COURSE 8

Vector spaces, subspaces

Let $(K, +, \cdot)$ be a field. Throughout this course this condition on K will always be valid.

Definition 1. Let K be a field. A **vector space over** K (or a K**-vector space**) is an Abelian group $(V, +)$ together with an external operation

$$\cdot : K \times V \to V\,, \quad (k, v) \mapsto k \cdot v\,,$$

satisfying the following axioms: for any $k, k_1, k_2 \in K$ and any $v, v_1, v_2 \in V$,
(L_1) $k \cdot (v_1 + v_2) = k \cdot v_1 + k \cdot v_2$;
(L_2) $(k_1 + k_2) \cdot v = k_1 \cdot v + k_2 \cdot v$;
(L_3) $(k_1 \cdot k_2) \cdot v = k_1 \cdot (k_2 \cdot v)$;
(L_4) $1 \cdot v = v$.
In this context, the elements of K are called **scalars**, the elements of V are called **vectors** and the external operation is called **scalar multiplication**. Sometimes a vector space is also called **linear space**.

We denote the fact that V is a vector space over K either by ${}_KV$ or by $(V, K, +, \cdot)$, since for a given field K, the addition on V and the external operation are the operations that determine the vector space structure of V.

Remark 2. In the definition of a vector space appear four operations, two denoted by the same symbol $+$ and two denoted by the same symbol $\cdot$. Of course, most of the time they are not the same, but we denote them identically for the sake of simplicity of writing. The nature of the elements involved when using these symbols tells us which is the operation. More precisely, if $+$ appears between two vectors, then it is the addition from V, if it appears between two scalars, it is the addition from K; if $\cdot$ appears between a scalar and a vector, then it is the scalar multiplication, otherwise, it appears between to scalars, hence it is the multiplication from K.

Examples 3. (a) If $V = \{0\}$ is a single element set, then we know that there is a unique Abelian group structure on V, defined by $0 + 0 = 0$. There is also a unique scalar multiplication, namely

$$k \cdot 0 = 0, \ \forall k \in K.$$

Thus, V is a vector space, called the **zero (null) vector space** and denoted by $\{0\}$.
(b) Let $n \in \mathbb{N}^*$ and

$$K^n = \{(x_1, \dots, x_n) \mid x_i \in K, \ i = \{1, \dots, n\}\}.$$

Define for any $(x_1, \dots, x_n), (y_1, \dots, y_n) \in K^n$ and for any $k \in K$,

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)\,,$$

$$k \cdot (x_1, \dots, x_n) = (kx_1, \dots, kx_n)\,.$$

Then K^n is a K-vector space.

For $n = 1$, we get that ${}_KK$ is a vector space (in particular, ${}_\mathbb{Q}\mathbb{Q}$, ${}_\mathbb{R}\mathbb{R}$ and ${}_\mathbb{C}\mathbb{C}$ are vector spaces).

(c) Let A be a subfield of the field K. Then K is a vector space over A, where the addition and the scalar multiplication are just the addition and the multiplication of elements in the field K.

In particular, ${}_{\mathbb{Q}}\mathbb{R}$, ${}_{\mathbb{Q}}\mathbb{C}$ and ${}_{\mathbb{R}}\mathbb{C}$ are vector spaces.

(d) Let V_2 be the set of all vectors (in the classical sense) in the plane with a fixed origin O. Then V_2 is a vector space over $\mathbb{R}$ (or a *real vector space*), where the addition is the usual addition of two vectors by the parallelogram rule and the external operation is the usual scalar multiplication of vectors by real scalars.

If we consider two coordinate axes Ox and Oy in the plane, each vector in V_2 is perfectly determined by the coordinates of its ending point. Therefore, the addition of vectors and the scalar multiplication of vectors by real numbers become:

$$(x, y) + (x', y') = (x + x', y + y'),$$

$$k \cdot (x, y) = (k \cdot x, k \cdot y).$$

Thus, one can identify the vector space $(V_2, \mathbb{R}, +, \cdot)$ with the vector space $(\mathbb{R}^2, \mathbb{R}, +, \cdot)$.

Similarly, one can consider the real vector space V_3 of all vectors in the space with a fixed origin and this vector space can be seen as the real vector space $\mathbb{R}^3$.

(e) Let $m, n \in \mathbb{N}^*$. The Abelian group $(M_{m,n}(K), +)$ of the $m \times n$ matrices over K is a K-vector space with the scalar multiplication

$$\alpha(a_{ij}) = (\alpha a_{ij}) \ (\alpha \in K,\ (a_{ij}) \in M_{m,n}(K)).$$

Let us notice that for $n \times n$ square matrices, besides the K-vector space structure, $M_n(K)$ also has a ring structure. Moreover, there is a certain connection between the scalar multiplication and the matrix multiplication given by

$$\alpha(A\,B) = (\alpha A)B = A(\alpha B),\ \forall \alpha \in K,\ \forall A, B \in M_n(K).$$

(f) $(K[X], K, +, \cdot)$ is a vector space, where the addition is the usual addition of polynomials and the scalar multiplication is defined as follows: if $f = a_0 + a_1 X + \cdots + a_n X^n \in K[X]$,

$$kf = (ka_0) + (ka_1)X + \cdots + (ka_n)X^n,\ \forall k \in K.$$

As in the previous example, $K[X]$ has also a ring structure which is connected to the vector space structure by the condition

$$\alpha(fg) = (\alpha f)g = f(\alpha g),\ \forall \alpha \in K,\ \forall f, g \in K[X].$$

