Counting

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Prerequisite

Definition: A set A has cardinality $n \in N$, in symbols, $|A| = n$, iff there is a **bijection** from {1, 2, . . . , n} to A. The special case when $n = 0$ is that $|\emptyset| = 0$. A set is finite iff it has cardinality n for some $n \in N$

Theorem. Let A and B be finite sets and f : from A to B be a function. If
1. f is a bijection, then $|A| = |B|$,
2. f is an injection, then $|A| \leq |B|$,
3. f is a surjection, then $|A| \geq |B|$

Introduction to Counting

- **C**ombinatorics, the study of arrangements of objects, is an important part of discrete mathematics.
- This subject was studied as long ago as the seventeenth century, when combinatorial questions arose in the study of gambling games.
- Enumeration, the counting of objects with certain properties, is an important part of combinatorics.
- We must count objects to solve many different types of problems. For instance, counting is used to determine the complexity of algorithms.
- Counting is also required to determine whether there are enough telephone numbers or Internet protocol addresses to meet demand.
- Recently, it has played a key role in mathematical biology, especially in sequencing DNA. Furthermore, counting techniques are used extensively when probabilities of events are computed.

The Basics of Counting

Introduction

- Suppose that a password on a computer system consists of six, seven, or eight characters. Each of these characters must be a digit or a letter of the alphabet. Each password must contain at least one digit. How many such passwords are there? The techniques needed to answer this question and a wide variety of other counting problems will be introduced in this section.

- Counting problems arise throughout mathematics and computer science. For example, we must count the successful outcomes of experiments and all the possible outcomes of these experiments to determine probabilities of discrete events. We need to count the number of operations used by an algorithm to study its time complexity

The Basics of Counting

- **Basic Counting Principles**
  We first present two basic counting principles, the **product rule** and the **sum rule**. Then we will show how they can be used to solve many different counting problems. The product rule applies when a procedure is made up of separate tasks.

THE PRODUCT RULE Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 n_2$ ways to do the procedure

Examples of THE PRODUCT RULE

EXAMPLE 1 A new company with just two employees, Sanchez and Patel, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?

Solution: The procedure of assigning offices to these two employees consists of assigning an office to Sanchez, which can be done in 12 ways, then assigning an office to Patel different from the office assigned to Sanchez, which can be done in 11 ways.

By the product rule, there are $12 \cdot 11 = 132$ ways to assign offices to these two employees.

- **EXAMPLE 2** The chairs of an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?
- **EXAMPLE 3** There are 32 microcomputers in a computer center. Each microcomputer has 24 ports. How many different ports to a microcomputer in the center are there?
- **EXAMPLE 4** How many different bit strings of length seven are there?
- **EXAMPLE 5** How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits (and no sequences of letters are prohibited, even if they are obscene)?
- **EXAMPLE 6** **Counting Functions** How many functions are there from a set with m elements to a set with n elements?
- **EXAMPLE 7** **Counting One-to-One Functions** How many one-to-one functions are there from a set with m elements to one with n elements?

Binary Strings

How many n −bit binary strings are there? If we let $B = \{0, 1\}$, then the set of n −bit binary strings is:

$B \times B \times \cdots \times B$ $\quad n$ terms

By the Product Rule, the number of binary strings is $|B|^n = 2^n$.

- **EXAMPLE 8 The Telephone Numbering Plan** The *North American numbering plan (NANP)* specifies the format of telephone numbers in the U.S., Canada, and many other parts of North America. A telephone number in this plan consists of 10 digits, which are split into a three-digit area code, a three-digit office code, and a four-digit station code. Because of signaling considerations, there are certain restrictions on some of these digits. To specify the allowable format, let X denote a digit that can take any of the values 0 through 9, let N denote a digit that can take any of the values 2 through 9, and let Y denote a digit that must be a 0 or a 1.

How many different North American telephone numbers are possible under the old plan and under the new plan?

