Partial Order Sets-Part-2

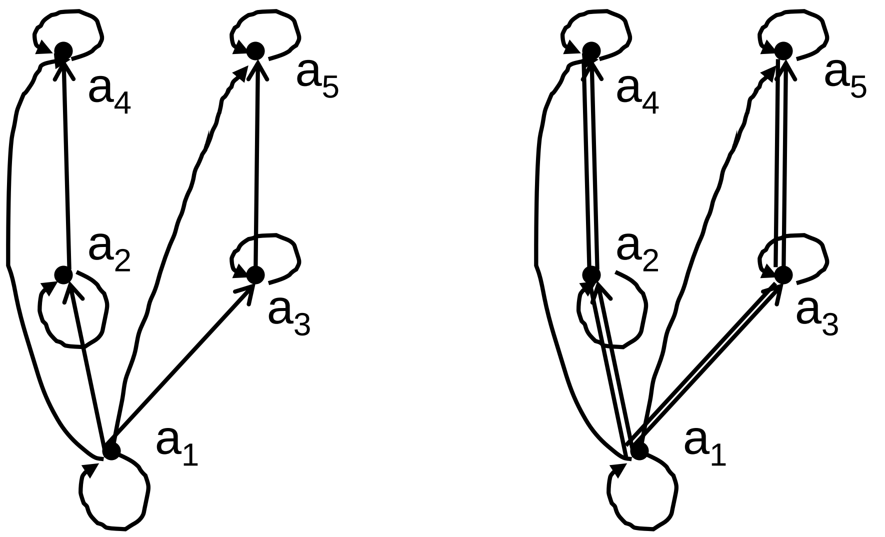
Maximal and Minimal Elements

Elements of posets that have certain extremal properties are important for many applications. An element of a poset is called maximal if it is not less than any element of the poset. That is, a is **maximal** in the poset $(S, \preccurlyeq)$ if there is no $b \in S$ such that $a \prec b$. Similarly, an element of a poset is called minimal if it is not greater than any element of the poset. That is, a is **minimal** if there is no element $b \in S$ such that $b \prec a$. Maximal and minimal elements are easy to spot using a Hasse diagram. They are the "top" and "bottom" elements in the diagram.

Hasse Diagrams

- Like relations and functions, partial orders have a convenient graphical representation: Hasse Diagrams
  - Consider the <u>digraph</u> representation of a partial order
  - Because we are dealing with a partial order, we know that the relation must be reflexive and transitive
  - Thus, we can simplify the graph as follows
    - Remove all self loops
    - Remove all transitive edges
    - Remove directions on edges assuming that they are oriented upwards
  - The resulting diagram is far simpler

Hasse Diagram: Example



EXAMPLE 14

Which elements of the poset $(\{2, 4, 5, 10, 12, 20, 25\}, |)$ are maximal, and which are minimal?

Solution:

The Hasse diagram in Figure 5 for this poset shows that the maximal elements are 12, 20, and 25, and the minimal elements are 2 and 5. As this example shows, a poset can have more than one maximal element and more than one minimal element

