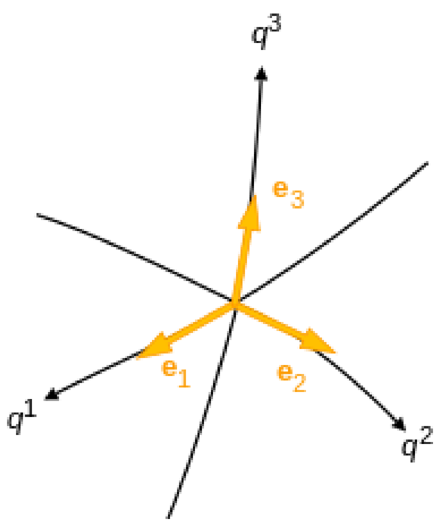
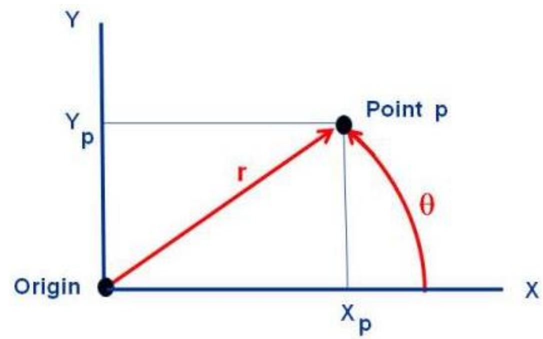
Theoretical Mechanics

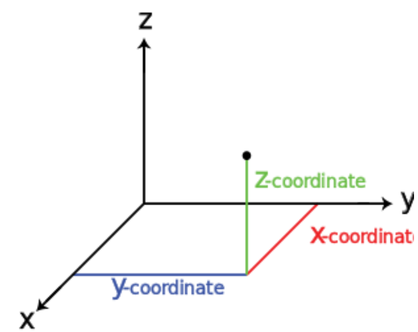
2. Kinematics of the material point in orthogonal curvilinear coordinates

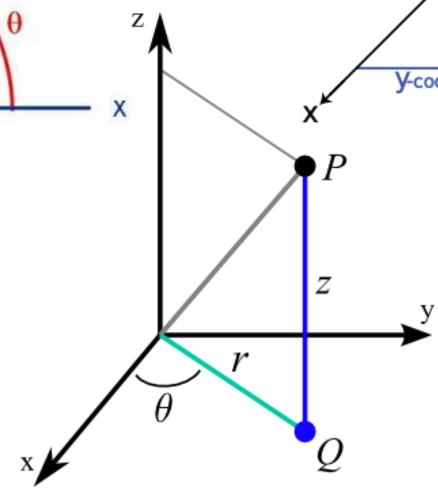
Curvilinear coordinates (q_1, q_2, q_3) are a coordinate system for Euclidean space in which the coordinate lines may be curved.

Examples: polar, rectangular, spherical, and cylindrical coordinate systems.









(2.3)

Theoretical Mechanics

We assume the existence of a vector function $\vec{r}\colon \Omega \subset R^3 \to D \subset R^3$ such that

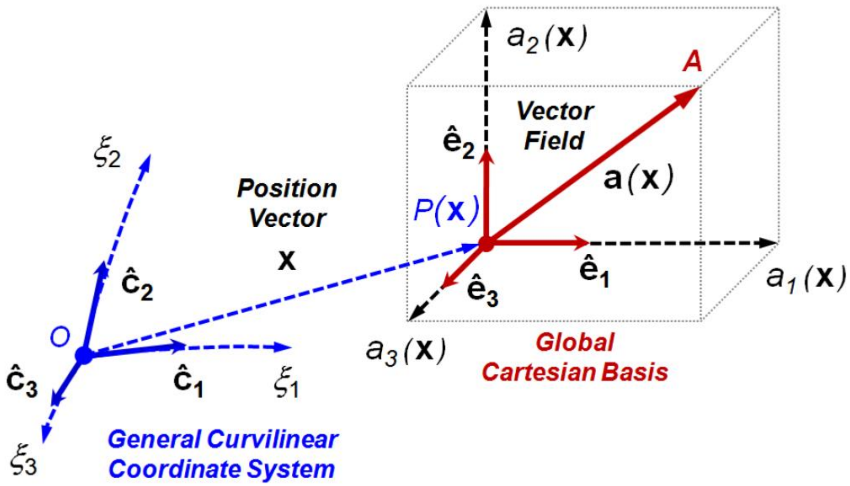
$$\Omega \ni (q_1, q_2, q_3) \xmapsto{r} \overrightarrow{r}(q_1, q_2, q_3) \in D \tag{2.1}$$

