LINEAR ALGEBRA (MAT2005)

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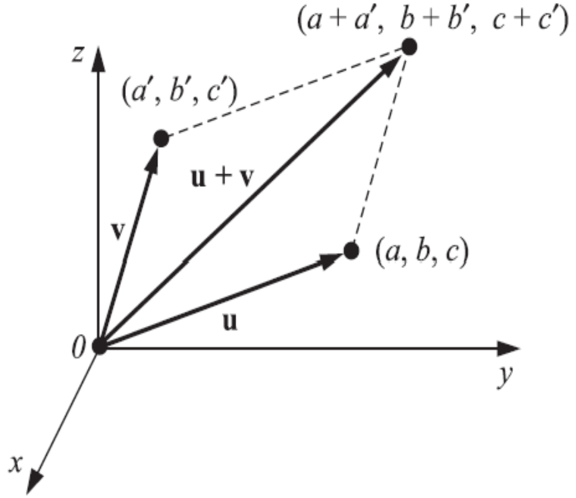
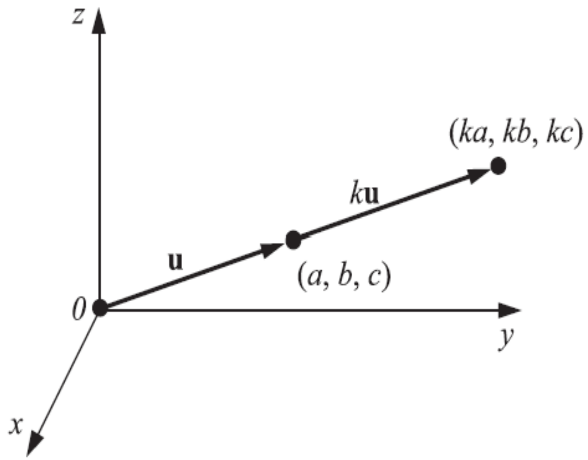
Vectors in Physics

Many physical quantities, such as temperature and speed, possess only "magnitude." These quantities can be represented by real numbers and are called *scalars*. On the other hand, there are also quantities, such as force and velocity, that possess both "magnitude" and "direction." These quantities, which can be represented by arrows having appropriate lengths and directions and emanating from some given reference point O, are called *vectors*.



(a) Vector Addition



(b) Scalar Multiplication

(i) ***Vector Addition:*** The resultant $\mathbf{u} + \mathbf{v}$ of two vectors $\mathbf{u}$ and $\mathbf{v}$ is obtained by the *parallelogram law*; that is, $\mathbf{u} + \mathbf{v}$ is the diagonal of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$. Furthermore, if (a, b, c) and (a', b', c') are the endpoints of the vectors $\mathbf{u}$ and $\mathbf{v}$, then $(a + a',\ b + b',\ c + c')$ is the endpoint of the vector $\mathbf{u} + \mathbf{v}$. These properties are pictured in Fig. 1-1(a).

(ii) ***Scalar Multiplication:*** The product $k\mathbf{u}$ of a vector $\mathbf{u}$ by a real number k is obtained by multiplying the magnitude of $\mathbf{u}$ by k and retaining the same direction if $k > 0$ or the opposite direction if $k < 0$. Also, if (a, b, c) is the endpoint of the vector $\mathbf{u}$, then (ka, kb, kc) is the endpoint of the vector $k\mathbf{u}$. These properties are pictured in Fig. 1-1(b).

1.2 Vectors in $\mathbf{R}^n$

The set of all n-tuples of real numbers, denoted by $\mathbf{R}^n$, is called *n-space*. A particular n-tuple in $\mathbf{R}^n$, say

$$u = (a_1, a_2, \ldots, a_n)$$

is called a *point* or *vector*. The numbers a_i are called the *coordinates*, *components*, *entries*, or *elements* of u. Moreover, when discussing the space $\mathbf{R}^n$, we use the term *scalar* for the elements of $\mathbf{R}$.

