GENERATING FUNCTIONS

- Generating functions are used to represent sequences efficiently by coding the terms of a sequence as coefficients of powers of a variable x in a formal power series.
- Generating functions can be used to solve many types of counting problems, such as the number of ways to select or distribute objects of different kinds, subject to a variety of constraints, and the number of ways to make change for a dollar using coins of different denominations.
- Generating functions can be used to solve recurrence relations by translating a recurrence relation for the terms of a sequence into an equation involving a generating function. This equation can then be solved to find a closed form for the generating function. From this closed form, the coefficients of the power series for the generating function can be found, solving the original recurrence relation.
- Generating functions can also be used to prove combinatorial identities by taking advantage of relatively simple relationships between functions that can be translated into identities involving the terms of sequences.

DEFINITION 1

- The *generating function for the sequence* $a_0, a_1, \ldots, a_k, \ldots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k$$

Remark: The generating function for $\{a_k\}$ given in Definition 1 is sometimes called the **ordinary generating function** of $\{a_k\}$ to distinguish it from other types of generating functions for this sequence.

Example:

Show that the generating function for the sequences $\{a_k\}$ with $a_k = 3$, $a_k = k + 1$, and $a_k = 2^k$ are $\sum_{k=0}^{\infty} 3x^k$, $\sum_{k=0}^{\infty}(k+1)x^k$, and $\sum_{k=0}^{\infty} 2^k x^k$ respectively.

Solution:

Since $a_k = 3^k$, implies $a_0 = 3^0 = 1; a_1 = 3^1 = 3, a_2 = 3^2 = 9$

Therefore, by the generating function $G(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k$, we have

$G(x) = 1 + 3 * x + 3^2 * x^2 + 3^3 * x^3 + \cdots + 3^k * x^k + .. = \sum_{k=0}^{\infty} 3^k x^k$

Similarly, we can get the other generating functions.

- We can define generating functions for finite sequences of real numbers by extending a finite sequence $a_0, a_1, \ldots, a_n$ into an infinite sequence by setting $a_{n+1} = 0$, $a_{n+2} = 0$, and so on.
- The generating function $G(x)$ of this infinite sequence $\{a_n\}$ is a polynomial of degree n because no terms of the form $a_j x^j$ with $j > n$ occur, that is,

$$G(x) = a_0 + a_1 x + \cdots + a_n x^n.$$

EXAMPLE 2

What is the generating function for the sequence 1, 1, 1, 1, 1, 1?

Solution:

The generating function of 1, 1, 1, 1, 1, 1 is $1 + x + x^2 + x^3 + x^4 + x^5$. we have

$$\frac{x^6 - 1}{x - 1} = 1 + x + x^2 + x^3 + x^4 + x^5.$$

when $x \neq 1$. Consequently, $G(x) = (x^6 - 1)/(x - 1)$ is the generating function of the sequence 1, 1, 1, 1, 1, 1.

Note: Because the powers of x are only place holders for the terms of the sequence in a generating function, we do not need to worry that $G(1)$ is undefined.

EXAMPLE 3

Let m be a positive integer. Let $a_k = C(m, k)$, for $k = 0, 1, 2, \ldots, m$. What is the generating function for the sequence $a_0, a_1, \ldots, a_m$?

Solution:

The generating function for this sequence is

$$G(x) = C(m, 0) + C(m, 1)x + C(m, 2)x^2 + \cdots + C(m, m)x^m.$$

The binomial theorem shows that $G(x) = (1 + x)^m$

Useful Facts About Power Series

- When generating functions are used to solve counting problems, they are usually considered to be **formal power series**. Questions about the convergence of these series are ignored.
- However, to apply some results from calculus, it is sometimes important to consider for which x the power series converges. The fact that a function has a unique power series around $x = 0$ will also be important.
- Generally, however, we will not be concerned with questions of convergence or the uniqueness of power series in our discussions.
- Readers familiar with calculus can consult textbooks on this subject for details about power series, including the convergence of the series we consider here.