In the old plan, the formats of the area code, office code, and station code are NYX, NNX, and $XXXX$, respectively, which generates $8 * 2 * 10 * 8 * 8 * 10 * 10 * 10 * 10 * 10$ numbers. Where as, the new plan, the formats of these codes are NXX, NXX, and $XXXX$, respectively, generates $8 * 10 * 10 * 8 * 10 * 10 * 10 * 10 * 10 * 10$ numbers..

- **EXAMPLE 9** What is the value of k after the following code, where $n_1, n_2, \ldots, n_m$ are positive integers, has been executed?

$k := 0$
for $i_1 := 1$ **to** n_1
    for $i_2 := 1$ **to** n_2
        .
        .
        .
        for $i_m := 1$ **to** n_m
            $k := k + 1$

EXAMPLE 10 Counting Subsets of a Finite Set Use the product rule to show that the number of different subsets of a finite set S is $2^{|S|}$.

The Cartesian product $A_1 \times A_2 \times \cdots \times A_m$ is done by choosing an element in A_1, an element in A_2, . . . , and an element in A_m. By the product rule it follows that

$$|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \cdot |A_2| \cdot \cdots \cdot |A_m|.$$

EXAMPLE 11 DNA and Genomes The hereditary information of a living organism is encoded using deoxyribonucleic acid (DNA), or in certain viruses, ribonucleic acid (RNA). DNA and RNA are extremely complex molecules, with different molecules interacting in a vast variety of ways to enable living process. For our purposes, we give only the briefest description of how DNA and RNA encode genetic information.

EXAMPLES of THE SUM RULE

THE SUM RULE If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task

EXAMPLE 12: Suppose that either a member of the mathematics faculty or a student who is a mathematics major is chosen as a representative to a university committee. How many different choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student?

EXAMPLE 13 A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?

Example:

Suppose that VIT graduates 60 majors in mechanical, 200 majors in EECS, and 40 majors in EEE. How many students graduate from VIT in these three departments?

Solution: Let A_1 be the set of mechanical majors, A_2 be the set of EECS majors, and A_3 be the set of EEE majors.

The set of graduating students in these three departments is $A_1 \cup A_2 \cup A_3$.

Assume for now that these sets are disjoint; that is, there are no double or triple majors. Then we can apply the Sum Rule to determine the total number of graduating students.

$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3|$
$= 60 + 200 + 40$
$= 300$

Note: For any finite set, B, and set, A, $|A \cap B| + |B - A| = |B|$.

EXAMPLE 14 What is the value of k after the following code, where $n_1, n_2, \ldots, n_m$ are positive integers, has been executed?

$$
\begin{array}{l}
k := 0 \\
\textbf{for } i_1 := 1 \textbf{ to } n_1 \\
\quad\quad k := k + 1 \\
\textbf{for } i_2 := 1 \textbf{ to } n_2 \\
\quad\quad k := k + 1 \\
\quad\quad\quad \cdot \\
\quad\quad\quad \cdot \\
\quad\quad\quad \cdot \\
\textbf{for } i_m := 1 \textbf{ to } n_m \\
\quad\quad k := k + 1
\end{array}
$$

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|$$

when $A_i \cap A_j = \phi$ for all i, j.

That is the above equality applies only when the sets in question are pairwise disjoint.

Example: Suppose statement labels in a programming language can be either a single letter or a letter followed by a digit. Find the number of possible labels.

Solution: Use the product rule. $26 + 26 \cdot 10 = 286$

More Complex Counting Problems

- Many counting problems cannot be solved using just the sum rule or just the product rule.
- However, many complicated counting problems can be solved using both of these rules in combination.
- We begin by counting the number of variable names in the programming language BASIC.
- Then we will count the number of valid passwords subject to a particular set of restrictions.

- **EXAMPLE 15**

In a version of the computer language BASIC, the name of a variable is a string of one or two alphanumeric characters, where uppercase and lowercase letters are not distinguished. (An *alphanumeric* character is either one of the 26 English letters or one of the 10 digits.) Moreover, a variable name must begin with a letter and must be different from the five strings of two characters that are reserved for programming use. How many different variable names are there in this version of BASIC?

