Kernel Collocation Excercise

Jonathan Ulmer (3545737)

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1 Projection Approach

Aim of this excrcise is to find solutions $u \in \mathcal{H}_k$ such that they satisfy the following system

$$
\begin{aligned}
-\nabla \cdot (a(x)\nabla u(x)) &= f(x) && \text{in} \quad \Omega && (1)\\
u(x) &= g_D(x) && \text{on} \quad \Gamma_D && (2)\\
(a(x)\nabla u(x)) \cdot \vec{n}(x) &= g_N && \text{on} \quad \Gamma_N && (3)
\end{aligned}
$$

we do this by projecting the system onto $\mathcal{H}_k(\Omega)$

$$
\begin{aligned}
\langle -\nabla \cdot (a(x)\nabla u(x)), \phi \rangle &= \langle f(x), \phi \rangle && \text{in} \quad \Omega, \phi \in \mathcal{H}_k && (4)\\
\langle u(x), \phi \rangle &= \langle g_D(x), \phi \rangle && \text{on} \quad \Gamma_D && (5)\\
\langle (a(x)\nabla u(x)) \cdot \vec{n}(x), \phi \rangle &= \langle g_N, \phi \rangle && \text{on} \quad \Gamma_N && (6)
\end{aligned}
$$

Let $\hat{X} := \left\{x_j\right\}_{j=1}^{n} \subset \mathbb{R}^d$. Since $\{k(x_i, \cdot)\}_{i=1}^{n}$ is the basis of a finite subspace in $\mathcal{H}_k$ it also has to hold

$$
\begin{aligned}
\langle -\nabla \cdot (a(x)\nabla u(x)), k(x_i, \cdot) \rangle &= \langle f(x), k(x_i, \cdot) \rangle && \text{in} \quad \Omega, x_i \in X && (7)\\
\langle u(x), k(x_i, \cdot) \rangle &= \langle g_D(x), k(x_i, \cdot) \rangle && \text{on} \quad \Gamma_D && (8)\\
\langle (a(x)\nabla u(x)) \cdot \vec{n}(x), k(x_i, \cdot) \rangle &= \langle g_N, k(x_i, \cdot) \rangle && \text{on} \quad \Gamma_N && (9)
\end{aligned}
$$

We assuming $f, g_D, g_N(\cdot, \vec{n}) \in \mathcal{H}_k$ i.e. $\langle f, k(x_i, \cdot) \rangle = f(x_i)$ etc. We search for a finite approximation $u_h \approx u$ such that it satisfies (7) where

$$
u_h(x) = \sum_{j=1}^{n} a_j k(x_j, x) \tag{10}
$$

correspondingly we are able to directly compute

$$
\begin{aligned}
\nabla_x u_h(x) &= \sum_{j=1}^{n} a_j \nabla_x k(x_j, x)\\
-\nabla_x \cdot (a(x)\nabla_x u_h(x)) &= -\nabla_x a(x) \cdot \nabla_x u(x) - a(x)\Delta_x u(x)\\
&= -\sum_{j=1}^{n} a_j \left(\nabla_x a(x) \cdot \nabla_x k(x_j, x) + a(x)\Delta_x k(x_j, x)\right)
\end{aligned}
$$

this leads to the following Linear system

$$-\sum_{j=1}^{n} a_j \left(\nabla_{x_i} a(x_i) \cdot \nabla_{x_i} k(x_j, x_i) + a(x_i)\Delta_{x_i} k(x_j, x_i)\right) = f(x_i) \qquad x_i \in \Omega, x_i \in X \tag{11}$$

$$\sum_{j=1}^{n} a_j k(x_j, x_i) = g_D(x_i) \qquad x_i \in \Gamma_D \tag{12}$$

$$\sum_{j=1}^{n} a_j \left(a(x_i)\nabla_{x_i} k(x_j, x_i) \cdot n_i\right) = g_N(x_i, n_i) \qquad x_i \in \Gamma_N \tag{13}$$

this corresponds directly with the System Matrix K, that we compute in julia using a GPU compatmible kernel that employs element wise notation

```
@kernel function system_matrix!(K ,@Const(X), a , ∇a ,k, ∇k, Δk  , sdf , grad_sdf
↪  , sdf_beta)
    Iᵢⱼ =  @index(Global , Cartesian)
    @inbounds xᵢ= SVector{2}(view(X, : , Iᵢⱼ[1])) # Essentially X[:,i]
    @inbounds xⱼ= SVector{2}(view(X, : , Iᵢⱼ[2])) # Essentially X[:,j]
    # poisson equation
    @inbounds K[Iᵢⱼ] = -a(xᵢ)*Δk(xⱼ ,xᵢ)- ∇a(xᵢ)⋅∇k(xⱼ ,xᵢ)
    if abs(sdf(xᵢ)) < 1e-10
        if sdf_beta(xᵢ) < 0
            # Neumann Boundary Condition
            @inbounds nᵢ= grad_sdf(xᵢ)
            @inbounds K[Iᵢⱼ] = a(xᵢ) * (nᵢ ⋅ ∇k( xⱼ , xᵢ ))
        else
          # Dirichlet Boundary
          @inbounds K[Iᵢⱼ] =k(xᵢ , xⱼ)
        end
    end
end
```