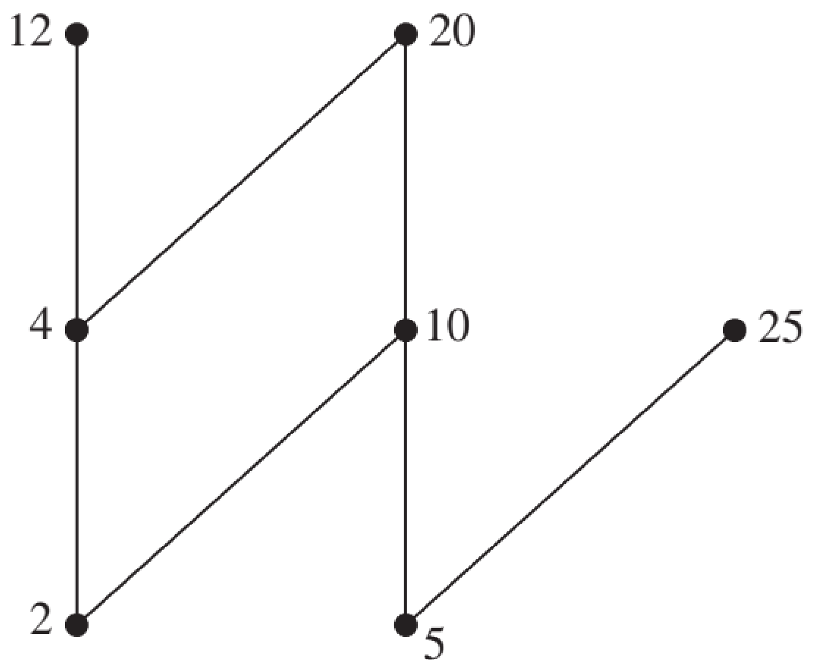


FIGURE 5 **The Hasse Diagram of a Poset.**

- Sometimes there is an element in a poset that is greater than every other element. Such an element is called the greatest element. That is, a is the **greatest element** of the poset $(S, \preccurlyeq)$ if $b \preccurlyeq a$ for all $b \in S$. The greatest element is unique when it exists [see Exercise 40(a)]. Likewise, an element is called the least element if it is less than all the other elements in the poset. That is, a is the **least element** of $(S, \preccurlyeq)$ if $a \preccurlyeq b$ for all $b \in S$. The least element is unique when it exists [see Exercise 40(b)].

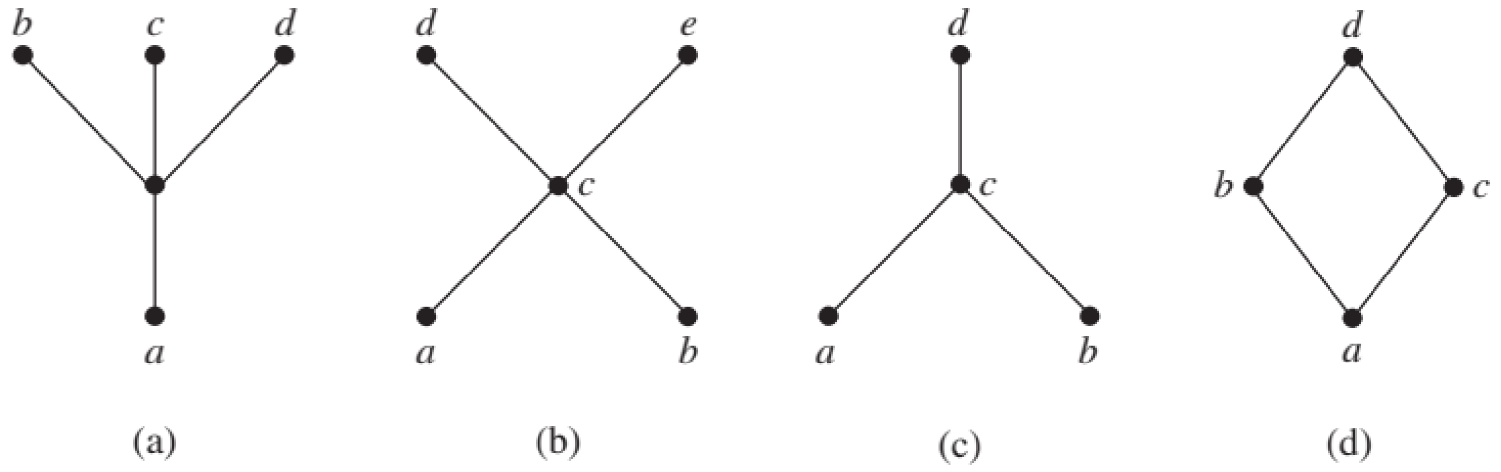


FIGURE 6 **Hasse Diagrams of Four Posets.**

EXAMPLE 15

Determine whether the posets represented by each of the Hasse diagrams in Figure 6 have a greatest element and a least element.

Solution:

The least element of the poset with Hasse diagram (a) is a. This poset has no greatest element.

The poset with Hasse diagram (b) has neither a least nor a greatest element.

The poset with Hasse diagram (c) has no least element. Its greatest element is d.

The poset with Hasse diagram (d) has least element a and greatest element d.

EXAMPLE 16

Let S be a set. Determine whether there is a greatest element and a least element in the poset $(P(S), \subseteq)$.

Solution: The least element is the empty set, because $\emptyset \subseteq T$ for any subset T of S. The set S is the greatest element in this poset, because $T \subseteq S$ whenever T is a subset of S. ◀

EXAMPLE 17

Is there a greatest element and a least element in the poset $(\mathbf{Z}^+, |)$?

Solution: The integer 1 is the least element because $1|n$ whenever n is a positive integer. Because there is no integer that is divisible by all positive integers, there is no greatest element. ◀

Sometimes it is possible to find an element that is greater than or equal to all the elements in a subset A of a poset $(S, \preccurlyeq)$. If u is an element of S such that $a \preccurlyeq u$ for all elements $a \in A$, then u is called an **upper bound** of A. Likewise, there may be an element less than or equal to all the elements in A. If l is an element of S such that $l \preccurlyeq a$ for all elements $a \in A$, then l is called a **lower bound** of A.

