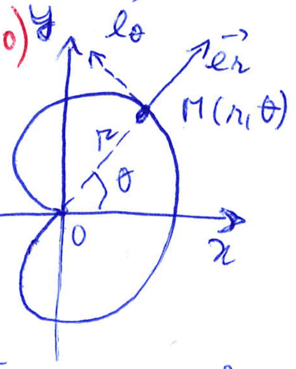
(1) A particle M moves on the cardioid $r = b(1+\cos\theta)$, $b>0$. It is known that its acceleration's direction intersect all the time the point $O(0,0)$. Find the speed, acceleration, normal and tangential acceleration and the radius of the curvature of the trajectory as functions of r.

Frenet:
$\vec{v} = v\cdot\vec{\tau} = \left(\frac{ds}{dt}, 0, 0\right)$

$\frac{d\vec{\tau}}{ds} = \frac{\vec{n}}{R}$

$\vec{a} = (\underbrace{\dot{v}}_{a_\tau}, \underbrace{\frac{v^2}{R}}_{a_n}, 0)$



Know: $\vec{a} \parallel \vec{e_r} \Rightarrow \vec{a} \perp \vec{e_\theta}$

$\Downarrow$

$a_\theta = Pr_{\vec{e}_\theta}\vec{a} = \vec{a}\cdot\vec{e}_\theta = 0$

But $a_\theta = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0 \Rightarrow r^2\dot{\theta} = c \in \mathbb{R}$

$\Rightarrow \boxed{\dot{\theta} = \frac{c}{r^2}}$ (1).

Speed: $v^2 = v_r^2 + v_\theta^2 = \dot{r}^2 + r^2\dot{\theta}^2 \underset{(1)}{=} \dot{r}^2 + r^2\cdot\frac{c^2}{r^4}$

$v_r = \dot{r}$
$v_\theta = r\dot{\theta}$

$\Rightarrow \boxed{v^2 = \dot{r}^2 + \frac{c^2}{r^2}}$ (2).

On the other hand $r = b(1+\cos\theta) \Rightarrow \dot{r} = -b\sin\theta\cdot\dot{\theta} \overset{(1)}{\Rightarrow}$

$\dot{r} = -\frac{bc}{r^2}\sin\theta$

$\dot{r}^2 = \frac{b^2c^2}{r^4}\sin^2\theta = \frac{b^2c^2}{r^4}(1-\cos^2\theta) = \frac{b^2c^2}{r^4}\left[1-\left(\frac{r-b}{b}\right)^2\right]$ (3).

Thus, from (2) and (3) we obtain:

$v^2 = \frac{b^2c^2}{r^4}\left[\cancel{1} - \frac{r^2}{b^2} + \frac{2r}{b} - \cancel{1}\right] + \frac{c^2}{r^2} = -\cancel{\frac{c^2}{r^2}} + \frac{2bc^2}{r^3} + \cancel{\frac{c^2}{r^2}}$

$\Rightarrow \boxed{v^2 = \frac{2bc^2}{r^3}}$ (4).

Acceleration: $a_\theta = 0 \Rightarrow \vec{a} = a_r\cdot\vec{e}_r \Rightarrow a = \ddot{r} - r\dot{\theta}^2 = \ddot{r} - r\frac{c^2}{r^4} \Rightarrow$

$\Rightarrow \boxed{a = \ddot{r} - \frac{c^2}{r^3}}$ (5)

We need to obtain $\ddot{r}$ as a function of r. We derive (2) with respect to t:

$v^2 = \dot{r}^2 + \frac{c^2}{r^2} \quad \Big| \frac{d}{dt}$

$\frac{dv^2}{dt} = 2\dot{r}\ddot{r} - \frac{2c^2}{r^3}\cdot\dot{r} = 2\dot{r}\underbrace{\left(\ddot{r} - \frac{c^2}{r^3}\right)}_{a}$

Cylindrical
$\vec{v}(\dot{r}, r\dot{\theta}, \dot{z})$
$\vec{a}(\underbrace{\ddot{r} - r\dot{\theta}^2}_{a_r}, \underbrace{\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})}_{a_\theta}, \underbrace{\ddot{z}}_{a_z})$

$(4) \Rightarrow -\frac{6bc^2}{r^4}\cdot\cancel{\dot{r}} = 2\cancel{\dot{r}}\cdot a \Rightarrow \boxed{a = -\frac{3bc^2}{r^4}}$ (6) "−" orientation

