Complements of Geometry

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Optional course for Maths and Computer Science

Week 6

Geometric inequalities

Theorem (Triangle's inequality)

Let a, b, c be the sides of a non-degenerate triangle. Then $a + b > c$, $b + c > a$ and $c + a > b$.

Applications

Problem 1. Prove that if ABC is a triangle and P a point in its interior, then $PA + PB + PC < AB + BC + CA$.

Problem 2. Let ABC be a triangle with side lengths a, b, and c. Prove that

$$abc \geq (b+c-a)(c+a-b)(a+b-c).$$

Problem 3. Show that in every triangle ABC, the following inequality holds
$$\frac{1}{4} < \frac{IA \cdot IB \cdot IC}{\ell_A \cdot \ell_B \cdot \ell_C} \leq \frac{8}{27},$$
where I is the incenter and ℓ_A, ℓ_B, ℓ_C are the lengths of the bisectors of the triangle ABC.

Some extremal properties in planar geometry

Proposition

Let $ABCD$ be a convex quadrilateral and denote by O the intersection of its diagonals. Then O is the unique point of in the plane such that the sum of the distances to the vertices of the quadrilateral is minimal.

Proof. We suppose that $P \neq O$ and show that

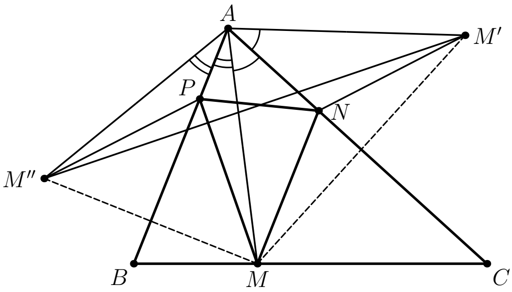
$$PA + PB + PC + PD > OA + OB + OC + OD.$$

A minimum property of the orthic triangle

The orthic triangle is the triangle determined by the feet of the altitudes in a given triangle.

Theorem (Fagnano's problem)

Let ABC be an acute triangle. Then, from all of the triangles which have vertices on the sides of ABC, the orthic triangle has the minimal perimeter.



We will show that $p_{\triangle MNP} \geq p_{\triangle M_1NP}$, where M_1 is the foot of the altitude from A. Let M' be the symmetric of M with respect to AC and M'' the symmetric of M with respect to AB.

We then have $NM = NM'$ and $PM = PM''$. It is easy to see that the perimeter of MNP is equal to $M''P + PN + NM'$.

The triangle $M'AM''$ is isosceles and $\angle M'AM'' = 2\angle A$.

Note that

$$p_{\triangle MNP} = M'N + NP + PM'' \geq M'M''.$$

We now determine the position of M on BC such that the length $M'M''$ is minimal.

From the cosine theorem we have:

$$\begin{aligned} M'M''^2 &= AM'^2 + AM''^2 - 2AM' \cdot AM'' \cos 2A \\ &= 2AM^2 - 2AM^2 \cos 2A \\ &= 2AM^2(1 - \cos 2A) \\ &= 4AM^2 \sin^2 A, \end{aligned}$$

therefore $M'M'' = 2AM \sin A$. It follows that $M'M''$ is minimal if and only if AM is minimal, hence M is the foot of the altitude from A. Therefore, $p_{\triangle MNP}$ is minimal if and only if M is the foot of the altitude from A.

Similarly, for the vertex B, we can deduce that N is the foot of the altitude from B, and for C, P must be the foot of the altitude from C. $\square$

A minimal property of the centroid (center of gravity)

Theorem

Let $\triangle ABC$, be a triangle. Then its centroid is the only point in the plane and M such that the sum

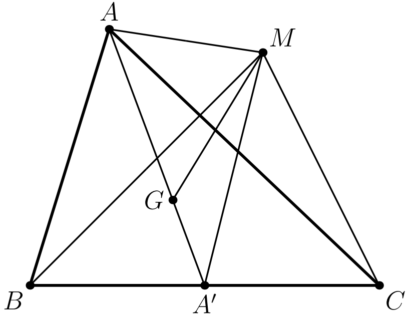
$$MA^2 + MB^2 + MC^2$$

is minimal.

Proof. First we show that for every point M in the plane we have the identity

$$MA^2 + MB^2 + MC^2 = 3MG^2 + GA^2 + GB^2 + GC^2,$$

(this is known as Leibniz identity).



Let A' be the midpoint of the segment $[BC]$. In the triangle AMA' we apply Stewart's theorem and we get

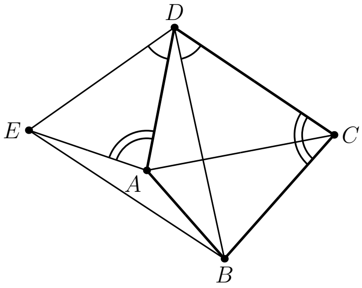
Ptolemy's inequality

Theorem

In every convex quadrilateral $ABCD$ the following inequality holds:

$$AC \cdot BD \leq AB \cdot CD + AD \cdot BC.$$

The above holds with equality if and only if $ABCD$ is cyclic.



Thank you very much for your attention!