- We now state some important facts about infinite series used when working with generating functions. A discussion of these and related results can be found in calculus texts.

EXAMPLE 4

The function $f(x) = \frac{1}{1-x}$ is the generating function of the sequence 1, 1, 1, 1, . . . , because

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots \text{ for } |x| < 1.$$

EXAMPLE 5

The function $f(x) = \frac{1}{1-ax}$ is the generating function of the sequence

$1, a, a^2, a^3, \ldots,$

Because $\frac{1}{1-ax} = 1 + a\,x + a^2x^2 + \cdots$ when $|ax| < 1,$

or

equivalently, when $|x| < \frac{1}{|a|}$ for $a \neq 0.$

THEOREM 1

- Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$. Then

$$f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k \text{ and}$$

$$f(x)g(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j} \right) x^k.$$

- **Remark:** Theorem 1 is valid only for power series that converge in an interval, as all series considered in this section do. However, the theory of generating functions is not limited to such series. In the case of series that do not converge, the statements in Theorem 1 can be taken as definitions of addition and multiplication of generating functions.

EXAMPLE 6

Let $f(x) = \frac{1}{(1-x)^2}$. Use Example 4 to find the coefficients $a_0, a_1, a_2, \ldots$ in the expansion $f(x) = \sum_{k=0}^{\infty} a_k x^k$

Solution:

From Example 4 we see that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots.$$

Hence, from Theorem 1, we have

$$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} 1\right) x^k = \sum_{k=0}^{\infty} (k+1) x^k.$$

DEFINITION 2

Let u be a real number and k a nonnegative integer. Then the *extended binomial coefficient* $\binom{u}{k}$ is defined by

$$\binom{u}{k} = \begin{cases} u(u-1)\cdots(u-k+1)/k! & \text{if } k > 0, \\ 1 & \text{if } k = 0. \end{cases}$$

Note: Here, we are finding the binomial coefficients for the real number u (need not be a positive integer always). If u is a positive integer then, our earlier binomial coefficients can be obtained as in earlier Section.

EXAMPLE 7

Find the values of the extended binomial coefficients $\binom{-2}{3}$ and $\binom{1/2}{3}$.

Solution: Taking $u = -2$ and $k = 3$ in Definition 2 gives us

$$\binom{-2}{3} = \frac{(-2)(-3)(-4)}{3!} = -4.$$

Similarly, taking $u = 1/2$ and $k = 3$ gives us

$$\binom{1/2}{3} = \frac{(1/2)(1/2-1)(1/2-2)}{3!}$$

$$= (1/2)(-1/2)(-3/2)/6$$

$$= 1/16.$$

EXAMPLE 8

When the top parameter is a negative integer, the extended binomial coefficient can be expressed in terms of an ordinary binomial coefficient. To see that this is the case, note that

$$\binom{-n}{r} = \frac{(-n)(-n-1)\cdots(-n-r+1)}{r!} \quad \text{by definition of extended binomial coefficient}$$

$$= \frac{(-1)^r n(n+1)\cdots(n+r-1)}{r!} \quad \text{factoring out } -1 \text{ from each term in the numerator}$$

$$= \frac{(-1)^r (n+r-1)(n+r-2)\cdots n}{r!} \quad \text{by the commutative law for multiplication}$$

$$= \frac{(-1)^r (n+r-1)!}{r!(n-1)!} \quad \text{multiplying both the numerator and denominator by } (n-1)!$$

$$= (-1)^r \binom{n+r-1}{r} \quad \text{by the definition of binomial coefficients}$$

$$= (-1)^r C(n+r-1, r). \quad \text{using alternative notation for binomial coefficients}$$

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THEOREM 2 THE EXTENDED BINOMIAL THEOREM

Let x be a real number with $|x| < 1$ and let u be a real number. Then

$$(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k.$$

Remark: When u is a positive integer, the extended binomial theorem reduces to the binomial theorem presented in Section 6.4, because in that case

$\binom{u}{k} = 0$ if $k > u$

EXAMPLE 9

- Find the generating functions for $(1 + x)^{-n}$ and $(1 - x)^{-n}$, where n is a positive integer, using the extended binomial theorem.

