

Bachelor Thesis

Jonathan Ulmer

December 29, 2023

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1 Cahn Hillard Equation Overview

Partial Differential Equation (PDE) solving the state of a 2 Phase Fluid[Wu_2022].
The form of the Cahn Hillard Equation used for the remainder of this thesis

is:

$$\phi_t(x, t) = \Delta\mu \quad (1)$$

$$\mu = -\varepsilon^2 \Delta\phi + W'(\phi) \quad (2)$$

where ϕ is the so-called phase field. Demarking the different states of the fluids through an Interval $I = [-1, 1]$ and where $\partial I = \{-1, 1\}$ represents full state of one fluid. $\varepsilon > 0$ is a positive constant

, and μ is the chemical potential[Wu_2022]. While the Cahn Hillard exist in a more general form taking the fluid's mobility $M(\Phi)$ into account, we will assume $M(\Phi) = 1$, simplifying the CH-Equations used in[Wu_2022][SHIN20117441] to what is stated above.

The Advantages of the Cahn Hillard Approach as compared to traditional fluid dynamics solvers are for example: “explicit tracking of the interface” [Wu_2022], as well as “evolution of complex geometries and topological changes [...] in a natural way” [Wu_2022]

1.1 TODO Derivation from paper

1.1.1 Free energy

The Cahn Hillard Equations can be motivated Using a **Ginzburg Landau** type free energy equation:

$$E^{\text{bulk}} = \int_{\Omega} \frac{\varepsilon^2}{2} |\nabla\phi|^2 + W(\phi) dx$$

where $W(\phi)$ denotes the (Helmholtz) free energy density of mixing.^{“a”} [Wu_2022] and will be approximated in further calculations as $W(\phi) = \frac{(1-\phi^2)^2}{4}$ as used in[SHIN20117441]

The chemical potential then follows as derivative of Energy in respect to time.

$$\mu = \frac{\delta E_{\text{bulk}}(\phi)}{\delta\phi} = -\varepsilon^2 \Delta\phi + W'(\phi)$$

1.1.2 Derivation by mass balance

The Cahn Hillard equation then can be motivated as follows: consider

$$\partial_t\phi + \nabla J = 0 \quad (3)$$

where \mathbf{J} is mass flux. eq:massbal then states that the change in mass balances the change of the phasefield. Using the no-flux boundry conditions:

$$J \cdot n = 0 \quad \partial\Omega \times (0, T) \quad (4)$$

$$\partial_n \phi = 0 \quad \partial\Omega \times (0, T) \quad (5)$$

conservation of mass follows see[Wu_2022].

Using:

$$J = -\nabla\mu \quad (6)$$

which conceptionally sets mass flux to equalize the potential energy gradient, leads to the formulation of the CH equations as stated above. Additionally the boundary conditions evaluate to:

$$\begin{aligned} -\nabla\mu &= 0 \\ \partial_n \phi &= 0 \end{aligned}$$

ie no flow leaves and potential on the border doesn't change. then for ϕ then follows:

$$\begin{aligned} \frac{d}{dt} E^{bulk}(\phi(t)) &= \int_{\Omega} (\varepsilon^2 \nabla \phi \cdot \nabla \partial_t \phi + W'(\phi) \partial_t \phi) \, dx \\ &= - \int_{\Omega} |\nabla \mu|^2 \, dx, \quad \forall t \in (0, T) \end{aligned}$$

hence the Free Energy is decreasing in time.

2 Baseline Multigrid solver:

As baseline for further experiments a multi grid method based on finite differences by[SHIN20117441]. is used.

2.1 Discretization:

it discretizes the phasefield and potential energy ϕ, μ into a grid wise functions ϕ_{ij}, μ_{ij} and defines the partial derivatives $D_x f_{ij}, D_y f_{ij}$ using the differential quotients:

$$D_x f_{i+\frac{1}{2}j} = \frac{f_{i+1j} - f_{ij}}{h} \quad D_y f_{ij+\frac{1}{2}} = \frac{f_{ij+1} - f_{ij}}{h} \quad (7)$$

for $\nabla f, \Delta f$ then follows:

$$\begin{aligned}\nabla_d f_{ij} &= (D_x f_{i+1j}, D_y f_{ij+1}) \\ \Delta_d f_{ij} &= \frac{D_x f_{i+\frac{1}{2}j} - D_x f_{i-\frac{1}{2}j} + D_y f_{ij+\frac{1}{2}} - D_y f_{ij-\frac{1}{2}}}{h} = \nabla_d \cdot \nabla_d f_{ij}\end{aligned}$$

the authors further adapt the discretized phasefield by the characteristic function of the domain Ω :

$$G(x, y) = \begin{cases} 1 & (x, y) \in \Omega \\ 0 & (x, y) \notin \Omega \end{cases}$$