EXAMPLE 18

Find the lower and upper bounds of the subsets $\{a, b, c\}, \{j, h\}$, and $\{a, c, d, f\}$ in the poset with the Hasse diagram shown in Figure 7.

Solution: The upper bounds of $\{a, b, c\}$ are e, f, j, and h, and its only lower bound is a. There are no upper bounds of $\{j, h\}$, and its lower bounds are a, b, c, d, e, and f. The upper bounds of $\{a, c, d, f\}$ are f, h, and j, and its lower bound is a.

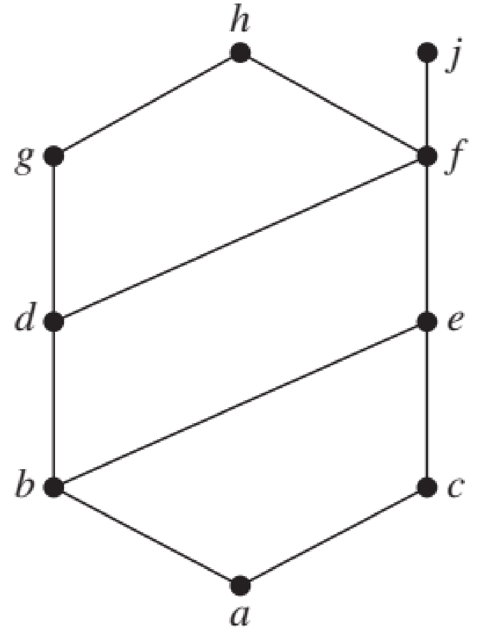


FIGURE 7 The Hasse Diagram of a Poset.

The element x is called the **least upper bound** of the subset A if x is an upper bound that is less than every other upper bound of A. Because there is only one such element, if it exists, it makes sense to call this element *the* least upper bound [see Exercise 42(a)]. That is, x is the least upper bound of A if $a \preccurlyeq x$ whenever $a \in A$, and $x \preccurlyeq z$ whenever z is an upper bound of A. Similarly, the element y is called the **greatest lower bound** of A if y is a lower bound of A and $z \preccurlyeq y$ whenever z is a lower bound of A. The greatest lower bound of A is unique if it exists [see Exercise 42(b)]. The greatest lower bound and least upper bound of a subset A are denoted by glb(A) and lub(A), respectively.

EXAMPLE 19

Find the greatest lower bound and the least upper bound of $\{b, d, g\}$, if they exist, in the poset shown in Figure 7.

Solution: The upper bounds of $\{b, d, g\}$ are g and h. Because $g \prec h$, g is the least upper bound. The lower bounds of $\{b, d, g\}$ are a and b. Because $a \prec b$, b is the greatest lower bound. ◀

EXAMPLE 20

Find the greatest lower bound and the least upper bound of the sets $\{3, 9, 12\}$ and $\{1, 2, 4, 5, 10\}$, if they exist, in the poset $(\mathbf{Z}^+, |)$.

Solution: An integer is a lower bound of $\{3, 9, 12\}$ if 3, 9, and 12 are divisible by this integer. The only such integers are 1 and 3. Because $1 \mid 3$, 3 is the greatest lower bound of $\{3, 9, 12\}$. The only lower bound for the set $\{1, 2, 4, 5, 10\}$ with respect to $|$ is the element 1. Hence, 1 is the greatest lower bound for $\{1, 2, 4, 5, 10\}$.

An integer is an upper bound for $\{3, 9, 12\}$ if and only if it is divisible by 3, 9, and 12. The integers with this property are those divisible by the least common multiple of 3, 9, and 12, which is 36. Hence, 36 is the least upper bound of $\{3, 9, 12\}$. A positive integer is an upper bound for the set $\{1, 2, 4, 5, 10\}$ if and only if it is divisible by 1, 2, 4, 5, and 10. The integers with this property are those integers divisible by the least common multiple of these integers, which is 20. Hence, 20 is the least upper bound of $\{1, 2, 4, 5, 10\}$. ◀

Lattices

A partially ordered set in which every pair of elements has both a least upper bound and a greatest lower bound is called a **lattice**. Lattices have many special properties. Furthermore, lattices are used in many different applications such as models of information flow and play an important role in Boolean algebra.

