Analytic Geometry

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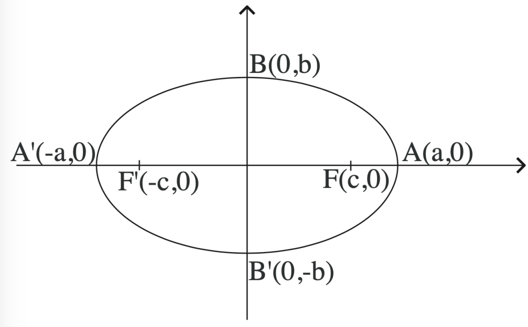
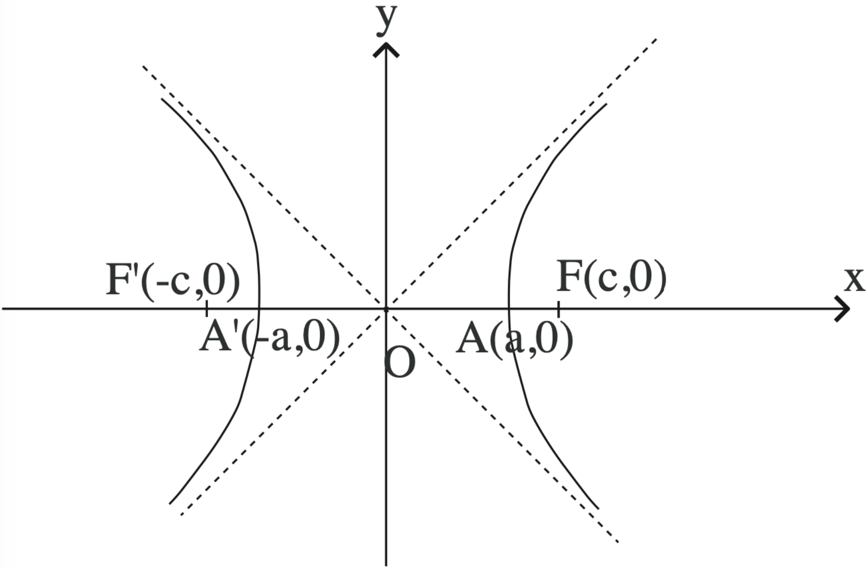
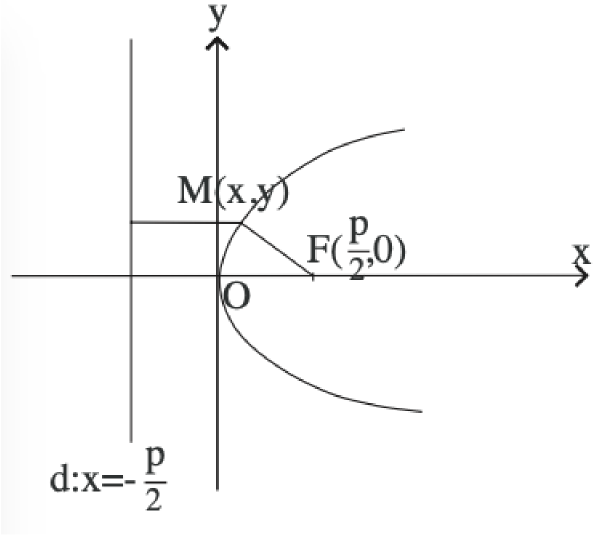
Recap... Conic sections

- We saw the geometric definitions of the circle, ellipse, hyperbola and parabola.
- By choosing convenient systems of coordinates, we derived "canonical equations" for each of these conic sections.




y
F'(-c,0)
F(c,0)
x
A'(-a,0)
O
A(a,0)


y
M(x,y)
F(p/2,0)
x
O
d:x=- p/2

- Let us fix a coordinate system. and an arbitrary conic section.
- One can find a rotation $\mathcal{R}$ and a translation $\mathcal{T}$ such that the image of the conic under $\mathcal{T} \circ \mathcal{R}$ is conic section (of the same type) in the canonical form.