`julia-async:50c35846-ece5-4961-8a3a-800152c4bb4f`

1.1 right hand side

The right hand side of the system is computed in a similar Fashion

```
@kernel function apply_function_colwise!(B ,@Const(X) , f , g_D , g_N , sdf  ,
↪ grad_sdf, sdf_beta)
    # boilerplate
    Iᵢ = @index(Global , Cartesian)
    @inbounds xᵢ= SVector{2}(view(X , : , Iᵢ[1]))
    # poisson equation

    @inbounds B[Iᵢ] = f(xᵢ)
    if abs(sdf(xᵢ)) < 1e-10
         if sdf_beta(xᵢ) < 0
             # Neumann Boundary Condition
             @inbounds nᵢ= grad_sdf(xᵢ)
             @inbounds B[Iᵢ] = g_N(xᵢ , nᵢ )
         else
            # Dirichlet Boundary
            @inbounds B[Iᵢ] = g_D(xᵢ)
         end
     end
end
```

1.2 Kernel Matrix for evaluation

```
@kernel function kernel_matrix!(A , @Const(X_test) , @Const(X_col) ,k , sdf)
    Iᵢⱼ =  @index(Global , Cartesian)
    @inbounds xᵢ= SVector{2}(view(X_test , : , Iᵢⱼ[1])) # Essentially X[:,1]
    @inbounds xⱼ= SVector{2}(view(X_col , : , Iᵢⱼ[2]))
    K = k(xᵢ , xⱼ)
    # Mask outside Domain with NaN
    if sdf(xᵢ) > 0
        K = NaN
        end
    @inbounds A[Iᵢⱼ] = K
end
```

2 Solver

Our implementation Provides some structs for convenience. A `PDESystem` that stores the Data functions of a diffusion PDE together with the kernel and its derrivatives $k, \nabla k, \Delta k$ and information about the Domain of the problem in form if its signed distance function and Its gradient (for normals at the boundary)

```
struct PDESystem
    k :: Function
    ∇k :: Function
    Δk :: Function
    a :: Function
    ∇a::Function
    f::Function
    g_D::Function
    g_N::Function
    sdf::Function
    grad_sdf::Function
    sdf_beta::Function
end
```

We provide a `PDESolver` that stores The PDE system, Collocation points $\hat{X}$, and the solution vector α in $u_h = \sum_{j=0}^{n} \alpha_j k(x_j, \cdot)$

```
struct PDESolver
    S::PDESystem
    X::AbstractMatrix
    α :: AbstractVector
end
```

`PDESolver` Provides methods for evaluation itselve on a test dataset

```
function (f::PDESolver)(X)
    dev = get_backend(X)
    print("Backend" , dev)
    K = KernelAbstractions.zeros(dev , Float32, size(X,2)  , size(f.X ,2))
    print("Size of the system Matrix:" , size(K))
    km! = kernel_matrix!( dev , 256 , size(K))
    km!(K, X , f.X , f.S.k , f.S.sdf )
return K * f.α , K
end
```

As well as a method to solve the approximation system and return a instance of `PDESystem`

```
function solve(S, X_col)
    dev = get_backend(X_col)
    K = KernelAbstractions.zeros(dev , Float32 , size(X_col , 2) , size(X_col ,
    ↪  2) )
    sys_matrix! = system_matrix!( dev , 256 , size(K))
```

```
    sys_matrix!(K ,X_col , S.a , S.∇a , S.k , S.∇k , S.Δk , S.sdf , S.grad_sdf ,
    ↪  S.sdf_beta  )
    B = get_boundary(S,X_col)
    α = K \ B
    return (PDESolver(S,X_col ,α) , K)
    end
```

```
function get_boundary(
    S,
    X
    )
    dev = get_backend(X)
    B = KernelAbstractions.zeros(dev , Float32 , size(X , 2))
    apply! = apply_function_colwise!(dev , 256 , size(B))
    apply!(B , X , S.f , S.g_D , S.g_N , S.sdf  , S.grad_sdf, S.sdf_beta)
    return B
    end
```

```
end
```

we tested our solver for compatibility with a CPU and A GPU Backend on a modern 16Core CPU and a RTX 4070. The GPU backend was ~8x faster for a system with 10000 DOF. (~120ms vs ~1.s)

