Recurrence Relations

Learning Objectives

- Learn about recurrence relations
- Learn the relationship between sequences and recurrence relations
- Explore how to solve recurrence relations by iteration
- Learn about linear homogeneous recurrence relations and how to solve them
- Become familiar with linear nonhomogeneous recurrence relations

Recurrence Relations

Definition:

A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer. A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation. (A recurrence relation is said to *recursively define* a sequence).

OR SIMPLY

A *recurrence relation*

- is an infinite sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$
- in which the formula for the n^{th} term a_n depends on one or more preceding terms,
- with a finite set of start-up values or **initial conditions**

Consider the following two sequences:

$$S_1 : \; 3, 5, 7, 9, \ldots$$
$$S_2 : \; 3, 9, 27, 81, \ldots$$

We can find a formula for the nth term of sequences S_1 and S_2 by observing the pattern of the sequences.

$$S_1 : \; 2 \cdot 1 + 1, 2 \cdot 2 + 1, 2 \cdot 3 + 1, 2 \cdot 4 + 1, \ldots$$
$$S_2 : \; 3^1, 3^2, 3^3, 3^4, \ldots$$

For S_1, $a_n = 2n + 1$ for $n \geq 1$, and for S_2, $a_n = 3^n$ for $n \geq 1$. This type of formula is called an **explicit formula** for the sequence, because using this formula we can directly find any term of the sequence without using other terms of the sequence. For example, $a_3 = 2 \cdot 3 + 1 = 7$.

Let S denote the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots .$$

For this sequence, the explicit formula is not obvious. If we observe closely, however, we find that the pattern of the sequence is such that any term after the second term is the sum of the preceding two terms. Now

$$\begin{aligned}
\text{3rd term} &= 2 = 1 + 1 = \text{1st term} + \text{2nd term} \\
\text{4th term} &= 3 = 1 + 2 = \text{2nd term} + \text{3rd term} \\
\text{5th term} &= 5 = 2 + 3 = \text{3rd term} + \text{4th term} \\
\text{6th term} &= 8 = 3 + 5 = \text{4th term} + \text{5th term} \\
\text{7th term} &= 13 = 5 + 8 = \text{5th term} + \text{6th term}
\end{aligned}$$

Hence, the sequence S can be defined by the equation

$$f_n = f_{n-1} + f_{n-2} \tag{8.1}$$

for all $n \geq 3$ and

$$\begin{aligned} f_1 &= 1, \\ f_2 &= 1. \end{aligned} \tag{8.2}$$

This sequence is called the Fibonacci sequence

EXAMPLES

EXAMPLE

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1, 2, 3, \ldots,$ and suppose that $a_0 = 2.$ What are $a_1, a_2,$ and a_3 ?

Solution: We see from the recurrence relation that $a_1 = a_0 + 3 = 2 + 3 = 5.$ It then follows that

$a_2 = 5 + 3 = 8$ and

$a_3 = 8 + 3 = 11.$

EXAMPLE

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$, and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3?

Solution: We see from the recurrence relation that

$a_2 = a_1 - a_0 = 5 - 3 = 2$ and

$a_3 = a_2 - a_1 = 2 - 5 = -3$.

We can find a_4, a_5, and each successive term in a similar way

Example:

Suppose that $\{a_n\}$ is the sequence of integers defined by $a_n = n!$, the value of the factorial function at the integer n, where $n = 1, 2, 3, \ldots$.

Since $n! = n((n-1)(n-2) \ldots 2 \cdot 1) = n(n-1)! = na_{n-1}$,

we see that the sequence of factorials satisfies the recurrence relation $a_n = n\, a_{n-1}$, together with the initial condition $a_1 = 1$.

Note: A **closed formula is an** explicit formula for the terms of the sequence, which can be obtained by solving a recurrence relation together with the initial conditions.

Fibonacci sequence

- Initial conditions:
  - $f_1 = 1, \; f_2 = 2$
- Recursive formula:
  - $f_{n+1} = f_{n-1} + f_n$ for $n \geq 3$
- First few terms:

n	1	2	3	4	5	6	7	8	9	10	11
f_n	1	2	3	5	8	13	21	34	55	89	144

Rabbits and the Fibonacci Numbers

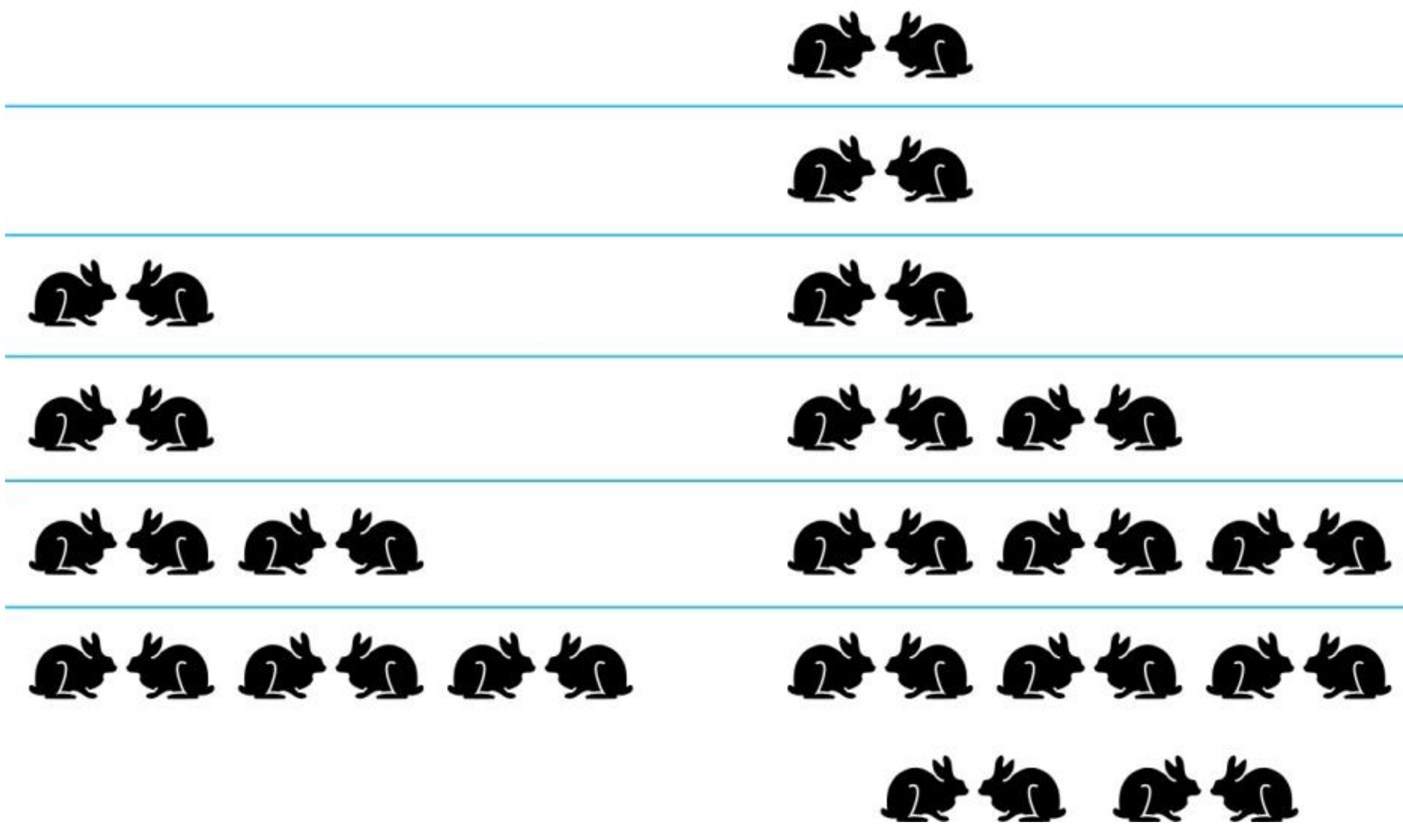
Example: A young pair of rabbits (one of each gender) is placed on an island. A pair of rabbits does not breed until they are 2 months old. After they are 2 months old, each pair of rabbits produces another pair each month. Find a recurrence relation for the number of pairs of rabbits on the island after n months, assuming that rabbits never die.

