Analytic Geometry

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Recap...

- A plane π in the 3-dimensional space can be uniquely determined by specifying a point $P_0(x_0, y_0, z_0)$ in the plane and a nonzero vector $\overline{n}(a, b, c)$, orthogonal to the plane. $\overline{n}$ is called the *normal vector* to the plane π.
- An arbitrary point $P(x, y, z)$ is contained into the plane π if and only if

$$\overline{n} \perp \overline{P_0 P},$$

or

$$\overline{n} \cdot \overline{P_0 P} = 0.$$

- But $\overline{P_0 P}(x - x_0, y - y_0, z - z_0)$ and one obtains the *normal* equation of the plane π containing the point $P_0(x_0, y_0, z_0)$ and of normal vector $\overline{n}(a, b, c)$.

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0. \tag{1}$$

Remark: The equation (1) can be written in the form $ax + by + cz + d = 0$.

Theorem

Given $a, b, c, d \in \mathbb{R}$, with $a^2 + b^2 + c^2 > 0$, the equation

$$ax + by + cz + d = 0 \qquad (2)$$

describes a plane in $\mathcal{E}_3$. This plane has $\overline{n}(a, b, c)$ as a normal vector.

Sketch: Let $P_0(x_0, y_0, z_0) \in \underbrace{\{(x,y,z) \in \mathbb{R}^3 : ax+by+cz+d=0\}}_{\pi}$.

Then $P(x,y,z) \in \pi \Leftrightarrow$

$ax+by+cz+d = 0 \Leftrightarrow$

$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0 \Leftrightarrow$

$\overline{n} \cdot \overline{P_0P} = 0$. We saw that such a set π is a plane with normal vector $\overline{n}$. $\boxtimes$

The analytic equation of the plane determined by a point and two nonparallel directions

$A(x_A, y_A, z_A) \in \pi$

$\overline{v}_i(p_i, q_i, r_i) \parallel \pi$ are l. independent, $\forall i = \overline{1,2}$.

$P(x, y, z) \in \pi \Longleftrightarrow \overline{AP}, \overline{v}_1, \overline{v}_2$ are l. dependent.

$\overline{AP} = \alpha . \overline{v}_1 + \beta . \overline{v}_2$ for some $\alpha, \beta \in \mathbb{R}$.

$$\pi : \quad \begin{vmatrix} x - x_A & y - y_A & z - z_A \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} = 0, \tag{3}$$

The analytic equation of the plane determined by three noncollinear points

$A(x_A, y_A, z_A)$, $B(\dots,\dots,\dots)$, $C(\dots,\dots,\dots) \in \pi$.

Idea is to use $A(x_A, y_A, z_A) \in \pi$ and $\overline{AB}(x_B - x_A, y_B - y_A, z_B - z_A)$

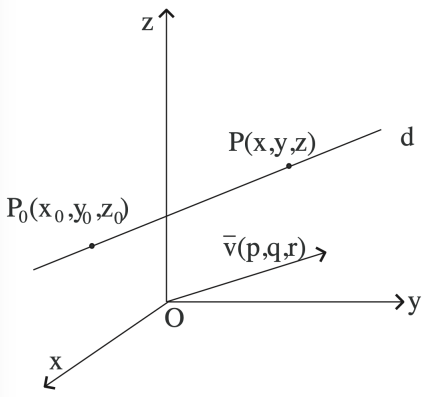
$\overline{AC}(x_C - x_A, y_C - y_A, z_C - z_A)$

$$\pi: \quad \begin{vmatrix} x & y & z & 1 \\ x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \end{vmatrix} = 0, \tag{4}$$

$$\pi: \begin{vmatrix} x - x_A & y - y_A & z - z_A \\ x_B - x_A & y_B - y_A & z_B - z_A \\ x_C - x_A & y_C - y_A & z_C - z_A \end{vmatrix} = 0$$

The line in Euclidean $3D$ space

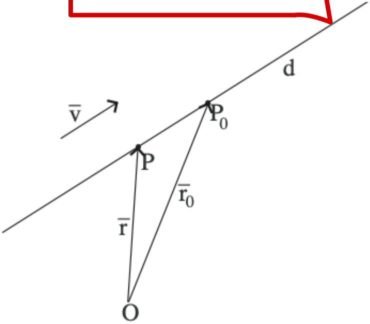
- As in the $2D$-space, a line d in the $3D$-space is completely determined by a point $P_0(x_0, y_0, z_0)$ on the line and a nonzero vector $\overline{v}(p, q, r)$, parallel to d.



