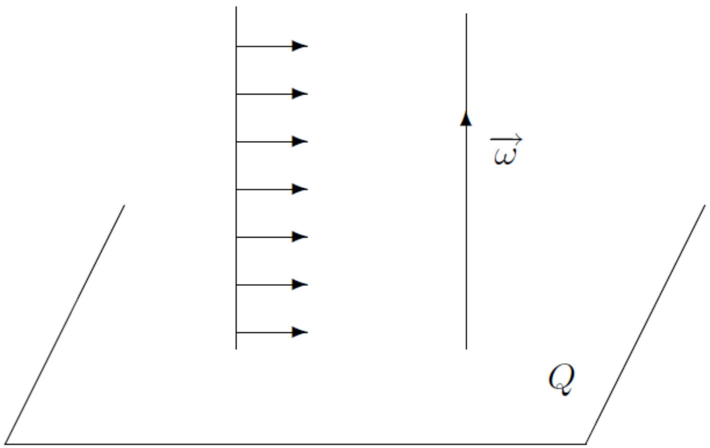
6. The plane parallel motion of the rigid body (Plane kinematics of rigid bodies)

The motion takes place in a Single Plane (Plane Motion) when all parts of the body move in parallel planes.

This is equivalent with:

$\vec{\omega}$ has a fixed direction

$$\vec{v}_0(t) \perp \vec{\omega}(t) \;\Leftrightarrow\; \vec{v}_0(t) \cdot \vec{\omega}(t) = 0 \qquad \forall\, t \in [t_0, T]$$



The rigid body points on a line parallel with $\vec{\omega}$ have the same velocity. Thus, the study is reduced to the study of a plane section of the rigid in its plane.

Theoretical Mechanics

Plane Motion

Translation

No rotation of any line in body. Motion of the body specified by motion of any point in the body ≈Motion of a single particle.

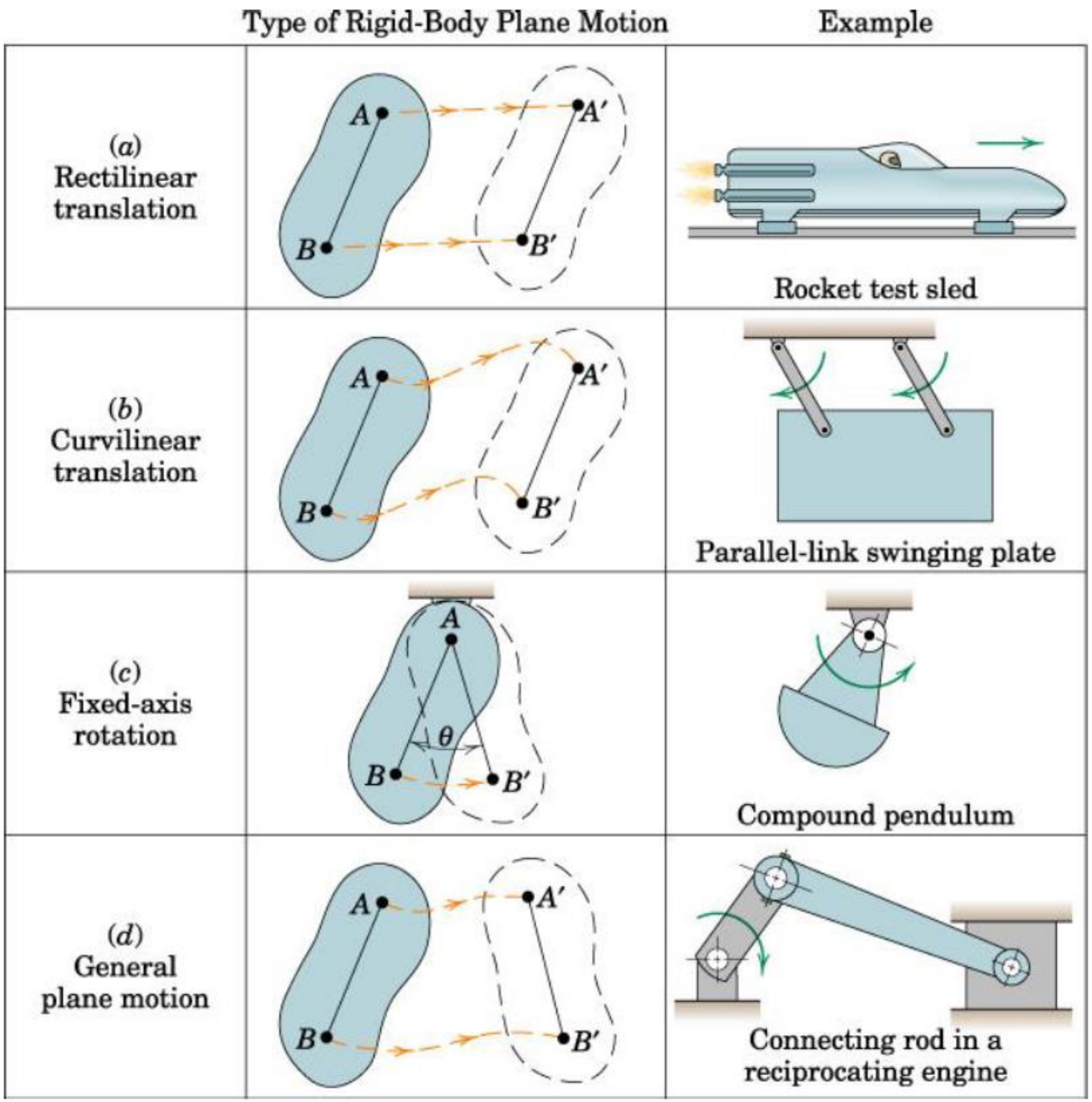
Rotation about a Fixed Axis

All particles move in circular paths about the axis of rotation. All lines perpendicular to the axis of rotationn rotate through the same angle.

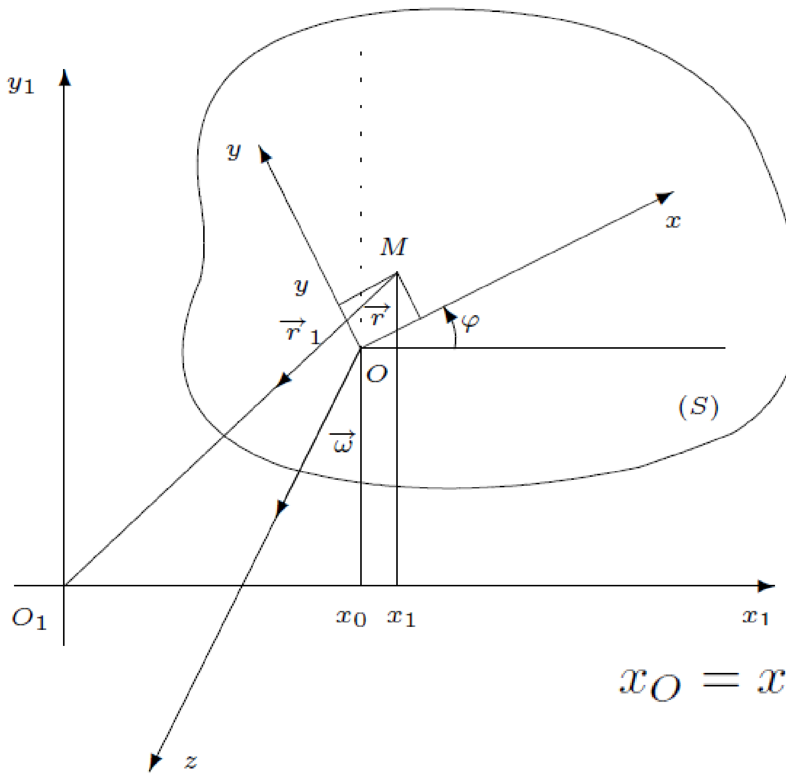
General Planar Motion

Combination of translation and rotation

The actual paths of all particles in the body are projected on a single plane of motion.