This is the original problem considered by Leonardo Pisano (Fibonacci) in the thirteenth century.

Rabbits and the Fibonacci Numbers (*cont.*)

Reproducing pairs (at least two months old)

Young pairs (less than two months old)



Month	Reproducing pairs	Young pairs	Total pairs
1	0	1	1
2	0	1	1
3	1	1	2
4	1	2	3
5	2	3	5
6	3	5	8

Modeling the Population Growth of Rabbits on an Island

Rabbits and the Fibonacci Numbers (*cont.*)

Solution: Let f_n be the number of pairs of rabbits after n months.

- There is $f_0 = 1$ pairs of rabbits on the island at the end of the first month.
- We also have $f_1 = 1$ because the pair does not breed during the first month.
- To find the number of pairs on the island after n months, add the number on the island after the previous month, f_{n-1}, and the number of newborn pairs, which equals f_{n-2}, because each newborn pair comes from a pair at least two months old.

Consequently the sequence $\{f_n\}$ satisfies the recurrence relation

$f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$ with the initial conditions $f_1 = 1$ and $f_2 = 1$.

The number of pairs of rabbits on the island after n months is given by the nth Fibonacci number.

The Tower of Hanoi

In the late nineteenth century, the French mathematician Édouard Lucas invented a puzzle consisting of three pegs on a board with disks of different sizes. Initially all of the disks are on the first peg in order of size, with the largest on the bottom.

- **Rules:** You are allowed to move the disks one at a time from one peg to another as long as a larger disk is never placed on a smaller.
- **Goal:** Using allowable moves, end up with all the disks on the second peg in order of size with largest on the bottom.

EXAMPLE Determine whether the sequence $\{a_n\}$, where $a_n = 3n$ for every nonnegative integer n, is a solution of the recurrence relation

$a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$. Answer the same question where $a_n = 2^n$ and where $a_n = 5$.

Solution: Suppose that $a_n = 3n$ for every nonnegative integer n.

Then, for $n \geq 2$,

we see that $2a_{n-1} - a_{n-2} = 2(3(n-1)) - 3(n-2) = 3n = a_n$.

Therefore, $\{a_n\}$, where $a_n = 3n$, is a solution of the recurrence relation.

Suppose that $a_n = 2^n$ for every nonnegative integer n.

Note that $a_0 = 1, a_1 = 2$, and $a_2 = 4$.

Because $2a_1 - a_0 = 2 * 2 - 1 = 3 \neq a_2$, we see that $\{a_n\}$, where $a_n = 2^n$, is not a solution of the recurrence relation.

(Continuation)

Suppose that $a_n = 5$ for every nonnegative integer n.

Then for $n \geq 2$,

we see that $a_n = 2a_{n-1} - a_{n-2} = 2 * 5 - 5 = 5 = an$.

Therefore, $\{a_n\}$, where $a_n = 5$, is a solution of the recurrence relation.

Compound interest

- Given
  - P = initial amount (principal)
  - n = number of years
  - r = annual interest rate
  - A_n = amount of money at the end of n years

At the end of:

- 1 year: $A_1 = P + r\,P = P(1+r)$
- 2 years: $A_2 = A_1 + r\,A_1 = A_1(1+r) = P(1+\mathrm{r})^2$
- 3 years: $A_3 = A_2 + r\,A_2 = A_2(1+r) = P(1+\mathrm{r})^3$

… ……….. ………………..

- Obtain the formula $A_n = P\,(1 + r)^{n}$

EXAMPLE

Compound Interest Suppose that a person deposits Rs.10,000 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 30 years?

Solution: To solve this problem, let P_n denote the amount in the account after n years.

Because the amount in the account after n years equals the amount in the account after $(n-1)$ years plus interest for the n^{th} year, we see that the sequence $\{P_n\}$ satisfies the recurrence relation

$$P_n = P_{n-1} + 0.11\, P_{n-1} = (1.11)\, P_{n-1}$$

(Continue)

The initial condition is $P_0 = 10{,}000$.
We can use an iterative approach to find a formula for P_n.

