Multiscale Finite Volume Method

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Outline

Numeral Problem I

Abstract

in our work, we solve the diffusion equation (1) with zero Dirichlet boundary and constant flux $f(x) = 1$. In order to investigate the effects of the multiscale method on the solution, we introduce a varied sample of diffusivity terms

$$\nabla \cdot (D(x)\nabla c) = f(x) \qquad \text{in } \Omega \tag{1}$$

$$c(x) = 0 \qquad \text{on } \partial\Omega \tag{2}$$

Our work provides a way to approximate a solution c to the PDE (1) with the Finite Volume method, and a multiscale adaptation.

Derivation of the 1D Finite Volume Method I

The Finite Volume method considers the differential equation in Integral form over disjunct $(Q_i \cap Q_j = \emptyset, i \neq j)$ reference cells Q_i, $\bigcup_{i=1}^{N} Q_i = \Omega$ and calculates the integral over them, with an integral over the reference cell boundaries using Stokes integration.

$$\int_{Q_i} \nabla \cdot (D(x)\nabla c) = \int_{Q_i} f(x)\,\mathrm{d}x \qquad i = 1, \ldots, N \tag{3}$$

$$\int_{\partial Q_i} D(x)\nabla c \cdot \vec{n}\,\mathrm{d}S = \int_{Q_i} f(x)\,\mathrm{d}x \qquad i = 1, \ldots, N \tag{4}$$

The Finite Volume Method then considers the solution piecewise constant on Q. This creates discontinuities on the cell boundaries, where the values are not uniquely defined. The Finite Volume method therefore introduces a numerical flux in the Ansatz and solves the integral over the flux instead. Since the assumed solution is constant we approximate the source term

Derivation of the 1D Finite Volume Method II

$f(\vec{x})$ with its value on the cell center x_i of Q_i and calculate the integrals directly.

$$\int_{\partial Q_i} g(c^+, c^-) \cdot \vec{n} \,\mathrm{d}S = \int_{Q_i} f(x) \,\mathrm{d}x \qquad i = 1, \ldots, N \tag{5}$$

$$\int_{\partial Q_i} g(c^+, c^-) \cdot \vec{n} \,\mathrm{d}S = |Q_i| f(x_i) \qquad i = 1, \ldots, N \tag{6}$$

Numerical flux approximation I

1D Flux

We employ the flux approximation introduced in the MMM Lecture Since we only investigated diffusion terms with an analytical representation, we are able to calculate this value directly.

$$g(c^+, c^-) = -D(x^{\frac{1}{2}+})\frac{c^+ - c^-}{h} \tag{7}$$

Furthermore, we introduce transmissitivities $T_\pm$ between both cells.

$$g(c^+, c^-) = T_\pm * \left(c^+ - c^-\right)$$
$$T_\pm = -D(x^{\frac{1}{2}+})\frac{1}{h}$$

Numerical flux approximation II

2D Flux

We define the flux term in 2 Dimensions very similar to those in one dimension.

$$g_x(c_{i+1,j}, c_{ij}) = -\Delta_y D(x_{i+\frac{1}{2},j}) \frac{c_{i+1,j} - c_{ij}}{\Delta_x} \tag{8}$$

$$g_y(c_{i,j+1}, c_{ij}) = -\Delta_x D(x_{i,j+\frac{1}{2}}) \frac{c_{i,j+1} - c_{ij}}{\Delta_y} \tag{9}$$

and in the same manner we introduce 2D transmissions T^x_{i+1j}, T^y_{ij+1}

$$g_x(c_{i+1j}, c_{ij}) = T^x_{i+1j}\,(c_{i+1j} - c_{ij})$$
$$g_y(c_{ij+1}, c_{ij}) = T^y_{ij+1}\,(c_{i+1j} - c_{ij})$$

Linearization I

We implemented our finite Volume solver on a rectangular grid. therefore the normals on the boundaries are constant, and the flux integral (6) simplifies to a sum

$$\sum_{n \in \partial Q} \vec{g}(c_{i+j+1}, c_{i+j}) \cdot \vec{n} = |Q_i| \overline{f}(x_i)$$

1D

- In one dimension there are only two outward normals $n \in \{-1, 1\}$,
- we use the 1D flux (7)

Linearization II

2D

- In two dimensions there are four outward cell normals

$$n_{\mathrm{north}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad n_{\mathrm{south}} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$n_{\mathrm{east}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad n_{\mathrm{west}} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