$$\overrightarrow{r} = \overrightarrow{r}(q_1, q_2, q_3)$$

$$x = x(q_1, q_2, q_3), \quad y = y(q_1, q_2, q_3), \quad z = z(q_1, q_2, q_3), \tag{2.2}$$

This function associates to each triplet $(q_1, q_2, q_3) \ni \Omega$ an unique position $\mathrm{M}(x, y, z) \in \mathrm{R}^3$ and vice versa.

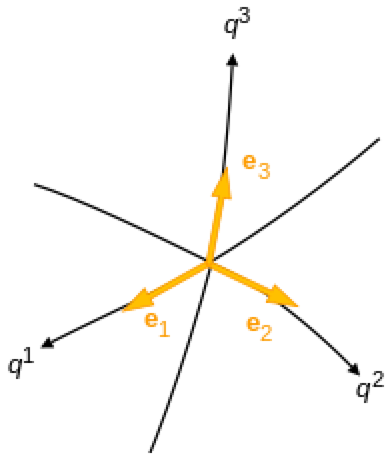
$$q_i = q_i(x, y, z),\ i = 1, 2, 3 \tag{2.3}$$



Thus, the application should be a diffeomorphism of class C^2 and in order to exist is necessary that:

$$J = \frac{\partial(x, y, z)}{\partial(q_1, q_2, q_3)} \neq 0 \text{ in } \Omega. \tag{2.4}$$

In this case the equation of motion are



$$q_i = q_i(t),\ t \in [t_0, T],\ i = 1, 2, 3. \tag{2.5}$$

Given two manifolds M and N, a differentiable map $f : M \to N$ is called a **diffeomorphism** if it is a bijection and its inverse $f^{-1} : N \to M$ is differentiable as well. If these functions are r times continuously differentiable, f is called a $\boldsymbol{C^r}$**-diffeomorphism**.

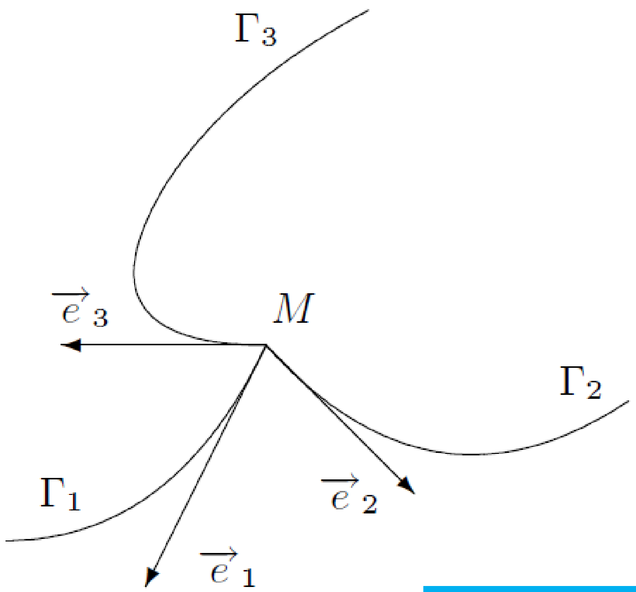
The curves:

