COURSE 12

Examples of other multi-step methods

1. THE BISECTION METHOD

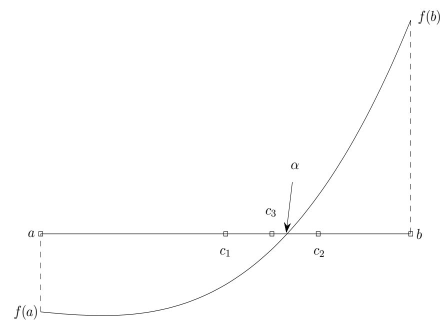
Let f be a given function, continuous on an interval $[a, b]$, such that

$$f(a)f(b) < 0. \tag{1}$$

By the Mean Value Theorem, it follows that there exists at least one zero α of f in (a, b).

The bisection method is based on halving the interval $[a, b]$ to determine a smaller and smaller interval within α must lie.

First we give the midpoint of $[a, b]$, $c = (a + b)/2$ and then compute the product $f(c)f(b)$. If the product is negative, then the root is in the interval $[c, b]$ and we take $a_1 = c$, $b_1 = b$. If the product is positive, then the root is in the interval $[a, c]$ and we take $a_1 = a$, $b_1 = c$. Thus, a new interval containing α is obtained.



Bisection method

The algorithm:

Suppose $f(a)f(b) \le 0$. Let $a_0 = a$ and $b_0 = b$.

for $n = 0, 1, ...,$ITMAX

$c \leftarrow \frac{a_n + b_n}{2}$

if $f(a_n)f(c) \le 0$, set $a_{n+1} = a_n, b_{n+1} = c$

else, set $a_{n+1} = c, b_{n+1} = b_n$

The process of halving the new interval continues until the root is located as accurately as desired, namely

$$\frac{|a_n - b_n|}{|a_n|} < \varepsilon,$$

where a_n and b_n are the endpoints of the n-th interval $[a_n, b_n]$ and ε is a specified precision. The approximation of the solution will be $\frac{a_n+b_n}{2}$.

Some other stopping criterions: $|a_n - b_n| < \varepsilon$ or $|f(a_n)| < \varepsilon$.

Example 1 *The function $f(x) = x^3 - x^2 - 1$ has one zero in $[1, 2]$. Use the bisection algorithm to approximate the zero of f with precision 10^{-4}.*

Sol. *Since $f(1) = -1 < 0$ and $f(2) = 3 > 0$, then (1) is satisfied. Starting with $a_0 = 1$ and $b_0 = 2$, we compute*

$$c_0 = \frac{a_0 + b_0}{2} = \frac{1+2}{2} = 1.5 \textit{ and } f(c_0) = 0.125.$$

Since $f(1.5)f(2) > 0$, *the function changes sign on* $[a_0, c_0] = [1, 1.5]$.

To continue, we set $a_1 = a_0$ *and* $b_1 = c_0$; *so*

$$c_1 = \frac{a_1 + b_1}{2} = \frac{1 + 1.5}{2} = 1.25 \text{ and } f(c_1) = -0.609375$$

Again, $f(1.25)f(1.5) < 0$ *so the function changes sign on* $[c_1, b_1] = [1.25,\ 1.5]$. *Next we set* $a_2 = c_1$ *and* $b_2 = b_1$. *Continuing in this manner we obtain a sequence* $(c_i)_{i>0}$ *which converges to* 1.465454, *the solution of the equation.*

2. THE METHOD OF FALSE POSITION

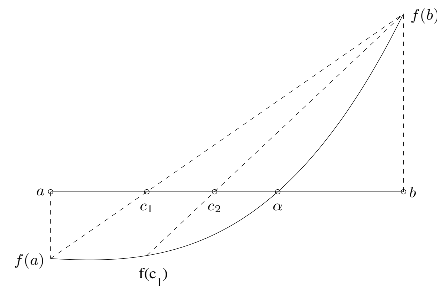
This method is also known as *regula falsi*, is similar to the Bisection method but has the advantage of being slightly faster than the latter. The function have to be continuous on $[a, b]$ with

$$f(a)f(b) < 0.$$

The point c is selected as point of intersection of the Ox-axis, and the straight line joining the points $(a,\ f(a))$ and $(b, f(b))$. From the equation of the secant line, it follows that

$$c = b - f(b)\frac{b-a}{f(b)-f(a)} = \frac{af(b)-bf(a)}{f(b)-f(a)} \tag{2}$$

Compute $f(c)$ and repeat the procedure between the values at which the function changes sign, that is, if $f(a)f(c) < 0$ set $b = c$, otherwise set $a = c$. At each step we get a new interval that contains a root of f and the generated sequence of points will eventually converge to the root.



Method of false position.

The algorithm:

Given a function f continuous on $[a_0, b_0]$, with $f(a_0)f(b_0) < 0$,

input: a_0, b_0

for $n = 0, 1, ..., ITMAX$

$c \leftarrow \frac{f(b_n)a_n - f(a_n)b_n}{f(b_n) - f(a_n)}$

if $f(a_n)f(c) < 0$, set $a_{n+1} = a_n, b_{n+1} = c$ **else** set $a_{n+1} = c, b_{n+1} = b_n$.

