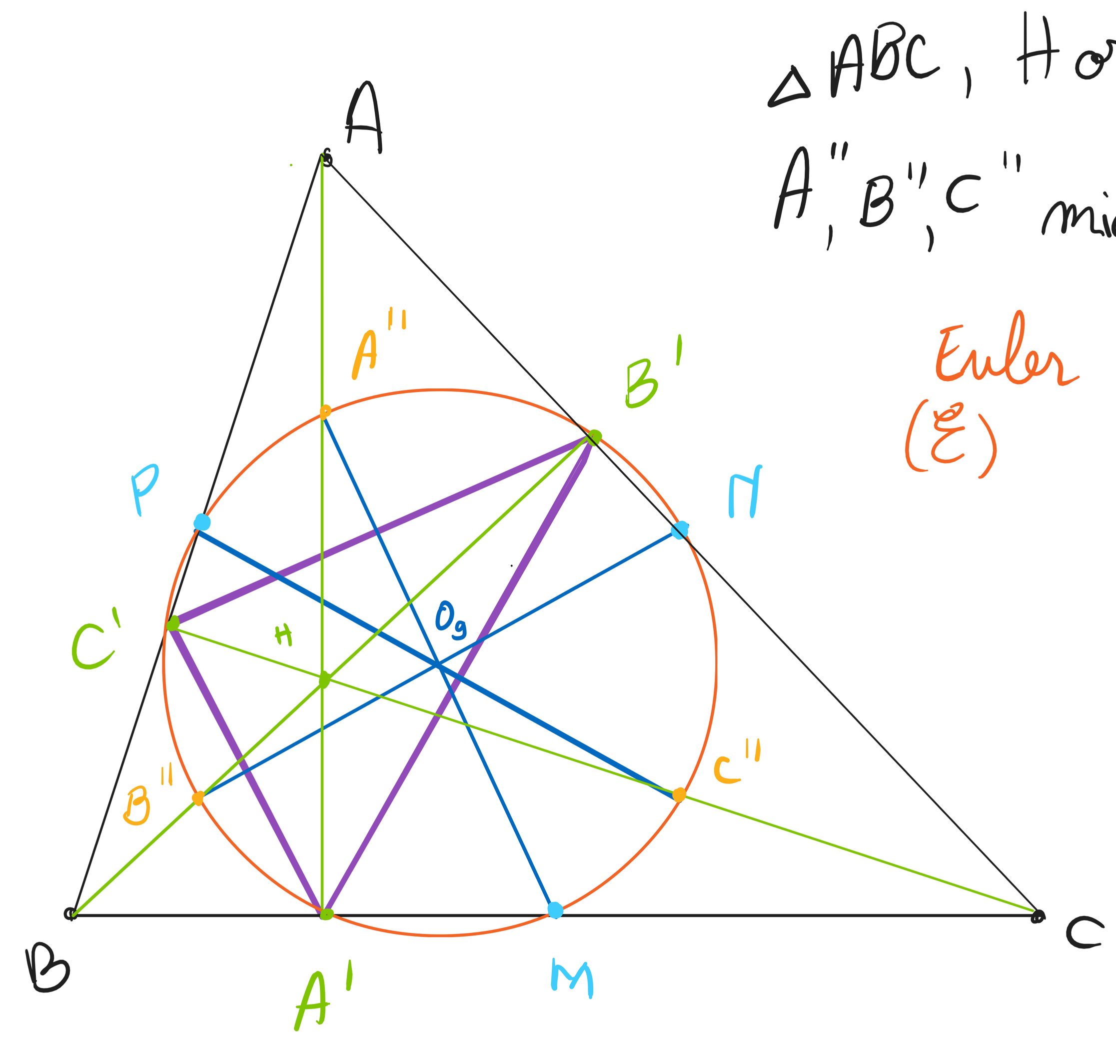


$\triangle ABC$, H orthocenter,

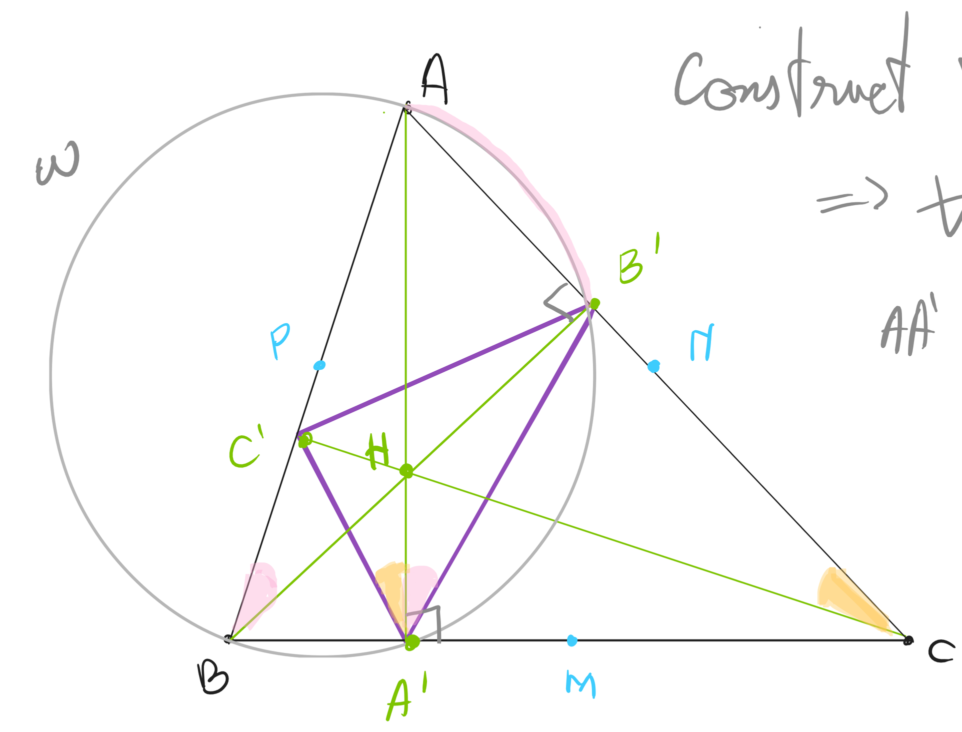
A'', B'', C'' midpts of AH, BH, CH resp

Euler circle $(\mathcal{E})$ (passes trough: → $A''M$, $B''N$, $C''P$ diameters

- A'', B'', C''
- M, N, P midpts of BC, AC, AB resp
- A', B', C' feet of the altitudes from A, B, C resp)

Orthic triangle $\triangle A'B'C'$

We need to prove that $A''M$, $B''N$, $C''P$ are the perpendicular bisectors of $\triangle A'B'C'$



Construct the circle ω of diameter $AB \Rightarrow$ center P

$\Rightarrow \forall$ point $X \in \omega$, $\widehat{AXB} = 90°$

AA', BB' heights of $\triangle ABC \Rightarrow \widehat{AA'B} = 90°$, $\widehat{AB'B} = 90°$ $\Big\} \Rightarrow$