- we use the 2D flux (8)

Diffusion I

We investigate of single and multiscale solvers with different Diffusion functions, that we introduce in the followin sections

1D I

Since the Aim of multiscale Finite Volume, is to improve the results for highly fluctuating diffusivities, we test with the following oscillating function

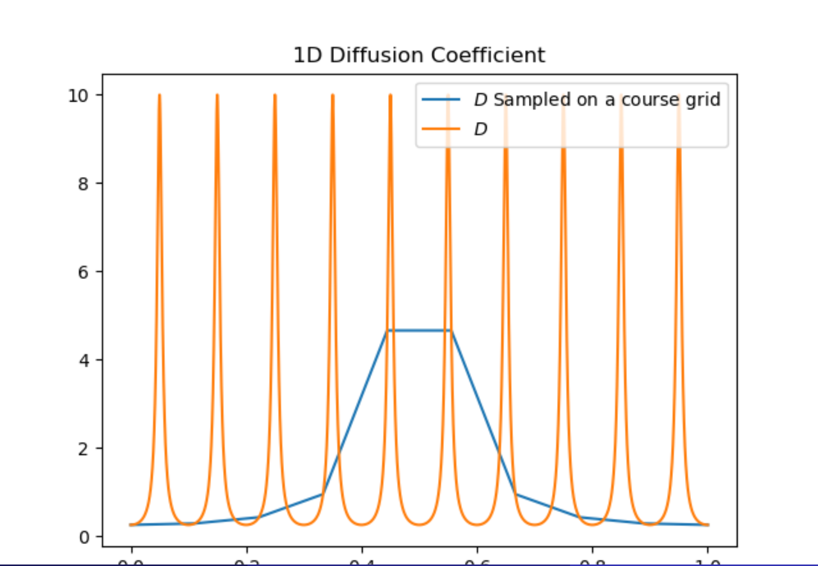
$$D(x) = \frac{1}{2 + 1.9\cos\left(\frac{2\pi x}{\epsilon}\right)}$$

Code

```
def oscillation(x, eps = 0.1):
    return 1 / (2+1.9 * np.cos(2 * np.pi* x / eps))
```

1D II

Diffusivity



2D Box Condition I

To test numerical stability of our methods we introduce a box constrain condition, that traps some concentration in the center.

2D Box Condition II

Code

```
alpha = 1.
gamma = 0.002

exp_kernel = lambda r: alpha * np.exp( - r / gamma)

def R(x,y , p=2):
    center = np.array([0.5,0.5])
    r = 0.2
    thicc = 0.03
    return np.maximum(0. , np.abs((np.abs(x -center[0])**p + np.abs(y
    ↪   - center[1])**p)**(1/p) - r) - thicc)

def box(x,y , p=2):
    return np.maximum(0.0005 , 1. -  exp_kernel(R(x,y , p=100)))
def circle(x,y , p=2):
    return np.maximum(0.0005 , 1. -  exp_kernel(R(x,y , p=2)))
def rhombus(x,y , p=2):
    return np.maximum(0.0005 , 1. -  exp_kernel(R(x,y , p=1)))
```

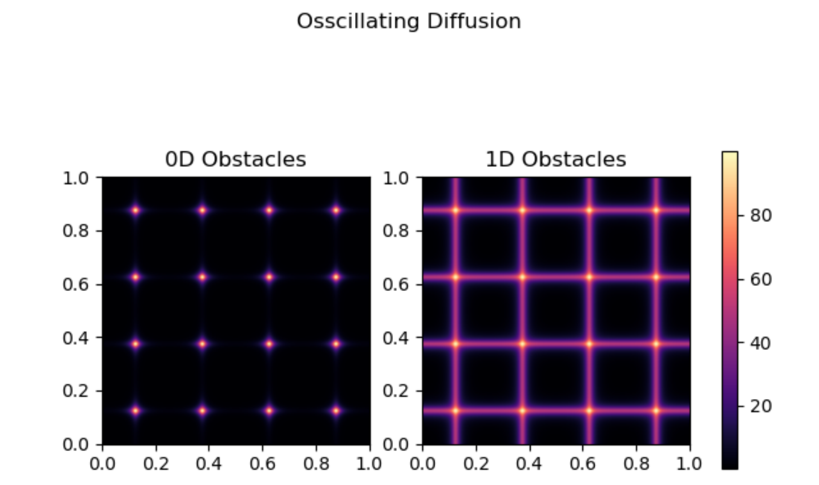
2D Oscillation I

Code

```
def osc2D_point(x,y , eps = 0.25):
    return oscillation(x, eps=eps) * oscillation(y, eps=eps)
def osc2D_line(x,y , eps = 0.25):
    return oscillation(x, eps=eps) + oscillation(y, eps=eps)
```