The vector equation of a line $\vartheta(\ldots, \ldots, \ldots, 0, \ldots)$

Suppose we fix an origin $O \in \mathcal{E}_3$. Saying that the point P belongs to the line d is equivalent to saying that the vectors $\overline{P_0P}$ and $\overline{v}$ are linearly dependent. The vector $\overline{P_0P}$ can be expressed as the difference $\overline{OP} - \overline{OP_0}$. The vectors $\overline{P_0P}$ and $\overline{v}$ are linearly dependent if and only if there exists $t \in \mathbb{R}$ such that $\overline{P_0P} = t \cdot \overline{v} \iff \overline{OP} - \overline{OP_0} = t \cdot \overline{v}$.

$$\boxed{\overline{OP} = \overline{OP_0} + t\overline{v}.} \tag{5}$$



The parametric equations of a line

- If $P(x, y, z)$ is an arbitrary point on the line d, then the vectors $\overline{P_0P}$ and $\overline{v}$ are linearly dependent in V_3 and there exists $t \in \mathbb{R}$, such that

$$\overline{P_0P} = t\overline{v}. \qquad , \overline{v}(p, q, r) \tag{6}$$

- Since $\overline{P_0P}(x - x_0, y - y_0, z - z_0)$, by decomposing (6) in components, one obtains the *parametric* equations of the line passing through $P_0(x_0, y_0, z_0)$ and parallel to $\overline{v}(p, q, r)$:

$$\frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

$$\begin{cases} x = x_0 + pt \\ y = y_0 + qt \\ z = z_0 + rt \end{cases}, \qquad t \in \mathbb{R}. \tag{7}$$

- The vector $\overline{v}(p, q, r)$ is called the *director* vector of the line d.

An example

$$\begin{cases} x = 2 + 2t \\ y = 3 + 100t \\ z = 7 \end{cases}, \quad t \in \mathbb{R}.$$

$$\overline{v}(2, 100, 0)$$

The symmetric equations

Suppose that $\overline{v}(p, q, r)$ is such that $p, q, r \in \mathbb{R}^*$.

- Expressing t three times in (7), one obtains the *symmetric* equations of the line d:

$$\frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}. \tag{8}$$

Remark: The director vector $\overline{v}$ is a nonzero vector, i.e. at least one of its components is different from zero. As in the 2-dimensional case, if $p = 0$, for instance, that means that $x = x_0$. Does this equation alone describe a line in space?

The equations of a line determined by two points

- A line d can be determined by two different points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ which belong to the line. In this case, a director vector for d can be taken as

$$\overline{P_1P_2}(x_2 - x_1, y_2 - y_1, z_2 - z_1).$$

In the particular case in which $x_2 \neq x_1$, $y_2 \neq y_1$ and $z_2 \neq z_1$, one can write the equations of this line in the following way

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}. \tag{9}$$

$$\begin{cases} \dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} \\ z = z_1 \end{cases}$$

$$\boxed{z_2 = z_1}$$

The lines as intersection of two planes

- Given two distinct and nonparallel planes $\pi_1 : A_1x + B_1y + C_1z + D_1 = 0$ and $\pi_2 : A_2x + B_2y + C_2z + D_2 = 0$ (the planes π_1 and π_2 are parallel when their normal vectors $\overline{n}_1(A_1, B_1, C_1)$ and $\overline{n}_2(A_2, B_2, C_2)$ are parallel, i.e. the rank of the matrix $\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix}$ is 1), they have an entire line d in common.
- Then, a line in 3-space can be determined as the intersection of two nonparallel planes:

$$d : \begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases} \tag{10}$$

with

$$\operatorname{rank} \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 2.$$

