Partial Order Relations

Definition:

A relation R on a set S is called a ***partial ordering* or *partial order*** if it is reflexive, antisymmetric, and transitive. A set S together with a partial ordering R is called a *partially ordered set*, or *poset*, and is denoted by (S, R). Members of S are called *elements* of the poset.

EXAMPLE 1

Show that the "greater than or equal" relation ($\geq$) is a partial ordering on the set of integers.

Solution:

Because $a \geq a$ for every integer a, $\geq$ is reflexive.

If $a \geq b$ and $b \geq a$, then $a = b$. Hence, $\geq$ is antisymmetric.

Finally, $\geq$ is transitive because $a \geq b$ and $b \geq c$ imply that $a \geq c$.

It follows that $\geq$ is a partial ordering on the set of integers and $(\boldsymbol{Z}, \geq)$ is a poset.

EXAMPLE 2

The divisibility relation $|$ is a partial ordering on the set of positive integers, because it is reflexive, antisymmetric, and transitive, as was shown in Section 9.1. We see that $(\mathbf{Z}^+, |)$ is a poset. Recall that ($\mathbf{Z}^+$ denotes the set of positive integers.) ◀

EXAMPLE 3

Show that the inclusion relation $\subseteq$ is a partial ordering on the power set of a set S.

Solution: Because $A \subseteq A$ whenever A is a subset of S, $\subseteq$ is reflexive. It is antisymmetric because $A \subseteq B$ and $B \subseteq A$ imply that $A = B$. Finally, $\subseteq$ is transitive, because $A \subseteq B$ and $B \subseteq C$ imply that $A \subseteq C$. Hence, $\subseteq$ is a partial ordering on $P(S)$, and $(P(S), \subseteq)$ is a poset. ◀

EXAMPLE 4

Let R be the relation on the set of people such that $x\,R\,y$ if x and y are people and x is older than y. Show that R is not a partial ordering.

Solution: Note that R is antisymmetric because if a person x is older than a person y, then y is not older than x. That is, if $x\,R\,y$, then $y \not R\, x$. The relation R is transitive because if person x is older than person y and y is older than person z, then x is older than z. That is, if $x\,R\,y$ and $y\,R\,z$, then $x\,R\,z$. However, R is not reflexive, because no person is older than himself or herself. That is, $x \not R\, x$ for all people x. It follows that R is not a partial ordering. ◀

DEFINITION 2

The elements a and b of a poset $(S,)$ are called *comparable* if either

$a \preccurlyeq b$ or $b \preccurlyeq a$.

When a and b are elements of S such that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, a and b are called *incomparable*.

EXAMPLE 5

In the poset $(\mathbf{Z}^+, |)$, are the integers 3 and 9 comparable? Are 5 and 7 comparable?

Solution: The integers 3 and 9 are comparable, because $3 \mid 9$. The integers 5 and 7 are incomparable, because $5 \nmid 7$ and $7 \nmid 5$. ◀

The adjective "partial" is used to describe partial orderings because pairs of elements may be incomparable. When every two elements in the set are comparable, the relation is called a **total ordering**.

DEFINITION 3

If $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a *totally ordered* or *linearly ordered set*, and $\preccurlyeq$ is called a *total order* or a *linear order*. A totally ordered set is also called a *chain*.

Example: 6

The poset $(\mathbf{Z}, \leq)$ is totally ordered, because $a \leq b$ or $b \leq a$ whenever a and b are integers.

Example: 7

The poset $(\mathbf{Z}^{+}, |)$ is not totally ordered because it contains elements that are incomparable, such as 5 and 7.

DEFINITION 4

$(S, \preccurlyeq)$ is a *well-ordered set* if it is a poset such that is $\preccurlyeq$a total ordering and every nonempty subset of S has a least element.

