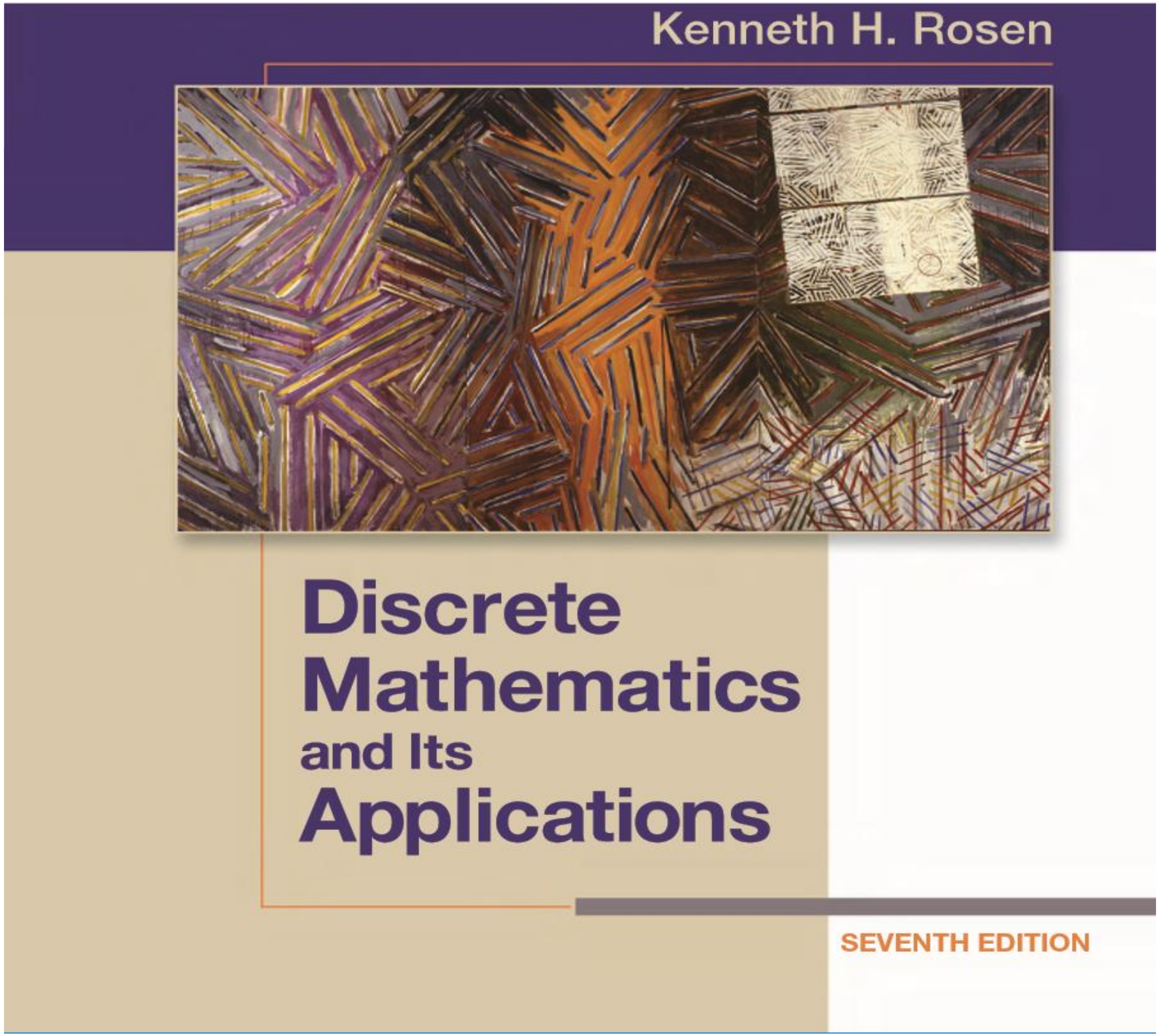
Discrete Mathematics and Graph Theory

MAT 1003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Discrete Mathematics and Its Applications
SEVENTH EDITION

Module 1- The Foundations: Logic and Proofs

1 **The Foundations: Logic and Proofs** **1**

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	2	9				4		
			5			1		
	4							
				4	2			
6							7	
5								
7			3					5
	1			9				
							6	

FIGURE 1 **A 9 x 9 Sudoku puzzle.**

A combinatorial circuit

FIGURE 1 **Basic logic gates.**

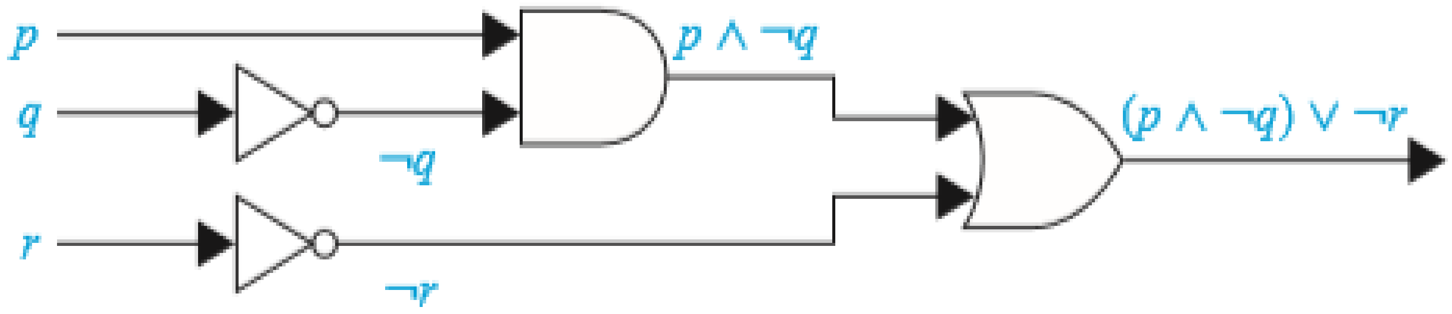


FIGURE 2 **A combinatorial circuit.**

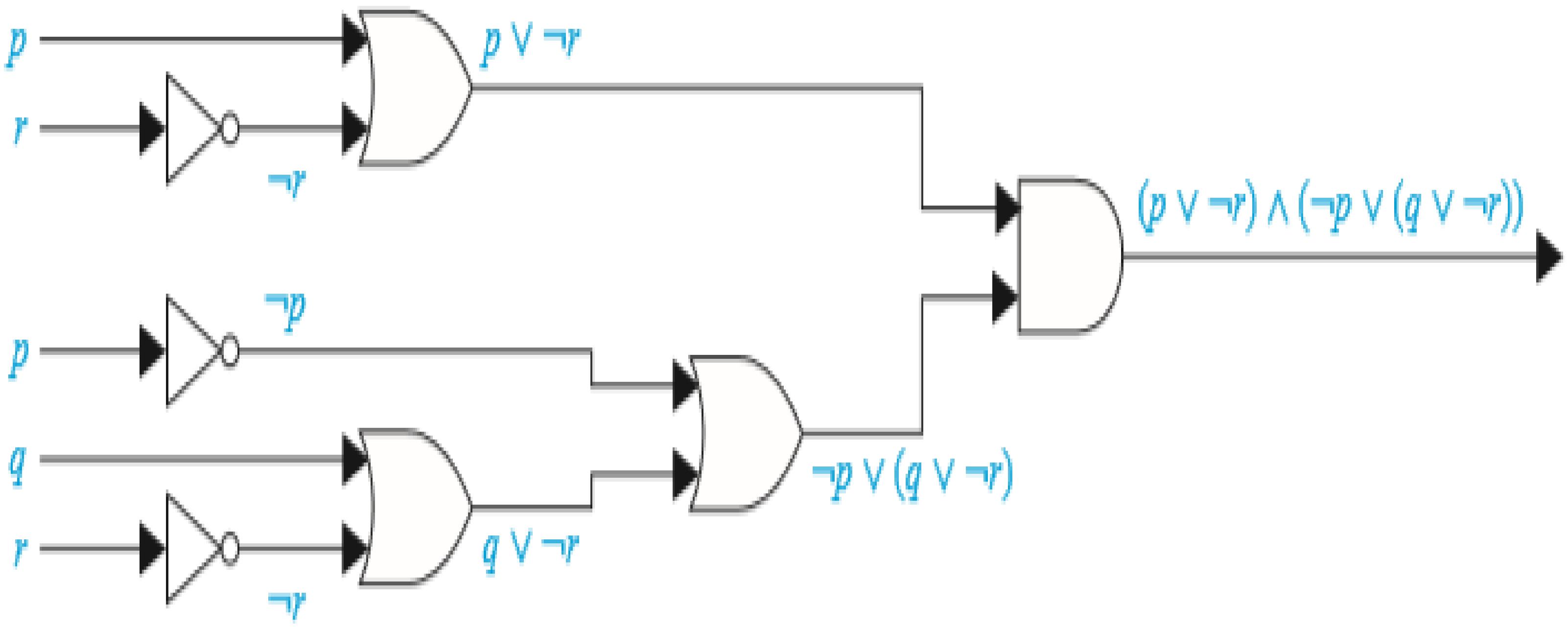


FIGURE 3 The circuit for $(p \vee \neg r) \wedge (\neg p \vee (q \vee \neg r))$.

