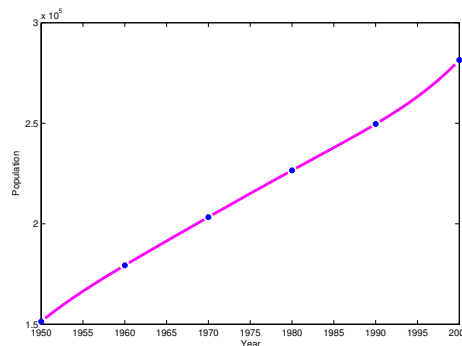


## COURSE 2

### 2.2. Lagrange interpolation

**Example 1** *A census of the population of the United States is taken every 10 years. The following table lists the population, in thousands of people, from 1950 to 2000.*

1950	1960	1970	1980	1990	2000
151326	179323	203302	226542	249633	281422



*Question: these data could be used to provide a reasonable estimate of the population in 1975? Answer: population in 1975 is 215042.*

Let  $[a, b] \subset \mathbb{R}$ ,  $x_i \in [a, b]$ ,  $i = 0, 1, \dots, m$  such that  $x_i \neq x_j$  for  $i \neq j$  and consider  $f : [a, b] \rightarrow \mathbb{R}$ .

**The Lagrange interpolation problem** (LIP) consists in determining the polynomial  $P$  of the smallest degree for which

$$P(x_i) = f(x_i), \quad i = 0, 1, \dots, m \quad (1)$$

i.e., the polynomial of the smallest degree which passes through the distinct points  $(x_i, f(x_i))$ ,  $i = 0, 1, \dots, m$ .

**Definition 2** *A solution of (LIP) is called **Lagrange interpolation polynomial**, denoted by  $L_m f$ .*

**Remark 3** *We have  $(L_m f)(x_i) = f(x_i)$ ,  $i = 0, 1, \dots, m$ .*

$L_m f \in \mathbb{P}_m$  ( $\mathbb{P}_m$  is the space of polynomials of at most  $m$ -th degree).

The Lagrange interpolation polynomial is given by

$$(L_m f)(x) = \sum_{i=0}^m \ell_i(x) f(x_i), \quad (2)$$

where by  $\ell_i(x)$  denote **the Lagrange fundamental interpolation polynomials**. We have

$$u(x) = \prod_{j=0}^m (x - x_j),$$

$$u_i(x) = \frac{u(x)}{x - x_i} = (x - x_0) \dots (x - x_{i-1})(x - x_{i+1}) \dots (x - x_m) = \prod_{\substack{j=0 \\ j \neq i}}^m (x - x_j)$$

and

$$\ell_i(x) = \frac{u_i(x)}{u_i(x_i)} = \frac{(x - x_0) \dots (x - x_{i-1})(x - x_{i+1}) \dots (x - x_m)}{(x_i - x_0) \dots (x_i - x_{i-1})(x_i - x_{i+1}) \dots (x_i - x_m)} = \prod_{\substack{j=0 \\ j \neq i}}^m \frac{x - x_j}{x_i - x_j}, \quad (3)$$

for  $i = 0, 1, \dots, m$ .

**Proposition 4** *We also have*

$$\ell_i(x) = \frac{u(x)}{(x - x_i)u'(x_i)}, \quad i = 0, 1, \dots, m. \quad (4)$$

**Proof.** We have  $u_i(x) = \frac{u(x)}{x - x_i}$ , so  $u(x) = u_i(x)(x - x_i)$ . We get  $u'(x) = u_i(x) + (x - x_i)u'_i(x)$ , whence it follows  $u'(x_i) = u_i(x_i)$ . So, as

$$\ell_i(x) = \frac{u_i(x)}{u_i(x_i)}$$

we get

$$\ell_i(x) = \frac{u_i(x)}{u_i(x_i)} = \frac{u(x)}{(x - x_i)u'(x_i)}, \quad i = 0, 1, \dots, m. \quad (5)$$

■

**Example 5** *a) Consider the nodes  $x_0, x_1$  and a function  $f$  to be interpolated.*

*b) Find the Lagrange polynomial that interpolates the data in the following table and find the approximative value of  $f(-0.5)$ .*

$x$	$-1$	$0$	$3$
$f(x)$	$8$	$-2$	$4$

*Sol.*

a) We have  $m = 1$ ,

$$u(x) = (x - x_0)(x - x_1)$$

$$u_0(x) = x - x_1$$

$$u_1(x) = x - x_0$$

$$\begin{aligned}(L_1 f)(x) &= l_0(x)f(x_0) + l_1(x)f(x_1) \\ &= \frac{x - x_1}{x_0 - x_1}f(x_0) + \frac{x - x_0}{x_1 - x_0}f(x_1),\end{aligned}$$

which is the line passing through the given points  $(x_0, f(x_0))$  and  $(x_1, f(x_1))$ .

