

Seminar 10 and 11 - 2024

1. A program returns a value according to a random variable X with $E(X) = m \in \mathbb{R}$ and $V(X) = \sigma^2$, $\sigma > 0$. Prove that X takes values in the interval $(m - 3\sigma, m + 3\sigma)$ with more than 88% probability.

A: By Chebyshev's inequality, $P(|X - E(X)| \geq 3\sigma) \leq \frac{1}{9\sigma^2}V(X)$, so $1 - P(|X - m| < 3\sigma) \leq \frac{1}{9}$. Hence, $P(-3\sigma < X - m < 3\sigma) \geq \frac{8}{9} = 0.\bar{8} > 0.88$.

2. The number of items produced in a factory during a day is a random variable with mean 50. If we consider a day, which event is more likely: E_1 : "the production is more than 100 items in this day" or E_2 : "the production is at most 100 items in this day"?

A: First, note that $P(E_1) = P(X > 100) = P(X \geq 101)$, while $P(E_2) = P(X \leq 100)$. Since $X = |X|$, by Markov's inequality, $P(E_1) = P(X \geq 101) \leq \frac{1}{101}E(X) = \frac{50}{101} < \frac{1}{2}$. Since $E_2 = \bar{E}_1$, we get $P(E_2) > \frac{1}{2}$, hence E_2 is more likely.

3. Let $(X_n)_{n \in \mathbb{N}^*}$ be a sequence of independent random variables with $Unif[a, b]$ distribution, where $a < b$. Define for each $n \in \mathbb{N}^*$

$$Y_n = \max\{X_1, \dots, X_n\} \quad \text{and} \quad Z_n = \min\{X_1, \dots, X_n\}.$$

Prove that $Y_n \xrightarrow{P} b$ and $Z_n \xrightarrow{P} a$.

A: From the definition of the uniform distribution, for each $n \in \mathbb{N}^*$, the distribution function of X_n is given by

$$F_{X_n}(x) = P(X_n \leq x) = \begin{cases} 0, & \text{if } x < a \\ \frac{x-a}{b-a}, & \text{if } a \leq x < b \\ 1, & \text{if } b \leq x. \end{cases}$$

Then, by the independence of the random variables in $(X_n)_{n \in \mathbb{N}^*}$, we have for every $x \in \mathbb{R}$

$$P(Y_n \leq x) = P(\max\{X_1, \dots, X_n\} \leq x) = P\left(\bigcap_{k=1}^n \{X_k \leq x\}\right) = \left(F_{X_1}(x)\right)^n.$$

For $x = b$ we get $P(Y_n \leq b) = 1$. Moreover, we have

$$\begin{aligned} P(|Y_n - b| > \varepsilon) &= P(b - Y_n > \varepsilon) = P(Y_n < b - \varepsilon) \\ &= \begin{cases} \left(1 - \frac{\varepsilon}{b-a}\right)^n, & \text{if } 0 < \varepsilon \leq b-a \\ 0, & \text{if } b-a < \varepsilon. \end{cases} \end{aligned}$$

Therefore, $\lim_{n \rightarrow \infty} P(|Y_n - b| > \varepsilon) = 0$ for each $\varepsilon > 0$. Observe that

$$Z_n = \min\{X_1, \dots, X_n\} = -\max\{-X_1, \dots, -X_n\}$$

with $-X_n \sim Unif[-b, -a]$ for each $n \in \mathbb{N}^*$. We apply the above result for the sequence $(-X_n)_{n \in \mathbb{N}^*}$ and we obtain $-Z_n \xrightarrow{P} -a$. It follows that $Z_n \xrightarrow{P} a$.

4. Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of Bernoulli random variables. Prove that $X_n \xrightarrow{P} 0$ if and only if $X_n \xrightarrow{L^2} 0$.
A: Let $p_n \in (0, 1)$ be such that $X_n \sim \text{Bernoulli}(p_n)$, $n \in \mathbb{N}$. From Example 32 in the course we have $X_n \xrightarrow{P} 0 \iff \lim_{n \rightarrow \infty} p_n = 0$. Since $E(X_n^2) = 1^2 \cdot p_n + 0^2 \cdot (1 - p_n) = p_n$, $X_n \xrightarrow{L^2} 0 \iff \lim_{n \rightarrow \infty} p_n = 0$.

5. Let $\lambda > 0$. A calling center has the following property, for every $n \in \mathbb{N}$, $n \geq 100$, during an hour interval $(0, 1]$: the calls arrive independently with at most one call in each time subinterval $(\frac{i}{n}, \frac{i+1}{n}]$, one call has probability $\frac{\lambda}{n}$ to occur, $i = 0, n-1$. Let's denote by X_n the corresponding total number of calls. Prove that $X_n \xrightarrow{d} X$, where $X \sim \text{Poiss}(\lambda)$.

A: Let $k \in \mathbb{N}$. For every $n \in \mathbb{N}$, $n \geq 100$, $n \geq k$, we have $X_n \sim \text{Bino}(n, \frac{\lambda}{n})$ and thus

$$P(X_n = k) = C_n^k \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} = \frac{\left(1 - \frac{\lambda}{n}\right)^n \lambda^k \frac{n}{k!} \frac{n-1}{n} \cdots \frac{n-k+1}{n}}{\left(1 - \frac{\lambda}{n}\right)^k}.$$

Hence, $\lim_{n \rightarrow \infty} P(X_n = k) = \frac{\lambda^k}{k!} \lim_{n \rightarrow \infty} \left(1 - \frac{\lambda}{n}\right)^n = \frac{\lambda^k}{k!} e^{-\lambda}$. Let F be the cumulative distribution function of X . For $x \in \mathbb{R}$, $F_{X_n}(x) = \sum_{k \leq x} P(X_n = k) \rightarrow F(x) = \sum_{k \leq x} \frac{\lambda^k}{k!} e^{-\lambda}$, as $n \rightarrow \infty$. So, $X_n \xrightarrow{d} X$.