EXAMPLE 8

The set of ordered pairs of positive integers, $\mathbf{Z}^+ \times \mathbf{Z}^+$, with $(a_1, a_2) \preccurlyeq (b_1, b_2)$ if $a_1 < b_1$, or if $a_1 = b_1$ and $a_2 \leq b_2$ (the lexicographic ordering), is a well-ordered set. The verification of this is left as Exercise 53. The set $\mathbf{Z}$, with the usual $\leq$ ordering, is not well-ordered because the set of negative integers, which is a subset of $\mathbf{Z}$, has no least element. ◀

THEOREM 1

THE PRINCIPLE OF WELL-ORDERED INDUCTION Suppose that S is a well-ordered set. Then $P(x)$ is true for all $x \in S$, if

INDUCTIVE STEP: For every $y \in S$, if $P(x)$ is true for all $x \in S$ with $x \prec y$, then $P(y)$ is true.

Lexicographic Order

- To figure out which of two words comes first in an English dictionary, you compare their letters one by one from left to right. If all letters have been the same to a certain point and one word runs out of letters, that word comes first in the dictionary.

- For example, *play* comes before *playhouse*. If all letters up to a certain point are the same and the next letters differ, then the word whose next letter is located earlier in the alphabet comes first in the dictionary. For instance, *playhouse* comes *before playmate*.

Lexicographic Order

More generally, if A is any set with a partial order relation, then a *dictionary* or *lexicographic* order can be defined on a set of strings over A as indicated in the following theorem.

Theorem [redacted]

Let A be a set with a partial order relation R, and let S be a set of strings over A. Define a relation $\preceq$ on S as follows:

For any two strings in S, $a_1a_2\cdots a_m$ and $b_1b_2\cdots b_n$, where m and n are positive integers,

1. If $m \le n$ and $a_i = b_i$ for all $i = 1, 2, \ldots, m$, then
$$a_1a_2\cdots a_m \preceq b_1b_2\cdots b_n.$$
2. If for some integer k with $k \le m$, $k \le n$, and $k \ge 1$, $a_i = b_i$ for all $i = 1, 2, \ldots, k-1$, and $a_k \ne b_k$, but $a_k \; R \; b_k$ then
$$a_1a_2\cdots a_m \preceq b_1b_2\cdots b_n.$$
3. If ε is the null string and s is any string in S, then $\epsilon \preceq s$.

If no strings are related other than by these three conditions, then $\preceq$ is a partial order relation.

Lexicographic Order

• Definition

The partial order relation of Theorem [illegible] is called the **lexicographic order for S** that corresponds to the partial order R on A.

Example 6 – *A Lexicographic Order*

- Let $A = \{x, y\}$ and let R be the following partial order relation on A:

$$R = \{(x, x), (x, y), (y, y)\}.$$

- Let S be the set of all strings over A, and denote by $\preceq$ the lexicographic order for S that corresponds to R.

- **a.** Is $x \preceq xx$? $x \preceq xy$? $xx \preceq xxx$? $yxy \preceq yxyxxx$?
- **b.** Is $x \preceq y$? $xx \preceq xyx$? $xxxy \preceq xy$? $yxyxxyy \preceq yxyxy$?
- **c.** Is $\epsilon \preceq x$? $\epsilon \preceq xy$? $\epsilon \preceq yyxy$?

Example 6 – Solution

- **a.** Yes in all cases, by relation (1) in theorem 8.5.1.

- **b.** Yes in all cases, by relation (2) in theorem 8.5.1.

- **c.** Yes in all cases, by relation (3) in theorem 8.5.1.

EXAMPLE 9

Determine whether $(3, 5) \prec (4, 8)$, whether $(3, 8) \prec (4, 5)$, and whether $(4, 9) \prec (4, 11)$ in the poset $(\mathbf{Z} \times \mathbf{Z}, \preccurlyeq)$, where $\preccurlyeq$ is the lexicographic ordering constructed from the usual $\leq$ relation on $\mathbf{Z}$.