In order to calculate the normal and tangential accelerations we consider the Frenet's frame.

$$a_t = \frac{dv}{dt} \Rightarrow a_t = \frac{dv}{dr} \cdot \frac{dr}{dt} = \frac{dv}{dr} \cdot \dot{r} = \frac{dv}{dr} \cdot \frac{dr}{d\theta} \cdot \frac{d\theta}{dt}$$

$$\frac{dv}{dr} \underset{(4)}{=} \frac{d}{dr}\left(\frac{\sqrt{2b} \cdot c}{r^{3/2}}\right) = -\frac{3}{2}\sqrt{2b}\, c \cdot \frac{1}{r^{5/2}}$$

$$\frac{dr}{d\theta} = \frac{d}{d\theta}(1 + b\cos\theta) = -b\sin\theta$$

$$\frac{d\theta}{dt} = \frac{c}{r^2}$$

$$\Rightarrow a_t = -\frac{3}{2}\sqrt{2bc} \cdot (-b\sin\theta) \cdot \frac{c}{r^{9/2}}$$

$$a_t^2 = \frac{9}{4} \cdot 2bc^2 (b^2 \sin^2\theta) \cdot \frac{c^2}{r^9} = \frac{9b^3c^4}{2}(1 - \cos^2\theta) \cdot \frac{1}{r^9} =$$

$$= \frac{9b^3c^4}{2r^9}\left[1 - \left(\frac{r}{b} - 1\right)^2\right]$$

$$\Rightarrow \boxed{a_t^2 = \frac{9b^3c^4}{2r^9}\left[\frac{2r}{b} - \frac{r^2}{b^2}\right]} \quad (7)$$

$$a_n^2 = a^2 - a_t^2 = \frac{9b^2c^4}{r^8} - \frac{9b^3c^4}{2r^9}\left[\frac{2r}{b} - \frac{r^2}{b^2}\right] \Rightarrow \boxed{a_n^2 = \frac{9bc^4}{2r^7}} \quad (8).$$

In order to find the radius of the curvature we use:

$$a_n = \frac{v^2}{R} \Rightarrow a_n^2 = \frac{v^4}{R^2} \Rightarrow \frac{9bc^4}{2r^7} = \frac{4b^2c^4}{r^6} \cdot \frac{1}{R^2} \quad \Big| : \frac{bc^4}{r^6}$$

$$\Rightarrow \frac{9}{2r} = 4b \cdot \frac{1}{R^2} \Rightarrow \boxed{R^2 = \frac{8br}{9}} \quad (9)$$

② The equations of motion of a particle are (in polar coordinates)

$$r = r_0 e^{\alpha t} \qquad , \alpha, \beta > 0$$
$$\theta = \beta \cdot t$$

Find the trajectory, the speed and the acceleration of the partic

Solution:

$$\left.\begin{array}{l} r = r_0 \cdot e^{\alpha t} \\ t = \dfrac{\theta}{\beta} \end{array}\right\} \Rightarrow r = r_0 e^{\frac{\alpha}{\beta}\theta}$$ – the logarithmic spiral.

$$\begin{cases} x = r_0 \cos\theta \, e^{\frac{\alpha}{\beta}\theta} \\ y = r_0 \sin\theta \, e^{\frac{\alpha}{\beta}\theta} \end{cases}$$

radial

transversal

$$\left.\begin{array}{l} v_r = \dot{r} = r_0 \alpha e^{\alpha t} \\ v_\theta = r\dot{\theta} = r\beta = r_0 \beta e^{\alpha t} \end{array}\right\} \Rightarrow v^2 = v_r^2 + v_\theta^2 = r_0^2 e^{2\alpha t}(\alpha^2 + \beta^2)$$

$$\boxed{a_r = \ddot{r} - r\dot{\theta}^2} = r_0 \alpha^2 e^{\alpha t} - r_0 e^{\alpha t} \cdot \beta^2 = r_0 e^{\alpha t}(\alpha^2 - \beta^2)$$