$\Rightarrow A', B' \in \omega$

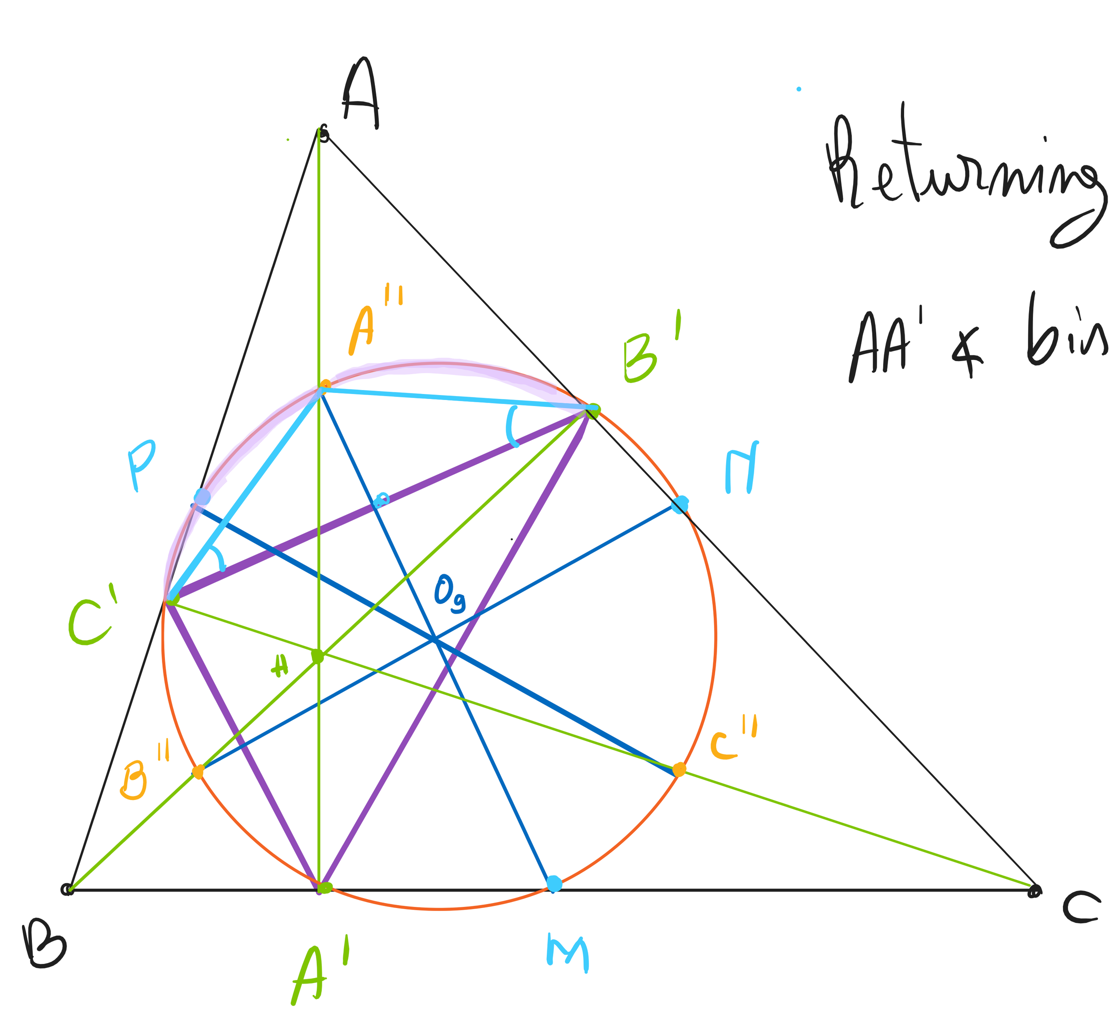
$\widehat{ABB'} = \widehat{AA'B}$ (they corresp. to the same arc $\overset{\frown}{AB'}$ of ω)

Similarly, with a circle of diameter AC, we obtain $\widehat{CA'A} = \widehat{C'CA}$

in $\Delta BC'H$ and $CB'H$: $\widehat{C'HB} = \widehat{B'HC}$ (opposite angles), $\widehat{HC'B} = \widehat{HB'C} = 90°$ $\Big\} \Rightarrow \underbrace{\widehat{C'BH}}_{=\widehat{ABB'}} = \underbrace{\widehat{HCB'}}_{=\widehat{C'CA}}$

$\Rightarrow \widehat{C'A'A} = \widehat{AA'B'} \Rightarrow$ in $\triangle A'B'C'$ AA' ∡ bisector of $\widehat{C'A'B'}$

Similarly, BB' ∡ bis of $\widehat{A'B'C'}$ and CC' ∡ bis of $\widehat{B'C'A'}$



Returning to the original figure:

AA' ∡ bis of $\widehat{C'A'B'} \Rightarrow$ the arcs of $\mathcal{E}$ that corresp to $\underbrace{\widehat{C'A'A}}_{=\widehat{C'A'A''}}$ and $\underbrace{\widehat{B'A'A}}_{=\widehat{B'A'A''}}$ are equal $\Rightarrow \overset{\frown}{C'A''} = \overset{\frown}{B'A''} \Rightarrow$

$\Delta A''C'B'$ is isosceles $\Rightarrow A'' \in$ perpendicular bisector of $C'B'$

$A', B', C' \in \mathcal{E} \Rightarrow \mathcal{E}$ is the circumcircle of $\triangle A'B'C' \Rightarrow$

$\Rightarrow O_9$ (the center of $\mathcal{E}$) is the $\cap$ of $\triangle A'B'C'$'s perp. bis $\Big\} \Rightarrow$

$\Rightarrow$ the diameter $A''M$ of $\mathcal{E}$ is the perpendicular bis. of $C'B'$

Similarly: $\begin{cases} B''N \text{ perp. bis. of } A'C' \\ C''P \text{ perp. bis. of } A'B' \end{cases}$