1.3 Vector Addition and Scalar Multiplication

Consider two vectors u and v in $\mathbf{R}^n$, say

$$u = (a_1, a_2, \ldots, a_n) \qquad \text{and} \qquad v = (b_1, b_2, \ldots, b_n)$$

Their *sum*, written $u + v$, is the vector obtained by adding corresponding components from u and v. That is,

$$u + v = (a_1 + b_1,\ a_2 + b_2,\ \ldots,\ a_n + b_n)$$

The *scalar product* or, simply, *product*, of the vector u by a real number k, written ku, is the vector obtained by multiplying each component of u by k. That is,

$$ku = k(a_1, a_2, \ldots, a_n) = (ka_1, ka_2, \ldots, ka_n)$$

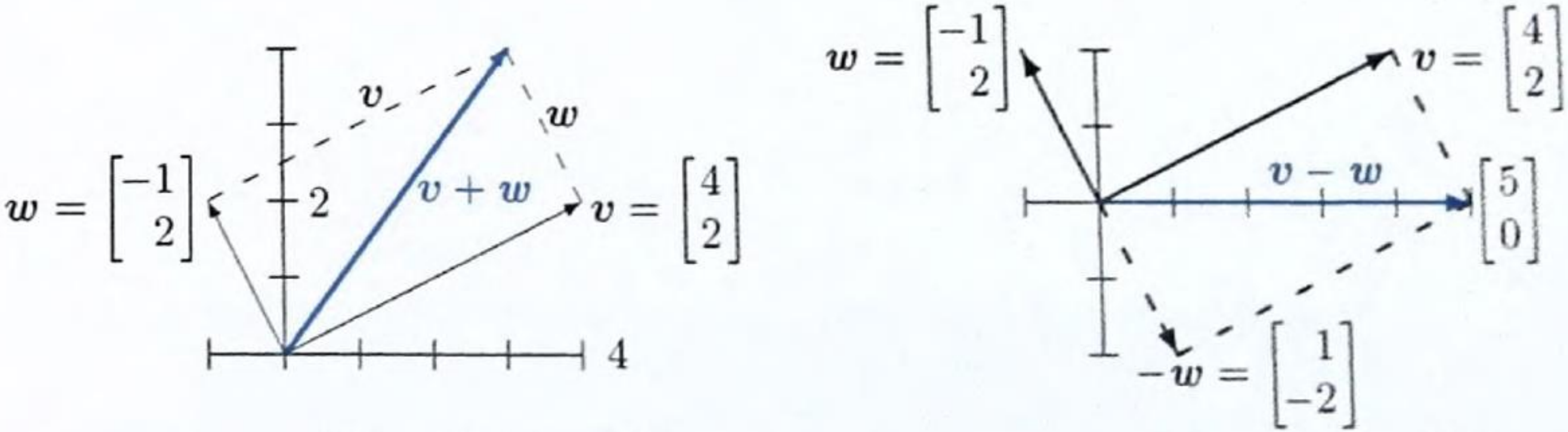
Observe that $u + v$ and ku are also vectors in $\mathbf{R}^n$. The sum of vectors with different numbers of components is not defined.

Linear Combination

Now suppose we are given vectors $u_1, u_2, \ldots, u_m$ in $\mathbf{R}^n$ and scalars $k_1, k_2, \ldots, k_m$ in $\mathbf{R}$. We can multiply the vectors by the corresponding scalars and then add the resultant scalar products to form the vector

$$v = k_1 u_1 + k_2 u_2 + k_3 u_3 + \cdots + k_m u_m$$

Such a vector v is called a *linear combination* of the vectors $u_1, u_2, \ldots, u_m$.



$$v + w = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad v - w = \begin{bmatrix} 4 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Figure 1.1: Vector addition $v + w = (3, 4)$ produces the diagonal of a parallelogram. The reverse of w is $-w$. The linear combination on the right is $v - w = (5, 0)$.