Consider the fixed frame $O_1x_1y_1$ and a mobile frame Oxy linked to a section *S* of the rigid body.The plane motion have 3 degree of fredom:



The coordinates of *S* in the fixed frame $O_1x_1y_1z_1$ are:

$$x_O,\ y_O,\ \varphi = (\widehat{Ox, O_1x_1}).$$

and the equation of motion for *S* are given by:

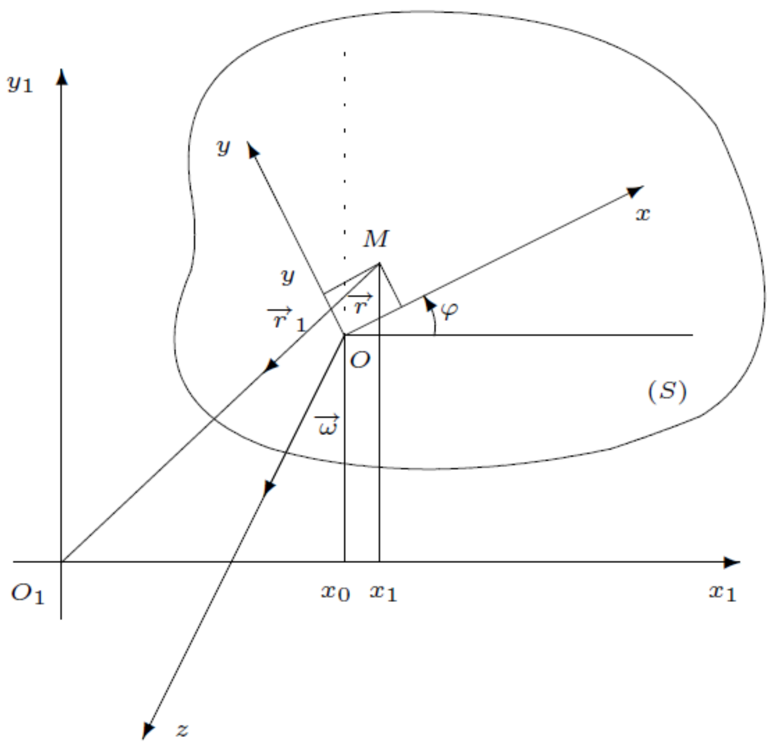
$$x_O = x_O(t),\ y_O = y_O(t),\ \varphi = \varphi(t),\ t \in [t_0, T]. \quad (6.1)$$

Consider $M \in (S);\ \vec{r}_1 = \overrightarrow{O_1M},\ \vec{r} = \overrightarrow{OM}.$

In $O_1x_1y_1$: $M(x_1, y_1)$

In Oxy : $M(x, y)$

Then, $\vec{r}_1 = \vec{r}_0 + \vec{r}$



$$\begin{cases} x_1 = x_O + x\cos\varphi - y\sin\varphi \\ y_1 = y_O + x\sin\varphi + y\cos\varphi. \end{cases} \qquad (6.2)$$

Taking into account the velocity

$$\vec{v}_M = \vec{v}_0 + \vec{\omega} \times \vec{r} \qquad (6.3)$$

we deduce that the motion of section *S* is a instantaneous roto - translation. It means that the finite motion of the rigid body between t_A and t_B is a succesion of instantanous roto - translations.

Moreover,

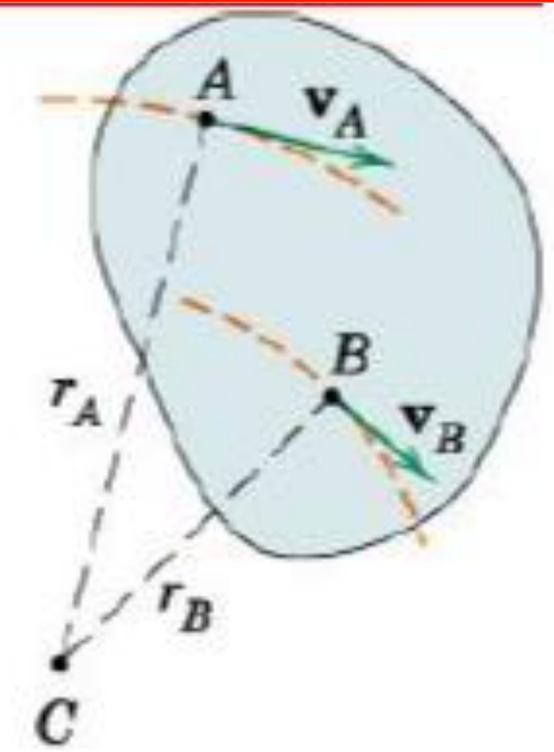
$$\vec{a}_M = \vec{a}_0 + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) = \vec{a}_0 + \dot{\vec{\omega}} \times \vec{r} - \omega^2 \vec{r}. \qquad (6.4)$$

Instantaneous Centre (IC) of Zero Velocity

We will show that if at the moment $t, \vec{\omega}(\mathrm{t}) \neq 0$ (i.e. the rigid body performs a rotation) there is a point $I \in S$ with the property $\vec{v}_I = 0$.
Thus, we have

$$\vec{v}_I = \vec{v}_0 + \vec{\omega} \times \overrightarrow{OI} = 0 \qquad (6.5)$$

Using (6.5) it is possible to determine the point I.
This point lies on the ***Instantaneous Axis of Zero Velocity*** (or instantaneous axis of rotation), and the intersection of this axis with the plane of motion is known as ***Instantaneous Centre of Zero Velocity.***
The **body** may be considered to be in **pure rotation** in a **circular path about** the Instantaneous Axis of Zero Velocity.
Locating the instantaneous centre of zero velocity is important to simplify the solution of many problems involving rigid body rotations.



Locating the Instantaneous Center (IC)

Assume that the directions of the absolute velocities of any points *A* and *B* on rigid body are known and are not parallel.

If *C* is a point about which *A* has an absolute circular motion at the instant considered, *C* must lie on the normal to $\mathbf{v}_A$ through *A*.

Similarly, point *B* can also have an absolute circular motion about *C* at the instant considered.

Intersection of two perpendiculars will give the absolute center of rotation at the instant considered.

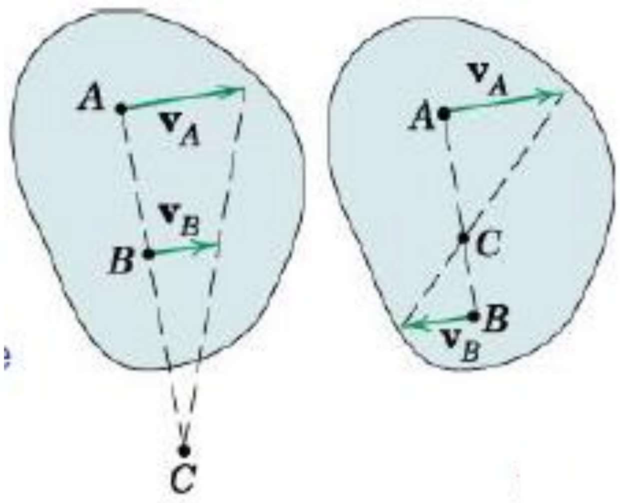
Point C is the instantaneous center of zero velocity and may lie on or off the body

Instantaneous center of zero velocity need not be a fixed point in the body or a fixed point in the plane.

Locating the Instantaneous Center (IC)

If the velocities of two points are parallel and the line joining the points is perpendicular to the direction of the velocity, the instantaneous center is located by direct proportion.

As the parallel velocities become equal in magnitude, the Instantaneous Centre moves farther away from the body and approaches infinity in the limit that body stops rotating and translates only.