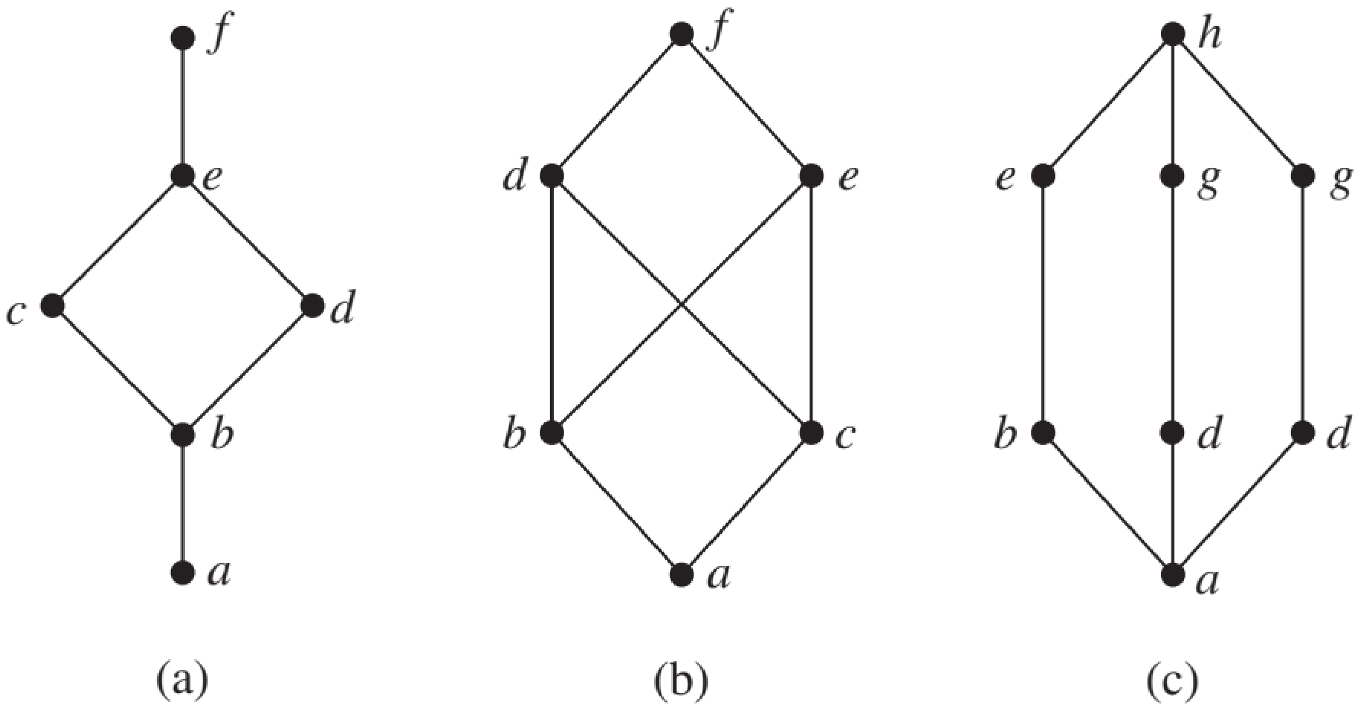


FIGURE 8 **Hasse Diagrams of Three Posets.**

EXAMPLE 21

Determine whether the posets represented by each of the Hasse diagrams in Figure 8 are lattices.

Solution: The posets represented by the Hasse diagrams in (a) and (c) are both lattices because in each poset every pair of elements has both a least upper bound and a greatest lower bound, as the reader should verify. On the other hand, the poset with the Hasse diagram shown in (b) is not a lattice, because the elements b and c have no least upper bound. To see this, note that each of the elements d, e, and f is an upper bound, but none of these three elements precedes the other two with respect to the ordering of this poset. ◀

EXAMPLE 22

Is the poset $(\mathbf{Z}^+, |)$ a lattice?

Solution: Let a and b be two positive integers. The least upper bound and greatest lower bound of these two integers are the least common multiple and the greatest common divisor of these integers, respectively, as the reader should verify. It follows that this poset is a lattice. ◀

EXAMPLE 23

Determine whether the posets $(\{1, 2, 3, 4, 5\}, |)$ and $(\{1, 2, 4, 8, 16\}, |)$ are lattices.

Solution: Because 2 and 3 have no upper bounds in $(\{1, 2, 3, 4, 5\}, |)$, they certainly do not have a least upper bound. Hence, the first poset is not a lattice.

Every two elements of the second poset have both a least upper bound and a greatest lower bound. The least upper bound of two elements in this poset is the larger of the elements and the greatest lower bound of two elements is the smaller of the elements, as the reader should verify. Hence, this second poset is a lattice. ◀

