Permutations and Combinations-Revisited

- A **permutation** of a set of objects is an ordering of those objects.
- Permutations reflect selections for which an ordering is important
- **DEFINITION**

A **permutation** of k objects from a set of size n is an ordered list of k of the n objects.

- The number of permutations of k objects from n is denoted $P(n, k)$.

(Listing Elements of a Set). The set $S = \{a, b, c, d, e\}$ is entered into the computer by listing its elements in some order between curly braces, such as S := {b, a, c, e, d} ;

Assuming that no element is listed more than once, in how many ways might this be done?

Solution. There are $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5! = 120$ ways to enter S.

(Lining Up). How many ways are there to put 8 children in a line to get ice cream?

THEOREM *The number of ways to put* n *distinct items in order is* $n!$.

How many different license plates consisting of 6 *distinct* digits (0 to 9) are
possible?

Solution. Any digit can be the first. Then there are 9 digits left from which to choose the second. Then there are 8 choices left for the third, and so on.

Hence, there are $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 151200$
6-digit license plates with distinct digits.

DEFINITION

A **combination** of k elements from a set of size n is a subset of size k.

If $S = \{a, b, c, d, e\}$ is the set of $n = 5$ objects under consideration and $k = 3$, then all of the possible combinations of 3 elements from S are listed in Figure 6.5. We see that the number of combinations of 3 objects from 5 is equal to 10.

$$\{a, b, c\} \quad \{a, b, d\} \quad \{a, b, e\} \quad \{a, c, d\} \quad \{a, c, e\}$$
$$\{a, d, e\} \quad \{b, c, d\} \quad \{b, c, e\} \quad \{b, d, e\} \quad \{c, d, e\}$$

Figure 6.5 The Combinations of Size 3 from $S = \{a, b, c, d, e\}$

Theorem:

Let $n, k \in \mathbb{Z}$ *with* $0 \le k \le n$. *Given a set of* n *distinct elements, the number of subsets of size* k *is given by the binomial coefficient*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

That is, $\binom{n}{k}$ *counts the number of combinations of* k *elements from a set of size* n.

Permutations vs. Combinations

It is important to distinguish between situations that require permutations and those that require combinations. For example, suppose we wish to select 2 cards from a deck of 52 distinct cards. A common incorrect analysis of this problem gives that there are

$$52 \cdot 51 = P(52, 2) = 2652$$

possible 2-card hands. The argument is that there are 52 choices for the first card and then 51 choices left for the second card. However, in that reasoning, it is assumed that the order of the cards is important. For example, the selections $3\heartsuit, 8\clubsuit$ and $8\clubsuit, 3\heartsuit$ would be counted as different. If it only matters which 2 cards are obtained, then there are

$$\binom{52}{2} = 1326$$

ways to select 2 cards from the deck. The desired outcome (the *set* of 2 cards) is what is being counted, not the process by which someone might obtain those cards (first one, then another).

Generalized Permutations and Combinations

Introduction

In many counting problems, elements may be used repeatedly. For instance, a letter or digit may be used more than once on a license plate. When a dozen donuts are selected, each variety can be chosen repeatedly. This contrasts with the counting problems discussed earlier in the chapter where we considered only permutations and combinations in which each item could be used at most once.

In this section we will show how to solve counting problems where elements may be used more than once

Permutations with Repetition

EXAMPLE 1 How many strings of length r can be formed from the uppercase letters of the English alphabet?

Solution: By the product rule, because there are 26 uppercase English letters, and because each letter can be used repeatedly, we see that there are 26^r strings of uppercase English letters of length r.

Permutations with Repetition

THEOREM 1

The number of r −permutations of a set of n objects with repetition allowed is n^r.

Proof: There are n ways to select an element of the set for each of the r positions in the r −permutation when repetition is allowed, because for each choice all n objects are available.

Hence, by the product rule there are n^r many r −permutations when repetition is allowed.

Combinations with Repetition

EXAMPLE 2 How many ways are there to select four pieces of fruit from a bowl containing apples, oranges, and pears if the order in which the pieces are selected does not matter, only the type of fruit and not the individual piece matters, and there are at least four pieces of each type of fruit in the bowl?

Solution: To solve this problem we list all the ways possible to select the fruit. There are 15 ways:

4 apples	4 oranges	4 pears
3 apples, 1 orange	3 apples, 1 pear	3 oranges, 1 apple
3 oranges, 1 pear	3 pears, 1 apple	3 pears, 1 orange
2 apples, 2 oranges	2 apples, 2 pears	2 oranges, 2 pears
2 apples, 1 orange, 1 pear	2 oranges, 1 apple, 1 pear	2 pears, 1 apple, 1 orange

The solution is the number of 4-combinations with repetition allowed from a three-element set, $\{apple, orange, pear\}$.

EXAMPLE 3 How many ways are there to select five bills from a cash box containing \$1 bills, \$2 bills, \$5 bills, \$10 bills, \$20 bills, \$50 bills, and \$100 bills? Assume that the order in which the bills are chosen does not matter, that the bills of each denomination are indistinguishable, and that there are at least five bills of each type.

