General theorems of dynamics

Momentum (quantity of motion)

$$\vec{H} = m \cdot \vec{v} \,; \quad \frac{d\vec{H}}{dt} = \vec{F} \,;$$

First integrals:

a) $\vec{F} = 0 \Rightarrow \vec{H} = \text{constant}.$

b) $\vec{F} \perp \vec{u} \Rightarrow \vec{H} \cdot \vec{u} = \text{constant}.$

Angular momentum (moment of momentum)

$$\vec{K}_O = \vec{r} \times \vec{H} = \vec{r} \times m\vec{v}$$

$$\frac{d\vec{K}_O}{dt} = M_O \vec{F} \text{, where } M_O \vec{F} = \vec{r} \times \vec{F}$$

Work

$\delta L = \vec{F} \cdot d\vec{r}$ (elementary work)

$\vec{F}(X, Y, Z)$, $d\vec{r}(dx, dy, dz)$ $\Rightarrow \delta L = X\,dx + Y\,dy + Z\,dz$

if $\vec{F} = \operatorname{grad} U \Rightarrow \vec{F}$ - is conservative $\Rightarrow \boxed{\delta L = dU}$ (*)

it is necessary that: $\dfrac{\partial X}{\partial y} = \dfrac{\partial Y}{\partial z}$, $\dfrac{\partial X}{\partial z} = \dfrac{\partial Z}{\partial x}$, $\dfrac{\partial Y}{\partial z} = \dfrac{\partial Z}{\partial y}$

U - is the force function

$V = -U$ is the potential function

(*) $\Rightarrow L_{AB} = \int_A^B dU = U(B) - U(A).$

Kinetic energy

$$T = \frac{1}{2} m v^2$$

$\boxed{\delta L = dT}$ - kinetic energy theorem

$\vec{F}$ - conservative $\Rightarrow dT = \delta L = dU \Rightarrow$

$$\Rightarrow T - U = h = T_0 - U_0$$

or

$\boxed{T + V = h}$ - conservation of mechanical energy

① In a force field defined by

$$X = \frac{z^2 - y^2}{(x+z)^2} ; \; Y = \frac{2y}{x+z} ; \; Z = \frac{x^2 - y^2}{(x+z)^2} \qquad (1)$$

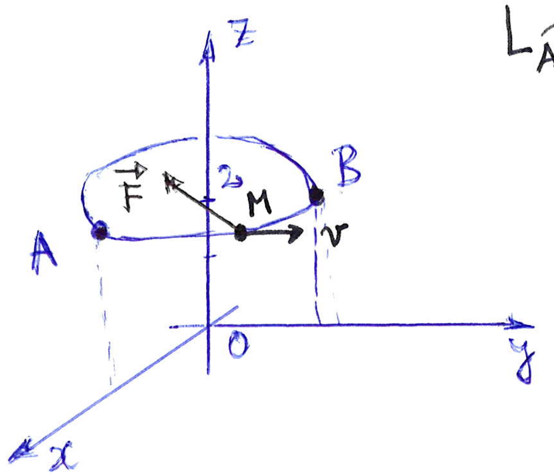
a material point M $(m=1)$ moves on the curve

$$(\mathcal{C}): \begin{cases} x = \cos\theta \\ y = \sin\theta \\ z = 2 . \end{cases} \qquad (2)$$

a) Find the work $L_{\widehat{AB}}$ when M describes the arc between $A(\theta = 0)$ and $B(\theta = \frac{\pi}{2})$.

b) Prove that a force function $U(x, y, z)$ exists and using it calculat $L_{\widehat{AB}}$ from a).

Solution:



$$L_{\widehat{AB}} = \int_{\widehat{AB}} X\,dx + Y\,dy + Z\,dz = \int_{\widehat{AB}} \delta L \qquad (3)$$

We use (2) in (1):

$$X = \frac{4 - \sin^2\theta}{(2+\cos\theta)^2}$$
$$Y = \frac{2\sin\theta}{2+\cos\theta} \qquad (4)$$
$$Z = \frac{\cos^2\theta - \sin^2\theta}{(2+\cos\theta)^2} .$$

On the other hand from (2) we have:

$$dx = -\sin\theta\, d\theta ; \; dy = \cos\theta\, d\theta ; \; dz = 0 \quad (5)$$