(Kaustubh Dasgupta http://www.iitg.ac.in/kd/homepage/index.html)

In order to find the position of the IC one multiply eq. (6.5) with $\vec{\omega}$:

$$\vec{v}_I = \vec{v}_0 + \vec{\omega} \times \overrightarrow{OI} = 0 \qquad (6.5)$$

$$\vec{\omega} \times \vec{v}_0 + \vec{\omega} \times (\vec{\omega} \times \overrightarrow{OI}) = 0 \quad (\vec{\omega} \perp \overrightarrow{OI})$$
$$\Leftrightarrow \; \vec{\omega} \times \vec{v}_0 - \omega^2 \overrightarrow{OI} = 0 \qquad (6.6)$$

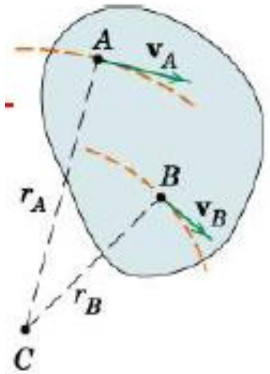
Thus, we obtain:

$$\overrightarrow{OI} = \frac{1}{\omega^2} \vec{\omega} \times \vec{v}_0, \quad |\overrightarrow{OI}| = \frac{v_0}{\omega}. \qquad (6.7)$$

If $M \in S$ the velocity is given by

$$\vec{v}_M = \underbrace{\vec{v}_I}_{=0} + \vec{\omega} \times \overrightarrow{IM} \;\Rightarrow\; \vec{v}_M = \vec{\omega} \times \overrightarrow{IM}. \qquad (6.8)$$

If magnitude of velocity at one of the points on the rigid body under general plane motion is known (v_A) the angular velocity of the body ω and linear velocity of every point in the body can be easily obtained using: $\omega = v_A / r_A$
Velocity of B is $v_B = r_B \omega = (r_B/r_A)\, v_A$.

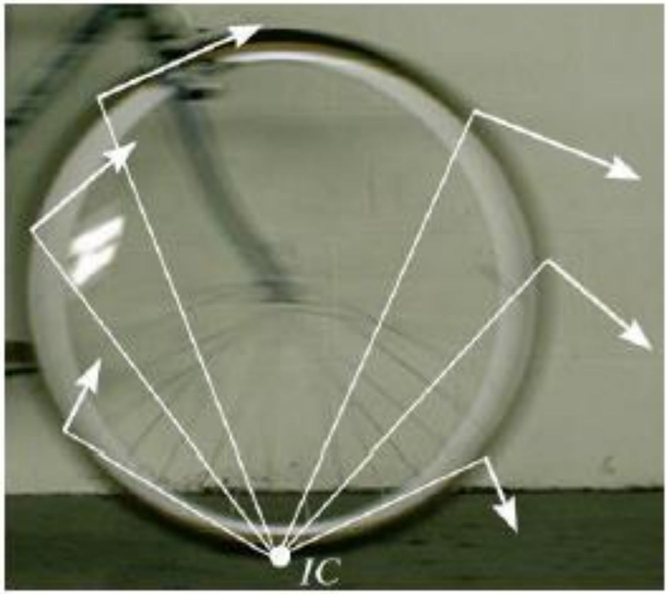


Once the IC is located, the direction of the instantaneous velocity of every point in the body is readily found since it must be perpendicular to the radial line joining the point in question with IC.

Velocity of all such points: $v = r\omega$

Note that the spokes are somewhat visible near IC, whereas at the top of the wheel they become blurred.

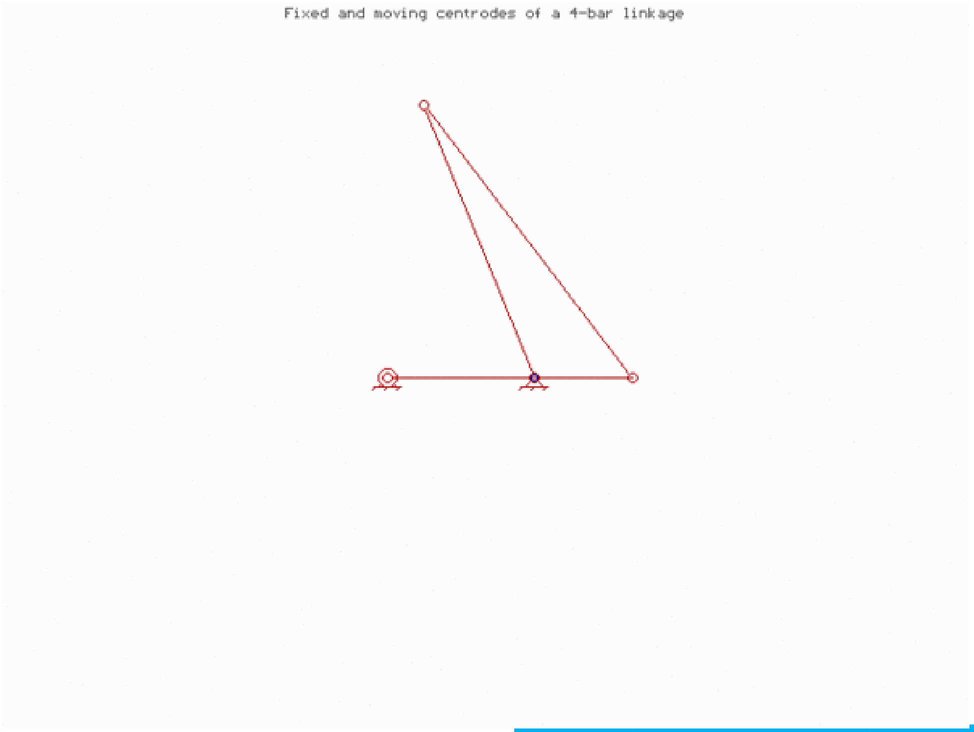
Also note how the direction of velocity varies at different point on the rim of the wheel.

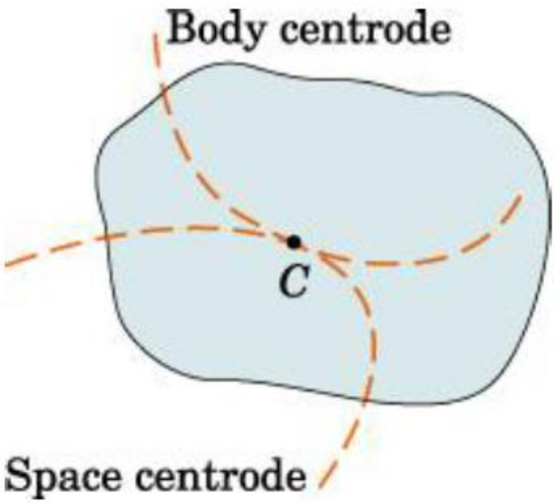


Locus of the ICs in space (in $O_1x_1y_1$) is known as **space centrode (baza)**

Locus of the ICs on the body (in Oxy) is known as **body centrode (rulanta)**

At the instant considered, the two curves are tangent at C
The body centrode rolls on the space centrode during the motion of the body.





Further, we deduce the equations of the space and body centrode:

$$\begin{aligned}
\overrightarrow{OI} &= \frac{1}{\omega^2}\overrightarrow{\omega} \times \overrightarrow{v}_0 \\
&= \frac{1}{\omega^2}\begin{vmatrix} \overrightarrow{i}_1 & \overrightarrow{j}_1 & \overrightarrow{k}_1 \\ 0 & 0 & \omega \\ \dot{x}_O & \dot{y}_O & 0 \end{vmatrix} \\
&= \frac{1}{\omega^2}\left(-\omega \dot{y}_O \overrightarrow{i}_1 + \omega \dot{x}_O \overrightarrow{j}_1\right) \\
&= \frac{1}{\omega}\left(-\dot{y}_O \overrightarrow{i}_1 + \dot{x}_O \overrightarrow{j}_1\right).
\end{aligned} \tag{6.9}$$

But $\overrightarrow{OI} = (x_1 - x_O, y_1 - y_O)$ and using (6.9) we have

$$\begin{cases} x_1 = x_O - \dfrac{\dot{y}_O}{\omega} \\ y_1 = y_O + \dfrac{\dot{x}_O}{\omega} \end{cases} \tag{6.10}$$

On the other hand,

$$\begin{cases} x_1 = x_1(\varphi) \\ y_1 = y_1(\varphi) \end{cases} ; \quad \begin{cases} x_O = x_O(\varphi) \\ y_O = y_O(\varphi) \end{cases} ; \quad \omega = \dot{\varphi}$$

$$\dot{y}_O = \frac{dy_O}{d\varphi}\dot{\varphi} = \omega\frac{dy_O}{d\varphi}, \; \dot{x}_0 = \omega\frac{dx_O}{d\varphi}$$

Thus

$$\begin{cases} x_1 = x_O - \dfrac{dy_O}{d\varphi} \\ y_1 = y_O + \dfrac{dx_O}{d\varphi} \end{cases} \tag{6.11}$$

Eqs. (6.11) are the equations of the space centrode (bazei) in the fixed frame of reference $O_1x_1y_1$.

