2 – Probabilistic robotics

Advanced Methods for Mapping and Self-localization in Robotics
MPC-MAP

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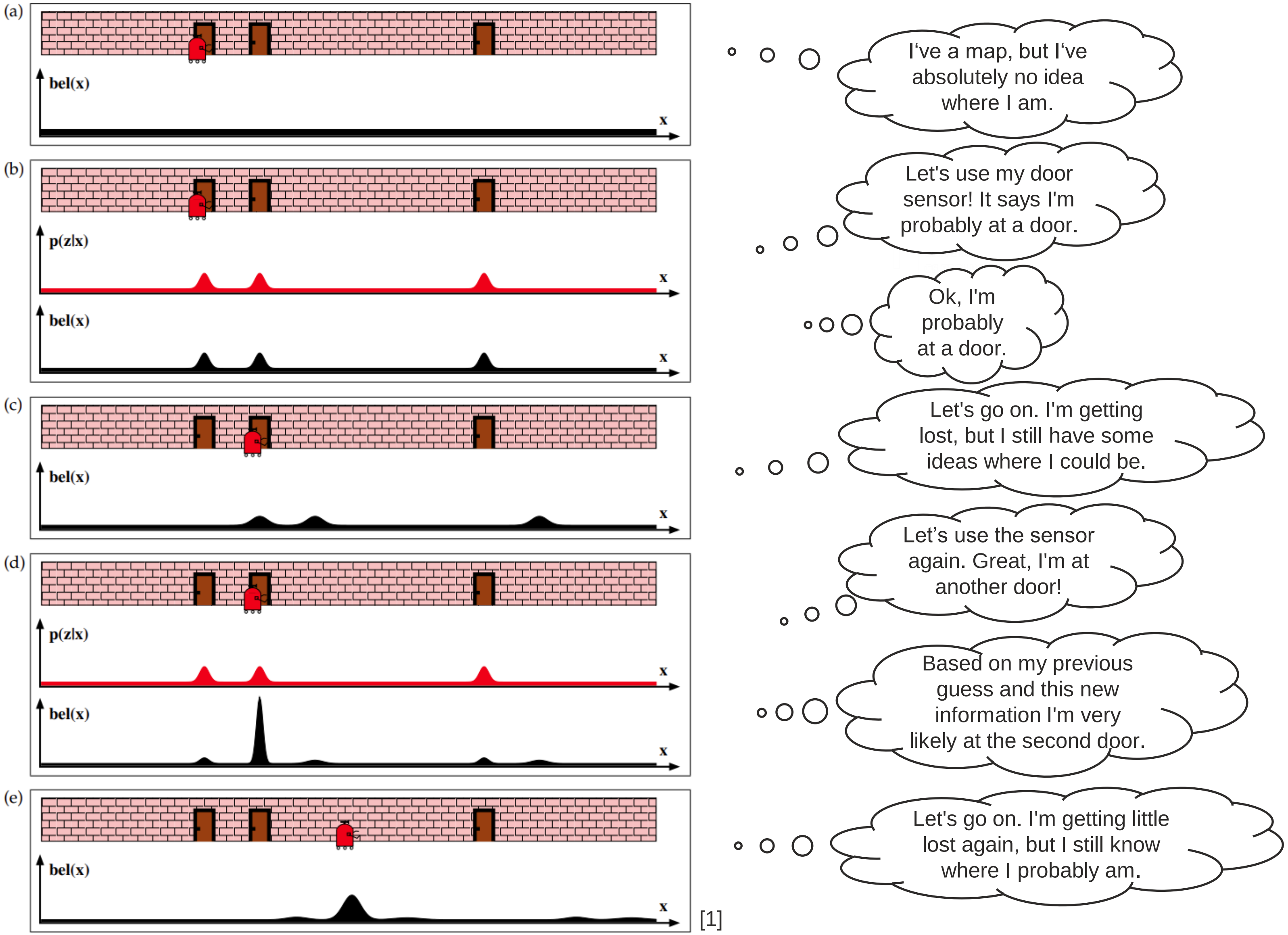
Probabilistic robotics

Key ideas of the probabilistic approach

1D localization example



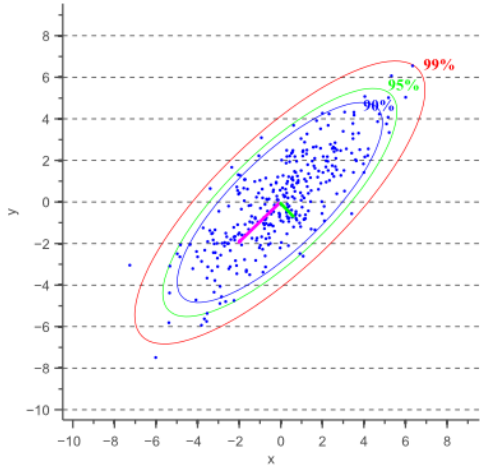
1. http://www.probabilistic-robotics.org/

Probabilistic robotics – the concept

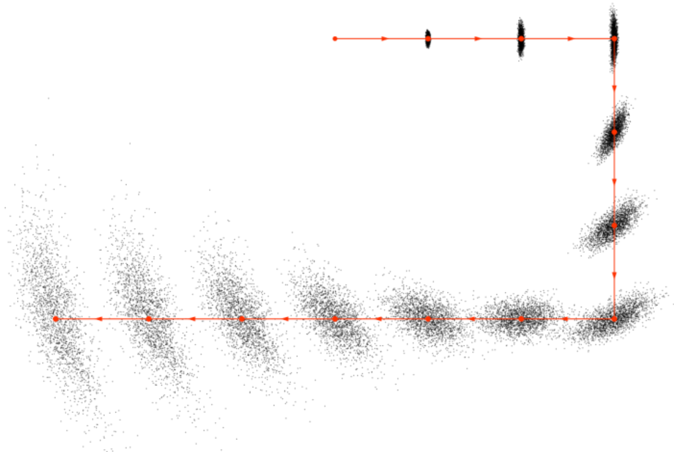
- **Probabilistic approach – nothing is deterministic, all phenomena are stochastic**