- *Solution:* Let V equal the number of different variable names in this version of BASIC. Let V_1 be the number of these that are one character long and V_2 be the number of these that are two characters long. Then by the sum rule, $V = V_1 + V_2$. Note that $V_1 = 26$, because a one-character variable name must be a letter. Furthermore, by the product rule there are $26 \cdot 36$ strings of length two that begin with a letter and end with an alphanumeric character. However, five of these are excluded, so $V2 = 26 \cdot 36 - 5 = 931$. Hence, there are $V = V_1 + V_2 = 26 + 931 = 957$ different names for variables in this version of BASIC.

EXAMPLE 16 Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

Solution: Let P be the total number of possible passwords, and let P_6, P_7, and P_8 denote the number of possible passwords of length 6, 7, and 8, respectively. By the sum rule, $P = P_6+P_7+P_8$. We will now find P_6, P_7, and P_8. Finding P_6 directly is difficult. To find P_6 it is easier to find the number of strings of uppercase letters and digits that are six characters long, including those with no digits, and subtract from this the number of strings with no digits. By the product rule, the number of strings of six characters is 36^6, and the number of strings with no digits is 26^6.

Hence, $P_6 = 36^6 - 26^6 = 2{,}176{,}782{,}336 - 308{,}915{,}776 = 1{,}867{,}866{,}560$. Similarly, we have

$P_7 = 36^7 - 26^7 = 78{,}364{,}164{,}096 - 8{,}031{,}810{,}176 = 70{,}332{,}353{,}920$ and $P_8=36^8 - 26^8=2{,}821{,}109{,}907{,}456 - 208{,}827{,}064{,}576 = 2{,}612{,}282{,}842{,}880$.

Therefore, $P = P_6 + P_7 + P_8 = 2{,}684{,}483{,}063{,}360$.

EXAMPLE 17 Counting Internet Addresses

In the Internet, which is made up of interconnected physical networks of computers, each computer (or more precisely, each network connection of a computer) is assigned an *Internet address*. In Version 4 of the Internet Protocol (IPv4), now in use,

Bit Number	0	1	2	3	4	8	16	24	31
Class A	0	netid				hostid			
Class B	1	0	netid				hostid		
Class C	1	1	0	netid				hostid	
Class D	1	1	1	0	Multicast Address				
Class E	1	1	1	1	0	Address			

FIGURE 1 **Internet Addresses (IPv4).**

- an address is a string of 32 bits. It begins with a *network number* (*netid*). The netid is followed by a *host number* (*hostid*), which identifies a computer as a member of a particular network.

Three forms of addresses are used, with different numbers of bits used for netids and hostids. **Class A addresses**, used for the largest networks, consist of 0, followed by a 7-bit netid and a 24-bit hostid. **Class B addresses**, used for medium-sized networks, consist of 10, followed by a 14-bit netid and a 16-bit hostid. **Class C addresses**, used for the smallest networks, consist of 110, followed by a 21-bit netid and an 8-bit hostid. There are several restrictions on addresses because of special uses: 1111111 is not available as the netid of a Class A network, and the hostids consisting of all 0s and all 1s are not available for use in any network. A computer on the Internet has either a Class A, a Class B, or a Class C address. (Besides Class A, B, and C addresses, there are also Class D addresses, reserved for use in multicasting when multiple computers are addressed at a single time, consisting of 1110 followed by 28 bits, and Class E addresses, reserved for future use, consisting of 11110 followed by 27 bits. Neither Class D nor Class E addresses are assigned as the IPv4 address of a computer on the Internet.)

How many different IPv4 addresses are available for computers on the Internet?

Solution: Let x be the number of available addresses for computers on the Internet, and let x_A, x_B, and x_C denote the number of ClassA, Class B, and Class C addresses available, respectively. By the sum rule, $x = x_A + x_B + x_C$.

To find x_A, note that there are $2^7 - 1 = 127$ Class A netids, recalling that the netid 1111111 is unavailable. For each netid, there are $2^{24} - 2 = 16{,}777{,}214$ hostids, recalling that the hostids consisting of all 0s and all 1s are unavailable. Consequently, $x_A = 127 \cdot 16{,}777{,}214 = 2{,}130{,}706{,}178$.