Stopping criterions: $|f(a_n)| \leq \varepsilon$ or $|a_n - a_{n-1}| \leq \varepsilon$, where ε is a specified tolerance value.

Remark 2 *The bisection and the false position methods converge at a very low speed compared to the secant method.*

Example 3 *The function $f(x) = x^3 - x^2 - 1$ has one zero in $[1, 2]$. Use the method of false position to approximate the zero of f with precision 10^{-4}.*

Sol. *A root lies in the interval $[1, 2]$ since $f(1) = -1$ and $f(2) = 3$. Starting with $a_0 = 1$ and $b_0 = 2$, we get using (2)*

$$c_0 = 2 - \frac{3(2-1)}{3-(-1)} = 1.25 \text{ and } f(c_0) = -0.609375.$$

Here, $f(c_0)$ has the same sign as $f(a_0)$ and so the root must lie on the interval $[c_0, b_0] = [1.25, 2]$. Next we set $a_1 = c_0$ and $b_1 = b_0$ to get the next approximation

$$c_1 = 2 - \frac{3-(2-1.25)}{3-(-0.609375)} = 1.37662337 \text{ and } f(c_1) = -0.2862640.$$

Now $f(x)$ *change sign on* $[c_1, b_1] = [1.37662337, 2]$. *Thus we set* $a_2 = c_1$ *and* $b_2 = b_{1.}$ *Continuing in this manner the iterations lead to the approximation* 1.465558.

Example 4 *Compare the false position method, the secant method and Newton's method for solving the equation* $x = cos x$, *having as starting points* $x_0 = 0.5$ *și* $x_1 = \pi/4$, *respectively* $x_0 = \pi/4$.

n	**(a)** x_n **False position**	**(b)** x_n **Secant**	**(c)** x_n **Newton**
0	0.5	0.5	0.5
1	0.785398163397	0.785398163397	0.785398163397
2	0.736384138837	0.736384138837	0.739536133515
3	0.739058139214	0.739058139214	0.739085178106
4	0.739084863815	0.739085149337	0.739085133215
5	0.739085130527	0.739085133215	
6	0.739085133188		
7	0.739085133215		

The extra condition from the false position method usually requires more computation than the secant method, and the simplifications from the secant method come with more iterations than in the case of Newton's method.

Example 5 *Consider the equation* $x^2 - x - 3 = 0$*. Give the next two iterations for approximating the solution of this equation using:*

a) Newton's method starting with $x_0 = 0$.

b) secant, false position and bisection methods starting with $x_0 = 0$ *and* $x_1 = 4$.

5.3. Numerical methods for solving nonlinear systems

Let $D \subseteq \mathbb{R}^n$, $f_i : D \to \mathbb{R}$, $i = 1, ..., n$ and the system

$$f_i(x_1, ..., x_n) = 0,\ i = 1, ..., n;\ (x_1, ..., x_n) \in D. \tag{3}$$

The system (3) can be written as

$$f(x) = 0,\ x \in D,\ \text{with } f = (f_1, ..., f_n).$$

5.3.1. Successive approximation method

We rewrite the system (3) as

$$x_i = \varphi_i(x_1, ..., x_n), \qquad i = 1, ..., n;\ (x_1, ..., x_n) \in D$$

or

$$x = \varphi(x), \quad \text{with } x = (x_1, ..., x_n) \in D \text{ and } \varphi = (\varphi_1, ..., \varphi_n), \tag{4}$$

where $\varphi_i : D \to \mathbb{R}$ are continuous functions on D such that for any point $(x_1, ..., x_n) \in D$ to have $(\varphi_1(x_1, ..., x_n), ..., \varphi_n(x_1, ..., x_n)) \in D$.

Considering the starting point $x^{(0)}$ we generate the sequence

$$x^{(0)}, x^{(1)}, ..., x^{(n)}, ... \tag{5}$$

with

$$x^{(m+1)} = \varphi(x^{(m)}),\ \ m = 0, 1, ...$$

If the sequence (5) is convergent and α is its limit, then α is the solution of system (3). We have

$$\lim_{m\to\infty} x^{(m+1)} = \varphi(\lim_{m\to\infty} x^{(m)}),$$

namely,

$$\alpha = \varphi(\alpha).$$

The convergence of the method, using Picard-Banach theorem: if $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ verifies the contraction condition

$$\|\varphi(x) - \varphi(y)\| \leq l\,\|x - y\|\,,\ \ x, y \in \mathbb{R}^n; 0 < \alpha < 1,$$

then there exists an unique element $\alpha \in \mathbb{R}^n$, which is solution of equation (4) and limit of the sequence (5). The approximation error is:

$$\left\| \alpha - x^{(n)} \right\| \leq \frac{l^n}{1-l} \left\| x^{(1)} - x^{(0)} \right\|.$$