b) We have  $m = 2$ . The Lagrange polynomial is

$$(L_2 f)(x) = l_0(x)f(x_0) + l_1(x)f(x_1) + l_2(x)f(x_2).$$

$u(x) = (x + 1)(x - 0)(x - 3)$  and it follows

$$\begin{aligned}l_0(x) &= \frac{(x - 0)(x - 3)}{(-1 - 0)(-1 - 3)} = \frac{1}{4}x(x - 3) \\l_1(x) &= \frac{(x + 1)(x - 3)}{(0 + 1)(0 - 3)} = -\frac{1}{3}(x + 1)(x - 3) \\l_2(x) &= \frac{(x + 1)(x - 0)}{(3 + 1)(3 - 0)} = \frac{1}{12}x(x + 1),\end{aligned}$$

The polynomial is

$$(L_2f)(x) = 2x(x - 3) + \frac{2}{3}(x + 1)(x - 3) + \frac{1}{3}x(x + 1).$$

and  $(L_2f)(-0.5) = 2.25$ .

**Remark 6** *Disadvantages of the form (2) of Lagrange polynomial: requires many computations and if we add or subtract a point we have to start with a complete new set of computations.*

Some calculations allow us to reduce the number of operations:

$$(L_m f)(x) = \frac{(L_m f)(x)}{1} = \frac{\sum_{i=0}^m l_i(x) f(x_i)}{\sum_{i=0}^m l_i(x)}.$$

Dividing the numerator and the denominator by

$$u(x) = \prod_{i=1}^m (x - x_i)$$

and denoting

$$A_i = \frac{1}{\prod_{j=0, j \neq i}^m (x_i - x_j)} = \frac{1}{u_i(x_i)}$$

one obtains

$$(L_m f)(x) = \frac{\sum_{i=0}^m \frac{A_i f(x_i)}{x - x_i}}{\sum_{i=0}^m \frac{A_i}{x - x_i}}, \quad (6)$$

called **the barycentric form** of *Lagrange interpolation polynomial*.



**Remark 7** *Formula (6) needs half of the number of arithmetic operations needed for (2) and it is easier to add or subtract a point.*

The Lagrange polynomial generates **the Lagrange interpolation formula**

$$f = L_m f + R_m f,$$

where  $R_m f$  denotes **the remainder (the error)**.

**Theorem 8** *Let  $\alpha = \min\{x, x_0, \dots, x_m\}$  and  $\beta = \max\{x, x_0, \dots, x_m\}$ . If  $f \in C^m[\alpha, \beta]$  and  $f^{(m)}$  is derivable on  $(\alpha, \beta)$  then  $\forall x \in (\alpha, \beta)$ , there exists  $\xi \in (\alpha, \beta)$  such that*

$$(R_m f)(x) = \frac{u(x)}{(m+1)!} f^{(m+1)}(\xi). \quad (7)$$

**Proof.** Consider

$$F(z) = \begin{vmatrix} u(z) & (R_m f)(z) \\ u(x) & (R_m f)(x) \end{vmatrix}.$$

From hypothesis it follows that  $F \in C^m[\alpha, \beta]$  and there exists  $F^{(m+1)}$  on  $(\alpha, \beta)$ .

We have

$$F(x) = 0, \quad F(x_i) = 0, \quad i = 0, 1, \dots, m,$$

as

$$u(x_i) = \prod_{j=0}^m (x_i - x_j) = 0$$

and

$$(R_m f)(x_i) = f(x_i) - (L_m f)(x_i) = f(x_i) - f(x_i) = 0,$$

so  $F$  has  $m + 2$  distinct zeros in  $(\alpha, \beta)$ . Applying successively the Rolle theorem it follows that:  $F$  has  $m + 2$  zeros in  $(\alpha, \beta) \Rightarrow F'$  has at least  $m + 1$  zeros in  $(\alpha, \beta) \Rightarrow \dots \Rightarrow F^{(m+1)}$  has at least one zero in  $(\alpha, \beta)$

So  $F^{(m+1)}$  has at least one zero  $\xi \in (\alpha, \beta)$ ,  $F^{(m+1)}(\xi) = 0$ .

We have

$$F^{(m+1)}(z) = \begin{vmatrix} u^{(m+1)}(z) & (R_m f)^{(m+1)}(z) \\ u(x) & (R_m f)(x) \end{vmatrix},$$

with

$$u(z) = \prod_{i=0}^m (z - z_i) \Rightarrow u^{(m+1)}(z) = (m+1)!,$$

and

$$\begin{aligned} (R_m f)^{(m+1)}(z) &= (f - (L_m f))^{(m+1)}(z) \\ &= f^{(m+1)}(z) - (L_m f)^{(m+1)}(z) = f^{(m+1)}(z) \end{aligned}$$

(as,  $L_m f \in \mathbb{P}_m$ ).