3 Kernel Implementation

As kernels we use Radial Basis Kernels (RBF) $k(x, x') := \phi(\frac{\|x-x'\|}{\gamma})$. That consist of a radial basis function ϕ as well as a scaling factor γ where ∇_x, Δ_x are the partial gradients and laplacians with respect to the second argument of $k(x_j, \cdot)$. for a radial basis function $\phi(r^2) \in C^2(\mathbb{R})$ and a corresponding RBF kernel they can be computed directly

$$\nabla_x k(x', x) = \phi'\left(\frac{\|x-x'\|}{\gamma}\right) \cdot \frac{x-x'}{\gamma\|x-x'\|} \tag{14}$$

$$\Delta_x k(x', x) = \frac{1}{\gamma^2}\phi''\left(\frac{\|x-x'\|}{\gamma}\right) + \frac{1}{\gamma}\frac{d-1}{\|x-x'\|} \cdot \phi'\left(\frac{\|x-x'\|}{\gamma}\right) \tag{15}$$

where d is the dimension of x. Note that those expressions are singular for $r = 0$. we prevent this numerically by using $\max(10^{-15}, r)$ instead of r.

```julia
using StaticArrays
@inline function k(ϕ::RBFType , ::Val{γ},x̂::SVector{N} ,x::SVector{N}) where {N ,
↪   γ , RBFType}
    r = max(1e-15,norm(x-x̂))
    ϕ(r/γ)
    end
@inline function ∇k(dϕ::dRBFType , ::Val{γ} ,x̂::SVector{N} ,x::SVector{N}) where
↪   {N , γ , dRBFType}
    r = max(1e-15,norm(x-x̂))
    (x-x̂)*dϕ(r/γ) *  1/(r*γ)
    end
@inline function Δk(d²ϕ::ddRBFType,  dϕ::dRBFType , ::Val{γ} ,x̂::SVector{N}
↪   ,x::SVector{N}) where {N , γ , ddRBFType , dRBFType}
    r = max(1e-15,norm(x-x̂))
    1/γ^2 * d²ϕ(r/γ)  +  1/γ * (N-1)/r *dϕ(r/γ)
    end
```

3.1 squared rbf

for a squared RBF the kernels $k(x,x') := \phi(\frac{\|x-x'\|^2}{\gamma})$ are simpler. and non singular

$$\nabla_x k(x',x) = \phi'\left(\frac{r^2}{\gamma}\right) \cdot \frac{x-x'}{\gamma} \tag{16}$$

$$\Delta_x k(x',x) = \frac{1}{\gamma}(4 * \frac{r^2}{\gamma^2}\phi''\left(\frac{r^2}{\gamma}\right) + 2d\phi'\left(\frac{r^2}{\gamma}\right)) \tag{17}$$

```julia
using StaticArrays
using LinearAlgebra
@inline function ksq(ϕ::RBFType , ::Val{γ},x̂::SVector{N} ,x::SVector{N}) where {N
↪   , γ , RBFType}
    r = dot(x-x̂,x-x̂)
    ϕ(r/γ)
    end
@inline function ∇ksq(dϕ::dRBFType , ::Val{γ} ,x̂::SVector{N} ,x::SVector{N})
↪   where {N , γ , dRBFType}
    r = dot(x-x̂,x-x̂)
    2/γ^2*(x-x̂)*dϕ(r/γ)
    end
@inline function Δksq(d²ϕ::ddRBFType,  dϕ::Function , ::Val{γ} ,x̂::SVector{N}
↪   ,x::SVector{N}) where {N , γ , ddRBFType , dRBFType}
    r = dot(x-x̂,x-x̂)
    1/γ^2 * 4*r * d²ϕ(r/γ)  + 2/γ*N*dϕ(r/γ)
    end
```

julia-async:70c08408-31c3-4adc-bc82-d8865b93b190

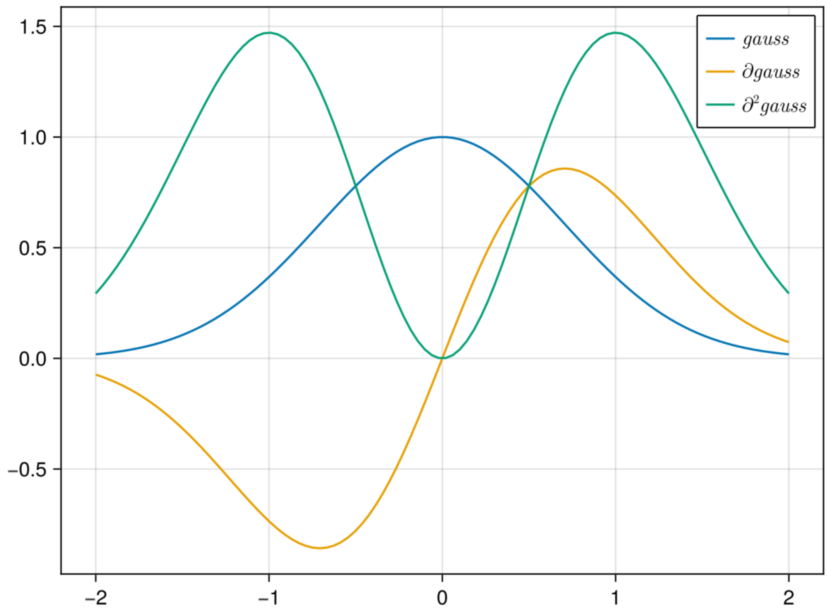
3.2 Gauss

```
using StaticArrays
@inline function rbf_gaussian(r)
    exp(-r)
    end
@inline function d_rbf_gaussian(r)
    -exp(-r)
    end
@inline function dd_rbf_gaussian(r)
    exp(-r)
    end
```

