SEMINARS 12+13

1) Give a necessary and sufficient condition for the vectors $v_1 = (a_1, b_1)$, $v_2 = (a_2, b_2)$ to form a basis for the $\mathbb{R}$-vector space $\mathbb{R}^2$. What does this condition mean from geometrical point of view? Using the condition established, find infinitely many bases for ${}_{\mathbb{R}}\mathbb{R}^2$. Is there any basis of $\mathbb{R}^2$ for which the coordinates of a vector $v = (x, y)$ are exactly x and y? Show that $v_1 = (1, 0)$ and $v_2 = (1, 1)$ form a basis of $\mathbb{R}^2$ and find the coordinates of $v = (x, y)$ in this basis.
Homework: Formulate and solve a similar problem for the $\mathbb{R}$-vector space $\mathbb{R}^3$.

2) Show that the vectors $(1, 2, -1)$, $(3, 2, 4)$, $(-1, 2, -6)$ from ${}_{\mathbb{R}}\mathbb{R}^3$ are linearly dependent and find a dependency relation between them.

3) Determine the values of $a \in \mathbb{R}$ for which the vectors $v_1 = (a, 1, 1)$, $v_2 = (1, a, 1)$, $v_3 = (1, 1, a)$ form a basis of ${}_{\mathbb{R}}\mathbb{R}^3$.

Homework: Which of the following systems of vectors from $\mathbb{R}^3$:
a) $((1, 0, -1), (2, 5, 1), (0, -4, 3))$;
b) $((2, -4, 1), (0, 3, -1), (6, 0, 1))$;
c) $((1, 2, -1), (1, 0, 3), (2, 1, 1))$;
d) $((-1, 3, 1), (2, -4, -3), (-3, 8, 2))$;
e) $((1, -3, -2), (-3, 1, 3), (-2, -10, -2))$
are bases for the $\mathbb{R}$-vector space $\mathbb{R}^3$?

4) Show that in the $\mathbb{R}$-vector space $M_2(\mathbb{R})$ the matrices

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ E_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},\ E_3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},\ E_4 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

form a basis and determine the coordinates of $A = \begin{pmatrix} -2 & 3 \\ 4 & -2 \end{pmatrix}$ in this basis.

5) In the $\mathbb{Q}$-vector space $\mathbb{Q}^3$ we consider the vectors

$$a = (-2, 1, 3),\ b = (3, -2, -1),\ c = (1, -1, 2),\ d = (-5, 3, 4),\ e = (-9, 5, 10).$$

Does the following equality $\langle a, b\rangle = \langle c, d, e\rangle$ hold?

6) In the $\mathbb{R}$-vector space $\mathbb{R}^4$ one considers the subspaces:
a) $S = \langle u_1, u_2\rangle$, with $u_1 = (1, 1, 0, 0)$, $u_2 = (1, 0, 1, 1)$,
$T = \langle v_1, v_2\rangle$, with $v_1 = (0, 0, 1, 1)$, $v_2 = (0, 1, 1, 0)$;
b) $S = \langle u_1, u_2, u_3\rangle$, with $u_1 = (1, 2, -1, -2)$, $u_2 = (3, 1, 1, 1)$, $u_3 = (-1, 0, 1, -1)$,
$T = \langle v_1, v_2\rangle$, with $v_1 = (-1, 2, -7, -3)$, $v_2 = (2, 5, -6, -5)$;
c) $S = \langle u_1, u_2\rangle$, with $u_1 = (1, 2, 1, 0)$, $u_2 = (-1, 1, 1, 1)$,
$T = \langle v_1, v_2\rangle$, with $v_1 = (2, -1, 0, 1)$, $v_2 = (1, -1, 3, 7)$.
Find a basis and the dimension for each of the subspaces S, T, $S + T$ and $S \cap T$.

