Dynamics of the relative motion of a particle

Relative motion:

$$\vec{a} = \vec{a}_t + \vec{a}_r + \vec{a}_c$$

transport acceleration — relative acceleration — Coriolis acceleration

Equation of motion:

$$m\cdot\vec{a}_r = \overbrace{\vec{F} - \underbrace{m\cdot\vec{a}_t}_{\vec{F}_t} - \underbrace{m\cdot\vec{a}_c}_{\vec{F}_c}}^{\vec{F}_r\text{ - relative force}}$$

direct applied forces — transport inertial force — centrifugal inertial force

where:

$$\vec{F}_c = -m\cdot\vec{a}_c = -2m\cdot\vec{\omega}\times\vec{v}_r$$

$$\vec{F}_t = -m\cdot\vec{a}_t = -m\left[\vec{a}_0 + \dot{\vec{\omega}}\times\vec{r} + \vec{\omega}\times(\vec{\omega}\times\vec{r})\right]$$

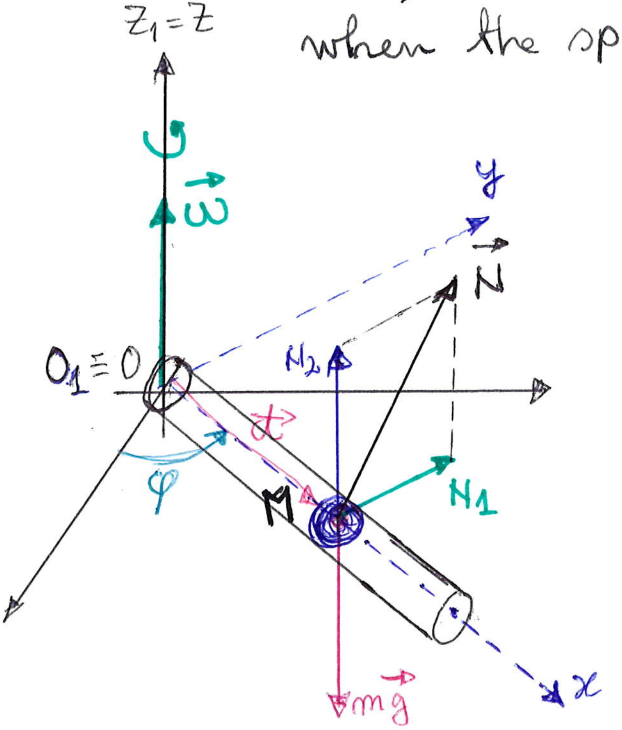
<u>Theorem</u> (of the relative motion) The equation of motion of a particle with respect to an inertial frame of reference maintains its form with respect to a non-inertial one if the given force is replaced by the force relative to the later frame.

$$m\cdot\vec{a}_r = \vec{F}_r$$

① A smooth horizontal cylinder of length $2a$ rotates about a vertical axis passing through the extremity of the cylinder with the angular velocity $\omega = const.$ Inside the cylinder there is a sphere of mass m. At the initial moment ($t=0$) the distance between the sphere and the rotation axis is a, and its velocity relative to the cylinder is zero.

Find: a) The relative equation of motion of the sphere along the cylinder and the reaction $\vec{N}$ of the cylinder upon the sphere.

b) The absolute trajectory, the absolute velocity $\vec{v}_a$, the relative velocity $\vec{v}_r$ and the time when the sphere leaves the cylinder.



a) <u>Remark</u>: The sphere can move inside the tube only in the Ox direction, it means we have two restrictions

$$y = 0, \quad z = 0$$

and thus, we have two normal reactions:

$$\vec{N}_1 = N_1 \cdot \vec{j}, \quad N_2 = N_2 \cdot \vec{k}. \quad (1)$$

The total normal reaction will be:

$$\vec{N} = \vec{N}_1 + \vec{N}_2 = N_1 \cdot \vec{j} + N_2 \cdot \vec{k}. \quad (2)$$

The equation of the relative motion is:

$$m\vec{a}_r = m\vec{g} + \vec{N} + \vec{F}_t + \vec{F}_c \quad (3)$$

where

$$\vec{F}_t = -m \cdot \vec{a}_t = -m\left[\underbrace{\cancel{\vec{a}_0}}_{=0} + (\vec{r} \times \underbrace{\cancel{\dot{\vec{\omega}}}}_{=0}) + \underbrace{\vec{\omega} \times (\vec{\omega} \times \vec{r})}_{-\omega^2 \cdot \vec{r}}\right] =$$

