

# Substitutions in Integrals

## 1 Euler's substitutions

Sometime when in the formulation of the function to be integrated we encounter

$$\sqrt{ax^2 + bx + c},$$

where  $a, b, c \in \mathbb{R}$ , we consider a new variable  $t$ , in one of the following cases:

$$\sqrt{ax^2 + bx + c} = \begin{cases} \pm\sqrt{ax} \pm t & \text{if } a > 0 \\ \pm x \cdot t \pm \sqrt{c} & \text{if } c > 0 \\ t(x - x_0) & \text{if } x_0 \text{ is a solution of the equation } ax^2 + bx + c = 0. \end{cases}$$

## 2 Weirstras' (trigonometric) substitutions

For functions in whose formulations are involved trigonometric functions, there is a usual substitution, namely:

$$tg \frac{x}{2} = t.$$

If we denote by  $R(\sin x, \cos x)$  the expression to be integrated, sometimes we may consider other substitutions, which might lead us faster to the expected solution. Hence:

- If  $R(-\sin x, \cos x) = -R(\sin x, \cos x)$ , then choose  $\cos x = t$ .
- If  $R(\sin x, -\cos x) = -R(\sin x, \cos x)$ , then choose  $\sin x = t$ .
- If  $R(-\sin x, -\cos x) = -R(\sin x, \cos x)$ , then choose  $tg x = t$ .

Recall the following trigonometric identities:

$$\cos^2 x = \frac{1}{1 + tg^2 x} \quad \sin^2 x = \frac{tg^2 x}{1 + tg^2 x}.$$

$$\sin x = \frac{2tg\frac{x}{2}}{1 + tg^2\frac{x}{2}} \quad \cos x = \frac{1 - tg^2\frac{x}{2}}{1 + tg^2\frac{x}{2}}$$

### 3 Other tigonometric substitution

Sometimes, when the integrating function contains square roots of second degree polynomials ( alternatively to using Euler's substitutions) we may pass to trigonometric functions, in the following situations:

- When  $\int R(x, \sqrt{r^2 - x^2})dx$  choose  $x = r \sin$  or  $x = r \cos t$ .
- When  $\int R(x, \sqrt{r^2 + x^2})dx$  choose  $x = rtgt$  or  $x = rctgt$ .
- When  $\int R(x, \sqrt{x^2 - r^2})dx$  choose  $x = \frac{r}{\cos x}$  or  $x = \frac{r}{\sin x}$ .

#### Exercise 1:

- $\int \frac{1}{1+\frac{1}{\sin x}}dx, \quad x \in (\pi, \pi);$
- $\int \frac{1}{3 \sin x + 4 \cos x}dx \quad x \in (\pi, \pi);$
- $\int \frac{\sqrt{9-x^2}}{x^2}dx, \quad x \in (-3, 3);$
- $\int \frac{1}{\sqrt{(x^2+1)^3}}dx, \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right);$
- $\int \frac{1}{\sqrt{(x^2-8)^3}}dx \quad x \in (-\sqrt{8}, \sqrt{8});$
- $\int \sqrt{2x - x^2}dx \quad x \in (0, 2);$
- $\int \sqrt{4 - x^2}dx \quad x \in (-2, 2);$
- $\int x\sqrt{1+x^2}dx.$

#### Exercise 2:

Determine

$$a) \int \frac{2x-1}{x^2-3x+2}dx, \quad x \in ]2, +\infty[;$$

$$b) \int \frac{4}{(x-1)(x+1)^2} dx, \quad x > 1;$$

$$c) \int \frac{1}{x^3 - x^4} dx, \quad x > 1;$$

$$d) \int \frac{2x+5}{x^2+5x+10}, \quad x \in \mathbb{R};$$

$$e) \int \frac{1}{x^2+x+1}, \quad x \in \mathbb{R}.$$

### Exercise 3:

Determine:

$$a) \quad I = \int \frac{1}{\sqrt{x+1} + \sqrt{x}} dx, \quad x \in ]0, +\infty[;$$

$$b) \quad I = \int \frac{1}{x + \sqrt{x-1}} dx, \quad x \in ]1, +\infty[.$$

### Exercise 4:

Determine:

$$a) \quad I = \int \frac{1}{1 + \sqrt{x^2 + 2x - 2}} dx, \quad x \in ]\sqrt{3} - 1, +\infty[;$$

$$b) \quad I = \int \frac{1}{(x+1)\sqrt{-4x^2 - x + 1}} dx, \quad x \in \left] \frac{-1 - \sqrt{17}}{8}, \frac{\sqrt{17} - 1}{8} \right[.$$

### Exercise 5:

Determine:

$$a) \int_1^2 \frac{1}{x^3 + x^2 + x + 1} dx; \quad b) \int_1^3 \frac{1}{x(x^2 + 9)} dx;$$

$$c) \int_{-1}^1 \frac{x^2 + 1}{x^4 + 1} dx; \quad d) \int_{-1}^1 \frac{x}{x^2 + x + 1} dx.$$

## Exercise 6:

Determine:

$$\begin{array}{ll} a) \int_{-3}^{-2} \frac{x}{(x+1)(x^2+3)} dx; & b) \int_0^1 \frac{x+1}{(x^2+4x+5)^2} dx; \\ c) \int_1^2 \frac{1}{x^3+x} dx; & d) \int_0^2 \frac{x^3+2x^2+x+4}{(x+1)^2} dx. e) \int_0^1 \frac{1}{(x+1)(x^2+4)} dx; \\ f) \int_2^3 \frac{2x^3+x^2+2x-1}{x^4-1} dx; & g) \int_0^1 \frac{x^3+2}{(x+1)^3} dx. \end{array}$$

## Exercise 7:

Determine:

$$\begin{array}{ll} a) \int_{-1}^1 \frac{1}{\sqrt{4-x^2}} dx; & b) \int_0^1 \frac{1}{\sqrt{x^2+x+1}} dx; \\ c) \int_{-1}^1 \frac{1}{\sqrt{4x^2+x+1}} dx; & d) \int_2^3 \frac{x^2}{(x^2-1)\sqrt{x^2-1}} dx. \end{array}$$

## Exercise 8:

Determine

$$\begin{array}{ll} a) \int_2^3 \sqrt{x^2+2x-7} dx; & b) \int_0^1 \sqrt{6+4x-2x^2} dx; \\ c) \int_0^{3/4} \frac{1}{(x+1)\sqrt{x^2+1}} dx; & d) \int_2^3 \frac{1}{x\sqrt{x^2-1}} dx. \end{array}$$

## Exercise 9:

Determine:

$$a) 2\sqrt{2} < \int_{-1}^1 \sqrt{x^2+4x+5} dx < 2\sqrt{10};$$

$$b) e^2(e-1) < \int_e^{e^2} \frac{x}{\ln x} dx < \frac{e^3}{2}(e-1).$$