Solution:

Refer Textbook, Example 3, Section 6.5 Combinations

Combinations with Repetition

THEOREM 2

There are

$$C(n+r-1,r) \ = \ C(n+r-1,n-r) = \frac{(n+r-1)!}{r!(n-r)!}$$

r −combinations from a set with n elements when repetition of elements is allowed.

TABLE 1 Combinations and Permutations With and Without Repetition.

Type	*Repetition Allowed?*	*Formula*
r-permutations	No	$\dfrac{n!}{(n-r)!}$
r-combinations	No	$\dfrac{n!}{r!\,(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\dfrac{(n+r-1)!}{r!\,(n-1)!}$

EXAMPLE 4

Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen? Assume that only the type of cookie, and not the individual cookies or
the order in which they are chosen, matters.

Solution: The number of ways to choose six cookies is the number of 6-combinations of a set with four elements. From Theorem 2 this equals $C(4 + 6 - 1, 6) = C(9, 6)$.

Since $C(9,6) = C(9,3) = \dfrac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3} = 84,$

there are 84 different ways to choose the six cookies

EXAMPLE 5

How many solutions does the equation

$$x_1 + x_2 + x_3 = 11$$

have, where x_1, x_2, and x_3 are nonnegative integers?

Solution:

Refer the textbook Example 5, Section 6.5 **Combinations with Repetition**

Permutations with Indistinguishable Objects

EXAMPLE 7 How many different strings can be made by reordering the letters of the word *SUCCESS*?

Solution: The given word contains three *S*s, two *C*s, one *U*, and one *E*. To determine the number of different strings that can be made by reordering the letters, first note that the three *S*s can be placed among the seven positions in $C(7, 3)$ different ways, leaving four positions free.

Then the two *C*s can be placed in $C(4, 2)$ ways, leaving two free positions. The *U* can be placed in $C(2, 1)$ ways, leaving just one position free.

Hence *E* can be placed in $C(1, 1)$ way.

Consequently, from the product rule, the number of different strings that can be made is

$$C(7,3)C(4,2)C(2,1)C(1,1) = \frac{7!}{3!\ 4!} * \frac{4!}{2!\ 2!} * \frac{2!}{1!\ 1!} * \frac{1!}{1!\ 0!} = 420$$

How many permutations of the word MISSISSIPPI are there?

'MISSISSIPPI' has

- 4 distinct letters: M,I,S,P
- with 1,4,4,2 (M,I,S,P)occurrences respectively
- Therefore, the number of permutations

$$\frac{11!}{1!4!4!2!}$$

THEOREM 3

The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, … , and n_k indistinguishable objects of type k, is

$$\frac{n!}{n_1!\, n_2! \cdots n_k!}.$$

DISTINGUISHABLE OBJECTS AND DISTINGUISHABLE BOXES

EXAMPLE 8 How many ways are there to distribute hands of 5 cards to each of four players from the standard deck of 52 cards?

Solution: We will use the product rule to solve this problem. To begin, note that the first player can be dealt 5 cards in $C(52,5)$ ways. The second player can be dealt 5 cards in $C(47,5)$ ways, because only 47 cards are left. The third player can be dealt 5 cards in $C(42,5)$ ways. Finally, the fourth player can be dealt 5 cards in $C(37,5)$ ways. Hence, the total number of ways to deal four players 5 cards each is

$$C(52,5)C(47,5)C(42,5)C(37,5) = \frac{52!}{47!\,5!} \cdot \frac{47!}{42!\,5!} \cdot \frac{42!}{37!\,5!} \cdot \frac{37!}{32!\,5!}$$
$$= \frac{52!}{5!\,5!\,5!\,5!\,32!}.$$

THEOREM 4

The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed into box i, $i = 1, 2, \ldots, k$, equals

$$\frac{n!}{n_1!\, n_2! \cdots n_k!}$$

INDISTINGUISHABLE OBJECTS AND DISTINGUISHABLE BOXES

EXAMPLE 9 How many ways are there to place 10 indistinguishable balls into eight distinguishable bins?
Solution: The number of ways to place 10 indistinguishable balls into eight bins equals the number of 10-combinations from a set with eight elements when repetition is allowed. Consequently,
there are

$$C(8+10-1, 10) = C(17, 10) = \frac{17!}{10!7!} = 19{,}448.$$

Note: There are $C(n+r-1, n-1)$ ways to place r indistinguishable objects into n distinguishable boxes.

DISTINGUISHABLE OBJECTS AND INDISTINGUISHABLE BOXES

EXAMPLE 10 How many ways are there to put four different employees into three indistinguishable offices, when each office can contain any number of employees?

INDISTINGUISHABLE OBJECTS AND INDISTINGUISHABLE BOXES Some counting problems can be solved by determining the number of ways to distribute indistinguishable objects into indistinguishable boxes. For example.

EXAMPLE 11 How many ways are there to pack six copies of the same book into four identical boxes, where a box can contain as many as six books?

Refer Textbook for solution Section 6.5