EXAMPLE 24

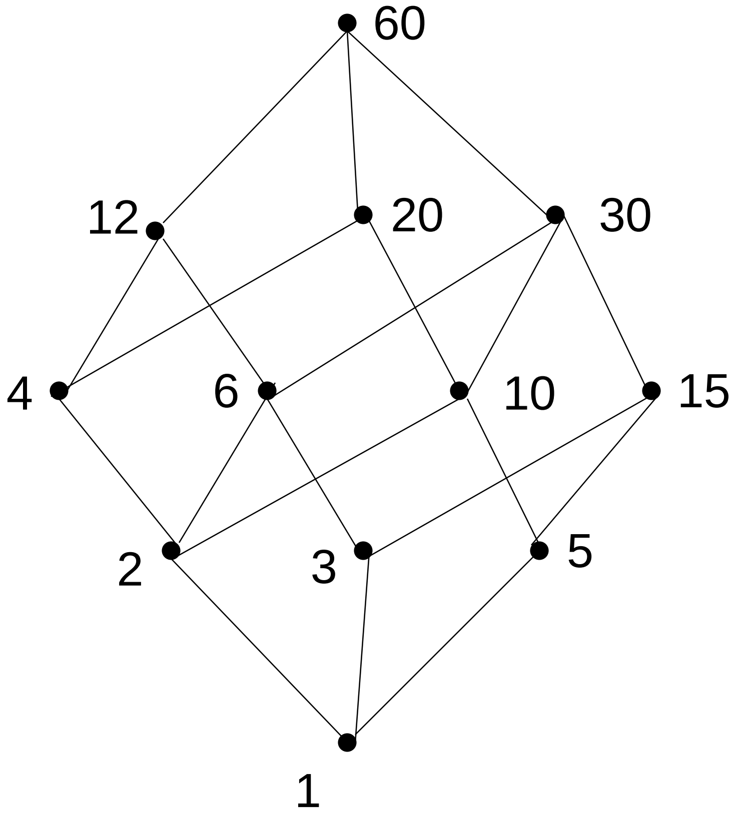
Determine whether $(P(S), \subseteq)$ is a lattice where S is a set.

Solution: Let A and B be two subsets of S. The least upper bound and the greatest lower bound of A and B are $A \cup B$ and $A \cap B$, respectively, as the reader can show. Hence, $(P(S), \subseteq)$ is a lattice. ◀

Hasse Diagrams: Example (1)

- Of course, you need not always start with the complete relation in the partial order and then trim everything.
- Rather, you can build a Hasse Diagram directly from the partial order
- Example: Draw the Hasse Diagram for the following partial ordering: $\{(a,b) \mid a|b\}$ on the set $\{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$ (these are the divisors of 60 which form the basis of the ancient Babylonian base-60 numeral system)

Hasse Diagram: Example (2)



Extremal Elements: Summary

We will define the following terms:

- A maximal/minimal element in a poset $(S, \preccurlyeq)$
- The maximum (greatest)/minimum (least) element of a poset $(S, \preccurlyeq)$
- An upper/lower bound element of a subset A of a poset $(S, \preccurlyeq)$
- The greatest lower/least upper bound element of a subset A of a poset $(S, \preccurlyeq)$

Extremal Elements: Maximal

- **Definition**: An element a in a poset $(S, \prec)$ is called maximal if it is not less than any other element in S. That is: $\neg(\exists b \in S\ (a \prec b))$
- If there is one unique maximal element a, we call it the maximum element (or the greatest element)

Extremal Elements: Minimal

- **Definition**: An element a in a poset (S, $\prec$) is called minimal if it is not greater than any other element in S. That is: $\neg(\exists b \in S\ (b \prec a))$
- If there is one unique minimal element a, we call it the minimum element (or the least element)

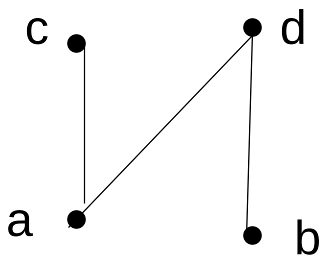
Extremal Elements: Upper Bound

- **Definition**: Let $(S,\prec)$ be a poset and let $A\subseteq S$. If u is an element of S such that $a \prec u$ for all $a\in A$ then u is an upper bound of A
- An element x that is an upper bound on a subset A and is less than all other upper bounds on A is called the least upper bound on A. We abbreviate it as lub.

Extremal Elements: Lower Bound

- **Definition**: Let $(S,\prec)$ be a poset and let $A\subseteq S$. If l is an element of S such that $l \prec a$ for all $a \in A$ then l is an <u>lower bound of A</u>
- An element x that is a lower bound on a subset A and is greater than all other lower bounds on A is called the <u>greatest lower bound on A</u>. We abbreviate it glb.