Note that $P_1 = (1+0.11)\,P_0 = (1.11)P_0$ as $r = 11\% = \frac{11}{100}$

$$P_2 = (1+0.11)\,P_1 = (1.11)^2\,P_0$$

$$P_3 = (1+0.11)\,P_2 = (1.11)^3 P_0$$

.. ….. ….

$$P_n = (1+0.11)\,P_{n-1} = (1.11)^n\,P_0.$$

When we insert the initial condition $P_0 = 10{,}000$, the formula

$P_n = (1.11)^n\, 10{,}000$ is obtained.
Inserting $n = 30$ into the formula $P_n = (1.11)^n\, 10{,}000$

shows that after 30 years the account contains

$$P_{30} = (1.11)^{30} * 10{,}000 = Rs.\, 2{,}28{,}922.97$$

Quiz: Recursions in Programming

```
int hello(int n)
{
        if (n==0)
                return 0;
        else
                printf("Hello World %d\n",n);
                hello(n-1);
}
```

1. What would the program do if I call hello(10)?

2. What if I call hello(-1)?

3. What if the order of printf() and hello() is reversed?

The Tower of Hanoi Puzzle

Example of Hanoi puzzle with three discs are illustrated in the figure (next slide). In the figure, we are find the minimum number of moves require to place all the discs on peg-*3* such that the large diameter one occupies the first place and the remaining discs sits by its dimeter (reduces the diameter sizes)

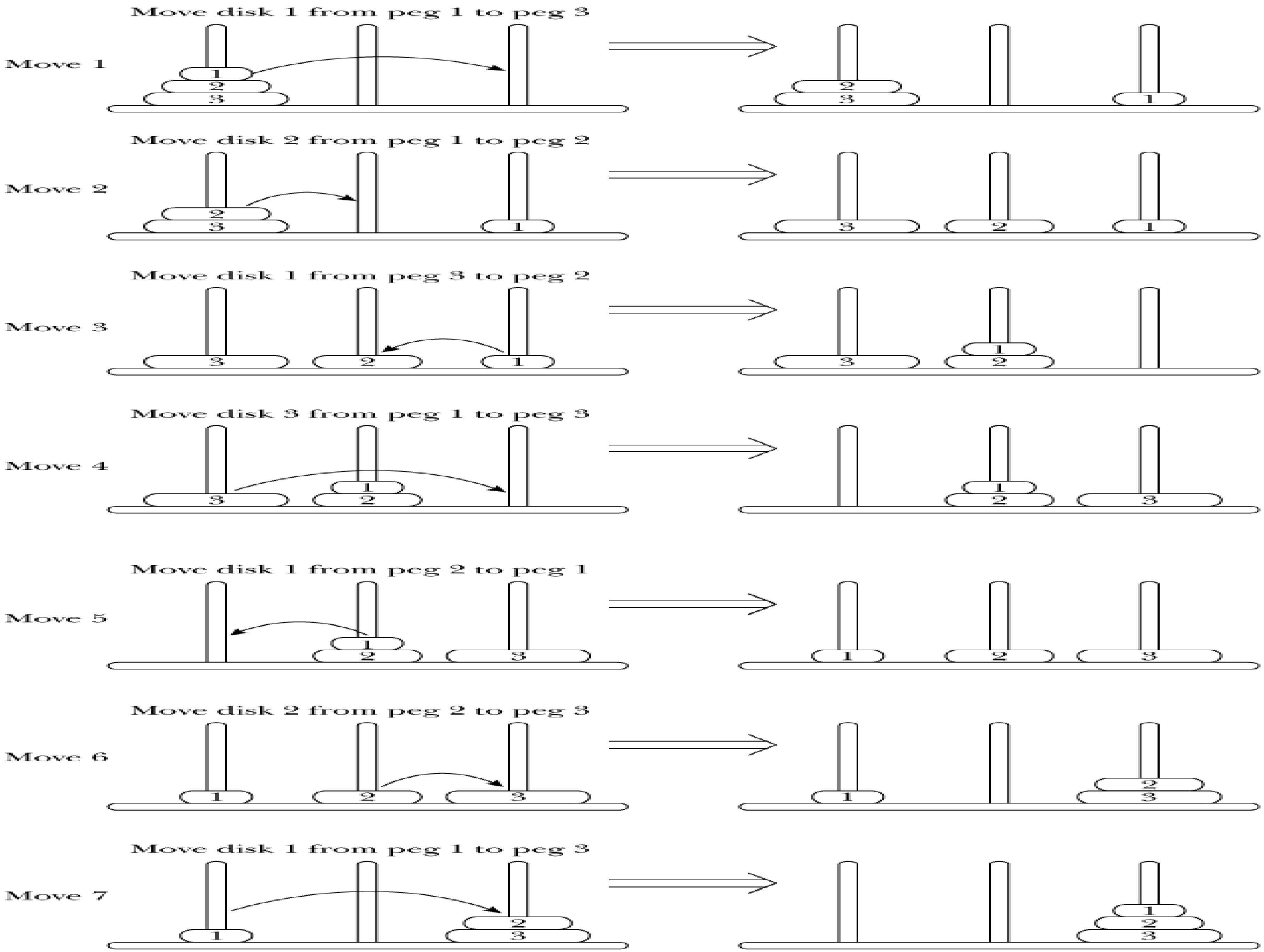


FIGURE 8.2 A solution to the Tower of Hanoi problem with three disks

EXAMPLE 2 (example number as per your textbook)

The Tower of Hanoi Puzzle

A popular puzzle of the late nineteenth century invented by the French mathematician Edouard Lucas, called the Tower of Hanoi, consists of three pegs mounted on a board together with disks of different sizes.

Initially these disks are placed on the first peg in order of size, with the largest on the bottom (as shown in Figure 2).

The rules of the puzzle allow disks to be moved one at a time from one peg to another as long as a disk is never placed on top of a smaller disk.

The goal of the puzzle is to have all the disks on the second peg in order of size, with the largest on the bottom.

Let H_n denote the number of moves needed to solve the Tower of Hanoi puzzle with n disks. Set up a recurrence relation for the sequence $\{H_n\}$.