- Uncertainty is in both:
  - **Perception** – sensor/measured data are not 100 % reliable; data from all real sensors are more or less noisy
  - **Action** – robot actions/movements are not 100 % reliable; all actions taken differ to some extent from the action desired

- All uncertain phenomena are **represented statistically**



Noisy measurement (e.g., GNSS positioning data) [1]



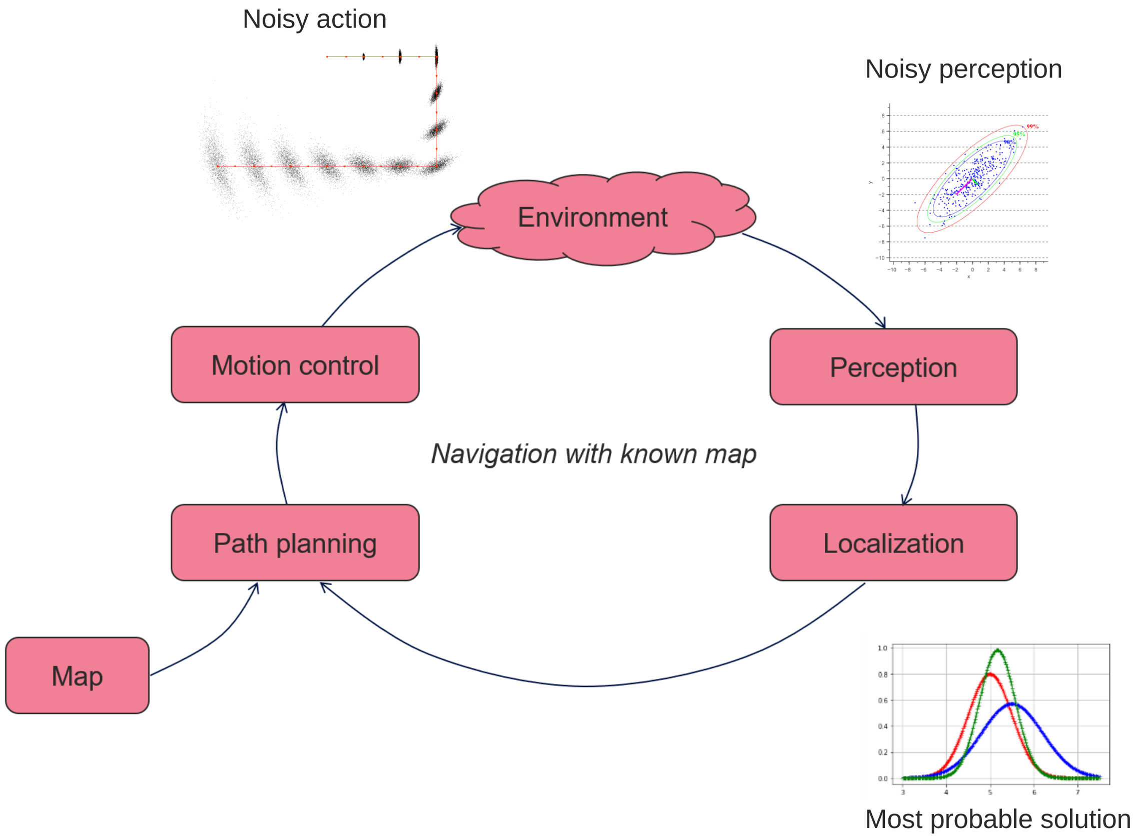
Noisy motion - propagation of error [2]

1. https://www.skyhook.com/blog/best-practices-measure-location-accuracy-geo-positioning-performance-metrics
2. https://en.wikipedia.org/wiki/Monte_Carlo_localization

Probabilistic robotics – the concept

- Robot **navigation** in the probabilistic concept **considers the uncertainties** to provide optimal solution.

- This is especially crucial for the **localization** task – what is the **most probable robot position** according **noisy sensor data**? And considering **known uncertain actions**?



Probability

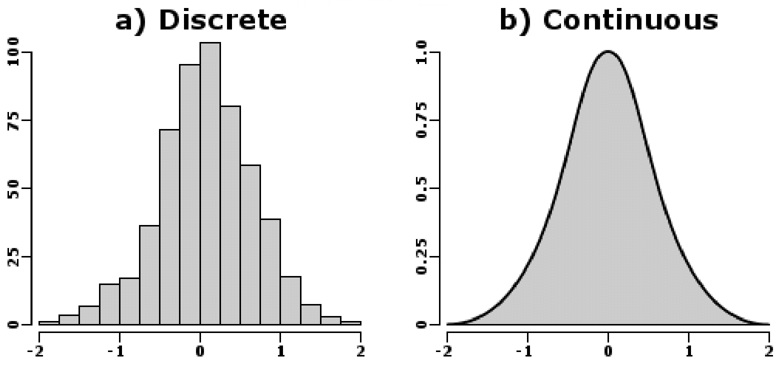
Basic terms and distribution types

Probability

- **Probability** expresses **how likely** a random **event occurs**
  - A number between **0 and 1**; higher value → higher probability