Using (6.11) and taking into account

$$\begin{cases} x_1 = x_O + x\cos\varphi - y\sin\varphi \\ y_1 = y_O + x\sin\varphi + y\cos\varphi \end{cases}$$

we have

$$\begin{cases} x\cos\varphi - y\sin\varphi = -\dfrac{dy_O}{d\varphi} \\ x\sin\varphi + y\cos\varphi = \dfrac{dx_O}{d\varphi} \end{cases}$$

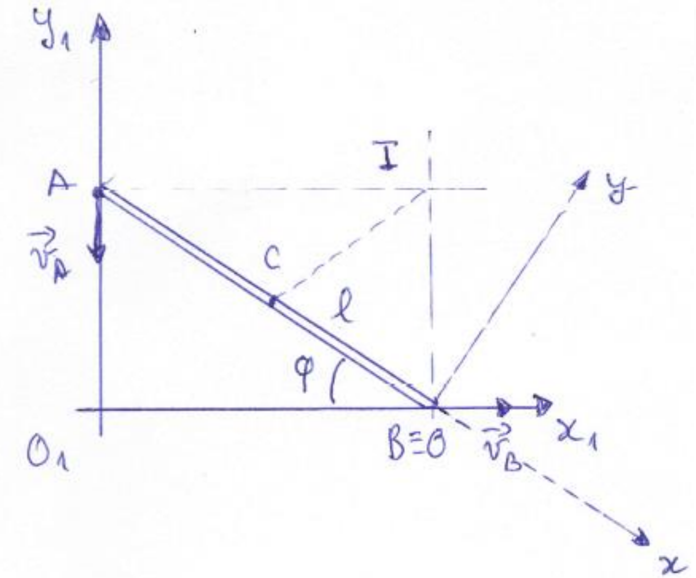
$$\begin{cases} x = \dfrac{dx_O}{d\varphi}\sin\varphi - \dfrac{dy_O}{d\varphi}\cos\varphi \\ y = \dfrac{dx_O}{d\varphi}\cos\varphi + \dfrac{dy_O}{d\varphi}\sin\varphi \end{cases} \qquad (6.12)$$

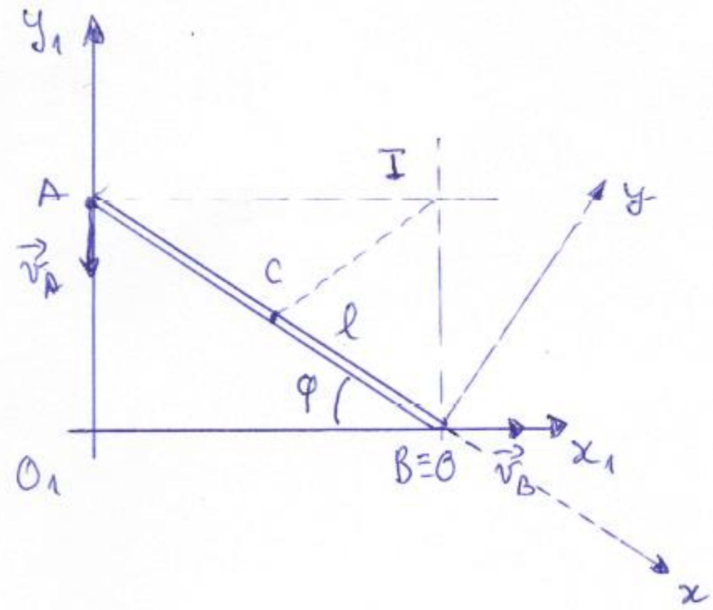
Eqs. (6.12) are the equations of the body centrode (rulantei) in the mobile frame of reference $O\,x\,y$.

① Consider the motion of a rigid bar $AB=2l$ in a plane sliping along O_1y_1 (vertical) and O_1x_1 (horizontal) axes. It is known that the point B (laying on the horizontal axis) has the velocity $\vec{v}_B$.

Find 1) the position of the IC, the equation of the space and body centrodes

2) the velocity of the point situated in the middle of the rigid bar





1) It is known that

$AI \perp \vec{v}_A$

$BI \perp \vec{v}_B$

Geometrical:

Space centrode

O_1BIA - rectangle $\Rightarrow$

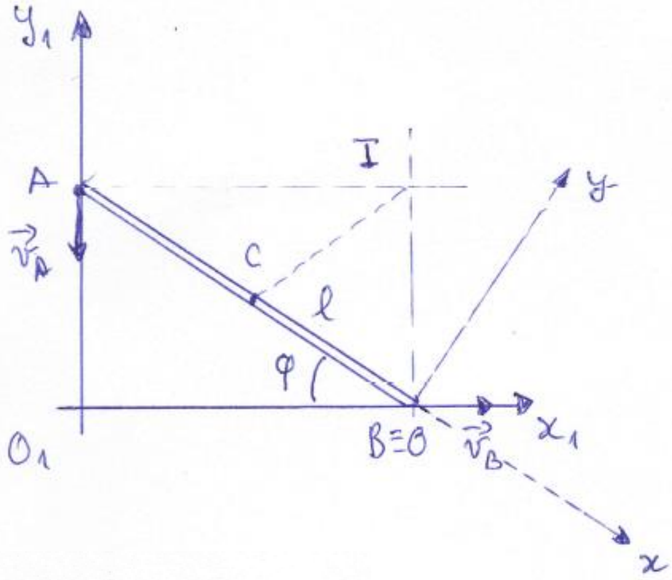
$\Rightarrow O_1I = \ell = \text{const} \Rightarrow$

$\Rightarrow$ the locus of I in $O_1x_1y_1$ is the circle $\mathcal{C}(O_1, \ell)$

Body centrode:

$m(\widehat{AIB}) = 90° \Rightarrow CI = \ell/2$ - constant

$\Rightarrow$ the locus of I in Oxy is the circle $\mathcal{C}(C, \ell/2)$



Analytical:

$$O(x_{10} = l\cos\varphi,\ y_{10} = 0)$$

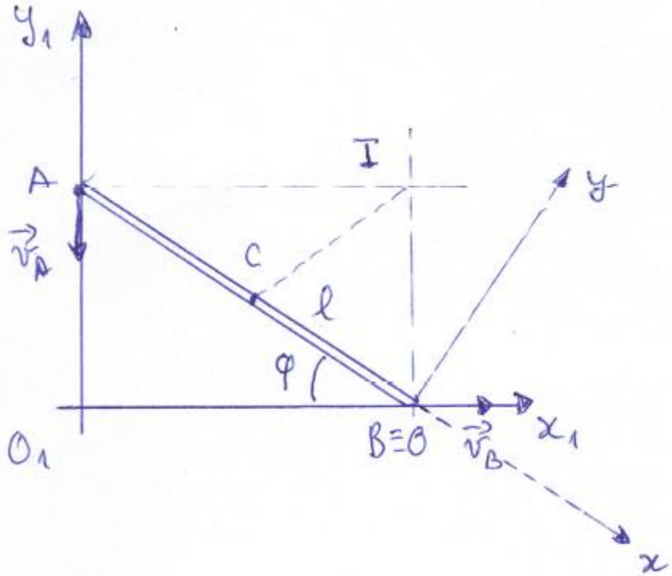
Space centrode:

$$\begin{cases} x_1 = x_0 - \dfrac{dy_{10}}{d\varphi} \\ y_1 = y_{10} + \dfrac{dx_{10}}{d\varphi} \end{cases} \Rightarrow \begin{cases} x_1 = l\cos\varphi \\ y_1 = -l\cos\varphi \end{cases}$$

$$\Rightarrow x_1^2 + y_1^2 = l \Rightarrow \mathcal{C}(O_1, l)$$

Body centrode:

$$\begin{cases} x = \dfrac{dx_{10}}{d\varphi}\sin\varphi - \dfrac{dy_{10}}{d\varphi}\cos\varphi \\ y = \dfrac{dx_1}{d\varphi}\cos\varphi + \dfrac{dy_{10}}{d\varphi}\sin\varphi \end{cases} \Rightarrow$$