An example

Let us find the locus of points $M(x, y) \in \mathcal{E}_2$ that satisfy the equation

$$3x^2 - 12x + 3y^2 - 6y - 12 = 0.$$

The general equation of a conic

Suppose we are given $\mathcal{E}_2$ and xOy a fixed system of coordinates associated to it. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial function, defined through

$$f(x,y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{10}x + 2a_{20}y + a_{00}, \tag{1}$$

with $a_{11}^2 + a_{12}^2 + a_{22}^2 \neq 0$.

The set $\Gamma = \{P(x,y) \in \mathcal{E}_2 : f(x,y) = 0\}$ is called *algebraic curve* of degree 2, or a *curve* in $\mathcal{E}_2$.

- We shall prove that, after making some changes of coordinates (given, in fact, by a rotation and, eventually, a translation of the original system), the equation $f(x, y) = 0$, i.e.

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{10}x + 2a_{20}y + a_{00} = 0$$

turns into an equation which can be identified to be of one of the conics already studied.

- To the curve $\Gamma \in \mathcal{E}_2$ defined by $f(x, y) = 0$, we associate the real numbers

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{10} \\ a_{12} & a_{22} & a_{20} \\ a_{10} & a_{20} & a_{00} \end{vmatrix}; \qquad \delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix}; \qquad I = a_{11} + a_{22}. \tag{2}$$

The effect of a rotation and a translation

If, in the original system of coordinates, a point P is of coordinates $P(x, y)$, then, after a rotation (around the origin O) and a translation by $\overline{v} = (x_0, y_0)^T$, the coordinates of its image $P'(x', y')$ are

$$\begin{cases} x' = ax - \varepsilon by + x_0 \\ y' = bx + \varepsilon ay + y_0 \end{cases}, \tag{3}$$

where $a^2 + b^2 = 1$ and $\varepsilon = \pm 1$.

Suppose that the point P belongs to the conic Γ, i.e. x, y satisfy

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{10}x + 2a_{20}y + a_{00} = 0.$$

Expressing x and y in terms of x' and y' from (3) and replacing them in the equation of the curve, one obtains

$$f'(x', y') = 0,$$

where f' is some polynomial of degree 2, in variables x' and y'. The latter defines (in the new coordinate system x', y') another conic Γ'.

It can be verified that the numbers Δ', δ' and I', associated to f', coincide with Δ, δ and, respectively, I.

The numbers Δ, δ and I are called the *metric invariants* of the conic Γ.

Centers of symmetry of a conic

Whenever we're given a conic Γ defined by $f(x, y) = 0$, where

$$f(x, y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{10}x + 2a_{20}y + a_{00},$$

understanding the centers of symmetry of Γ is very important.

Theorem

The point $M_0(x_0, y_0)$ is the center of symmetry of the conic Γ if and only if (x_0, y_0) is a critical point of the function f.

Proof

Let Γ be given by the zeros of $f : \mathbb{R}^2 \to \mathbb{R}$, as above. After a translation

$$\begin{cases} x = x_0 + x' \\ y = y_0 + y' \end{cases},$$

the equation $f(x, y) = 0$ becomes

$$a_{11}x'^2 + 2a_{12}x'y' + a_{22}y'^2 + f'_x(x_0, y_0)x' + f'_y(x_0, y_0)y' + f(x_0, y_0) = 0. \quad (4)$$

Suppose that the point M_0 is the center of symmetry of the conic Γ. Then, the origin $(0, 0)$ is the center of symmetry in the new system. Hence, both $P(x', y')$ and $s_O(P)(-x', -y')$ belong to the conic, when $P \in \Gamma$.

