Series of real numbers - part 1

Geometric series

We recall that:

$$\lim_{n\to\infty} a^n = \begin{cases} +\infty & : a > 1 \\ 1 & : a = 1 \\ 0 & : a \in (-1,1) \\ \nexists & : a \leq -1 \end{cases}$$

Each series of the type

$$\sum_{n\geq m} q^{n-1},$$

for $q \in \mathbb{R}$ is a **geometric series**. Then:

$$\sum_{n=1}^{\infty} q^{n-1} = \begin{cases} 0 & : q = 0 \\ \frac{1}{1-q} & : q \in (-1,1)\backslash\{0\} \\ +\infty & : q \geq 1 \end{cases}$$

In the case when $q \leq -1$, the geometric series does not posses a sum.

Proof:

We study the general term for the sequence of partial sums, given $n \geq 1$.

$$s_n = q^0 + q^1 + ... + q^{n-1} = \frac{1-q^n}{1-q}.$$

Case 1: $q = 0$. In this case $s_n = 0, \forall n \geq 1$, therefore its limit 0 as well, thus the sum of the series is 0.

Case 2: $q \in (-1,1)\backslash\{0\}$. Then: $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1-q^n}{1-q} = \frac{1-0}{1-q} = \frac{1}{1-q}$.

Case 3: $q > 1$. Then: $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1-q^n}{1-q} = \frac{1-\infty}{1-q} = \frac{-\infty}{1-q} = \infty$ because $1 - q < 0$.

Case 4: $q \leq -1$. We notice that

$$\lim_{n\to\infty q^n}$$

does not exist, therefore there does not exists the limit $\lim_{n\to\infty} s_n$ as well. Thus, in this case the geometric series does not have a sum.

Telescopic series

If the general term of degree n of the series of real numbers $\sum_{n\geq m} x_n$ is defined as the difference of two successive terms of a sequence of real numbers $(a_n)_{n\geq m}$, i.e.

$$x_n = a_n - a_{n+1}, \forall n \geq m,$$

it is called a **telescopic series**. If the sequence $(a_n)_{n\geq m}$ has the limit

$$l = \lim_{n\to\infty} a_n,$$

then the series $\sum_{n\geq m} x_n$ has a sum, and it:

$$\sum_{n=m}^{\infty} x_n = a_m - l.$$

Proof: We write the general term of degree n of the sequence of partial sums of the series:

$$s_n = x_m + x_{m+1} + ... + x_n = a_m - a_{m+1} + a_{m+1} - a_{m+2} + ... + a_{n-1} - a_n + a_n - a_{n+1} =$$

$$= a_m - a_{n+1}.$$

In conclusion

$$s_n = a_m - a_{n+1}, \forall n \geq m.$$

We notice that a_m is a constant, thus, due to the fact that the sequence $(a_n)_{n\geq m}$ has a limit, the sequence $(s_n)_{n\geq m}$ has a limit as well, and it is:

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} (a_m - a_{n+1}) = a_m - \lim_{n\to\infty} a_{n+1} = a_m - l.$$

Since the limit of the sequence of partial sums is the sum o the series, we reach the conclusion that:

$$\sum_{n=m}^{\infty} x_n = a_m - l.$$

Operations with convergent series

1. Let $\sum_{n\geq 1} x_n$ and $\sum_{n\geq 1} y_n$ be two series of real numbers which have the sums:

$$\sum_{n=1}^{\infty} x_n = x, \quad \sum_{n=1}^{\infty} y_n = y.$$

and let $a, b \in \mathbb{R}$. If $ax + by$ is not an not determined case, then the series

$$\sum_{n\geq 1} ax_n + by_n$$

has a sum and it is:

$$\sum_{n=1}^{\infty}(ax_n + by_n) = ax + by.$$

2. In the case of the sum of a series it is very important to notice the first term of the summation. Thus, if the series $\sum x_n$ has a sum, and if $p \in \mathbb{N}$, then

$$\sum_{n=p}^{\infty} x_n = \sum_{n=1}^{\infty} x_n - \Big(x_1 + x_2 + ... + x_{p-1}\Big).$$

Exercises

Exercise 1: Compute the sums for the following geometric series (if they exists):

$$a) \sum_{n\geq 3} \frac{7}{9^n}, \quad b) \sum_{n\geq 4} \frac{3^{n-3} + (-4)^{n+3}}{5^n}, \quad c) \sum_{n\geq 5} e^n, \quad d) \sum_{n\geq 2} \Big(-\frac{1}{\pi}\Big)^n \quad e) \sum_{n\geq 3} (-4)^n.$$

Exercise 2: Compute the sums of the following telescopic series:

$$a) \sum_{n\geq 1} \frac{1}{4n^2 - 1}, \quad b) \sum_{n\geq 1} \frac{1}{\sqrt{n} + \sqrt{n+1}}, \quad c) \sum_{n\geq 5} \frac{1}{n(n+1)(n+2)}$$

$$d) \sum_{n\geq 1} \ln\Big(1 + \frac{1}{n}\Big), \quad e) \sum_{n\geq 2} \frac{\ln\big(1 + \frac{1}{n}\big)}{\ln\big(n^{\ln(n+1)}\big)}.$$