julia-async:f3f21d6e-71df-4891-8781-f0dd47c6dd10

```
using GLMakie
X = range(-2 , 2 , 100)
using LinearAlgebra

fig = Figure()
ax = Axis(fig[1,1])
lines!(ax , X ,x->    rbf_gaussian(x^2), label=L"gauss")
lines!(ax , X ,x-> -2x* d_rbf_gaussian(x^2) , label=L"\partial gauss")
lines!(ax , X ,x-> 4x^2* dd_rbf_gaussian(x^2) , label=L"\partial^2 gauss")
axislegend(ax)
save("images/gauss-rbf.png",fig )
```



3.3 Cardinal B_3 Spline

$$B_d(r) = \sum_{n=0}^{4} \frac{(-1)^n}{d!} \binom{d+1}{n} \left(r + \frac{d+1}{2} - n\right)_+^d$$

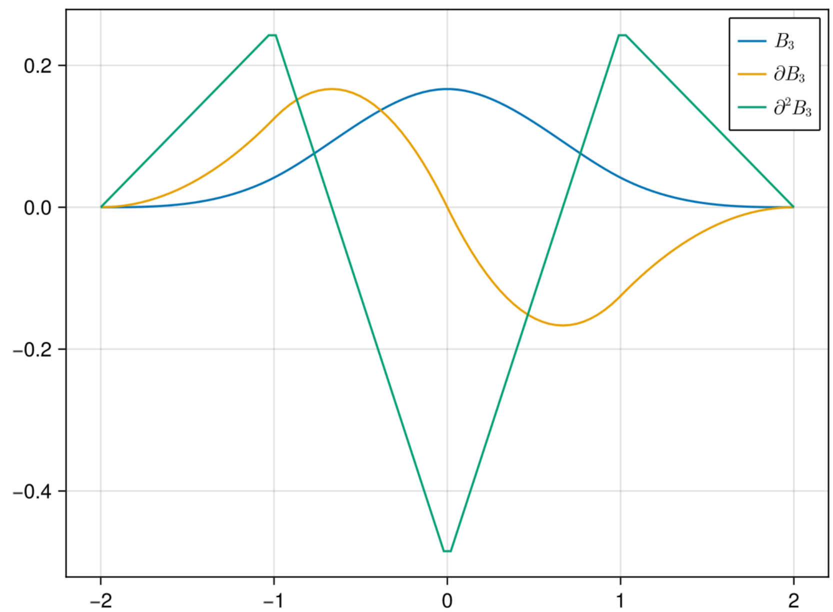
```
function B_3(r)
r_prime = r+2
    return 1/24 * (
    1 *max(0, (r_prime - 0))^3
    -4*max(0, (r_prime - 1))^3
    +6*max(0, (r_prime - 2))^3
    -4*max(0, (r_prime - 3))^3
    +1*max(0, (r_prime - 4))^3
    )
end
function d_B_3(r)
r_prime = r+2
    return 1/8 * (
    1 *max(0, (r_prime - 0))^2
    -4*max(0, (r_prime - 1))^2
    +6*max(0, (r_prime - 2))^2
    -4*max(0, (r_prime - 3))^2
    +1*max(0, (r_prime - 4))^2
    )
```

```
end
function dd_B_3(r)
r_prime = r+2
    return 1/4 * (
    1 *max(0, (r_prime - 0))
    -4*max(0, (r_prime - 1))
    +6*max(0, (r_prime - 2))
    -4*max(0, (r_prime - 3))
    +1*max(0, (r_prime - 4))
    )
end
```

```
using GLMakie
using LaTeXStrings
X = range(-2 , 2 , 100)
Y = range(-2 , 2 , 100)

fig = Figure()
ax = Axis(fig[1,1])

lines!(ax , X , B_3 , label=L"B_3")
lines!(ax , X , d_B_3 , label=L"\partial B_3")
lines!(ax , X , dd_B_3 , label=L"\partial^2 B_3")
axislegend(ax)
save("images/b-spline.png",fig )
```



3.4 Thin Plate

$$T(r^2) = \frac{1}{2} r \ln r$$
$$T(r) = r^2 \ln r$$

note that the thin plate kernel is not p.d. it is c.p.d. However this seemed a non issue numerically

```
function thin_plate(r)
    r == 0.0 && return 0.0
    return 0.5* r *  log(r)
end

function d_thin_plate(r)
    r == 0.0 && return 0.0
    return 0.5 * log(r) + 1
end

function dd_thin_plate(r)
    r == 0.0 && return 0.0
    return 0.5 * 1/r
end
```