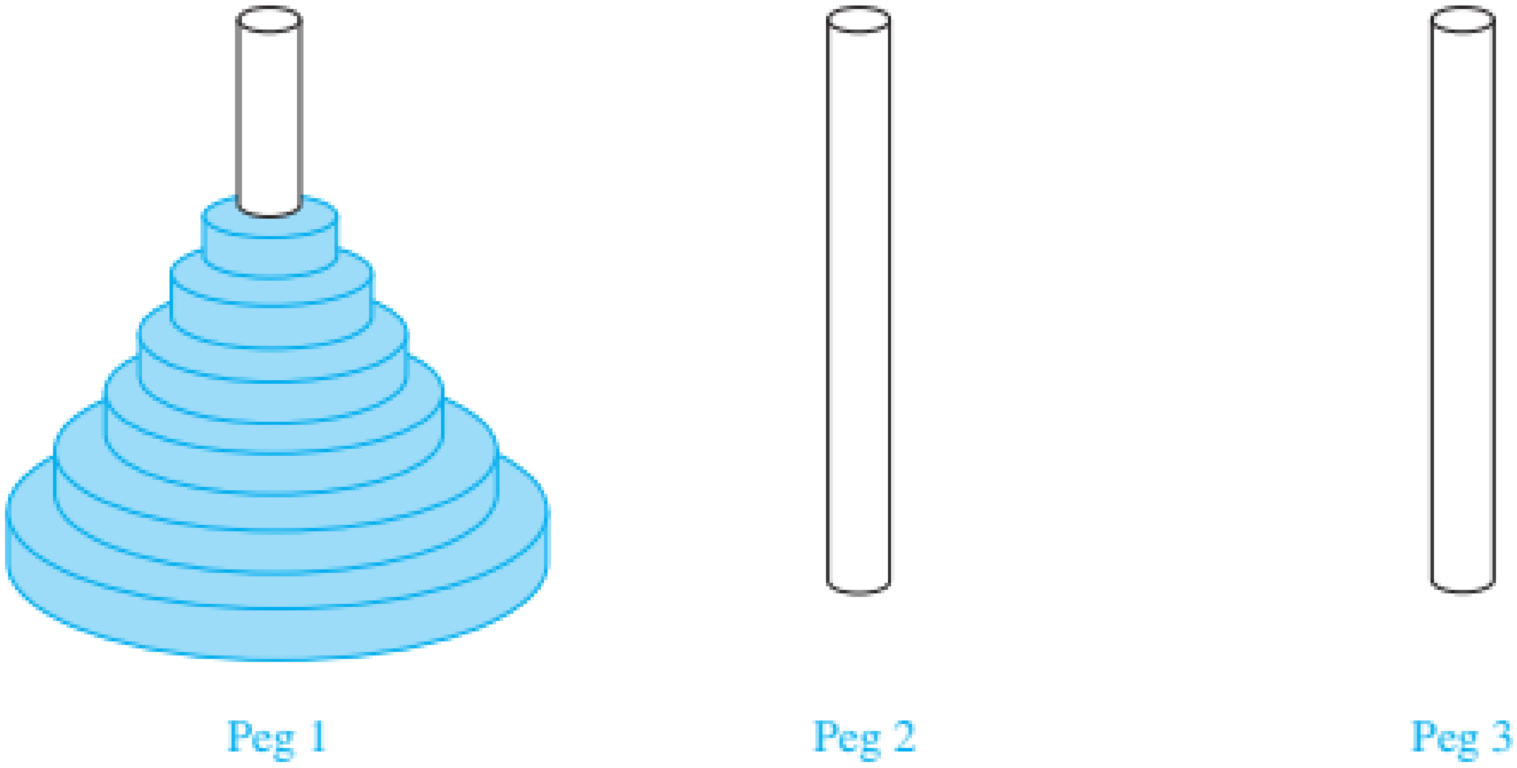


FIGURE 2 **The initial position in the Tower of Hanoi.**

(Continue)

Solution:

Begin with n disks on peg 1. We can transfer the top $n - 1$ disks, following the rules of the puzzle, to peg 3 using H_{n-1} moves (see Figure 3 for an illustration of the pegs and disks at this point). We keep the largest disk fixed during these moves. Then, we use one move to transfer the largest disk to the second peg.

Finally, we transfer the $n - 1$ disks on peg 3 to peg 2 using H_{n-1} moves, placing them on top of the largest disk, which always stays fixed on the bottom of peg 2. This shows that we can solve the Tower of Hanoi puzzle for n disks using $2\, H_{n-1} + 1$ moves.

We now show that we cannot solve the puzzle for n disks using fewer that $2\,H_{n-1} + 1$ moves.

Note that when we move the largest disk, we must have already moved the $(n-1)$ smaller disks onto a peg other than peg 1. Doing so requires at least H_{n-1} moves. Another move is needed to transfer the largest disk.

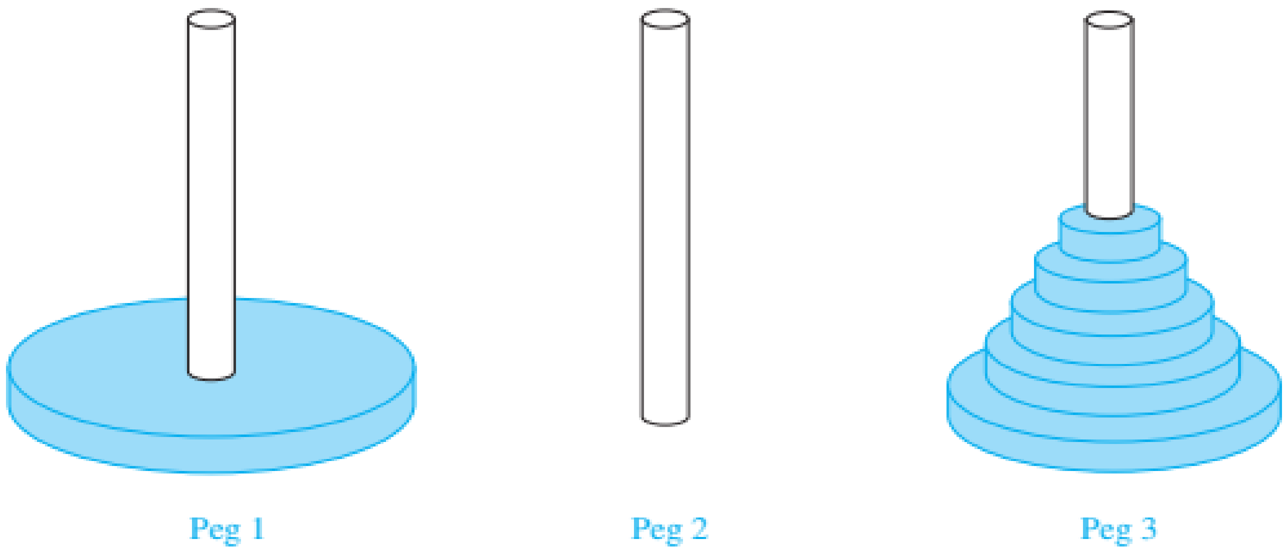


FIGURE 3 **An intermediate position in the Tower of Hanoi.**

- Finally, at least H_{n-1} more moves are needed to put the $n-1$ smallest disks back on top of the largest disk. Adding the number of moves required gives us the desired lower bound.