To find x_B and x_C, note that there are $2^{14} = 16{,}384$ Class B netids and $2^{21} = 2{,}097{,}152$ Class C netids.

For each Class B netid, there are $2^{16} - 2 = 65{,}534$ hostids, and for each Class C netid, there are $2^8 - 2 = 254$ hostids, recalling that in each network the hostids consisting of all 0s and all 1s are unavailable. Consequently,

$x_B = 1{,}073{,}709{,}056$ and $x_C = 532{,}676{,}608$.

We conclude that the total number of IPv4 addresses available is

$$x = x_A + x_B + x_C = 2{,}130{,}706{,}178 + 1{,}073{,}709{,}056 + 532{,}676{,}608 = 3{,}737{,}091{,}842.$$

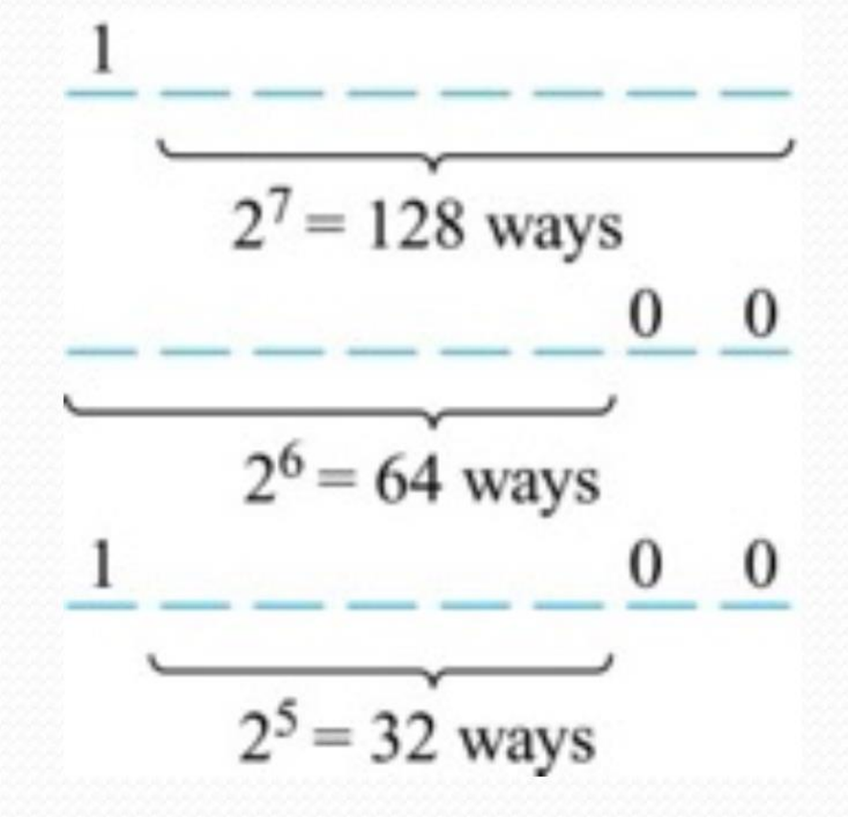
The Subtraction Rule (Inclusion–Exclusion for Two Sets)

- THE SUBTRACTION RULE If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.
  The subtraction rule is also known as the **principle of inclusion–exclusion**, especially when it is used to count the number of elements in the union of two sets. Suppose that A_1 and A_2 are sets. Then, there are $|A_1|$ ways to select an element from A_1 and $|A_2|$ ways to select an element from A_2
- There are $|A1 \cup A2|$ ways to select an element in either A_1 or in A_2 , and $|A_1 \cap A_2|$ ways to select an element common to both sets, we have $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

EXAMPLE 18 How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

Solution: We can construct a bit string of length eight that either starts with a 1 bit or ends with the two bits 00, by constructing a bit string of length eight beginning with a 1 bit or by constructing a bit string of length eight that ends with the two bits 00.

