subseteq A$ imply A=B , $\subseteq$ is antisymmetric.

Transitive: $A \subseteq B$ and B $\subseteq C$ imply $A \subseteq C$, $\subseteq$ is transitive .

It follows that $\subseteq$ is a partial ordering on P(S) and (P(S), $\subseteq$) is a poset.

Example: Show that the set Z^+ of all positive integers under divisibility relation forms a poset.

Solution: Reflexive: n/n for all $n \in Z^+$, / is reflexive.

Antisymmetric: n/m and m/n imply n=m, / is antisymmetric.

Transitive: n/m and m/p imply n/p, / is transitive.

It follows that / is a partial ordering on Z^+ and (Z^+, /) is a poset.

COMPARABLE AND INCOMPARABLE

Comparability: The elements *a* and *b* of a poset ($S,\preccurlyeq$) are *comparable* if either $a \preccurlyeq b$ or $b \preccurlyeq a$. or **we can say that a and b are comparable if either "a" is related to" b " i.e ., (a,b) $\in$ R or " b" is related to "a" i.e., (b,a) $\in$ R.**

Incomparable: When *a* and *b* are elements of *S* so that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, then *a* and *b* are called i*ncomparable*.

Example: In the poset (Z+, |), are the integers 3 and 9 comparable? Are 5 and 7 comparable?

Solution: The integers 3 and 9 are comparable, because 3 | 9. The integers 5 and 7 are incomparable, because 5 | 7 and 7 | 5.

REPRESENTATION OF RELATION

There are many ways of representing relations on finite sets. For visualising of information about relations, graphical methods are particularly useful. To do mathematical calculations, it is often more convenient to represent them as matrices. Since relations are sets, the methods to represent sets are also available to represent relation.

MATRIX REPRESENTATION

Let A={a_1, a_2, a_3,...........a_n} and B={ b_1,b_2,b_3,.............b_n} are finite sets containing m and n elements respectively and let R be a relation from A to B .Then R can be represented by $m \times n$ matrix M_R=[m_{ij}] where

$$m_{ij} = \begin{cases} 1, & if\ (a_i, b_j) \in \mathrm{R} \\ 0 & if\ (a_i, b_j) \notin \mathrm{R} \end{cases}$$

Then matrix M_R is called the matrix of R.

Example: Let R be the relation from set A={1,3,4} on itself and defined by R={(1,1), (1,3), (3,3), (4,4)} then the matrix of R can be found as follows:

Solution: Let M_R denotes the matrix of R. Number of rows in M_R=Number of elements in A=3.Since the relation from the set A on itself, the number of column in M_R is also 3. So M_R is 3x3 matrix.

$M_R = \begin{array}{c c} & \begin{array}{ccc} 1 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$	(1,1) $\in$ R so entry is 1 (1,3) $\in$ R so entry is 1 (1,4) $\notin$ R so entry is 0 (3,1) $\notin$ so entry is 0 (3,3) $\in$ so entry is 1 (3,4) $\notin$ so entry is 0 (4,,1) $\notin$ so entry is 0 (4,3) $\notin$ so entry is 0

	(4,4) $\in$ so entry is 1

Example: Let A= {a_1, a_2, a_3} and B= {b_1, b_2, b_3, b_4} which ordered pairs are in the relation R represented by the matrix

$$M_R = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Solution: Since R contains of those ordered pairs (a_i,b_j) with m_{ij} .It follows that

R={ (a_1,b_2),(a_2,b_1),(a_2,b_3),(a_2,b_4),(a_3,b_1), (a_3,b_3) }

Remarks: The matrix representing a relation can be used to determine whether the relation has various properties i.e. reflexive, irreflexive, symmetric, asymmetric, antisymmetric and transitive.

