

COURSE 7

Elementary operations on matrices. Applications

Let K be a field, $m, n \in \mathbb{N}^*$ and $A = (a_{ij}) \in M_{m,n}(K)$.

Definition 1. By an **elementary operation on the rows (columns)** of a matrix we understand one of the following:

- (I) the interchange of two rows (columns).
- (II) multiplying a row (column) by a non-zero element $\alpha \in K$.
- (III) multiplying a row (column) by an element $\alpha \in K$ (also called scalar) and adding the result to another row (column).

Application 1. Computing determinants: see the seminars.

We remind that these operations can be used for computing determinants and none of them turns a non-zero determinant into a zero and, also, none of them turns a zero determinant into a non-zero one. One can easily deduce from this fact that performing these operations on a matrix A we get a matrix B for which maximum number of columns (rows) we can choose from the columns (rows) of such that none of them is a linear combination of the others is the same, hence A and B have the same rank.

This remark can be used for developing another algorithm for finding the rank of a matrix.

Application 2. Computing the rank of a matrix:

For finding the rank of a matrix $A \in M_{m,n}(K)$ we may proceed by performing certain elementary operations on the rows and columns of A and on the matrices that we successively get this way. Our purpose is to transform A (by using elementary operations) into a matrix B for which the only non-zero elements are on the main diagonal, preferably at the beginning (if such elements exist). This way, from A we get a matrix

$$B = \begin{pmatrix} a_{11} & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & a_{22} & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{rr} & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}$$

with all $a_{11}, a_{22}, \dots, a_{rr}$ non-zero. The matrices A and B have the same rank which is r .

Example 2. see the seminar ...

Application 3. Solving systems of linear equations by using Gauss elimination algorithm.

Let K be a field and let us consider the system

[illegible]

over K with the augmented matrix \overline{A} . This algorithm is based on the fact that

- (i) interchanging of two equations of (1),
- (ii) multiplying an equation of (1) by a non-zero element $\alpha \in K$,
- (iii) multiplying an equation of (1) by $\alpha \in K$ and adding the resulted equation to another one,

are operations which lead us to systems which are equivalent to (1). Since all these operations act on the coefficients and constant terms of the system, it is quite obvious that these operations can be performed as elementary row operations on the system augmented matrix.

Thus, we can infer that providing elementary row operations on the augmented matrix of (1), we get the augmented matrix of an equivalent system. **Gaussian elimination** (also known as **row reduction**) is an algorithm which uses row elementary operations on some matrices resulted from \overline{A} in order to get a matrix with a number zero entries at the beginning of each row which strictly increases while we descend in the matrix (matrix known as **echelon matrix** or **echelon form**). This procedure corresponds to a partial elimination of some unknowns to get an equivalent system which can be easier solved.

Definition 3. A matrix $A \in M_{mn}(K)$ is in an **echelon form** with $k \geq 1$ non-zero rows if:

- (1) the rows $1, \dots, k$ are non-zero and the rows $k+1, \dots, m$ consists only of 0;
- (2) if $N(i)$ is the number of zeros at the beginning of the row i ($i \in \{1, \dots, k\}$), then

$$0 \leq N(1) < N(2) < \dots < N(k).$$

A k non-zero rows echelon form with $N(1) = 0$, $N(2) = 1$, $N(3) = 2, \dots$, $N(k) = k - 1$ is called **trapezoidal form**.

Remarks 4. a) Any matrix can be brought to an echelon form by elementary row operations.

b) A square matrix of size n is invertible if and only if it can be brought to a trapezoidal form with n non-zero rows by using only elementary row operations (such a matrix is called **triangular form (matrix)**).

c) A square matrix of size n is invertible if and only if it can be brought to the identity matrix I_n by using only elementary row operations.

Thus, the purpose of Gauss elimination is to successively use elementary operations on the rows of the augmented matrix \overline{A} of (1) in order to bring it to an echelon form B . If we manage to do this, then B is the augmented matrix of an equivalent system. In some forms of Gauss elimination,

notice the advantage we have when we form the equivalent system of (1) provided by B . This algorithm is known as **Gauss-Jordan elimination**.

b) Moreover, we can bring the augmented matrix of a consistent system to the following form:

$$B = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & a'''_{1,k+1} & \dots & a'''_{1n} & b'''_1 \\ 0 & 1 & 0 & \dots & 0 & a'''_{2,k+1} & \dots & a'''_{2n} & b'''_2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & a'''_{k,k+1} & \dots & a'''_{kn} & b'''_k \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Now, it is straightforward to express the main unknowns as linear combinations of the side unknowns.

