Problem 2

2) For the helix: $r(t) = (R \cdot \cos t, R \cdot \sin t, \ell \cdot t)$, where $R, \ell > 0$ fixed

Compute T, N, B, k, ρ, τ

T = unit tangent vector

N = principal normal vector

k = curvature

ρ = radius of curvature

τ = torsion

B = binormal vector

Tangent vector – T:

$$\boxed{\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}}$$

$$\vec{r}(t) = (R\cos t, R\sin t, \ell t)$$

$$\vec{r}\,'(t) = (-R\sin t, R\cos t, \ell)$$

$$\|\vec{r}\,'(t)\| = \sqrt{(-R\sin t)^2 + (R\cos t)^2 + \ell^2} = \sqrt{R^2(\sin^2 t + \cos^2 t) + \ell^2} = \sqrt{R^2 + \ell^2}$$

$$\boxed{\vec{T}(t) = \frac{1}{\sqrt{R^2+\ell^2}}(-R\sin t, R\cos t, \ell)}$$

Normal vector – N:

$$\boxed{\vec{N}(t) = \frac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|}}$$

$$\vec{T}\,'(t) = \frac{d}{dt}\left(\frac{-R\sin t}{\sqrt{R^2+\ell^2}}, \frac{R\cos t}{\sqrt{R^2+\ell^2}}, \frac{\ell}{\sqrt{R^2+\ell^2}}\right) = \frac{1}{\sqrt{R^2+\ell^2}}(-R\cos t, -R\sin t, 0)$$

$$\|\vec{T}\,'(t)\| = \frac{1}{\sqrt{R^2+\ell^2}}\sqrt{(-R\cos t)^2 + (-R\sin t)^2} = \frac{1}{\sqrt{R^2+\ell^2}}\sqrt{R^2\cos^2 t + R^2\sin^2 t} =$$

$$= \frac{1}{\sqrt{R^2+\ell^2}}\sqrt{R^2(\cos^2 t + \sin^2 t)} = \frac{R}{\sqrt{R^2+\ell^2}}$$

$$\boxed{\vec{N}(t) = \frac{1}{\cancel{\sqrt{R^2+\ell^2}}}(-R\cos t, -R\sin t, 0) \cdot \frac{\cancel{\sqrt{R^2+\ell^2}}}{R} = \frac{(-R\cos t, -R\sin t, 0)}{R} = (-\cos t, -\sin t, 0)}$$

Binormal vector: $\boxed{\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)}$

$$\vec{T}(t) = \frac{1}{\sqrt{R^2+\ell^2}}(-R\sin t, R\cos t, \ell)$$

$$\vec{N}(t) = (-\cos t, -\sin t, 0)$$

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(T) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{-R\sin t}{\sqrt{R^2+\ell^2}} & \frac{R\cos t}{\sqrt{R^2+\ell^2}} & \frac{\ell}{\sqrt{R^2+\ell^2}} \\ -\cos t & -\sin t & 0 \end{vmatrix} =$$

$$= 0 + \frac{R\sin^2 t}{\sqrt{R^2+\ell^2}}\vec{k} - \frac{\ell\cos t}{\sqrt{R^2+\ell^2}}\vec{j} + \frac{R\cos^2 t}{\sqrt{R^2+\ell^2}}\vec{k} + \frac{\ell\sin t}{\sqrt{R^2+\ell^2}}\vec{i} - 0 =$$

$$= \vec{k}\,\frac{R\sin^2 t + R\cos^2 t}{\sqrt{R^2+\ell^2}} - \frac{\ell\cos t}{\sqrt{R^2+\ell^2}}\vec{j} + \frac{\ell\sin t}{\sqrt{R^2+\ell^2}}\vec{i} =$$

$$= \frac{\ell\sin t}{\sqrt{R^2+\ell^2}}\vec{i} - \frac{\ell\cos t}{\sqrt{R^2+\ell^2}}\vec{j} + \frac{R}{\sqrt{R^2+\ell^2}}\vec{k}$$

$$\boxed{\vec{B}(t) = \left(\frac{\ell\sin t}{\sqrt{R^2+\ell^2}}, -\frac{\ell\cos t}{\sqrt{R^2+\ell^2}}, \frac{R}{\sqrt{R^2+\ell^2}}\right)}$$

Curvature: $\boxed{k = \frac{\|\vec{T}'\|}{\|\vec{r}'\|}}$

Radius of curvature: $\boxed{\rho = \frac{1}{k}}$

$$k = \frac{R}{\sqrt{R^2+\ell^2}} \cdot \frac{1}{\sqrt{R^2+\ell^2}} = \frac{R}{R^2+\ell^2}$$

$$\rho = \frac{1}{\frac{R}{R^2+\ell^2}} = \frac{R^2+\ell^2}{R}$$

$$\boxed{k = \frac{R}{R^2+\ell^2}}$$

$$\boxed{\rho = \frac{R^2+\ell^2}{R}}$$

Torsion: $\boxed{\tau = \dfrac{-\vec{N}(t)\cdot\vec{B}'(t)}{\|\vec{r}'\|}}$ $\quad\vec{N}(t)\cdot\vec{B}'(t)$ – dot product

$$\vec{B}(t)=\frac{1}{\sqrt{R^2+e^2}}\left(e\sin t,-e\cos t,R\right)$$

$$\vec{B}'(t)=\frac{d}{dt}\left(\frac{1}{\sqrt{R^2+e^2}}\left(e\sin t,-e\cos t,R\right)\right)=\frac{1}{\sqrt{R^2+e^2}}\left(e\cos t,e\sin t,0\right)$$

$$\vec{N}(t)=(-\cos t,-\sin t,0)$$

dot product

$$\vec{N}(t)\cdot\vec{B}'(t)=(-\cos t,-\sin t,0)\cdot\frac{1}{\sqrt{R^2+e^2}}\left(e\cos t,e\sin t,0\right)=$$

$$=\frac{1}{\sqrt{R^2+e^2}}\left(-e\cos^2 t,-e\sin^2 t\right)=\frac{1}{\sqrt{R^2+e^2}}\left(-e\left(\cos^2 t+\sin^2 t\right)\right)=\frac{-e}{\sqrt{R^2+e^2}}$$

$$\|\vec{r}'(t)\|=\sqrt{R^2+e^2}$$

$$\tau=-\frac{-e}{\sqrt{R^2+e^2}}\cdot\frac{1}{\sqrt{R^2+e^2}}=\frac{-e}{R^2+e^2}$$

$$\boxed{\tau=\frac{-e}{R^2+e^2}}$$