1. **Reflexive:** If all the elements in the main diagonal of the matrix representation of relation are 1 i.e., $m_{ii} = 1$, then the relation is reflexive. For example

$$M_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. **Irreflexive :** If all the elements in the main diagonal of the matrix representation of relation are 0 i.e., $m_{ii} = 0$, then the relation is reflexive. For example

$$M_R = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

3. **Symmetric:** If the representative matrix of a relation is symmetric with respect to the main diagonal, i.e., $m_{ij} = m_{ji}$ for all value of *i* and *j* then the relation is symmetric. (i.e., M_R=M_R^T).
4. **Antisymmetric:** A relation is antisymmetric if and only if $m_{ij} = 1$ necessitates that $m_{ij} = 0$.The following matrices illustrate the notions of symmetry and antisymmetric.
5. **Transitive:** There is no simple way to test whether a relation R is a transitive by examining the matrix M_R. A relation R is transitive if and only if its matrix M_R=[m_{ij}] has the property if $m_{ij} = 1$ and $m_{jk} = 1$then $m_{ik} = 1$This statement simply means R is transitive if M_R.M_R has 1 in position of *i ,k*. Thus, transitivity of R means that if M_R^2=M_R.M_R has a 1 in any position then M_R must have a 1 in the same position. Thus, R is transitive if and only if M_R^2+M_R=M_R.

Example: Let A= {1,2,3,4} and let R be a relation on A whose matrix is

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$. Show that R is transitive.

Solution : $M^2_R = M_R.M_R$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} . \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

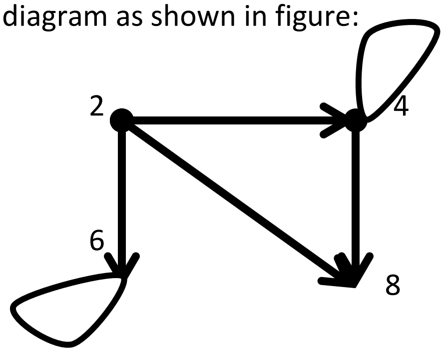
$$M^2_R + M_R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} = M_R$$

Therefore, relation R is transitive.

GRAPHICAL REPRESENTATION

Let A and B are two finite sets and R is a relation from A to B. For graphical representation of a relation on a set, each element of the set is represented by a point. These points are called nodes or vertices. An arc is drawn from each point to its related point. If the par $x \in A, y \in B$ is in relation, the corresponding nodes are connected by arcs called edges or arcs. The arcs start at the first element of the pair, and they go to the second element of the pair. The direction is indicated by an arrow. All arcs with an arrow are called directed arcs. The resulting pictorial representation of R is called a directed graph or diagraph of R. An edge of the form (a, a) is represented using an arc from the vertex a back to itself. Such an edge is called loop.

For example: Let A= {2,4,6} and B= {4,6,8} and R be the relation from the set A to the set B given by **xRy** means x is a factor of y, then R= {(2,4), (2,6), (2,8), (4,4), (6,6), (4,8)}. This relation R from A to B is represented by the arrow diagram as shown in figure:



Remarks:

1. The directed graph representing a relation can be used to determine whether the relation has various properties

(i) A relation is **reflexive** if and only if there is a loop at every vertex of the directed graph, so that ordered pair of the form (a,a) occurs in the relation. If no vertex has a loop then the relation is **irreflexive.**

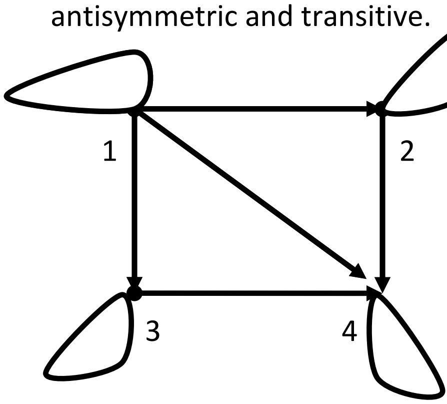
(ii) A relation **is symmetric** if and only if for every edge between distinct vertices in its digraph there is an edge in the opposite direction, s that (b, a) is in the relation whenever (a, b) is in relation. A relation is **antisymmetric** if no two distinct points in the diagraph have an edge going between them in both directions.

(iii) A relation is **transitive** if and only if whenever there is a directed edge from to vertex a to a vertex b and from a vertex b to vertex c , then there is also a directed edge from a to c.

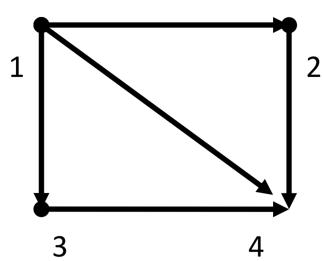
2. Every diagraph determines a relation, so that we may recover R from the graph. Domain and Range can be found easily if the relation is represented by a graph. Every node with an outgoing arc belongs to the domain and every node with an incoming arc belongs to the range.