Solution: By the extended binomial theorem, it follows that

$$(1+x)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} x^k.$$

Using Example 8, which provides a simple formula for $\binom{-n}{k}$, we obtain

$$(1+x)^{-n} = \sum_{k=0}^{\infty} (-1)^k C(n+k-1, k) x^k.$$

Replacing x by $-x$, we find that

$$(1-x)^{-n} = \sum_{k=0}^{\infty} C(n+k-1, k)x^k.$$

Some useful Identities

TABLE 1 Useful Generating Functions.

$G(x)$	a_k
$(1+x)^n = \sum_{k=0}^{n} C(n,k)x^k$ $= 1 + C(n,1)x + C(n,2)x^2 + \cdots + x^n$	$C(n,k)$
$(1+ax)^n = \sum_{k=0}^{n} C(n,k)a^k x^k$ $= 1 + C(n,1)ax + C(n,2)a^2x^2 + \cdots + a^n x^n$	$C(n,k)a^k$
$(1+x^r)^n = \sum_{k=0}^{n} C(n,k)x^{rk}$ $= 1 + C(n,1)x^r + C(n,2)x^{2r} + \cdots + x^{rn}$	$C(n,k/r)$ if $r \mid k$; 0 otherwise

$\dfrac{1-x^{n+1}}{1-x} = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$	1 if $k \leq n$; 0 otherwise
$\dfrac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$	1
$\dfrac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k = 1 + ax + a^2x^2 + \cdots$	a^k
$\dfrac{1}{1-x^r} = \sum_{k=0}^{\infty} x^{rk} = 1 + x^r + x^{2r} + \cdots$	1 if $r \mid k$; 0 otherwise

$\dfrac{1}{(1-x)^2} = \sum_{k=0}^{\infty}(k+1)x^k = 1 + 2x + 3x^2 + \cdots$	$k+1$
$\dfrac{1}{(1-x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)x^k$ $= 1 + C(n,1)x + C(n+1,2)x^2 + \cdots$	$C(n+k-1,k) = C(n+k-1,n-1)$
$\dfrac{1}{(1+x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)(-1)^k x^k$ $= 1 - C(n,1)x + C(n+1,2)x^2 - \cdots$	$(-1)^k C(n+k-1,k) = (-1)^k C(n+k-1,n-1)$

$\dfrac{1}{(1-ax)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)a^k x^k = 1 + C(n,1)ax + C(n+1,2)a^2x^2 + \cdots$	$C(n+k-1,k)a^k = C(n+k-1,n-1)a^k$
$e^x = \sum_{k=0}^{\infty} \dfrac{x^k}{k!} = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$	$1/k!$
$\ln(1+x) = \sum_{k=1}^{\infty} \dfrac{(-1)^{k+1}}{k} x^k = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots$	$(-1)^{k+1}/k$

Note: The series for the last two generating functions can be found in most calculus books when power series are discussed.

Counting Problems and Generating Functions

- Generating functions can be used to solve a wide variety of counting problems. In particular, they can be used to count the number of combinations of various types.
- In previous classes, we developed techniques to count the r −combinations from a set with n elements when repetition is allowed and additional constraints may exist.
- Such problems are equivalent to counting the solutions to equations of the form

  $$e_1 + e_2 + \cdots + e_n = C,$$

  where C is a constant and each e_i is a nonnegative integer that may be subject to a specified constraint.
- Generating functions can also be used to solve counting problems of this type, as Examples 10–12 show

EXAMPLE 10

Find the number of solutions of $e_1 + e_2 + e_3 = 17$,

where e_1, e_2, and e_3 are nonnegative integers with

$2 \leq e_1 \leq 5, 3 \leq e_2 \leq 6$, and $4 \leq e_3 \leq 7$.