2D Oscillation II

Diffusion



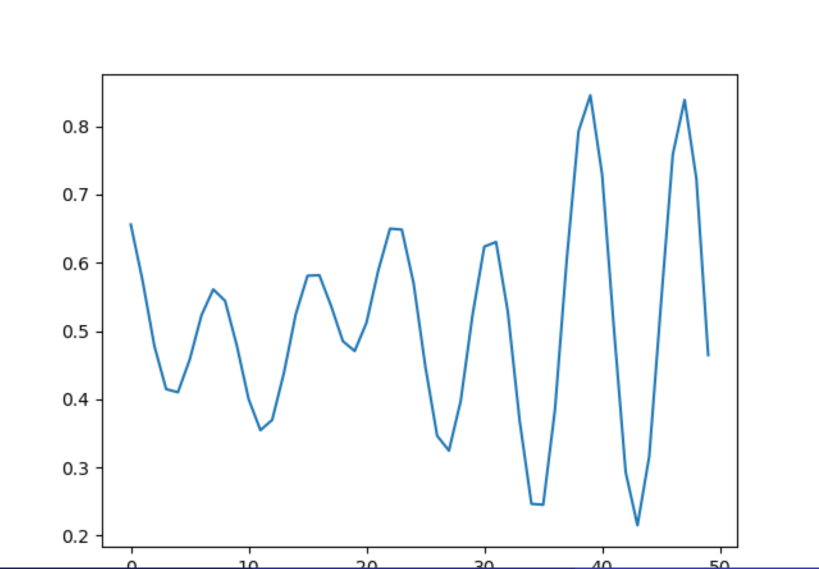
1D Noise I

Code

```
def noise1D(x,scale=10.  , frequencies=5):
    s = lambda x ,f , a , o: a* np.sin(f*2*np.pi*(x + o))
    coeffs = np.random.rand(frequencies,3)
    res = np.zeros(len(x))
    for i in range(frequencies):
        res += s(x, scale *coeffs[i,0] ,coeffs[i,1] , coeffs[i,2] )
    res = res / (2*np.sum(coeffs[:,1])) + 0.5
    return res
```

1D Noise II

Diffusion



Noise 2D I

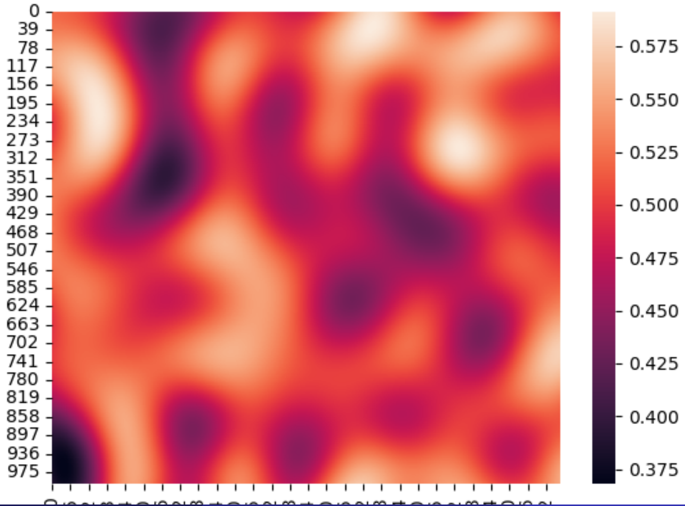
Code

```
def noise2D(x,y , scale=1. , frequencies=50):
    s = lambda x ,f , a , o: a* np.sin(f*2*np.pi*(x + o))
    coeffs = np.random.rand(frequencies,6)
    res = np.zeros(len(x))
    for i in range(frequencies):
        theta = np.pi *coeffs[i,5]
        x_prime = x * np.cos(theta) - y * np.sin(theta)
        y_prime = x * np.cos(theta) - y * np.sin(theta)
        res += 0.5*(s(x_prime, scale *coeffs[i,0] ,coeffs[i,1] ,
        ↪   coeffs[i,2] ) + s(y_prime, scale *coeffs[i,1] ,coeffs[i,3]
        ↪   , coeffs[i,4] ))

    res = res / (2*np.sum(coeffs[:,1])) + 0.5
    return res
    return
```

