Section Summary
Euler and Hamilton Paths

- Euler Paths and Circuits
- Hamilton Paths and Circuits
- Applications of Hamilton Circuits

Paths (Review)

Definitions:

- A ***path*** of *length* n (a nonnegative integer) from u to v in an undirected graph G is a sequence of n edges $e_1, e_2 \dots, e_n$ of G for which there exists a sequence $x_0 = u, x_1, \dots, x_{n-1}, x_n = v$ of vertices such that $e_i = \{x_{i-1}, x_i\}$ for $i = 1, \dots, n$.
  - When the graph is simple, we denote this path by its vertex sequence $x_0, x_1, \dots, x_n$ (since listing the vertices uniquely determines the path).

- A ***circuit*** is a path which begins and ends at the same vertex ($u = v$) and has length greater than zero.

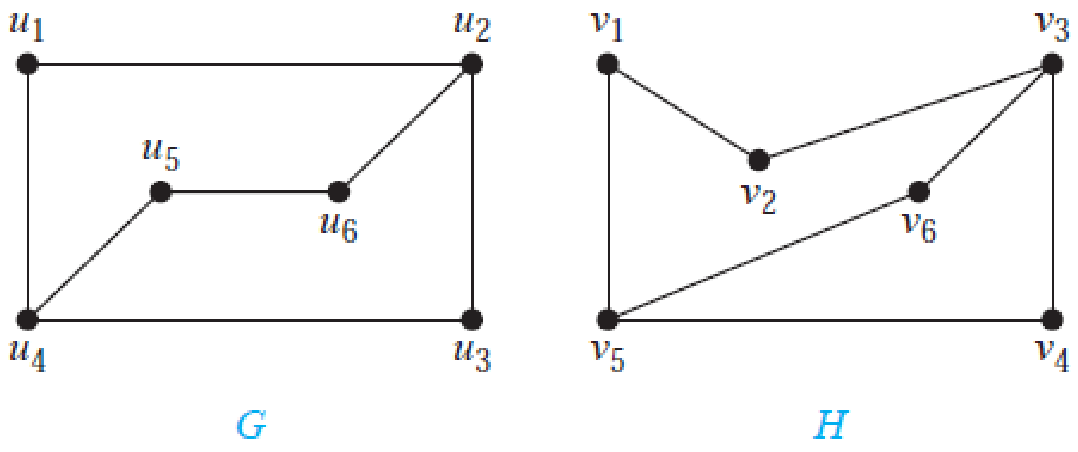
- The path or circuit is said to *pass through* the vertices $x_1, x_2, \dots, x_{n-1}$ and *traverse* the edges $e_1, \dots, e_n$.

- *Simple* path or circuit is *simple* if it does not contain the same edge more than once.

Note: This terminology is readily extended to. directed graphs, by taking $e_i=(x_{i-1}, x_i)$

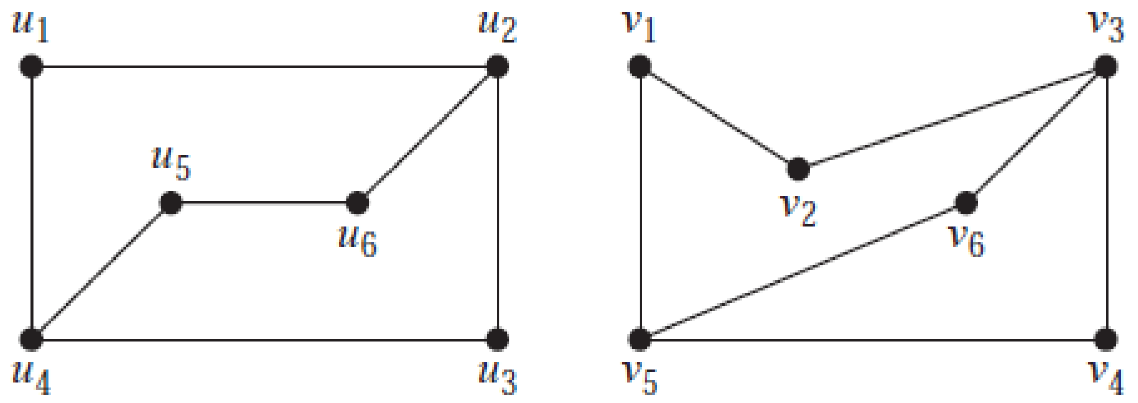
Paths and Isomorphism

- Determine whether the graphs G and H displayed in figure are isomorphic.



- Both G and H have
  - six vertices and seven edges.
  - four vertices of degree two and two vertices of degree three.
- It is also easy to see that the subgraphs of G and H consisting of all vertices of degree two and the edges connecting them are isomorphic (verify??).
- Because G and H agree with respect to these invariants, it is reasonable to try to find an isomorphism f.

- $f(u1) = v6$, $f(u2) = v3$,
- $f(u3) = v4$, $f(u4) = v5$,
- $f(u5) = v1$, $f(u6) = v2$.



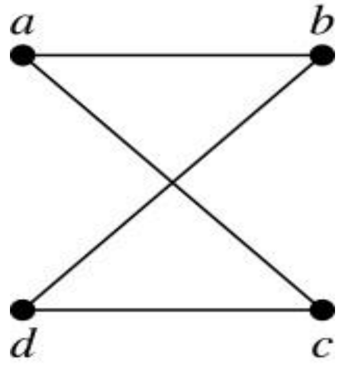
G $\qquad$ H

$$
\mathbf{A}_G = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array}
\begin{array}{c}
\begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\
\begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0
\end{bmatrix}
\end{array},
\qquad
\mathbf{A}_H = \begin{array}{c} \\ v_6 \\ v_3 \\ v_4 \\ v_5 \\ v_1 \\ v_2 \end{array}
\begin{array}{c}
\begin{array}{cccccc} v_6 & v_3 & v_4 & v_5 & v_1 & v_2 \end{array} \\
\begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0
\end{bmatrix}
\end{array}.
$$

Counting Paths between Vertices

Example: How many paths of length four are there from a to d in the graph G.

G



$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$ adjacency matrix of G

Solution: The adjacency matrix of G (ordering the vertices as a, b, c, d) is given above. Hence the number of paths of length four from a to d is the (1, 4)th entry of $\mathbf{A}^4$

The eight paths are as [None is simple]:

$$\mathbf{A}^4 = \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix}$$

a, b, a, b, d	*a, b, a, c, d*
a, b, d, b, d	*a, b, d, c, d*
a, c, a, b, d	*a, c, a, c, d*
a, c, d, b, d	*a, c, d, c, d*

The Euler cycle (or tour) problem:
Is it possible to traverse each of the edges of a graph exactly once, starting and ending at the same vertex?

The Hamiltonian cycle problem:
Is it possible to traverse each of the vertices of a graph exactly once, starting and ending at the same vertex?

Euler and Hamilton Paths

Introduction

- Can we travel along the edges of a graph starting at a vertex and returning to it by traversing each edge of the graph exactly once?
- Similarly, can we travel along the edges of a graph starting at a vertex and returning to it while visiting each vertex of the graph exactly once?

Although these questions seem to be similar, the first question, which asks whether a graph has an Euler circuit, can be easily answered simply by examining the degrees of the vertices of the graph, while the second question, which asks whether a graph has a Hamilton circuit, is quite difficult to solve for most graphs.

In this section we will study these questions and discuss the difficulty of solving them. Although both questions have many practical applications in many different areas, both arose in old puzzles. We will learn about these old puzzles as well as modern practical applications.

Euler Paths and Circuits

The town of Königsberg, Prussia (now called Kaliningrad and part of the Russian republic), was divided into four sections by the branches of the Pregel River. These four sections included the two regions on the banks of the Pregel, Kneiphof Island, and the region between the two branches of the Pregel. In the eighteenth century seven bridges connected these regions. Figure 1 depicts these regions and bridges.

The townspeople took long walks through town on Sundays. They wondered whether it was possible to start at some location in the town, travel across all the bridges once without crossing any bridge twice, and return to the starting point

The Swiss mathematician Leonhard Euler solved this problem. His solution, published in 1736, may be the first use of graph theory. (For a translation of Euler's original paper see [BiLlWi99].) Euler studied this problem using the multigraph obtained when the four regions are represented by vertices and the bridges by edges. This multigraph is shown in Figure 2.

