Numerical methods
on the Cahn-Hilliard Equation

Jonathan Ulmer

June 27, 2024

Table of Contents

Introduction

The Cahn-Hilliard equation

$M(\phi) : [-1, 1] \to \mathbb{R}^+$	Mobility coefficient
$W(\phi) : [-1, 1] \to \mathbb{R}^+$	Double well potential
$\varepsilon > 0 \in \mathbb{R}$	Interface coefficient
$\phi : \Omega \times (0, T) \to \mathbb{R}^d$	Phase-field variable
$\mu : \Omega \times (0, T) \to \mathbb{R}^d$	Chemical potential variable

Cahn-Hilliard Equation:

$$\begin{aligned} \partial_t \phi(x, t) &= \nabla \cdot (M(\phi) \nabla \mu), \\ \mu &= -\varepsilon^2 \Delta \phi + W'(\phi), \end{aligned} \tag{1}$$

- phase field equation for two phase flow
- diffuse interface equation
- gives position of phases
- constant mobility $M(\phi) \equiv 1$
- polynomial potential $W(\phi) = \frac{1}{4}\phi^2(1 - \phi^2)$
- 0 Neumann boundary conditions

Baseline solver

- Implicit in time
- discretized on NxN grid
- uses multi-grid scheme

b	collects all terms not dependant on ϕ_{ij}^{n+1}
DL	Jacobian of L
L	Implicit terms of the discrete CH equation
Ω_d	discrete version of the computational domain Ω

- solves equation of type

$$DL \cdot \begin{pmatrix} \phi_{ij}^{n+1} \\ \mu_{ij}^{n+\frac{1}{2}} \end{pmatrix} = b \tag{3}$$

- with Gauss Seidel iteration
- for every point $(i, j) \in \Omega_d$
- on two grid scales
- multiple times per sub-iteration

Relaxation

Relaxed Cahn Hilliard equation

$$\begin{aligned} \partial_t \phi^\alpha &= \Delta \mu \,, \\ \mu &= \varepsilon^2 \alpha (c^\alpha - \phi^\alpha) + W'(\phi). \end{aligned} \tag{4}$$

where $\alpha < 0$ is a relaxation parameter

Additional elliptical system

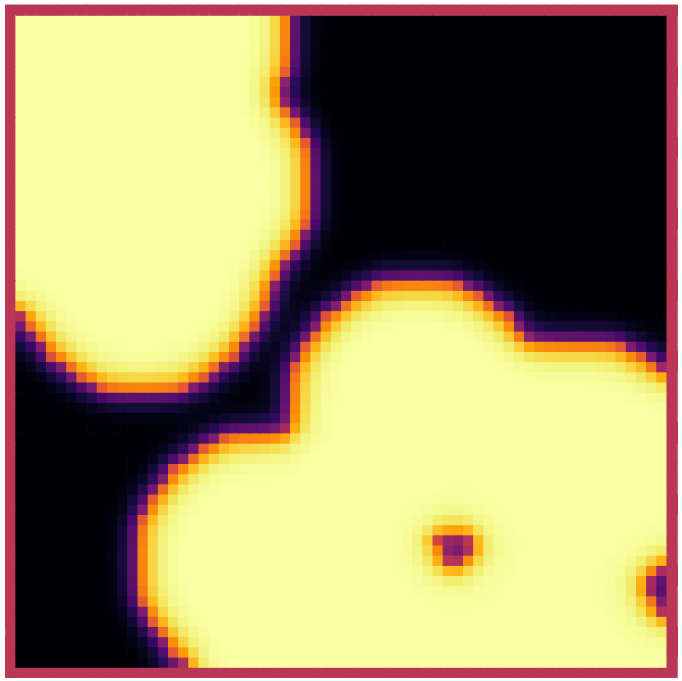
$$-\Delta c^\alpha + \alpha c^a = \alpha \phi^\alpha, \tag{5}$$