Continuation of the proof

Thus,

$$\begin{cases} a_{11}x'^2 + 2a_{12}x'y' + a_{22}y'^2 + f'_x(x_0, y_0)x' + f'_y(x_0, y_0)y' + f(x_0, y_0) = 0 \\ a_{11}x'^2 + 2a_{12}x'y' + a_{22}y'^2 - f'_x(x_0, y_0)x' - f'_y(x_0, y_0)y' + f(x_0, y_0) = 0 \end{cases}$$

and $f'_x(x_0, y_0)x' + f'_y(x_0, y_0)y' = 0$, for any $P(x', y') \in \Gamma$. Then,

$$\begin{cases} f'_x(x_0, y_0) = 0 \\ f'_y(x_0, y_0) = 0 \end{cases},$$

and $(x_0, y_0) \in C(f)$.
Conversely, if $(x_0, y_0) \in C(f)$, then $f'_x(x_0, y_0) = 0$ and $f'_y(x_0, y_0) = 0$, the equation (4) becomes $a_{11}x'^2 + 2a_{12}x'y' + a_{22}y'^2 + f(x_0, y_0) = 0$ and, obviously, $O(0, 0)$ is center of symmetry, i.e. $M_0(x_0, y_0)$ is the center of symmetry for Γ.

$\square$

Classification algorithm

The centers of symmetry of a conic can give an algorithm to classify the conics. The critical points of f are given by the solutions of the system of equations

$$\begin{cases} f'_x(x_0, y_0) = 0 \\ f'_y(x_0, y_0) = 0 \end{cases} \iff \begin{cases} a_{11}x_0 + a_{12}y_0 + a_{10} = 0 \\ a_{12}x_0 + a_{22}y_0 + a_{20} = 0 \end{cases},$$

with $\delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix}$. Let $r =$rank $\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$ be the rank of the coefficient matrix and $r' =$ rank $\begin{pmatrix} a_{11} & a_{12} & a_{10} \\ a_{12} & a_{22} & a_{20} \end{pmatrix}$, the rank of the extended matrix of the system.

Classification algorithm

- If $\delta \neq 0$, then $r = r' = 2$ and Γ has a unique center of symmetry. The conics with this property are: the circle, the ellipse, the hyperbola, a pair of concurrent lines, a point, the empty set. The equation of Γ becomes

$$a_{11}x'^2 + 2a_{12}x'y' + a_{22}y'^2 + \frac{\Delta}{\delta} = 0. \tag{5}$$

- If $\delta = 0$ and $\Delta \neq 0$, then $r = 1$, $r' = 2$ and Γ has no center of symmetry. The parabola has this property.
- If $\delta = 0$ and $\Delta = 0$, then $r = r' = 1$ and Γ has an entire line of centers of symmetry. The conics with this property are: two parallel lines, two identical lines, the empty set.

The sign of δ

The conics can be further classified using the sign of $\delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix}$.

- If $\delta > 0$, the conic is said to be of *elliptical genus*;
- If $\delta < 0$, then the conic is of *hyperbolical genus*;
- If $\delta = 0$, then the conic is of *parabolical genus*.

A remark

A particular case of conics is obtained if $\delta \neq 0$ and, in

$$a_{11}x'^2 + 2a_{12}x'y' + a_{22}y'^2 + \frac{\Delta}{\delta} = 0,$$

we have $a_{12} = 0$ and $a_{11} = a_{22} = a \neq 0$. Then, the equation becomes

$$ax'^2 + ay'^2 + \frac{\Delta}{\delta} = 0 \Longleftrightarrow x'^2 + y'^2 = -\frac{\Delta}{a\delta}.$$

- If $-\dfrac{\Delta}{a\delta} < 0$, then $\Gamma = \emptyset$;
- If $-\dfrac{\Delta}{a\delta} = 0$, then $\Gamma = \{M_0(x_0, y_0)\}$; the conic is degenerated to one point: its center of symmetry;
- If $-\dfrac{\Delta}{a\delta} > 0$, then Γ is the circle centered at M_0 and of radius $\sqrt{-\dfrac{\Delta}{a\delta}}$.