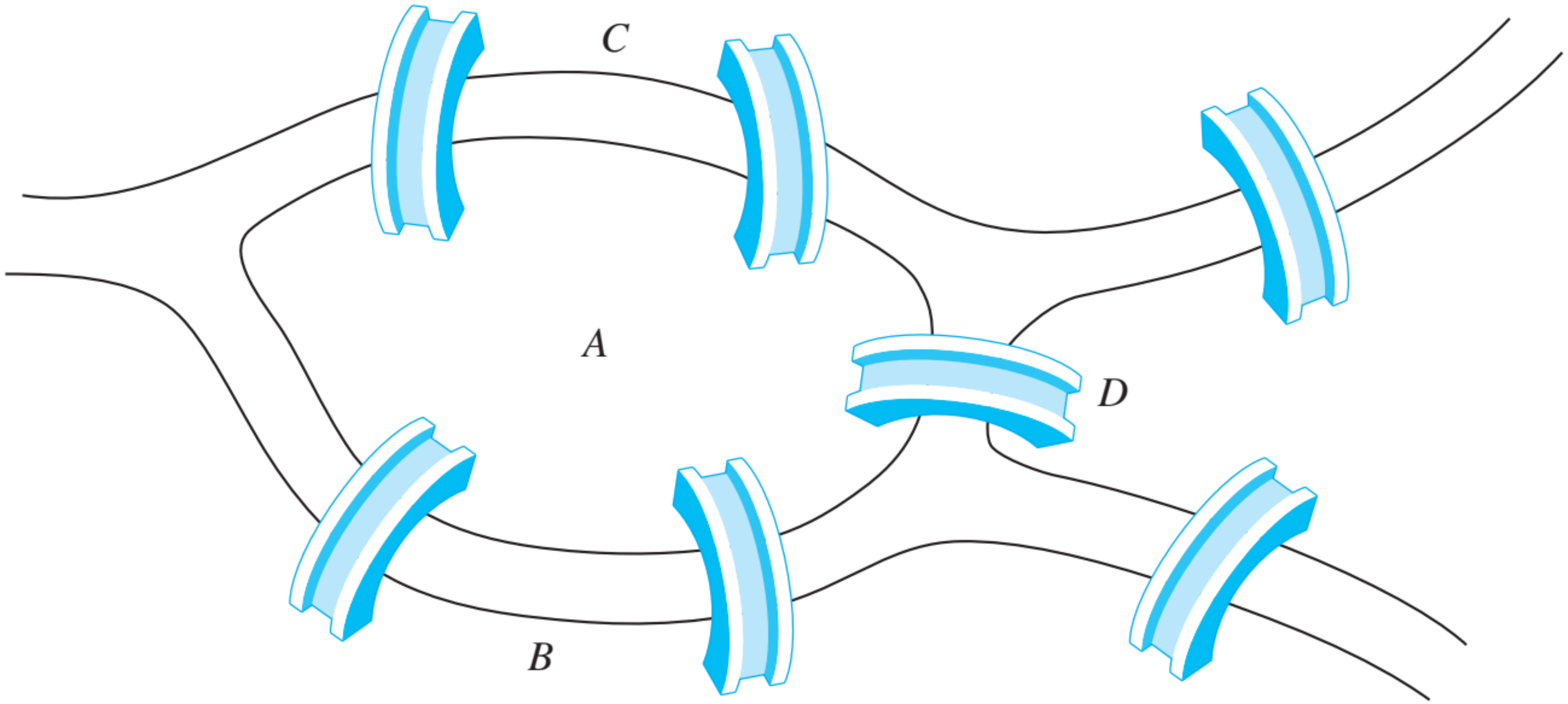


FIGURE 1 **The Seven Bridges of Königsberg.**

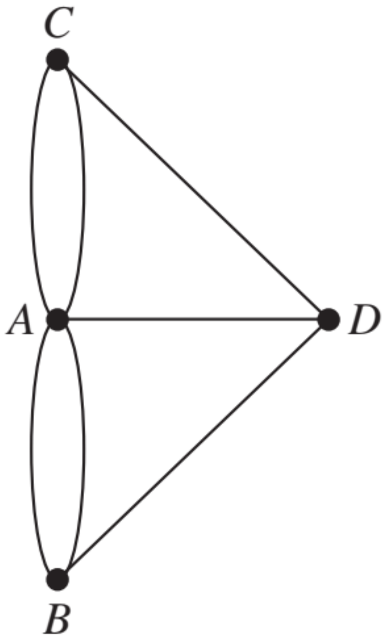


FIGURE 2 **Multigraph Model of the Town of Königsberg.**

- The problem of traveling across every bridge without crossing any bridge more than once can be rephrased in terms of this model. The question becomes: Is there a simple circuit in this multigraph that contains every edge?

DEFINITION 1

An *Euler circuit* in a graph G is a simple circuit containing every edge of G. An *Euler path* in G is a simple path containing every edge of G.

Examples 1 and 2 illustrate the concept of Euler circuits and paths.

EXAMPLE 1 Which of the undirected graphs in Figure 3 have an Euler circuit? Of those that do not, which have an Euler path?

Solution:

The graph G_1 has an Euler circuit, for example, a, e, c, d, e, b, a. Neither of the graphs G_2 or G_3 has an Euler circuit (the reader should verify this). However, G_3 has an Euler path, namely, a, c, d, e, b, d, a, b. G_2 does not have an Euler path (as the reader should verify).

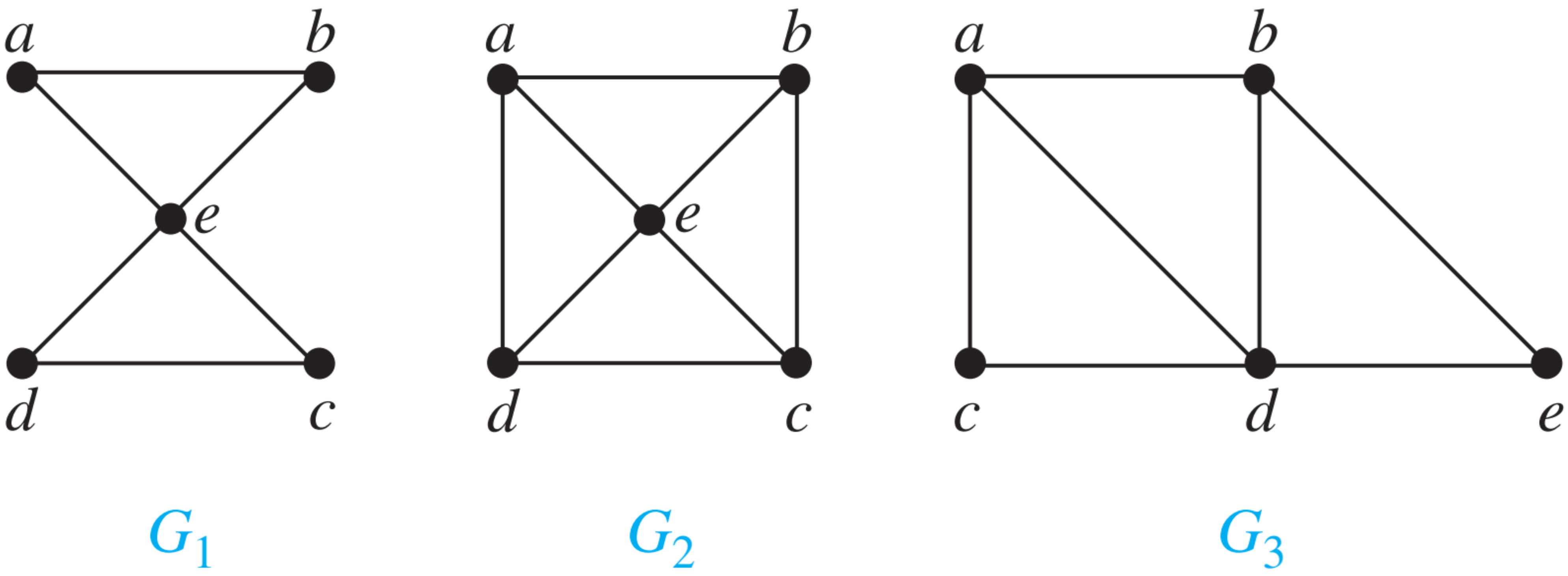
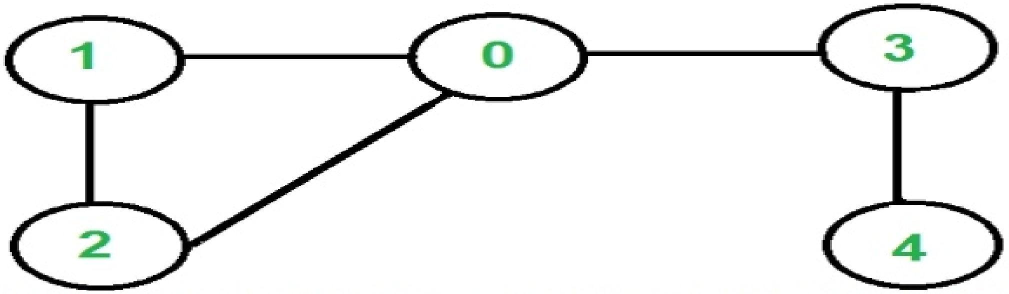


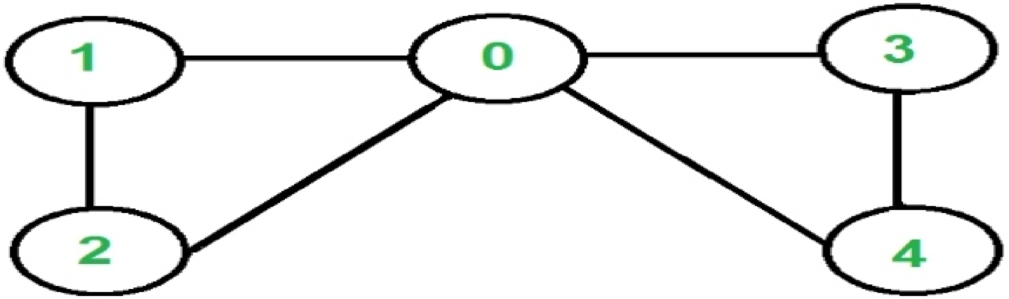
FIGURE 3 **The Undirected Graphs G_1, G_2, and G_3.**

Eulerian path and circuit for undirected graph

Eulerian Path is a path in graph that visits every edge exactly once. Eulerian Circuit is an Eulerian Path which starts and ends on the same vertex.



The graph has Eulerian Paths, for example "4 3 0 1 2 0", but no Eulerian Cycle. Note that there are two vertices with odd degree (4 and 0)



The graph has Eulerian Cycles, for example "2 1 0 3 4 0 2" Note that all vertices have even degree

EXAMPLE 2

Which of the directed graphs in Figure 4 have an Euler circuit? Of those that do not, which have an Euler path?

Solution:

The graph H_2 has an Euler circuit, for example, $a, g, c, b, g, e, d, f, a$. Neither H_1 nor H_3 has an Euler circuit (as the reader should verify). H_3 has an Euler path, namely, c, a, b, c, d, b, but H_1 does not (as the reader should verify).