6. Let $(X_n)_{n \in \mathbb{N}^*}$ be a sequence of independent random variables with $\text{Unif}[0, 1]$ distribution. Define for each $n \in \mathbb{N}^*$

$$Y_n = \max\{X_1, \dots, X_n\} \quad \text{and} \quad Z_n = \min\{X_1, \dots, X_n\}.$$

Prove that $Y_n \xrightarrow{L^2} 1$ and $Z_n \xrightarrow{L^2} 0$.

A: Let $F(x) = \begin{cases} 0, & x < 0 \\ x, & x \in [0, 1] \\ 1, & x > 1 \end{cases}$ be the cumulative distribution function of $\text{Unif}[0, 1]$.

As in the solution of problem 3,

$$F_{Y_n}(x) = P(Y_n \leq x) = P\left(\bigcap_{k=1}^n X_k \leq x\right) = (F(x))^n, \quad x \in \mathbb{R}.$$

So, for $n \in \mathbb{N}^*$, $f_{Y_n}(x) = \begin{cases} nx^{n-1}, & x \in [0, 1] \\ 0, & x \notin [0, 1] \end{cases}$ is a density function for Y_n . Hence, $E((Y_n - 1)^2) =$

$E(Y_n^2) - 2E(Y_n) + 1 = \int_0^1 nx^{n+1}dx - 2 \int_0^1 nx^n dx + 1 = \frac{n}{n+2} - \frac{2n}{n+1} + 1 \rightarrow 0$, $n \rightarrow \infty$, and thus $Y_n \xrightarrow{L^2} 1$.
Next,

$$F_{Z_n}(x) = P(Z_n \leq x) = 1 - P\left(\bigcap_{k=1}^n X_k > x\right) = 1 - (1 - F(x))^n, \quad x \in \mathbb{R}.$$

Hence, for $n \in \mathbb{N}^*$, $f_{Z_n}(x) = \begin{cases} n(1-x)^{n-1}, & x \in [0, 1] \\ 0, & x \notin [0, 1] \end{cases}$ is a density function for Z_n . Therefore, $E(Z_n^2) =$

$\int_0^1 nx^2(1-x)^{n-1}dx = n \left(-x^2 \frac{(1-x)^n}{n} \Big|_0^1 - \frac{2}{n} x \frac{(1-x)^{n+1}}{n+1} \Big|_0^1 - \frac{2}{n(n+1)} \frac{(1-x)^{n+2}}{n+2} \Big|_0^1 \right) = \frac{2}{(n+1)(n+2)} \rightarrow 0$, $n \rightarrow \infty$, and thus $Z_n \xrightarrow{L^2} 0$.

7. Consider a sequence of distinct coins such that the probability of getting a head with the n th coin is $\frac{1}{n}$, $n \in \mathbb{N}^*$. Let X_n be 1, if the toss of the n th coin shows a head, and 0, otherwise. Do we have $X_n \xrightarrow{a.s.} 0$?

A: Let's compute $P(\{\omega \in \Omega : \lim_{n \rightarrow \infty} X_n(\omega) = 0\})$.

For every $\omega \in \Omega$, since $X_n(\omega) \in \{0, 1\}$, $\lim_{n \rightarrow \infty} X_n(\omega) = 0$ if and only if $\exists m \in \mathbb{N}^*$ such that $X_n(\omega) = 0, \forall n \geq m$.

For $m \in \mathbb{N}^*$, let $A_m = \{\omega \in \Omega : X_n(\omega) = 0, \forall n \geq m\}$. So,

$$P(\{\omega \in \Omega : \lim_{n \rightarrow \infty} X_n(\omega) = 0\}) = P\left(\bigcup_{m=1}^{\infty} A_m\right) = \lim_{m \rightarrow \infty} P(A_m),$$

in view of Theorem 4 from the course, because $(A_m)_{m \in \mathbb{N}^*}$ is an increasing sequence of events.

Next, let $m \in \mathbb{N}^*$. Let $B_k = \bigcap_{n=m}^k (X_n = 0)$, $k \geq m$. Since $(B_k)_{k \geq m}$ is a decreasing sequence of events, by Theorem 4 from the course,

$$\begin{aligned} P(A_m) &= P\left(\bigcap_{n=m}^{\infty} (X_n = 0)\right) = \lim_{k \rightarrow \infty} P(B_k) = \lim_{k \rightarrow \infty} P\left(\bigcap_{n=m}^k (X_n = 0)\right) \\ &= \lim_{k \rightarrow \infty} P(X_m = 0, X_{m+1} = 0, \dots, X_k = 0) \\ &= \lim_{k \rightarrow \infty} P(X_m = 0)P(X_{m+1} = 0) \dots P(X_k = 0) \\ &= \lim_{k \rightarrow \infty} \frac{m-1}{m} \frac{m}{m+1} \dots \frac{k-1}{k} = \lim_{k \rightarrow \infty} \frac{m-1}{k} = 0, \end{aligned}$$

where we use the independence of the random variables. Hence, $P(\{\omega \in \Omega : \lim_{n \rightarrow \infty} X_n(\omega) = 0\}) = 0$ and thus we don't have $X_n \xrightarrow{a.s.} 0$.