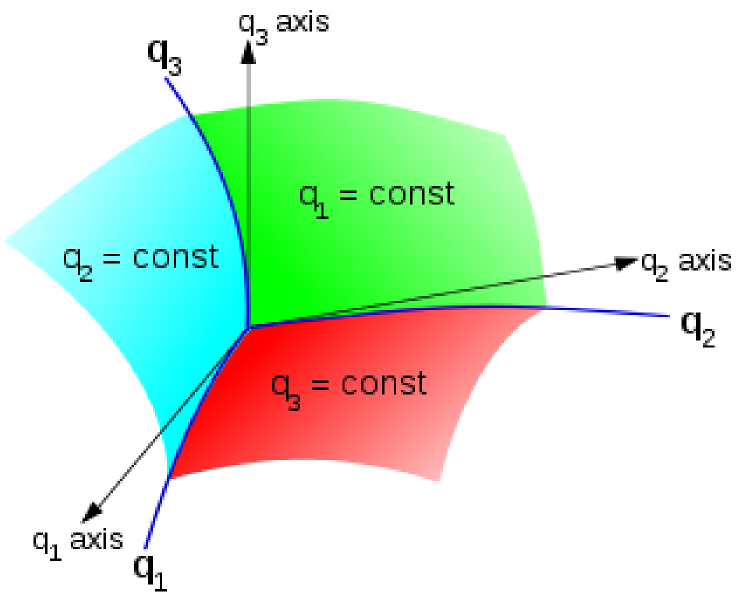
$\Gamma_{1:}$ • q_1 =variabil, q_2 = constant, q_3 =constant

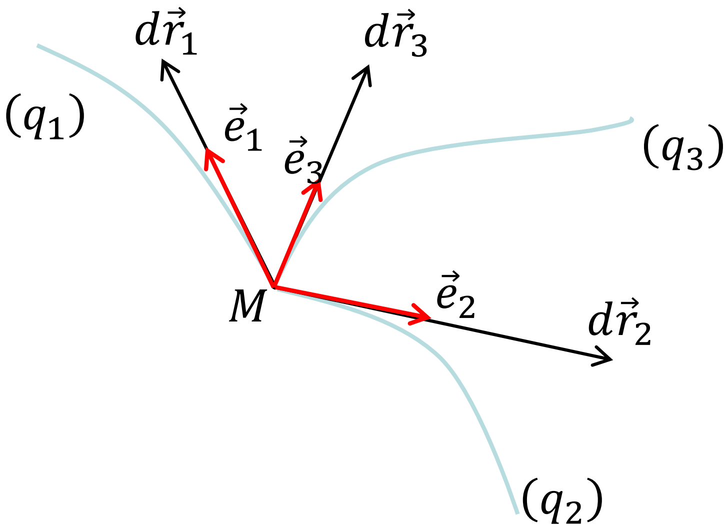
Γ_2: • q_2 =variabil, q_3 = constant, q_1 =constant

Γ_3: • q_3 =variabil, q_1 = constant, q_2 =constant

are called *curves of coordinates*. In a similar way the *surfaces of coordinates* are obtained.







The elementary displacement on

$$(\Gamma_1): \ \overrightarrow{r}(q_1 = \text{var.}, q_2 = \text{const.}, q_3 = \text{const.}) := \overrightarrow{r}_1(q_1).$$

is: $$d\overrightarrow{r}_1 = \frac{d\overrightarrow{r}_1}{dq_1}dq_1 = \frac{\partial \overrightarrow{r}}{\partial q_1}dq_1$$

and the length of the curve arc covered by particle M is given by:

$$ds_1 = d\overrightarrow{r}_1| = \left|\frac{\partial \overrightarrow{r}}{\partial q_1}\right| dq_1 = \frac{ds_1}{dq_1}dq_1.$$

Taking into account that $d\vec{r}$ is tangent to the curve Γ_1 then

$$\overrightarrow{e}_1 = \frac{1}{\left|\frac{\partial \overrightarrow{r}}{\partial q_1}\right|}\frac{\partial \overrightarrow{r}}{\partial q_1} = \frac{1}{H_1}\frac{\partial \overrightarrow{r}}{\partial q_1}, \quad H_1 := \left|\frac{\partial \overrightarrow{r}}{\partial q_1}\right|.$$

Generally, we have:

$$\overrightarrow{e}_i = \frac{1}{H_i}\frac{\partial \overrightarrow{r}}{\partial q_i}, \quad H_i = \left|\frac{\partial \overrightarrow{r}}{\partial q_i}\right| = \left(\left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2 + \left(\frac{\partial z}{\partial q_i}\right)^2\right)^{\frac{1}{2}} \qquad (2.6)$$

where H_i are the coefficients of Lamé corresponding to the curves Γ_i $(\mathrm{i} = 1,2,3)$.