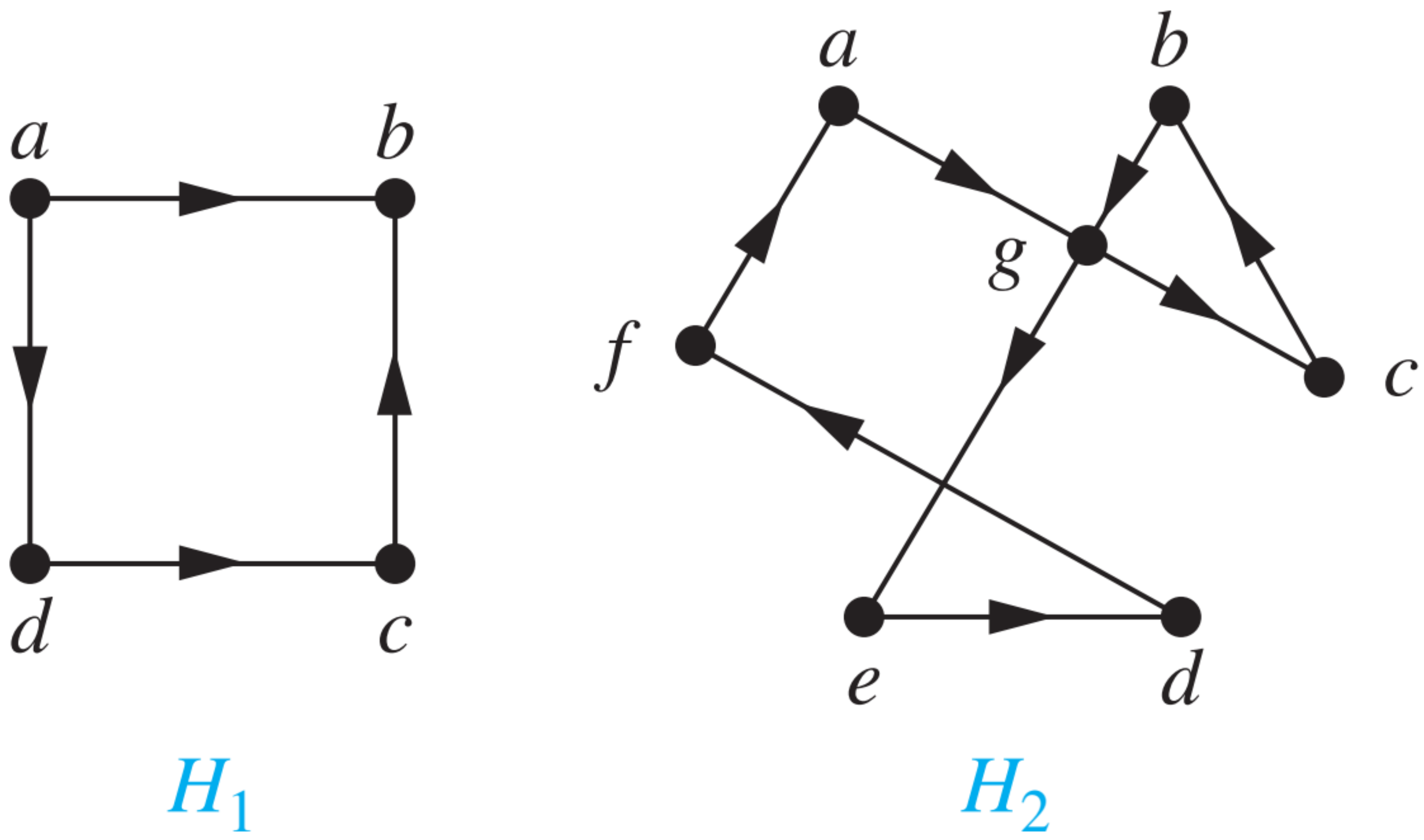


FIGURE 4 The Directed Graphs H_1, H_2, and H_3.

NECESSARY AND SUFFICIENT CONDITIONS FOR EULER CIRCUITS AND PATHS

There are simple criteria for determining whether a multigraph has an Euler circuit or an Euler path. Euler discovered them when he solved the famous Königsberg bridge problem. We will assume that all graphs discussed in this section have a finite number of vertices and edges.

- An Euler circuit begins with a vertex a and continues with an edge incident with a, say $\{a, b\}$. The edge $\{a, b\}$ contributes one to deg(a).
- Each time the circuit passes through a vertex it contributes two to the vertex's degree.
- Finally, the circuit terminates where it started, contributing one to deg(a). Therefore deg(a) must be even.

- We conclude that the degree of every other vertex must also be even.
- By the same reasoning, we see that the initial vertex and the final vertex of an Euler path have odd degree, while every other vertex has even degree. So, a graph with an Euler path has exactly two vertices of odd degree.
- In the next slide we will show that these necessary conditions are also sufficient conditions. That is , is it a necessary condition for the existence of an Euler circuit also sufficient? That is, must an Euler circuit exist in a connected multigraph if all vertices have even degree? This question can be settled affirmatively with a construction.

Suppose that G is a connected multigraph with at least two vertices and the degree of every vertex of G is even. We will form a simple circuit that begins at an arbitrary vertex a of G, building it edge by edge.

Let $x_0 = a$. First, we arbitrarily choose an edge $\{x_0, x_1\}$ incident with a which is possible because G is connected. We continue by building a simple path $\{x_0, x_1\}, \{x_1, x_2\}, \ldots, \{x_{n-1}, x_n\}$, successively adding edges one by one to the path until we cannot add another edge to the path. This happens when we reach a vertex for which we have already included all edges incident with that vertex in the path. For instance, in the graph G in Figure 5 we begin at a and choose in succession the edges $\{a, f\}, \{f, c\}, \{c, b\}$, and $\{b, a\}$.

The path we have constructed must terminate because the graph has a finite number of edges, so we are guaranteed to eventually reach a vertex for which no edges are available to add to the path.

The path begins at a with an edge of the form $\{a, x\}$, and we now show that it must terminate at a with an edge of the form $\{y, a\}$. To see that the path must terminate at a, note that each time the path goes through a vertex with even degree, it uses only one edge to enter this vertex, so because the degree must be at least two, at least one edge remains for the path to leave the vertex. Furthermore, every time we enter and leave a vertex of even degree, there are an even number of edges incident with this vertex that we have not yet used in our path. Consequently, as we form the path, every time we enter a vertex other than a, we can leave it. This means that the path can end only at a.

Note: The path we have constructed may use all the edges of the graph, or it may not if we have returned to a for the last time before using all the edges.

An Euler circuit has been constructed if all the edges have been used. Otherwise, consider the subgraph H obtained from G by deleting the edges already used and vertices that are not incident with any remaining edges. When we delete the circuit a, f, c, b, a from the graph in Figure 5, we obtain the subgraph labeled as H.

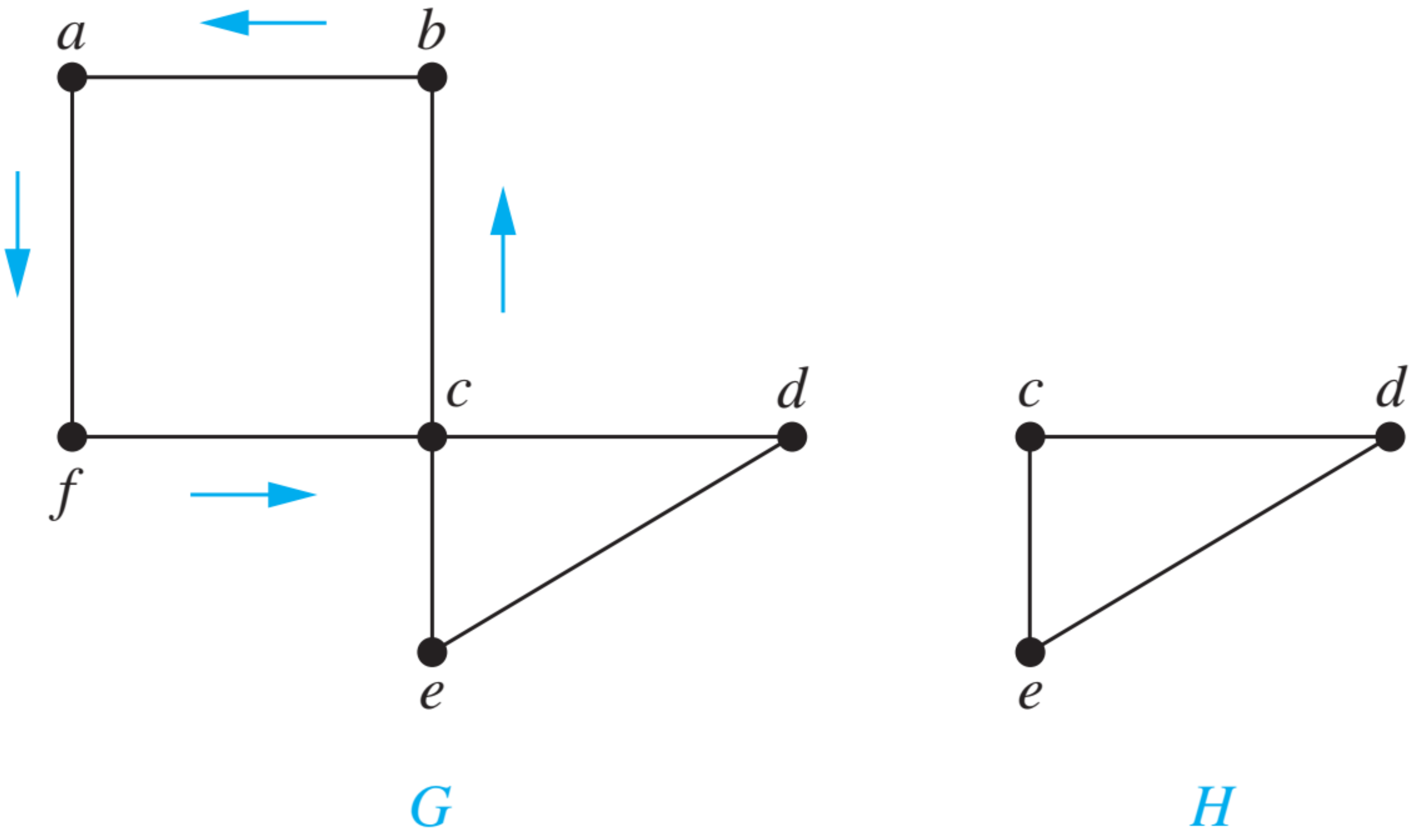


FIGURE 5

Constructing an Euler Circuit in G.

THEOREM 1

A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.

We can now solve the Königsberg bridge problem. Because the multigraph representing these bridges, shown in Figure 2, has four vertices of odd degree, it does not have an Euler circuit. There is no way to start at a given point, cross each bridge exactly once, and return to the starting point.