Extremal Elements: Example 1



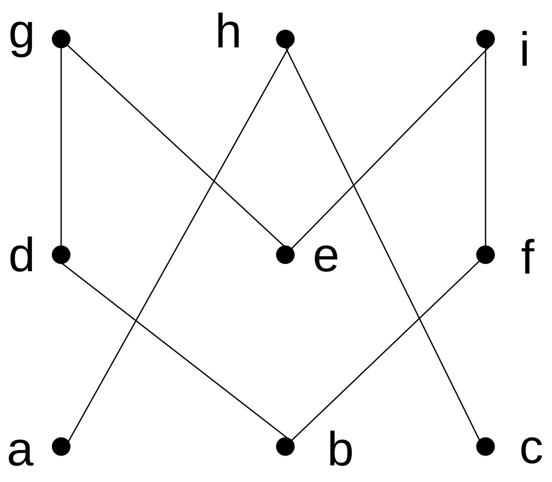
What are the minimal, maximal, minimum, maximum elements?

- Minimal: {a,b}
- Maximal: {c,d}
- There are no unique minimal or maximal elements, thus no minimum or maximum

Extremal Elements: Example 2

Give lower/upper bounds & glb/lub of the sets:

{d,e,f}, {a,c} and {b,d}



{d,e,f}

- Lower bounds: ∅, thus no glb
- Upper bounds: ∅, thus no lub

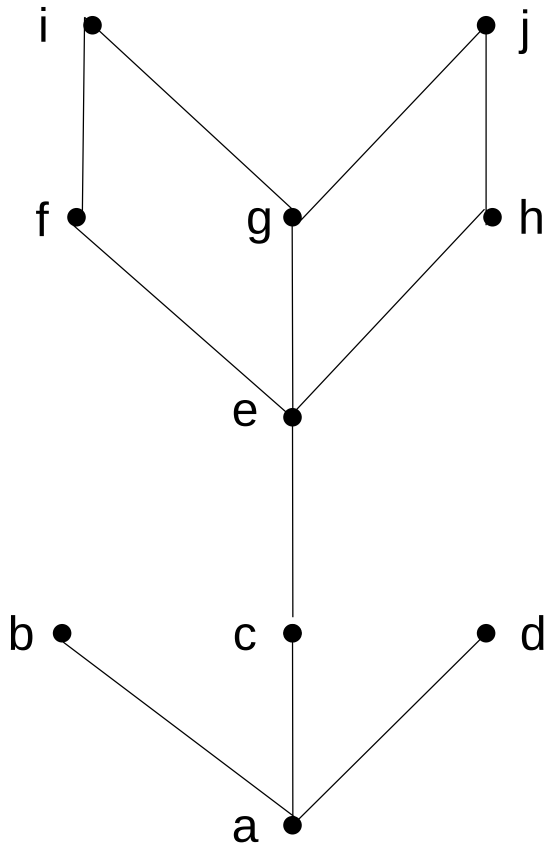
{a,c}

- Lower bounds: ∅, thus no glb
- Upper bounds: {h}, lub: h

{b,d}

- Lower bounds: {b}, glb: b
- Upper bounds: {d,g}, lub: d because $d \prec g$

Extremal Elements: Example 3



- Minimal/Maximal elements?
  - Minimal & Minimum element: a
  - Maximal elements: b,d,i,j
- Bounds, glb, lub of {c,e}?
  - Lower bounds: {a,c}, thus glb is c
  - Upper bounds: {e,f,g,h,i,j}, thus lub is e
- Bounds, glb, lub of {b,i}?
  - Lower bounds: {a}, thus glb is c
  - Upper bounds: ∅, thus lub DNE

Outline

- Motivating example
- Definitions
  - Partial ordering, comparability, total ordering, well ordering
- Principle of well-ordered induction
- Lexicographic orderings
- Hasse Diagrams
- Extremal elements
- **Lattices**
- Topological Sorting

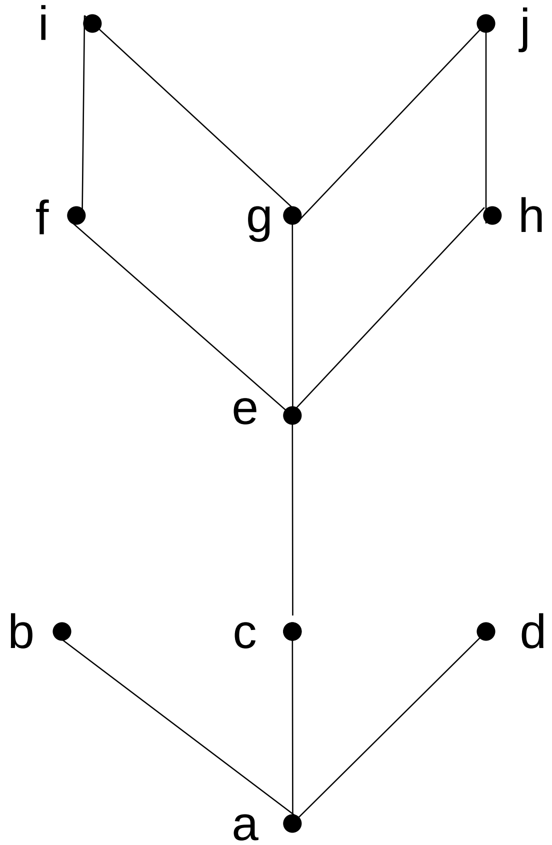
Lattices

- A special structure arises when every pair of elements in a poset has an lub and a glb
- **Definition**: A lattice is a partially ordered set in which every pair of elements has both
  - a least upper bound and
  - a greatest lower bound

Lattices: Example 1

- Is the example from before a lattice?

