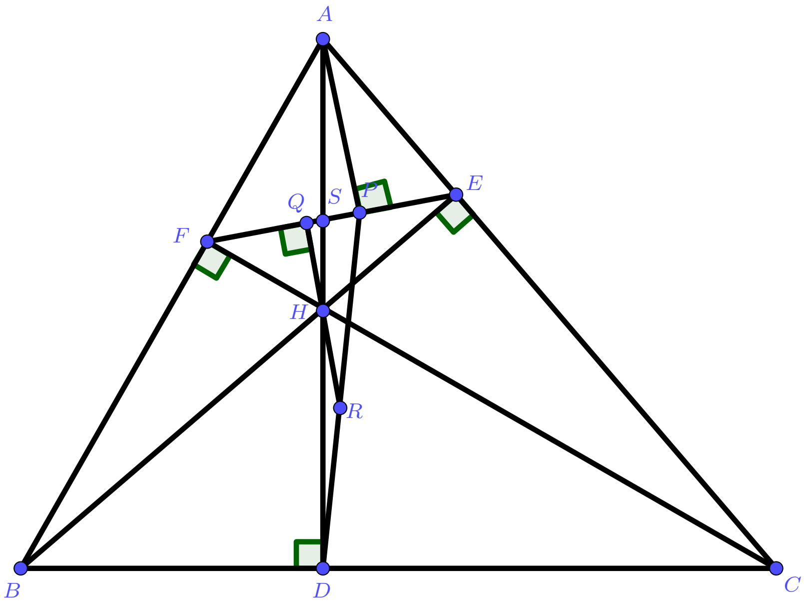
**Presentation 1 from Seminar 2:**

In a non-isosceles acute angled triangle $\triangle ABC$, consider the heights $AD$, $BE$, $CF$ and let $H$ be the orthocenter. $AD$ and $EF$ intersect at $S$. Draw $AP \perp EF$ and $HQ \perp EF$ with $P, Q \in EF$. If the lines $DP$ and $QH$ intersect at $R$, show that $HQ = HR$.




$$\left.\begin{array}{c} HQ \perp EF \\ AP \perp EF \end{array}\right| \implies \left.\begin{array}{c} HQ \parallel AP \\ R \in HQ \end{array}\right| \implies HR \parallel AP.$$

From $HQ \parallel AP$, by the Fundamental Theorem of Similar Triangles, we have:

$$\triangle HQS \sim \triangle APS \implies \frac{HQ}{AP} = \frac{HS}{AS} = \frac{SQ}{SP}. \tag{1}$$

Similarly, from $HR \parallel AP$, by the Fundamental Theorem of Similar Triangles, we have:

$$\triangle DHR \sim \triangle DAP \implies \frac{HR}{AP} = \frac{HD}{AD} = \frac{DR}{DP}. \tag{2}$$

From (1) and (2), we can observe that:

If $HQ = HR \iff \frac{HQ}{AP} = \frac{HR}{AP} \iff \frac{HS}{AS} = \frac{HD}{AD}$. (3)

We look for a triangle in which we can find one of those ratios. The triangle we can work with is $\triangle AHC$.

We apply Menelaus' theorem in $\triangle AHC$ for the transversal $F - S - E$. We get:

$$\frac{AE}{EC} \cdot \frac{FC}{FH} \cdot \frac{HS}{AS} = 1. \tag{4}$$

From (3) and (4), we get:

$$\frac{AE}{EC} \cdot \frac{FC}{FH} \cdot \frac{HD}{AD} = 1. \tag{5}$$

Now, if we show that (5) is true, then $HQ = HR$.

We can express each ratio in terms of areas. We get:
$\frac{HD}{AD} = \frac{HD \cdot BC \cdot \frac{1}{2}}{AD \cdot BC \cdot \frac{1}{2}} = \frac{[\triangle HBC]}{[\triangle ABC]}$,

$\frac{FC}{FH} = \frac{BA \cdot FC \cdot \frac{1}{2}}{BA \cdot FH \cdot \frac{1}{2}} = \frac{[\triangle ABC]}{[\triangle AHB]}$,

$\frac{AE}{EC} = \frac{AE \cdot BH \cdot \frac{1}{2}}{CE \cdot BH \cdot \frac{1}{2}} = \frac{[\triangle AHB]}{[\triangle HBC]}$.
Now, if we substitute this relations in (5), we get that $\frac{AE}{EC} \cdot \frac{FC}{FH} \cdot \frac{HD}{AD} = 1$.
Hence $HQ = HR$.