$\boxed{\vec{r} = x \cdot \vec{i}}$

$$= -m(-\omega^2 \cdot x \cdot \vec{i}) = m\omega^2 x \cdot \vec{i}. \quad (4)$$

$$\vec{F}_c = -m\vec{a}_c = -m \cdot 2\vec{\omega} \times \frac{\partial \vec{r}}{\partial t} = -2m\,(\vec{\omega} \times \dot{x}\vec{i}) = -2m\omega \cdot \dot{x}\,(\vec{k}_0 \times \vec{i}) \Rightarrow$$

$$\vec{r} = x \cdot \vec{i} \quad , \quad \vec{v}_r = \frac{\partial \vec{r}}{\partial t} = \dot{x}$$

$$\vec{F}_c = -2m\omega\dot{x}\vec{j} \qquad (5)$$

Next we consider eq. (3) on the Oxyz axis:

$$\begin{cases} Ox: \; m\ddot{x} = m\omega^2 x \\ Oy: \; 0 = N_1 - 2m\omega\dot{x} \\ Oz: \; 0 = N_2 - mg \end{cases} \Rightarrow \begin{cases} x(t) = C_1 e^{\omega t} + C_2 e^{-\omega t} \\ N_1 = 2m\omega\dot{x} \\ N_2 = mg \end{cases} \qquad (6)$$

$$\ddot{x} - \omega^2 x = 0 \Rightarrow \rho^2 - \omega^2 = 0 \Rightarrow \rho_{1,2} = \pm\omega$$

At $t=0$ (initial conditions):

$$(7) \quad \begin{cases} x(0) = a \\ \dot{x}(0) = 0 \end{cases} \Rightarrow \begin{cases} C_1 + C_2 = a \\ \omega C_1 - \omega C_2 = 0 \end{cases} \Rightarrow C_1 = C_2 = \frac{a}{2} \Rightarrow$$

$$\Rightarrow \left.\begin{aligned} x(t) &= \frac{a}{2}\left(e^{\omega t} + e^{-\omega t}\right) = a \cdot ch(\omega t) \\ y(t) &= 0 \\ z(t) &= 0 \end{aligned}\right\} (8) \text{ equations of motion (relative).}$$

Next from (6b) and (6c) we have:

$$N_1 = 2m\omega \frac{a}{2}\omega\left(e^{\omega t} - e^{-\omega t}\right) \Rightarrow N_1 = 2m\omega^2 a\, sh(\omega t) \quad (9)$$

$$N_2 = mg$$

$$\Rightarrow \boxed{\vec{N} = 2m\omega^2 a\; sh(\omega t) \cdot \vec{j} + mg\vec{k}} \quad (10)$$

b) The absolute motion of the sphere is composed by a rotation and a translation.

Let be
$$\left.\begin{aligned} \rho &= \rho(t) = x(t) = a\; ch(\omega t) \\ \varphi &= \varphi(t) = \omega \cdot t \end{aligned}\right\} \Rightarrow \boxed{\rho = ch(\varphi)} \quad (11)$$

$$\left(\frac{d\varphi}{dt} = \omega \Rightarrow \varphi = \omega \cdot t + \underset{0}{\underset{\shortparallel}{\varphi_0}}\right)$$

the absolute trajectory of the sphere (spiral).

The absolute velocity:

$$\vec{v}_a = \vec{v}_r + \vec{v}_t = \dot{x}\vec{i} + (\vec{\omega} \times \vec{r}) = \dot{x}\vec{i} + (\omega\vec{k} \times x\cdot\vec{i}) = \dot{x}\vec{i} + \omega x\vec{j}$$

$$\Rightarrow \vec{v}_a = a\omega\, sh(\omega t)\cdot\vec{i} + a\omega\, ch(\omega t)\cdot\vec{j} = a\omega\left[sh(\omega t)\cdot\vec{i} + ch(\omega t)\vec{j}\right] \quad (12)$$

$$\dot{x}(t) = a\omega\,\frac{e^{\omega t} - e^{-\omega t}}{2} = a\omega\, sh(\omega t).$$

The relative velocity:

$$\vec{v}_r = \dot{x}\vec{i} = a\omega\, sh(\omega t)\cdot\vec{i}. \quad (13)$$

$$|\vec{v}_a| = a\omega\sqrt{sh^2(\omega t) + ch^2(\omega t)} \underset{ch^2 - sh^2 = 1}{=} a\omega\sqrt{2ch^2(\omega t) - 1}$$

$$|\vec{v}_r| = a\omega\sqrt{ch^2(\omega t) - 1}$$

When the sphere leaves the tube, we have:

$$x(t_e) = 2a \underset{(8)}{=} a\, ch(\omega t_e) \Rightarrow \boxed{ch(\omega t_e) = 2} \quad (14)$$

Thus

$$v_a(t_e) = a\cdot\omega\sqrt{8-1} = a\omega\sqrt{7}$$

$$v_r(t_e) = a\omega\sqrt{4-1} = a\omega\sqrt{3}$$

In order to find the exit time, t_e, we have to solve eq. (14).