PRATICE EXAMPLE

1. Draw the directed graph that represents the relation R= {(1,1), (2,2), (1,2), (2,3), (3,2), (3,1), (3,3)} on X.
2. Determine whether the relation for the directed graph shown in figure are reflexive, symmetric, antisymmetric and transitive.



3. Write the relation as set of ordered pairs from the diagraph as shown in figure



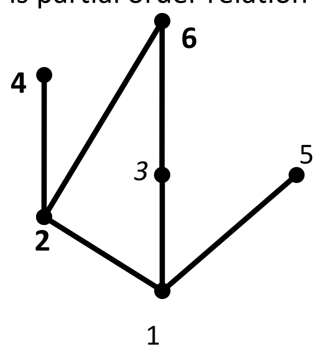
REPRESENTATION AND HASSE DIAGRAM

HASSE DIAGRAM: A partial order relation on X can be represented by means of a diagram known as Hasse diagram of X.

Remark: This gives a method of representing finite posets which works well for posets with relatively few elements. We represent the element of X by points and if y is an immediate successor of x we take y at a higher level than x and join x and y by a straight line. A diagram formed as above is known as Hasse diagram. Thus there will not be any horizontal lines in the diagram of a poset.

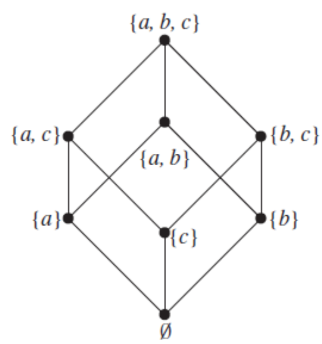
Example: Let X={1,2,3,4,5,6}, then / is partial order relation on X. Draw the Hasse diagram of (X,/)

Solution:



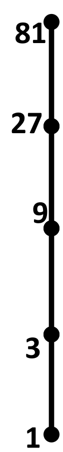
Example: Draw the Hasse diagram for the partial ordering { (A,B): A$\subseteq$**B}** on the power set P(S) where S= { a,b,c}

Solution: Here P(S)={ $\varnothing$,{a}, {b},{c}, {a, b},{a, c},{b, c}, {a, b, c}}. The Hasse diagram of the poset (P(S), $\subseteq$) is shown below



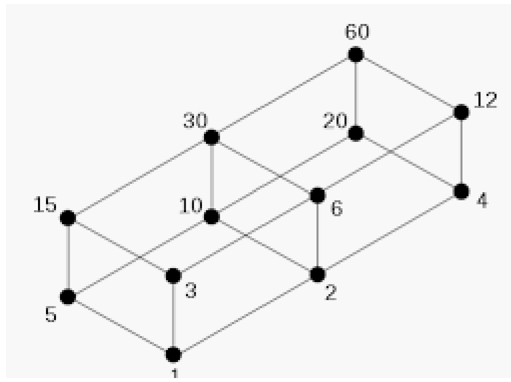
Example: Let A= {1,3,9,27,81}, draw the Hasse diagram of the Poset (A, /)

Solution:

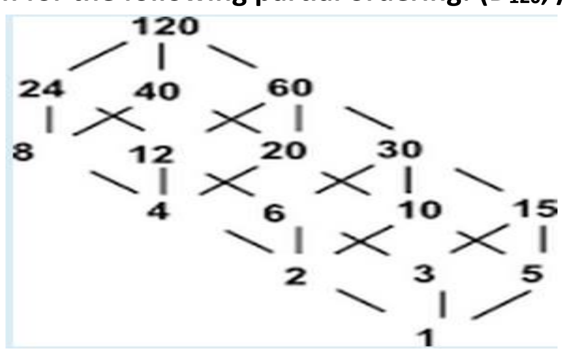


Example: Draw the Hasse Diagram for the following partial ordering: {(a,b) /a∈A, b∈A, a|b } on the set A={1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60}.

Solution:



Example: Draw the Hasse Diagram for the following partial ordering: (D_{120}, /).