- **Probability distribution**
  - *„Mathematical* ***function*** *that gives the* ***probabilities of occurrence*** *of different possible* ***outcomes*** *for an experiment.“* [1]
  - *„Mathematical* ***description of a random phenomenon*** *in terms of its* ***sample space*** *and the* ***probabilities*** *of events.“* [1]

  - **Discrete** space → **probability mass function** (PMF)
  - **Continuous** space → **probability density function** (PDF) or cumulative distribution function (CDF)

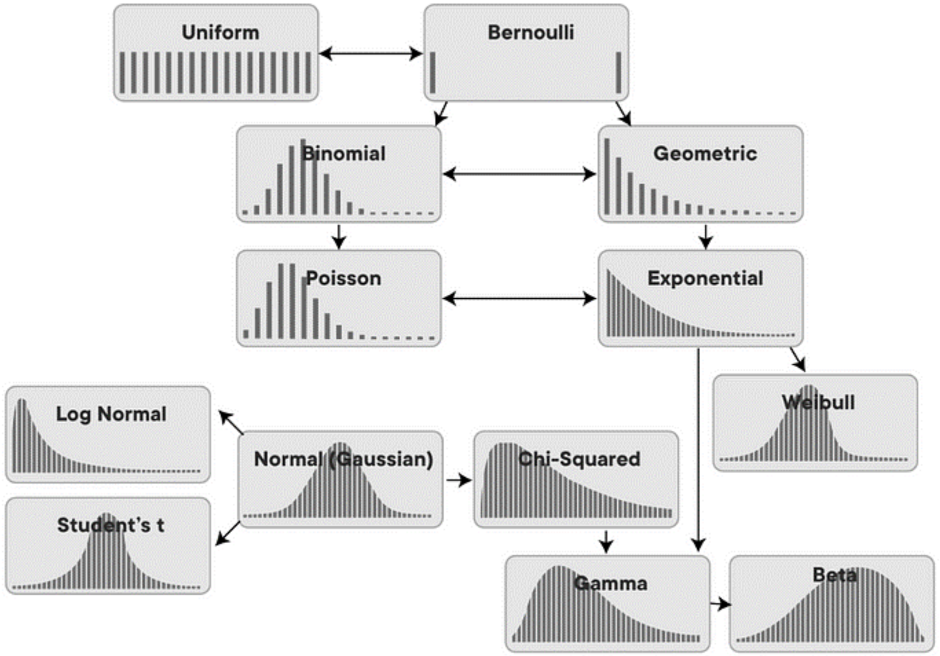


PMF (left) and PDF (right) examples [1]

1. https://medium.com/analytics-vidhya/probability-distributions-444e7babf2e1

Probability mass function (PMF)

- **Probability mass function** (PMF) – probability that a **discrete random variable** is exactly equal to some value
  - $p_X(x) = P(X = x)$
- The probabilities sum up to 1
  - $\sum_x p_x(x) = 1$
- Values are non-zero, positive
  - $p_X(x) \geq 0$

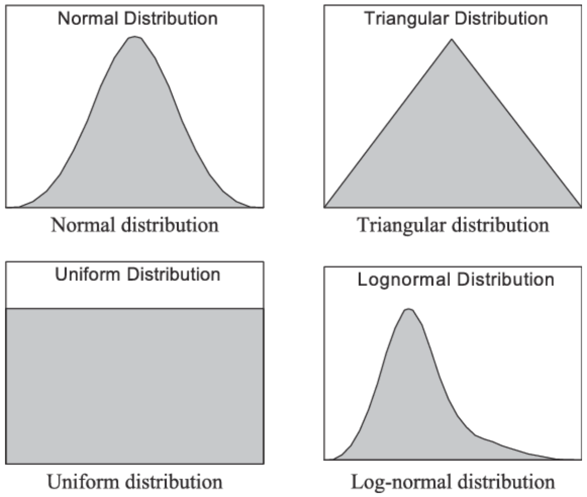


Various probability mass functions (PMF) [1]

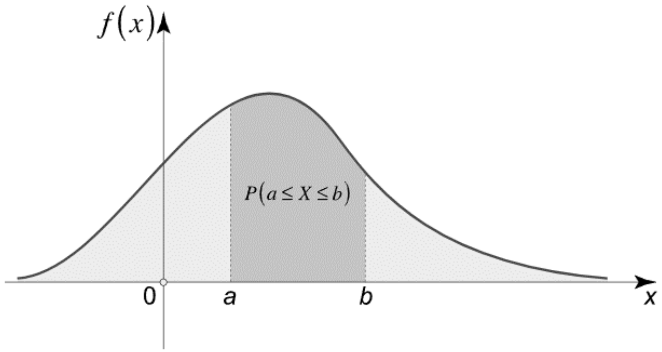
1. https://medium.com/@amanatulla1606/understanding-probability-distributions-in-statistics-and-its-application-in-machine-learning-e62751e83c39

Probability density function (PDF)

- **Probability density function** (PDF) – a function f_X describing the infinitesimal probability of any given value of a **continuous random variable** X
- The **probability** P of a **particular value** equals **0**
  - $P(X) = f_X(x) = 0$
- The **probability** of a **particular range** of value equals to the **integral** of the PDF over the range
  - The area between the PDF and horizontal axis over the range
  - $P(a \leq X \leq b) = \int_a^b f_X(x)\,dx$
- PDF is nonnegative everywhere, integrates to 1
  - $P(-\infty \leq X \leq \infty) = \int_{-\infty}^{\infty} f_X(x)\,dx = 1$



PDF examples [1]