Solution: Because $3 < 4$, it follows that $(3, 5) \prec (4, 8)$ and that $(3, 8) \prec (4, 5)$. We have $(4, 9) \prec (4, 11)$, because the first entries of $(4, 9)$ and $(4, 11)$ are the same but $9 < 11$. ◀

EXAMPLE 10

Note that $(1, 2, 3, 5) \prec (1, 2, 4, 3)$, because the entries in the first two positions of these 4-tuples agree, but in the third position the entry in the first 4-tuple, 3, is less than that in the second 4-tuple, 4. (Here the ordering on 4-tuples is the lexicographic ordering that comes from the usual "less than or equals" relation on the set of integers.) ◀

In Figure 1 the ordered pairs in $\mathbf{Z}^+ \times \mathbf{Z}^+$ that are less than $(3, 4)$ are highlighted. A lexicographic ordering can be defined on the Cartesian product of n posets $(A_1, \preccurlyeq_1)$, $(A_2, \preccurlyeq_2), \ldots, (A_n, \preccurlyeq_n)$. Define the partial ordering $\preccurlyeq$ on $A_1 \times A_2 \times \cdots \times A_n$ by

$$(a_1, a_2, \ldots, a_n) \prec (b_1, b_2, \ldots, b_n)$$

if $a_1 \prec_1 b_1$, or if there is an integer $i > 0$ such that $a_1 = b_1, \ldots, a_i = b_i$, and $a_{i+1} \prec_{i+1} b_{i+1}$. In other words, one n-tuple is less than a second n-tuple if the entry of the first n-tuple in the first position where the two n-tuples disagree is less than the entry in that position in the second n-tuple.