- How can this English sentence be translated into a logical expression?

"You can access the Internet from campus only if you are a computer science major or you are not a freshman."

- Let p: You can access the Internet from campus

  q: you are a computer science major

  r: you are a freshman

  A logical expression of above statement is

$$p \rightarrow (q \vee \sim r)$$

Module 1- Foundations: Logic and Proofs

- PROPOSITION

A proposition is a declarative sentence i.e.,

either true(T) or false(F), but not both.

Example:

1. Lucknow is in India.	True(T)
2. Bangalore is in Gujarat.	False(F)
3. Where are you going?	Doesn't give conclusion
4. Go to bed.	Doesn’t give conclusion

Prepositional Logic – Definition

A proposition is a collection of declarative statements that has either a truth value "true” or a truth value "false".

A propositional consists of propositional variables and connectives. We denote the propositional variables by capital letters (A, B, etc). The connectives connect the propositional variables.

Some examples of Propositions are given below

- "Man is Mortal", it returns truth value "TRUE"
- "12 + 9 = 3 – 2", it returns truth value "FALSE"

The following is not a Proposition –

- "A is less than 2".

It is because unless we give a specific value of A, we cannot say whether the statement is true or false.

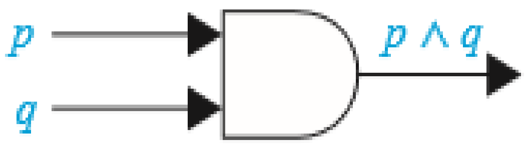
Basic logical operations

- 1. Conjunction (AND, $P \wedge Q$)
- 2. Disjunction (OR, $P \vee Q$)
- 3. Negation (NOT, $\sim P$)
- 4. Conditional Statement ($P \rightarrow Q$)
- 5. Bi Conditional Statement ($P \leftrightarrow Q$)

- Consider:
  - (i) P: Kolkata is in Orissa and Q: 4+4= 9 [F and F = F]
  - (ii) P: Kolkata is in Orissa and Q: 4+4= 8 [F and T = F]
  - (iii) P: Kolkata is in West Bengal and Q: 4+4= 9 [T and F = F]
  - (iv) P: Kolkata is in West Bengal and Q: 4+4= 8 [T and T = T]

(multiplication)

P	Q	$P \wedge Q$
F	F	F
F	T	F
T	F	F
T	T	T



AND gate

Conjunction (AND, $P \wedge Q$)

- If two propositions can be combined by the word 'and' then we get the conjunction of P AND Q.
- The statement $P \wedge Q$ has a truth value T whenever both P and Q have truth value T, otherwise it has truth value is False (F).

P	Q	$P \wedge Q$
F	F	F
F	T	F
T	F	F
T	T	T

- Consider:

(i) P: Kolkata is in Orissa or Q: 4+4= 9 [F or F = F]

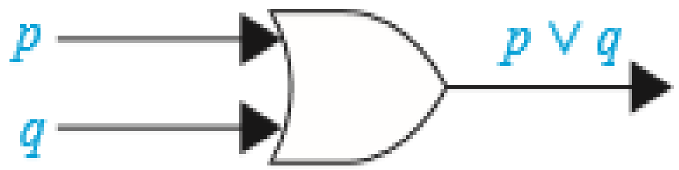
(ii) P: Kolkata is in Orissa or Q: 4+4= 8 [F or T = T]

(iii) P: Kolkata is in West Bengal or Q: 4+4= 9 [T or F = T]

(iv) P: Kolkata is in West Bengal or Q: 4+4= 8 [T or T = T]

(addition)

P	Q	$P \vee Q$
F	F	F
F	T	T
T	F	T
T	T	T



OR gate

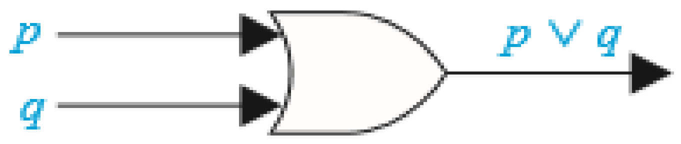
Disjunction (OR, $P \vee Q$)

- If two propositions can be combined by the word 'or' then we get the disjunction of two propositions i.e., P OR Q , $P \vee Q$.
- The truth value of $P \vee Q$ depends only on the truth values of P and Q.

Thus, if P and Q are false, the $P \vee Q$ is false(F),

otherwise $P \vee Q$ is true (T).

P	Q	$P \vee Q$
F	F	F
F	T	T
T	F	T
T	T	T



OR gate

Negation (NOT , $\sim P$)

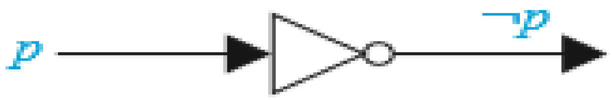
- The negation of a statement is formed by introducing the word ‘not’ at a proper place in the statement or by prefixing the statement with the phrase ‘It is not the case that …’ or ‘It is false that ….’

$\sim P$ is read as not P

If the truth value of statement P is True then negation of P is False.

If the truth value of statement P is False then negation of P is True.

P	$\sim P$
T	F
F	T



Inverter

Example:

- P : Tanuj speaks English

  Q : Tanuj speaks hindi

$P \wedge Q$: Tanuj speaks English and hindi.

$P \vee Q$: Tanuj speaks English or hindi.

$\sim P$: It is false that Tanuj speaks English.

$\sim Q$: It is not the case that Tanuj speaks hindi.

- P: The function is differentiable.

  Q: The function is continuous.

  $P \rightarrow Q$: If the function is differentiable then it is continuous.

Conditional statement(Implication statement)

- A compound proposition obtained by combining two given propositions by using the words ‘if’ and ‘then’ at appropriate places is called a conditional statement.

Q is called the consequent(conclusion)

$$P \rightarrow Q$$

P is called the antecedent(hypothesis)

$P \rightarrow Q$ is read as

- (i) P only if Q
- (ii) P implies Q
- (iii) P is sufficient for Q
- (iv) Q if P
- (v) If P, then Q
- (vi) If P,Q

$$P \rightarrow Q$$

- The statement

$P \rightarrow Q$ has a truth value F

when Q has the truth value F and P has the truth value T; otherwise, it has truth value T.

P	Q	$P \rightarrow Q$
F	F	T
F	T	T
T	F	F
T	T	T

Bi-Conditional statement

- If P and Q are any two statements, then the statement $P \leftrightarrow Q$, read as "P if and only if Q" is called a Bi-Conditional statement.

  The statement $P \leftrightarrow Q$ has a truth value T

  whenever both P and Q have identical truth values,

  otherwise, its truth value is F.

Example: Two lines are parallel if and only if they have the same slope.

P	Q	$P \leftrightarrow Q$
F	F	T
F	T	F
T	F	F
T	T	T

Find the truth table for

$$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$$

P	Q	R	$Q \vee R$	$P \wedge (Q \vee R)$	$P \wedge Q$	$P \wedge R$	$(P \wedge Q) \vee (P \wedge R)$
F	F	F	F	F	F	F	F
F	F	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	T	T	T	F	F	F	F
T	F	F	F	F	F	F	F
T	F	T	T	T	F	T	T
T	T	F	T	T	T	F	T
T	T	T	T	T	T	T	T

Prove that $\sim (P \vee Q)$ is equivalent to $(\sim P) \wedge (\sim Q)$

P	Q	$P \vee Q$	$\sim (P \vee Q)$
F	F	F	T
F	T	T	F
T	F	T	F
T	T	T	F

P	Q	$\sim P$	$\sim Q$	$(\sim P) \wedge (\sim Q)$
F	F	T	T	T
F	T	T	F	F
T	F	F	T	F
T	T	F	F	F

Tautology

A compound statement that is always true, irrespective of the truth values of the propositions that occur in it, is called a tautology.