Solution:

The number of solutions with the indicated constraints is the coefficient of x^{17} in the expansion of

$$(x^2 + x^3 + x^4 + x^5)(x^3 + x^4 + x^5 + x^6)(x^4 + x^5 + x^6 + x^7).$$

This follows because we obtain a term equal to x^{17} in the product by picking a term in the first sum x^{e_1}, a term in the second sum x^{e_2}, and a term in the third sum x^{e_3}, where the exponents e_1, e_2, and e_3 satisfy the equation $e_1 + e_2 + e_3 = 17$ and the given constraints. (Cont…)

It is not hard to see that the coefficient of x^{17} in this product is *3*.

Hence, there are three solutions.

Note that the calculating of this coefficient involves about as much work as enumerating all the solutions of the equation with the given constraints. However, the method that this illustrates often can be used to solve wide classes of counting problems with special formulae, as we will see. Furthermore, a computer algebra system can be used to do such computations.

Determine the coefficient of x^{15} in

$$f(x) = (x^2 + x^3 + x^4 + \cdots)^4.$$

$$(x^2 + x^3 + x^4 + \cdots) = x^2(1 + x + x^2 + \cdots) = \frac{x^2}{1-x}$$

$$f(x) = (x^2/(1-x))^\wedge 4 = \frac{x^8}{(1-x)^4}$$

Hence the solution is the coefficient of x^7 in $(1-x)^{-4}$ which is

$C(-4,7)(-1)7 = C(10,7)$.

Find the number of ways to select R balls from a pile of 2 red, 2 green and 2 blue balls

$$x^{e_1} x^{e_2} x^{e_3}$$

$$e_1 + e_2 + e_3 = R$$

EXAMPLE 11

In how many different ways can eight identical cookies be distributed among three distinct children if each child receives at least two cookies and no more than four cookies?

EXAMPLE 12 (See Page no 543)

Use generating functions to determine the number of ways to insert tokens worth \$1, \$2, and \$5 into a vending machine to pay for an item that costs r dollars in both the cases when the order in which the tokens are inserted does not matter and when the order does matter.

For example, there are two ways to pay for an item that costs \$3 when the order in which the tokens are inserted does not matter: inserting three \$1 tokens or one \$1 token and a \$2 token. When the order matters, there are three ways: inserting three \$1 tokens, inserting a \$1 token and then a \$2 token, or inserting a \$2 token and then a \$1 token.

EXAMPLE 13 (See Page no: 544)

Use generating functions to find the number of k −combinations of a set with n elements. Assume that the binomial theorem has already been established.

EXAMPLE 14 (See Page no: 545)

Use generating functions to find the number of r −combinations from a set with n elements when repetition of elements is allowed.

EXAMPLE 15 (See Page no: 546)

Use generating functions to find the number of ways to select r objects of n different kinds if we must select at least one object of each kind.

Using Generating Functions to Solve Recurrence Relations

EXAMPLE 16

Solve the recurrence relation $a_k = 3a_{k-1}$ for $k = 1, 2, 3, \ldots$ and initial condition $a_0 = 2$.

Solution: Let $G(x)$ be the generating function for the sequence $\{a_k\}$, that is, $G(x) = \sum_{k=0}^{\infty} a_k x^k$.

First note that

$$xG(x) = \sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=1}^{\infty} a_{k-1} x^k.$$

Using the recurrence relation, we see that

$G(x) - 3\,x\,G(x) =$

$$\sum_{k=0}^{\infty} a_k x^k - 3 \sum_{k=1}^{\infty} a_{k-1} x^k = a_0 + \sum_{k=1}^{\infty} (a_k - 3a_{k-1}) x^k$$

Since $a_0 = 2$ and $a_k = 3a_{k-1}$.

Thus, $G(x) - 3\,x\,G(x) = 2 + 0$.

