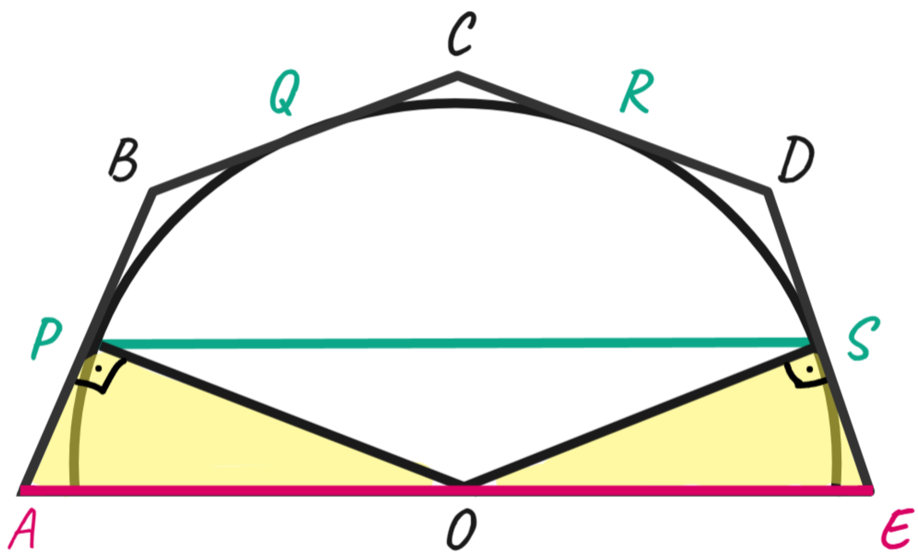
Seminar 3 – Presentation 2

Let $ABCDE$ be a convex pentagon such that $AB + CD = BC + DE$ and a circle ω with center O on the side AE is tangent to the sides AB, BC, CD and DE at points P, Q, R and S, respectively. Prove that the lines PS and AE are parallel.



Proof. Given $AB + CD = BC + DE$ (1)

We have:

$$AB = AP + PB$$

$$CD = CR + RD$$

$$BC = BQ + QC$$

$$DE = DS + SE$$

$$\Rightarrow \ (1) \ \Leftrightarrow \ AP + PB + CR + RD = BQ + QC + DS + SE$$

We will use the fact that tangents from a given point to a circle are equal to transform the given metric condition into something more approachable.

$PB = BQ \rightarrow$ tangents from B

$QC = CR \rightarrow$ tangents from C

$RD = DS \rightarrow$ tangents from D

$$\Rightarrow \ (1) \ \Leftrightarrow AP = SE$$

Now, observing ΔAOP and ΔEOS , we can prove that these are congruent by using the Side-Angle-Side Theorem:

$$AP = SE\ , OP = OS \text{ (radius)}, \measuredangle APO = \measuredangle ESO = 90°$$

(a tangent to a circle is perpendicular to the radius at the point of tangency)

$$\Rightarrow \Delta AOP \equiv \Delta EOS$$

Hence, their altitudes are equal:

$$d(P, AE) = d(S, AE).$$

$$\therefore PS \parallel AE\ . \ \blacksquare$$