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Complements of geometry

P4.

a) Let $\vec{r}: I \to \mathbb{R}^3$ be a particle path such that $\|\vec{r}\,'(t)\| = 1 \ \forall t \in I$ and for which $k(t) > 0, \ \forall t \in I$.

Show that $\vec{r}$ is contained in one plane if and only if $\tau(t) = 0 \ \forall t \in I$.

We write $\beta(s)$ for the arc-length parametrisation of this curve. $(\Rightarrow)$

If β is planar (i.e included in one plane)

$\Rightarrow$ there exist points p and a normal vector n such that $(\beta(s) - p) \cdot n = 0$

Differenciating 2 times: $\beta'(s) \cdot n = \beta''(s) \cdot n = 0$

$$\Rightarrow T(s) \cdot n = k(s) N(s) \cdot n = 0 \quad \forall s$$

$\Rightarrow n$ is orthogonal to $T(s)$ and $N(s)$

$$\left.\begin{array}{l} B(t) \text{ is also orthogonal to } T(s) \text{ and } N(s) \\ \text{we are working in } \mathbb{R}^3 \end{array}\right\} \Rightarrow B(s) = \pm \frac{n}{\|n\|}$$

$\Rightarrow B'(s) = 0 \Rightarrow \tau(s) = 0 \quad \forall s$

$(\Leftarrow)$

$\tau(s) = 0 \quad \forall s \Rightarrow B'(s) = 0 \Rightarrow B(s)$ is constant $\forall s$

We want to see $(\beta(s) - \beta(0)) \cdot B(s) = 0$ because that implies tat β is planar

We define $f(s) = (\beta(s) - \beta(0)) \cdot B(s)$

$$\Rightarrow f'(s) = \beta'(s) \cdot B(s) = T(s) \cdot B(s) = 0$$

$$\left.\begin{array}{l} \Rightarrow f(s) \text{ constant} \\ f(0) = 0 \end{array}\right\} \Rightarrow f(s) = 0 \quad \forall s$$

$$\Rightarrow \beta \text{ planar}$$

b) Show that if $\tau(t) = 0$ $\forall t \in I$ and $k(t)$ is constant

$\Rightarrow$ $Im(\vec{r})$ is part of a circle

β arc-length parametrisation

As $\tau(t) = 0 \xRightarrow{a)} \beta$ is planar

We are going to consither the curve

$$\gamma(s) = \beta(s) + \frac{1}{k} N(s) \quad , \quad k = k(s) \text{ curvature of } \beta(s)$$

We know that k is constant

$$\Rightarrow \gamma'(s) = \beta'(s) + \frac{1}{k} N'(s)$$

$$= T(s) + \frac{1}{k}(-kT(s) + \underbrace{\tau}_{=0} B(s))$$

$$= T(s) - T(s)$$

$$= 0$$

$\gamma(s)$ is constant , $\gamma(s) = p \in \mathbb{R}^n$

If we rearrange it and take norms, we get:

$$\| \beta(s) - \gamma(s) \| = \| \tfrac{1}{k} N(s) \| = \frac{1}{k}$$

$$\Rightarrow \| \beta(s) - p \| = 1/k$$

$\Rightarrow$ $\beta(s)$ lies in a circle with center p and radius $1/k$

c) Illustrate (a) and (b) in the particular case

$\vec{r}: \mathbb{R} \to \mathbb{R}^3$, $\vec{r}(t) = \left(\frac{4}{5}\cos(t),\ 1 - \sin(t),\ -\frac{3}{5}\cos(t) \right)$

$$\vec{r}\,'(t) = \left(-\frac{4}{5}\sin t,\ -\cos t,\ \frac{3}{5}\sin t \right.$$

$$\vec{r}\,''(t) = \left(-\frac{4}{5}\cos t,\ \sin t,\ \frac{3}{5}\cos t \right)$$

- Unit tangent vector : $T(t) = \vec{r}\,'(t)$ (note $\|\vec{r}(t)\| = 1$
- Curvature $\kappa(t) = \|\vec{r}\,''(t)\| = \sqrt{\frac{16}{25}\cos^2 t + \sin^2(t) + \frac{9}{25}\cos^2 t}$

$$= \sqrt{\sin^2 t + \cos^2 t} = \sqrt{1} = 1$$

the curvature is constant $\kappa(t) = 1$

- Torsion, $\tau(t)$

$$N(t) = \frac{\vec{r}\,''(t)}{\|\vec{r}\,''(t)\|} \Rightarrow N(t) = \vec{r}\,''(t)$$

$\|\vec{r}\,''(t)\| = 1$

$$B(t) = T(t) \times N(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\frac{4}{5}\cdot s & -c & \frac{3}{5}\cdot s \\ -\frac{4}{5}\cdot c & s & \frac{3}{5}\cdot c \end{vmatrix} = \dots = -\frac{3}{5}\vec{i} - \frac{4}{5}\vec{k}$$

$\Rightarrow B(t) = \left(-\frac{3}{5}, 0, -\frac{4}{5} \right)$ constant $\Rightarrow \tau(t) = 0$

$\Rightarrow$ the given curve lies in a plane and is part of a circle.