A bit string of length eight that begins with 1 is $2^7 = 128$ ways. Similarly, we can construct a bit string of length eight ending with the two bits 00, is $2^6 = 64$ ways. Number of bit strings of length eight that start with a 1 bit and end with bits 00 is $2^5 = 32$ ways to construct such a string.
Consequently, the number of bit strings of length eight that begin with a 1 or end with a 00, equals $128 + 64 - 32 = 160$.



EXAMPLE 19 A computer company receives 350 applications from computer graduates for a job planning a line of new Web servers. Suppose that 220 of these applicants majored in computer science, 147 majored in business, and 51 majored both in computer science and in business. How many of these applicants majored neither in computer science nor in business?

Solution: Let A_1 be the set of students who majored in computer science and A_2 the set of students who majored in business. Then $A_1 \cup A_2$ is the set of students who majored in computer science or business (or both), and $A_1 \cap A_2$ is the set of students who majored both in computer science and in business.

$| A_1 \cup A_2 | = | A_1 | + | A_2 | - |A_1 \cap A_2| = 220 + 147 - 51 = 316.$
We conclude that $350 - 316 = 34$ of the applicants majored neither in computer science nor in
business.

The Division Rule

THE DIVISION RULE There are n/d ways to do a task if it can be done using a procedure that can be carried out in n ways, and for every way w, exactly d of the n ways correspond to way w.

We now restate the division rule in terms of sets: "If the finite set A is the union of n pairwise disjoint subsets each with d elements, then $n = |A|/d$." We can also formulate the division rule in terms of functions: "If f is a function from A to B where A and B are finite sets, and that for every value $y \in B$ there are exactly d values $x \in A$ such that $f(x) = y$ (in which case, we say that f is d-to-one), then $|B| = |A|/d$."

Theorem (The division rule). If $f : X \to Y$ is k-to-1, then $|Y| = \frac{|X|}{k}$.

The division rule In how many ways can we place two identical rooks on an 8 by 8 chessboard so that they occupy different rows and different columns?

Let's first count the number of configurations for two different rooks. Each configuration can then be represented by a sequence (r1r, r1c, r2r, r2c) indicating the row and column of the first and second rook, respectively.

By the generalized product rule, the set of configurations $C_{different}$ has size $|C_{different}| = 8 \cdot 8 \cdot 7 \cdot 7 = (8 \cdot 7)^2$. Now let $C_{identical}$ be the set of configurations when the two rooks are identical. We won't count the number of elements of $C_{identical}$ directly but take advantage of what we know already. Each configuration in $C_{identical}$ can be naturally represented by a pair of sequences in $C_{different}$.

For example, the configuration in $C_{identical}$ in which one rook is at position (1, 1) and the second one is at (2, 3) is represented by the pair of sequences (1, 1, 2, 3) and (2, 3, 1, 1) in $C_{different}$. Since each element in $C_{identical}$ is represented by exactly two elements in $C_{different}$, the set $C_{different}$ must be exactly twice as large as $C_{identical}$ and so the desired number of configurations is

$$| C_{identical} | = \frac{|C_{different}|}{2} = \frac{(8 \cdot 7)^2}{2}.$$

- Here is a general description of this type of counting argument. A function $f : X \to Y$ is k –to-1 if for every $y \; in \; Y$, the number of $x \in X$ such that $f(x) = y$ is exactly k: $|\{x \in X : f(x) = y\}| = k$ for every $y \in Y$. If we want to count the size of Y and have a k-to-1 function from X to Y where X is a set whose size we know, we can conclude that Y has size $\frac{|X|}{k}$.

EXAMPLE 20 How many different ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left neighbor and the same right

Solution: Number the seats around the table from 1 to 4 proceeding clockwise.

There are four ways to select the person for seat 1, 3 for seat 2, 2, for seat 3, and one way for seat 4.

Thus there are $4! = 24$ ways to order the four people.

But since two seatings are the same when each person has the same left and right neighbor, for every choice for seat 1, we get the same seating.

Therefore, by the division rule, there are $24/4 = 6$ different seating arrangements.