SPECIAL ELEMENT IN POSETS

MAXIMAL ELEMENT: An element a in the poset is called maximal element of P if a $a \prec x$ for no x in P, that is no element of P strictly succeeds a. **OR** If in poset, an element is related to any other element.

MINIMAL ELEMENT: An element b in P is called minimal element of P if $x \prec b$ for no x in P. **OR** If in a poset no element is related to an element.

Remark: Maximal and minimal element are easy to spot in the Hasse diagram, they are the top and bottom elements in the diagram. That is, a maximal element has no connections leading up and minimal element has no connection leading down.

MAXIMUM (GREATEST ELEMENT): Let $(P,\leq)$ be a poset. An element a$\in$P is the greatest element of P if $x \leq a$ for all x$\in$P i.e., every element in P precedes a. OR An element of poset is said to be maximum if it is maximal and every element is related to it.

MINIMUM ELEMENT (LEAST ELEMENT) : Let $(P,\leq)$ be a poset. An element a$\in$P is the least element of P if $a \leq x$ for all x$\in$P i.e., every element in P succeeds a. **OR** an element of poset is said to be minimum if it is minimal and every element is related to it.

Remarks: The following points are to be noted

1. A poset may not have maximal element. For instance, the natural numbers under usual $\leq$have no maximal element.
2. A poset have more than one maximal or minimal element. In poset {2,3,4,6} under divisibility, 4nad 6 are both maximal or minimal elements.
3. Maximal element may not be the greatest element. In above 4 and 6 are maximal but neither 4 nor 6 is the greatest element.
4. A poset may have a maximal element but no minimal elements, or minimal element but no maximal elements. For example, the poset (Z^-, $\leq$) has a maximal element but no minimal element, whereas the poset (Z^+, $\leq$) has minimal element but no maximal element.

UPPER BOUND: Let B be a subset of a poset $(P,\leq)$.An element u belong to A is called an upper bound of B if u succeeds every element of B i.e., x u for all x B.

LOWER BOUND: An element l A is called a lower bound of B if l precedes every element of B i.e., l x for all x B.

LEAST UPPER BOUND: An element ***a*** belong to A is called least upper bound (lub) of B if ***a*** is an upper bound of B and $a \leq a'$ whenever a' is the upper bound of B.

GREATEST LOWER BOUND: An element a A is called the greatest lower bound (glb) of B if a is lower bound of B and $a' \leq a$, whenever a' is a lower bound of B.

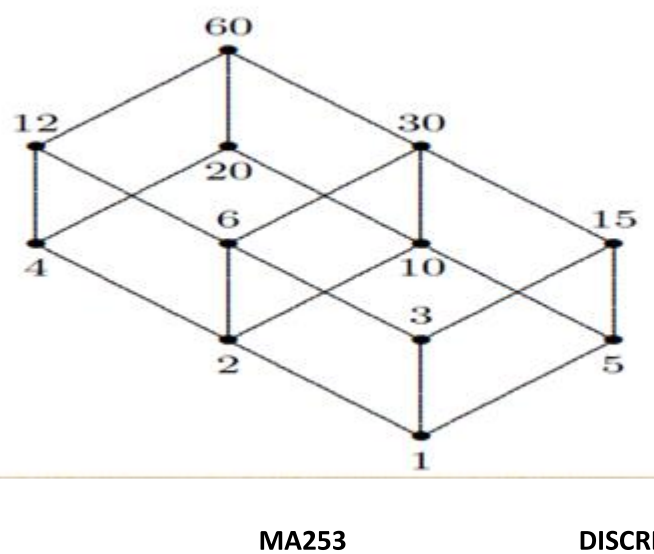
JOIN SEMI LATTICE : In a poset if **LUB/JOIN/ SUPREMUM/** $\vee$ exist if every pair of elements then poset is called join semi lattice.

MEET SEMI LATTICE: In a poset of **GLB/MEET/INFIMUM/**$\wedge$ exist for every pair of elements then poset is called meet semi lattice.

LATTICE: A poset is called lattice if it is both MEET and JOIN SEMI LATTICE.