- $[P \wedge (P \longrightarrow Q)] \longrightarrow Q$ is a tautology

P	Q	$P \longrightarrow Q$	$P \wedge (P \longrightarrow Q)$	$[P \wedge (P \longrightarrow Q)] \longrightarrow Q$
F	F	T	F	T
F	T	T	F	T
T	F	F	F	T
T	T	T	T	T

Prove that $(P \longrightarrow Q) \Leftrightarrow (\backsim P \vee Q)$ is a tautology.

P	Q	$P \longrightarrow Q$	$\backsim P$	$\sim P \vee Q$	$(P \longrightarrow Q) \leftrightarrow (\backsim P \vee Q)$
F	F	T	T	T	T
F	T	T	T	T	T
T	F	F	F	F	T
T	T	T	F	T	T

Prove that, for any propositions P,Q,R the compound proposition $[(P \longrightarrow Q) \wedge (Q \longrightarrow R)] \longrightarrow (P \longrightarrow R)$ is a tautology.

P	Q	R	$P \longrightarrow Q$	$Q \longrightarrow R$	$(P \longrightarrow Q) \wedge (Q \longrightarrow R)$	$P \longrightarrow R$	$[(P \longrightarrow Q) \wedge (Q \longrightarrow R)] \longrightarrow (P \longrightarrow R)$
F	F	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	T	T	T	T	T	T	T
T	F	F	F	T	F	F	T
T	F	T	F	T	F	T	T
T	T	F	T	F	F	F	T
T	T	T	T	T	T	T	T

(1). Prove that, for any proposition P,Q,R the compound proposition

$$\{P \longrightarrow (Q \longrightarrow R)\} \longrightarrow \{(P \longrightarrow Q) \longrightarrow (P \longrightarrow R)\}$$

is a tautology.

(2) Prove that the following are tautology.

(i) $\backsim (P \vee \backsim Q) \longrightarrow \backsim P$

(ii) $\backsim P \longrightarrow (P \longrightarrow Q)$

Contradiction

- A compound statement that is always false, irrespective of the truth value of the propositions that occur in it, is called a contradiction.

Example (1). Prove that $P \wedge (\backsim P)$ is a contradiction.

P	$\backsim P$	$P \wedge (\backsim P)$
F	T	F
T	F	F

But $P \vee (\backsim P)$ is a tautology.

Example (2) Prove that $P \wedge (\backsim P \wedge Q)$ is a contradiction.

Contingency

A statement (proposition) that is neither tautology nor a contradiction is called a contingency.

Converse:

The proposition $\mathrm{Q} \longrightarrow P$ is called the <u>converse</u> of $\mathrm{P} \longrightarrow Q$.

Contrapositive:

The proposition $\backsim \mathrm{Q} \longrightarrow \backsim P$ is called the <u>contrapositive</u> of $\mathrm{P} \longrightarrow Q$.

Inverse:

The proposition $\backsim \mathrm{P} \longrightarrow \backsim Q$ is called the <u>inverse</u> of $\mathrm{P} \longrightarrow Q$.

Example:
“If today is a holiday, then I will go for a movie”

here let P: Today is a holiday.

Q: I will go for a movie.

(i) Converse: $Q \longrightarrow P$

“If I go for a movie, then today is a holiday”.

(ii) Inverse: $\backsim P \longrightarrow \backsim Q$

“If today is not a holiday, then I will not go for a movie”.

(iii) Contrapositive: $\backsim Q \longrightarrow \backsim P$

“If I do not go for a movie, then today is not a holiday”.

What is the truth value of each of the following statements?

a. 2 is even or 4 is odd.

b. 2 is even and 4 is odd.

c. 2 is odd or 4 is odd.

d. 2 is odd and 4 is odd.

e. If 2 is odd, then 4 is odd.

f. If 2 is even, then 4 is odd.

g. If 2 is odd, then 4 is even.

h. If 2 is odd and 4 is even, then $2 < 4$.

Example: Let A, B, and C be the following statements:
A: Roses are red.
B: Violets are blue.
C: Sugar is sweet.
Translate the following compound statements into symbolic notation.

1. Roses are red and violets are blue.
2. Roses are red, and either violets are blue or sugar is sweet.
3. Whenever violets are blue, roses are red and sugar is sweet.
4. Roses are red only if violets aren't blue or sugar is sour.
5. Roses are red and, if sugar is sour, then either violets aren't blue or sugar is sweet.

Equivalence of formulas

- Two propositions P and Q are said to be logically equivalent whenever P and Q have identical truth values i.e., the bi-conditional $P \leftrightarrow Q$ is a tautology and is donated by $P \Leftrightarrow Q$.

Example:

$(i)\ \backsim(\backsim P) \Leftrightarrow P$

P	$\backsim P$	$\backsim(\backsim P)$	$\backsim(\backsim P) \leftrightarrow P$
F	T	F	T
T	F	T	T

(ii) $P \wedge P \Leftrightarrow P$

P	P	$P \wedge P$	$P \wedge P \leftrightarrow P$
F	F	F	T
T	T	T	T

- (iii) $(P \longrightarrow Q) \Leftrightarrow (\sim P) \vee Q$

P	Q	$(P \longrightarrow Q)$	$(\sim P)$	$(\sim P) \vee Q$	$(P \longrightarrow Q) \leftrightarrow (\sim P) \vee Q$
F	F	T	T	T	T
F	T	T	T	T	T
T	F	F	F	F	T
T	T	T	F	T	T

- *(iv)* $[(P \wedge (\sim P)) \vee Q] \Leftrightarrow Q$

- (V) $P \vee (\sim P) \Leftrightarrow Q \vee (\sim Q)$
- (vi) $[(P \vee Q) \longrightarrow R] \Leftrightarrow [(P \longrightarrow R) \wedge (Q \longrightarrow R)]$
- (vii) $P \vee (Q \wedge R) \Leftrightarrow [(P \vee Q) \wedge (P \vee R)]$

Consider the compound propositions:
1. 15 is not divisible by 6 or 7.
2. 15 is not divisible by 6 and 15 is not divisible by 7.
Now what you can say about above two propositions?

Both propositions are constructed from two simple propositions:
a. p: 15 is divisible by 6.
b. q: 15 is divisible by 7.

Now, the compound proposition (1) is presented as $\neg(p \vee q)$ and the proposition (2) is by $(\neg p) \wedge (\neg q)$.

The laws of logic

- Let P, Q and R be any three propositions

(1). Law of double negation : $\sim(\sim P) \Leftrightarrow P$

(2). Idempotent laws: $P \vee P \Leftrightarrow P$, $P \wedge P \Leftrightarrow P$

(3). Identity laws: $P \vee F \Leftrightarrow P$, $P \wedge T \Leftrightarrow P$

(4). Inverse laws: : $P \vee \sim P \Leftrightarrow T$, $P \wedge \sim P \Leftrightarrow F$

(5). Domination laws: $P \vee T \Leftrightarrow T$, $P \wedge F \Leftrightarrow F$

(6). Commutative laws: $P \vee Q \Leftrightarrow Q \vee P$, $P \wedge Q \Leftrightarrow Q \wedge \mathrm{P}$

(7). Absorption laws:

$$P \vee (P \wedge Q) \Leftrightarrow P$$

P	Q	$P \wedge Q$	$P \vee (P \wedge Q)$
F	F	F	F
F	T	F	F
T	F	F	T
T	T	T	T

$$P \wedge (P \vee Q) \Leftrightarrow P$$

P	Q	$P \vee Q$	$P \wedge (P \vee Q)$
F	F	F	F
F	T	T	F
T	F	T	T
T	T	T	T

VIT AP

(8). DeMorgan laws

$(i) \backsim (P \vee Q) \Leftrightarrow \backsim P \wedge \backsim Q$

P	Q	$P \vee Q$	$\backsim (P \vee Q)$	$\backsim P$	$\backsim Q$	$\backsim P \wedge \backsim Q$
F	F	F	T	T	T	T
F	T	T	F	T	F	F
T	F	T	F	F	T	F
T	T	T	F	F	F	F

(ii) $\backsim (P \wedge Q) \Leftrightarrow \backsim P \vee \backsim Q$

(9). Associate laws

(i) $P \vee (Q \vee R) \Leftrightarrow (P \vee Q) \vee R.$

(ii) $P \wedge (Q \wedge R) \Leftrightarrow (P \wedge Q) \wedge R.$

(10). Distributive laws

(i) $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$

(ii) $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$

Law for the negation of a conditional

(i) $\backsim (P \longrightarrow Q) \Leftrightarrow P \wedge (\backsim Q)$

P	Q	$P \longrightarrow Q$	$\backsim (P \longrightarrow Q)$	$\backsim Q$	$P \wedge (\backsim Q)$
F	F	T	F	T	F
F	T	T	F	F	F
T	F	F	T	T	T
T	T	T	F	F	F