1.4 Dot (Inner) Product

Consider arbitrary vectors u and v in $\mathbf{R}^n$; say,

$$u = (a_1, a_2, \ldots, a_n) \qquad \text{and} \qquad v = (b_1, b_2, \ldots, b_n)$$

The *dot product* or *inner product* or *scalar product* of u and v is denoted and defined by

$$u \cdot v = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

That is, $u \cdot v$ is obtained by multiplying corresponding components and adding the resulting products. The vectors u and v are said to be *orthogonal* (or *perpendicular*) if their dot product is zero—that is, if $u \cdot v = 0$.

EXAMPLE 1.3

(a) Let $u = (1, -2, 3)$, $v = (4, 5, -1)$, $w = (2, 7, 4)$. Then,

$$u \cdot v = 1(4) - 2(5) + 3(-1) = 4 - 10 - 3 = -9$$
$$u \cdot w = 2 - 14 + 12 = 0, \qquad v \cdot w = 8 + 35 - 4 = 39$$

Thus, u and w are orthogonal.

(b) Let $u = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$ and $v = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$. Then $u \cdot v = 6 - 3 + 8 = 11$.

(c) Suppose $u = (1, 2, 3, 4)$ and $v = (6, k, -8, 2)$. Find k so that u and v are orthogonal.

First obtain $u \cdot v = 6 + 2k - 24 + 8 = -10 + 2k$. Then set $u \cdot v = 0$ and solve for k:

$$-10 + 2k = 0 \qquad \text{or} \qquad 2k = 10 \qquad \text{or} \qquad k = 5$$

Norm (Length) of a Vector

The *norm* or *length* of a vector u in $\mathbf{R}^n$, denoted by $\|u\|$, is defined to be the nonnegative square root of $u \cdot u$. In particular, if $u = (a_1, a_2, \ldots, a_n)$, then

$$\|u\| = \sqrt{u \cdot u} = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

That is, $\|u\|$ is the square root of the sum of the squares of the components of u. Thus, $\|u\| \geq 0$, and $\|u\| = 0$ if and only if $u = 0$.

A vector u is called a *unit* vector if $\|u\| = 1$ or, equivalently, if $u \cdot u = 1$. For any nonzero vector v in $\mathbf{R}^n$, the vector

$$\hat{v} = \frac{1}{\|v\|} v = \frac{v}{\|v\|}$$

is the unique unit vector in the same direction as v. The process of finding $\hat{v}$ from v is called *normalizing* v.

EXAMPLE 1.4

(a) Suppose $u = (1, -2, -4, 5, 3)$. To find $\|u\|$, we can first find $\|u\|^2 = u \cdot u$ by squaring each component of u and adding, as follows:

$$\|u\|^2 = 1^2 + (-2)^2 + (-4)^2 + 5^2 + 3^2 = 1 + 4 + 16 + 25 + 9 = 55$$

Then $\|u\| = \sqrt{55}$.

(b) Let $v = (1, -3, 4, 2)$ and $w = (\frac{1}{2}, -\frac{1}{6}, \frac{5}{6}, \frac{1}{6})$. Then

$$\|v\| = \sqrt{1 + 9 + 16 + 4} = \sqrt{30} \qquad \text{and} \qquad \|w\| = \sqrt{\frac{9}{36} + \frac{1}{36} + \frac{25}{36} + \frac{1}{36}} = \sqrt{\frac{36}{36}} = \sqrt{1} = 1$$

Thus w is a unit vector, but v is not a unit vector. However, we can normalize v as follows:

$$\hat{v} = \frac{v}{\|v\|} = \left(\frac{1}{\sqrt{30}}, \frac{-3}{\sqrt{30}}, \frac{4}{\sqrt{30}}, \frac{2}{\sqrt{30}}\right)$$

This is the unique unit vector in the same direction as v.

Distance, Angles, Projections

The *distance* between vectors $u = (a_1, a_2, \ldots, a_n)$ and $v = (b_1, b_2, \ldots, b_n)$ in $\mathbf{R}^n$ is denoted and defined by

$$d(u, v) = \|u - v\| = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + \cdots + (a_n - b_n)^2}$$

One can show that this definition agrees with the usual notion of distance in the Euclidean plane $\mathbf{R}^2$ or space $\mathbf{R}^3$.