  We conclude that $H_n = 2\,H_{n-1} + 1$.

  The initial condition is $H_1 = 1$, because one disk can be transferred from peg 1 to peg 2, according to the rules of the puzzle, in one move.

  We can use an iterative approach to solve this recurrence relation. Note that

$$H_n = 2H_{n-1} + 1 \quad = 2(2H_{n-2} + 1) + 1$$

$$= 2^2H_{n-2} + 2 + 1$$

$$= 2^2(2H_{n-3} + 1) + 2 + 1$$

$$= 2^3H_{n-3} + 2^2 + 2 + 1$$

$$\vdots$$

$$= 2^{n-1}H_1 + 2^{n-2} + 2^{n-3} + \ldots . + 2 + 1$$

$$= 2^{n-1} + 2^{n-2} + 2^{n-3} + \ldots . + 2 + 1$$ *because* $H_1 = 1$

$$= 2^n - 1$$ *using the formula for the sum of the terms of a geometric series*

There was a myth created with the puzzle. Monks in a tower in Hanoi are transferring 64 gold disks from one peg to another following the rules of the puzzle. They move one disk each day. When the puzzle is finished, the world will end.

Using this formula for the 64 gold disks of the myth,

$$2^{64} - 1 = 18{,}446{,}744{,}073{,}709{,}551{,}615$$

days are needed to solve the puzzle, which is more than 500 billion years.

Reve's puzzle (proposed in 1907 by Henry Dudeney) is similar but has 4 pegs. There is a well-known unsettled conjecture for the the minimum number of moves needed to solve this puzzle. (*see Exercises* 38-45)

A problem in Economics

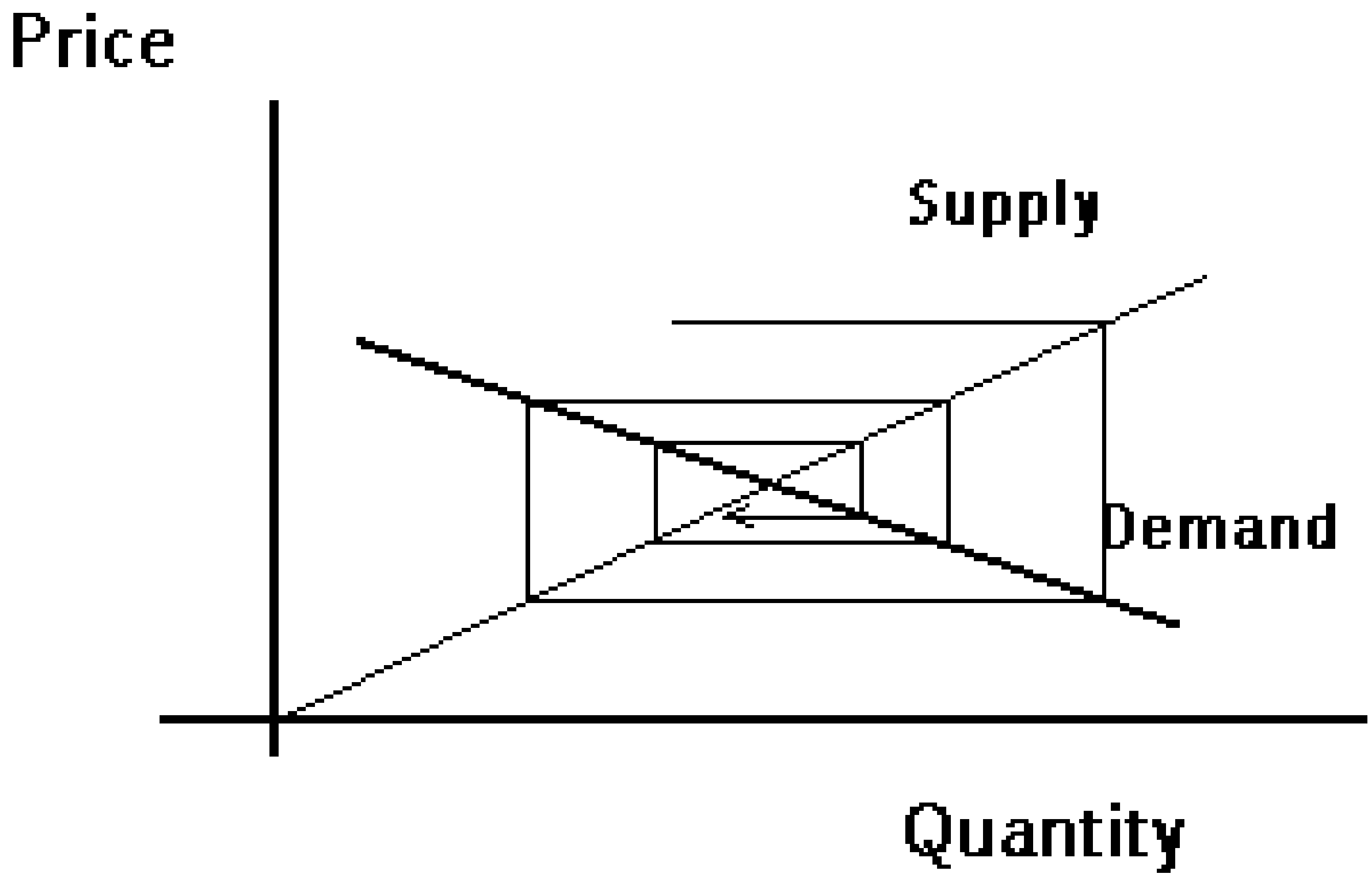
- Demand equation: $p = a - b\,q$
- Supply equation: $p = kq$
- There is a time lag as supply reacts to changes in demand
- Use discrete time intervals as $n = 0, 1, 2, 3, \ldots$
- Given the time delayed equations