(ii) $\backsim (P \vee Q) \Leftrightarrow (\backsim P) \wedge (\backsim Q)$

(iii) $\backsim (P \wedge Q) \Leftrightarrow (\backsim P) \vee (\backsim Q)$

(iv) $(P \longrightarrow Q) \Leftrightarrow (\backsim P) \vee Q$

Example 1

Prove the logical equivalence

$$(P \longrightarrow Q) \wedge [\backsim Q \ \wedge \ (R \vee \backsim Q)] \Leftrightarrow \backsim (Q \vee P)$$

Solution:

$(P \longrightarrow Q) \wedge [\backsim Q \ \wedge \ (R \vee \backsim Q)]$

$\Leftrightarrow (P \longrightarrow Q) \wedge [\backsim Q \ \wedge \ (\backsim Q \vee R)]$ (by commutative law)

$\Leftrightarrow (P \longrightarrow Q) \wedge [\backsim Q\ ]$ (by absorption law $P \wedge \ (P \ \vee Q) \ \Leftrightarrow P$)

$\Leftrightarrow \backsim [(P \longrightarrow Q) \longrightarrow Q]$ $(U \ \wedge \ (\backsim V) \ \Leftrightarrow \backsim (U \ \longrightarrow V)$

$\Leftrightarrow \backsim [\backsim (P \longrightarrow Q) \vee Q]$ $((P \longrightarrow Q) \Leftrightarrow \backsim P \vee Q\ )$

$\Leftrightarrow \backsim [(P \wedge \backsim Q) \vee Q]$

$\Leftrightarrow \backsim [Q \vee (P \wedge \backsim Q)\ ]$ (by commutative law)

$\Leftrightarrow \backsim [(Q \vee P) \wedge (Q \vee \backsim Q)\ ]$ (by distributive law)

$\Leftrightarrow \backsim [(Q \vee P) \wedge T\ ]$ (by inverse law $(Q \vee \backsim Q) \Leftrightarrow T$)

$\Leftrightarrow \backsim (Q \vee P)$ (by identity law $P \wedge T \ \Leftrightarrow P$)

$\Leftrightarrow \ R.H.S$

- Show that

$$\left[(P \vee Q) \wedge \sim \left(\sim P \wedge (\sim Q \vee \sim R)\right)\right] \vee (\sim P \wedge \sim Q) \vee (\sim P \wedge \sim R)$$

is a tautology using laws(without using truth tables)

$$\sim(\sim P \wedge (\sim Q \vee \sim R)) \Leftrightarrow \sim(\sim P \wedge \sim(Q \wedge R))$$

$\Leftrightarrow \sim(\sim P) \vee \sim(\sim(Q \wedge R))$ (by De Morgan's Law)

$\Leftrightarrow P \vee (Q \wedge R)$ (by double negation)

$\Leftrightarrow (P \vee Q) \wedge (P \vee R)$ (by distributive law)

$$[(P \vee Q) \wedge \sim(\sim P \wedge (\sim Q \vee \sim R))] \Leftrightarrow (P \vee Q) \wedge [(P \vee Q) \wedge (P \vee R)]$$

$\Leftrightarrow [(P \vee Q) \wedge (P \vee Q)] \wedge (P \vee R)$

(by associative law)

$\Leftrightarrow (P \vee Q) \wedge (P \vee R)$ -------(1)

(by idempotent law $u \wedge u \Leftrightarrow u$)

Now

$$(\sim P \wedge \sim Q) \vee (\sim P \wedge \sim R) \Leftrightarrow \sim (P \vee Q) \vee \sim (P \vee R)$$

(by DeMorgon's Laws)

$$\Leftrightarrow \sim [(P \vee Q) \wedge (P \vee R)] \quad \text{----}(2)$$

Substituting (1) and (2) in

$$\left[(P \vee Q) \wedge \sim \left(\sim P \wedge (\sim Q \vee \sim R)\right)\right] \vee (\sim P \wedge \sim Q) \vee (\sim P \wedge \sim R)$$

$$\Leftrightarrow [(P \vee Q) \wedge (P \vee R)] \vee \left[\sim [(P \vee Q) \wedge (P \vee R)]\right]$$

$$\Leftrightarrow T \qquad (u \vee \sim u \Leftrightarrow T)$$

$\therefore$ *it is a tautology*

Satisfiable

A compound proposition is **satisfiable** if there is an assignment of truth values to its variables that makes it true. When no such assignments exists, that is, when the compound proposition is false for all assignments of truth values to its variables, the compound proposition is **unsatisfiable**.

Duality law

Two formulas S and S^* are said to be **duals** of each other if one can be obtained from the other by replacing ∧ by ∨ , ∨ by ∧ , each T by F, and each F by T.

- Ex: Prove that $\left[(\backsim P \vee Q) \wedge \left(P \wedge (P \wedge Q)\right)\right] \Leftrightarrow P \wedge Q$.

  Hence deduce that $\left[(\backsim P \wedge Q) \vee \left(P \vee (P \vee Q)\right)\right] \Leftrightarrow P \vee Q$.

Solution:

$$\left[(\backsim P \vee Q) \wedge \left(P \wedge (P \wedge Q)\right)\right] \Leftrightarrow (\backsim P \vee Q) \wedge \left((P \wedge Q)\right)$$

Rules of Inference

Let $P_1, P_2, P_3, \ldots\ P_n\ \ be$ & C be propositions

A compound proposition of the form

$$(P_1\ \land P_2 \land P_3 \land \cdots\ P_n\,) \longrightarrow C$$

is called an **argument**.

Here $P_1, P_2, P_3, \ldots\ P_n$ are called the **premises** of the argument

And C a c**onclusion** of the argument.

In tabular form

$$
\begin{array}{l}
P_1 \\
P_2 \\
P_3 \\
. \\
. \\
. \\
P_n \\
\hline
\therefore C
\end{array}
$$

- If the conjunction of the premises $(P_1 \wedge P_2 \wedge P_3 \wedge \cdots \ P_n)$ is true in atleast one possible situations, then the premises $P_1, P_2, P_3, \ldots \ P_n$ of an argument is said to be consistent.

Ex: The premises $(P \vee Q)$ *and* $\backsim P$ in the argument are consistence.

$$\begin{array}{l} (P \vee Q) \\ \quad \backsim P \\ \hline \therefore r \end{array}$$

P	Q	$P \vee Q$	$\backsim P$	$(P \vee Q) \wedge \backsim P$
F	F	F	T	F
F	T	T	T	T
T	F	T	F	F
T	T	T	F	F

- $(\boldsymbol{P} \vee \boldsymbol{Q}) \wedge \backsim \boldsymbol{P}$ is true when P is false and Q is true.

Hence <u>consistent</u>

- If the conjunction of the premises $(P_1 \wedge P_2 \wedge P_3 \wedge \cdots P_n)$ is false in every possible situation, then the premises $P_1, P_2, P_3, \ldots P_n$ of an argument is said to be inconsistent.

Ex: The premises P and $(\backsim P \wedge Q)$ in the argument

$$\begin{array}{l} P \\ (\backsim P \wedge Q) \\ \hline \therefore r \end{array}$$

is inconsistent as the results are false for all situations.

Example

- Check the validity of the following argument:

"If I try and I have a talent, then I will become scientist

If I become scientist, then I will be happy.

Therefore, If I will not be happy then I did not try hard or I do not have talent".

Valid and invalid arguments

- An argument with premises $P_1, P_2, P_3, \ldots\ P_n$ and conclusion C is said to be valid if whenever each of the premises $P_1, P_2, P_3, \ldots\ P_n$ is true, then the conclusion C is likewise true.

- The conclusion is true only in the case of a valid argument.

- An argument is valid if and only if the conjunction of the premises implies the conclusion.

  in this case $(P_1 \wedge P_2 \wedge P_3 \wedge \cdots\ P_n) \longrightarrow C$ is a tautology.

Method of testing the validity of an argument

(1). Construct a truth table of all premises and conclusion.

(2). Identify the rows called critical rows in which all premises are true.

In case no such row is found, the argument is invalid.

(3). In each critical row, find whether the conclusion is true.

In such case the argument is valid, otherwise invalid.

However, if at least one critical row contains false conclusion, then the argument is invalid.

Example 1

- Test the validity of the argument

If the morning is fine, I go for a walk.

