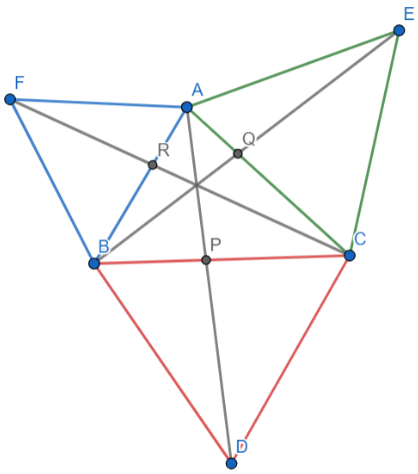
Seminar 2 - Presentation 2

Given a triangle ABC, draw equilateral triangles ABF, BCD, ACE outwards based on AB, BC, AC respectively. Show that AD, BE and CF are concurrent.



Proof. Seeing the word '**concurrent**', and studying the diagram, we suspect that this problem can be proved using *Ceva's theorem.*

Let $AD \cap BC = \{P\}$

$BE \cap AC = \{Q\}$

$CF \cap AB = \{R\}$

We are to show that:

(BP/CP)*(CQ/AQ)*(AR/BR) = 1

We are going to express the above ratios using more familiar terms – AREAS:

We start with the first ratio: BP/CP = [△ABP] / [△ACP]

But BP/CP = [△BDP] / [△CDP]

Noticing that if we were to multiply the ratios expressed as the area of those triangles, we wouldn't get the result that we need. So we try to look in another triangle to see if the ratios match.

Let's look in the triangles formed by those four ones mentioned above:

$\triangle ABP + \triangle BDP = \triangle ABD$

And

$\triangle ACP + \triangle CDP = \triangle ACD$

we notice that : BP/CP=[△ABD] / [△ACD] (1)

(1) Can be further expressed using *the sinus theorem*:
[△ABD] / [△ACD]=[1/2*AB*BD*sin(ABD)] / [1/2*AC*CD*sin(ACD)]

Knowing that △BCD is equilateral, we can simplify BD with CD because BD=CD, and we also rewrite the angle ABD and ACD as the sum of the angles of the initial triangle and 60° from the equilateral triangles.

- ⇨ [△ABD] / [△ACD] = [AB*sin(ABC+60°)] / [AC*sin(ACB+60°)]
- ⇨ BP / CP = [AB*sin(ABC+60°)] / [AC*sin(ACB+60°)]

Now, we expressed the first ratio based on elements of the triangle ABC, which is the common element of these three equilateral triangles.

Similarly, CQ / AQ = [BC*sin(ACB+60°)] / [AB*sin(BAC+60°] and

AR / BR = [AC*sin(BAC+60°)] / [BC*sin(ABC+60°)]

If we multiply the ratios again, we will get that

(BP/CP)*(CQ/AQ)*(AR/BR) = 1 and hence by the converse of Ceva's theorem; AD, BE and CF are concurrent.