ANALYTIC GEOMETRY, PROBLEM SET 11

1. Problems with the hyperbola and parabola

1. Find the intersection points between the line $d_2 : 2x - y - 10 = 0$ and the hyperbola $\mathcal{H} : \frac{x^2}{20} - \frac{y^2}{5} - 1 = 0$.

2. Find the area of the triangle determined by the asymptotes of the hyperbola $\mathcal{H} : \frac{x^2}{4} - \frac{y^2}{9} - 1 = 0$ and the line $d : 9x + 2y - 24 = 0$.

3. Find the equation of the parabola having the focus $F(-7, 0)$ and the director line $x - 7 = 0$.

4. Find the equation of the tangent line(s) to:

(1) the hyperbola $\mathcal{H} : \frac{x^2}{20} - \frac{y^2}{5} - 1 = 0$, orthogonal to the line $d_2 : 4x + 3y - 7 = 0$;
(2) the parabola $\mathcal{P} : y^2 - 8x = 0$, parallel to $d_3 : 2x + 2y - 3 = 0$.

5. Find the equations of the tangent line(s) to:

(1) the hyperbola $\mathcal{H} : \frac{x^2}{3} - \frac{y^2}{5} - 1 = 0$ passing through $P_2(1, -5)$;
(2) the parabola $\mathcal{P} : y^2 - 36x = 0$, passing through $P_3(2, 9)$.

6. Consider the hiperbola $x^2 - \frac{y^2}{4} = 1$ and denote by F_1, F_2 its foci. Find the locus of all points M, situated on the hyperbola such that

(a) The angle $\angle F_1MF_2$ is right;
(b) The angle $\angle F_1MF_2$ is equal to $60°$.

7. From the point $P(-3, 12)$ we draw tangents to the parabola $y^2 = 10x$. Compute the distance from the point P to the chord of the parabola which is formed by the two contact points.

8. Find a relation between the coordinates of the point $P_0(x_0, y_0)$ such that there is no tangent from this point to the hiperbola $\frac{x^2}{4} - \frac{y^2}{9} = 1$.

9. Write down the formula for the isometry $\text{Rot}_{90} : \mathcal{E}_2 \to \mathcal{E}_2$ which represents the rotation of center O (origin) and angle $90°$ in the trigonometric sense. Find the equation of the image under Rot_{90} of:

(a) The hyperbola $\frac{x^2}{4} - \frac{y^2}{9} = 1$;
(b) The parabola $y^2 - 8x = 0$.

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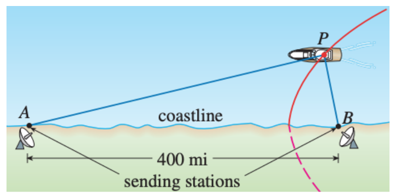
Do the same for $t_{\overline{v}} \circ \mathrm{Rot}_{90}$, where $t_{\overline{v}} : \mathcal{E}_2 \to \mathcal{E}_2$ is the translation by $\overline{v}(1,0)$.

9. In the LORAN (Long Range Navigation) radio navigation system, two radio stations located at A and B transmit simultaneous signals to a ship or an aircraft located at P. The onboard computer converts the time difference in receiving these signals into a distance difference $|PA| - |PB|$, and this, according to the definition of a hyperbola, locates the ship or aircraft on one branch of a hyperbola (see the figure). Suppose that station B is located 400 mi due east of station A on a coastline. A ship received the signal from B 1200 micro-seconds (μs) before it received the signal from A.

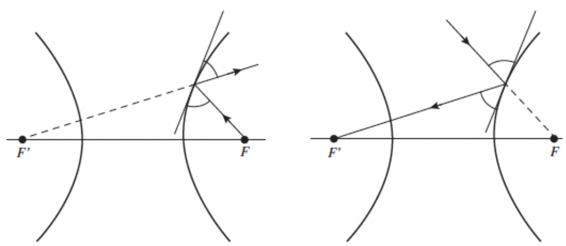
(a) Assuming the radio signals travel at a speed of 0.2 miles per μs, find an equation of the hyperbola on which the ship lies.

(b) If the ship is due north of B, how far off the coastline is the ship?



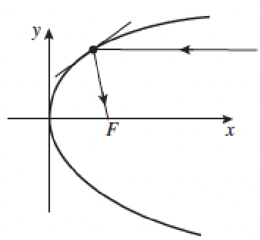
2. The reflection (optical) properties of the hyperbola and parabola

The hyperbola and the parabola have reflective properties similar to those of the ellipse.



Theorem 1 (Reflection property of the hyperbola)**.** *Light coming from one focus of a hyperbolic mirror is reflected at the hyperbola in such a way that the light appears to have come from the other focus.*

Also, light going towards one focus of a hyperbolic mirror is reflected at the mirror towards the other focus.



Theorem 2 (Reflection property of the parabola)**.** *Incoming light parallel to the axis of a parabolic mirror is reflected at the parabola to pass through the focus.*

Conversely, light incoming from the focus of a parabolic mirror is reflected at the parabola to give a beam of light parallel to the axis of the parabola.