I do not go for a walk.

____

∴ The morning is not fine.

- Solution:

Let P: the morning is fine

Q: I go for a walk

<u>In symbolic form</u>

$$P \longrightarrow Q$$

$$\backsim Q$$

$$\therefore \backsim P$$

		PREMISES		CONCLUTION
P	Q	$P \longrightarrow Q$	$\backsim Q$	$\backsim P$
F	F	T	T	T
F	T	T	F	T
T	F	F	T	F
T	T	T	F	F

The critical row here row1 contains true conclusions and hence the argument is valid.

Example 2

Test the validity of the following argument

If two sides of a triangle are equal, then the opposite angels are equal

Two sides of a triangle are not equal

____

$\therefore$ The opposite angles are not equal

- Solution:

Let P: two sides of a triangle are equal

Q: the opposite angels are equal

In symbolic form

$P \longrightarrow Q$

$\backsim P$

$\therefore \backsim Q$

		PREMISES		CONCLUTION		
P	Q	$P \longrightarrow Q$	$\backsim P$	$\backsim Q$	$[(P \longrightarrow Q) \wedge \backsim P]$	$[(P \longrightarrow Q) \wedge \backsim P] \longrightarrow \backsim Q$
F	F	T	T	T	T	T
F	T	T	T	F	T	F
T	F	F	F	T	F	T
T	T	T	F	F	F	T

The critical rows here are row1, row 2.

Here the critical row2 conclusion contains false.

Also $[(P \longrightarrow Q) \wedge \backsim P] \longrightarrow \backsim Q$ is <u>not a tautology</u>.

hence the argument is invalid.

TABLE 1 **Rules of Inference.**

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore\ q$	$(p \land (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore\ \neg p$	$(\neg q \land (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore\ p \to r$	$((p \to q) \land (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \lor q$ $\neg p$ $\therefore\ q$	$((p \lor q) \land \neg p) \to q$	Disjunctive syllogism
p $\therefore\ p \lor q$	$p \to (p \lor q)$	Addition
$p \land q$ $\therefore\ p$	$(p \land q) \to p$	Simplification
p q $\therefore\ p \land q$	$((p) \land (q)) \to (p \land q)$	Conjunction
$p \lor q$ $\neg p \lor r$ $\therefore\ q \lor r$	$((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$	Resolution

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Example 3

Test the validity of the following argument:

If I study, I will not fail in the examination.

If I do not watch TV in the evenings, I will study.

I failed in the examination.

____

Therefore, I must have watched TV in the evenings.

Solution:

Let p: I study

q: I fail in the examination

r: I watch TV in the evenings.

In symbolic form the argument is

$p \longrightarrow \backsim q$

$\backsim r \longrightarrow p$

q

____

$\therefore r$

We can write logical expression as

$$(p \longrightarrow \backsim q) \wedge (\backsim r \longrightarrow p) \wedge q \Leftrightarrow r$$

Without using truth tables

- $(p \longrightarrow \backsim q) \wedge (\backsim r \longrightarrow p) \wedge q$

$\Leftrightarrow [\backsim\backsim q \longrightarrow \backsim p] \wedge (\backsim p \longrightarrow \backsim\backsim r) \wedge q$ (by contrapositive)

$\Leftrightarrow (q \longrightarrow \backsim p) \wedge (\backsim p \longrightarrow r) \wedge q$ $(\backsim\backsim u \Leftrightarrow u)$

$\Leftrightarrow (q \longrightarrow r) \wedge q$ (by rule of syllogism)

$\Leftrightarrow r$ (by Modus Ponens rule)

This argument is valid.

Example 4

- Consider the argument

I will get grade A in this course or I will not graduate.

If I do not graduate, I will join the army.

I got grade A

____

Therefore, I will not join the army.

Is this a valid argument ?

Example 5

- Test the validity of the argument

$$p \longrightarrow q$$
$$r \longrightarrow s$$
$$\backsim q \vee \backsim s$$
$$\overline{\qquad\qquad\qquad}$$
$$\therefore \backsim (p \wedge r)$$

(answer is VALID)

Example 6

- Test the validity of the argument

$$\begin{array}{l} p \longrightarrow q \\ r \longrightarrow s \\ p \vee r \\ \hline \therefore (q \vee s) \end{array}$$

(answer is VALID)

Example 7

Show that the premises $p \longrightarrow q, p \longrightarrow r, q \longrightarrow \backsim r, r$ are consistent whereas $p \longrightarrow q, p \longrightarrow r, q \longrightarrow \backsim r,\ p$ are inconsistent.

Predicate calculus

- The area of logic that deals with predicates and quantifiers is called the predicate calculus.

Predicate

- “ x is greater than 5”

variable predicate

the statement is denoted by P(x)

when substituted a particular value for x, **P(x) becomes a proposition**.

with given x=3, P(3) has a truth value F

x=9, P(9) has a truth value T

- “$x = y + 2$”

the statement is denoted as Q(x, y)

for $x = 1, y = 2$ Q(x, y) has a truth value F

for $x = 2, y = 0$ Q(x, y) has a truth value T

- “$x + y = z$”

Quantifiers

- Quantification expresses the extent to which a predicate is true over a range of elements.

Ex: Let P(x): $x + 5 > 2$

for all x in N i.e., $\forall x \in N$(set of all natural numbers)

P(x) is true.

Ex: for every integer x , $x^2$is a non-negative integer.

Ex: There exist a real number whose square is equal to itself.

The words *all* , *every, some, there exists* are associated with the idea of a quantity. Such words are called quantifiers.

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Types of quantifiers

- There are two types of quantifiers

    (1). Universal quantifier

    (2). Existential quantifier

- The universal quantification of P(x) for a particular domain is the proposition that asserts that P(x) is **true** for all values of x in this domain.

Universal quantifier

- The universal quantification of $P(x)$ is the statement
  "$P(x)$ for all values of x in the domain."
- The notation $\forall x\, P(x)$ denotes the universal quantification of $P(x)$.
  Here $\forall$ is called the universal quantifier.
  We read $\forall x\, P(x)$ as "for all x $P(x)$" or "for every x $P(x)$."

$\forall x\, P(x)$ $P(x)$ is true for every x

An element for which $P(x)$ is false is called a counter example of $\forall x\, P(x)$.

Universal quantification can be identified with

- For every
- For all
- All of
- For each
- Given any
- For arbitrary
- For any

- When all the elements in the domain can be listed—
  say, $x_1, x_2, x_3, \dots \dots x_n$
  it follows that the universal quantification ∀x P(x) is the same as the conjunction P(x_1)∧P(x_2)∧ ⋯∧P(x_n)
  because this conjunction is true **if and only if** P(x_1), P(x_2), ..., P(x_n) are all true.

Existential quantifier

- The existential quantification of $P(x)$ is the proposition

"There exists an element x in the domain such that $P(x)$."

We use the notation $\exists x\ P(x)$ for the existential quantification of $P(x)$.

Here $\exists$ is called the existential quantifier.

$\exists x$ $P(x)$ There is an x for which $P(x)$ is true

Existential quantification can be identified with

- For some
- For at least one
- There is one

- When all elements in the domain can be listed
  —say, $x_1, x_2, x_3, \ldots \ldots x_n$

  the existential quantification ∃x P(x) is the same as the disjunction $P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n)$,

  because this disjunction is true <u>if and only</u> if at least one of $P(x_1)$, $P(x_2)$,…,$P(x_n)$ is true.

Example 1

- For the universe of all integers, let

  $p(x)$: $x>0$, $q(x)$: x is even

  $r(x)$: x is a perfect square $s(x)$: x is divisible by 3

  $t(x)$: x is divisible by 7

  write down the following quantified statements in symbolic form:

  (i) At least one integer is even

  (ii) There exist a positive integer that is even

  (iii) Some even integers are divisible by 3

  (iv) If x is even and a perfect square, then x is not divisible by 3.

  (v) If x is odd or is not divisible by 7, then x is divisible by 3.

- At least one integer is even $\exists x, q(x)$

- There exist a positive integer that is even $\exists x, [p(x) \wedge q(x)]$

- Some even integers are divisible by 3 $\exists x, [q(x) \wedge s(x)]$

- If x is even and a perfect square, then x is not divisible by 3.

$$\forall x, \big[[q(x) \wedge r(x)] \longrightarrow \backsim s(x)\big]$$

- If x is odd or is not divisible by 7, then x is divisible by 3.

$$\forall x, \big[[\backsim q(x) \vee \backsim t(x)] \longrightarrow s(x)\big]$$

- Example 1
- Express each of these statements using predicates and quantifiers.