Then, using (3), (4) and (5) we obtain:

$$L_{\widehat{AB}} = \int_0^{\frac{\pi}{2}} \left[\frac{4 - \sin^2\theta}{(2+\cos\theta)^2} (-\sin\theta) + \frac{2\sin\theta}{2+\cos\theta} \cdot \cos\theta \right] \cdot d\theta =$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin^3\theta - 4\sin\theta + 2\sin\theta\cos\theta(2+\cos\theta)}{(2+\cos\theta)^2} d\theta =$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin\theta}{(2+\cos\theta)^2} \left[\sin^2\theta - 4 + 4\cos\theta + 2\cos^2\theta \right] d\theta =$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin\theta}{(2+\cos\theta)^2}\left(\cos^2\theta + 4\cos\theta - 3\right) d\theta =$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin\theta}{(2+\cos\theta)^2}\left[(\cos\theta+2)^2 - 7\right] d\theta = \int_0^{\frac{\pi}{2}} \sin\theta \, d\theta - \int_0^{\frac{\pi}{2}} \frac{7\sin\theta}{(2+\cos\theta)^2} d\theta =$$

$$= -\cos\theta \Big|_0^{\frac{\pi}{2}} - 7\int_0^{\frac{\pi}{2}} d\left(\frac{1}{2+\cos\theta}\right) = 1 - 7 \cdot \frac{1}{2+\cos\theta}\Big|_0^{\frac{\pi}{2}} =$$

$$= 1 - \frac{7}{2} + \frac{7}{3} = -\frac{1}{6}.$$

<u>Remark</u>: The work $L_{AB} < 0 \Rightarrow$ the force $\vec{F}$ and the displacement $d\vec{r}$ have "different signs" ($\vec{F} \cdot d\vec{r} < 0$).

b) $\exists\, U(x,y,z)$ such that $\vec{F}(x,y,z) = \operatorname{grad} U = \frac{\partial U}{\partial x}\vec{i} + \frac{\partial U}{\partial y}\vec{j} + \frac{\partial U}{\partial z}\vec{k}$?

Conditions: (of existence) $\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x}$; $\frac{\partial Y}{\partial z} = \frac{\partial Z}{\partial y}$; $\frac{\partial X}{\partial z} = \frac{\partial Z}{\partial x}$.

$$\frac{\partial X}{\partial y} = \frac{-2y}{(x+z)^2} ; \quad \frac{\partial Y}{\partial x} = \frac{-2y}{(x+z)^2} \Rightarrow \boxed{\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x}} \qquad (6)$$

$$\frac{\partial Y}{\partial z} = -\frac{2y}{(x+z)^2} ; \quad \frac{\partial Z}{\partial y} = \frac{-2y}{(x+z)^2} \Rightarrow \boxed{\frac{\partial Y}{\partial z} = \frac{\partial Z}{\partial y}} \qquad (7)$$

$$\frac{\partial X}{\partial z} = \frac{2z(x+z)^2 - 2(z^2-y^2)(x+z)}{(x+z)^4} = \frac{2xz + 2z^2 - 2z^2 + 2y^2}{(x+z)^3} = \frac{2xz + 2y^2}{(x+z)^3}$$

$$\frac{\partial Z}{\partial x} = \frac{2xz + 2y^2}{(x+z)^3} \Rightarrow \boxed{\frac{\partial X}{\partial z} = \frac{\partial Z}{\partial x}} \qquad (8)$$

From (6),(7),(8) $\Rightarrow \exists\, U(x,y,z)$ - force function and

$$\frac{\partial U}{\partial x} = X = \frac{z^2-y^2}{(x+z)^2} ; \quad \frac{\partial U}{\partial y} = Y = \frac{2y}{x+z} ; \quad \frac{\partial U}{\partial z} = \frac{x^2-y^2}{(x+z)^2} = Z \qquad (9)$$

Next, we integrate $(9)_2$:

$$\frac{\partial U}{\partial y} = \frac{2y}{x+z} \Rightarrow U = \frac{y^2}{x+z} + \varphi(x,z) \quad (10)$$