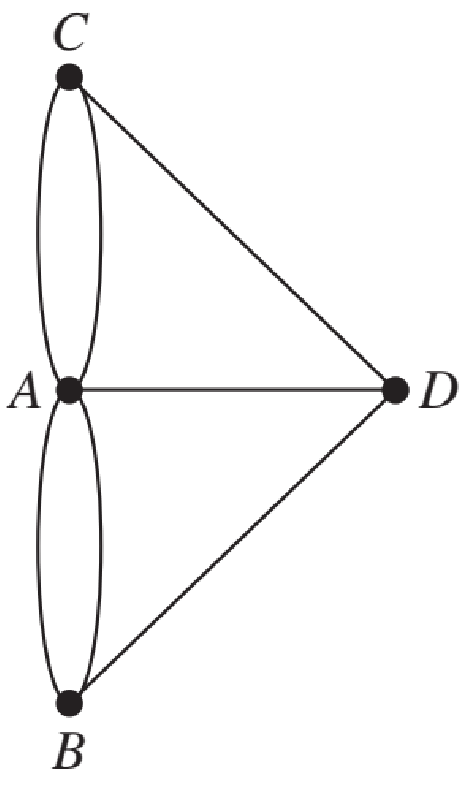


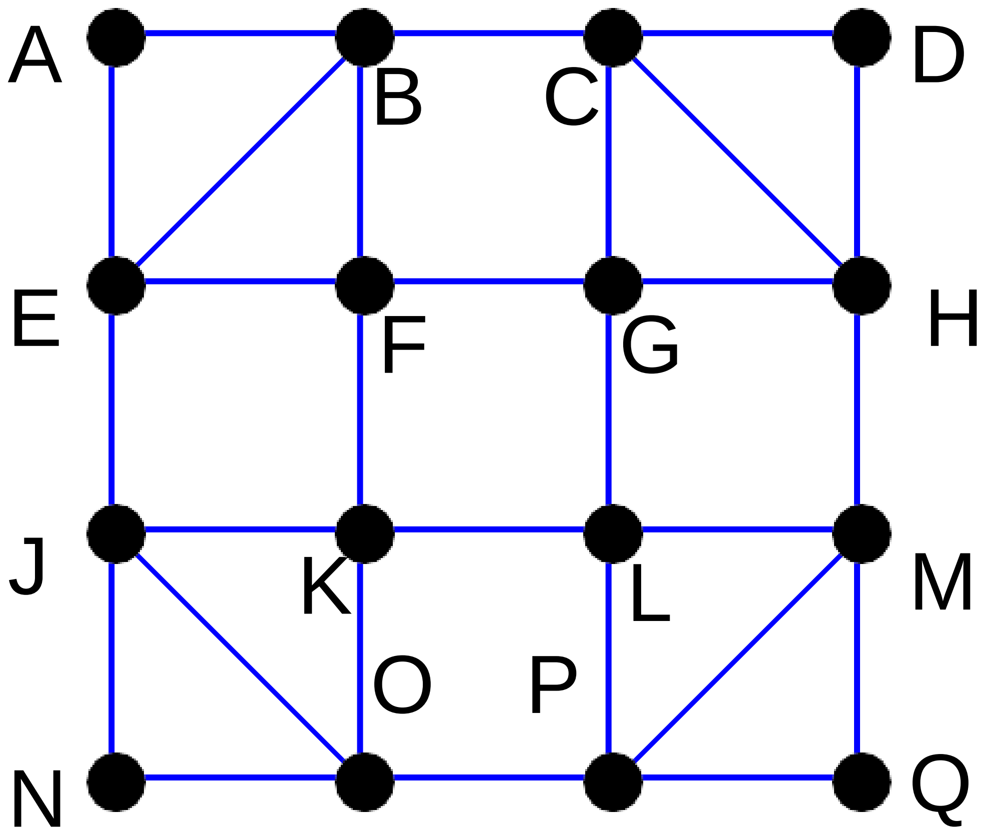
FIGURE 2 **Multigraph Model of the Town of Königsberg.**

Algorithm 1 gives the constructive procedure for finding Euler circuits given in the discussion preceding Theorem 1. (Because the circuits in the procedure are chosen arbitrarily, there is some ambiguity. We will not bother to remove this ambiguity by specifying the steps of the procedure more precisely.)

ALGORITHM 1 Constructing Euler Circuits.

procedure *Euler*(G: connected multigraph with all vertices of
    even degree)
circuit := a circuit in G beginning at an arbitrarily chosen
    vertex with edges successively added to form a path that
    returns to this vertex
H := G with the edges of this circuit removed
while H has edges
    subcircuit := a circuit in H beginning at a vertex in H that
        also is an endpoint of an edge of *circuit*
    H := H with edges of *subcircuit* and all isolated vertices
        removed
    circuit := *circuit* with *subcircuit* inserted at the appropriate
        vertex
return *circuit* {*circuit* is an Euler circuit}

Fleury's Algorithm: Exercise



Applications of Euler Paths and Circuits

- Euler paths and circuits can be used to solve many practical problems such as finding a path or circuit that traverses each
  - street in a neighborhood,
  - road in a transportation network,
  - connection in a utility grid,
  - link in a communications network.
- Other applications are found in the
  - layout of circuits,
  - network multicasting,
  - molecular biology, where Euler paths are used in the sequencing of DNA.

Applications : Chinese postman problem

- The problem of finding a circuit in a graph with the fewest edges that traverses every edge at least once is known as the *Chinese postman problem* in honor of Guan Meigu, who posed it in 1962.

Example 3 shows how Euler paths and circuits can be used to solve a type of puzzle.

EXAMPLE 3

Many puzzles ask you to draw a picture in a continuous motion without lifting a pencil so that no part of the picture is retraced. We can solve such puzzles using Euler circuits and paths. For example, can *Mohammed's scimitars*, shown in Figure 6, be drawn in this way, where the drawing begins and ends at the same point?

Solution: We can solve this problem because the graph G shown in Figure 6 has an Euler circuit. It has such a circuit because all its vertices have even degree. We will use Algorithm 1 to construct an Euler circuit. First, we form the circuit $a, b, d, c, b, e, i, f, e, a$. We obtain the subgraph H by deleting the edges in this circuit and all vertices that become isolated when these edges are removed. Then we form the circuit $d, g, h, j, i, h, k, g, f, d$ in H. After forming this circuit we have used all edges in G. Splicing this new circuit into the first circuit at the appropriate place produces the Euler circuit $a, b, d, g, h, j, i, h, k, g, f, d, c, b, e, i, f, e, a$. This circuit gives a way to draw the scimitars without lifting the pencil or retracing part of the picture. ◀

Another algorithm for constructing Euler circuits, called Fleury's algorithm, is described in the premble to Exercise 50.

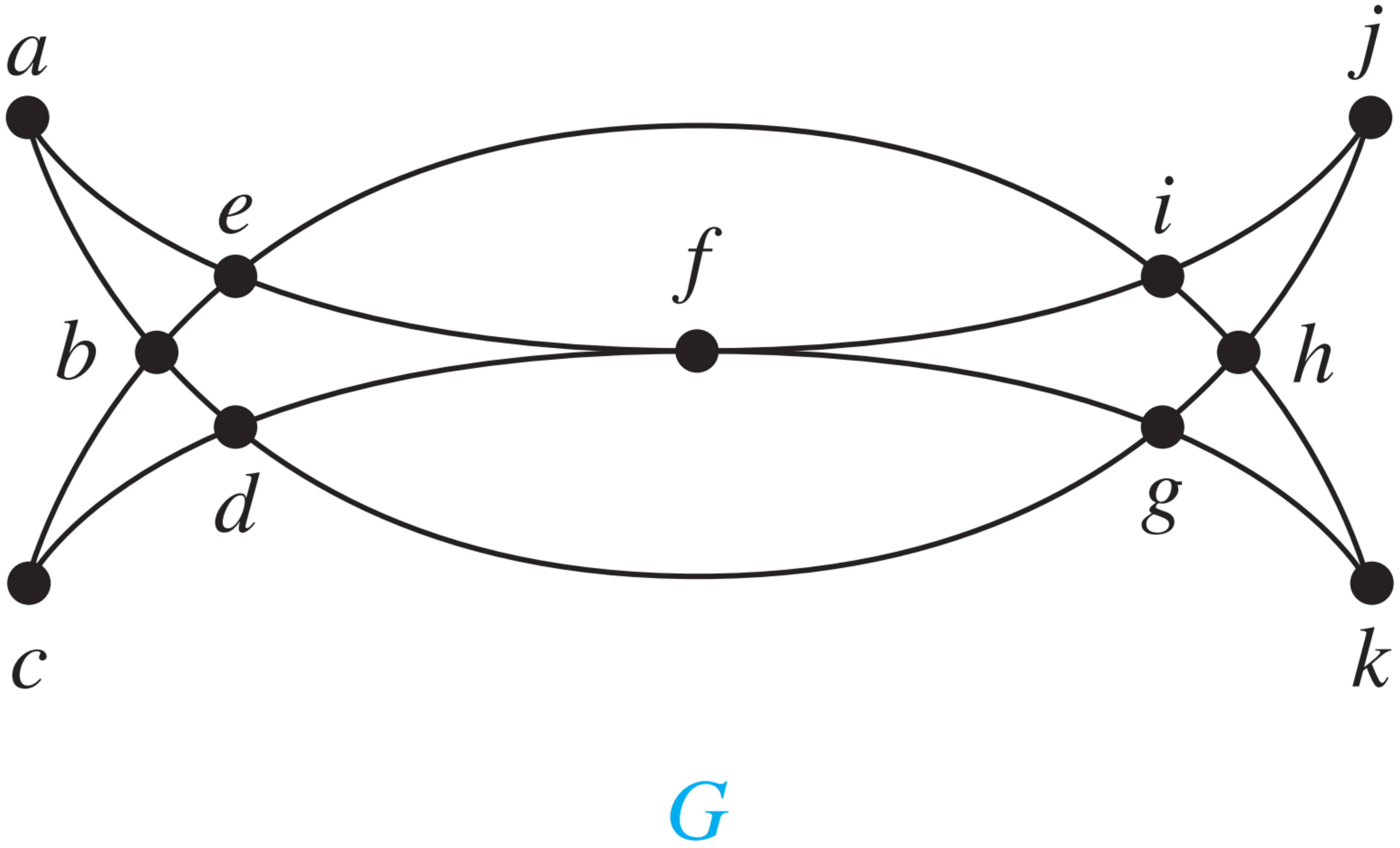


FIGURE 6 **Mohammed's Scimitars.**

THEOREM 2.