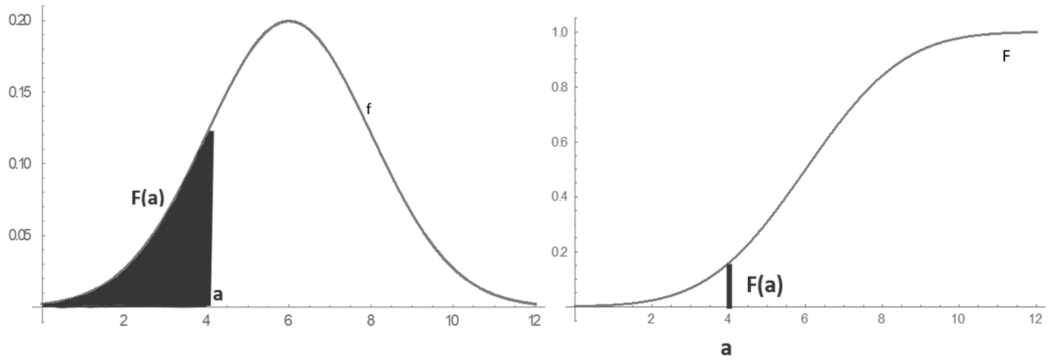


Probability equals to the area under the PDF [2]

1. https://www.researchgate.net/publication/269949106_A_Study_on_the_Economic_Evaluation_for_a_Feasibility_Study_in_Molybdenum_Mineral_Processing
2. https://math24.net/probability-density-function.html

Cumulative distribution function (CDF)

- **Cumulative distribution function** (CDF) – $F_X(x)$ represent a cumulative representation of the PDF
- Describes the probability that the random variable is no larger than a given value
  - The area under the PDF from $-\infty$ to x
  - $F_X(x) = \int_{-\infty}^{x} f_X(u)\, du$

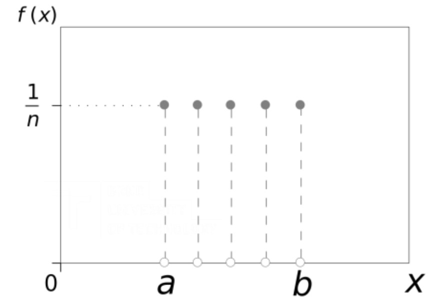


PDF and corresponding CDF [1]

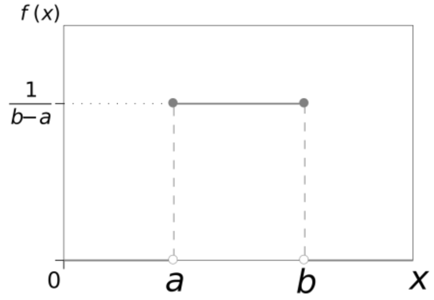
1. https://en.wikipedia.org/wiki/Probability_distribution

Uniform distribution

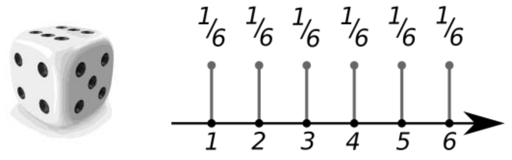
- **Uniform** distribution – all hypotheses will occur with **equal probability**
- **Discrete space –** a finite number of values has equal probability
  - $p(x) = \frac{1}{n}$ $n = b - a + 1$ $\mathrm{k} \in \{a, a+1, \cdots, b-1, b\}$ $b \geq a$
- **Continuous space –** all intervals of the same length within the bounds are equally probable
  - $f(x) = \begin{cases} \frac{1}{b-a} & \text{for } x\epsilon[a,b] \\ 0 & \text{otherwise} \end{cases}$ $-\infty < a < b < \infty$
- Examples
  - Rolling the dice
  - **Initial guess** in the **global localization** task



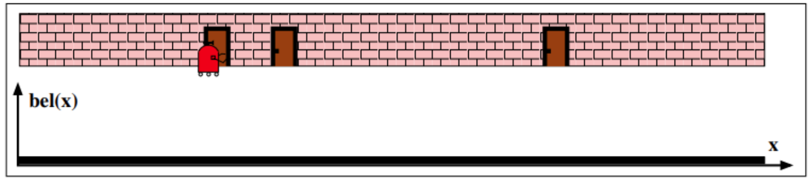
PMF of the discrete uniform distribution example [1]



PDF of the continuous uniform distribution example [2]



Dice rolling – typical discrete uniform distribution

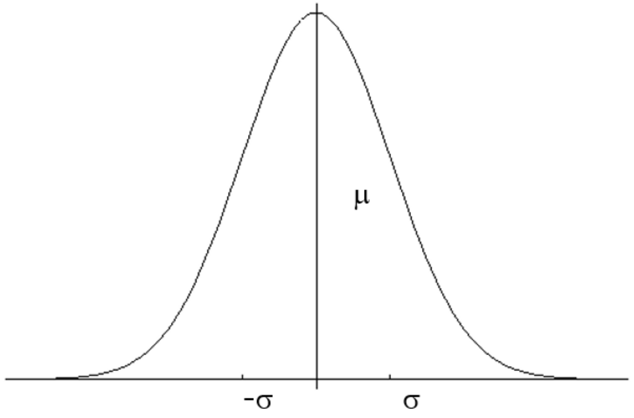
(a)


Initial guess with the uniform distribution [3]

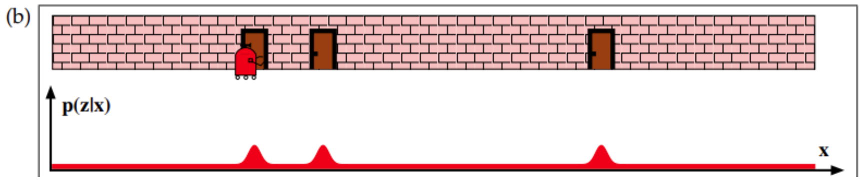
1. https://en.wikipedia.org/wiki/Discrete_uniform_distribution
2. https://en.wikipedia.org/wiki/Continuous_uniform_distribution
3. http://www.probabilistic-robotics.org/