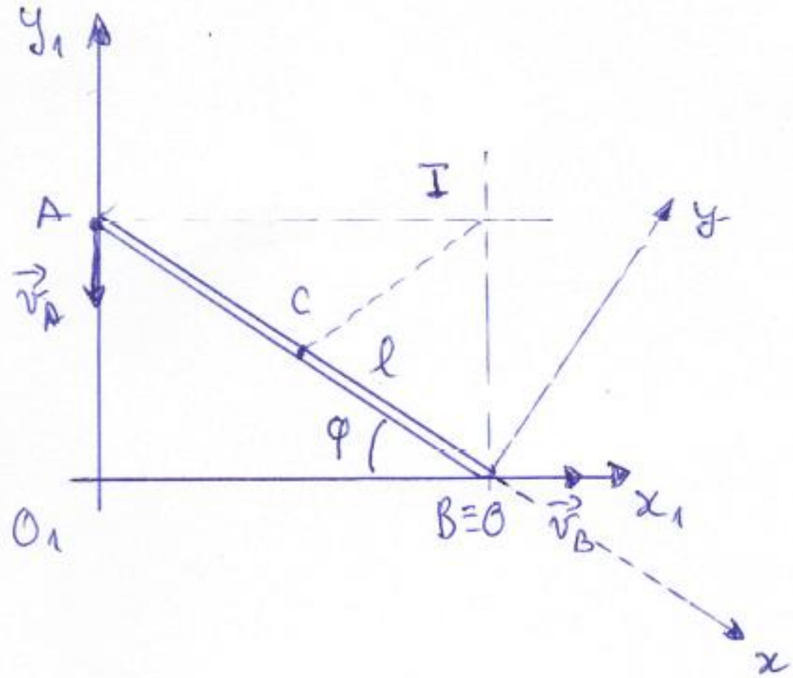
$$\begin{cases} x = -\ell \sin^2\varphi \\ y = \ell \cos\varphi \sin\varphi \end{cases}$$

$$y^2 = \ell^2 \sin^2\varphi\,(1 - \sin^2\varphi) = \ell^2\left(-\frac{x}{\ell}\right)\left[1 - \left(-\frac{x}{\ell}\right)\right] =$$

$$= -\ell^2 \cdot \frac{x}{\ell}\left(\frac{x}{\ell} + 1\right) = -\ell^2\left[\frac{x^2}{\ell^2} + \frac{x}{\ell}\right]$$

$$\Rightarrow \quad x^2 + y^2 + x\ell = 0 \quad \Rightarrow \left(x + \frac{\ell}{2}\right)^2 + y^2 = \frac{\ell^2}{4}$$

$$\Rightarrow \mathcal{C}\left(C, \frac{\ell}{2}\right)$$



2) $\vec{v}_C = \vec{\omega} \times \vec{IC}$

$\vec{\omega} \perp \vec{IC}$

$\Rightarrow v_C = \omega \cdot IC = \omega \cdot \frac{\ell}{2}$

On the other hand

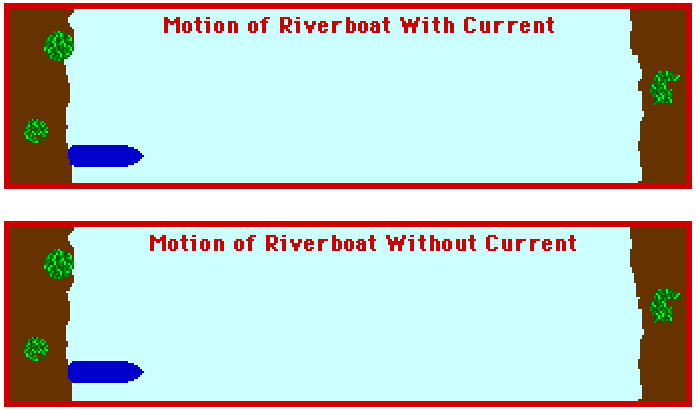
$v_B = \omega \cdot IB = \omega \cdot \ell \sin\varphi \Rightarrow$

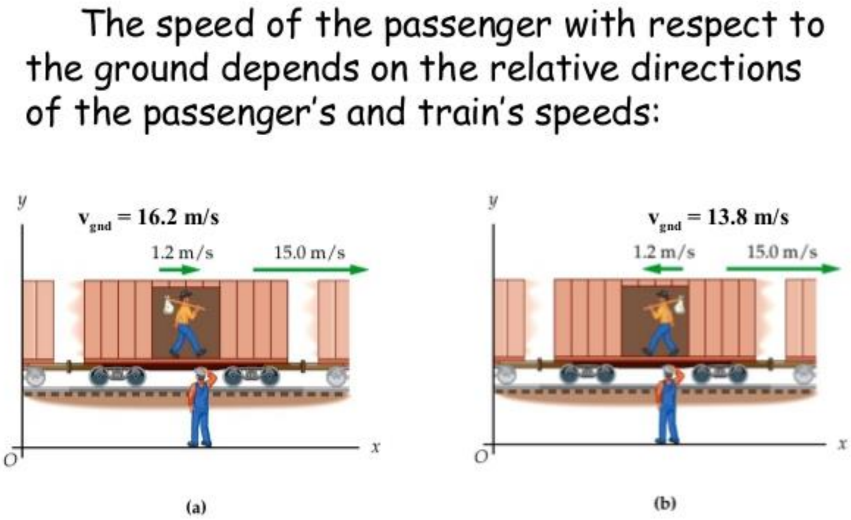
$\Rightarrow \omega = \frac{v_B}{\ell \sin\varphi}$

Thus, $\Rightarrow$ $v_C = \frac{v_B}{\cancel{\ell} \sin\varphi} \cdot \frac{\cancel{\ell}}{2} = \frac{v_B}{2 \sin\varphi}$

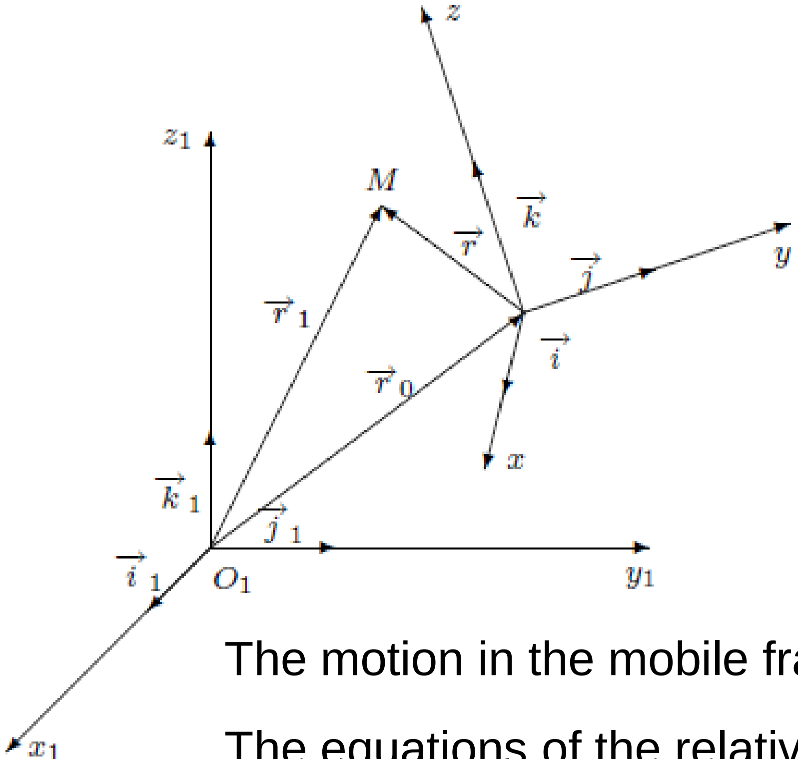
7. Relative motion of the particle

Consider a particle *M* moving in a refernce frame *Oxyz.* The reference frame *Oxyz* is moving relative to a fixed frame $O_1x_1y_1z_1$.