$$p_n = a - b\,q_n \text{ (demand)}$$
$$p_n = k\,q_{n+1} \text{ (supply)}$$

- The recurrence relation obtained is

$$p_{n+1} = a - \frac{b}{k}\,p_n$$

Economic cobweb with a stabilizing price



Supply: p = kq   Demand: p = a - bq

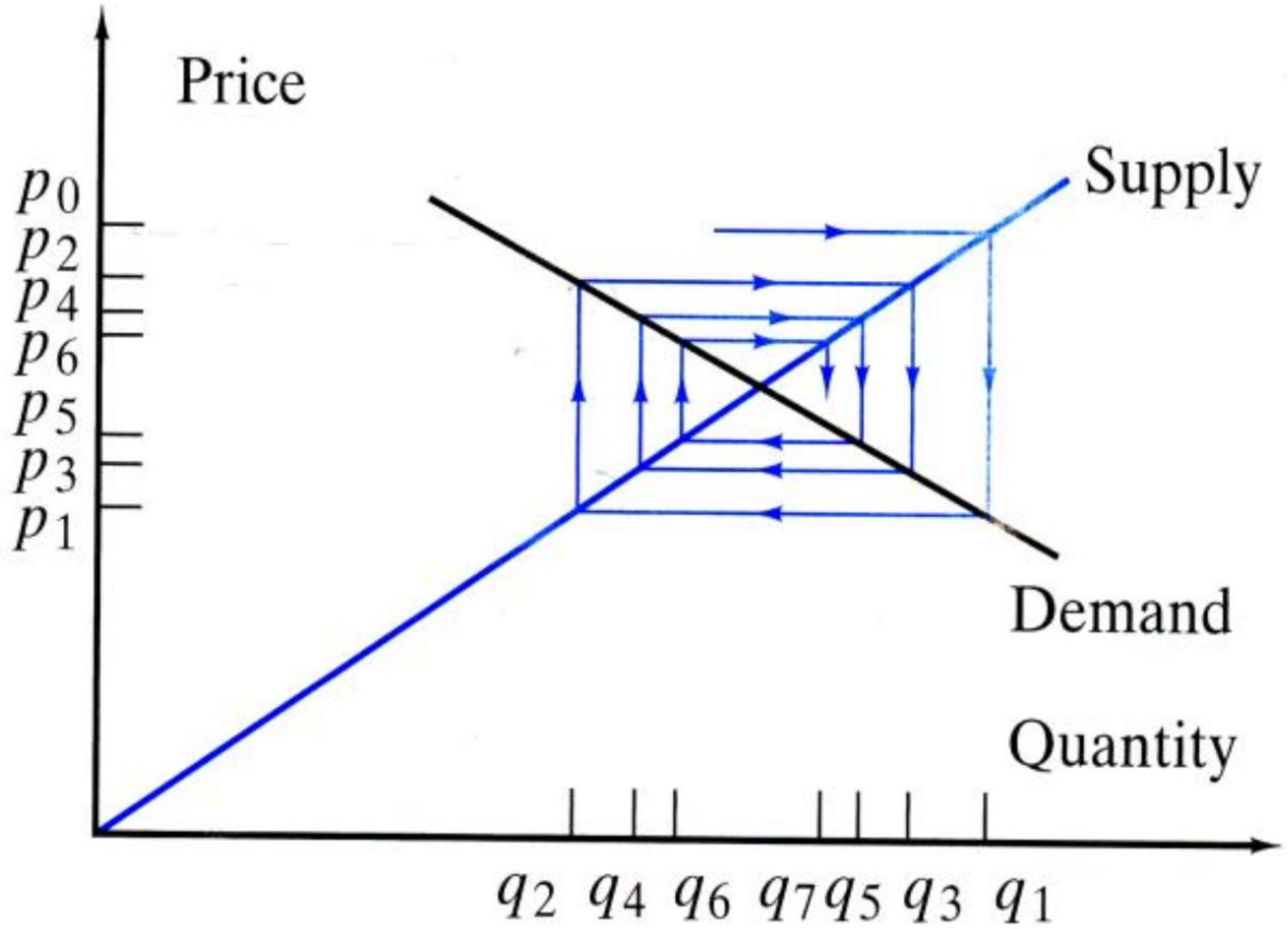


Figure 2 A cobweb with a stabilizing price.