However, for a general motion the elementary displacement on the trajectory is:

$$d\vec{r} = \sum_{i=1}^{3} d\vec{r}_i = \sum_{i=1}^{3} \frac{\partial \vec{r}}{\partial q_i} dq_i = \sum_{i=1}^{3} H_i dq_i \vec{e}_i \qquad (2.7)$$

$$ds = |d\vec{r}| = \sqrt{\sum_{i=1}^{3} (H_i dq_i)^2}\,; \quad ds_i = H_i q_i, i = 1,2,3 \qquad (2.8)$$

Velocity in curvilinear coordinates

Taking into account that $\vec{r} = \vec{r}(q_1, q_2, q_3)$ and $q_i = q_i(t),\ t \in [t_0, T],\ i = 1, 2, 3$

one obtain:

$$\vec{v} = \frac{d\vec{r}}{dt} = \sum_{i=1}^{3} \frac{\partial \vec{r}}{\partial q_i} \dot{q}_i = \sum_{i=1}^{3} H_i \dot{q}_i \vec{e}_i$$

or

$$\vec{v} = \sum_{i=1}^{3} H_i \dot{q}_i \vec{e}_i = \sum_{i=1}^{3} v_i \vec{e}_i \qquad (2.8)$$

where

$$v_i = H_i \dot{q}_i, \quad (i = 1,2,3) \qquad (2.9)$$

are the components of the velocity in the frame $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$.

Acceleration in curvilinear coordinates

Consider the components of the acceleration:

$$a_i := pr_{\vec{e}_i}\vec{a} = \vec{a}\cdot\vec{e}_i = \vec{a}\cdot\frac{1}{H_i}\frac{\partial\vec{r}}{\partial q_i}$$

$$= \frac{1}{H_i}\frac{d\vec{v}}{dt}\cdot\frac{\partial\vec{r}}{\partial q_i} = \frac{1}{H_i}\left[\frac{d}{dt}\left(\vec{v}\cdot\frac{\partial\vec{r}}{\partial q_i}\right) - \vec{v}\cdot\frac{d}{dt}\left(\frac{\partial\vec{r}}{\partial q_i}\right)\right]$$

We have
$$\vec{v} = \sum_{k=1}^{3}\frac{\partial\vec{r}}{\partial q_k}\dot{q}_k \;\Rightarrow\; \frac{\partial\vec{r}}{\partial q_i} = \frac{\partial\vec{v}}{\partial\dot{q}_i}$$

$$\vec{r} = \vec{r}(q_1, q_2, q_3) \;\Rightarrow\; \frac{\partial}{\partial\dot{q}_i}\left(\frac{\partial\vec{r}}{\partial q_k}\right) = 0$$

$$\frac{d}{dt}\left(\frac{\partial\vec{r}}{\partial q_i}\right) = \sum_{k=1}^{3}\frac{\partial}{\partial q_k}\left(\frac{\partial\vec{r}}{\partial q_i}\right)\dot{q}_k = \sum_{k=1}^{3}\frac{\partial^2\vec{r}}{\partial q_k\partial q_i}\dot{q}_k$$

$$= \sum_{k=1}^{3}\frac{\partial^2\vec{r}}{\partial q_i\partial q_k}\dot{q}_k = \frac{\partial}{\partial q_i}\left(\sum_{k=1}^{3}\frac{\partial\vec{r}}{\partial q_k}\dot{q}_k\right) = \frac{\partial\vec{v}}{\partial q_i},\;\; i = 1, 2, 3$$