That is, $G(x) - 3\,x\,G(x) = (1-3x)G(x) = 2 \Rightarrow G(x) = \frac{2}{(1-3x)}$.

Using the identity $\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k$, from Table 1, we have

$$G(x) = 2 \sum_{k=0}^{\infty} 3^k x^k = \sum_{k=0}^{\infty} 2 * 3^k x^k .$$

Consequently, $a_k = 2 * 3^k$

Example with another method/approach

The sequence of numbers 1,1,2,3,5,8,13, ... is known as the Fibonacci sequence. We can describe the sequence in terms of a recurrence relation as : $a_0 = 1, a_1 = 1$ and

$a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$.

How do we find a formula for a_n?

Let $G(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$ be the generating function for the sequence.

Then $G(x) = 1 + x + \sum_{n=2}^{\infty} a_n x^n$

$$= 1 + x + \sum_{n=2}^{\infty} (a_{n-1} + a_{n-2}) x^n$$

$$= 1 + x + (a_1 x^2 + a_2 x^3 + \cdots) + (a_0 x^2 + a_1 x^3 + a_2 x^4 + \cdots)$$

$$= 1 + x + x * (a_1 x + a_2 x^2 + \cdots.) + x^2 (a_0 + a_1 x + a_2 x^2 + \cdots.)$$

$$= 1 + x + x(G(x) - 1) + x^2 G(x)$$

Therefore, $G(x) = 1 + x + x(G(x) - 1) + x^2 G(x)$.

From this, we obtain $G(x) = \frac{1}{1-x-x^2} = -\frac{1}{x^2+x-1}$

What is the expansion for $\frac{-1}{x^2+x-1}$? The roots of $-x^2+x+1=0$ are $x=\frac{-1\pm\sqrt{5}}{2}$. Let $\alpha_1=\frac{-1+\sqrt{5}}{2}$ and $\alpha_2=\frac{-1-\sqrt{5}}{2}$. Then

$$G(x)=\frac{-1}{x^2+x-1}=\frac{-1}{(x-\alpha_1)(x-\alpha_2)}=\frac{1}{\alpha_1-\alpha_2}\left(\frac{1}{x-\alpha_2}-\frac{1}{x-\alpha_1}\right).$$

Given that $\alpha_1-\alpha_2=\sqrt{5}$ it follows that $G(x)=\frac{1}{\sqrt{5}}(\frac{1}{x-\alpha_2}-\frac{1}{x-\alpha_1})$. We also have

$$\begin{aligned}
\frac{1}{x-\alpha_1} &= -\frac{1}{\alpha_1}\left(\frac{1}{1-\frac{x}{\alpha_1}}\right)=-\frac{1}{\alpha_1}\left(1+\frac{x}{\alpha_1}+\frac{x^2}{\alpha_1^2}+\cdots\right)\\
\frac{1}{x-\alpha_2} &= -\frac{1}{\alpha_2}\left(\frac{1}{1-\frac{x}{\alpha_2}}\right)=-\frac{1}{\alpha_2}\left(1+\frac{x}{\alpha_2}+\frac{x^2}{\alpha_2^2}+\cdots\right).
\end{aligned}$$

Since $G(x) = \frac{-1}{(x-\alpha_1)(x-\alpha_2)}$, it follows that $a_n = \frac{1}{\sqrt{5}}\left(\frac{1}{\alpha_1^{n+1}} - \frac{1}{\alpha_2^{n+1}}\right)$, $n = 0, 1, 2, 3, \ldots$.

Example Solve for a_n given that $a_0 = 0$, $a_1 = 6$ and $a_n = -3a_{n-1} + 10a_{n-2} + 3 \cdot 2^n$, for $n \geq 2$.