A connected multigraph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree.

EXAMPLE 4

Which graphs shown in Figure 7 have an Euler path?

Solution:

G_1 contains exactly two vertices of odd degree, namely, b and d. Hence, it has an Euler path that must have b and d as its endpoints. One such Euler path is d, a, b, c, d, b.

Similarly, G_2 has exactly two vertices of odd degree, namely, b and d. So it has an Euler path that must have b and d as endpoints. One such Euler path is $b, a, g, f, e, d, c, g, b, c, f, d$.

G_3 has no Euler path because it has six vertices of odd degree.

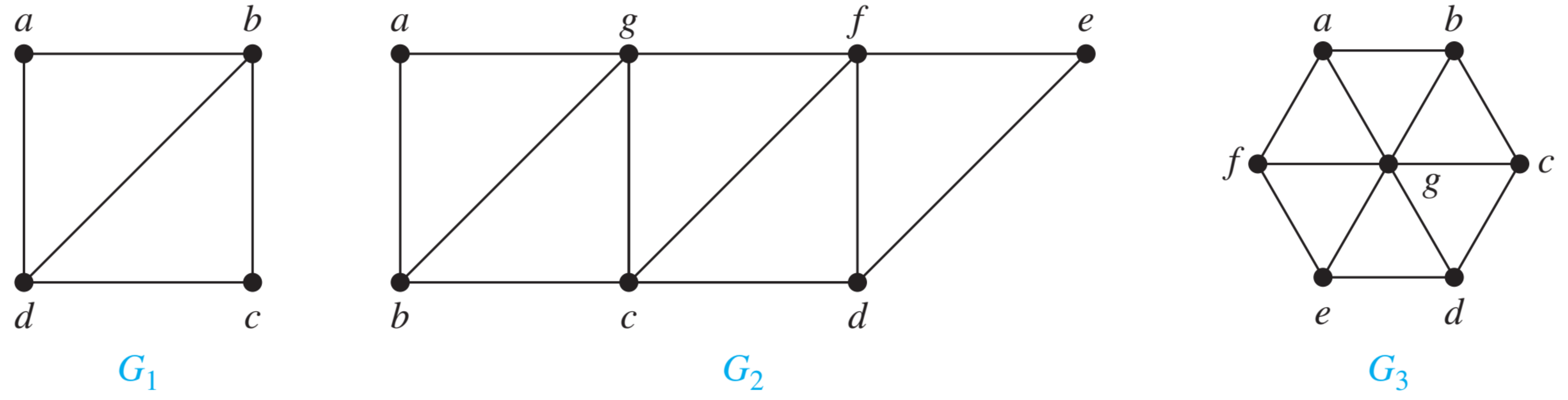
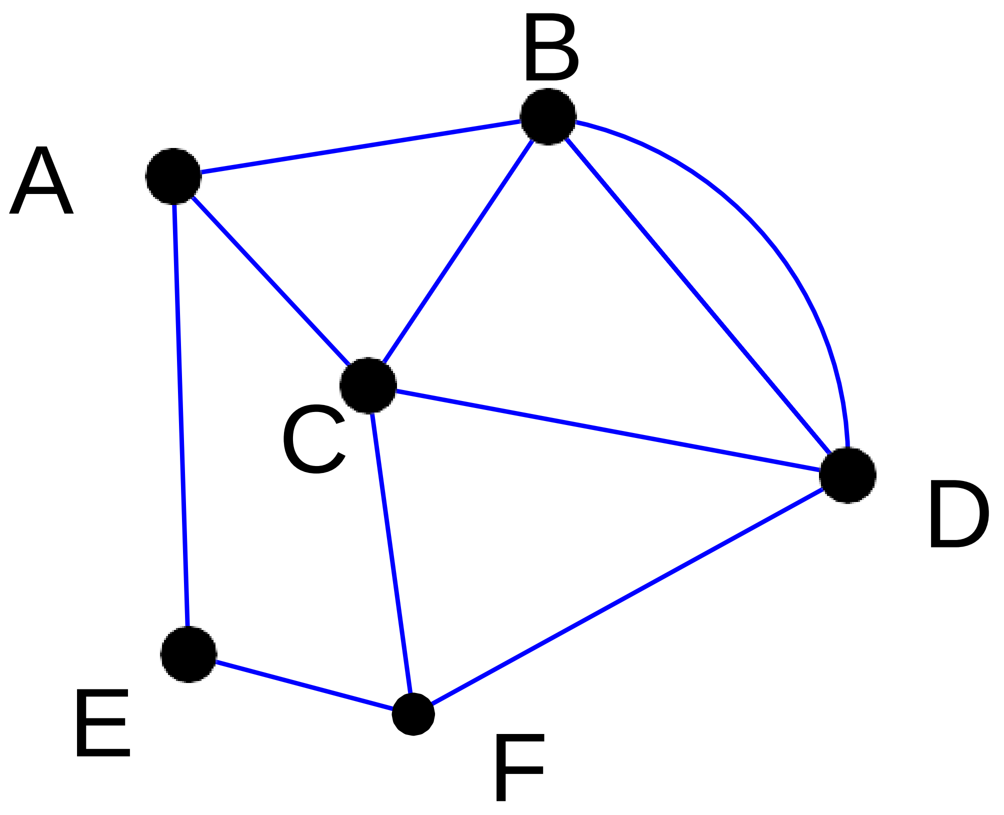


FIGURE 7 **Three Undirected Graphs.**

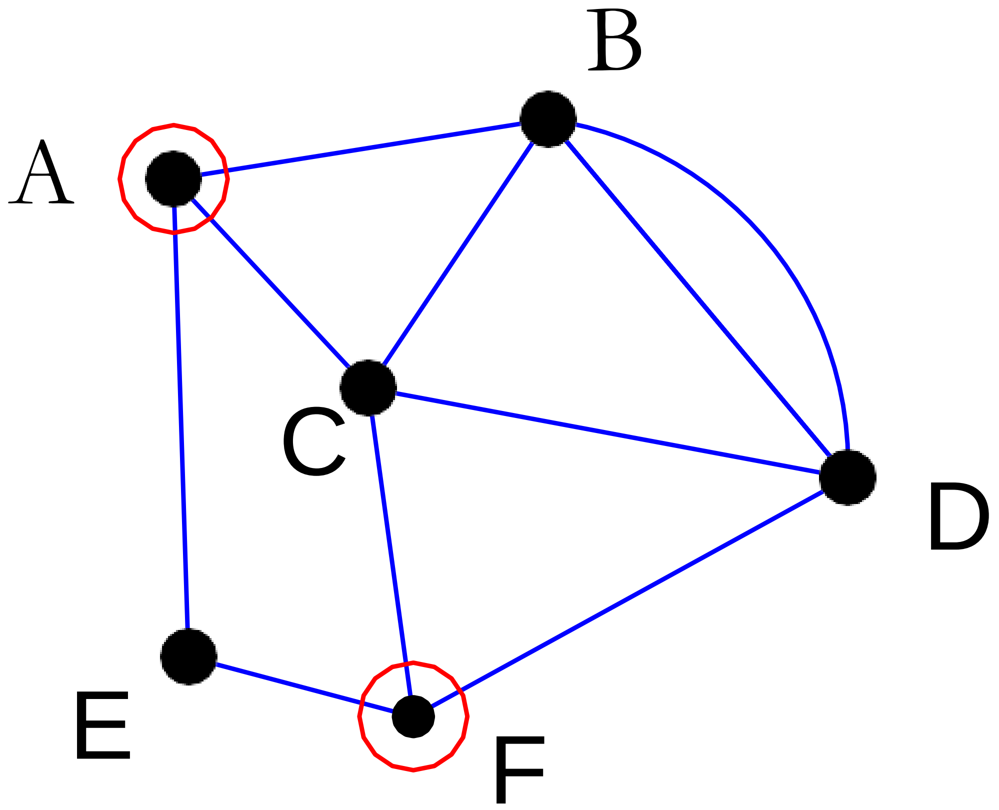
Finding Euler Circuits and Paths

Problem: Find an Euler circuit in the graph below.



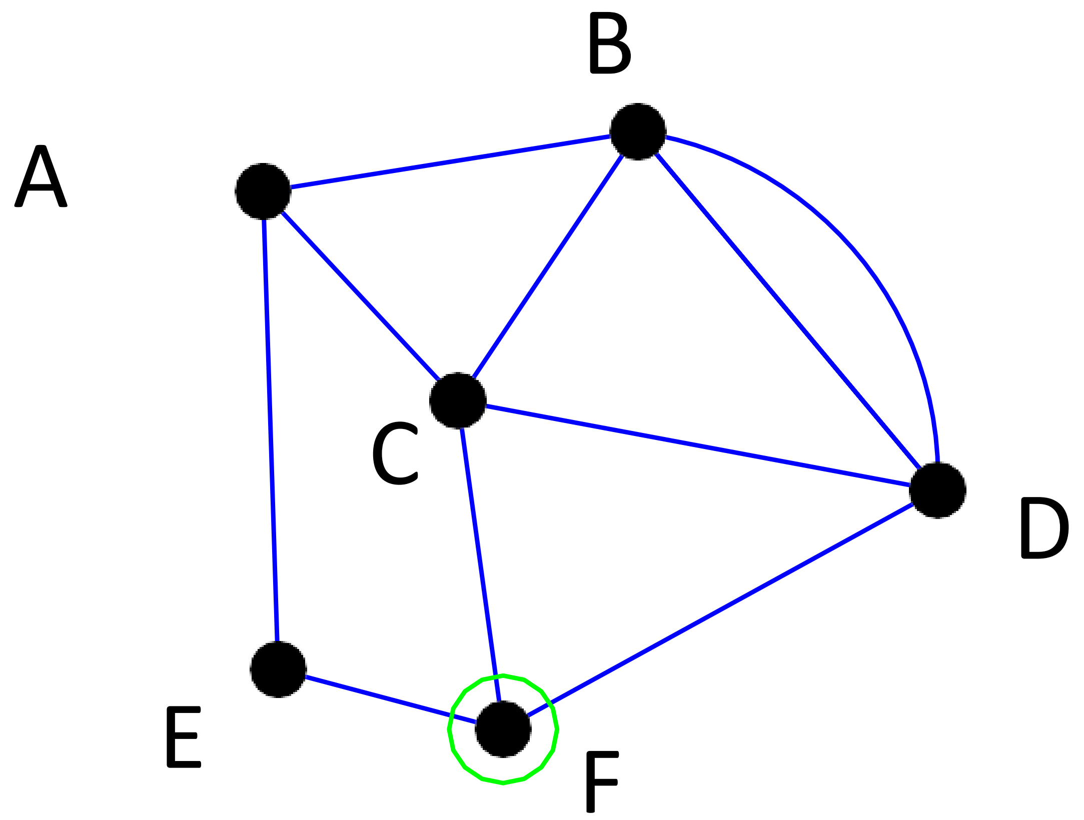
Finding Euler Circuits and Paths

There are two odd vertices, A and F. Let's start at F.



