Calculus on $\mathbb{R}$ Lect. dr. ANCA GRAD
Homework 7 **2022**

Continuous Functions

Exercise 1: By using the defintion, verify if the following function

$$f:\mathbb{R}\to\mathbb{R},\quad f(x)=x^2$$

is continuous.

Exercise 2:
a) By using the ε and δ characterization theorem, prove that the function

$$f:\mathbb{R}\to\mathbb{R},\quad f(x)=\begin{cases} x & : x\in\mathbb{Q} \\ -x & : x\in\mathbb{R}\setminus\mathbb{Q}, \end{cases}$$

is continuous at 0.

b) By using the definition, prove that the function from **a)** has not other continuity points, except for 0, thus $\forall x\in\mathbb{R}\setminus\{0\}$, f is discontinuous at x.

Exercise 3: Study the continuity of the functions:

a) $f:(-\infty,0]\to\mathbb{R}$,

$$f(x)=\begin{cases} \sin x & : x\in(-\infty,0) \\ 7 & : x=0, \end{cases}$$

b)$f:[-1,2]\cup\{4\}\to\mathbb{R}$,

$$f(x)=\begin{cases} 2x+3 & : x\in[-1,2] \\ 0 & : x=4. \end{cases}$$

Exercise 4: Study the continuity of the functions:

a)$f:\mathbb{R}\to\mathbb{R}$,

$$f(x)=\begin{cases} \dfrac{\sin x^2}{|x|} & : x\neq 0 \\ 0 & : x=0. \end{cases}.$$

b)$f:\mathbb{R}\to\mathbb{R}$,

$$f(x)=\begin{cases} e^{x^{-1}} & : x\in(0,\infty) \\ 0 & : x=0, \\ x^2+2x+\sin x & : x\in(-\infty,0). \end{cases}.$$

c)$f:\mathbb{R}\to\mathbb{R}$,

$$f(x)=\begin{cases} \sin x & : x\in\mathbb{Q} \\ \cos x & : x\in\mathbb{R}\setminus\mathbb{Q}, \end{cases}.$$

d) $f : [-2, 1] \cup \{3\} \to \mathbb{R}$,

$$f(x) = \begin{cases} \cos(\pi x) & : x \in [-2, 0] \\ 1 + \sin x & : x = (0, 1] \\ 2 & : x = 3. \end{cases}$$

Exercise 5: Study, by discussing the parameter $a \in \mathbb{R}$, the continuity of the following functions:

a) $f : [1, 3] \to \mathbb{R}$,

$$f(x) = \begin{cases} \sqrt{a^2 - 2ax + x^2} & : x \in [1, 2] \\ \\ 3a + 2x & : x \in (2, 3]. \end{cases}$$

b) $f : (0, \pi) \to \mathbb{R}$,

$$f(x) = \begin{cases} e^{3x} & : x \in (0, 1] \\ \\ a\dfrac{\sin(x-1)}{x^2 - 5x + 4} & : x \in (1, \pi). \end{cases}$$

Exercise 6: Let $0 < a < b \in \mathbb{R}$ and $f : \mathbb{R}\backslash\{0, 1\} \to \mathbb{R}$, defined by:

$$f(x) = \left(\frac{b^x - a^x}{x(b-a)}\right)^{\frac{1}{x-1}}, \forall x \in \mathbb{R}\backslash\{0, 1\}.$$

- a) Prove that f is a continuous function.
- b) Prove that there exists a continuous function $F : \mathbb{R} \to \mathbb{R}$ such that $F(x) = f(x), \forall x \in \mathbb{R}\backslash\{0, 1\}$.
- c) Compute $\lim\limits_{x \to -\infty} F(x)$ and $\lim\limits_{x \to \infty} F(x)$.

Theory briefing

General Hypotheses

$$\begin{cases} \emptyset \neq D \subseteq \mathbb{R} \\ f : D \to \mathbb{R} \\ x_0 \in D. \end{cases}$$

Definition:

The function f is said to be **continuous at** x_0 if

$$\forall (x_n) \subseteq D \quad \text{with} \quad \lim_{n \to \infty} x_n = x_0 \Longrightarrow \lim_{n \to \infty} f(x_n) = f(x_0).$$

Characterization theorem with neighborhoods:

f is continuous at x_0 if and only if

$$\forall V \in \mathcal{V}(f(x_0)), \quad \exists U \in \mathcal{V}(x_0) \quad \text{such that} \quad f(x) \in V,$$

Characterization theorem with ε and δ:

f is continuous at x_0 if and only if

$\forall \varepsilon > 0, \quad \exists \delta > 0,$ such that $\quad \forall x \in D \quad$ with $\quad |x - x_0| < \delta, \quad$ it holds $\quad |f(x) - f(x_0)| < \varepsilon.$

Theorem characterizing the connection between limits and continuity:

Let $x_0 \in D \cap D' = D \backslash IzoD$. The following statements are true:

1. f is continuous at în $x_0 \Longrightarrow \exists \lim\limits_{x \to x_0} f(x) = f(x_0 - 0) = f(x_0 + 0) = f(x_0)$.
2. If $\begin{cases} \exists f(x_0 - 0) \\ \exists f(x_0 + 0) \\ f(x_0 - 0) = f(x_0 + 0) = f(x_0) \end{cases} \Longrightarrow f$ is continuous at x_0.

Remark: By using the definition, it can be easily proved that all elementary functions are continuous on their greatest definition domain. Therefore, unless you are explicitly required to prove the continuity of such elementary functions, you may assume by default that they are continuous.