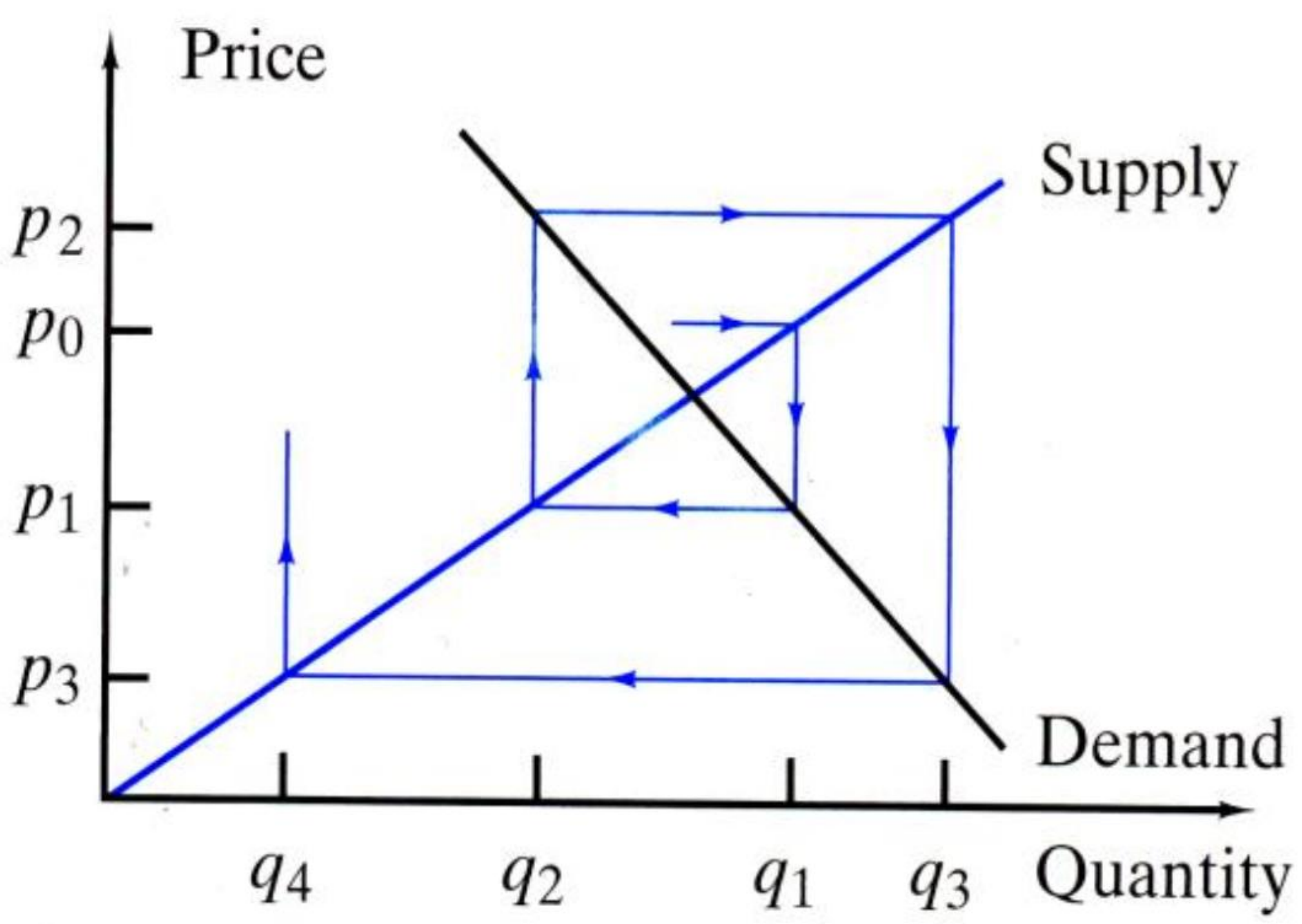


Figure 3 A cobweb with increasing price swings.

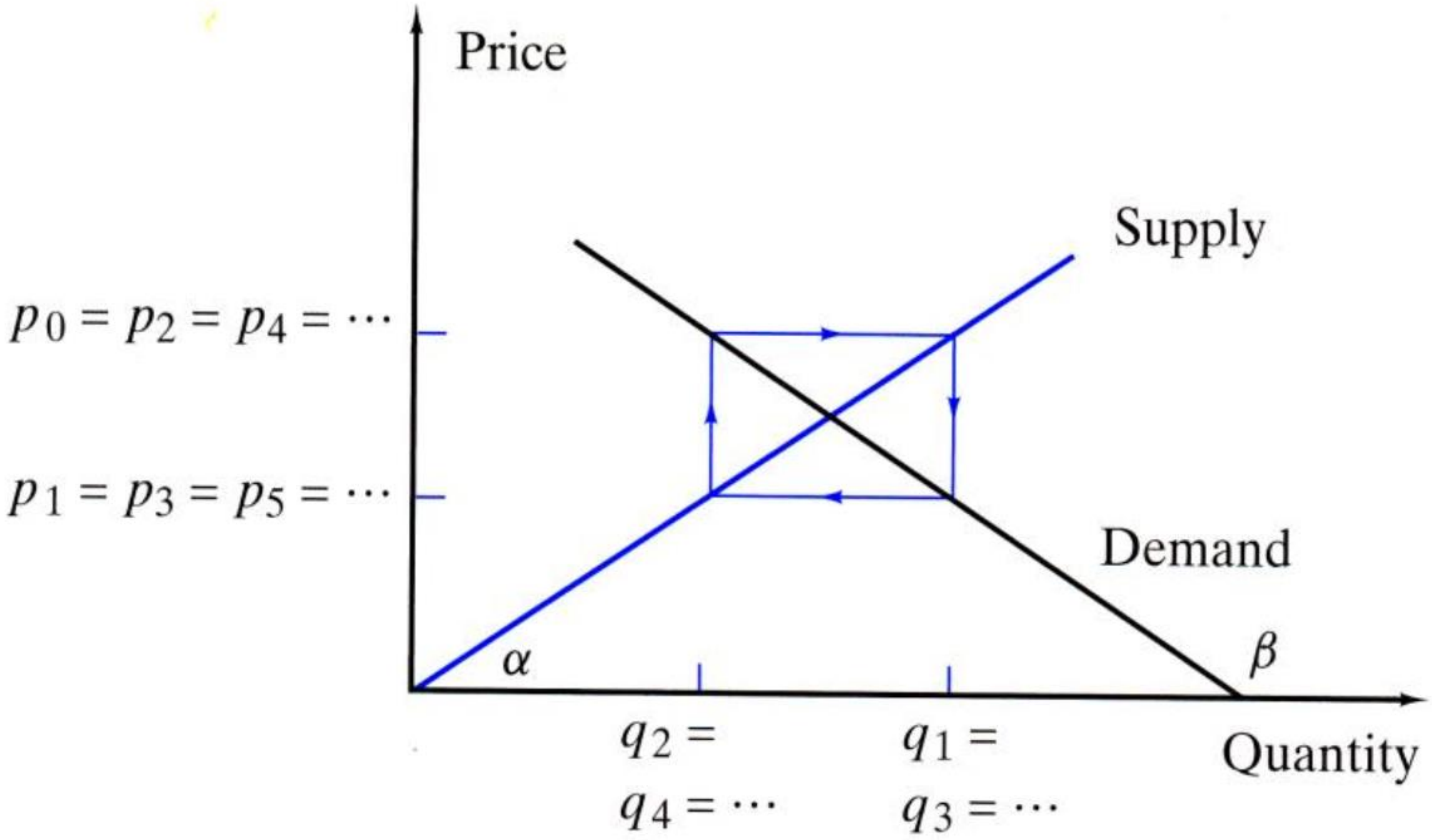


Figure 4 A cobweb with a fluctuating price.