Consider a particle *M* moving in a reference frame *Oxyz* and the reference frame *Oxyz* is moving relative to a fixed frame $O_1x_1y_1z_1$.



$$\vec{r}_1 = \overrightarrow{O_1M},\ \vec{r} = \overrightarrow{OM}$$

The motion in the frame $O_1x_1y_1z_1$ is called ***absolute motion***.

The equations of the absolute motion are:

$$\vec{r}_1 = \vec{r}_1(t),\ t \in [t_0, T] \qquad (7.1)$$

The motion in the mobile frame $Oxyz$ is called ***relative motion***.

The equations of the relative motion are:

$$\vec{r} = \vec{r}(t),\ t \in [t_0, T] \qquad (7.2)$$

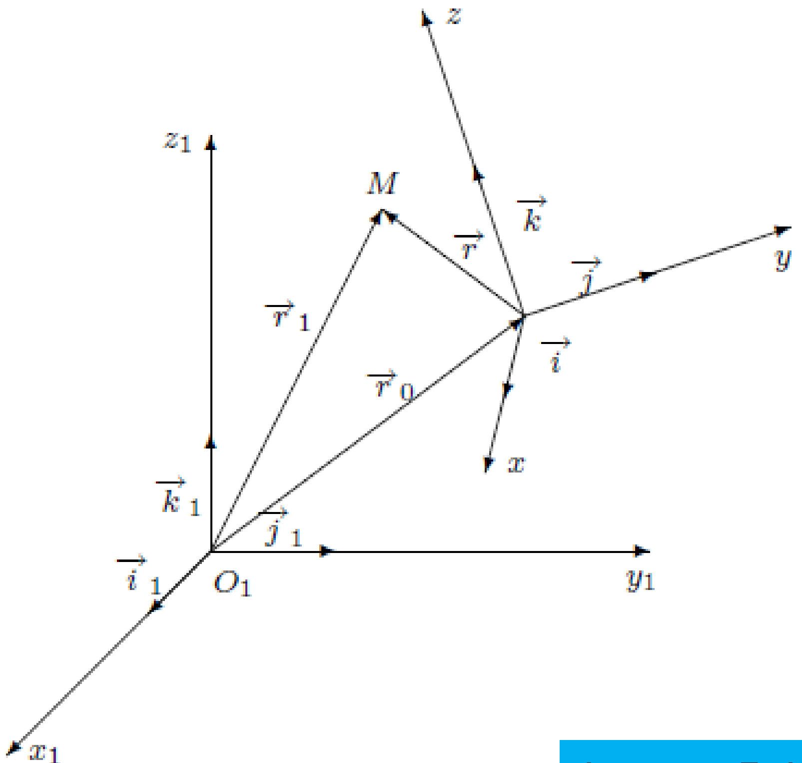
Consider the position vector of *M:* $\overrightarrow{r}_1 = \overrightarrow{r}_0 + \overrightarrow{r} = \overrightarrow{r}_0 + x\overrightarrow{i} + y\overrightarrow{j} + z\overrightarrow{k}$. (7.3)

Velocity: $$\overrightarrow{v}_{abs} := \frac{d\overrightarrow{r}_1}{dt} = \underbrace{\frac{d\overrightarrow{r}_0}{dt}}_{=\overrightarrow{v}_0} + \left(x \underbrace{\frac{d\overrightarrow{i}}{dt}}_{=\overrightarrow{\omega}\times\overrightarrow{i}} + y \underbrace{\frac{d\overrightarrow{j}}{dt}}_{=\overrightarrow{\omega}\times\overrightarrow{j}} + z \underbrace{\frac{d\overrightarrow{k}}{dt}}_{=\overrightarrow{\omega}\times\overrightarrow{k}} \right) +$$

$$\left(\frac{dx}{dt}\overrightarrow{i} + \frac{dy}{dt}\overrightarrow{j} + \frac{dz}{dt}\overrightarrow{k} \right)$$

$$= (\overrightarrow{v}_0 + \overrightarrow{\omega} \times \overrightarrow{r}) + (\dot{x}\overrightarrow{i} + \dot{y}\overrightarrow{j} + \dot{z}\overrightarrow{k})$$

(7.4)



The term

$$\vec{v}_{rel} := \left(\frac{d\vec{r}}{dt}\right)_{rel} = \dot{x}\,\vec{i} + \dot{y}\,\vec{j} + \dot{z}\,\vec{k} = \left(\frac{d\vec{r}}{dt}\right)_{\vec{i},\vec{j},\vec{k}=\overrightarrow{const}} \tag{7.5}$$

is the relative velocity of point *M* (the velocity in $Oxyz$ supposed fixed)

The term

$$\vec{v}_{tran} = \vec{v}_0 + \vec{\omega} \times \vec{r} \tag{7.6}$$

is the transport velocity of point *M* (the velocity of the point M considered fixed in the moving frame of reference $Oxyz$). The vector $\vec{\omega}$ is the angular velocity of $Oxyz$ about O.

Using (7.4) -(7.6) we can give the first **Coriolis formula**:

$$\vec{v}_{abs} = \vec{v}_{rel} + \vec{v}_{tran} \tag{7.7}$$

The absolute velocity is the sum of the relative and transport velocity.

Acceleration

$$\vec{v}_{abs} = \vec{v}_{rel} + \vec{v}_{tran} \;\Rightarrow\; \vec{a}_{abs} = \frac{d}{dt}(\vec{v}_{rel} + \vec{v}_0 + \vec{\omega} \times \vec{r}) =$$

$$\left(\left(\frac{d\vec{v}_{rel}}{dt}\right)_{rel} + \vec{\omega} \times \vec{v}_{rel}\right) + \left(\vec{a}_0 + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \underbrace{\left(\frac{d\vec{r}}{dt}\right)_{abs}}_{=\frac{d\vec{r}}{dt}}\right) =$$

$$(\vec{a}_{rel} + \vec{\omega} \times \vec{v}_{rel}) + \left(\vec{a}_0 + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \left(\underbrace{\left(\frac{d\vec{r}}{dt}\right)_{rel}}_{=\vec{v}_{rel}} + \vec{\omega} \times \vec{r}\right)\right)$$

$$= \vec{a}_{rel} + 2\vec{\omega} \times \vec{v}_{rel} + (\vec{a}_0 + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})). \qquad (7.8)$$