Methods of graphical representation

Let Γ be the conic defined through $f(x, y) = 0$, namely

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{10}x + 2a_{20}y + a_{00} = 0$$

and Δ, δ and I be the invariants of Γ, given by (2). We saw that, after a translation, the equation of Γ has been reduced to

$$a_{11}x'^2 + 2a_{12}x'y' + a_{22}y'^2 + \frac{\Delta}{\delta} = 0.$$

- If $a_{12} = 0$, then one only makes the translation, and Γ has the form

$$a_{11}x'^2 + a_{22}y'^2 + \frac{\Delta}{\delta} = 0.$$

- If $a_{12} \neq 0$, then, before making the translation, one makes a rotation (which will cancel the a_{12}) and, after, the translation.

Finding the rotation. The eigenvalues method

Take the matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$, whose determinant is δ. The eigenvalues of this matrix are the solutions of its characteristic equation:

$$\det\,(A - \lambda I_2) = 0 \Leftrightarrow \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{12} & a_{22-\lambda} \end{vmatrix} \Leftrightarrow \lambda^2 - I\lambda + \delta = 0. \qquad (6)$$

The discriminant of the last equation is

$$I^2 - 4\delta = (a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}^2) = (a_{11} - a_{22})^2 + 4a_{12}^2 > 0, (\ a_{12} \neq 0),$$

so that the equation (6) has two real and distinct solutions, λ_1 and λ_2.

Let $\overline{a}_1(u_1, v_1)$ and $\overline{a}_2(u_2, v_2)$ be the eigenvectors associated to the eigenvalues λ_1 respectively λ_2. Their components are, thus, given by the solutions of the systems

$$\begin{cases} (a_{11} - \lambda_1)u_1 + a_{12}v_1 = 0 \\ a_{12}u_1 + (a_{22} - \lambda_1)v_1 = 0 \end{cases}, \quad \text{respectively} \quad \begin{cases} (a_{11} - \lambda_2)u_2 + a_{12}v_2 = 0 \\ a_{12}u_2 + (a_{22} - \lambda_2)v_2 = 0 \end{cases}$$

Choose representatives such that $|\overline{a}_1| = |\overline{a_2}| = 1$ and such that the 2×2 matrix

$$R = (\overline{a}_1 | \overline{a_2})$$

has determinant $\det R = 1$.
Now write

$$\begin{pmatrix} x \\ y \end{pmatrix} = R \cdot \begin{pmatrix} x' \\ y' \end{pmatrix}.$$

We can now write the equation of the conic as

$$\lambda_1 x'^2 + \lambda_2 y'^2 + 2a'_{10}x' + 2a'_{20}y' + a'_{00} = 0,$$

which is equivalent to

$$\lambda_1 \left(x' + \frac{a'_{10}}{\lambda_1} \right)^2 + \lambda_2 \left(y' + \frac{a'_{20}}{\lambda_2} \right)^2 + a'_{00} - \frac{a'^2_{10}}{\lambda_1} - \frac{a'^2_{20}}{\lambda_2} = 0.$$

After a translation of equations

$$\begin{cases} x'' = x' + \dfrac{a'_{10}}{\lambda_1} \\ y'' = y' + \dfrac{a'_{20}}{\lambda_2} \end{cases},$$

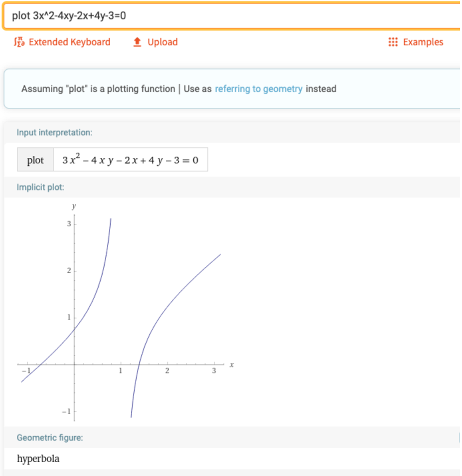
one obtains the canonical equation of the conic

$$\lambda_1 x''^2 + \lambda_2 y''^2 + a = 0,$$

where $a := a'_{00} - \dfrac{a'^2_{10}}{\lambda_1} - \dfrac{a'^2_{20}}{\lambda_2}$.