Example 6 *Choosing* $x^{(0)} = (0.1, 0.1, -0.1)$, *solve the system*

$$\begin{cases} 3x_1 - \cos(x_2 x_3) - \frac{1}{2} & = & 0 \\ x_1^2 - 81(x_2 + 0.1)^2 + \sin x_3 + 1.06 & = & 0 \\ e^{-x_1 x_2} + 20x_3 + \frac{10\pi - 3}{3} & = & 0 \end{cases}.$$

Sol. We have

$$\begin{cases} x_1^{(1)} = \frac{1}{3}\cos(x_2^{(0)} x_3^{(0)}) + \frac{1}{6} \\ x_2^{(1)} = \frac{1}{9}\sqrt{(x_1^{(0)})^2 + \sin x_3^{(0)} + 1.06} - 0.1 \\ x_3^{(1)} = -\frac{1}{20} e^{-x_1^{(0)} x_2^{(0)}} - \frac{10\pi - 3}{60} \end{cases},$$

and

$$\begin{cases} x_1^{(m)} = \frac{1}{3}\cos(x_2^{(m-1)}x_3^{(m-1)}) + \frac{1}{6} \\ x_2^{(m)} = \frac{1}{9}\sqrt{(x_1^{(m-1)})^2 + \sin x_3^{(m-1)} + 1.06} - 0.1 \\ x_3^{(m)} = -\frac{1}{20}e^{-x_1^{(m-1)}x_2^{(m-1)}} - \frac{10\pi-3}{60} \end{cases} .$$

The sequence converges to (0.5,0,-0.5236).

5.3.2. Newton's method for solving nonlinear systems

Consider the system (3) written as

$$f(x) = 0,\ x \in D, D \subseteq \mathbb{R}^n.$$

Let $\alpha \in D$ be a solution of this equation and $x^{(p)}$ an approximation of it.

The Newton's method for nonlinear systems:

$$x^{(p+1)} = x^{(p)} - J^{-1}(x^{(p)})f(x^{(p)}),\ p = 0, 1, ... \tag{6}$$

where

$$J(x^{(p)}) = f'(x^{(p)}) = \left(\frac{\partial f_i}{\partial x_i}(x^{(p)})\right)_{i,\ j=1,...,n}$$

is the Jacobian matrix.

If the sequence $(x^{(p)})_{p\in\mathbb{N}}$ is convergent and α is its limit then by (6) it follows that α is solution of the system. Regarding the convergence of the sequence we have:

Theorem 7 *Let $f : D \subseteq R^n \to R^n$ and consider a given norm $\|\cdot\|$ on R^n. If*

- *there exists a solution $\alpha \in D$, such that $f(\alpha) = 0$;*

- f *is differentiable on* D, *with* f' *Lipschitz continuous, i.e.,* $\exists L > 0$ *s.t.*

$$\left\| f'(x) - f'(y) \right\| \leq L \|x - y\|, \forall x, y \in D;$$

- *the Jacobian of* f *is nonsingular at* α*:* $\exists f'(\alpha)^{-1} : R^n \to R^n$,

then there exists an open neighborhood $D_0 \subseteq D$ *of* α *such that for any initial approximation* $x_0 \in D_0$ *the sequence generated by the Newton's method remains in* D_0, *converges to the solution* α *and there exists a constant* $K > 0$ *such that*

$$\left\| x_{k+1} - \alpha \right\| \leq K \|x_k - \alpha\|^2, \ \forall k \geq 0.$$

Example 8 *Solve the system*

$$\begin{cases} x_1^3 + 3x_2^2 - 21 = 0 \\ x_1^2 + 2x_2 + 2 = 0 \end{cases}$$

using $x^{(0)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\varepsilon = 10^{-6}$.

We have

$$x^{(p+1)} = x^{(p)} - J^{-1}(x^{(p)})f(x^{(p)}),\ \ p = 0, 1, ... \tag{7}$$

We compute

$$J(x) = f'(x) = \left(\frac{\partial}{\partial x_i} f_i(x)\right)_{i,\ j=1,...,n} = \begin{pmatrix} 3x_1^2 & 6x_2 \\ 2x_1 & 2 \end{pmatrix}$$

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} x_1^3 + 3x_2^2 - 21 \\ x_1^2 + 2x_2 + 2 \end{pmatrix}.$$

We have

$$x^{(p+1)} = x^{(p)} - J^{-1}(x^{(p)})f(x^{(p)}), \tag{8}$$

i.e.,

$$\begin{aligned} x^{(1)} &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 3 & -6 \\ 2 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 + 3 - 21 \\ 1 - 2 + 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 0.(1) & 0.(3) \\ -0.(1) & 0.1(6) \end{pmatrix} \begin{pmatrix} -17 \\ 1 \end{pmatrix} \end{aligned}$$

and it follows

$$x^{(1)} \approx \begin{pmatrix} 2.55 \\ -3.05 \end{pmatrix}.$$

Continuing in this way we obtain the approx. solution $\begin{pmatrix} 1.64 \\ -2.35 \end{pmatrix}$