The *angle* θ between nonzero vectors u, v in $\mathbf{R}^n$ is defined by

$$\cos\theta = \frac{u \cdot v}{\|u\|\|v\|}$$

EXAMPLE 1.5

(a) Suppose $u = (1, -2, 3)$ and $v = (2, 4, 5)$. Then

$$d(u, v) = \sqrt{(1-2)^2 + (-2-4)^2 + (3-5)^2} = \sqrt{1 + 36 + 4} = \sqrt{41}$$

To find $\cos\theta$, where θ is the angle between u and v, we first find

$$u \cdot v = 2 - 8 + 15 = 9, \qquad \|u\|^2 = 1 + 4 + 9 = 14, \qquad \|v\|^2 = 4 + 16 + 25 = 45$$

Then

$$\cos\theta = \frac{u \cdot v}{\|u\|\|v\|} = \frac{9}{\sqrt{14}\sqrt{45}}$$

Also,

$$\operatorname{proj}(u, v) = \frac{u \cdot v}{\|v\|^2} v = \frac{9}{45}(2, 4, 5) = \frac{1}{5}(2, 4, 5) = \left(\frac{2}{5}, \frac{4}{5}, 1\right)$$

DEFINITION *A unit vector* $\boldsymbol{u}$ *is a vector whose length equals one*. Then $\boldsymbol{u} \cdot \boldsymbol{u} = \mathbf{1}$.

Unit vector $u = v / \|v\|$ **is a unit vector in the same direction as v.**

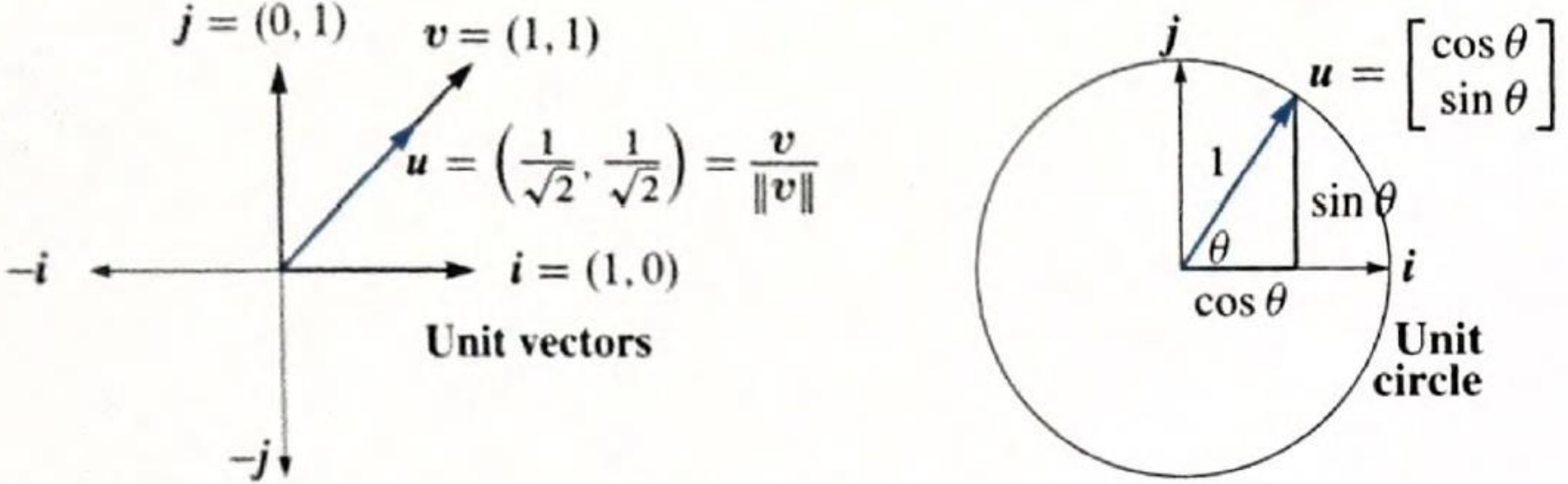


Figure 1.7: The coordinate vectors i and j. The unit vector u at angle 45° (left) divides $v = (1, 1)$ by its length $\|v\| = \sqrt{2}$. The unit vector $u = (\cos\theta, \sin\theta)$ is at angle θ.