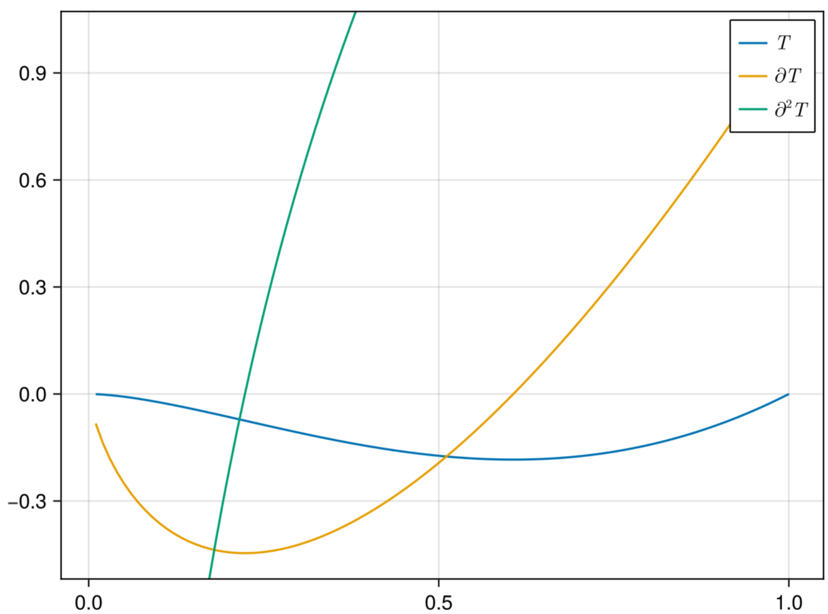
```
dd_thin_plate (generic function with 1 method)
```

```
using GLMakie
X = range(0 , 1 , 100)
Y = range(-5 , 5 , 100)

fig = Figure()
ax = Axis(fig[1,1])

lines!(ax , X , x-> x^2 * log(x),  label=L"T")
lines!(ax , X , x-> 2x * log(x) + x , label=L"\partial T")
lines!(ax , X , x-> 2log(x) + 3 , label=L"\partial^2 T")
axislegend(ax)

save("images/plate-spline.png",fig )
```



4 PDE

To use our PDE solver we include all our modules

```
using Revise
using LinearAlgebra
includet("src/pdesolver.jl")
includet("src/domains.jl")
```

```julia
includet("src/rbf.jl")
using .PDESolvers
using .Domains
using .RadialBasisKernels
```

and generate a set of collocation and test points. If a functional CUDA GPU is available, we move the data to the GPU. The solver will then attempt so solve on the GPU. Anoyingly all functions have to be known at compile time, when using the GPU backend.

```julia
using CUDA
dev = CUDA.functional() ? cu : Array
#dev = Array
X = range(0 , 1 , 20)
Y = range(0 , 1 , 20)
X_col = [ [x,y] for x in X , y in Y]
X_col = reduce(vcat ,X_col )
X_col = reshape(X_col, 2,:) |> dev
X_t = range(0 , 1 , 100)
Y_t = range(0 , 1 , 100)
X_test = [ [x,y] for x in X_t , y in Y_t]
X_test = reduce(vcat , X_test)
X_test = reshape(X_test, 2,:) |> dev
X_lol = rand(2,400) |> dev


size(X_col)
```

```
(2 400)
```

4.1 PDE Poisson

with $a(x) = 1, g_D(x) = 0$ and $\Gamma_N = \emptyset$ this method is able to model the poisson equation

$$-\Delta u(x) = f(x) \qquad \text{in} \quad \Omega \tag{18}$$
$$u(x) = 0 \qquad \text{on} \quad \Gamma_D \tag{19}$$

```julia
using StaticArrays
function domain(x::SVector{2})
    return sdf_square(x , 0.5 , SVector(0.5,0.5))
end
function ∇domain(x::SVector{2})
    return sdf_square_grad(x , 0.5 , SVector(0.5,0.5))
end
function sdf_β(x::SVector{2})
```

```
    return sdf_square(x , 0. , SVector(-1.,-1) )
end

a(x::SVector{2}) = 1
∇a(x::SVector{2}) = SVector{2}(0.,0.)
f(x::SVector{2}) =2 * (x[1]+x[2] - x[1]^2 - x[2]^2)
g_D(x::SVector{2})= 0
g_N(x::SVector{2} , n::SVector{2}) = 0
```