7) a) Let $\varphi \in \mathbb{R}$. Show that the plane rotation

$$h : \mathbb{R}^2 \to \mathbb{R}^2,\ h(x, y) = (x\cos\varphi - y\sin\varphi, x\sin\varphi + y\cos\varphi),$$

is an automorphism of $\mathbb{R}^2$. Write the matrix of h of $\mathbb{R}^2$ in the standard basis (i.e. the basis $E = (e_1, e_2)$, with $e_1 = (1,0)$, $e_2 = (0,1)$).
b) Show that $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (x,-y)$ (the symmetry with respect to Ox) and $g : \mathbb{R}^2 \to \mathbb{R}^2$, $g(x,y) = (-x,y)$ (the symmetry with respect to Oy) are automorphisms of $\mathbb{R}^2$. Find the matrices of f, g, $f-g$, $f+2g$ and $g \circ f$ in the standard basis.

8) Let $f : \mathbb{R}^2 \to \mathbb{R}^3$, $f(x,y) = (x+y, 2x-y, 3x+2y)$. Show that f is an $\mathbb{R}$-linear map, that $B = ((1,2),(-2,1))$ and $B' = ((1,-1,0),(-1,0,1),(1,1,1))$ are bases for $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively, then determine the matrix of f in the pair of bases (B, B').

9) Let $A = \begin{pmatrix} 1 & 1 & 3 \\ 0 & 5 & -1 \\ 2 & 7 & -3 \end{pmatrix}$, $B = (v_1, v_2, v_3)$ be a basis of the $\mathbb{R}$-vector space $\mathbb{R}^3$, the vectors

$$u_1 = v_1 + 2v_2 + v_3,\ u_2 = v_1 + v_2 + 2v_3,\ u_3 = v_1 + v_2$$

and $f \in End_{\mathbb{R}}(\mathbb{R}^3)$.
a) Show that $B' = (u_1, u_2, u_3)$ is a basis of ${}_{\mathbb{R}}\mathbb{R}^3$.
b) Provided that $[f]_B = A$, find the matrix $[f]_{B'}$.
c) Provided that $[f]_{B'} = A$, find the matrix $[f]_B$.

10) Let V, V' be $\mathbb{R}$-vector spaces, $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$ bases in V and V', respectively and $f : V \to V'$ a $\mathbb{R}$-linear map with

$$[f]_{a,b} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Determine:
i) $f(v)$ for an arbitrary $v \in V$;
ii) the dimensions of $\operatorname{Im} f$ and $\operatorname{Ker} f$;
iii) the matrix $[f]_{a',b'}$ when $a' = (a_1, a_1+a_2, a_1+a_2+a_3)$ and $b' = (b_1, b_1+b_2, b_1+b_2+b_3)$.

11) Let V, V' be $\mathbb{R}$-vector spaces, $B = (v_1, v_2, v_3)$ a basis for V, $B' = (v_1', v_2', v_3')$ a basis for V' and $f : V \to V'$ a $\mathbb{R}$-linear map with

$$[f]_{B,B'} = \begin{pmatrix} 0 & -1 & 5 \\ 1 & 0 & 0 \\ 0 & 1 & -5 \end{pmatrix}.$$

Determine:
i) the dimension and a basis for each of the spaces $\operatorname{Im} f$ and $\operatorname{Ker} f$;
ii) $[f]_{B,E'}$ when $V' = \mathbb{R}^3$, $v_1' = (1,0,0)$, $v_2' = (0,1,1)$, $v_3' = (0,0,1)$ and E' is the standard basis of $\mathbb{R}^3$;
iii) $f(x)$ for $x = 2v_1 - v_2 + 3v_3$, under the circumstances of ii).

13) Let $f \in End_{\mathbb{Q}}(\mathbb{Q}^4)$ with the matrix in the standard basis

$$a) \begin{pmatrix} 1 & 2 & 1 & 2 \\ 3 & 2 & 3 & 2 \\ -1 & -3 & 0 & 4 \\ 0 & 4 & -1 & -3 \end{pmatrix};\quad b) \begin{pmatrix} 0 & 1 & 2 & 3 \\ -1 & 2 & 1 & 0 \\ 3 & 0 & -1 & -2 \\ 5 & -3 & -1 & 1 \end{pmatrix}.$$

Determine a basis for each of the $\mathbb{Q}$-vector spaces $\operatorname{Ker} f$, $\operatorname{Im} f$, $\operatorname{Ker} f + \operatorname{Im} f$ and $\operatorname{Ker} f \cap \operatorname{Im} f$.