Presentation 2 - Seminar 6

Tămășilă Amalia Mirabela

Problem statement

(a) Let $\triangle ABC$ be an equilateral triangle. Let P be a point on the minor arc AB of its circumcircle. Prove that $PC = PA + PB$.

(b) In a regular heptagon $ABCDEFG$, prove that:

$$\frac{1}{AB} = \frac{1}{AC} + \frac{1}{AD}.$$

Solution

(a)

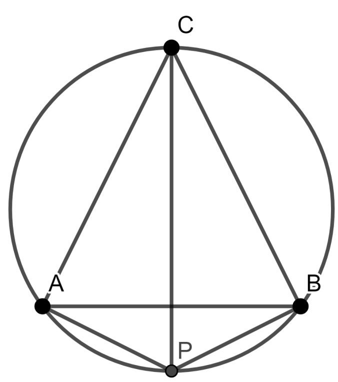

Figure 1: $\triangle ABC$ - equilateral triangle
P - point on minor arc AB
$APCD$ - quadrilateral

We see that:

$$\left.\begin{array}{l} APBC - \text{cyclic quadrilateral} \\ AB,\ PC - \text{diagonals} \end{array}\right| \Rightarrow \text{we can use the equality case of Ptolemy's theorem}$$

$$\Rightarrow \quad PC \cdot AB = PB \cdot AC + PA \cdot BC \quad \Big| \quad : AB = AC = BC$$
$$\Rightarrow \quad PC = PA + PB.$$

(b)

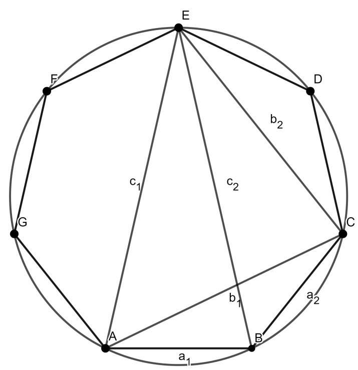

Figure 2: $ABCDEFG$ - regular heptagon

We take:

$$\left.\begin{array}{l} ABCE - \text{cyclic quadrilateral} \\ AC,\ BE - \text{diagonals} \end{array}\right| \Rightarrow \text{we can use the equality case of Ptolemy's theorem}$$

$$\Rightarrow \quad AC \cdot BE = AB \cdot CE + AE \cdot BC$$

Since the heptagon is regular we can make some observations and notations:
$AB = BC = a_1 = a_2 = a$
$AC = CE = b_1 = b_2 = b$
$AE = BE = c_1 = c_2 = c$

So:

$$b \cdot c = a \cdot b + a \cdot c \quad \Big| \quad : abc$$
$$\Rightarrow \quad \frac{bc}{abc} = \frac{ab}{abc} + \frac{ac}{abc}$$
$$\Rightarrow \quad \frac{1}{a} = \frac{1}{c} + \frac{1}{b}$$

We have that:

$$\frac{1}{AB} = \frac{1}{AE} + \frac{1}{AC}$$

Observe that AE = AD.

$$\Rightarrow \quad \frac{1}{AB} = \frac{1}{AD} + \frac{1}{AC}.$$