To account for boundry conditions and arbitrary shaped domains. The authors [SHIN20117441] then define the discrete CH Equation adapted for Domain, as:

$$\begin{aligned}\frac{\phi_{i+1j} - \phi_{ij}}{\Delta t} &= \nabla_d \cdot (G_{ij} \nabla_d \mu_{ij}^{n+1}) \\ \mu_{ij}^{n+1} &= 2\phi_{ij}^{n+1} - \varepsilon^2 \nabla_d \cdot (G_{ij} \nabla_d \phi_{ij}^{n+1}) + W'(\phi_{ij}^n) - 2\phi_{ij}^n\end{aligned}$$

and derive the iteration operator $L(\phi^{n+1}, \mu^{n+\frac{1}{2}}) = (\zeta^n, \psi^n)$

$$L \begin{pmatrix} \phi^{n+1} \\ \mu^{n+\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} \frac{\phi^{n+1}}{\Delta t} - \nabla_d \cdot (G_{ij} \nabla_d \mu^{n+\frac{1}{2}}) \\ \varepsilon^2 \nabla_d \cdot (G_{ij} \nabla_d \phi_{ij}^{n+1}) - 2\phi_{ij}^{n+1} + \mu_{ij}^{n+\frac{1}{2}} \end{pmatrix}$$

initialized as

$$(\zeta^n, \psi^n) = \begin{pmatrix} \frac{\phi_{ij}^{n+1}}{\Delta t} \\ W'(\phi_{ij}^n) - 2\phi_{ij}^n \end{pmatrix}$$

the algorithm is then defined as:

Wherein SMOOTH consists of point-wise Gauß Seidel Relaxation , by solving L for $\bar{\phi}, \bar{\mu}$ with the initial guess for ζ^n, ψ^n .

2.2 adaptations to the simplified problem

even tough this work uses rectangular domains, the adaptation of the algorithm is simplified by the domain indicator function, as well as 0 padding, in order to correctly include the boundary conditions of the CH equation. Therefore the internal representation of the adapted algorithm considers phasefield and potential field ϕ, μ as 2D arrays of shape $(N_X + 2, N_Y + 2)$ in order to accommodate padding. Where N_X and N_Y are the number of

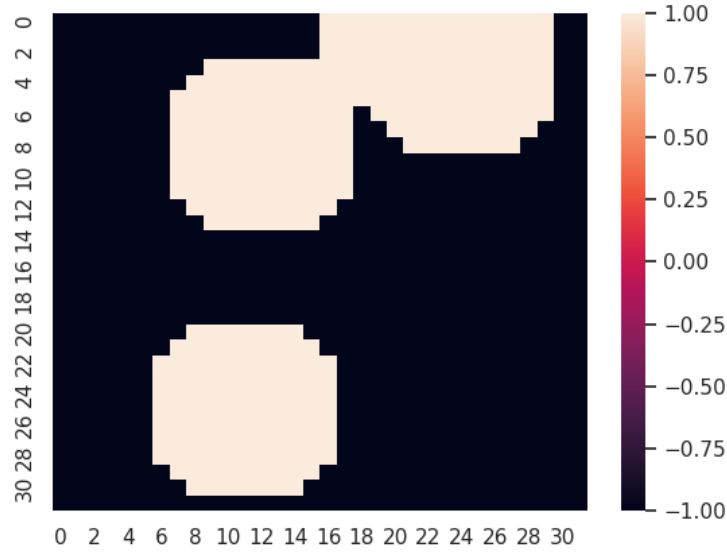
steps in x- / y-Direction respectively. Hence, we define the discrete domain function as:

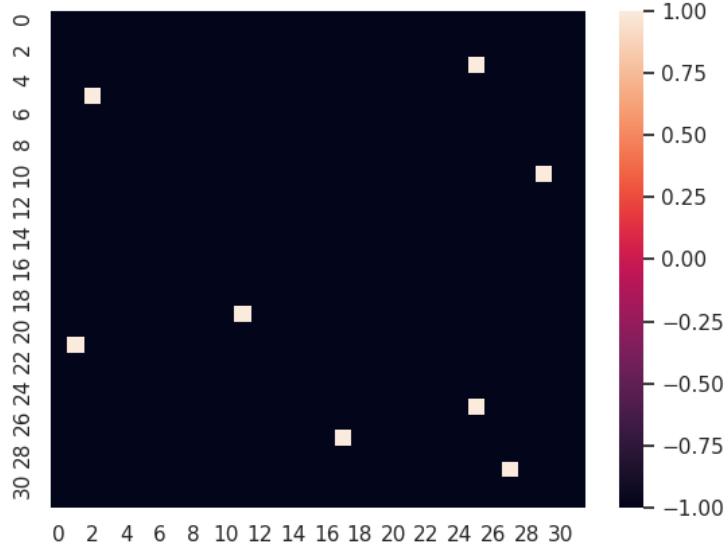
$$G_{ij} = \begin{cases} 1 & (i, j) \in [1, N_x + 1] \times [1, N_y + 1] \\ 0 & \text{else} \end{cases}$$

2.3 tests_{data}:

2.3.1 squares

For testing and later training , a multitude o different phasefields where used. notably an assortment of randomly placed circles, squares, and arbitrary generated values





2.4 Tests

```

1 test_phase = tu.k_spheres_phase(4,7)
2 solver = tu.setup_solver(test_phase)
3 solver.solve(4,10)

```

```

1 plt.figure()
2 sns.heatmap(solver.phase_small)

```

3 Relaxed Problem

In effort to decrease the order of complexity, the following relaxation to the classical CH Equation is proposed:

$$\begin{aligned}
 \partial_t \phi &= \Delta \mu \\
 \mu &= \varepsilon^2 (c^\alpha - \phi^\alpha) + W'(\phi)
 \end{aligned}$$

that in turn requires an additional PDE to be solved each time-step to calculate c . here c is the solution of the following elliptical PDE

$$-\Delta c^\alpha + \alpha c^a = \alpha \phi^\alpha$$

3.1 TODO relaxed operators:

the multi-grid solver proposed earlier is then adapted to the relaxed Problem by replacing the differential operators by their discrete counterparts as defined in #TODO and expanding them

3.1.1 L Relaxed

for the reformulation of the iteration in terms of Operator L then follows:

$$L \begin{pmatrix} \phi^{n+1} \\ \mu^{n+1} \end{pmatrix} = \begin{pmatrix} \frac{\phi_{ij}^{n+1,m}}{\Delta t} - \nabla_d \cdot (G_{ji} \nabla_d \mu_{ji}^{n+\frac{1}{2},m}) \\ \varepsilon^2 (c^\alpha - (\phi_{ij}^{n+1,m})^\alpha) - 2\phi_{ij}^{n+1,m} - \mu_{ji}^{n+\frac{1}{2},m} \end{pmatrix}$$

3.1.2 SMOOTH

and correspondingly the SMOOTH operation expands to:

$$SMOOTH(\phi_{ij}^{n+1,m}, \mu_{ji}^{n+\frac{1}{2},m}, L_h, \zeta^n, \psi^n)$$

$$\begin{aligned} \bar{\mu}_{ji}^{n+\frac{1}{2},m} &= \frac{\phi_{ij}^{n+1,m}}{\Delta t} - \zeta_{ij}^n \\ &\quad - \frac{1}{h^2} \left(G_{i+\frac{1}{2}j} \mu_{i+1j}^{n+\frac{1}{2},m} + G_{i-1j} \mu_{i-1j}^{n+\frac{1}{2},m} + G_{ij+1} \mu_{ij+1}^{n+\frac{1}{2},m} + G_{ij-1} \mu_{ij-1}^{n+\frac{1}{2},m} \right) \\ &\quad \cdot (G_{i+1j} + G_{i-1j} + G_{ij+1} + G_{ij-1})^{-1} \\ \varepsilon^2 (\bar{\phi}_{ij}^{n+1,m})^\alpha + 2\phi_{ij}^{n+1,m} &= \varepsilon^2 c^\alpha - \mu_{ji}^{n+\frac{1}{2},m} - \psi_{ij} \end{aligned}$$