Application 4. Computing the inverse of a matrix: Let K be a field, $n \in \mathbb{N}^*$ and let us consider $A = (a_{ij}) \in M_n(K)$ a matrix with $d = \det A \neq 0$. We remind that the matrix equation

$$A \begin{pmatrix} x_{1j} \\ x_{2j} \\ \vdots \\ x_{nj} \end{pmatrix} = \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix} \quad (2)$$

is an equivalent form of a (consistent) Cramer system and that its unique solution is

$$\begin{pmatrix} x_{1j} \\ x_{2j} \\ \vdots \\ x_{nj} \end{pmatrix} = A^{-1} \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix}.$$

Let us take $j = 1$ and $\begin{pmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. Then $\begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix}$ is the first column of the matrix A^{-1} ,

i.e.

$$\begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix} = A^{-1} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} d^{-1}\alpha_{11} \\ d^{-1}\alpha_{12} \\ \vdots \\ d^{-1}\alpha_{1n} \end{pmatrix}$$

(we remind that in our previous courses we denoted by α_{ij} the cofactor of a_{ij}). Of course,

$$\begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix} = I_n \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix}$$

By means of Gauss-Jordan algorithm, one deduces that the augmented matrix of the system (2) can be brought by elementary row operations to the following form

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & d^{-1}\alpha_{11} \\ 0 & 1 & 0 & \dots & 0 & d^{-1}\alpha_{12} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & d^{-1}\alpha_{1n} \end{pmatrix}.$$

Taking, successively, $j = 2$ and $\begin{pmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{n2} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, then $j = 3$ and $\begin{pmatrix} b_{13} \\ b_{23} \\ \vdots \\ b_{n3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, j = n$

and $\begin{pmatrix} b_{1n} \\ b_{2n} \\ \vdots \\ b_{nn} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$, we form the corresponding systems (2) and we use the Gauss-Jordan

algorithm to solve them. We perform exactly the same elementary operations as in the case $j = 1$ on the rows of each augmented matrix of a resulted system in order to bring the system matrix to the form I_n . We get:

$$\begin{pmatrix} 1 & 0 & \dots & 0 & d^{-1}\alpha_{21} \\ 0 & 1 & \dots & 0 & d^{-1}\alpha_{22} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & d^{-1}\alpha_{2n} \end{pmatrix}, \begin{pmatrix} 1 & 0 & \dots & 0 & d^{-1}\alpha_{31} \\ 0 & 1 & \dots & 0 & d^{-1}\alpha_{32} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & d^{-1}\alpha_{3n} \end{pmatrix}, \dots, \begin{pmatrix} 1 & 0 & \dots & 0 & d^{-1}\alpha_{n1} \\ 0 & 1 & \dots & 0 & d^{-1}\alpha_{n2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & d^{-1}\alpha_{nn} \end{pmatrix},$$

respectively. The constant terms column and, consequently, the solution of each system we solved is the column 2 of A^{-1} , column 3 of A^{-1} , ..., column n of A^{-1} , respectively.

Since we performed the same row operations on each of the previously mentioned n systems, we can solve all of them using the same algorithm. This way one can find an algorithm for computing the inverse of the matrix A : we start from the $n \times 2n$ matrix resulted by attaching the matrices A and I_n

$$(A \mid I_n) = \left(\begin{array}{cccc|cccc} a_{11} & a_{12} & \dots & a_{1n} & 1 & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & a_{2n} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & 0 & 0 & \dots & 1 \end{array} \right) \in M_{n,2n}(K)$$

and we perform successive elementary row operations (and only row operations) on this matrix and on the matrices successively resulted from this in order to transform the left size block into I_n . Remark 8 c) of the previous course ensures us that this is possible (if and only if A is invertible). The resulted matrix is:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & \dots & 0 & d^{-1}\alpha_{11} & d^{-1}\alpha_{21} & \dots & d^{-1}\alpha_{n1} \\ 0 & 1 & \dots & 0 & d^{-1}\alpha_{12} & d^{-1}\alpha_{22} & \dots & d^{-1}\alpha_{n2} \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 & d^{-1}\alpha_{1n} & d^{-1}\alpha_{2n} & \dots & d^{-1}\alpha_{nn} \end{array} \right) = (I_n \mid A^{-1})$$

Thus, the right side block of the resulted matrix is the exactly the inverse matrix of A .

Definition 7. A square matrix resulted from the identity matrix after performing only one elementary operation is called **elementary matrix**.