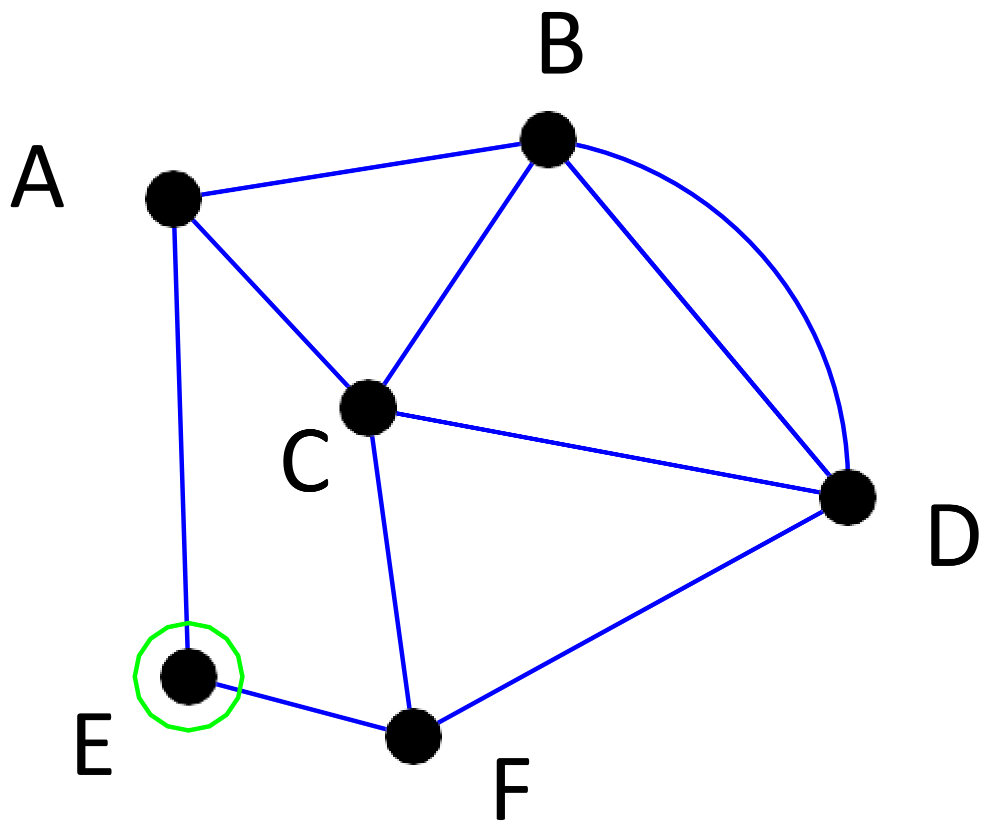
Finding Euler Circuits and Paths

Start walking at F. When you use an edge, delete it.