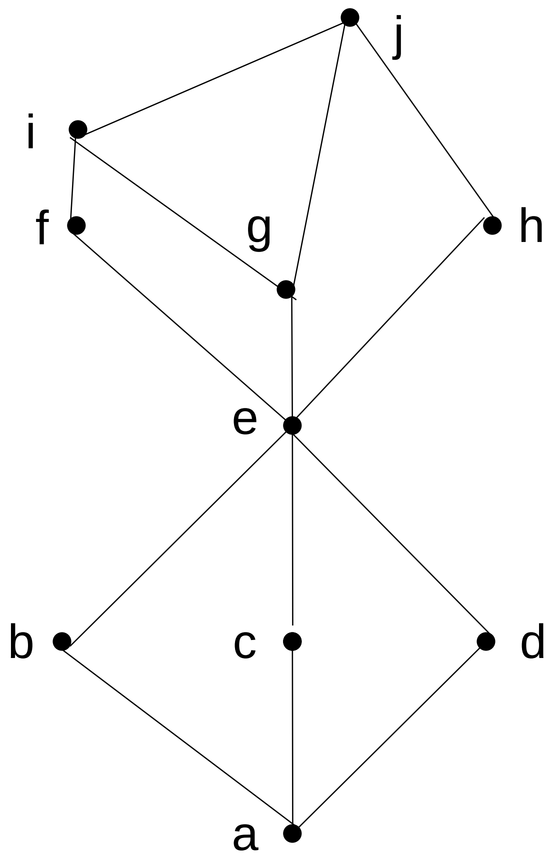
- **No, because the pair {b,c} does not have a least upper bound**



Lattices: Example 2

- What if we modified it as shown here?

- **Yes, because for any pair, there is an lub & a glb**



A Lattice Or Not a Lattice?

- To show that a partial order is not a lattice, it suffices to find a pair that does not have an lub or a glb (i.e., a counter-example)
- For a pair not to have an lub/glb, the elements of the pair must first be <u>incomparable</u> (Why?)
- You can then view the upper/lower bounds on a pair as a sub-Hasse diagram: If there is no minimum element in this sub-diagram, then it is not a lattice

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- **Topological Sorting**

Topological Sorting

- Let us return to the introductory example of Avery Hall renovation. Now that we have got a partial order model, it would be nice to actually create a concrete schedule
- That is, given a <u>partial order</u>, we would like to transform it into a <u>total order</u> that is <u>compatible</u> with the partial order
- A total order is <u>compatible</u> if it does not violate any of the original relations in the partial order
- Essentially, we are simply <u>imposing an order on incomparable</u> elements in the partial order

Topological Sorting: Preliminaries (1)

- Before we give the algorithm, we need some tools to justify its correctness
- **Fact**: Every finite, nonempty poset $(S,\prec)$ has a minimal element
- We will prove the above fact by a form of *reductio ad absurdum*

Topological Sorting: Preliminaries (2)

- **Proof**:
  - Assume, to the contrary, that a nonempty finite poset $(S,\prec)$ has no minimal element. In particular, assume that a_1 is not a minimal element.
  - Assume, w/o loss of generality, that $|S|=n$
  - If a_1 is not minimal, then there exists a_2 such that $a_2 \prec a_1$
  - But a_2 is also not minimal because of the above assumption
  - Therefore, there exists a_3 such that $a_3 \prec a_2$. This process proceeds until we have the last element a_n. Thus, $a_n \prec a_{n-1} \prec \ldots \prec a_2 \prec a_1$
  - Finally, by definition a_n is the minimal element **QED**

Topological Sorting: Intuition

- The idea of topological sorting is
  - We start with a poset $(S, \prec)$
  - We remove a minimal element, choosing arbitrarily if there is more than one. Such an element is guaranteed to exist by the previous fact
  - As we remove each minimal element, one at a time, the set S shrinks
  - Thus we are guaranteed that the algorithm will <u>terminate</u> in a finite number of steps
  - Furthermore, the order in which the elements are removed is a total order: $a_1 \prec a_2 \prec \dots \prec a_{n-1} \prec a_n$
- Now, we can give the algorithm itself