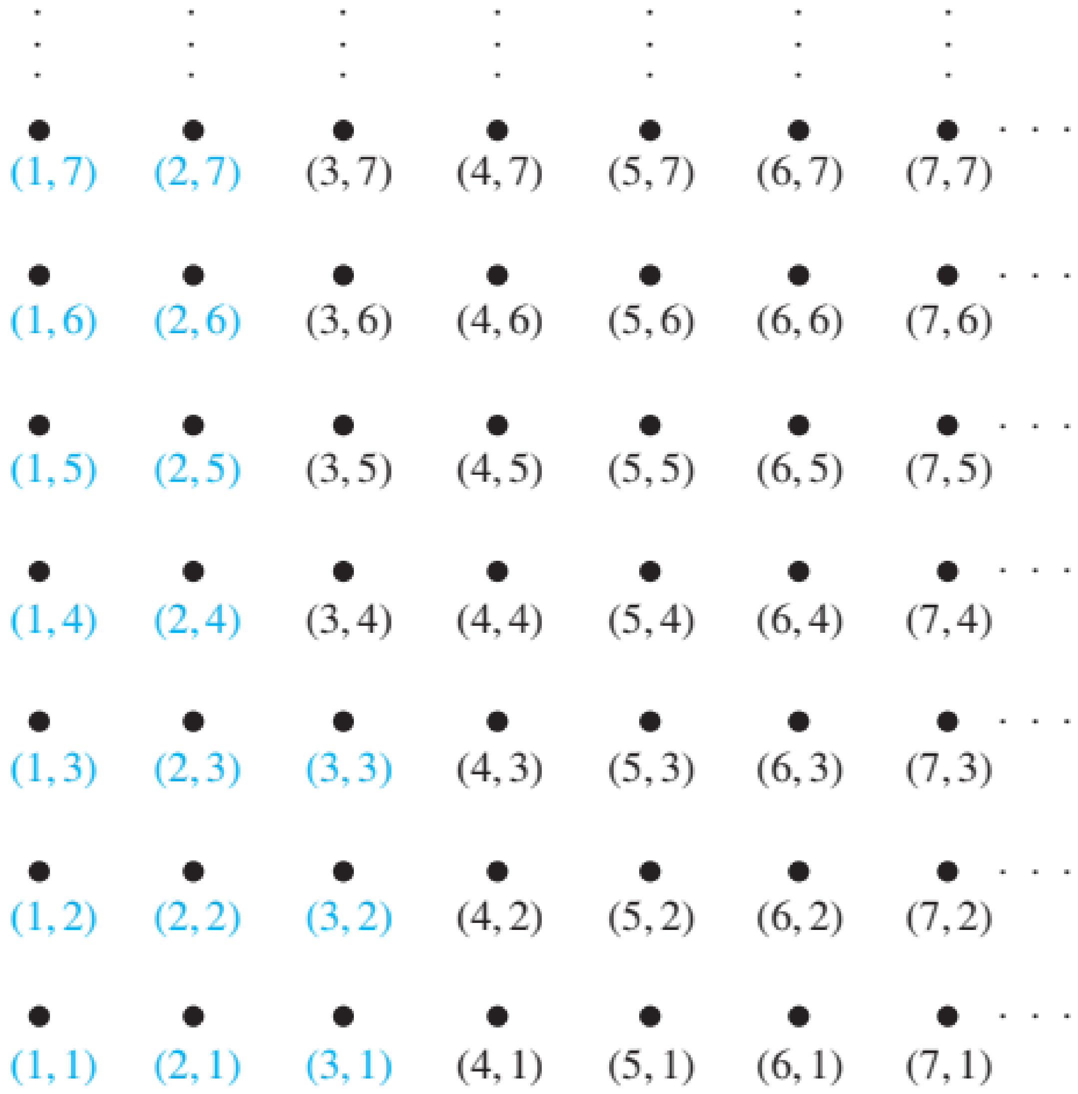

FIGURE 1 **The Ordered Pairs Less Than (3, 4) in Lexicographic Order.**

EXAMPLE 11

Consider the set of strings of lowercase English letters. Using the ordering of letters in the alphabet, a lexicographic ordering on the set of strings can be constructed. A string is less than a second string if the letter in the first string in the first position where the strings differ comes before the letter in the second string in this position, or if the first string and the second string agree in all positions, but the second string has more letters. This ordering is the same as that used in dictionaries. For example,

$$discreet \prec discrete,$$

because these strings differ first in the seventh position, and $e \prec t$. Also,

$$discreet \prec discreetness,$$

because the first eight letters agree, but the second string is longer. Furthermore,

$$discrete \prec discretion,$$

because

$$discrete \prec discreti.$$

Hasse Diagrams

- If S is finite, we can visually depict a partially ordered set $(S,\preceq)$ by using a **hasse diagram**. Each of the elements of S is represented by a dot, called a **node**, or **vertex**, of the diagram. If x is an immediate predecessor of y, then the node for y is placed above the node for x and the two nodes are connected by a straight-line segment.

EXAMPLE 10

Consider $\wp(\{1, 2\})$ under the relation of set inclusion. This is a partially ordered set, a restriction of the partially ordered set $(\wp(\mathbb{N}), \subseteq)$. The elements of $\wp(\{1, 2\})$ are $\varnothing$, $\{1\}$, $\{2\}$, and $\{1, 2\}$. The binary relation $\subseteq$ consists of the following ordered pairs:

$$(\varnothing, \varnothing), (\{1\}, \{1\}), (\{2\}, \{2\}), (\{1, 2\}, \{1, 2\}), (\varnothing, \{1\}),$$
$$(\varnothing, \{2\}), (\varnothing, \{1, 2\}), (\{1\}, \{1, 2\}), (\{2\}, \{1, 2\})$$

The Hasse diagram of this partially ordered set appears in Figure 5.2. Note that although $\varnothing$ is not an immediate predecessor of $\{1, 2\}$, it is a predecessor of $\{1, 2\}$ (shown in the diagram by the chain of upward line segments connecting $\varnothing$ with $\{1, 2\}$).