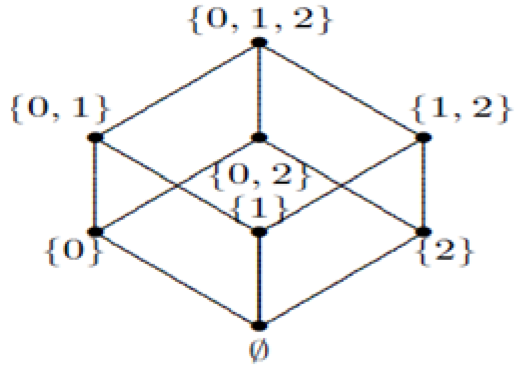
Example: Show that the following posets are lattices, and interpret their meets and joins:

(a) The poset of the divisors of 60, ordered by divisibility.



The poset consisting of all the divisors of 60 is a lattice as every pair of elements has both a meet and a join.

(b). The poset of the subsets of {0, 1, 2}, ordered by the subset relation.



Solution: The poset of the subsets of {0, 1, 2}, ordered by the subset relation as every pair of elements has both a meet and a join.

Example : The Hasse diagram of a poset (S, R) is given below. Check that it is a lattice or not.

Solution: we take pair {e, c},

Upper bound of e is e itself and Upper bound of c is c itself. Common upper bound of {e, c} does not exist. Therefore, LUB of {e, c} does not exist.

Thus, the poset (S, R) is not a Joint semi lattice.

Therefore, the poset (S, R) is not a lattice.

Note: Similarly, you can take pair {c, a}.

PROPERTIES OF LATTICE

Let L be a lattice. Then, for every a, b and c in L

1. **Commutative :** $a \vee b = b \vee a; \quad a \wedge b = b \wedge a$
2. **Associativity:** $a \vee (b \vee c) = (a \vee b) \vee c; \; a \wedge (b \wedge c) = (a \wedge b) \wedge c$
3. **Impotence:** $a \vee a = a; \quad a \wedge a = a$
4. **Absorption:** $a \vee (a \wedge b) = a; \; a \wedge (a \vee b) = a$
5. If a ≤ b, then **(a) a ∨ c ≤ b ∨ c; (b) a ∧ c ≤ b ∧ c**
6. a ≤ c and b ≤ c if and only if a ∨ b ≤ c
7. c ≤ a and c ≤ b if and only if c ≤ a ∧b
8. If a ≤b and c ≤d, then **(a) a∨c ≤ b∨d (b) a ∧ c ≤ b∧d**

SUBLATTICE : A nonempty subset L' of a lattice L is called a sub lattice of L if for all a,b ∈ L' ⟹ a ∨ b ∈ L' , a ∧ b ∈ L' i.e. the algebra(L', ∨, ∧) is a sublattice of (L, ∨, ∧) iff L' is closed under both operations ∨ and ∧.

Remark:

1. From the definition it follows that a sub lattice itself is a lattice and
2. Every singleton of a lattice L is a sub lattice. However, any subset of L, which is a lattice need not be a sub lattice.

Example: Let T={ ∅, {a}, {c}, {a,c}, {a,b,c}} be a subset of P(S) for S={a, b, c} and L=(P(s), ∩, ∪) be a lattice. Show that T is a sub lattice of L.

Solution: Here P(S)={∅, {a}, {b}, {c}, {a,b}, {a, c}, {b, c}, {a,b,c}}. $T \subseteq P(S)$ is a closed under $\cap$ and $\cup$, **i. e.** for any set A and B in T, the $A \cap B$ and $A \cup B$ is in T. Therefor, T is a sub lattice.

Example: Let T={ ∅, {a}, {b}, {a, c}, {a, b, c}} be a subset of P(S) for S={a, b, c} and L=(P(s), ∩, ∪) be a lattice. Show that T is not a sub lattice of L.

Solution: Here P(S)={∅, {a}, {b},{c}, {a,c}, {b,c}, {a,b,c}}.

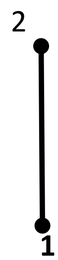
$T \subseteq P(S)$ is not closed under ∪, beause A={a} and B={b} in T, For A={a} and B={b}, the $A \cup B$ is not in a T. Therefor, T is not a sub lattice.

SOME SPECIAL LATTICES

COMPLETE LATTICE: A lattice is called complete if each of its non-empty subsets has a least upper bound and greatest lower bound.

Example: Check whether lattice (D_2, /) is a complete lattice.

