# Seminar 1

## Combinatorics

**1.** *The basic principle of counting:* If $m$ experiments that are to be performed are such that the first one may result in any of $n_1$ possible outcomes, and if, for each of these $n_1$ possible outcomes, there are $n_2$ possible outcomes of the second experiment, and if, for each of the possible outcomes of the first two experiments, there are $n_3$ possible outcomes of the third experiment, and if..., then there is a total of $n_1 \cdot n_2 \cdot \ldots \cdot n_m$ possible outcomes of the $m$ experiments.
*Example:* In how many ways can a person choose to wear 3 shirts, 2 jeans and 4 pairs of shoes?
A: $3 \cdot 2 \cdot 4 = 24$.

**2.** *Arrangements of $n$ objects taken $k$ ($n \geq k$)*: selections of $k$ **distinct** and **ordered** items from $n$ distinct given objects.

$$\begin{aligned} A_n^k &= \text{number of arrangements of } n \text{ objects taken } k \\ &= n \cdot (n-1) \cdot \ldots \cdot (n-k+1) = \frac{n!}{(n-k)!} \end{aligned}$$

*Example:* How many codes with 3 distinct digits can be generated with the digits: 0, 1, 2, 3, 4?
A: $A_5^3 = \frac{5!}{2!} = 60$.

**3.** *Permutations of $n$ objects*: arrangements of $n$ objects taken $n$.

$$P_n = \text{number of permutations of } n \text{ objects} = A_n^n = n!$$

*Remark:* By convention, $0! = 1$.
*Example:* In how many ways can 4 persons sit on a bench? A: $P_4 = 4!$

**4.** *Combinations of $n$ objects taken $k$ ($n \geq k$)*: selections of $k$ **distinct** and **not ordered** items from $n$ distinct given objects, i.e., subsets with $k$ elements of a set with $n$ elements.

$$\begin{aligned} C_n^k &= \text{number of combinations of } n \text{ objects taken } k \\ &= \frac{A_n^k}{k!} = \frac{n!}{k!(n-k)!} \end{aligned}$$

*Remark:* $C_n^k = C_n^{n-k}$.
*Example:* How many football teams can be formed out of a group of 15 persons? A: $C_{15}^{11} = C_{15}^4 = 1365$.

**5.** *Number of functions* from a set $A$ with $k$ elements to a set $B$ with $n$ elements: $n^k$ ($k, n \in \mathbb{N}^*$).
*Remark:* A function can be identified with selections of $k$ items that are **not** necessarily distinct (i.e., an item can be chosen multiple times), but **ordered**, from $n$ distinct objects. Hence, we can say that the *functions* are *arrangements with repetitions.*
*Example:* How many codes with 3 digits can be generated with the digits: 0, 1, 2, 3, 4? A: $5^3 = 125$.

**6.** *Permutations with repetitions:* Consider $n$ objects sucht that $n_1$ are alike, $n_2$ are alike,...,$n_m$ are alike ($n, m \in \mathbb{N}^*, m \leq n; n_1, \ldots, n_m \in \mathbb{N}, n_1 + \ldots + n_m = n$). The number of distinct permutations of these $n$ objects is

$$\frac{n!}{n_1! \cdot n_2! \cdot \ldots \cdot n_m!}$$

*Example:* In how many ways can we order 3 white marbles, 4 black marbles and 2 gray marbles on a line?
Example: ● ○ ● ○ ● ○ ● ● ● A: $\frac{9!}{3!4!2!} = 1260$.
*Example:* In how many ways can we order 3 white marbles and 4 black marbles on a line?
Example: ○ ● ○ ○ ● ● ● A: $\frac{7!}{3!4!} = 35$.
*Remark:* In the above example we get the same value as $C_7^3 = C_7^4$, because the number of permutations with repetitions for $m = 2$ types of objects is the same as the number of ways to choose the distinct and not ordered positions (i.e., combinations), in each permutation, for the objects of the first type (the remaining positions are automatically left for the second type objects).