Solution Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function for $a_0, a_1, a_2, \ldots$. Then

$$\begin{aligned} G(x) &= 6x + \sum_{n=2}^{\infty} a_n x^n \\ &= 6x + \sum_{n=2}^{\infty} (-3a_{n-1} + 10a_{n-2} + 3 \cdot 2^n)x^n \end{aligned}$$

$$
\begin{aligned}
&= 6x - 3\sum_{n=2}^{\infty} a_{n-1}x^n + 10\sum_{n=2}^{\infty} a_{n-2}x^n - 3\sum_{n=2}^{\infty} 2^n x^n \\
&= 6x - 3x(a_1x + a_2x^2 + \cdots) + 10x^2(a_0 + a_1x + a_2x^2 + \cdots) + 3((2x)^2 + (2x)^3 + (2x)^4 + \cdots) \\
&= 6x - 3xG(x) + 10x^2G(x) + 3\cdot(2x)^2(1 + (2x) + (2x)^2 + \cdots).
\end{aligned}
$$

From the above, we see that $G(x) = 6x + G(x)(-3x + 10x^2) + 12x^2 \cdot \frac{1}{1-2x}$.
Therefore,

$$
\begin{aligned}
G(x)(1 + 3x - 10x^2) &= 6x + \frac{12x^2}{1-2x} \\
G(x) &= \frac{6x}{1+3x-10x^2} + \frac{12x^2}{(1-2x)(1+3x-10x^2)}.
\end{aligned}
$$

Note that $1 + 3x - 10x^2 = (1-2x)(1+5x)$. Therefore

$$G(x) = \frac{6x}{(1-2x)(1+5x)} + \frac{12x^2}{(1-2x)^2(1+5x)}.$$

Using partial fractions, we obtain that

$$\frac{1}{(1-2x)(1+5x)} = \frac{2}{7} \cdot \frac{1}{(1-2x)} + \frac{5}{7} \cdot \frac{1}{1+5x}.$$

From this, we also obtain that

$$\frac{1}{(1-2x)^2(1+5x)} = \frac{2}{7} \cdot \frac{1}{(1-2x)^2} + \frac{10}{49} \cdot \frac{1}{1-2x} + \frac{25}{49} \frac{1}{1+5x}.$$

Therefore

$$G(x) = 6x\left(\frac{2}{7}\frac{1}{1-2x} + \frac{5}{7}\frac{1}{1+5x}\right) + 12x^2\left(\frac{2}{7}\frac{1}{(1-2x)^2} + \frac{10}{49}\frac{1}{1-2x} + \frac{25}{49}\frac{1}{1+5x}\right).$$

The coeff. of x^n in $6x\left(\frac{2}{7}\frac{1}{1-2x} + \frac{5}{7}\frac{1}{1+5x}\right)$ equals $6\times$ coeff. of x^{n-1} in $\frac{2}{7}\frac{1}{1-2x} + \frac{5}{7}\frac{1}{1+5x}$ which equals $6\left(\frac{2}{7}\cdot 2^{n-1} + \frac{5}{7}\cdot(-5)^{n-1}\right)$.

The coeff. of x^n in $12x^2\left(\frac{2}{7}\frac{1}{(1-2x)^2} + \frac{10}{49}\frac{1}{1-2x} + \frac{25}{49}\frac{1}{1+5x}\right)$ equals $12\times$ coeff. of x^{n-2} in $\frac{2}{7}\frac{1}{(1-2x)^2} + \frac{10}{49}\frac{1}{1-2x} + \frac{25}{49}\frac{1}{1+5x}$ which equals $12\left(\frac{2}{7}(n-1)\cdot 2^{n-2} + \frac{10}{49}2^{n-2} + \frac{25}{49}(-5)^{n-2}\right)$.