Solution: The Hasse diagram of (D_2, /)



The **nonempty subset** of D_2 are {1}, {2}, {1,2}.

To check D_2 is a complete lattice, we find least upper bound and greatest lower bound of every non empty subset of D_2.

Non Empty Subset of D_2	Least upper bound	Greatest lower bound
{1}	1	1
{2}	2	2
{1,2}	2	1

Since the least upper bound and greatest lower bound exist for every non empty subset of D_2, **therefore it is a complete lattice.**

DISTRIBITIVE LATTICE: A lattice $(L, \vee, \wedge)$ is called a distributive lattice if for any a,b,c $\in$ L

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

Remark: Note that both the equalities are equivalent to one another, hence to check whether the lattice is distributive or not, it is sufficient to verify one of them.

If L is not distributive, we say that L is non-distributive.

Note: The distributive property holds when

1. any two of the elements a, b and c are equal or
2. when any one of the elements is 0 or 1.

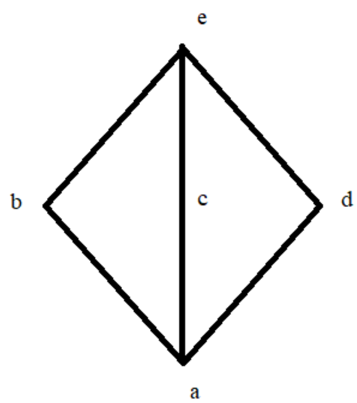
BOUNDED LATTICE: A lattice L is said to be bounded if it has a greatest (maximum) element and a least (minimum) element which are denoted by 1(I) and 0(O) respectively.

COMPLEMENT OF ELEMENT: Let L be bounded lattice with greatest element 1 and least element 0, and let a in L. An element b in L is called a complement of a if a ∨ b = 1 and a ∧ b =0

Note: 0'/0^c = 1 and 1' /1^c= 0

COMPLEMENTED LATTICE: A lattice L is said to be complemented if it is bounded and every element in it has a complement.

Example: The Hasse diagram of a lattice (S, R) is given below. Check that it is a complemented lattice or not.



Element	Complement
a	e
b	c and d(**not unique**)
c	b and d(**not unique**)
d	b and c(**not unique**)
Complement of an element need not to be unique.	

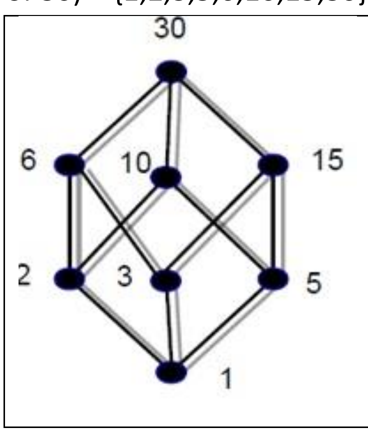
Maximum element is e and minimum element is a. Therefore, it is a bounded lattice. It is a complemented lattice because each element has /have its complement as shown in table. It is a complemented lattice.

Uniqueness of Complements in Distributive Lattices: If (L, ≤) is a bounded distributive lattice with minimum 0 and maximum 1, then complements are unique, provided they exist.

Corollary: Every element in a complemented distributive lattice has a unique complement.

Example: Prove that D_{30} is a complemented distributive lattice.

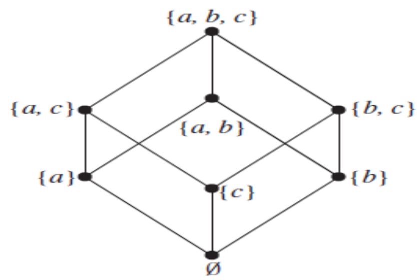
Solution: D_{30} (divisors of 30) = {1,2,3,5,6,10,15,30}



Element	Its Complement
1	30
2	15
3	10
5	6
6	5
10	3
15	2
30	1

Each element has complement and we know that in lattice at most one complement of each element exist then it is distributive. Therefore, D_{30} is complemented distributive lattice.