1. Proposal Since the resulting system no longer is linear, (albeit simpler in Dimension), we propose a newton method to solve second equation (in conjunction with the first one) hopefully solving this converges faster than the original multiple SMOOTH Iterations. The iteration

solves for $\phi_{ij}^{n+1,m} = x$ as free variable. Therefore it follows for $F(x)$

$$\begin{aligned} F(x) &= \varepsilon^2 x^\alpha + 2x - \varepsilon^2 c^\alpha + y + \psi_{ij} \\ y &= \frac{x}{\Delta t} - \zeta_{ij}^n \\ &\quad - \frac{1}{h^2} \left(G_{i+\frac{1}{2}j} \mu_{i+1j}^{n+\frac{1}{2},m} + G_{i-1j} \mu_{i-1j}^{n+\frac{1}{2},m} + G_{ij+1} \mu_{ij+1}^{n+\frac{1}{2},m} + G_{ij-1} \mu_{ij-1}^{n+\frac{1}{2},m} \right) \\ &\quad \cdot (G_{i+1j} + G_{i-1j} + G_{ij+1} + G_{ij-1})^{-1} \end{aligned}$$

And the derivative for the iteration is

$$\begin{aligned} \frac{d}{dx} F(x) &= \alpha \varepsilon^2 x^{\alpha-1} + 2 + \frac{d}{dx} y \\ \frac{d}{dx} y &= \frac{1}{\Delta t} \end{aligned}$$

3.2 Elliptical PDE:

on order to solve the relaxed CH Equation the following PDE as to be solved in Each additional time step: or in terms of the characteristic function:

$$-\nabla \cdot (G \nabla c^\alpha) + \alpha c^\alpha = \alpha \phi^\alpha$$

In a similar manner to the first solver this PDE is solved with a finite difference scheme using the same discretisations as bevore:

3.2.1 Discretization

the Discretization of the PDE expands the differential operators in the same way

$$\begin{aligned} &-\nabla_d \cdot (G_{ij} \nabla_d c_{ij}^\alpha) + \alpha c_{ij}^\alpha = \alpha \phi_{ij}^\alpha \\ &-\left(\frac{1}{h} (G_{i+\frac{1}{2}j} \nabla c_{i+\frac{1}{2}j}^\alpha + G_{ij+\frac{1}{2}} \nabla c_{ij+\frac{1}{2}}^\alpha) \right. \\ &\quad \left. - (G_{i-\frac{1}{2}j} \nabla c_{i-\frac{1}{2}j}^\alpha + G_{ij-\frac{1}{2}} \nabla c_{ij-\frac{1}{2}}^\alpha) \right) + \alpha c_{ij}^\alpha = \alpha \phi_{ij}^\alpha \\ &\quad - \frac{1}{h^2} (G_{i+\frac{1}{2}j} (c_{i+1j}^\alpha - c_{ij}^\alpha) \\ &\quad \quad + G_{ij+\frac{1}{2}} (c_{ij+1}^\alpha - c_{ij}^\alpha) \\ &\quad \quad + G_{i-\frac{1}{2}j} (c_{i-1j}^\alpha - c_{ij}^\alpha) \\ &\quad \quad + G_{ij-\frac{1}{2}} (c_{ij-1}^\alpha - c_{ij}^\alpha)) + \alpha c_{ij}^\alpha = \alpha \phi_{ij}^\alpha \end{aligned}$$

proposed simple solver: Let F, dF be:

$$\begin{aligned}
F(x) = & -\frac{1}{h^2} (\\
& G_{i+\frac{1}{2}j} c_{i+1j}^\alpha + G_{i-\frac{1}{2}j} c_{i-1j}^\alpha \\
& + G_{ij+\frac{1}{2}} c_{ij+1}^\alpha + G_{ij-\frac{1}{2}} c_{ij-1}^\alpha) \\
& + \frac{1}{h^2} (G_{i+\frac{1}{2}j} + G_{i-\frac{1}{2}j} + G_{ij+\frac{1}{2}} + G_{ij-\frac{1}{2}}) x^\alpha \\
& + \alpha x^\alpha - \alpha \phi_{ij}^\alpha
\end{aligned}$$

and $dF(x)$

$$dF(x) = - \cdot (G_{i+\frac{1}{2}j} + G_{i-\frac{1}{2}j} + G_{ij+\frac{1}{2}} + G_{ij-\frac{1}{2}}) \alpha x^{\alpha-1} + \alpha^2 x^{\alpha-1}$$

solve equation for c_{ij}^α and then iteratively update the entire phasefield using the already updated values:

4 References