Here

$$\boxed{\vec{a}_{abs} := \frac{d\vec{v}_{abs}}{dt}}$$

is the absolute acceleration,

$$\boxed{\vec{a}_{rel} := \left(\frac{d\vec{v}_{rel}}{dt}\right)_{rel} = \ddot{x}\,\vec{i} + \ddot{y}\,\vec{j} + \ddot{z}\,\vec{k} \;\left(= \left(\frac{d\vec{v}_{rel}}{dt}\right)_{\vec{i},\,\vec{j},\,\vec{k}=\overrightarrow{const}}\right)} \tag{7.9}$$

is the relative acceleration,

$$\boxed{\vec{a}_{tran} = \vec{a}_0 + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})} \tag{7.10}$$

is the transport acceleration, and

$$\boxed{\vec{a}_c = 2\vec{\omega} \times \vec{v}_{rel}} \tag{7.11}$$

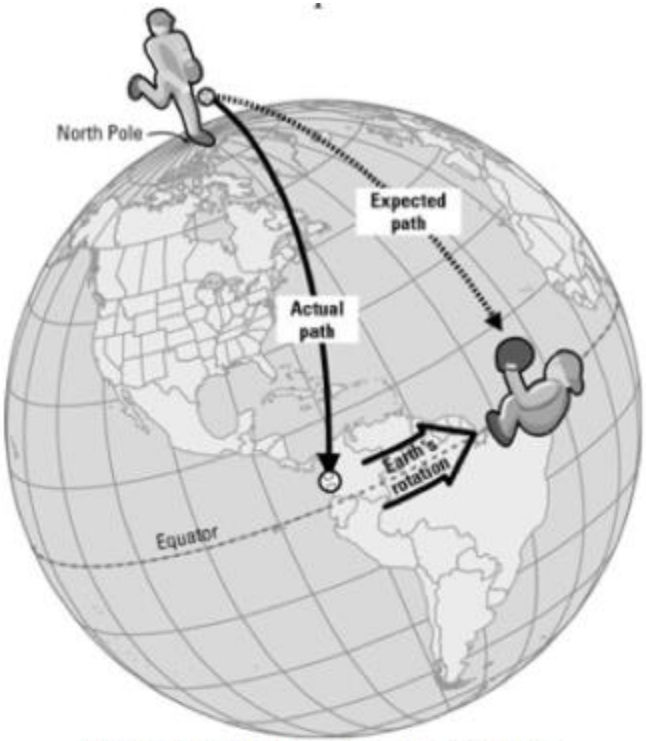
is the **Coriolis acceleration.**

The second Coriolis formula is given by:

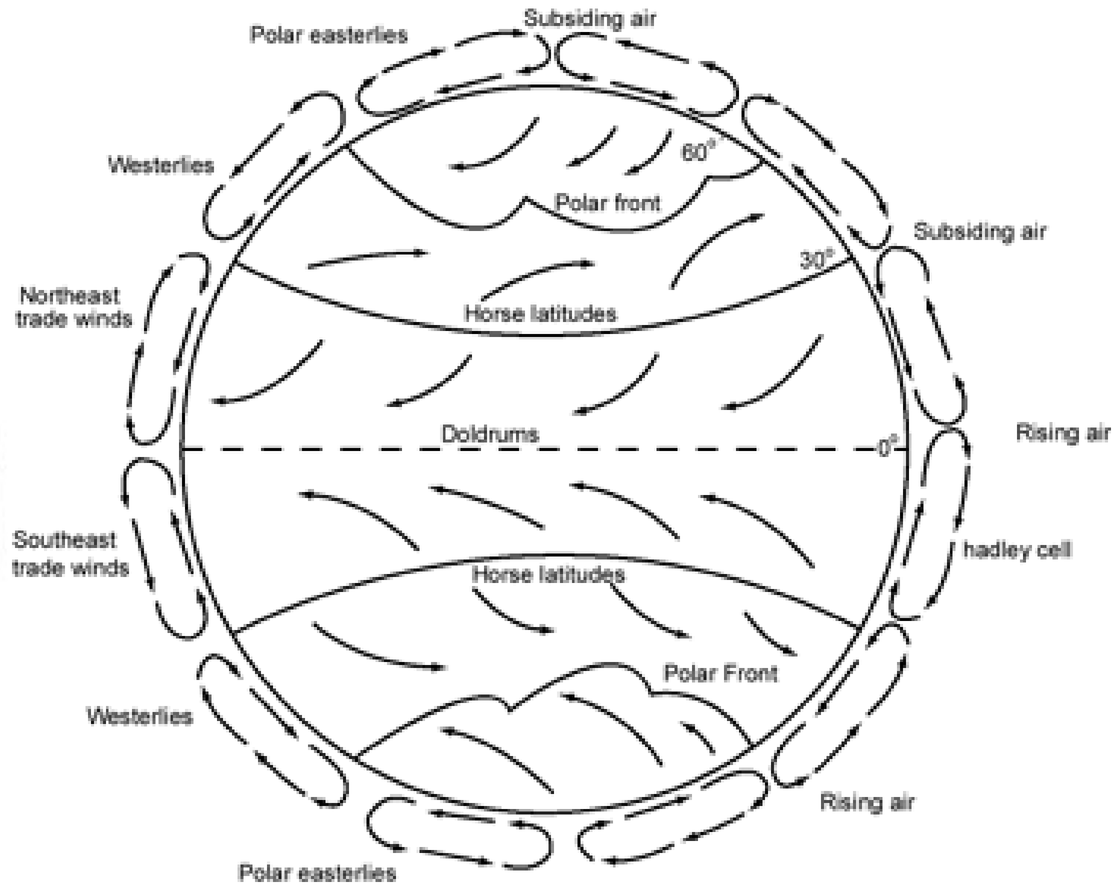
$$\vec{a}_{abs} = \vec{a}_{rel} + \vec{a}_{tran} + \vec{a}_c \tag{7.12}$$

Theoretical Mechanics

Corriolis effect

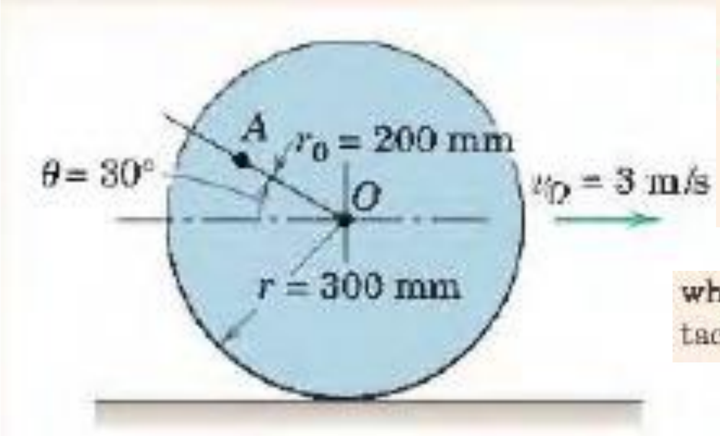


Wiley, Composition Services Graphics



Example

The wheel of radius $r = 300$ mm rolls to the right without slipping and has a velocity $v_O = 3$ m/s of its center O. Calculate the velocity of point A on the wheel for the instant represented.



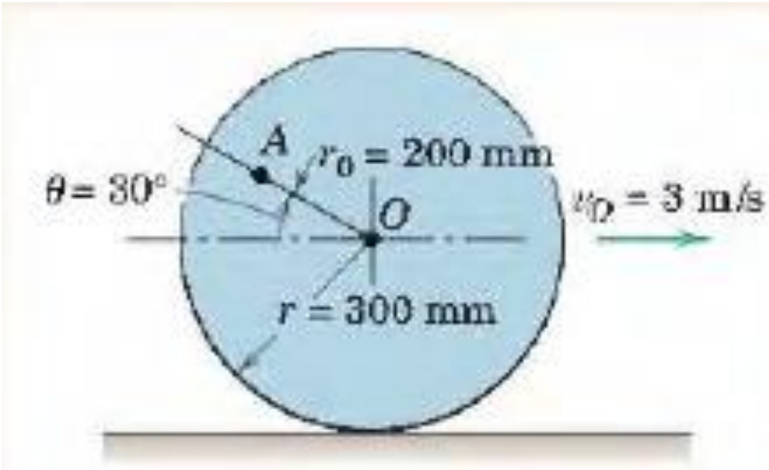
Solution I (Scalar-Geometric). The center O is chosen as the reference point for the relative-velocity equation since its motion is given. We therefore write