Noise 2D II

Difusion



Program Structure I

For convenience in Explanation and Execution, we bundle all required information for solving a 1D system into a python class, which is structured as follows

Program Structure II

Class Structure

```
class FVSolver:
   N : int
   resolution : int
   h : np.float64
   x : NDArray[np.float64]
   D : Callable
   f : NDArray[np.float64]
   c : NDArray[np.float64]
   micro_basis : NDArray[np.float64]
   _T : NDArray[np.float64]

<<Init>>
<<Assemble Matrix>>
<<Boundary>>
<<Solve>>
<<Microscale Transmissions>>
<<Reconstruct Microscale Solution>>
```

Program Structure III

Initialization

```
def __init__(self , N :int , D :Callable  , domain=(0.,1.))->None:
    self.h = (domain[1] - domain[0]) / N
    self.N = N
    self.D = D
    self.x = np.linspace(domain[0] , domain[1] , N)
    self._T =  -1/self.h * D((self.x[:-1] + self.x[1:])/2)
    self.f = self.h* np.ones(N)
```

Solving

```
def solve(self):
   self.c = spsolve(self._A.tocsr() , self.f)
   return self.c
```

Program Structure IV

Boundary

```
def set_boundary(self , bc=(0.,0.)):
    self.f[0] = bc[0]
    self.f[-1] = bc[1]
```

Assembly of the linear system

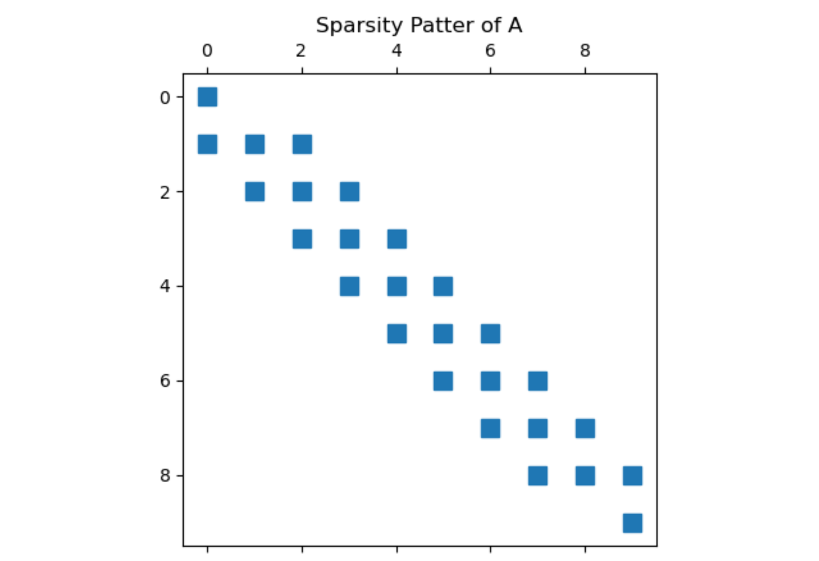
Program Structure V

Matrix Assembly

```python
def assemble_matrix(self)-> None:
    diagp1 = np.zeros(self.N)
    diagp1[2:] =  self._T[1:]
    diagm1 = np.zeros(self.N)
    diagm1[:-2] =  self._T[:-1]
    diag0 = np.ones(self.N)
    diag0[1:-1] = -1 * (self._T[1:] + self._T[:-1])
    self._A = spdiags([diagm1 , diag0 , diagp1] , np.array( [-1, 0,
    ↪   1] ))
```

Program Structure VI

Sparsity Pattern of the linear system



Reference I

Solution of the 2D Laplace equation:

$$-\Delta u(x,y) = f(x,y) \qquad \text{in} \quad \Omega \tag{10}$$
$$u(x,y) = 0 \qquad \text{on} \quad \Gamma_D \tag{11}$$

where $f(x,y) = 2 * (x + y - x^2 - y^2)$ the analytical solution is

$$u(x,y) = x * (1 - x) * y * (1 - y)$$