The relative positions of two lines

- Let d_1 and d_2 be two lines in $\mathcal{E}_3$, of director vectors $\overline{v}_1(p_1, q_1, r_1) \neq \overline{0}$, respectively $\overline{v}_2(p_2, q_2, r_2) \neq \overline{0}$. The parametric equations of these lines are

$$d_1 : \begin{cases} x = x_1 + p_1 t \\ y = y_1 + q_1 t \\ z = z_1 + r_1 t \end{cases}, \quad t \in \mathbb{R};$$

and

$$d_2 : \begin{cases} x = x_2 + p_2 s \\ y = y_2 + q_2 s \\ z = z_2 + r_2 s \end{cases}, \quad s \in \mathbb{R}.$$

- The set of the intersection points of d_1 and d_2 is given by the set of the solutions (t, s) of the system of equations

$$\begin{cases} x_1 + p_1 t = x_2 + p_2 s \\ y_1 + q_1 t = y_2 + q_2 s \\ z_1 + r_1 t = z_2 + r_2 s \end{cases}. \tag{11}$$

$$\underbrace{\begin{pmatrix} p_1 & -p_2 \\ q_1 & -q_2 \\ r_1 & -r_2 \end{pmatrix}}_{A} \begin{pmatrix} t \\ s \end{pmatrix} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix}$$

$$\bar{A} := \left(A \,\middle|\, \begin{matrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{matrix} \right)$$

$$\text{rank}(A) = \text{rank}(\bar{A})$$

- The set of the intersection points of d_1 and d_2 is given by the set of the solutions (t, s) of the system of equations

$$\begin{cases} x_1 + p_1 t = x_2 + p_2 s \\ y_1 + q_1 t = y_2 + q_2 s \\ z_1 + r_1 t = z_2 + r_2 s \end{cases} . \tag{11}$$

$$\begin{pmatrix} p_1 & -p_2 \\ q_1 & -q_2 \\ r_1 & -r_2 \end{pmatrix} \begin{pmatrix} t \\ s \end{pmatrix} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix}$$

- If $\mathrm{rank} A = \mathrm{rank} \overline{A} = 2$, then the system (11) has a unique solution.
- If $\mathrm{rank} A = \mathrm{rank} \overline{A} = 1$, then the system (11) has infinitely many solutions.
- If $1 = \mathrm{rank} A < \mathrm{rank} \overline{A} = 2$, then the system (11) does not have solutions.
- If $2 = \mathrm{rank} A < \mathrm{rank} \overline{A} = 3$, then the system (11) does not have solutions.

- If the system (11) has a unique solution (t_0, s_0), then the lines d_1 and d_2 have exactly one intersection point P_0, corresponding to t_0 (or s_0). One says that the lines are *concurrent* (or *incident*); $\{P_0\} = d_1 \cap d_2$.
- The vectors $\overline{v}_1$ and $\overline{v}_2$ are in that case linearly independent.

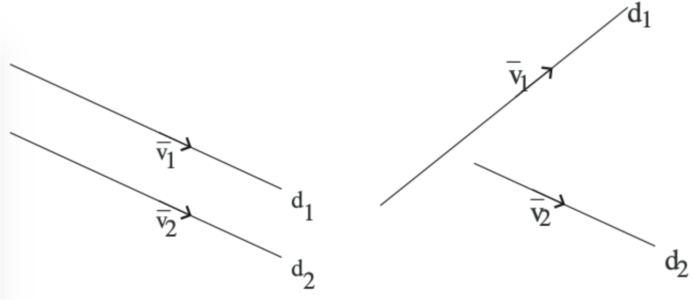
- If the system (11) has infinitely many solutions, then the two lines have infinitely many points in common, so they coincide. We say that these lines are *identical*; $d_1 = d_2$. There exists $\alpha \in \mathbb{R}^*$ such that $\overline{v}_1 = \alpha \overline{v}_2$ (their director vectors are linearly dependent) and any arbitrary point of d_1 belongs to d_2 (and vice-versa).