COSINE FORMULA If v and w are nonzero vectors then $\dfrac{v \cdot w}{\|v\|\,\|w\|} = \cos\theta$.

SCHWARZ INEQUALITY $|v \cdot w| \leq \|v\|\,\|w\|$

TRIANGLE INEQUALITY $\|v + w\| \leq \|v\| + \|w\|$

Example 5 Find $\cos\theta$ for $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $w = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and check both inequalities.

Solution The dot product is $v \cdot w = 4$. Both v and w have length $\sqrt{5}$. The cosine is $4/5$.

$$\cos\theta = \frac{v \cdot w}{\|v\|\,\|w\|} = \frac{4}{\sqrt{5}\sqrt{5}} = \frac{4}{5}.$$

By the Schwarz inequality, $\boldsymbol{v} \cdot \boldsymbol{w} = 4$ is less than $\|\boldsymbol{v}\| \|\boldsymbol{w}\| = 5$. By the triangle inequality, side $3 = \|\boldsymbol{v} + \boldsymbol{w}\|$ is less than side $1 +$ side 2. For $\boldsymbol{v} + \boldsymbol{w} = (3, 3)$ the three sides are $\sqrt{18} < \sqrt{5} + \sqrt{5}$. Square this triangle inequality to get $18 < 20$.

Example

Write $v = (2, -5, 3)$ as a linear combination of

$$u_1 = (1, -3, 2), u_2 = (2, -4, -1), u_3 = (1, -5, 7).$$

Find the equivalent system of linear equations and then solve. First,

$$\begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix} = x \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + y \begin{bmatrix} 2 \\ -4 \\ -1 \end{bmatrix} + z \begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix} = \begin{bmatrix} x + 2y + z \\ -3x - 4y - 5z \\ 2x - y + 7z \end{bmatrix}$$

Set the corresponding entries equal to each other to obtain

$$\begin{aligned} x + 2y + z &= 2 \\ -3x - 4y - 5z &= -5 \\ 2x - y + 7z &= 3 \end{aligned} \quad \text{or} \quad \begin{aligned} x + 2y + z &= 2 \\ 2y - 2z &= 1 \\ -5y + 5z &= -1 \end{aligned} \quad \text{or} \quad \begin{aligned} x + 2y + z &= 2 \\ 2y - 2z &= 1 \\ 0 &= 3 \end{aligned}$$

The third equation, $0x + 0y + 0z = 3$, indicates that the system has no solution. Thus, v cannot be written as a linear combination of the vectors u_1, u_2, u_3.

Example

Find k so that u and v are orthogonal, where:

(a) $u = (1, k, -3)$ and $v = (2, -5, 4)$,

(b) $u = (2, 3k, -4, 1, 5)$ and $v = (6, -1, 3, 7, 2k)$.

Compute $u \cdot v$, set $u \cdot v$ equal to 0, and then solve for k:

(a) $u \cdot v = 1(2) + k(-5) - 3(4) = -5k - 10$. Then $-5k - 10 = 0$, or $k = -2$.

(b) $u \cdot v = 12 - 3k - 12 + 7 + 10k = 7k + 7$. Then $7k + 7 = 0$, or $k = -1$.

2.2 Matrices

A *matrix A over a field K* or, simply, a *matrix A* (when K is implicit) is a rectangular array of scalars usually presented in the following form:

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

The *rows* of such a matrix A are the m horizontal lists of scalars:

$$(a_{11}, a_{12}, \ldots, a_{1n}), \quad (a_{21}, a_{22}, \ldots, a_{2n}), \quad \ldots, \quad (a_{m1}, a_{m2}, \ldots, a_{mn})$$

and the *columns* of A are the n vertical lists of scalars:

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \ldots \\ a_{m1} \end{bmatrix}, \quad \begin{bmatrix} a_{12} \\ a_{22} \\ \ldots \\ a_{m2} \end{bmatrix}, \quad \ldots, \quad \begin{bmatrix} a_{1n} \\ a_{2n} \\ \ldots \\ a_{mn} \end{bmatrix}$$

Note that the element a_{ij}, called the *ij-entry* or *ij-element*, appears in row i and column j. We frequently denote such a matrix by simply writing $A = [a_{ij}]$.

