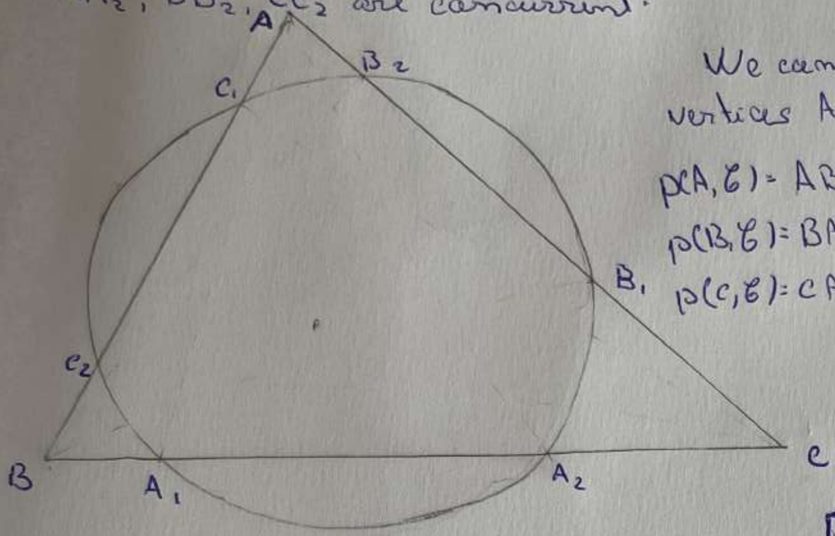
• Let ABC be a triangle and let $\mathcal{C}$ be a circle intersecting the sides BC, CA, AB in the points $A_1, A_2, B_1, B_2, C_1, C_2$.

? Show that the lines AA_1, BB_1, CC_1 are concurrent if and only if AA_2, BB_2, CC_2 are concurrent.



We compute the power of the vertices A, B, C w.r.t the circle $\mathcal{C}$

$$\left.\begin{aligned} p(A,\mathcal{C}) &= AB_1 \cdot AB_2 = AC_1 \cdot AC_2 \\ p(B,\mathcal{C}) &= BA_1 \cdot BA_2 = BC_1 \cdot BC_2 \\ p(C,\mathcal{C}) &= CA_1 \cdot CA_2 = BC_1 \cdot BC_2 \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \begin{cases} \dfrac{C_1A}{B_1A} = \dfrac{B_2A}{C_2A} \\[2ex] \dfrac{A_1B}{C_1B} = \dfrac{C_2B}{A_2B} \\[2ex] \dfrac{B_1C}{A_1C} = \dfrac{A_2C}{B_2C} \end{cases} \quad (\cdot)$$

$$\frac{A_1B}{C_1B} \cdot \frac{B_1C}{A_1C} \cdot \frac{C_1A}{B_1A} = \frac{A_2C}{B_2C} \cdot \frac{B_2A}{C_2A} \cdot \frac{C_2B}{A_2B}$$

We rearrange the terms to get:

$$\frac{A_1B}{A_1C} \cdot \frac{B_1C}{B_1A} \cdot \frac{C_1A}{C_1B} = \frac{A_2C}{A_2B} \cdot \frac{C_2B}{C_2A} \cdot \frac{B_2A}{B_2C}$$

Since AA_2, BB_2, CC_2 are concurrent $\Rightarrow$

$\Rightarrow \dfrac{A_2B}{A_2C} \cdot \dfrac{B_2C}{B_2A} \cdot \dfrac{C_2A}{C_2B} = 1$ (Ceva's th.)

$\Rightarrow \dfrac{A_1C}{A_1B} \cdot \dfrac{C_1B}{C_1A} \cdot \dfrac{B_1A}{B_1C} = 1 \Rightarrow$

$\Rightarrow AA_2, BB_2, CC_2$ are concurrent (reciprocal of Ceva's th.) $\Rightarrow$

$\Rightarrow AA_1, BB_1, CC_1$ are concurrent since products from each side are equal and if one is equal to 1 the other is also equal to 1