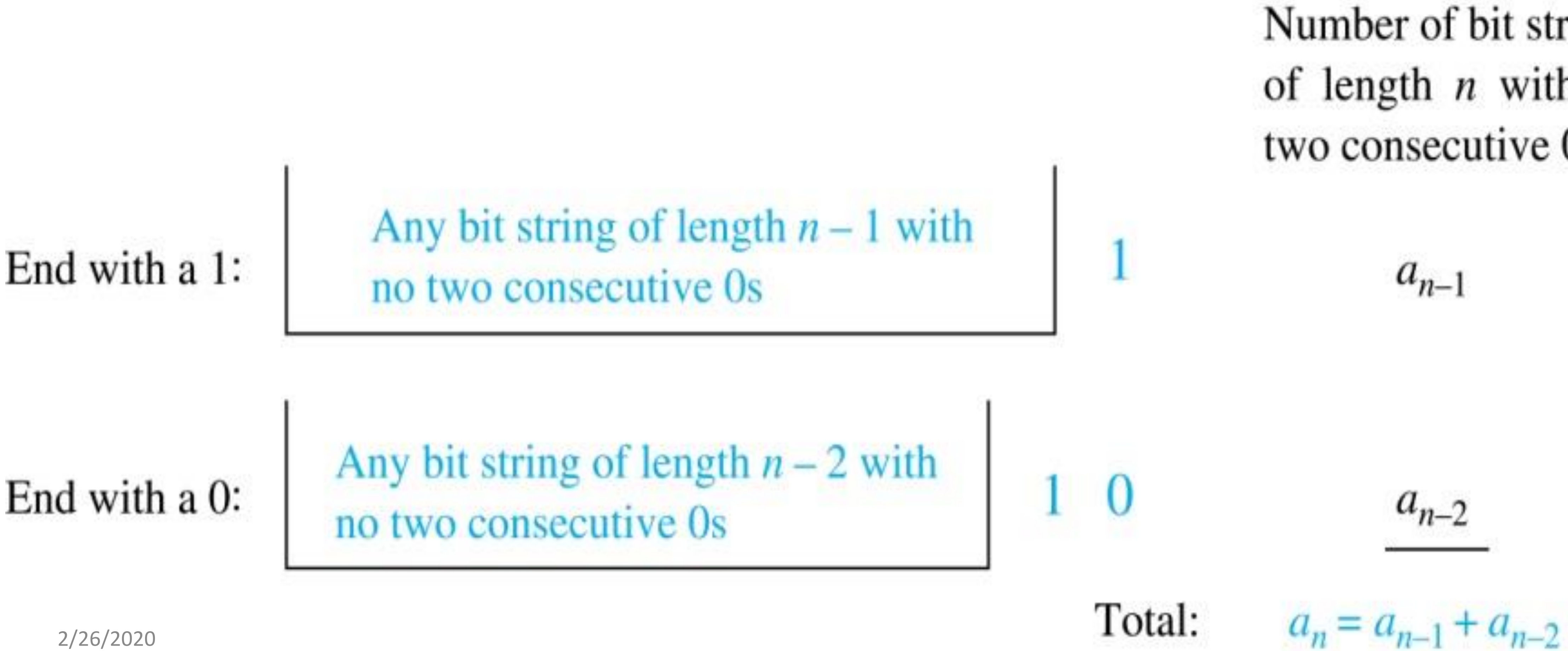
Counting Bit Strings

Example 3: Find a recurrence relation and give initial conditions for the number of bit strings of length n without two consecutive 0s. How many such bit strings are there of length five?

Solution: Let a_n denote the number of bit strings of length n without two consecutive 0s. To obtain a recurrence relation for $\{a_n\}$ note that the number of bit strings of length n that do not have two consecutive 0s is the number of bit strings ending with a 0 plus the number of such bit strings ending with a 1.

- Now assume that $n \geq 3$. The bit strings of length n ending with 1 without two consecutive 0s are the bit strings of length $n-1$ with no two consecutive 0s with a 1 at the end. Hence, there are a_{n-1} such bit strings.

The bit strings of length n ending with 0 without two consecutive 0s are the bit strings of length $n - 2$ with no two consecutive 0s with 10 at the end.
Hence, there are a_{n-2} such bit strings.
We conclude that $a_n = a_{n-1} + a_{n-1}$ for $n \geq 3$.



Bit Strings (continued)

The initial conditions are:

- $a_1 = 2$, since both the bit strings 0 and 1 do not have consecutive 0s.
- $a_2 = 3$, since the bit strings 01, 10, and 11 do not have consecutive 0s, while 00 does.

To obtain a_5, we use the recurrence relation three times to find that:

- $a_3 = a_2 + a_1 = 3 + 2 = 5$
- $a_4 = a_3 + a_2 = 5 + 3 = 8$
- $a_5 = a_4 + a_3 = 8 + 5 = 13$

Note that $\{a_n\}$ satisfies the same recurrence relation as the Fibonacci sequence. Since $a_1 = f_3$ and $a_2 = f_4$, we conclude that $a_n = f_{n+2}$.

EXAMPLE 4

Codeword Enumeration A computer system considers a string of decimal digits a valid codeword if it contains an even number of 0 digits. For instance, 1230407869 is valid, whereas 120987045608 is not valid. Let a_n be the number of valid n-digit codewords. Find a recurrence relation for a_n.

Counting the Ways to Parenthesize a Product

Example 5: Find a recurrence relation for C_n, the number of ways to parenthesize the product of $(n+1)$ numbers,, to specify the order of multiplication. $x_0 \cdot x_1 \cdot x_2 \cdot \cdots \cdot x_n$

For example, $C_3 = 5$, since all the possible ways to parenthesize 4 numbers are

$((x_0 \cdot x_1) \cdot x_2) \cdot x_3$, $(x_0 \cdot (x_1 \cdot x_2)) \cdot x_3$,
$(x_0 \cdot x_1) \cdot (x_2 \cdot x_3)$, $x_0 \cdot ((x_1 \cdot x_2) \cdot x_3)$,
$x_0 \cdot (x_1 \cdot (x_2 \cdot x_3))$

Continue

Counting the Ways to Parenthesize a Product (Cont..)

Solution: Note that however parentheses are inserted in $x_0 \cdot x_1 \cdot x_2 \cdot \cdots \cdot x_n$, one "$\cdot$" operator remains outside all parentheses. This final operator appears between two of the $n+1$ numbers, say x_k and x_{k+1}. Since there are C_k ways to insert parentheses in the product $x_0 \cdot x_1 \cdot x_2 \cdot \cdots \cdot x_k$ and C_{n-k-1} ways to insert parentheses in the product $x_{k+1} \cdot x_{k+2} \cdot \cdots \cdot x_n$, we have

$$C_n = C_0 C_{n-1} + C_1 C_{n-2} + \cdots + C_{n-2} C_1 + C_{n-1} C_0 = \sum_{k=0}^{n-1} C_k C_{n-k-1}$$

The initial conditions are $C_0 = 1$ and $C_1 = 1$.

The sequence $\{C_n\}$ is the sequence of **Catalan Numbers**. This recurrence relation can be solved using the method of generating functions; see Exercise 41 in Section 8.4.