Example: Prove that P(S) **is a bounded and distributive lattice under** $\cap$ and $\cup$**, where S= {a, b, c}.**

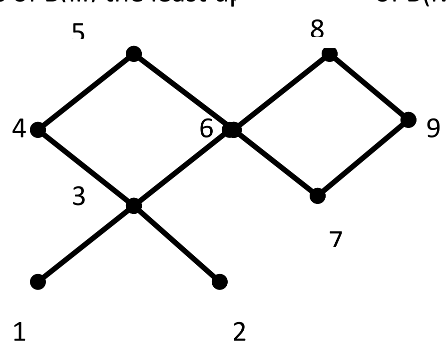


Solution: We know that (P(S), ∩, ∪) is a lattice. Here P(S)={ ∅,{a}, {b},{c}, {a, b},{a, c},{b, c}, {a, b, c}}. Maximum element is {a, ,b, c} and minimum element is ∅. Therefore P(S) is a bounded under $\cap$ and $\cup$.
Also we know that P(S) satisfier the distributive relation
a $\cup$ $(b \cap c) =$ (a $\cup$ $b)$ $\cap$ (a $\cup$ $c)$, therefore it is a distributive lattice

PREVIOUS YEARS UNIVERSITY QUESTION FOR

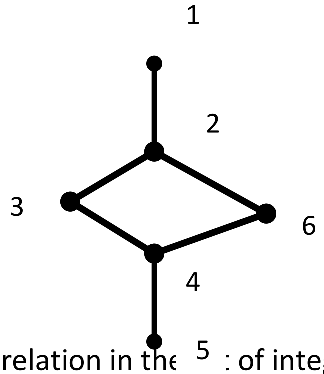
Chapter: Relation and Lattice

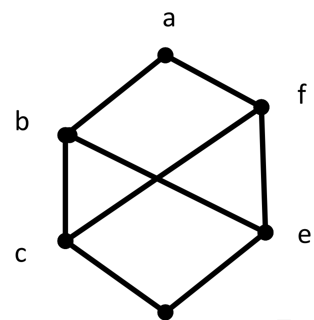
1. Give an example of a relation which is symmetric as well as antisymmetric.
2. Define: Minimal and Maximal elements of a poset. Find the minimal and maximal element of the poset ({ 2,4,6,12,20},/)
3. Define GLB and LUB of a set in Poset. Find all possible upper bounds and lower bounds of the set { {1,2},{1,4},{3,4}} , a subset of the Poset $(\{1\},\{2\},\{4\},\{1,2\},\{1,4\},\{2,4\},\{3,4\},\{1,3,4\},\{2,3,4\},\subseteq)$
4. Define Bounded Lattice.
5. Give an example of a non-empty, symmetric, transitive relation on the set {1,2} that is not reflexive.
6. Consider the following Hasse diagram and the set B= {3,4,6}. Find, if they exist, (i) all upper bounds of B (ii) All lower bounds of B(iii) the least upper bound of B(iv) greatest lower bound of B



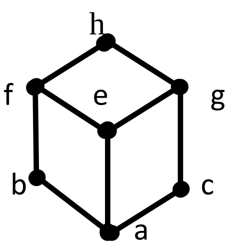
7. Draw the Hasse diagram and find maximal, minimal maximum and minimum element of the poset { 2,7,14,28,56,84} with the relation $a \le b$ if and only if " a" divides "b".

8. Draw the Hasse diagram of the set of all divisor of 42 with respect to the division relation. Is this lattice bounded or not? Justify your answer.
9. Define Partially Ordered Set and determine whether the posets represented by each of the following Hasse diagram are Lattices or not by justifying your answer.

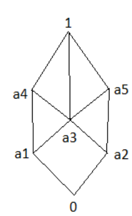




10. If R be a relation in the set of integers Z defined by R={ (x,y): $x \in Z$, $y \in Z$,x-y is divisible by 6} .Then prove that R is an equivalence relation. Find the equivalence class of 5.
11. Define Complemented Lattice and Distributive lattice. Find the complement of each of the element in given Hasse diagram. Is it a distributive lattice?



12. Draw the Hasse Diagram of (D_{48}, /). Find the Maximal, Minimal, Maximum and Minimum element of the poset.
13. Consider the lattice L given in the following figure



14. Is $L_1 = \{0, a_1, a_2, 1\}$ a sub lattice of L?

    Find the complements of a_1, a_2 and a_3, if exist.
15. For the Hasse diagram given in the following figure, find maximal, minimal, greatest and least elements, if exists.