An example

Let us consider the conic Γ, given by $3x^2 - 4xy - 2x + 4y - 3 = 0$.



The associated invariants of Γ are

$$\Delta = \begin{vmatrix} 3 & -2 & -1 \\ -2 & 0 & 2 \\ -1 & 2 & -3 \end{vmatrix} = 8; \qquad \delta = \begin{vmatrix} 3 & -2 \\ -2 & 0 \end{vmatrix} = -4; \qquad I = 3+0 = 3.$$

Since $\delta \neq 0$, then Γ has a unique center of symmetry. Its coordinates are given by the solution of the system of equations

$$\begin{cases} 6x - 4y - 2 = 0 \\ -4x + 4 = 0 \end{cases},$$

so that the center of symmetry is $C(1,1)$.

Since $\delta < 0$, the conic is of hyperbolical genus.

Let us identify the rotation matrix

The eigenvalues λ_1 and λ_2 are the solutions of the equation $\lambda^2 - 3\lambda - 4 = 0$, hence $\lambda_1 = -1$ and $\lambda_2 = 4$.

Let us determine the eigenvectors associated to λ_1 and λ_2.

$$\lambda_1 = -1 \qquad \begin{cases} 4u_1 - 2v_1 = 0 \\ -2u_1 + v_1 = 0 \end{cases} \Leftrightarrow \overline{a}_1(\alpha, 2\alpha),\ \alpha \in \mathbb{R}^* \ \Rightarrow\ \overline{a}_1\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$

$$\lambda_2 = 4 \qquad \begin{cases} -u_2 - 2v_2 = 0 \\ -2u_2 - 4v_2 = 0 \end{cases} \Leftrightarrow \overline{a}_2(-2\beta, \beta),\ \beta \in \mathbb{R}^* \ \Rightarrow\ \overline{a}_2\left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$$

The matrix of the rotation is given by $R = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$, and det $R = 1$. The equations of the rotation are

$$\begin{pmatrix} x \\ y \end{pmatrix} = R \begin{pmatrix} x' \\ y' \end{pmatrix} \Longleftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \Longleftrightarrow$$

$$\begin{cases} x = \frac{1}{\sqrt{5}}x' - \frac{2}{\sqrt{5}}y' \\ y = \frac{2}{\sqrt{5}}x' + \frac{1}{\sqrt{5}}y' \end{cases}.$$

Let us replace these in the original equation $3x^2 - 4xy - 2x + 4y - 3 = 0$.

Replacing in the equation of the conic, one obtains

$$-x'^2 + 4y'^2 + \frac{6}{\sqrt{5}}x' + \frac{8}{\sqrt{5}}y' - 3 = 0,$$

or

$$-\left(x' - \frac{3}{\sqrt{5}}\right)^2 + 4\left(y' + \frac{1}{\sqrt{5}}y'\right)^2 - 2 = 0.$$

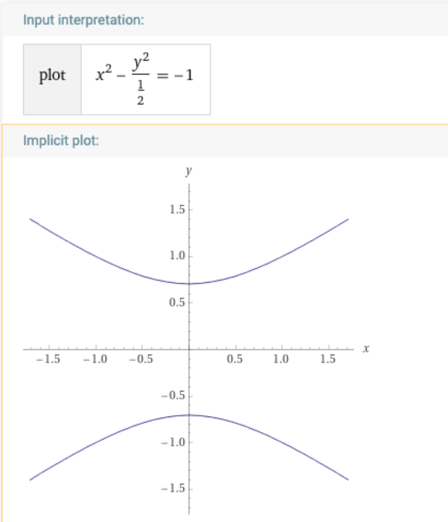
After a translation of equations $\begin{cases} x'' = x' - \dfrac{3}{\sqrt{5}} \\ y'' = y' + \dfrac{1}{\sqrt{5}} \end{cases}$, the conic is of the form $-x''^2 + 4y''^2 - 2 = 0$, so that the canonic equation of the given conic is

$$\frac{x''^2}{2} - \frac{y''^2}{\frac{1}{2}} = -1,$$

and it is a hyperbola.