Putting the above together, we obtain that

$$a_n = 6\left(\frac{2}{7}\cdot 2^{n-1} + \frac{5}{7}\cdot(-5)^{n-1}\right) + 12\left(\frac{2}{7}(n-1)\cdot 2^{n-2} + \frac{10}{49}2^{n-2} + \frac{25}{49}(-5)^{n-2}\right),\ n \geq 2.$$

EXAMPLE 17

Suppose that a valid codeword is an n-digit number in decimal notation containing an even number of 0s. Let a_n denote the number of valid codewords of length n. In Example 4 of Section 8.1 we showed that the sequence $\{a_n\}$ satisfies the recurrence relation

$$a_n = 8\,a_{n-1} + 10^{n-1}$$

And the initial condition $a_1 = 9$. Using generating functions to find an explicit formula for a_n.

Solution:

To make our work with generating functions simpler, we extend this sequence by setting $a_0 = 1$; when we assign this value to a_0 and use the recurrence relation, we have $a_1 = 8a_0 + 10^0 = 8 + 1 = 9$, which is consistent with our original initial condition.

(It also makes sense because there is one code word of length 0—the empty string.) (Cont…)

We multiply both sides of the recurrence relation by x^n to obtain

$$a_n x^n = 8\, a_{n-1}\, x^n + 10^{n-1} x^n.$$

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function of the sequence $a_0, a_1, a_2, \ldots$.

We sum both sides of the last equation starting with $n = 1$, to find that

$$G(x) - 1 = \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} (8a_{n-1}x^n + 10^{n-1}x^n)$$

$$= 8 \sum_{n=1}^{\infty} a_{n-1}x^n + \sum_{n=1}^{\infty} 10^{n-1}x^n$$

$$= 8x \sum_{n=1}^{\infty} a_{n-1}x^{n-1} + x \sum_{n=1}^{\infty} 10^{n-1}x^{n-1}$$ (Cont…)

$$= 8x\sum_{n=0}^{\infty} a_n x^n + x\sum_{n=0}^{\infty} 10^n x^n = 8\,x\,G(x) + \ x/(1\ -\ 10\,x)$$

where we have used Example 5 to evaluate the second summation. Therefore, we have $G(x) - 1 = 8\,x\,G(x) + \frac{x}{(1-10\,x)}$.

Solving for $G(x)$ we get,

$$G(x) = \frac{1-9x}{(1-8x)(1\ -\ 10\,x)}$$

Expanding the right-hand side of this equation into partial fractions (as is done in the integration of rational functions studied in calculus) gives

$$G(x) = \frac{1}{2}\left(\frac{1}{1-8\,x} + \frac{1}{1-10\,x}\right).$$

(Cont…)

Using Example 5 twice (once with $a = 8$ and once with $a = 10$) gives

$$G(x) = \frac{1}{2}\left(\sum_{n=0}^{\infty} 8^n x^n + \sum_{n=0}^{\infty} 10^n x^n\right) = \sum_{n=0}^{\infty} \frac{1}{2}(8^n + 10^n)x^n$$

Consequently, we have shown that

$$a_n = \frac{1}{2}(8^n + 10^n).$$

Proving Identities via Generating Functions

EXAMPLE 18

Use generating functions to show that

$\sum_{k=0}^{n} C(n,k)^2 = C(2n,n)$ whenever n is a positive integer.

Solution:

First note that by the binomial theorem $C(2n,n)$ is the coefficient of x^n in $(1 + x)^{2n}$.

However, we also have

$$(1+x)^{2x} = [(1+x)^n]^2$$

$$= [C(n,0) + C(n,1)x + C(n,2)x^2 + \cdots + C(n,n-1)x^{n-1} + C(n,n)x^n]^2.$$

The coefficient of x^n in this expression is

$$C(n,0)C(n,n) + C(n,1)C(n,n-1) + C(n,2)C(n,n-2) + \cdots$$
$$+ C(n,n)C(n,0).$$

This equals $\sum_{k=0}^{n} C(n,k)^2$, since $C(n,n-1) = C(n,k)$.

Because bothe $C(2n,n)$ and $\sum_{k=0}^{n} C(n,k)^2$ arepresent the coefficeint of x^n in $(1+x)^{2n}$, they must be equal.