A matrix with m rows and n columns is called an *m by n* matrix, written $m \times n$. The pair of numbers m and n is called the *size* of the matrix. Two matrices A and B are *equal*, written $A = B$, if they have the same size and if corresponding elements are equal. Thus, the equality of two $m \times n$ matrices is equivalent to a system of mn equalities, one for each corresponding pair of elements.

A matrix with only one row is called a *row matrix* or *row vector*, and a matrix with only one column is called a *column matrix* or *column vector*. A matrix whose entries are all zero is called a *zero matrix* and will usually be denoted by 0.

2.3 Matrix Addition and Scalar Multiplication

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two matrices with the same size, say $m \times n$ matrices. The *sum* of A and B, written $A + B$, is the matrix obtained by adding corresponding elements from A and B. That is,

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{bmatrix}$$

The *product* of the matrix A by a scalar k, written $k \cdot A$ or simply kA, is the matrix obtained by multiplying each element of A by k. That is,

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & \dots & ka_{2n} \\ \dots & \dots & \dots & \dots \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{bmatrix}$$

THEOREM 2.1: Consider any matrices A, B, C (with the same size) and any scalars k and k'. Then

(i)	$(A+B)+C = A+(B+C)$,	(v)	$k(A+B) = kA + kB$,
(ii)	$A+0 = 0+A = A$,	(vi)	$(k+k')A = kA + k'A$,
(iii)	$A+(-A) = (-A)+A = 0$,	(vii)	$(kk')A = k(k'A)$,
(iv)	$A+B = B+A$,	(viii)	$1 \cdot A = A$.

2.5 Matrix Multiplication

The product of matrices A and B, written AB, is somewhat complicated. For this reason, we first begin with a special case.

The product AB of a row matrix $A = [a_i]$ and a column matrix $B = [b_i]$ with the same number of elements is defined to be the scalar (or 1×1 matrix) obtained by multiplying corresponding entries and adding; that is,

$$AB = [a_1, a_2, \ldots, a_n] \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{k=1}^{n} a_k b_k$$

We emphasize that AB is a scalar (or a 1×1 matrix). The product AB is not defined when A and B have different numbers of elements.

THEOREM 2.2: Let A, B, C be matrices. Then, whenever the products and sums are defined,

(i) $(AB)C = A(BC)$ (associative law),

(ii) $A(B + C) = AB + AC$ (left distributive law),

(iii) $(B + C)A = BA + CA$ (right distributive law),

(iv) $k(AB) = (kA)B = A(kB)$, where k is a scalar.

We note that $0A = 0$ and $B0 = 0$, where 0 is the zero matrix.

2.6 Transpose of a Matrix

The *transpose* of a matrix A, written A^T, is the matrix obtained by writing the columns of A, in order, as rows. For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \qquad \text{and} \qquad [1, -3, -5]^T = \begin{bmatrix} 1 \\ -3 \\ -5 \end{bmatrix}$$

In other words, if $A = [a_{ij}]$ is an $m \times n$ matrix, then $A^T = [b_{ij}]$ is the $n \times m$ matrix where $b_{ij} = a_{ji}$.

Observe that the tranpose of a row vector is a column vector. Similarly, the transpose of a column vector is a row vector.

The next theorem lists basic properties of the transpose operation.

THEOREM 2.3: Let A and B be matrices and let k be a scalar. Then, whenever the sum and product are defined,

(i) $(A + B)^T = A^T + B^T$, (iii) $(kA)^T = kA^T$,

(ii) $(A^T)^T = A$, (iv) $(AB)^T = B^T A^T$.