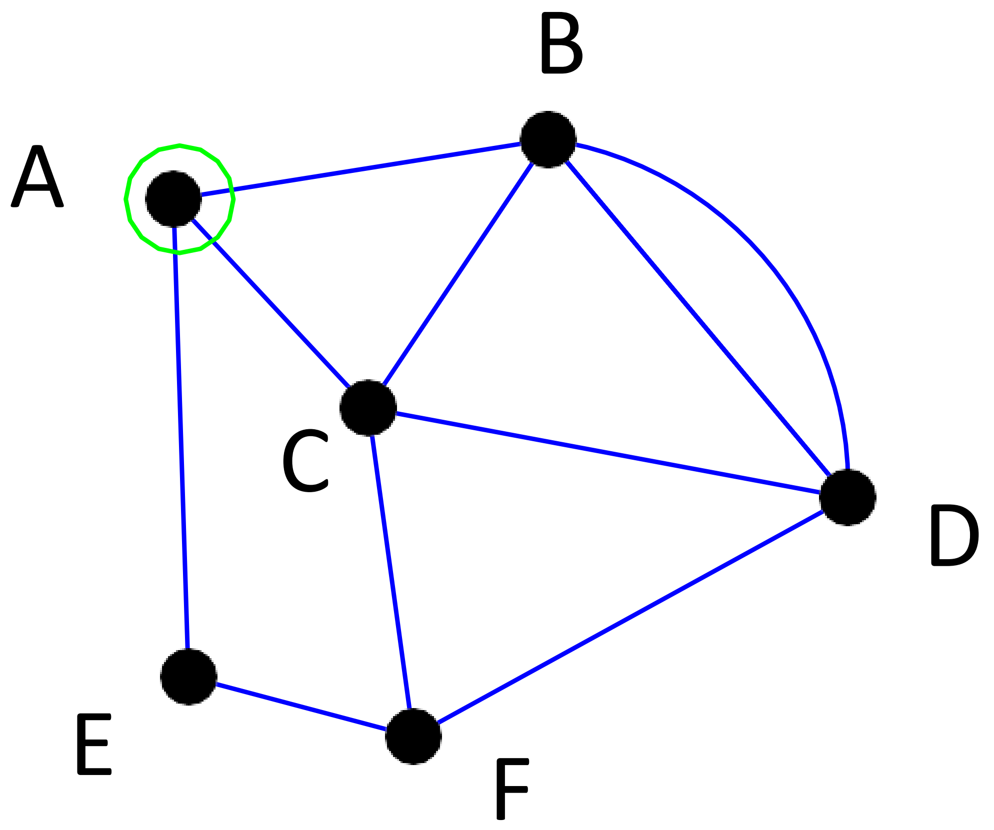
Finding Euler Circuits and Paths

Path so far: FE



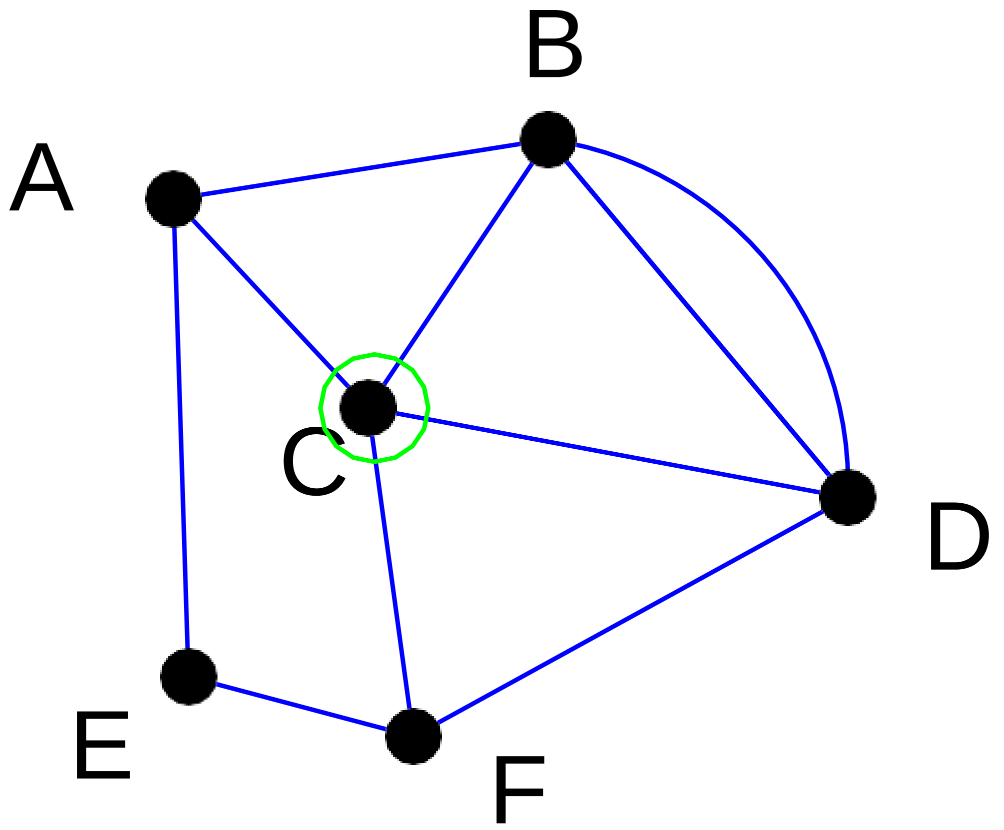
Finding Euler Circuits and Paths

Path so far: FEA



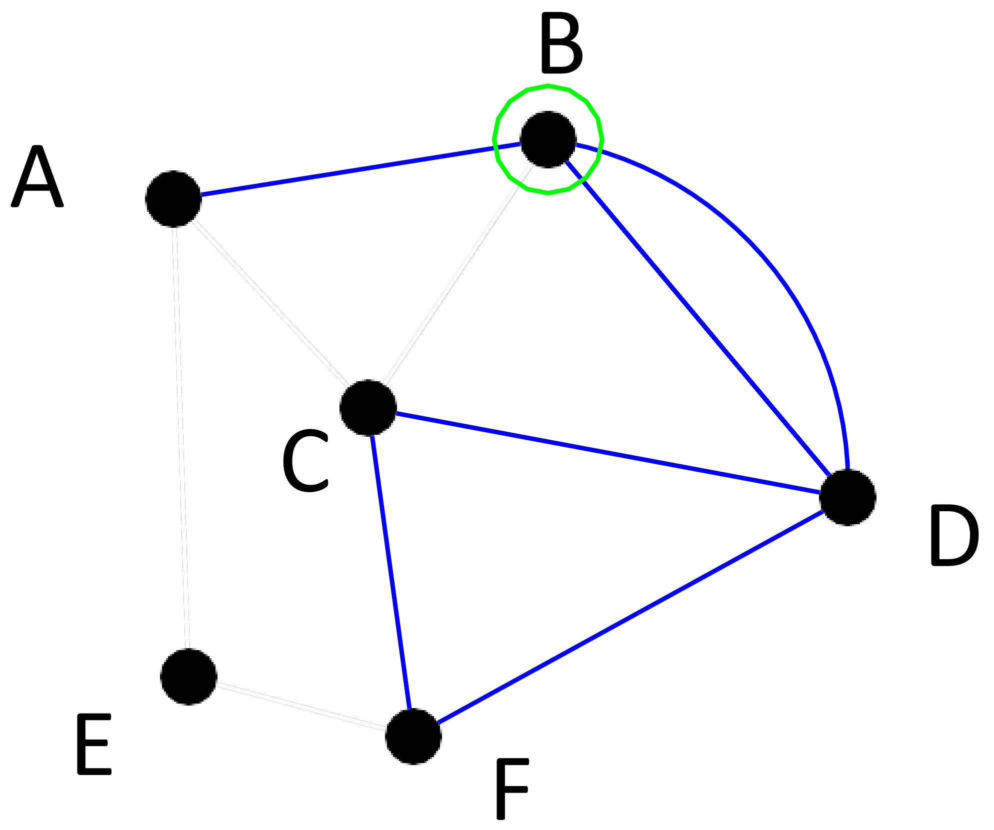
Finding Euler Circuits and Paths

Path so far: FEAC



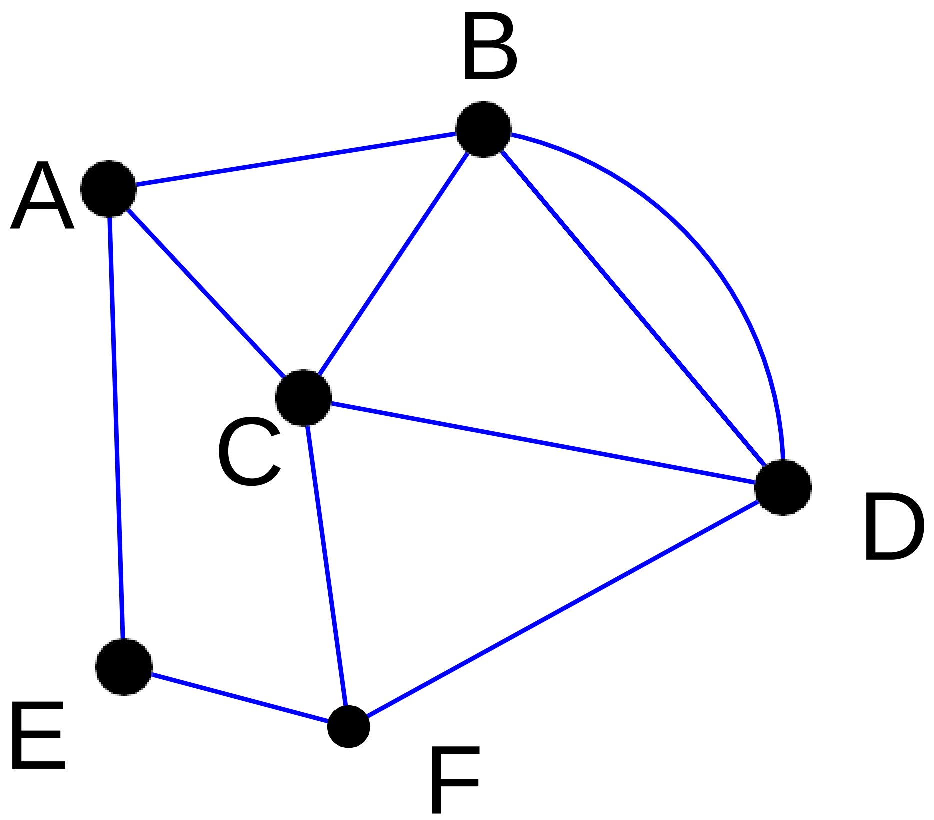
Finding Euler Circuits and Paths

Path so far: FEACB



Finding Euler Circuits and Paths

Euler Path: FEACBDCFDBA (at last)



Hamilton Paths and Circuits

We have developed necessary and sufficient conditions for the existence of paths and circuits that contain every edge of a multigraph exactly once. Can we do the same for simple paths and circuits that contain every vertex of the graph exactly once?

DEFINITION 2

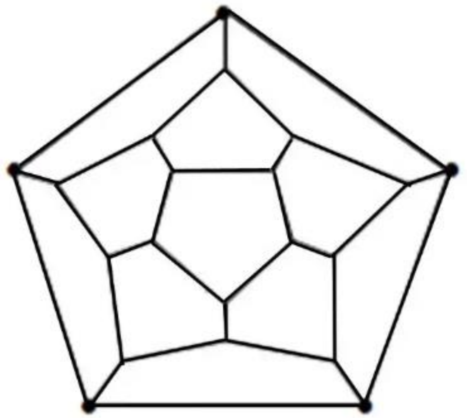
- A simple path in a graph G that passes through every vertex exactly once is called a *Hamilton path*, and
- A simple circuit in a graph G that passes through every vertex exactly once is called a *Hamilton circuit*.

That is, the simple path $x_0, x_1, \ldots, x_{n-1}, x_n$ in the graph $G = (V, E)$ is a Hamilton path if $V = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$ and $x_i \neq x_j$ for $0 \leq i < j \leq n$,

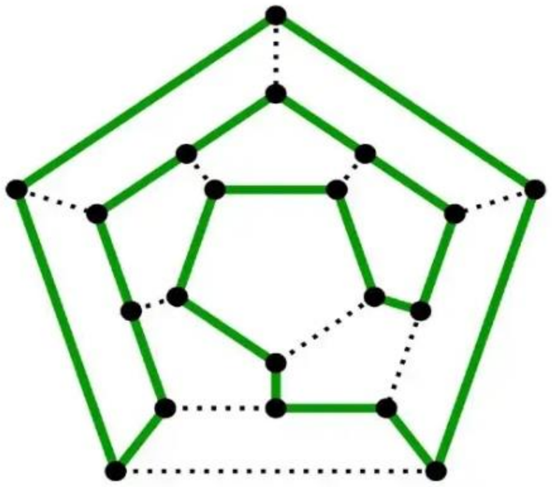
and the simple circuit $x_0, x_1, \ldots, x_{n-1}, x_n, x_0$ (with $n > 0$) is a Hamilton circuit if $x_0, x_1, \ldots, x_{n-1}, x_n$ is a Hamilton path

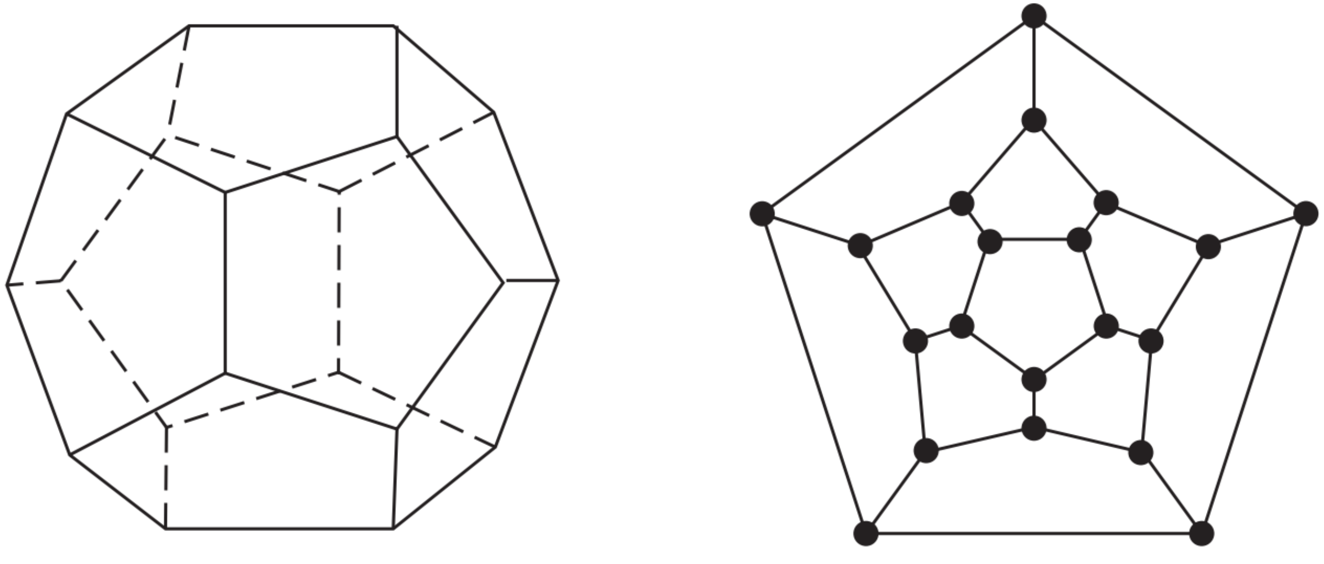
This terminology comes from a game, called the *Icosian puzzle*, invented in 1857 by the Irish mathematician Sir William Rowan Hamilton. It consisted of a wooden dodecahedron [a polyhedron with 12 regular pentagons as faces, as shown in Figure 8(a)], with a peg at each vertex of the dodecahedron, and string. The 20 vertices of the dodecahedron were labeled with different cities in the world. The object of the puzzle was to start at a city and travel along the edges of the dodecahedron, visiting each of the other 19 cities exactly once, and end back at the first city. The circuit traveled was marked off using the string and pegs.