We have  $F^{(m+1)}(\xi) = 0$ , for  $\xi \in (\alpha, \beta)$ , so

$$F^{(m+1)}(\xi) = \begin{vmatrix} (m+1)! & f^{(m+1)}(\xi) \\ u(x) & (R_m f)(x) \end{vmatrix} = 0,$$

i.e.,  $(m+1)!(R_m f)(x) = u(x)f^{(m+1)}(\xi)$ ,

whence  $(R_m f)(x) = \frac{u(x)}{(m+1)!} f^{(m+1)}(\xi)$ . ■

**Corrolary 9** *If  $f \in C^{m+1}[a, b]$  then*

$$|(R_m f)(x)| \leq \frac{|u(x)|}{(m+1)!} \|f^{(m+1)}\|_{\infty}, \quad x \in [a, b]$$

where  $\|\cdot\|_{\infty}$  denotes the uniform norm, and  $\|f\|_{\infty} = \sup_{x \in [a, b]} |f(x)|$ .

**Example 10** *If we know that  $\lg 2 = 0.301$ ,  $\lg 3 = 0.477$ ,  $\lg 5 = 0.699$ , find  $\lg 76$ . Study the approximation error.*

**Example 11** *Which is the limit of the error for computing  $\sqrt{115}$  using Lagrange interpolation formula for the nodes  $x_0 = 100$ ,  $x_1 = 121$  and  $x_2 = 144$ ? Find the approximative value of  $\sqrt{115}$ .*

## The Aitken's algorithm

Let  $[a, b] \subset \mathbb{R}$ ,  $x_i \in [a, b]$ ,  $i = 0, 1, \dots, m$  such that  $x_i \neq x_j$  for  $i \neq j$  and consider  $f : [a, b] \rightarrow \mathbb{R}$ .

Usually, for a practical approximation problem, for a given function  $f : [a, b] \rightarrow \mathbb{R}$  we have to find the approximation of  $f(\alpha)$ ,  $\alpha \in [a, b]$  with an error not greater than a given  $\varepsilon > 0$ .

If we have enough information about  $f$  and its derivatives, we use the inequality  $|(R_m f)(x)| \leq \varepsilon$  to find  $m$  such that  $(L_m f)(\alpha)$  approximates  $f(\alpha)$  with the given precision.

We may use the condition  $\frac{|u(x)|}{(m+1)!} \|f^{(m+1)}\|_{\infty} \leq \varepsilon$ , but it should be known  $\|f^{(m+1)}\|_{\infty}$  or a majorant of it.

A practical method for computing the Lagrange polynomial is **the Aitken's algorithm**. This consists in generating the table:

$x_0$	$f_{00}$				
$x_1$	$f_{10}$	$f_{11}$			
$x_2$	$f_{20}$	$f_{21}$	$f_{22}$		
$x_3$	$f_{30}$	$f_{31}$	$f_{32}$	$f_{33}$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$x_m$	$f_{m0}$	$f_{m1}$	$f_{m2}$	$f_{m3}$	$\dots f_{mm}$

where

$$f_{i0} = f(x_i), \quad i = 0, 1, \dots, m,$$

and

$$f_{i,j+1} = \frac{1}{x_i - x_j} \left| \begin{array}{cc} f_{jj} & x_j - x \\ f_{ij} & x_i - x \end{array} \right|, \quad i = 0, 1, \dots, m; j = 0, \dots, i - 1.$$

For example,

$$\begin{aligned} f_{11} &= \frac{1}{x_1 - x_0} \begin{vmatrix} f_{00} & x_0 - x \\ f_{10} & x_1 - x \end{vmatrix} \\ &= \frac{1}{x_1 - x_0} [f_{00}(x_1 - x) - f_{10}(x_0 - x)] \\ &= \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1) = (L_1 f)(x), \end{aligned}$$

so  $f_{11}$  is the value in  $x$  of Lagrange polynomial for the nodes  $x_0, x_1$ .  
We have

$$f_{ii} = (L_i f)(x),$$

$L_i f$  being Lagrange polynomial for the nodes  $x_0, x_1, \dots, x_i$ .

So  $f_{11}, f_{22}, \dots, f_{ii}, \dots, f_{mm}$  is a sequence of approximations of  $f(x)$ .

If the interpolation procedure is convergent then the sequence is also convergent, i.e.,  $\lim_{m \rightarrow \infty} f_{mm} = f(x)$ . By Cauchy convergence criterion it follows

$$\lim_{i \rightarrow \infty} |f_{ii} - f_{i-1,i-1}| = 0.$$

This could be used as a stopping criterion, i.e.,

$$\left| f_{ii} - f_{i-1,i-1} \right| \leq \varepsilon, \quad \text{for a given precision } \varepsilon > 0.$$

Recommendation is to sort the nodes  $x_0, x_1, \dots, x_m$  with respect to the distance to  $x$ , such that

$$|x_i - x| \leq |x_j - x| \quad \text{if } i < j, \quad i, j = 1, \dots, m.$$

**Example 12** *Approximate  $\sqrt{115}$  with precision  $\varepsilon = 10^{-3}$ , using Aitken's algorithm.*