(g) Let A be a non-empty set. Denote

$$K^A = \{f \mid f : A \to K\}.$$

Then $(K^A, K, +, \cdot)$ is a vector space, where the addition and the scalar multiplication are defined as follows: for any $f, g \in K^A$, for any $k \in K$, we have $f + g \in K^A$, $kf \in K^A$, where

$$(f+g)(x) = f(x) + g(x)\,,\ (kf)(x) = kf(x)\,, \forall x \in A\,.$$

As a particular case, we obtain the vector space $(\mathbb{R}^{\mathbb{R}}, \mathbb{R}, +, \cdot)$ of real functions of a real variable.
(i) If V_1 and V_2 are K-vector spaces, one defines on the Cartesian product $V_1 \times V_2$ the following operations: for any $(x_1, x_2), (x_1', x_2') \in V_1 \times V_2$ and $\alpha \in K$

$$(x_1, x_2) + (x_1', x_2') = (x_1 + x_1', x_2 + x_2'),$$

$$\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2).$$

This way $V_1 \times V_2$ becomes a K- vector space, called the **direct product** of ${}_K V_1$ and ${}_K V_2$.

Next we give some computation rules in a vector space. Notice that we denote by 0 both the zero scalar and the zero vector.

Theorem 4. Let V be a vector space over K. Then for any $k, k', k_1, \ldots, k_n \in K$ and for any $v, v', v_1, \ldots, v_n \in V$ we have:

(i) $k \cdot 0 = 0 \cdot v = 0$;
(ii) $k(-v) = (-k)v = -kv,\ (-k)(-v) = kv$;
(iii) $k(v - v') = kv - kv',\ (k - k')v = kv - k'v$;
(iv) $(k_1 + \cdots + k_n)v = k_1 v + \cdots + k_n v,\ k(v_1 + \cdots + v_n) = kv_1 + \cdots + kv_n$.

Proof. □

Theorem 5. Let V be a vector space over K and let $k \in K$ and $v \in V$. Then

$$kv = 0 \Leftrightarrow k = 0 \text{ or } v = 0\,.$$

Proof. □

Definition 6. Let V be a vector space over K and let $S \subseteq V$. Then S is a **subspace** of V if:
(1) S is closed with respect to the addition of V and to the scalar multiplication, that is,

$$\forall x, y \in S\,,\quad x + y \in S\,,$$

$$\forall k \in K\,,\ \forall x \in S\,,\ kx \in S\,.$$

(2) S is a vector space over K with respect to the induced operations of addition and scalar multiplication.

We denote by $S \leq_K V$ the fact that S is a subspace of the vector space V over K.

Remark 7. If $S \leq_K V$ then S contains the zero vector of V, i.e. $0 \in S$.

We have the following **characterization theorem for subspaces**.

Theorem 8. Let V be a vector space over K and let $S \subseteq V$. The following conditions are equivalent:
1) $S \leq_K V$.
2) The following conditions hold for S:
  α) $S \neq \emptyset$;
  β) $\forall x, y \in S\,,\ x + y \in S$;
  γ) $\forall k \in K\,,\ \forall x \in S\,,\ kx \in S\,.$
3) The following conditions hold for S:
  α) $S \neq \emptyset$;
  δ) $\forall k_1, k_2 \in K\,,\ \forall x, y \in S\,,\ k_1 x + k_2 y \in S\,.$

Proof.

$\square$

Remark 9. (1) One can replace α) in the previous theorem with $0 \in S$.
(2) If $S \leq_K V$, $k_1, \dots, k_n \in K$ and $x_1, \dots, x_n \in S$ then $k_1 x_1 + \dots + k_n x_n \in S$.

Examples 10. (a) Every non-zero vector space V over K has two subspaces, namely $\{0\}$ and V. They are called the **trivial subspaces**.
(b) Let

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}\,,$$

$$T = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}\,.$$

Then S and T are subspaces of the real vector space $\mathbb{R}^3$.

(c) Let $n \in \mathbb{N}$ and let

$$K_n[X] = \{f \in K[X] \mid \deg f \leq n\}.$$

Then $K_n[X]$ is a subspace of the polynomial vector space $K[X]$ over K.
d) Let $I \subseteq \mathbb{R}$ be an interval. The set $\mathbb{R}^I = \{f \mid f : I \to \mathbb{R}\}$ is a $\mathbb{R}$-vector space with respect to the following operations

$$(f+g)(x) = f(x) + g(x),\ (\alpha f)(x) = \alpha f(x)$$

with $f, g \in \mathbb{R}^I$ and $\alpha \in \mathbb{R}$. The subsets

$$C(I, \mathbb{R}) = \{f \in \mathbb{R}^I \mid f \text{ continuous on } I\},\ D(I, \mathbb{R}) = \{f \in \mathbb{R}^I \mid f \text{ derivable on } I\}$$

are subspaces of $\mathbb{R}^I$ since they are nonempty and

$$\alpha, \beta \in \mathbb{R},\ f, g \in C(I, \mathbb{R}) \Rightarrow \alpha f + \beta g \in C(I, \mathbb{R});$$

$$\alpha, \beta \in \mathbb{R},\ f, g \in D(I, \mathbb{R}) \Rightarrow \alpha f + \beta g \in D(I, \mathbb{R}).$$