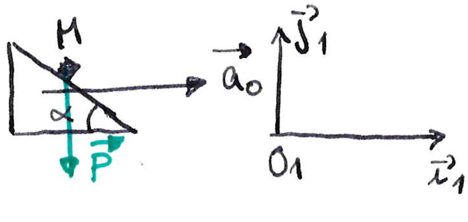
$$\frac{e^{\omega t_e} + e^{-\omega t_e}}{2} = 2 \Rightarrow e^{\omega t_e} + \frac{1}{e^{\omega t_e}} = 4 \Rightarrow u + \frac{1}{u} = 4$$

$$\Rightarrow u^2 - 4u + 1 = 0 \Rightarrow u_{1,2} = 2 \pm \sqrt{3} \Rightarrow e^{\omega t_e} = 2 \pm \sqrt{3} \Rightarrow$$

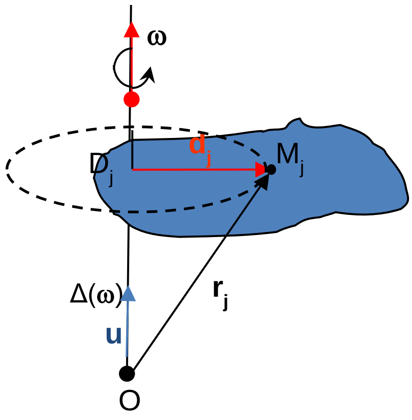
$$\Rightarrow \omega t_e = \ln(2 \pm \sqrt{3}) \Rightarrow \omega t_e = \ln(2+\sqrt{3}) \Rightarrow \boxed{t_e = \frac{1}{\omega}\ln(2+\sqrt{3})} \quad (15)$$

for „–" we obtain a negative time.

(home-work) ② A particle M of weight $\vec{P} = m\vec{g}$ is moving with friction (the friction coefficient is f) on the face of a triangular prism. The prism has a translation motion with the acceleration $\vec{a}_0$ in a fixed system $O_1\vec{i}_1\vec{j}_1$.
Find the relative acceleration of M and the pressure exerted by M on the prism's face. The angle between the prism's face and $O_1\vec{i}_1$ is α.



2) Find the expression of the kinetic energy of a rotating rigid system (rigid body), (S) about a fixed axis $\Delta(O,\vec{u})$ with an angular velocity $\vec{\omega}$, where $\vec{u}$ is the unit vector of the axis Δ.



Consider the rigid discrete system:

$$(S): M_j(m_j), \vec{r}_j = \overrightarrow{OM_j}, j = 1, \dots, N$$

The velocity $\vec{v}_j$ of the point M_j is given by:

$$\vec{v}_j = \vec{\omega} \times \vec{r}_j \qquad (1)$$

Let be $D_j = pr_\Delta M_j$. Thus, we have

$$\vec{r}_j = \overrightarrow{OD_j} + \vec{d}_j \qquad (2)$$

where $\vec{d}_j = \overrightarrow{D_jM_j}$. Using (1) and (2) we get:

$$\vec{v}_j = \vec{\omega} \times (\overrightarrow{OD_j} + \vec{d}_j) \underbrace{=}_{\vec{\omega} \parallel \overrightarrow{OD}_j} \vec{\omega} \times \vec{d}_j => \vec{v}_j = \vec{\omega} \times \vec{d}_j \qquad (3)$$

Now we can calculate the kinetic energy

$$T = \frac{1}{2}\sum_{j=1}^{N} m_j v_j^2 = \frac{1}{2}\sum_{j=1}^{N} m_j(\vec{\omega} \times \vec{d}_j)^2 = \frac{1}{2}\sum_{j=1}^{N} m_j \omega^2 d_j^2 \underbrace{\sin^2(\vec{\omega}, \vec{d}_j)}_{=1 (\vec{\omega} \perp \vec{d}_j)} = \frac{1}{2}\omega^2 \sum_{j=1}^{N} m_j d_j^2 = \frac{1}{2} I(\Delta)\omega^2$$

where

$$I(\Delta) = \sum_{j=1}^{N} m_j d_j^2$$

is the moment of inertia of the system (S) with respect to Δ.

Therefore, the kinetic energy of the rigid body rotating about the axis Δ is given by:

$$\boxed{T = \frac{1}{2} I(\Delta)\omega^2}$$