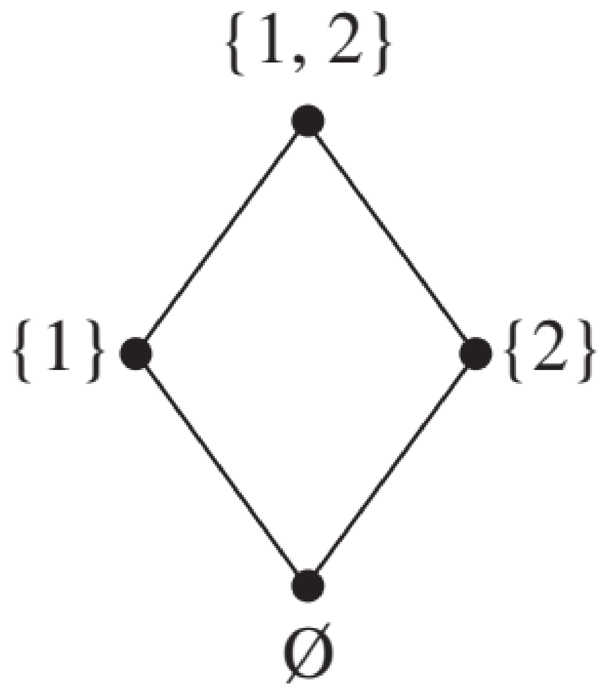

Figure 5.2

REMINDER

Two nodes in a Hasse diagram should never be joined by a horizontal line.

Many edges in the directed graph for a finite poset do not have to be shown because they must be present. For instance, consider the directed graph for the partial ordering $\{(a, b) \mid a \leq b\}$ on the set $\{1, 2, 3, 4\}$, shown in Figure 2(a). Because this relation is a partial ordering, it is reflexive, and its directed graph has loops at all vertices. Consequently, we do not have to show these loops because they must be present; in Figure 2(b) loops are not shown. Because a partial ordering is transitive, we do not have to show those edges that must be present because of transitivity. For example, in Figure 2(c) the edges $(1, 3)$, $(1, 4)$, and $(2, 4)$ are not shown because they must be present. If we assume that all edges are pointed "upward" (as they are drawn in the figure), we do not have to show the directions of the edges; Figure 2(c) does not show directions.

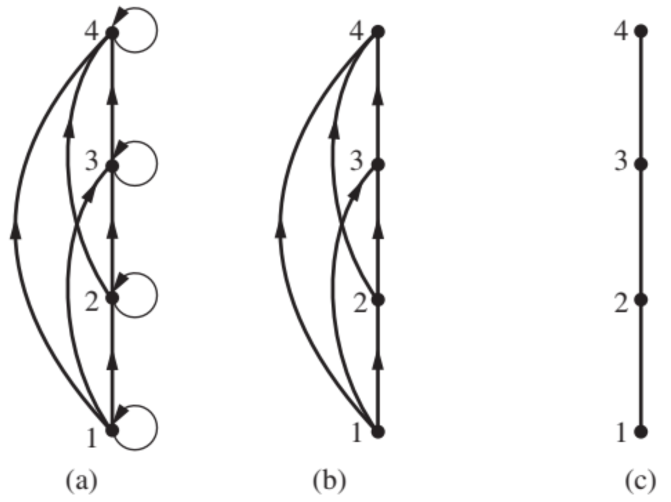

FIGURE 2 **Constructing the Hasse Diagram for ({1, 2, 3, 4}, ≤).**

Let $(S, \preccurlyeq)$ be a poset. We say that an element $y \in S$ **covers** an element $x \in S$ if $x \prec y$ and there is no element $z \in S$ such that $x \prec z \prec y$. The set of pairs (x, y) such that y covers x is called the **covering relation** of $(S, \preccurlyeq)$. From the description of the Hasse diagram of a poset, we see that the edges in the Hasse diagram of $(S, \preccurlyeq)$ are upwardly pointing edges corresponding to the pairs in the covering relation of $(S, \preccurlyeq)$. Furthermore, we can recover a poset from its covering relation, because it is the reflexive transitive closure of its covering relation. (Exercise 31 asks for a proof of this fact.) This tells us that we can construct a partial ordering from its Hasse diagram.

Draw the Hasse diagram representing the partial ordering $\{(a, b) \mid a \text{ divides } b\}$ on $\{1, 2, 3, 4, 6, 8, 12\}$.

Solution: Begin with the digraph for this partial order, as shown in Figure 3(a). Remove all loops, as shown in Figure 3(b). Then delete all the edges implied by the transitive property. These are (1, 4), (1, 6), (1, 8), (1, 12), (2, 8), (2, 12), and (3, 12). Arrange all edges to point upward, and delete all arrows to obtain the Hasse diagram. The resulting Hasse diagram is shown in Figure 3(c).