Normal distribution

- **Normal** distribution = **Gaussian** distribution – type of **continuous** probability distribution
- Important in natural sciences – measurement errors caused by many independent processes posses nearly normal distribution
- Notation: $p(x) \sim \mathcal{N}(x; \mu, \sigma^2)$
- PDF: $\boxed{f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}$
- Parametric distribution defined by:
  - **Mean** (expectation) μ
  - **Standard deviation** σ or **variance** σ^2
- Examples
  - Natural random processes
  - Approximate distribution of many **sensors in robotics**



PDF of normal distribution [1]

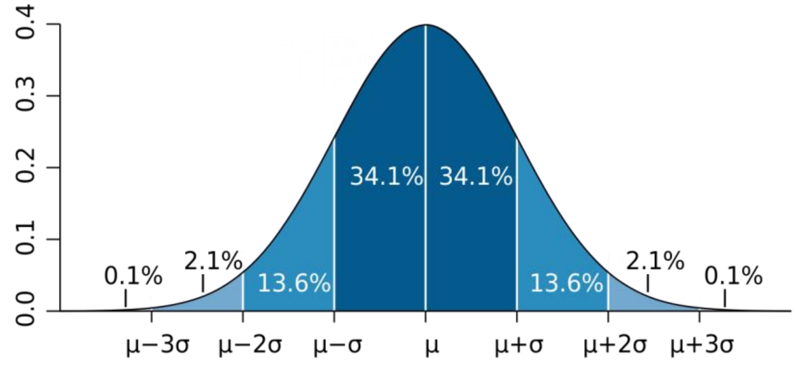


Sensor measurements are often near normal [1]

1. http://www.probabilistic-robotics.org/

Normal distribution – probabilities

- In the measurement field the uncertainty is often expresses by an interval of possible values occurring with certain probability defined via standard deviation σ
- variable $=$ value $\pm$ inverval typically meaning 1σ interval
- Well known probabilities for the normal distribution
  - $P(\mu - \sigma \leq X \leq \mu + \sigma) = 68{,}3\ \%$
  - $P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) = 95{,}4\ \%$
  - $P(\mu - 3\sigma \leq X \leq \mu + 3\sigma) = 99{,}7\ \%$

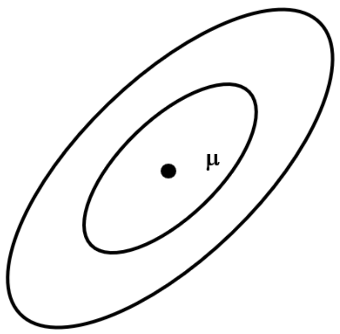


PDF of normal distribution with probabilities for various intervals defined with σ [1]

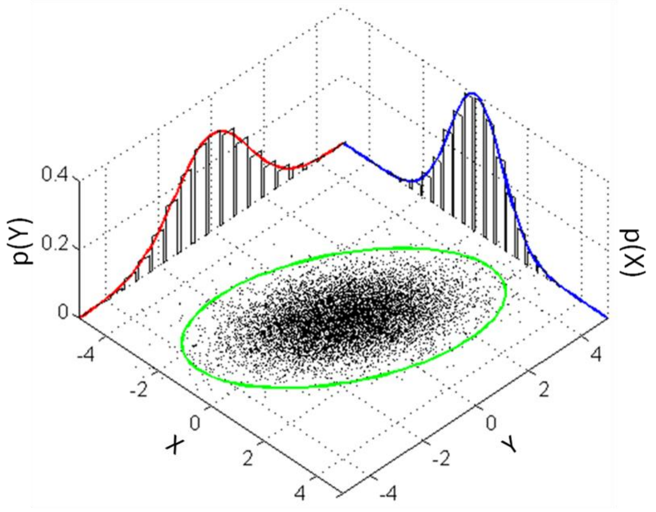
1. https://en.wikipedia.org/wiki/Normal_distribution

Multivariate normal distribution

- Normal distribution over vectors is called ***multivariate*** (x is a vector)
- Often used in localization algorithms when the space vector is multidimensional
- Notation: $p(x) \sim \mathcal{N}(x; \mu, \Sigma)$
- PDF: $p(x) = \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\{-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\}$
- Parameters:
  - **Mean vector** (expectation vector) μ
  - **Covariance matrix** Σ



[1]

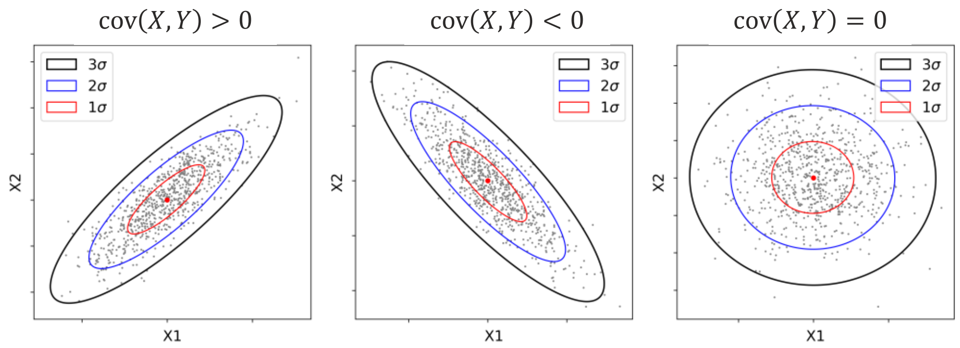


Normal distribution of two variables is often illustrated with ellipse in 2D space [2]