A final plot

The rotated and translated hyperbola is now



Finding the angle of rotation

One can determine the angle of rotation of the coordinate axes. Let Γ be a conic given associated to

$$f(x, y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{10}x + 2a_{20}y + a_{00},$$

with $a_{12} \neq 0$.

Theorem

The angle θ of the rotation r_θ is given by the equation

$$(a_{11} - a_{22}) \sin 2\theta = 2a_{12} \cos 2\theta. \tag{7}$$

Proof

The matrix R is, actually, the matrix of r_θ, so that

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The equations of the rotation are

$$\begin{pmatrix} x \\ y \end{pmatrix} = R\begin{pmatrix} x' \\ y' \end{pmatrix} \iff \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} \iff \begin{cases} x = x'\cos\theta - y' \\ y = x'\sin\theta + y' \end{cases}$$

Replacing in the equation of the conic, one obtains

$$f(x'\cos\theta - y'\sin\theta, x'\sin\theta + y'\cos\theta) = 0 \Leftrightarrow$$

$$\Leftrightarrow a_{11}(x'\cos\theta - y'\sin\theta)^2 + 2a_{12}(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) +$$

$$+a_{22}(x'\sin\theta + y'\cos\theta)^2 + 2a_{10}(x'\cos\theta - y'\sin\theta) +$$

$$+2a_{20}(x'\sin\theta + y'\cos\theta) + a_{00} = 0.$$

The proof continued

The coefficient of $x'y'$ in this equation is

$$(a_{11} - a_{22}) \sin 2\theta - 2a_{12} \cos 2\theta.$$

But we saw that the effect of the rotation is to cancel this coefficient. Thus

$$(a_{11} - a_{22}) \sin 2\theta - 2a_{12} \cos 2\theta = 0. \quad \square$$

Example

Let us take the conic $x^2 + xy + y^2 - 6x - 16 = 0$. The invariants are

$$\Delta = \begin{vmatrix} 1 & \frac{1}{2} & -3 \\ \frac{1}{2} & 1 & 0 \\ -3 & 0 & -16 \end{vmatrix} = -21; \qquad \delta = \begin{vmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{vmatrix} = \frac{3}{4}; \qquad I = 1 + 1 = 2.$$

Since $\delta \neq 0$ and $\delta > 0$, then the conic has a unique center of symmetry and is of elliptic genus. The coordinates of the center of symmetry are given by

$$\begin{cases} 2x + y - 6 = 0 \\ x + 2y = 0 \end{cases},$$

so that the center of symmetry is $C(4, -2)$.

The angle of rotation is given by

$$(1-1)\sin 2\theta = \cos 2\theta \iff \cos 2\theta = 0 \iff \theta = \frac{\pi}{4}.$$

The eigenvalues λ_1 and λ_2 are the solutions of the equation $\lambda^2 - 2\lambda + \frac{3}{4} = 0$, hence $\lambda_1 = \frac{3}{2}$ and $\lambda_2 = \frac{1}{2}$.