We use (10) in $(9)_1 \Rightarrow -\frac{y^2}{(x+z)^2} + \frac{\partial \varphi}{\partial x} = \frac{z^2-y^2}{(x+z)^2} \Rightarrow \frac{\partial \varphi}{\partial x} = \frac{z^2}{(x+z)^2}$ (11)

and then (10) in $(9)_3 \Rightarrow -\frac{y^2}{(x+z)^2} + \frac{\partial \varphi}{\partial z} = \frac{x^2-y^2}{(x+z)^2} \Rightarrow \frac{\partial \varphi}{\partial z} = \frac{x^2}{(x+z)^2}$ (12)

Integrate (11) $\Rightarrow \varphi(x,z) = \frac{-z^2}{x+z} + \psi(z)$ (13)

In order to find $\psi(z)$ we use (13) in (12):

$$\frac{-2z(x+z)+z^2}{(x+z)^2} + \psi'(z) = \frac{x^2}{(x+z)^2} \Rightarrow$$

$$\Rightarrow \psi'(z) = 1 \Rightarrow \boxed{\psi(z) = z + C} \quad (14)$$

Thus, from (10), (13) and (14) we get:

$$U(x,y,z) = \frac{y^2}{x+z} - \frac{z^2}{x+z} + z + \underset{0}{\underset{\|}{\cancel{C}}} = \frac{y^2+xz}{x+z} \quad (15)$$

Next, $U(\theta) = \frac{\sin^2\theta + 2\cos\theta}{2+\cos\theta}$

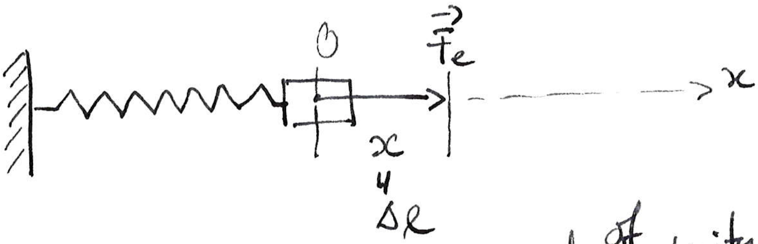
The work can be calculated:

$$L_{\widehat{AB}} = \int_{\widehat{AB}} dU = U(B) - U(A) = \left.\frac{\sin^2\theta + 2\cos\theta}{2+\cos\theta}\right|_0^{\frac{\pi}{2}} =$$

$$= \frac{1}{2} - \frac{2}{3} = -\frac{1}{6}.$$

Harmonic Oscillator

Consider a mass on a spring. Deforming the spring with $x = \Delta l$ an elastic force $\vec{F} = -kx \cdot \vec{i}$ is trying to restore the initial position.



coefficient of elasticity.

$$F_x = -kx, \quad F_y = 0, \quad F_z = 0.$$

The force function is: $U = \int F_x \, dx + C = -\int kx + C =$

$$= -\frac{1}{2} kx^2 + \underset{0}{\underset{\shortparallel}{\cancel{C'}}}.$$

Thus, the potential energy is: $V = -U = \frac{1}{2} kx^2.$

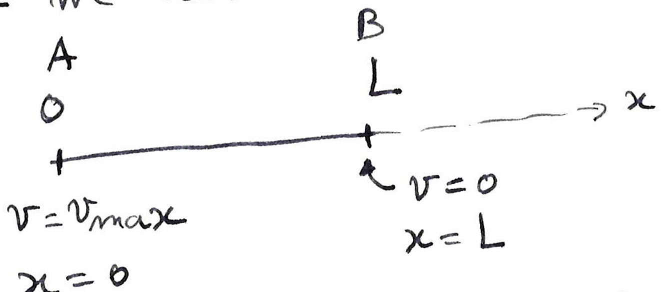
and the work is given by:

$$L_{AB} = U_B - U_A = -\frac{1}{k}\left(x_B^2 - x_A^2\right)$$

The total energy is:

$$T = \frac{1}{2} m v^2 + \frac{1}{2} k x^2.$$

In order to find the maximum velocity for a given elongation l we have:



$$T_A = T_B \Rightarrow \frac{1}{2} m v_{max}^2 + 0 = 0 + \frac{1}{2} k L^2 \Rightarrow$$

$$\Rightarrow v_{max}^2 = \frac{k}{m} L^2 \Rightarrow \underline{v_{max} = \sqrt{\frac{k}{m}}\, L}$$