1. A passenger on an airline qualifies as an elite flyer if the passenger flies more than 25,000 miles in a year or takes more than 25 flights during that year.

2. A man qualifies for the marathon if his best previous time is less than 3 hours and a woman qualifies for the marathon if her best previous time is less than 3.5 hours.

- A passenger on an airline qualifies as an elite flyer if the passenger flies more than 25,000 miles in a year or takes more than 25 flights during that year.

Solution:

Let E(x) : Person x qualifies as an elite flyer in a given year.

F(x, y) : Person x flies more than y miles in a given year

S(x, y) : Person x takes more than y flights in a given year

The symbolic form is

$$\forall x, \left(\left(F(x, 25000) \vee \left(S(x, 25) \right) \right) \longrightarrow E(x) \right)$$

(elite means best trained)

- A man qualifies for the marathon if his best previous time is less than 3 hours and a woman qualifies for the marathon if her best previous time is less than 3.5 hours.

Let Q(x) : Person x qualifies for the marathon.(race)

M(x) : person x is a man

T(x, y) : Person x has run a marathon in less than y hours

$$\forall x, \left[\left(M(x) \wedge T(x,3)\right) \wedge (\backsim M(x) \wedge T(x,3.5)\right] \longrightarrow Q(x)$$

- A student must take at least 60 course hours, or at least 45 course hours and write a master's thesis, and receive a grade not lower than a B in all required courses, to receive a master's degree.

Let M : The student received a masters degree

H(x) : The student took at least x course hours

T : The student wrote a thesis.

G(B, y): The person got grade B or higher in a course y.

The symbolic form is

$$\left[\left(H(60) \vee (H(45) \wedge T)\right) \wedge \ \forall y, G(B, y)\right] \longrightarrow M$$

- There is a student who has taken more than 21 credit hours in a semester and received all A’s.

Let T(x, y): Person x took more than y credit hours

G(x, P): Person x earned grade point average P.

The symbolic form is

$$\exists x, [T(x, 21) \ \wedge G(x, 4.0)]$$

Example 2

Express each of these system specifications using predicates, quantifiers and logical connectives

- At least one mail message can be saved if there is a disk with more than 10 kilobytes of free space.
- Whenever there is an active alert, all queued messages are transmitted.
- The diagnostic monitor tracks the status of all systems except the main console.
- Each participant on the conference call whom the host of the call did not put on a special list was billed.

Solution:

- At least one mail message can be saved if there is a disk with more than 10 kilobytes of free space.

Let $F(x, y)$: Disc x has more than y kilobytes of free space.

$S(x)$: Mail message x can be saved.

$$\big(\exists x, F(x, 10)\big) \longrightarrow \exists x, S(x)$$

- Whenever there is an active alert, all queued messages are transmitted.

Let $A(x)$: Alert x is active

$Q(x)$: Message x is queued

$T(x)$: Message x is transmitted

$$(\exists x, A(x)) \longrightarrow \forall x, \left(Q(x) \longrightarrow T(x)\right)$$

- The diagnostic monitor tracks the status of all systems except the main console.

let $T(x)$: The diagnostic monitor tracks the status of system x.

$$\forall x, (x \neq main\ console) \longrightarrow T(x)$$

- Each participant on the conference call whom the host of the call did not put on a special list was billed.

Let $L(x)$: The host of the conference call put participant x on a special list.

$B(x)$: participant x was billed.

$$\forall x, \left(\backsim L(x) \longrightarrow B(x) \right)$$

Logical equivalence involving quantifiers

- $\forall x, \left(P(x) \wedge Q(x)\right) \quad \Leftrightarrow (\forall x, P(x)) \wedge (\forall x, Q(x))$
- $\exists x, \left(P(x) \vee Q(x)\right) \quad \Leftrightarrow (\exists x, P(x)) \vee (\exists x, Q(x))$
- $\exists x, \left(P(x) \longrightarrow Q(x)\right) \Leftrightarrow \exists x, (\backsim P(x) \vee Q(x))$

- $\backsim [\forall x, P(x)] \equiv \exists x, [\backsim P(x)]$

- $\backsim [\exists x, Q(x)] \equiv \forall x, [\backsim Q(x)]$

- $\backsim \forall x \left(P(x) \longrightarrow Q(x)\right) \equiv \exists x \,(P(x) \land \backsim Q(x))$

Two rules of inference

1. Rule of universal instantiation

   If an open statement P(x) is known to be true

   for all x in a universe S and if c ∈ S, the P(c) is true.

2. Rule of universal generalization

   If an open statement P(x) is proved to be true

   for any (arbitrary) x chosen from a set S,

   then the quantified statement $\forall x \in S, P(x)$ is true.

TABLE 2 Rules of Inference for Quantified Statements.

Rule of Inference	*Name*
$\dfrac{\forall x\, P(x)}{\therefore\ P(c)}$	Universal instantiation
$\dfrac{P(c) \text{ for an arbitrary } c}{\therefore\ \forall x\, P(x)}$	Universal generalization
$\dfrac{\exists x\, P(x)}{\therefore\ P(c) \text{ for some element } c}$	Existential instantiation
$\dfrac{P(c) \text{ for some element } c}{\therefore\ \exists x\, P(x)}$	Existential generalization

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TABLE 1 Rules of Inference.

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$(p \land (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$(\neg q \land (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \land (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \lor q$ $\neg p$ $\therefore q$	$((p \lor q) \land \neg p) \to q$	Disjunctive syllogism
p $\therefore p \lor q$	$p \to (p \lor q)$	Addition
$p \land q$ $\therefore p$	$(p \land q) \to p$	Simplification
p q $\therefore p \land q$	$((p) \land (q)) \to (p \land q)$	Conjunction
$p \lor q$ $\neg p \lor r$ $\therefore q \lor r$	$((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$	Resolution

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Example 1

- Test the validity of the argument

All men are mortal

Socrates is a man

____

$\therefore$ Socrates is mortal

Solution:

Let S denote the set all men,

P(x): x is mortal (living things. Ex: people)

a: Socrates

The give argument is

$$\forall x \in S, P(x)$$
$$a \in S$$
$$\overline{\therefore P(a)}$$

Since the statement $\forall x \in S, P(x)$ and $a \in S$

By <u>Rule of universal instantiation</u> $P(a)$ is true.

Hence the given argument is valid.

TABLE 1 Rules of Inference.

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore\ q$	$(p \land (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore\ \neg p$	$(\neg q \land (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore\ p \to r$	$((p \to q) \land (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \lor q$ $\neg p$ $\therefore\ q$	$((p \lor q) \land \neg p) \to q$	Disjunctive syllogism
p $\therefore\ p \lor q$	$p \to (p \lor q)$	Addition
$p \land q$ $\therefore\ p$	$(p \land q) \to p$	Simplification
p q $\therefore\ p \land q$	$((p) \land (q)) \to (p \land q)$	Conjunction
$p \lor q$ $\neg p \lor r$ $\therefore\ q \lor r$	$((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$	Resolution

Example 2

- Test the validity of argument

No engineering student of First or Second Semester studies Logic.

Anil is an engineering student who studies Logic.

$\therefore$ Anil is not in the Second Semester

Solution:

- Let the universe be the set of all engineering students.

$P(x)$: Engineering student x is in the First semester

$Q(x)$: Engineering student x is in the Second semester.