The equations of the rotation are

$$\begin{pmatrix} x \\ y \end{pmatrix} = R \begin{pmatrix} x' \\ y' \end{pmatrix} \iff \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \iff$$

$$\begin{cases} x = x'\dfrac{\sqrt{2}}{2} - y'\dfrac{\sqrt{2}}{2} \\ y = x'\dfrac{\sqrt{2}}{2} + y'\dfrac{\sqrt{2}}{2} \end{cases},$$

and the conic becomes

$$\frac{3}{2}x'^2 + \frac{1}{2}y'^2 - 3\sqrt{2}x' + 3\sqrt{2}y' - 16 = 0,$$

or

$$\frac{3}{2}(x' - \sqrt{2})^2 + \frac{1}{2}(y' + 3\sqrt{2})^2 - 28 = 0.$$

After a translation of equations $\begin{cases} x'' = x' - \sqrt{2} \\ y'' = y' + 3\sqrt{2} \end{cases}$, one obtains the reduced equation of the conic

$$\frac{x''^2}{\frac{56}{3}} + \frac{y''^2}{56} = 1,$$

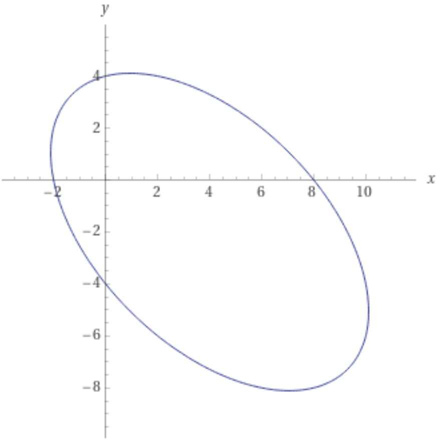
the conic being an ellipse.

Initial ellipse

Input interpretation:

plot	$x^2 + x\,y + y^2 - 6\,x - 16 = 0$

Implicit plot:

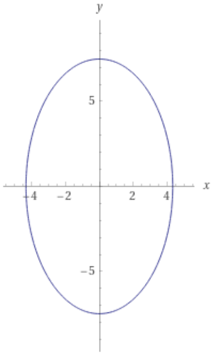


Ellipse after rotation by $\pi/4$ and translation

Input interpretation:

plot	$\frac{x^2}{\frac{56}{3}} + \frac{y^2}{56} = 1$

Implicit plot:



Conclusions

We can put together all the considerations we have made. Let Γ be the conic given by the zeros of the polynomial function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x, y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{10}x + 2a_{20}y + a_{00},$$

with $a_{11}^2 + a_{12}^2 + a_{22}^2 \neq 0$. Let

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{10} \\ a_{12} & a_{22} & a_{20} \\ a_{10} & a_{20} & a_{00} \end{vmatrix}; \qquad \delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix}; \qquad I = a_{11} + a_{22}.$$

be the invariants of Γ.

Conclusions...

<table>
<tr><th colspan="3">Conditions</th><th>The curve</th><th>Transformations</th></tr>
<tr><td rowspan="3">$\Delta = 0$</td><td colspan="2">$\delta > 0$</td><td>$\Gamma = \{(x_0, y_0)\}$</td><td rowspan="7">If $a_{12} = 0$, one makes a translation;

If $a_{12} \neq 0$, one makes a rotation of angle θ, given by $(a_{11} - a_{22}) \sin 2\theta = 2a_{12} \cos 2\theta$, and, eventually, a translation.</td></tr>
<tr><td colspan="2">$\delta = 0$</td><td>two parallel lines, two identical lines, or the empty set</td></tr>
<tr><td colspan="2">$\delta < 0$</td><td>two concurrent lines; if $I = 0$, then the lines are orthogonal.</td></tr>
<tr><td rowspan="4">$\Delta \neq 0$</td><td rowspan="2">$\delta < 0$</td><td>$I\Delta < 0$</td><td>ellipse</td></tr>
<tr><td>$I\Delta > 0$</td><td>$\emptyset$</td></tr>
<tr><td colspan="2">$\delta = 0$</td><td>parabola</td></tr>
<tr><td colspan="2">$\delta < 0$</td><td>hyperbola; if $I = 0$, then the hyperbola is equilateral.</td></tr>
</table>

Thank you very much for your attention!