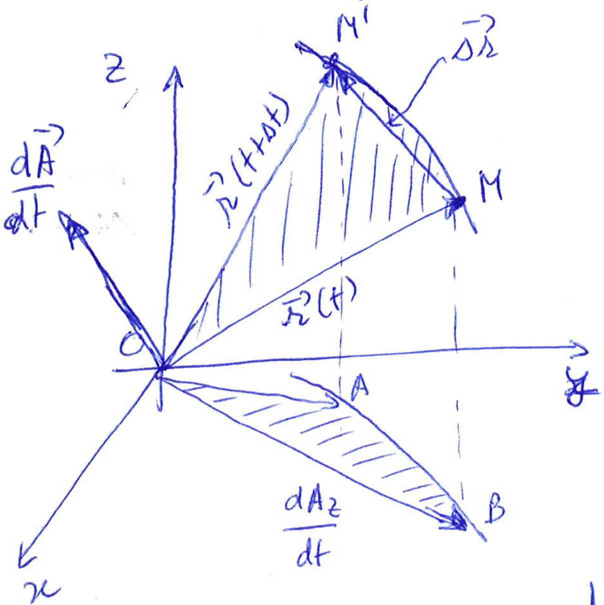
$$\boxed{a_\theta = \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)} = \frac{1}{r}\left(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}\right) = 2\dot{r}\dot{\theta} + r\underset{0}{\underset{\parallel}{\ddot{\theta}}} =$$

$$= 2 r_0 \alpha \beta e^{\alpha t}$$

$$a^2 = a_r^2 + a_\theta^2 = r_0^2 e^{2\alpha t}\left[(\alpha^2 - \beta^2)^2 + 4\alpha^2\beta^2\right] =$$
$$= r_0^2 e^{2\alpha t}(\alpha^2 + \beta^2)^2.$$

Areal velocity

Areal velocity (Viteza areolară) (also called sector or sectorial velocity) is the rate at wich area is swept out by a particle as it moves along a curve.



The oriented area:

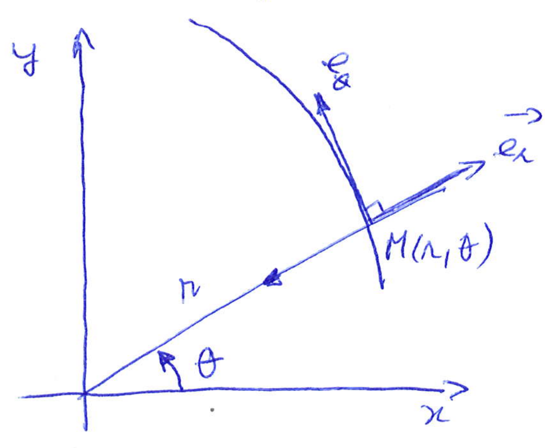
$$\vec{\Delta A} = \frac{1}{2}\vec{r} \times \vec{\Delta r} \;\Big|\; : \Delta t \quad (\Delta t \to 0)$$

(areal velocity of M) $$\frac{d\vec{A}}{dt} = \frac{1}{2}\vec{r} \times \vec{v}$$

$$\frac{d\vec{A}}{dt}\left(\frac{dA_x}{dt}, \frac{dA_y}{dt}, \frac{dA_z}{dt}\right) = \frac{1}{2}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ \dot{x} & \dot{y} & \dot{z} \end{vmatrix} =$$

$$\Leftrightarrow \frac{dA_x}{dt} = \frac{1}{2}(y\dot{z} - z\dot{y}); \quad \frac{dA_y}{dt} = \frac{1}{2}(z\dot{x} - x\dot{z}); \quad \frac{dA_z}{dt} = \frac{1}{2}(x\dot{y} - y\dot{x})$$

③ A material point M moves in a plane such that its speed is proportional ~~with~~ its position vector length at power $(n-1)$ and ~~the~~ the areal velocity is constant. Find the acceleration and the trajectory.



$$\frac{d\vec{A}}{dt} = \frac{1}{2}(\vec{r} \times \vec{v}) \overset{\text{in plane}}{=} \left(0, 0, \frac{dA_z}{dt}\right)$$

$$\frac{dA_z}{dt} = \frac{1}{2}(x\dot{y} - y\dot{x})$$

$x = r\cos\theta$, $\quad \dot{x} = \dot{r}\cos\theta - r\sin\theta\cdot\dot{\theta}$