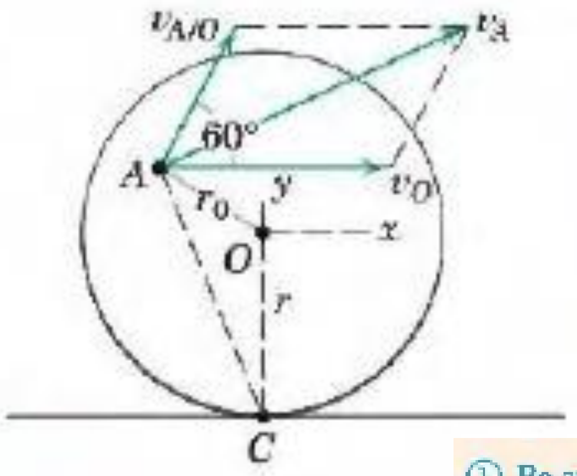
$$\mathbf{v}_A = \mathbf{v}_O + \mathbf{v}_{A/O}$$

where the relative-velocity term is observed from the translating axes x-y attached to O. The angular velocity of AO is

$$\omega = v_O/r = 3/0.3 = 10 \text{ rad/s.}$$

$[v_{A/O} = r_0\dot{\theta}]$ $\qquad v_{A/O} = 0.2(10) = 2$ m/s





$[v_{A/O} = r_0\dot{\theta}]$ $\qquad v_{A/O} = 0.2(10) = 2$ m/s

① which is normal to AO as shown. The vector sum $\mathbf{v}_A$ is shown on the diagram and may be calculated from the law of cosines. Thus,

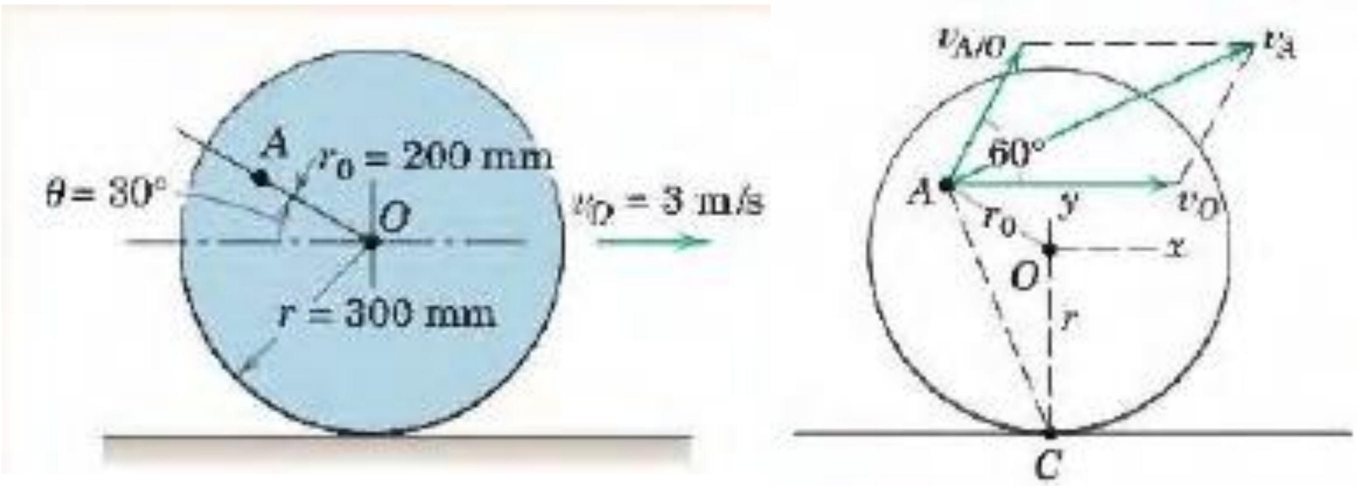
② $$v_A^2 = 3^2 + 2^2 + 2(3)(2)\cos 60° = 19\ (\text{m/s})^2 \qquad v_A = 4.36 \text{ m/s} \qquad \textit{Ans.}$$

The contact point C momentarily has zero velocity and can be used alternatively as the reference point, in which case, the relative-velocity equation becomes $\mathbf{v}_A = \mathbf{v}_C + \mathbf{v}_{A/C} = \mathbf{v}_{A/C}$ where

$$v_{A/C} = \overline{AC}\omega = \frac{\overline{AC}}{\overline{OC}} v_O = \frac{0.436}{0.300}(3) = 4.36 \text{ m/s} \qquad v_A = v_{A/C} = 4.36 \text{ m/s}$$

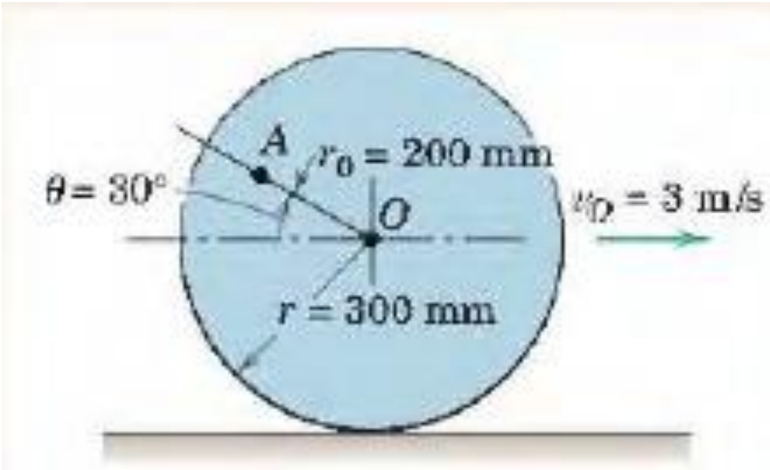
① Be sure to visualize $v_{A/O}$ as the velocity which A appears to have in its circular motion relative to O.

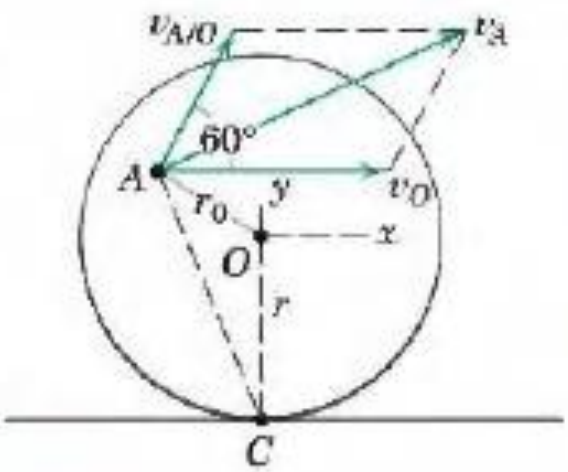
② The vectors may also be laid off to scale graphically and the magnitude and direction of v_A measured directly from the diagram.



The distance $\overline{AC} = 436$ mm is calculated separately. We see that $\mathbf{v}_A$ is normal to (3) AC since A is momentarily rotating about point C.

(3) The velocity of any point on the wheel is easily determined by using the contact point C as the reference point. You should construct the velocity vectors for a number of points on the wheel for practice.





Solution II (Vector). We will now use Eq. 5/6 and write

$$\mathbf{v}_A = \mathbf{v}_O + \mathbf{v}_{A/O} = \mathbf{v}_O + \boldsymbol{\omega} \times \mathbf{r}_0$$

where

④ $$\boldsymbol{\omega} = -10\mathbf{k} \text{ rad/s}$$

$$\mathbf{r}_0 = 0.2(-\mathbf{i}\cos 30° + \mathbf{j}\sin 30°) = -0.1732\mathbf{i} + 0.1\mathbf{j} \text{ m}$$

$$\mathbf{v}_O = 3\mathbf{i} \text{ m/s}$$

④ The vector $\boldsymbol{\omega}$ is directed into the paper by the right-hand rule, whereas the positive z-direction is out from the paper; hence, the minus sign.

We now solve the vector equation

$$\mathbf{v}_A = 3\mathbf{i} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -10 \\ -0.1732 & 0.1 & 0 \end{vmatrix} = 3\mathbf{i} + 1.732\mathbf{j} + \mathbf{i}$$

$$= 4\mathbf{i} + 1.732\mathbf{j} \text{ m/s} \qquad \textit{Ans.}$$

The magnitude $v_A = \sqrt{4^2 + (1.732)^2} = \sqrt{19} = 4.36$ m/s and direction agree with the previous solution.