1. http://www.probabilistic-robotics.org/
2. https://en.wikipedia.org/wiki/Multivariate_normal_distribution

Covariance and covariance matrix

- ***Covariance*** is a measure of **the joint variability of two random variables**
  - The mean of the product of X and Y random variable deviations from their means
  - $\mathrm{cov}(X,Y) = E[(X - E[X])(Y - E[Y])]$ expectation $E[X] \approx \mathrm{mean}(X)$
  - Alternative notations: σ_{XY} or $\sigma(X,Y)$
  - **Covariance with itself** is **variance**: $\mathrm{cov}(X,X) = \mathrm{var}(X) \equiv \sigma^2(X)$
  - Zero covariance means **independent variables**
  - Normalized covariance → **correlation**
- ***Covariance matrix*** contains covariance between each pair of elements
  - Matrix dimension: dimensionality of the state x squared
  - Key matrix properties: square, symmetric, quadratic
  - $\boxed{\Sigma = \begin{pmatrix} \sigma_1^2 & \cdots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \cdots & \sigma_n^2 \end{pmatrix}}$
  - **Main diagonal contains variances**



The covariance sign and the „shape“ of data/ellipses [1]

```
process_noise_covariance: [0.05, 0,    0,    0,    0,    0,    0,     0,     0,    0,    0,    0,    0,    0,    0,
                           0,    0.05, 0,    0,    0,    0,    0,     0,     0,    0,    0,    0,    0,    0,    0,
                           0,    0,    0.06, 0,    0,    0,    0,     0,     0,    0,    0,    0,    0,    0,    0,
                           0,    0,    0,    0.03, 0,    0,    0,     0,     0,    0,    0,    0,    0,    0,    0,
                           0,    0,    0,    0,    0.03, 0,    0,     0,     0,    0,    0,    0,    0,    0,    0,
                           0,    0,    0,    0,    0,    0.06, 0,     0,     0,    0,    0,    0,    0,    0,    0,
                           0,    0,    0,    0,    0,    0,    0.025, 0,     0,    0,    0,    0,    0,    0,    0,
                           0,    0,    0,    0,    0,    0,    0,     0.025, 0,    0,    0,    0,    0,    0,    0,
                           0,    0,    0,    0,    0,    0,    0,     0,     0.04, 0,    0,    0,    0,    0,    0,
                           0,    0,    0,    0,    0,    0,    0,     0,     0,    0.01, 0,    0,    0,    0,    0,
                           0,    0,    0,    0,    0,    0,    0,     0,     0,    0,    0.01, 0,    0,    0,    0,
                           0,    0,    0,    0,    0,    0,    0,     0,     0,    0,    0,    0.02, 0,    0,    0,
                           0,    0,    0,    0,    0,    0,    0,     0,     0,    0,    0,    0,    0.01, 0,    0,
                           0,    0,    0,    0,    0,    0,    0,     0,     0,    0,    0,    0,    0,    0.01, 0,
                           0,    0,    0,    0,    0,    0,    0,     0,     0,    0,    0,    0,    0,    0,    0.015]
```

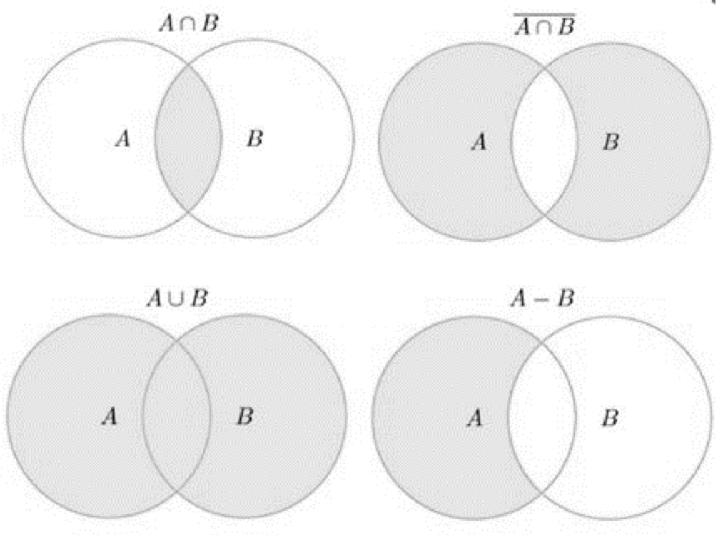
Covariance matrix describing noise characteristics of a process (robot motion) [2]

1. https://geostatisticslessons.com/lessons/errorellipses
2. https://answers.ros.org/question/353624/what-is-the-default-noise-parameters-in-sensor-inputs-in-robot_localization/

Combining probabilities

- The probabilities can be combined in different ways according to the phenomenon we are describing

Event	Probability
A	$P(A) \in [0,1]$
not A	$P(A^{\complement}) = 1 - P(A)$
A or B	$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ $P(A \cup B) = P(A) + P(B) \qquad \text{if A and B are mutually exclusive}$
A and B	$P(A \cap B) = P(A\|B)P(B) = P(B\|A)P(A)$ $P(A \cap B) = P(A)P(B) \qquad \text{if A and B are independent}$
A given B	$P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{P(B\|A)P(A)}{P(B)}$