4.1.1 Plotting Utility

```
using LaTeXStrings
function plot(fig , i,::Val{γ} , limits , errors, rbf , d_rbf , dd_rbf) where γ
        @inline k_rbf(x,y) = @inline k( rbf ,Val(γ), x,y)
        @inline ∇k_rbf(x,y) =@inline ∇k(d_rbf,Val(γ), x,y)
        @inline Δk_rbf(x,y) =@inline Δk(dd_rbf , d_rbf ,Val(γ), x,y)
        S_gauss = PDESystem(k_rbf , ∇k_rbf , Δk_rbf , a, ∇a , f, g_D ,g_N , domain
         ↪  , ∇domain , sdf_β )
        solution , K = solve(S_gauss ,X_col);
        ax = Axis(fig[1,i] , title=L"$\gamma=%$γ$ Condition %$(cond(K))",
         ↪  aspect=DataAspect())
        sol , K_t = solution(X_test)
        push!(errors , norm(Array(sol) - u.(eachcol(Array(X_test))) , Inf))
        sol = reshape(Array(sol) , size(X_t,1) , :)
        hm = heatmap!(ax , X,Y, sol , colorrange=limits)
        return fig
end
function plotsq(fig , i,::Val{γ} , limits , errors, rbf , d_rbf , dd_rbf) where γ
        @inline k_rbf(x,y) = @inline ksq( rbf ,Val(γ), x,y)
        @inline ∇k_rbf(x,y) =@inline ∇ksq(d_rbf,Val(γ), x,y)
        @inline Δk_rbf(x,y) =@inline Δksq(dd_rbf , d_rbf ,Val(γ), x,y)
        S_gauss = PDESystem(k_rbf , ∇k_rbf , Δk_rbf , a, ∇a , f, g_D ,g_N , domain
         ↪  , ∇domain , sdf_β )
        solution , K = solve(S_gauss ,X_col);
        ax = Axis(fig[1,i] , title=L"$\gamma=%$γ$ Condition %$(cond(K))",
         ↪  aspect=DataAspect())
        sol , K_t = solution(X_test)
        push!(errors , norm(Array(sol) - u.(eachcol(Array(X_test))) , Inf))
        sol = reshape(Array(sol) , size(X_t,1) , :)
        hm = heatmap!(ax , X,Y, sol , colorrange=limits)
        return fig
end
```

```
plotsq (generic function with 1 method)
```

4.1.2 Results

```
using GLMakie
fig = Figure(size=(2600,400))
limits = (0, 0.06)
errors = Vector{Float32}()
u(x , y) = x * (1-x) * y* ( 1- y)
u(x) = u(x[1] , x[2])
for (i,gamma) in enumerate([Val(0.1) , Val(0.075) , Val(0.05) , Val(0.025)])
plotsq(fig , i,gamma , limits , errors , rbf_gaussian , d_rbf_gaussian ,
↪  dd_rbf_gaussian)
end
ax = Axis(fig[1,0] , title="Exact sollution" , aspect=DataAspect())
hm = heatmap!(ax,X_t,Y_t,u , colorrange=limits)
Colorbar(fig[:, end+1], hm)
ax = Axis(fig[1,end+1] , title=L"$L^\infty$ Error" , xlabel=L"\gamma" ,
↪  ylabel=L"|u_h - u |_\infty")
lines!(ax , [0.01 , 0.0075 , 0.005 , 0.0025] , errors)
save("images/gauss-kernel.png",fig )
```

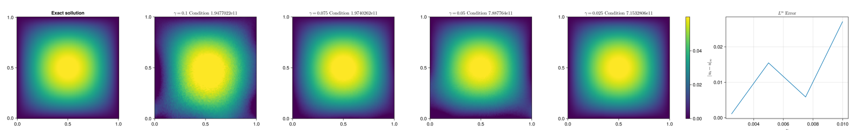
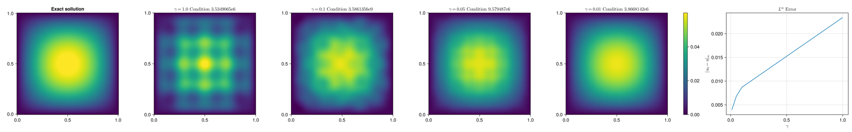
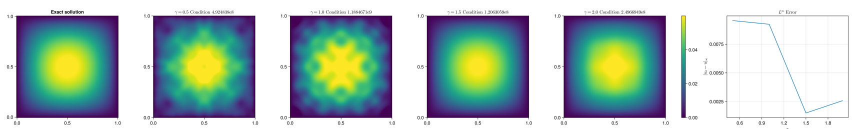


Figure 1: gauss kernel with various values for γ

```
using GLMakie
fig = Figure(size=(2600,400))
limits = (0, 0.06)
errors = Vector{Float32}()
for (i,gamma) in enumerate([Val(1.0) , Val(0.1) , Val(0.05) , Val(0.01)])
plotsq(fig , i,gamma , limits , errors , thin_plate , d_thin_plate ,
↪  dd_thin_plate)
end
u(x , y) = x * (1-x) * y* ( 1- y)
u(x) = u(x[1] , x[2])
ax = Axis(fig[1,0] , title="Exact sollution" , aspect=DataAspect())
hm = heatmap!(ax,X_t,Y_t,u , colorrange=limits)
Colorbar(fig[:, end+1], hm)
ax = Axis(fig[1,end+1] , title=L"$L^\infty$ Error" , xlabel=L"\gamma" ,
↪  ylabel=L"|u_h - u |_\infty")
lines!(ax , [1.,0.1,0.05,0.01] , errors)
save("images/thin-plate-kernel.png",fig )
```



```julia
using GLMakie
fig = Figure(size=(2600,400))
limits = (0, 0.06)
errors = Vector{Float32}()
for (i,gamma) in enumerate([Val(0.5) , Val(1.0) , Val(1.5) , Val(2.0)])
plot(fig , i,gamma , limits , errors , B_3 , d_B_3 , dd_B_3)
end
u(x , y) = x * (1-x) * y* ( 1- y)
u(x) = u(x[1] , x[2])
ax = Axis(fig[1,0] , title="Exact sollution" , aspect=DataAspect())
hm = heatmap!(ax,X_t,Y_t,u , colorrange=limits)
Colorbar(fig[:, end+1], hm)
ax = Axis(fig[1,end+1] , title=L"$L^\infty$ Error" , xlabel=L"\gamma" ,
    ylabel=L"|u_h - u |_\infty")
lines!(ax , [0.5,1.,1.5,2.] , errors)
save("images/B3-spline-kernel.png",fig )
```