R(x): Engineering student x studies logic

a : Anil

The given argument is

$$\forall x, [(P(x) \vee Q(x)) \longrightarrow (\backsim R(x))]$$

$$\mathrm{R}(a)$$

$$\overline{\therefore \backsim Q(a)}$$

- $\{\forall x, [(P(x) \vee Q(x)) \longrightarrow (\backsim R(x))]\} \wedge R(a)$

$\longrightarrow [(P(a) \vee Q(a))] \longrightarrow (\backsim R(a)) \wedge R(a)$

(Rule of universal instantiation)

$\longrightarrow R(a) \wedge [R(a) \longrightarrow \backsim [P(a) \vee Q(a)]]$

(by using commutative law and contrapositive)

$\longrightarrow \backsim [P(a) \vee Q(a)]$ (by the Modus Pones Rule $[P \wedge (P \longrightarrow Q)] \longrightarrow Q$)

$\longrightarrow \backsim P(a) \wedge \backsim Q(a)$ (by De Morgan Law)

$\longrightarrow \backsim Q(a) \wedge \backsim P(a)$ (by commutative law)

$\longrightarrow \backsim Q(a)$ (by rule of simplification)

Hence the are is valid. $\Longleftrightarrow$

- When a hypothetical statement $p \longrightarrow q$ is such that q is true whenever p is true, we say that p implies q i.e. $p \implies q$

- Example $[p \wedge (p \longrightarrow q)] \implies q$

  here p and $p \longrightarrow q$ are true the q is true

p	q	$p \longrightarrow q$	$p \wedge (p \longrightarrow q)$	q	$[p \wedge (p \longrightarrow q)] \longrightarrow q$
F	F	T	F	F	T
F	T	T	F	T	T
T	F	F	F	F	T
T	T	T	T	T	T

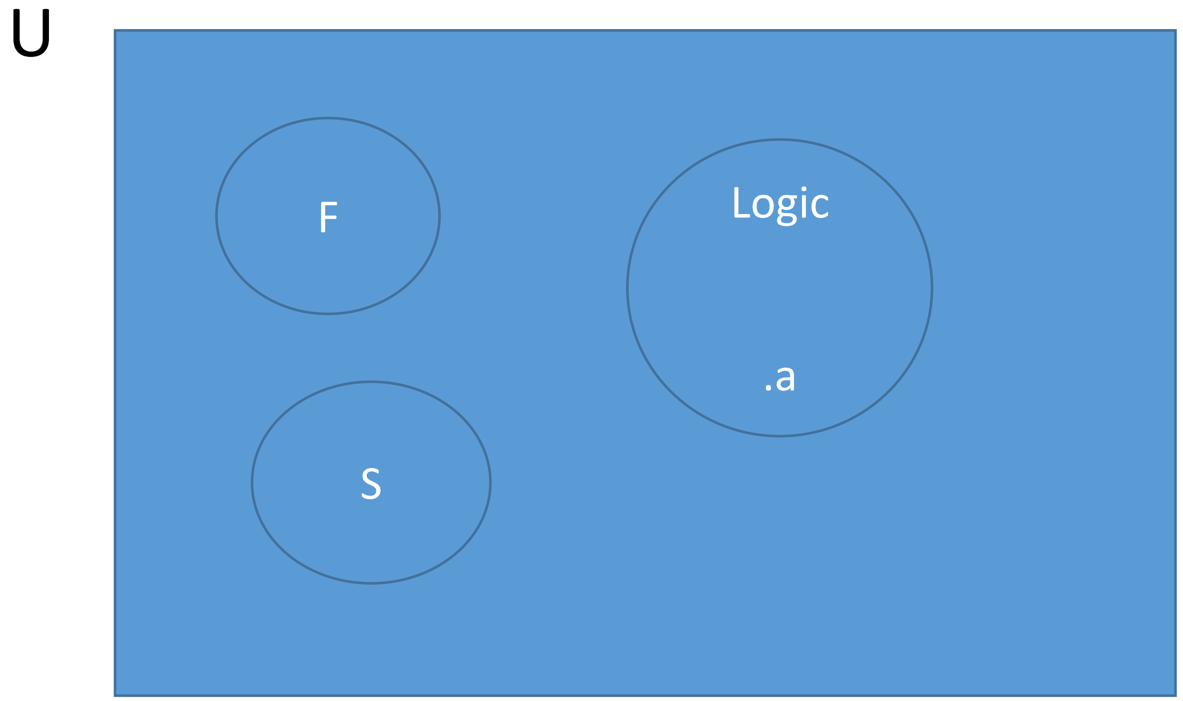
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TABLE 1 **Rules of Inference.**

Rule of Inference	*Tautology*	*Name*
$\begin{array}{rl} & p \\ & p \to q \\ \hline \therefore & q \end{array}$	$(p \wedge (p \to q)) \to q$	Modus ponens
$\begin{array}{rl} & \neg q \\ & p \to q \\ \hline \therefore & \neg p \end{array}$	$(\neg q \wedge (p \to q)) \to \neg p$	Modus tollens
$\begin{array}{rl} & p \to q \\ & q \to r \\ \hline \therefore & p \to r \end{array}$	$((p \to q) \wedge (q \to r)) \to (p \to r)$	Hypothetical syllogism
$\begin{array}{rl} & p \vee q \\ & \neg p \\ \hline \therefore & q \end{array}$	$((p \vee q) \wedge \neg p) \to q$	Disjunctive syllogism
$\begin{array}{rl} & p \\ \hline \therefore & p \vee q \end{array}$	$p \to (p \vee q)$	Addition
$\begin{array}{rl} & p \wedge q \\ \hline \therefore & p \end{array}$	$(p \wedge q) \to p$	Simplification
$\begin{array}{rl} & p \\ & q \\ \hline \therefore & p \wedge q \end{array}$	$((p) \wedge (q)) \to (p \wedge q)$	Conjunction
$\begin{array}{rl} & p \vee q \\ & \neg p \vee r \\ \hline \therefore & q \vee r \end{array}$	$((p \vee q) \wedge (\neg p \vee r)) \to (q \vee r)$	Resolution

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Example 3

- Prove the following argument is valid.

$$\forall x, [P(x) \vee Q(x)]$$
$$\exists x, \backsim P(x)$$
$$\forall x, [\backsim Q(x) \vee R(x)]$$
$$\forall x, [S(x) \longrightarrow \backsim R(x)]$$

$$\therefore\ \exists x, \backsim S(x)$$

- Given

$\forall x, [P(x) \vee Q(x)] \wedge [\exists x, \backsim P(x)]$

$\longrightarrow \{P(a) \vee Q(a)\} \wedge \{\backsim P(a)\}$ for some a in the universe

$\longrightarrow Q(a)$ by the rule of disjunctive syllogism

Now

$\{\forall x, [P(x) \vee Q(x)]\} \wedge \{\exists x, [\backsim P(x)]\} \wedge \{\forall x, [\backsim Q(x) \vee R(x)]\}$

$\longrightarrow Q(a) \wedge [\backsim Q(a) \vee R(a)]$ (Rule of universal instantiation)

$\longrightarrow R(a)$ by the rule of disjunctive syllogism

Now

$\{\forall x, [P(x) \vee Q(x)]\} \wedge \{\exists x, [\backsim P(x)]\} \wedge \{\forall x, [\backsim Q(x) \vee R(x)]\} \wedge \{\forall x, [S(x) \longrightarrow \backsim R(x)]\}$

$\longrightarrow R(a) \wedge [S(a) \longrightarrow \backsim R(a)]$ (Rule of universal instantiation)

$\longrightarrow \backsim S(a)$ by Modus Tollens rule $[\backsim q \wedge (p \longrightarrow q)] \longrightarrow \backsim p$

$\longrightarrow \exists x, \backsim S(x)$ (Rule of existential generalization)

VIT AP

Introduction to proofs

- A theorem may be the universal quantification of a conditional statement with one or more premises and a conclusion.

- A proof is a valid argument that establishes the truth of a theorem.
- Complicated proofs are usually easier to understand when they are proved using a series of lemmas, where each lemma is proved individual
- A corollary is a theorem that can be established directly from a theorem that has been proved.
- A conjecture is a statement that is being proposed to be a true statement, usually on the basis of some partial evidence, a heuristic argument, or the intuition of an expert.

Principle of Mathematical Induction

- Let P(n) be a statement defined for all integers $n \geq n_0$

The given statement is true if we can prove that

1. If $P(n_0)$ is true.
2. If P(K) is true for some $k > n_0$, then P(K+1) must be true.

Then, P(n) is true for all $n \geq n_0$.

This result is called the Principle of Mathematical Induction.

Example

- Prove by the principle of mathematical induction that

$11^{n+2} + 12^{2n+1}$ is divisible by 133.

Solution:

Let $P(n) = 11^{n+2} + 12^{2n+1}$

when n=1, $P(1) = 11^{(1+2)} + 12^{(2(1)+1)} = 3059$

P(1) is divisible by 133.

Les us assume that P(n) is true for n=k, $k \geq 1$ and

$\therefore$ P(k) is true.

hence $11^{k+2} + 12^{2k+1}$ is divisible by 133.

- $p(k+1) = 11^{k+3} + 12^{2k+3}$

$$= 11\left(11^{k+2}\right) + 12^2\left(12^{2k+1}\right)$$
$$= 11\left(11^{k+2}\right) + 144\left(12^{2k+1}\right)$$
$$= 11\left(11^{k+2}\right) + (11 + 133)\left(12^{2k+1}\right)$$
$$= 11\left(11^{k+2} + 12^{2k+1}\right) + 133\left(12^{2k+1}\right)$$
$$= 11(133m) + 133\left(12^{2k+1}\right)$$
$$= 133\left(11m + 12^{2k+1}\right)$$

which is divisible by 133,

∴ By principle of mathematical induction,

the statement is true for $n \geq 1$

Methods of proof

(1). Direct proof

(2). Indirect proof

- Given a set of hypothesis $P_1, P_2, \ldots P_n$ which can lead to a conclusion C

  i.e., $(P_1 \land P_2 \land \cdots \land P_n) \longrightarrow C$ is a tautology

  such a proof is called direct proof.