Tree Diagrams

Tree Diagrams

Counting problems can be solved using **tree diagrams**. A tree consists of a root, a number of branches leaving the root, and possible additional branches leaving the endpoints of other branches. (We will study trees in detail in Chapter 11.) To use trees in counting, we use a branch to represent each possible choice. We represent the possible outcomes by the leaves, which are

EXAMPLE 21 How many bit strings of length four do not have two consecutive 1s?

Solution: The tree diagram in Figure 2 displays all bit strings of length four without two consecutive 1s. We see that there are eight bit strings of length four without two consecutive 1s.

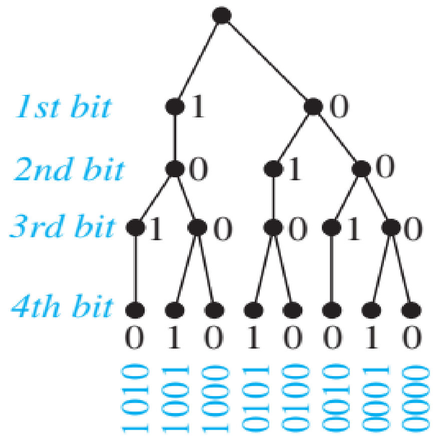


FIGURE 2 **Bit Strings of Length Four without Consecutive 1s.**

EXAMPLE 22 A playoff between two teams consists of at most five games. The first team that wins three games wins the playoff. In how many different ways can the playoff occur?

Solution: The tree diagram in Figure 3 displays all the ways the playoff can proceed, with the winner of each game shown. We see that there are 20 different ways for the playoff to occur.