Remarks 8. (and examples ...)

a) The elementary matrices resulted by switching rows (columns):

$$\begin{pmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \text{last} & 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{pmatrix}$$

have the determinant -1 .

b) The elementary matrices resulted by multiplying a row (column) with $\alpha \in K^*$:

$$\begin{pmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & \alpha & \dots & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{pmatrix}$$

have the determinant α .

c) The elementary matrices resulted by multiplying a row (column) by $\alpha \in K$ and adding the result to another row (column):

$$\begin{pmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 1 & \dots & \alpha & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{pmatrix}$$

have the determinant 1.

For the elementary matrices resulted by switching rows (columns), we have:

$$\begin{pmatrix} 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{pmatrix} = \begin{pmatrix} 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{pmatrix}$$

For the elementary matrices resulted by multiplying a row (column) with $\alpha \in K^*$, we have:

$$\begin{pmatrix} 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & \alpha & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & \alpha^{-1} & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{pmatrix} = \begin{pmatrix} 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{pmatrix}$$

For the elementary matrices resulted by multiplying a row (column) by $\alpha \in K$ and adding the result to another row (column), we have:

$$\begin{pmatrix} 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 1 & \dots & \alpha & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 1 & \dots & -\alpha & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{pmatrix} = \begin{pmatrix} 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{pmatrix}$$

Therefore, we can state the following:

Lemma 9. The inverse of an elementary matrix is also an elementary matrix.

Lemma 10. Let $m, n \in \mathbb{N}^*$. Any elementary operation on a matrix $A = (a_{ij}) \in M_{m,n}(K)$ is the result of the multiplication of A with an elementary matrix. More precisely, any elementary operation on the rows (columns) of A results by multiplying A on the left (right) side with the elementary matrix resulted by performing the same elementary operation on I_m (I_n , respectively).

Proof. We check this property for rows. For columns — HOMEWORK.

Let us switch the rows i and j of I_m and let us multiply the resulted elementary matrix with

A. The matrix

$$\begin{pmatrix} 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} a_{11} & \dots & a_{1i} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{i1} & \dots & a_{ii} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{j1} & \dots & a_{ji} & \dots & a_{jj} & \dots & a_{jn} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1} & \dots & a_{mi} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix}$$

is

$$\begin{pmatrix} a_{11} & \dots & a_{1i} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{j1} & \dots & a_{ji} & \dots & a_{jj} & \dots & a_{jn} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{i1} & \dots & a_{ii} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1} & \dots & a_{mi} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix}$$

which is exactly the matrix resulted from A by switching the rows i and j .

Let $\alpha \in K^*$, let us multiply the i 'th row of I_m by α and let us multiply the resulted elementary matrix with A . The matrix

$$\begin{pmatrix} 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & \alpha & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} a_{11} & \dots & a_{1i} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{i1} & \dots & a_{ii} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{j1} & \dots & a_{ji} & \dots & a_{jj} & \dots & a_{jn} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1} & \dots & a_{mi} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix}$$

is

$$\begin{pmatrix} a_{11} & \dots & a_{1i} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \alpha a_{i1} & \dots & \alpha a_{ii} & \dots & \alpha a_{ij} & \dots & \alpha a_{in} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{j1} & \dots & a_{ji} & \dots & a_{jj} & \dots & a_{jn} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1} & \dots & a_{mi} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix}$$

which is exactly the matrix resulted from A by multiplying the i 'th row by α .

Let $\alpha \in K$, let us take the elementary matrix that we get from I_m after multiplying the j 'th row by α and adding the result to the i 'th row, and let us multiply this elementary matrix with

A. The matrix

$$\begin{pmatrix} 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 1 & \dots & \alpha & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} a_{11} & \dots & a_{1i} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{i1} & \dots & a_{ii} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{j1} & \dots & a_{ji} & \dots & a_{jj} & \dots & a_{jn} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1} & \dots & a_{mi} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix}$$

is

$$\begin{pmatrix} a_{11} & \dots & a_{1i} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{i1} + \alpha a_{j1} & \dots & a_{ii} + \alpha a_{ji} & \dots & a_{ij} + \alpha a_{jj} & \dots & a_{in} + \alpha a_{jn} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{j1} & \dots & a_{ji} & \dots & a_{jj} & \dots & a_{jn} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1} & \dots & a_{mi} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix}$$

which is exactly the matrix that we get from A after multiplying the j 'th row by α and adding the result to the i 'th row. \square

Corollary 11. Any invertible matrix is a product of elementary matrices.

Theorem 12. Let $n \in \mathbb{N}^*$. For any matrices $A, B \in M_n(K)$ we have $\det(AB) = \det A \cdot \det B$.

Proof.

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