- **Example 1-** Does the following graph have a Hamiltonian Circuit?



- **Solution-** Yes, the above graph has a Hamiltonian circuit. The solution is –





(a) (b)

FIGURE 8 **Hamilton's "A Voyage Round the World" Puzzle.**

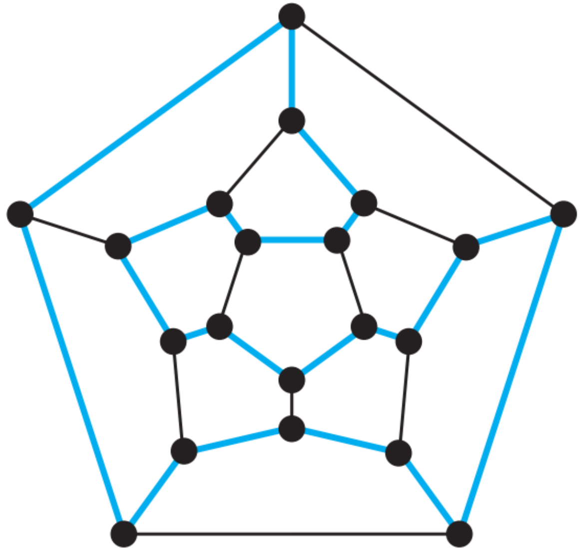


FIGURE 9 **A Solution to the "A Voyage Round the World" Puzzle.**

EXAMPLE 5

Which of the simple graphs in Figure 10 have a Hamilton circuit or, if not, a Hamilton path?

Solution:

G_1 has a Hamilton circuit: a, b, c, d, e, a. There is no Hamilton circuit in G_2 (this can be seen by noting that any circuit containing every vertex must contain the edge $\{a, b\}$ twice), but G_2 does have a Hamilton path, namely, a, b, c, d. G_3 has neither a Hamilton circuit nor a Hamilton path, because any path containing all vertices must contain one of the edges $\{a, b\}$, $\{e, f\}$, and $\{c, d\}$ more than once.

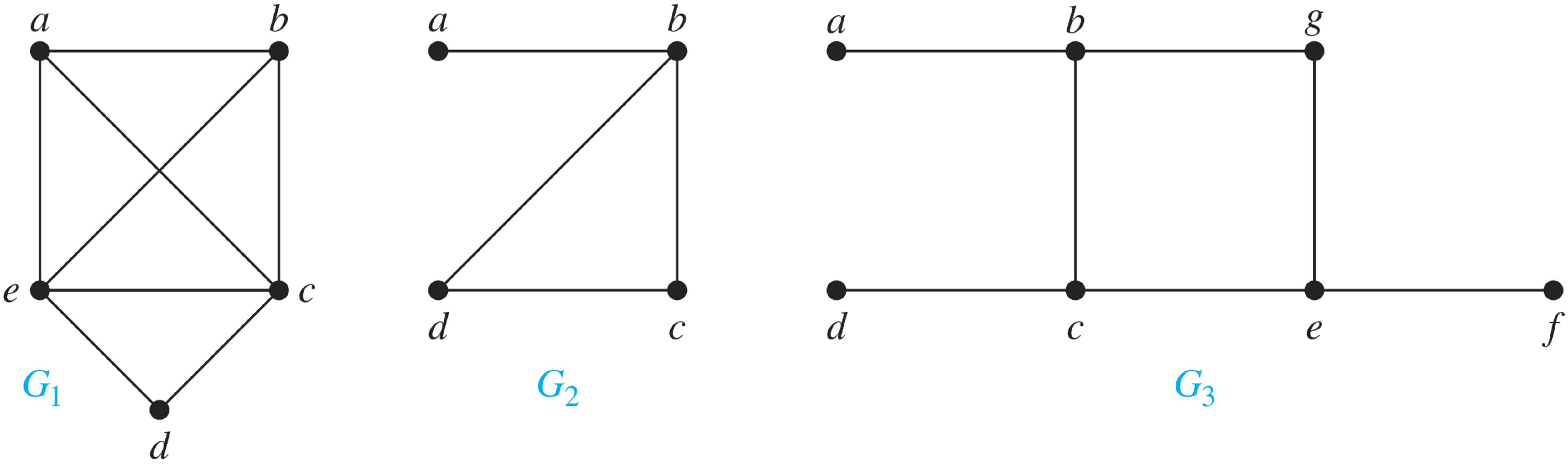
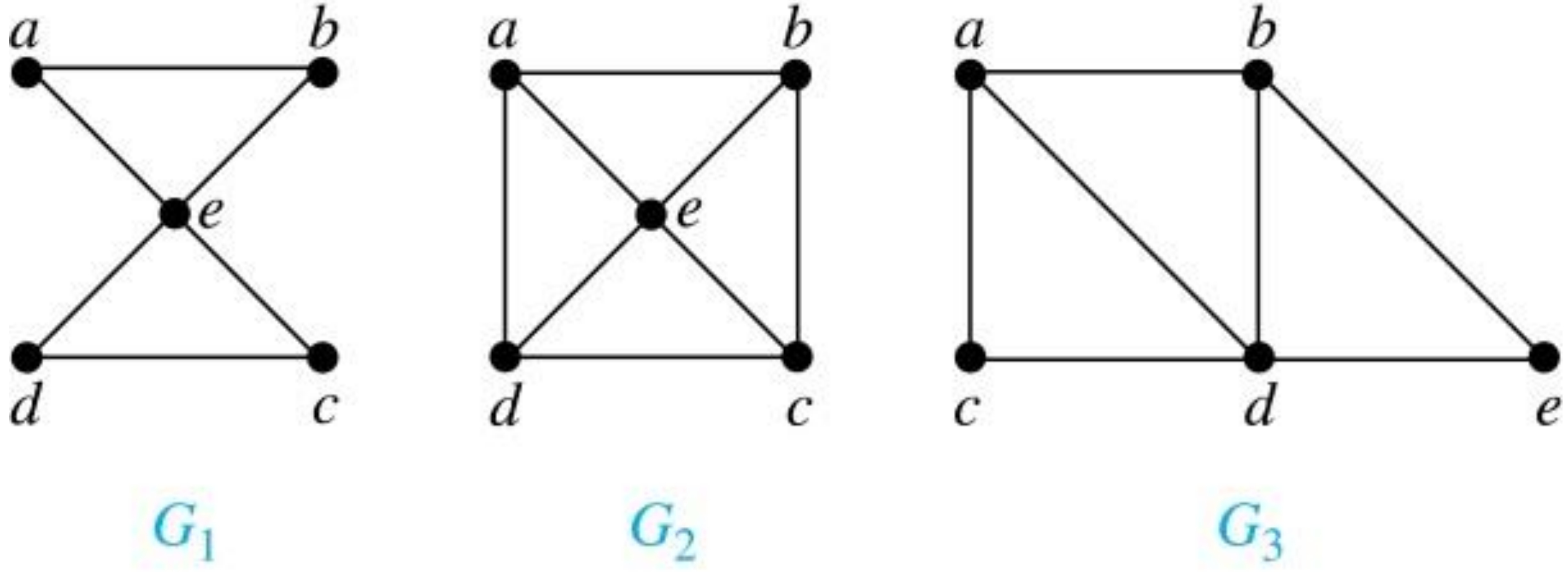


FIGURE 10 **Three Simple Graphs.**

Hamilton Paths and Circuits (*continued*)

Example: Which of these simple graphs has a Hamilton circuit or, if not, a Hamilton path?



Solution:

- G_1 has a Hamilton path: a, b, e, d, c.
- G_2 has a Hamilton circuit: a,d,c,e,b,a, also has a Hamilton path : a, b, c, d, e.
- G_3 has a Hamilton circuit: a,b,e,d,c,a also a Hamilton path: a,b,e,d,c.

Finding Hamilton Circuits

- Unlike the Euler circuit problem, finding Hamilton circuits is hard.

- There is no simple set of necessary and sufficient conditions, and no simple algorithm.

Properties to look for ...

- No vertex of degree 1
- If a node has degree 2, then both edges incident to it must be in any Hamilton circuit.
- No smaller circuits contained in any Hamilton circuit (the start/endpoint of any smaller circuit would have to be visited twice).

A Sufficient Condition

➢DIRAC'S THEOREM

If G be a connected simple graph with n vertices with $n \geq 3$ such that the degree of every vertex in G is $\geq n/2$, then G has a Hamilton circuit.

➢ORE'S THEOREM

If G be a connected simple graph with n vertices with $n \geq 3$ such that $\deg(u) + \deg(v) \geq n$ for every pair of non-adjacent vertices u and v in G, then G has a Hamilton circuit.

Applications of Hamilton Paths and Circuits

- Applications that ask for a path or a circuit that visits each intersection of a city, each place pipelines intersect in a utility grid, or each node in a communications network exactly once, can be solved by finding a Hamilton path in the appropriate graph.
- The famous *traveling salesperson problem* (*TSP*) asks for the shortest route a traveling salesperson should take to visit a set of cities. This problem reduces to finding a Hamilton circuit such that the total sum of the weights of its edges is as small as possible.
- A family of binary codes, known as *Gray codes*, which minimize the effect of transmission errors, correspond to Hamilton circuits in the n-cube Q_n. (*See the text for details*.)

Travelling Salesman Problem

A Hamilton circuit or path may be used to solve practical problems that require visiting "vertices", such as:

- road intersections
- pipeline crossings
- communication network nodes

A classic example is the Travelling Salesman Problem – finding a Hamilton circuit in a complete graph such that the total weight of its edges is minimal.

Summary

Property	Euler	Hamilton
Repeated visits to a given node allowed?	Yes	No
Repeated traversals of a given edge allowed?	No	No
Omitted nodes allowed?	No	No
Omitted edges allowed?	No	Yes

Applications of Hamilton Circuits

(Refer Textbook)