Topological Sorting: Algorithm

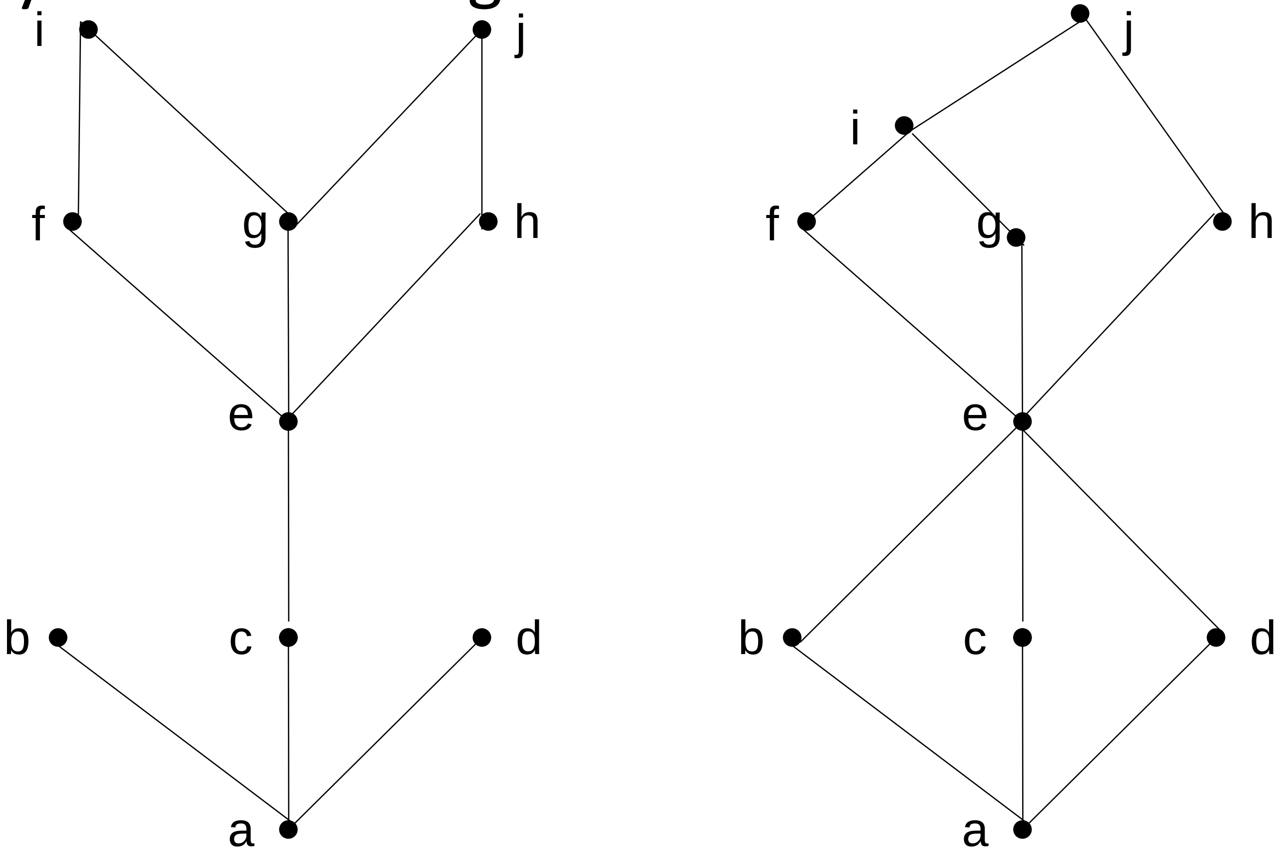
Input: (S, $\prec$) a poset with $|S|=n$

Output: A total ordering $(a_1, a_2, \dots, a_n)$

1. $k \leftarrow 1$
2. **While** S **Do**
3. $\quad a_k \leftarrow$ a minimal element in S
4. $\quad S \leftarrow S \setminus \{a_k\}$
5. $\quad k \leftarrow k+1$
6. **End**
7. **Return** $(a_1, a_2, \dots, a_n)$

Topological Sorting: Example

- Find a compatible ordering (topological ordering) of the poset represented by the Hasse diagrams below



Summary

- Definitions
  - Partial ordering, comparability, total ordering, well ordering
- Principle of well-ordered induction
- Lexicographic orderings
  - Idea, on $A_1 \times A_2$, $A_1 \times A_2 \times \ldots \times A_n$, S^t (strings)
- Hasse Diagrams
- Extremal elements
  - Minimal/minimum, maximal/maximum, glb, lub
- Lattices
- Topological Sorting