$y = r\sin\theta$, $\quad \dot{y} = \dot{r}\sin\theta + r\cos\theta\cdot\dot{\theta}$

$$\text{constanta} \Rightarrow \frac{dA_z}{dt} = \frac{1}{2}\left[r\cos\theta(\cancel{\dot{r}\sin\theta} + r\cos\theta\,\dot{\theta}) - r\sin\theta(\cancel{\dot{r}\cos\theta} - r\sin\theta\cdot\dot{\theta})\right] =$$

$$= \frac{1}{2}\left[r^2\cos^2\theta\cdot\dot{\theta} + r^2\sin^2\theta\,\dot{\theta}\right] = \frac{1}{2}r^2\dot{\theta} \Rightarrow \boxed{r^2\dot{\theta} = C}, \; C \in \mathbb{R} \quad (1)$$

$$\Downarrow$$

$$\dot{\theta} = \frac{C}{r^2}$$

On the other hand,

$$\vec{a}(a_r, a_\theta) \qquad \boxed{a_r = \ddot{r} - r\dot{\theta}^2 \overset{(1)}{=} \ddot{r} - \frac{c^2}{r^3}} \ (2) \Rightarrow \boxed{a = a_r = \ddot{r} - \frac{c^2}{r^3}} \ (3)$$

$$a_\theta = \frac{1}{r}\frac{d}{dt}(\underbrace{r^2\dot{\theta}}_{= c}) = 0$$

Moreover $\boxed{v^2 = \dot{r}^2 + r^2\dot{\theta}^2 = \dot{r}^2 + r^2 \cdot \frac{c^2}{r^4} = \dot{r}^2 + \frac{c^2}{r^2}}$ (4)

$$\frac{dv^2}{dt} = 2\dot{r}\ddot{r} - 2\dot{r}\frac{c^2}{r^3} = 2\dot{r}\underbrace{\left(\ddot{r} - \frac{c^2}{r^3}\right)}_{a} \underset{(3)}{\Rightarrow} a = \frac{1}{2\dot{r}}\frac{dv^2}{dt} =$$

$$= \frac{1}{2} \cdot \frac{dt}{dr} \cdot \frac{dv^2}{dt} = \frac{1}{2} \cdot \frac{dv^2}{dr}.$$

We know that $\quad v = h_n \cdot r^{n-1} \Rightarrow v^2 = h_n^2 r^{2n-2}$

Thus, $\boxed{a = \frac{1}{2} \cdot \frac{d}{dr}\left[h_n^2 \cdot r^{2(n-1)}\right] = \frac{1}{2} \cdot 2(n-1) \cdot h_n^2 \cdot r^{2n-3} = k^2(n-1) \cdot r^{2n-3}}$ (5)

In order to find the trajectory we use:

$$\left.\begin{array}{l} v^2 = \dot{r}^2 + \frac{c^2}{r^2} \\ \dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta} \cdot \frac{d\theta}{dt} = \frac{dr}{d\theta} \cdot \frac{c}{r^2} \end{array}\right\} \Rightarrow v^2 = \left(\frac{dr}{d\theta}\right)^2 \cdot \frac{c^2}{r^4} + \frac{c^2}{r^2}$$

$$h_n^2 r^{2n-2} = \frac{c^2}{r^2}\left[1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right] \Rightarrow 1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2 = \frac{h_n^2 r^{2n}}{c^2}$$

$$\Rightarrow \left(\frac{dr}{d\theta}\right)^2 = \frac{h_n^2 r^{2n+2}}{c^2} - r^2 \Rightarrow \frac{dr}{d\theta} = \pm\sqrt{\frac{h_n^2 r^{2n+2}}{c^2} - r^2}$$

$$\Rightarrow \pm\frac{dr}{\sqrt{\frac{h_n^2 r^{2n+2}}{c^2} - r^2}} = d\theta \Rightarrow \pm\int_{r_0}^{r} \frac{dr}{\frac{k}{c} r^{n+1}\sqrt{1 - \frac{c^2}{k^2}\frac{1}{r^{2n}}}} = \theta - \theta_0$$

$$\Rightarrow \pm\frac{1}{n}\int_{r_0}^{r} \frac{-d\left(\frac{c}{k}\frac{1}{r^n}\right)}{\sqrt{1 - \frac{c^2}{k^2} \cdot \frac{1}{r^{2n}}}} = \theta - \theta_0 \Rightarrow \theta - \theta_0 = \pm\frac{1}{n}\arcsin\left(\frac{c}{k}\frac{1}{r^n}\right).$$

$$\Rightarrow \boxed{r^n = \frac{c}{k}\,\frac{1}{\cos[n(\theta - \theta_0)]}} \ (6)$$

trajectory.

$$\boxed{\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a} + C = -\arccos\frac{x}{a} + C.}$$