- The most important event in the **probabilistic robotics** is **A given B**

Bayes' theorem

The theorem and its application in robotics

Bayes' theorem

- Statistical theory based on the Bayesian interpretation of probability
- **Probability** is based on **prior knowledge** of an event and related **evidence**
- The probability expresses a degree of **belief** in an event

$$\boxed{P(H|E) = \frac{P(E|H)P(H)}{P(E)}} = \frac{P(E|H)P(H)}{P(E|H)P(H) + P(E|\neg H)P(\neg H)} = \eta P(E|H)P(H)$$

- H **hypothesis** whose probability may be affected by evidence
- E **evidence** (observation) – new data that were not used in computing the prior probability
- $P(H)$ **prior** probability – the estimate of the probability of the hypothesis H before the data E
- $P(H|E)$ **posterior** probability – is the probability of H after E is observed
- $P(E|H)$ **likelihood** – probability of observing E given H
- $P(E)$ model evidence, probability of seeing evidence independently on H

Recommended resource: https://www.youtube.com/watch?v=HZGCoVF3YvM

Bayes' theorem

Likelihood

Posterior probability

Prior probability

$$P(H|E) = \frac{P(E|H)P(H)}{P(E)}$$

Model evidence

Bayes' theorem

Sensor model

New estimate

Old estimate

$$P(H|E) = \frac{P(E|H)P(H)}{P(E)}$$

Normalization

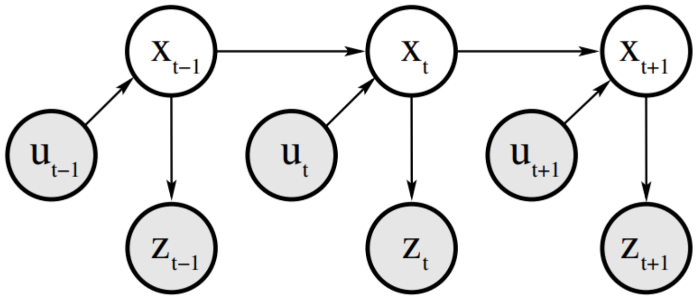
Bayes filter

- **Bayes filter** – standard approach for the **state estimation** in robotics
- General algorithm for calculating **belief** from **measurement** z and **control data** u
- **Recursive**, includes the previous **state** x_{t-1}
- Each period includes **prediction** and **correction** step
- Bayes filter forms the basis for widely used algorithms:
  - **Kalman filter** – Gaussian filter, belief is represented via normal distribution
  - **Particle filter** – Monte Carlo filter, belief is represented by a set of particles

Prediction (control update)

Algorithm Bayes_filter($bel(x_{t-1}), u_t, z_t$**):**
    for all x_t do
        $\boxed{\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1})\, bel(x_{t-1})\, dx}$
        $\boxed{bel(x_t) = \eta\, p(z_t \mid x_t)\, \overline{bel}(x_t)}$
    endfor
    return $bel(x_t)$

Correction (measurement update)



The evolution of controls, states and measurements [1]

Bayes filter

Prior (initial) belief $\overline{bel}(x_0)$

Measurement $p(z_1|x_1)$

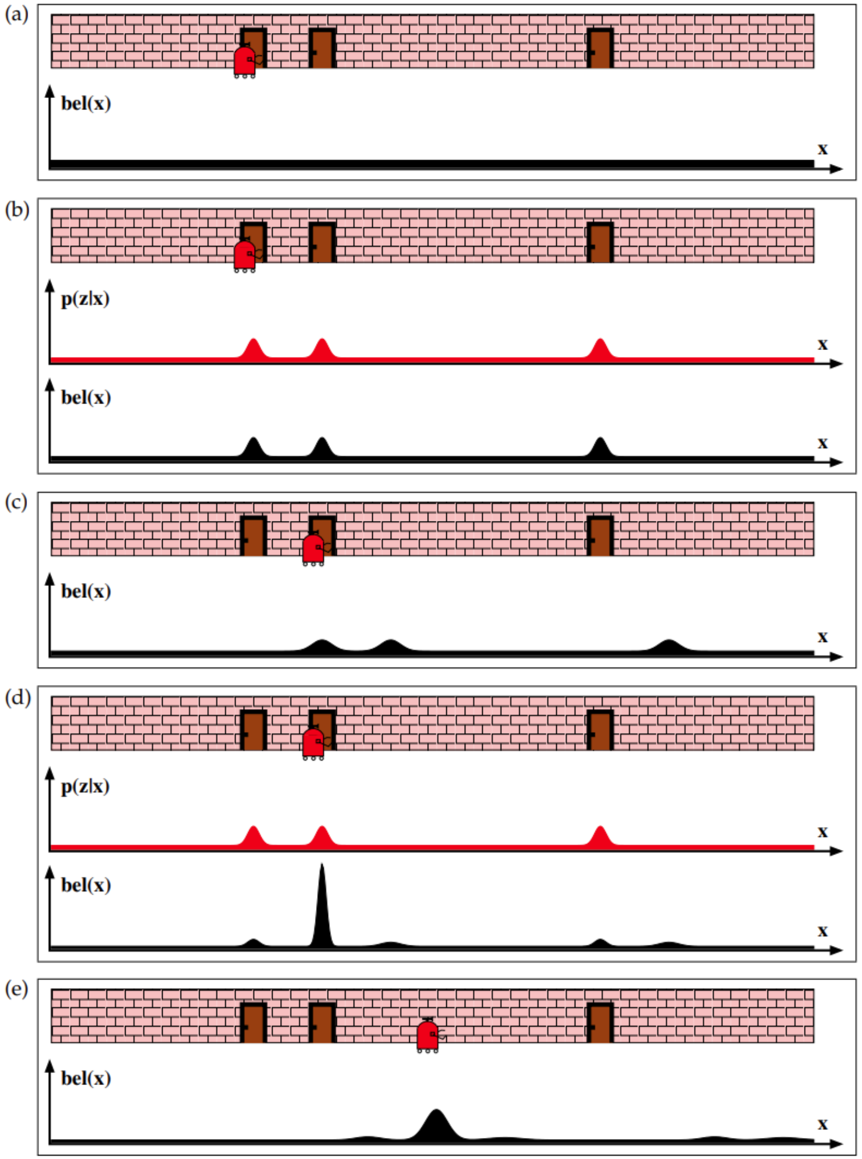
Correction, posterior belief $bel(x_1)$

Prediction, posterior belief $\overline{bel}(x_2)$

Measurement $p(z_2|x_2)$

Correction, posterior belief $bel(x_2)$

Prediction, posterior belief $\overline{bel}(x_3)$



[1]