4.2 Diffusion PDE

we evaluate the diffusion PDE with

$$
\begin{aligned}
a(x) &= x_1 + 2 \\
f(x) &= -\alpha \|x\|^{\alpha-2} * (3x_1 + 4) - \alpha * (\alpha - 2) * (x_1 + 2) * \|x\|^{\alpha-3} \\
g_D(x) &= \|x\|^{\alpha} \\
g_N(x, n) &= \alpha \|x\|^{\alpha-2} * (x_1 + 2) * x \cdot n
\end{aligned}
$$

where

```julia
using StaticArrays
a(x::SVector{2}) = x[1] + 2
∇a(x::SVector{2}) = SVector{2}(1.,0.)
α = 0.5
β = 0.2
f(x::SVector{2} , ::Val{α}) where α = - α*norm(x ,2)^(α - 2)*(3x[1] +4) - α*(α
    -2) * (x[1] + 2) * norm(x,2)^(α - 3)
```

```julia
g_D(x::SVector{2} , ::Val{α}) where α = norm(x,2)^α
g_N(x::SVector{2} , n::SVector{2} , ::Val{α}) where α = α*
↪  norm(x,2.)^(α-2.)*(x[1] +2.) * x ⋅ n
f(x) = f(x,Val(α))
g_D(x) = g_D(x,Val(α))
g_N(x, n) = g_N(x , n,Val(α))
function sdf_β(x::SVector{2})
    return sdf_square(x , β , SVector(-1.,-1) )
end
```

```
sdf_β (generic function with 1 method)
```

And select a collocation set filtered to be inside the domain

```julia
X = range(-1 , 1 , 11)
Y = range(-1 , 1 , 11)
X_col = [ [x,y] for x in X , y in Y]
X_col = reduce(vcat ,X_col )
X_col = reshape(X_col, 2,:)
X_col = filter(x -> sdf_L(SVector{2}(x)) <= 0 , eachcol(X_col))
X_col = reduce(hcat , X_col) |> dev
X_t = range(-1.1 , 1.1 , 100)
Y_t = range(-1.1, 1.1 , 100)
X_test = [ [x,y] for x in X_t , y in Y_t]
X_test = reduce(vcat , X_test)
X_test = reshape(X_test, 2,:) |> dev
size(X_col)
```

```
(2 96)
```

4.2.1 Plotting utility

```julia
using LaTeXStrings
function plotdiff(fig , i,::Val{γ} , limits ,  rbf , d_rbf , dd_rbf) where γ
        @inline k_rbf(x,y) = @inline k( rbf ,Val(γ), x,y)
        @inline ∇k_rbf(x,y) =@inline ∇k(d_rbf,Val(γ), x,y)
        @inline Δk_rbf(x,y) =@inline Δk(dd_rbf , d_rbf ,Val(γ), x,y)
        S_gauss = PDESystem(k_rbf , ∇k_rbf , Δk_rbf , a, ∇a , f, g_D ,g_N , sdf_L
        ↪  , sdf_L_grad , sdf_β )
        solution , K = solve(S_gauss ,X_col);
        ax = Axis(fig[1,i] , title=L"$\gamma=%$γ$ Condition %$(cond(K))",
        ↪  aspect=DataAspect())
        sol , K_t = solution(X_test)
        push!(errors , norm(Array(sol) - u.(eachcol(Array(X_test))) , Inf))
        sol = reshape(Array(sol) , size(X_t,1) , :)
        hm = heatmap!(ax , X,Y, sol , colorrange=limits)
```

```
        return fig
end
function plotsqdiff(fig , i,::Val{γ} , limits , rbf , d_rbf , dd_rbf) where γ
        @inline k_rbf(x,y) = @inline ksq( rbf ,Val(γ), x,y)
        @inline ∇k_rbf(x,y) =@inline ∇ksq(d_rbf,Val(γ), x,y)
        @inline Δk_rbf(x,y) =@inline Δksq(dd_rbf , d_rbf ,Val(γ), x,y)
        S_gauss = PDESystem(k_rbf , ∇k_rbf , Δk_rbf , a, ∇a , f, g_D ,g_N , sdf_L
        ↪  , sdf_L_grad , sdf_β )
        solution , K = solve(S_gauss ,X_col);
        ax = Axis(fig[1,i] , title=L"$\gamma=%$γ$ Condition %$(cond(K))",
        ↪  aspect=DataAspect())
        sol , K_t = solution(X_test)
        push!(errors , norm(Array(sol) - u.(eachcol(Array(X_test))) , Inf))
        sol = reshape(Array(sol) , size(X_t,1) , :)
        hm = heatmap!(ax , X,Y, sol , colorrange=limits)
        return fig
end
```

```
plotsqdiff (generic function with 1 method)
```