- requires solving an additional equation for c
- two dependant equations
- two one dimensional second order equations
- solved similar to the baseline equation

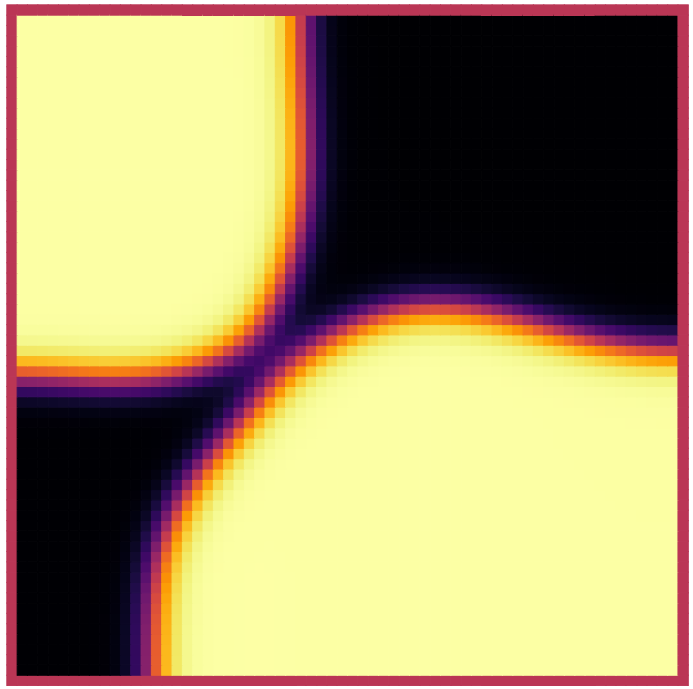
Numerical Experiments

Energy

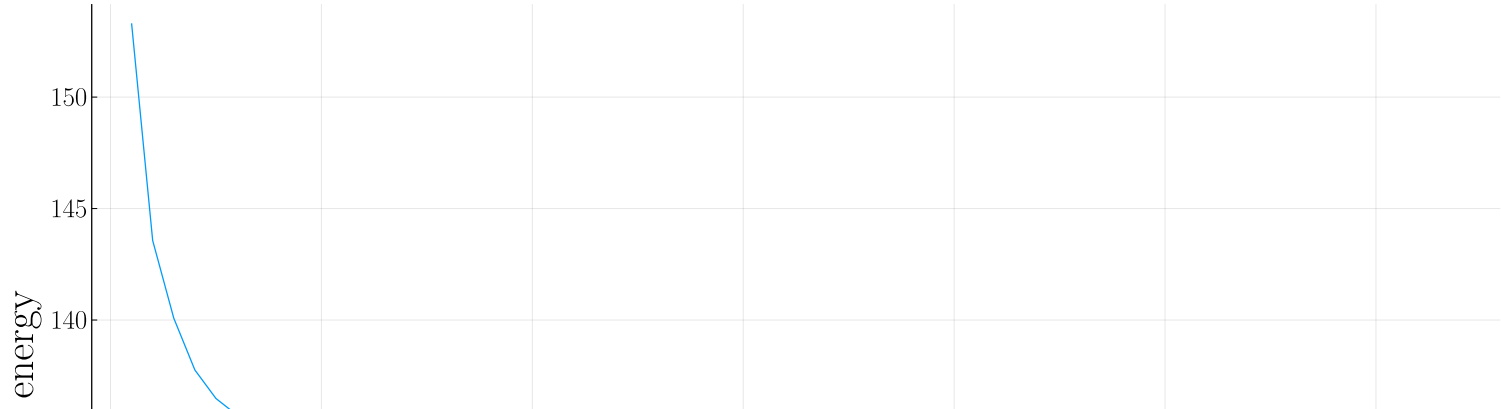
initial condition



after 64 time-steps



Discrete Helmholtz Energy E_d^{bulk}



Comparison

images/relaxed-comparison.gif
images/relaxed-anim.gif
images/iteration.gif

Conclusion