## Problems

**1.** How many different 7-place license plates are possible in Romania, given that the digit groups 00 and 000 are not allowed, $Q$ cannot be used at all and all 3-letter groups starting with $O$ or $I$ are forbidden? Examples of 7-place license plates in Romania: CJ-01-YYR, IF-12-XYZ, B-101-ABC.

A: There are 41 counties in Romania, so for them there are $41 \cdot 99 \cdot 23 \cdot 25 \cdot 25 = 58\,348\,125$ license plates possible. For Bucharest there are $999 \cdot 23 \cdot 25 \cdot 25 = 14\,360\,625$ (7-place) license plates possible. Total: $58\,348\,125 + 14\,360\,625 = 72\,708\,750$.

**2.** In how many ways can we put 5 maths books, 3 programming books and 4 novels on a shelf, given that each book has a different author and

a) the books of the same type are grouped together?

b) only the novels are necessarily grouped together?

A: a) First we put together the books of the same type in 3 tomes, then we permute the tomes on the shelf. So, the answer is $5!3!4!3! = 103\,680$.

b) First we put together the novels in a tome, then we permute the remaining books and this tome on the shelf. So, the answer is $4!(5+3+1)! = 8\,709\,120$.

**3.** X goes to the theater with 10 girls and 9 boys. In how many ways can they occupy an entire row such that X seats between two girls?

A: $18 \cdot A_{10}^2 \cdot 17! = 576\,213\,633\,515\,520\,000$, where 18 is the number of ways to choose the seat for $X$ ($X$ can't occupy the first seat or the last seat), $A_{10}^2$ is number of ways to choose the girls that seat near $X$ (the order is important, because one girl seats to the left and the other one to the right of $X$) and 17! is the number of ways to choose (i.e., permute) the seats for the remaining persons.

**4.** a) How many 5-place codes contain exactly 3 decimal digits and 2 capital latin letters? Examples: $01DD0$, $A21Z0$, $999XX$.

b) How many 5-place codes contain exactly 3 distinct decimal digits and 2 distinct capital latin letters? Example: $A21Z0$.

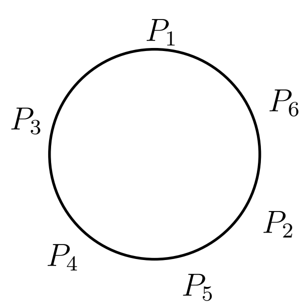
A: a) We first choose the positions for the digits in $C_5^3$ ways. The number of ways to put the digits on the chosen positions is the number of functions from these positions to the set of digits: $10^3$. Similarly, the number of ways to put the letters on the remaining positions is $26^2$. So, the answer is $C_5^2 \cdot 10^3 \cdot 26^2 = 6\,760\,000$.

b) Reasoning in a similar way, but by considering arrangements instead of functions, we get: $C_5^2 \cdot A_{10}^3 \cdot A_{26}^2 = 4\,680\,000$.

**5.** How many anagrams does the word MISSISSIPPI have?

A: Number of permutations with repetitions: $\frac{11!}{1!4!4!2!} = 34650$.

**6.** In how many ways can 6 persons sit at a round table? Example:




A: There are 6! linear permutations of the persons. Any circular permutation of the persons can be identified with 6 unique distinct linear permutations. So, *the number of linear permutations is 6 times the number of circular permutations.* Hence, the answer is $\frac{6!}{6} = 5! = 120$.

**7.** a) In how many distinct ways can the characters: A, A, A, B, B, 0, 0, 0, 1 be ordered on a line? Example: 1 B A 0 A 0 0 B A.

b) In how many distinct ways can the characters: A, A, A, B, B, 0, 0, 0, 1 be ordered on a circle? Example:

A
0 0
A 1
A 0
B B

A: a) $\frac{9!}{3!2!3!1!}$=5040 permutations with repetitions of the characters, if we order the characters linearly.
b) Any circular order of the characters can be identified with 9 unique distinct linear orders (they are distinct because each one has 1 on a different position). So, *the number of linear orders is 9 times the number of circular orders*. Hence, the answer is $\frac{1}{9} \cdot \frac{9!}{3!2!3!1!} = 560$.