4.2.2 Results

```
using GLMakie
fig = Figure(size=(1800,400))
limits = (-0.6, 0)
for (i,gamma) in enumerate([Val(0.1) , Val(0.075) , Val(0.05) , Val(0.025)])
plotsqdiff(fig , i,gamma , limits , rbf_gaussian , d_rbf_gaussian ,
↪  dd_rbf_gaussian)
end
Colorbar(fig[:, end+1], hm)
save("images/gauss-kernel-diff.png",fig )
```

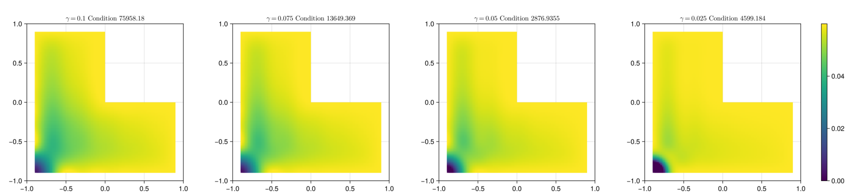


Figure 2: gauss kernel with various values for γ for a diffusive system

```
fig = Figure(size=(1800,400))
limits = (-0.6, 0)
for (i,gamma) in enumerate([Val(0.1) , Val(0.075) , Val(0.05) , Val(0.025)])
plotsqdiff(fig , i,gamma , limits , thin_plate , d_thin_plate , dd_thin_plate)
```

```
end
Colorbar(fig[:, end+1], hm)
save("images/thin-plate-kernel-diff.png",fig )
```

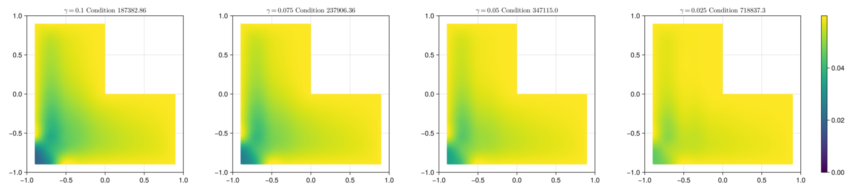


Figure 3: thin plate kernel with various values for γ for a diffusive system

```
using GLMakie
fig = Figure(size=(1800,400))
limits = (-0.6, 0)
for (i,gamma) in enumerate([Val(1.0) , Val(1.25) , Val(1.5) , Val(1.75)])
plotsqdiff(fig , i,gamma , limits , B_3 , d_B_3 , dd_B_3)
end
Colorbar(fig[:, end+1], hm)
save("images/B3-spline-kernel-diff.png",fig )
```

5 Domains

We define our domains using signed distance functions (SDF) and their gradients. The SDF of a unit square centered on $c \in \mathbb{R}^n$ is given by the L^∞ norm

$$\mathrm{sdf}(x) = \|x - c\|_\infty$$

The gradient was calculated analytically and imlpemented as:

```
function sdf_square(x::SVector , r::Float64 , center::SVector)
    return norm(x-center,Inf) .- r
end
function sdf_square_grad(x::SVector{2}, r::Float64, center::SVector{2})
    d = x - center
    if abs(d[1]) > abs(d[2])
        return SVector(sign(d[1]), 0.0)
    elseif abs(d[2]) > abs(d[1])
        return SVector(0.0, sign(d[2]))
    else
        # Subgradient: pick any valid direction; here we average the two
```

```
        return normalize(SVector(sign(d[1]), sign(d[2])))
    end
end
```

The L shaped Domain can be described by intersecting 2 square SDF centered on $c = (0, 0)$

$$sdf(\Omega_1 \ \Omega_2) = \max(sdf(\Omega_1), -sdf(\Omega_2))$$

```
function sdf_L(x::SVector{2})
    return max(sdf_square(x , 1. , SVector(0,0)) , - sdf_square(x, 1. ,
    ↪  SVector(1.,1.)))
end

function ∇sdf_L(x::SVector{2})
    ForwardDiff.gradient(sdf_L , x)
    return
end


function sdf_L_grad(x::SVector{2})
    f1 = sdf_square(x, 1.0, SVector(0.0, 0.0))
    f2 = -sdf_square(x, 1.0, SVector(1.0, 1.0))

    if f1 > f2
        return sdf_square_grad(x, 1.0, SVector(0.0, 0.0))
    elseif f2 > f1
        return -sdf_square_grad(x, 1.0, SVector(1.0, 1.0))  # negative because of
        ↪  the minus
    else
        # Subgradient — average of both directions
        g1 = sdf_square_grad(x, 1.0, SVector(0.0, 0.0))
        g2 = -sdf_square_grad(x, 1.0, SVector(1.0, 1.0))
        return normalize(g1 + g2)
    end
end
```