Filter iteration 1

Filter iteration 2

Filter iteration 3

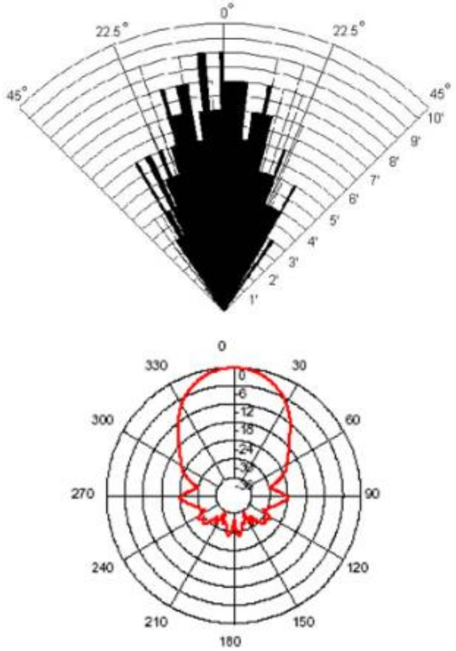
1. http://www.probabilistic-robotics.org/

Sensors

Range and pose sensors

- Provide information about the **distance** and the (eventually) **azimuth** of the object with respect to robot’s coordinate system
- The most common types:
  - **LiDAR** – narrow laser beam, time-of-flight
    - Mapping, localization, obstacle detection
    - 1D, 2D and 3D
  - **Ultrasonic** – wide „beam“, time-of-flight
    - Obstacle detection
    - 1D only
  - **Radar** – wide „beam“, time-of-flight, Doppler
    - Obstacle detection, velocity measurement
    - 1D (2D, 3D)

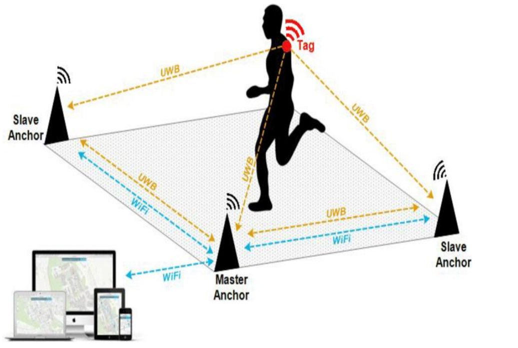
2D LiDAR scan



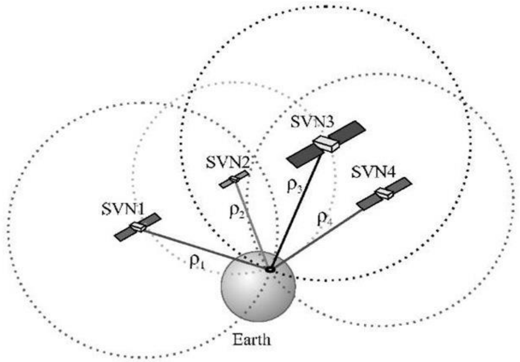
Ultrasonic sensor characteristics

External pose sensors

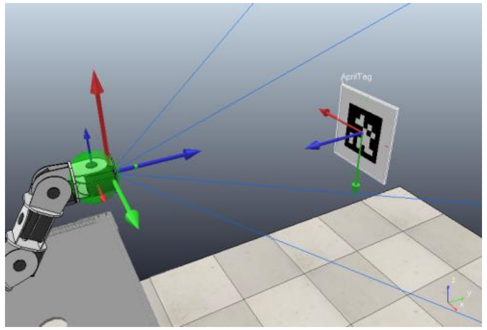
- Determines **position** or **orientation** or both position and orientation – **pose** within an external coordinate system
- The most common types:
  - **GNSS** – Global Navigation Satellite System
    - Global, world-wide localization
    - Position only, orientation – dual-antenna RTK systems
  - **Motion capture** and **beacon localization**
    - Local localization
    - Based on cameras and reflective markers or on radio beacons and receivers
  - **Fiducial markers**
    - Local localization
    - Relative pose of graphical marker and camera



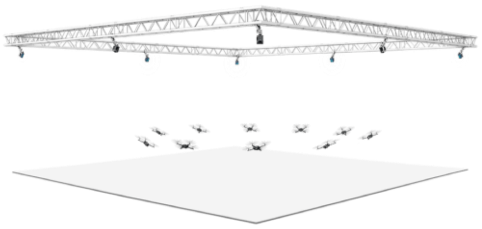
Beacon loclalization (UWB, BT, ...) [3]



GNSS [1]



Fiducial markers/tags [4]



Motion Capture [2]

1. http://what-when-how.com/
2. https://deva90sapmc8w.cloudfront.net/roboticsPrime17w_8-450h.png
3. https://www.researchgate.net/publication/336832935
4. https://haowensh.com/assets/img/2020/kdc/frame_confirmation.png

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