Direct proof

- The direct method of proving a conditional $p \longrightarrow q$ has the following lines of argument:

1. Hypothesis: First assume that p is true.
2. Analysis: Staring with the hypothesis and employing the rules/laws of logic and known facts, infer that q is true.
3. Conclusion: $p \longrightarrow q$ is true.

Example: 1

- Write the following argument using quantifiers, variables and predicates. Prove the validity of the argument.

  All healthy persons take fruit juice in breakfast.

  Anil does not take fruit juice in breakfast.

  Anil is not a healthy person.

Solution:

- $\forall x$ if *x* is a healthy person, then *x* takes fruit juice in breakfast.

Let P(x) : x is a healthy person.

Q(x): x takes fruit juice in breakfast.

a : Anil,

The symbolic form is

$$\begin{array}{l} \forall x,\ [P(x) \longrightarrow Q(x)] \\ \backsim Q(a) \\ \hline \therefore\ \backsim P(a) \end{array}$$

By Universal Modus Tollends, the argument is true.

Example 2

Prove that the product of two odd integers is an odd integer.

Solution:

Let m and n be two odd integers,

$m=2p+1$ and $n=2q+1$, where p and q are integers

$$\begin{aligned} \text{product} &= (2p+1)(2q+1) \\ &= 4pq + 2(p+q) + 1 \\ &= 2[2pq + p + q] + 1, \end{aligned}$$

which is an odd integer.

Indirect proof

- Proofs that are not direct are called indirect proofs

There are two kinds of indirect proofs:

1. Proof by contraposition
2. Proof by contradiction

Proof by contraposition

- To prove $p \longrightarrow q$ by contraposition

  we have to show that

$$\backsim q \longrightarrow \backsim p$$

Example 1

- Prove that if $n^2$is even, then n is even, where n be an integer.

solution:

Let p: $n^2$is even

q: n is even

to prove that $p \longrightarrow q$

by proof of contrapositive to show that $\backsim q \longrightarrow \backsim p$

Let n is not even $\Longrightarrow n$ is odd.

here $n = 2m + 1$

$$n^2 = (2m+1)^2 = 4m^2 + 4m + 1 = 2(2m^2 + 2m) + 1 = \overset{\text{even}}{2r} + 1$$

$\therefore\ n^2$ is odd

Which is a contrapositive of the given statement.

Hence, if $n^2$is even, the n is even.

Example 2

- Prove that if $(m+n) \geq 73$, then $m \geq 37$ or $\mathrm{n} \geq 37$,
  where m and n are positive integers.

Solution:

Let p: $(m+n) \geq 73$

q: $(m \geq 37$ or $\mathrm{n} \geq 37)$

Given $p \longrightarrow q$ by contraposition to prove $\backsim q \longrightarrow \backsim p$

i.e., (not $m \geq 37$ and not $\mathrm{n} \geq 37) \rightarrow$ not $(m+n \geq 73)$

Take(not $m \geq 37$ and not $\mathrm{n} \geq 37)$

$\Longrightarrow m \leq 36$ and $n \leq 36$

$\Longrightarrow m+n \leq (36+36)$

$\Longrightarrow m+n \leq 72$

$\Longrightarrow \mathit{not}\ (m+n) \geq 73$

Hence the proof.

Proof by contradiction

- The lines of argument in this method of proof of the statement $p \longrightarrow q$ as follows:

1. Hypothesis: First assume that $p \longrightarrow q$ is false. i.e., assume p is true and q is false.
2. Analysis: Staring with the hypothesis that q is false and employing the rules/laws of logic and known facts, infer that p is false. This contradicts that the assumption is true.
3. Conclusion: Because of the contradiction arrived in the analysis, $p \longrightarrow q$ is true.

Proof by contradiction

- To prove statement $p \longrightarrow q$ by contradiction

  Here we assume just the negative (opposite) of the given statement.

If this assumption leads to a contradiction, then our assumption is false.

Hence, the given statement is true.

Such a proof is called a <u>proof by contradiction</u>.

Example

- Prove that there is <u>no</u> rational number ${}^{p}/_{q}$ whose square is 2, i.e., show that $\sqrt{2}$ is an irrational number.
- <u>Solution</u>:

  Let us assume a rational number $({}^{p}/_{q})$ whose square is 2

$\Longrightarrow \left(\frac{p}{q}\right)^2 = 2$ where p and q are integers and have no common factor, $q \neq 0$

$\Longrightarrow p^2 = 2q^2$ i.e., p^2 is even $\Longrightarrow p$ is even

$let\ \ p = 2m$

$\therefore\ p^2 = 4\,m^2 = 2q^2$

$\Longrightarrow q^2 = 2m^2$

$\therefore\ q^2$ is <u>even</u> and q is <u>even</u>.

Here p and q are even and have a common factor 2, which is a contradiction to the assumption that p and q have no common factors.

Hence, our assumption that $\sqrt{2}$ is an rational number is wrong.

$\therefore \sqrt{2}$ is an irrational number.

Example:

- Prove that if $(3n + 2)$ is even, then n is even,

  where n is an integer

  using
  - a proof by contraposition.
  - a proof by contradiction.

Solution:

n is an integer, if 3n + 2 is even, then n is even

(i). By contraposition i.e., to show that $\backsim q \longrightarrow \backsim p$

to prove

If n is not even then (3n+2) is not even

Or

If n is odd then (3n+2) is odd.

Let n is odd i.e., n = 2k+1

Now, 3n+2 = 3(2k+1) +2 = 6k +5 = 2(3k+2) +1

$\therefore (3n+2)$ is odd

Hence, If (3n+2) is even, then n is even.

Solution:

n is an integer, if $3n + 2$ is even, then n is even

(i). By contradiction,

Let us assume n is odd i.e., $n=2k+1$

$3n+2=3(2k+1)+2 = 6k+5 = 2(3k+2)+1$

$\therefore$ $(3n+2)$ is odd.

which a contradiction to hypothesis that $(3n+2)k$ is even.

Hence, by contradiction,

if $3n + 2$ is even, then n is even

- Give

  (i) a direct proof

  (ii) an indirect proof and

" If n is an odd integer, then $n + 11$ is an even integer."

Practise problems

- Test the validity of the argument

  I will be a physicist or I will be a mathematician

  I will not be a mathematician

  ∴ I will be a physicist

- Is the following argument valid?

  If income tax rates are lowered, income tax collection increases

  Income tax collection increases

  ∴ Income tax rates are lowered

- Test whether the following is a valid argument

If I study, then I do not fail in the examination.
If I do not fail in the examination, my father gifts a two-wheeler to me.
____
∴ If I study then my father gifts a two wheeler to me.

- Test whether the following is a valid argument

If Nixon is not re-elected, then Tulsa will lose its air base.
Nixon will be re-elected if and only if Tulsa votes for him.
If Tulsa keeps its air base, Nixon will be re-elected.
____
∴ Nixon will be re-elected.

Practice problems

For each of these arguments determine whether the argument is correct or incorrect and explain why.

a) All students in this class understand logic. Xavier is a student in this class. Therefore, Xavier understands logic.

b) Every computer science major takes discrete mathematics. Natasha is taking discrete mathematics. Therefore, Natasha is a computer science major.

c) All parrots like fruit. My pet bird is not a parrot. Therefore, my pet bird does not like fruit.

d) Everyone who eats granola everyday is healthy. Linda is not healthy. Therefore, Linda does not eat granola every day.

Answers:

- A) This is correct, using universal instantiation and modus ponens.
- B) This is invalid. After applying universal instantiation, it contains the fallacy of affirming the conclusion.
- C)) This is invalid. After applying universal instantiation, it contains the fallacy of denying the hypothesis.
- D) This is valid by universal instantiation and modus tollens.

Example

Express each of these system specifications using predicates, quantifiers and logical connectives

- At least one mail message can be saved if there is a disk with more than 10 kilobytes of free space.
- Whenever there is an active alert, all queued messages are transmitted.
- The diagnostic monitor tracks the status of all systems except the main console.
- Each participant on the conference call whom the host of the call did not put on a special list was billed.