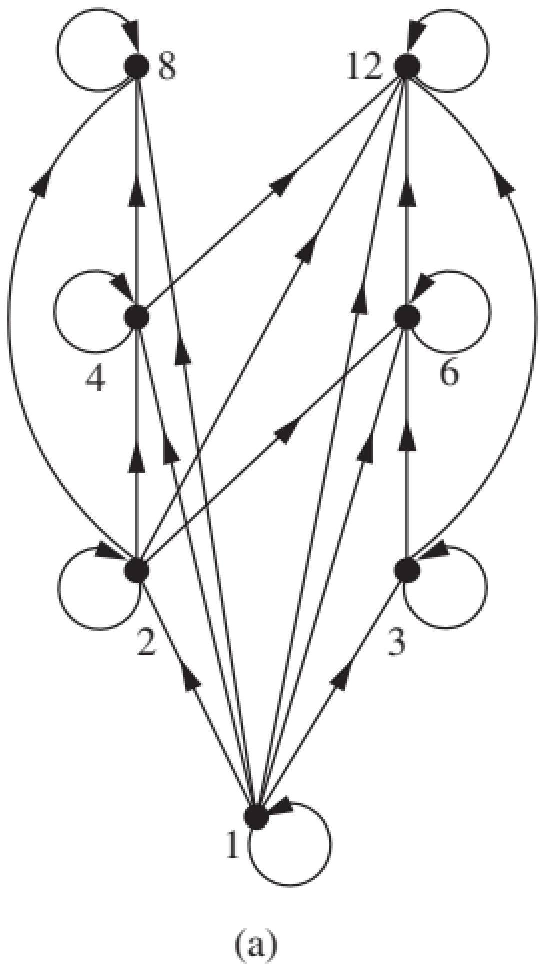

FIGURE 3 **Constructing the Hasse Diagram of ({1, 2, 3, 4, 6, 8, 12}, |).**

EXAMPLE 13

Draw the Hasse diagram for the partial ordering $\{(A, B) \mid A \subseteq B\}$ on the power set $P(S)$ where $S = \{a, b, c\}$.

Solution: The Hasse diagram for this partial ordering is obtained from the associated digraph by deleting all the loops and all the edges that occur from transitivity, namely, $(\emptyset, \{a, b\})$, $(\emptyset, \{a, c\})$, $(\emptyset, \{b, c\})$, $(\emptyset, \{a, b, c\})$, $(\{a\}, \{a, b, c\})$, $(\{b\}, \{a, b, c\})$, and $(\{c\}, \{a, b, c\})$. Finally all edges point upward, and arrows are deleted. The resulting Hasse diagram is illustrated in Figure 4. ◀

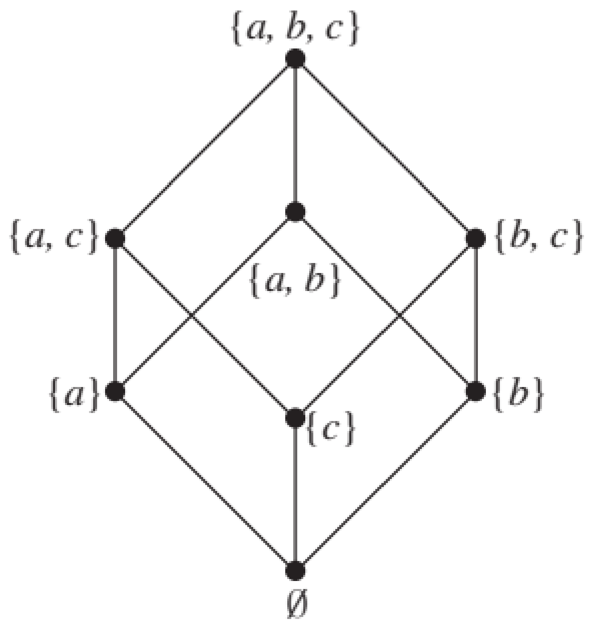

FIGURE 4 The Hasse Diagram of $(P(\{a, b, c\}), \subseteq)$.