Suppose the system

$$\begin{cases} x_1 + p_1 t = x_2 + p_2 s \\ y_1 + q_1 t = y_2 + q_2 s \\ z_1 + r_1 t = z_2 + r_2 s \end{cases} \tag{12}$$

is not compatible.

- If $\mathrm{rank} A = 1 < 2 = \mathrm{rank} \overline{A}$, the vectors $\overline{v}_1(p_1, q_1, r_1)$ and $\overline{v}_2(p_2, q_2, r_2)$ are linearly dependent. In this case, the lines are *parallel*; $d_1 \parallel d_2$.
- If $\mathrm{rank} A = 2 < 3 = \mathrm{rank} \overline{A}$, then $\overline{v}_1(p_1, q_1, r_1)$ and $\overline{v}_2(p_2, q_2, r_2)$ are linearly independent. One deals with *skew* lines (nonparallel and nonincident); $d_1 \cap d_2 = \emptyset$ and $d_1 \nparallel d_2$.


$\bar{v}_1$
$\bar{v}_2$
d_1
d_2
$\bar{v}_1$
d_1
$\bar{v}_2$
d_2

Relative position of two planes

- Let

$$\pi_1 : a_1x + b_1y + c_1z + d_1 = 0, \qquad \overline{n}_1(a_1, b_1, c_1) \neq \overline{0}$$

and

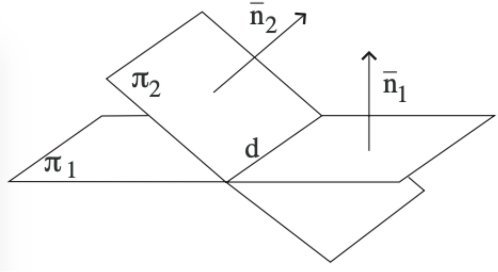
$$\pi_2 : a_2x + b_2y + c_2z + d_2 = 0, \qquad \overline{n}_2(a_2, b_2, c_2) \neq \overline{0}$$

be two planes, having the normal vectors $\overline{n}_1$, respectively $\overline{n}_2$.

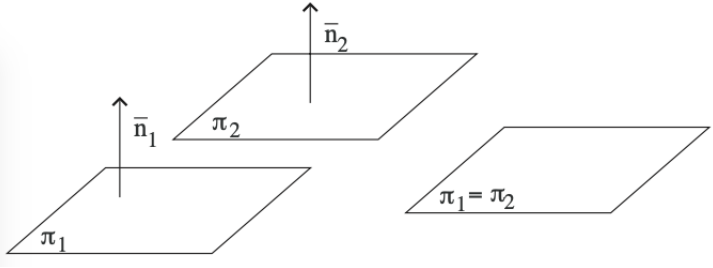
- The intersection of these planes is given by the solution of the system of equations

$$\begin{cases} \pi_1 : a_1x + b_1y + c_1z + d_1 = 0 \\ \pi_2 : a_2x + b_2y + c_2z + d_2 = 0 \end{cases} . \tag{13}$$

$$\bar{A} = \left(\begin{array}{ccc|c} a_1 & b_1 & c_1 & -d_1 \\ a_2 & b_2 & c_2 & -d_2 \end{array}\right)$$


$\bar{n}_2$
π_2
$\bar{n}_1$
d
π_1

- If $\operatorname{rank}\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} = 2$, then the system (13) is compatible and the planes have a line in common. They are *incident*; $\pi_1 \cap \pi_2 = d$.
- If $\operatorname{rank}\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} = 1$, then the rows of the matrix are linearly dependent, which means that the normal vectors of the planes are linearly dependent. There are two possible situations:


$\bar{n}_2$
$\bar{n}_1$
π_2
π_1
$\pi_1 = \pi_2$

- If $\mathrm{rank}(A) = 1 < \mathrm{rank}(\overline{A}) = 2$, then the system (13) is not compatible, and the planes are *parallel*; $\pi_1 \parallel \pi_2$.
- If $\mathrm{rank}(A) = \mathrm{rank}(\overline{A}) = 1$, then the planes are *identical*; $\pi_1 = \pi_2$.

The problem set for this week has already been posted!

Thank you very much for your attention!