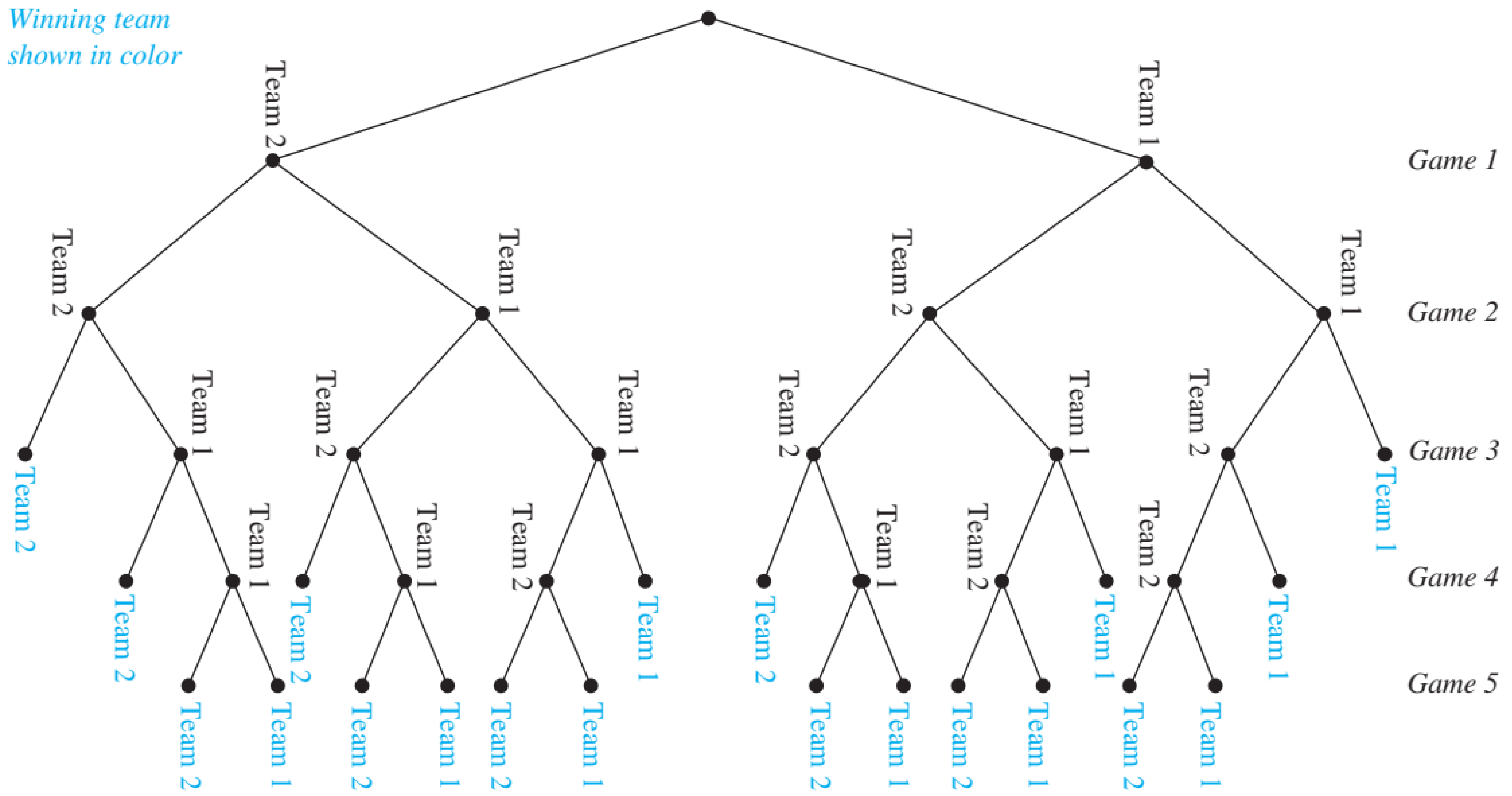


FIGURE 3 **Best Three Games Out of Five Playoffs.**

EXAMPLE 23 Suppose that "I Love New Jersey" T-shirts come in five different sizes: S, M, L, XL, and XXL. Further suppose that each size comes in four colors, white, red, green, and black, except for XL, which comes only in red, green, and black, and XXL, which comes only in green and black. How many different shirts does a souvenir shop have to stock to have at least one of each available size and color of the T-shirt?

Solution: The tree diagram in Figure 4 displays all possible size and color pairs. It follows that the souvenir shop owner needs to stock 17 different T-shirts. ◀

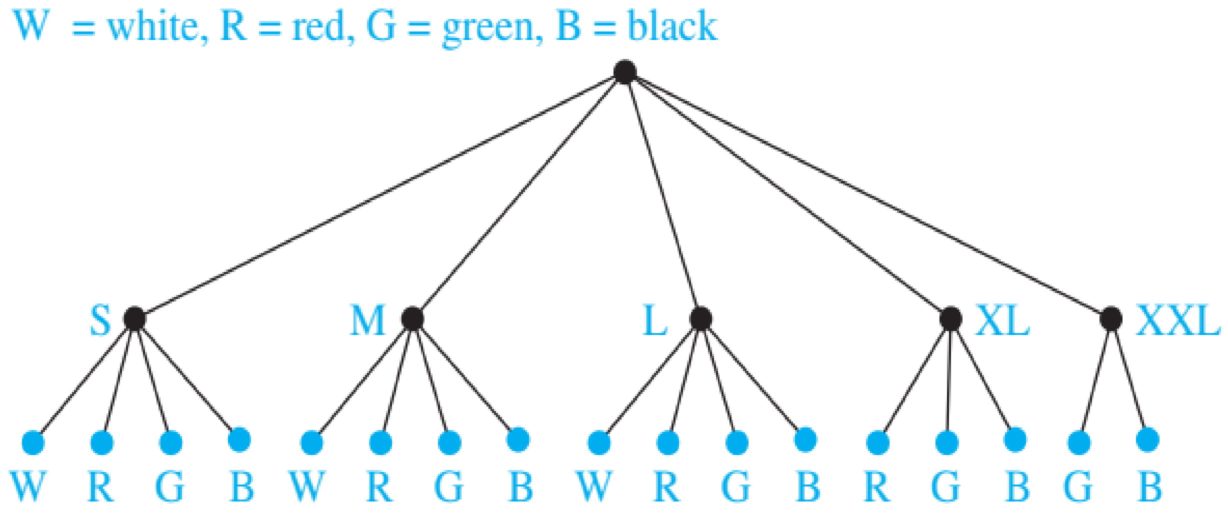


FIGURE 4 **Counting Varieties of T-Shirts.**