Finally, one obtain

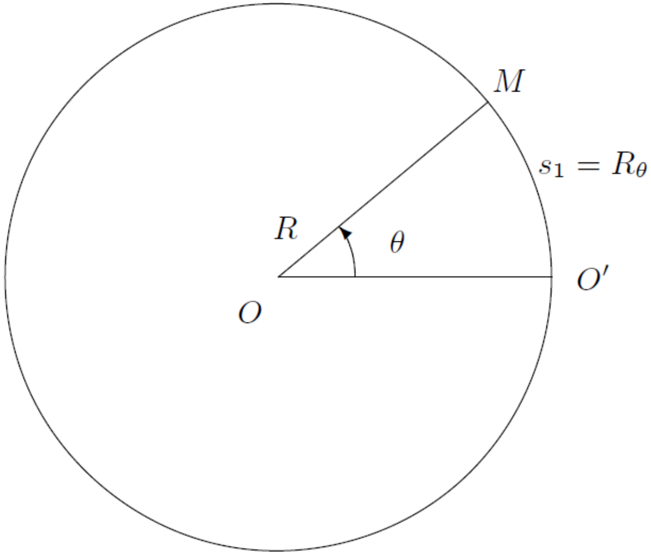
$$a_i = \frac{1}{2H_i}\left[\frac{d}{dt}\left(\frac{\partial v^2}{\partial \dot{q}_i}\right) - \frac{\partial v^2}{\partial q_i}\right], \; i = 1, 2, 3. \qquad (2.10)$$

Remark: If the curve of coordinate is a straight line or a circle of radius *R* then we have $H_{\text{line}} = 1$ and $H_{\text{circle}} = R$.

Indeed, for Γ_1 line we have $dq_1 = ds_1 \Rightarrow$

$$\underbrace{ds_1}_{|d\overrightarrow{r}_1|} = \underbrace{\left|\frac{d\overrightarrow{r}}{\partial q_1}\right|}_{\left|\frac{d\overrightarrow{r}_1}{dq_1}\right|} dq_1 = H_1 dq_1$$

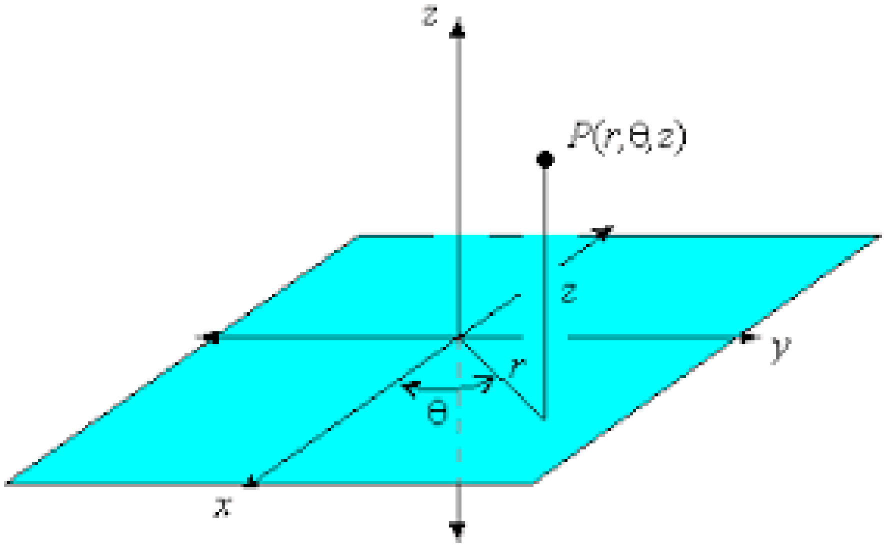
For Γ_1 circle we have $ds_1 = Rd\theta$, and thus



$$H_1 = \frac{ds_1}{dq_1} = \begin{cases} 1, & \Gamma_1 \ (s_1 = q_1) \\ R, & \Gamma_1 \ (s_1 = R\theta) \end{cases}$$

Particular curvilinear coordinates –cylindrical coordinates

(q_1, q_2, q_3) : $r = OM', \ \theta = (\widehat{Ox, OM'}), \ z.$



Equations of the motion are:

$$r = r(t), \quad \theta = \theta(t), \quad z = z(t), \quad t \in [t_0, T]$$

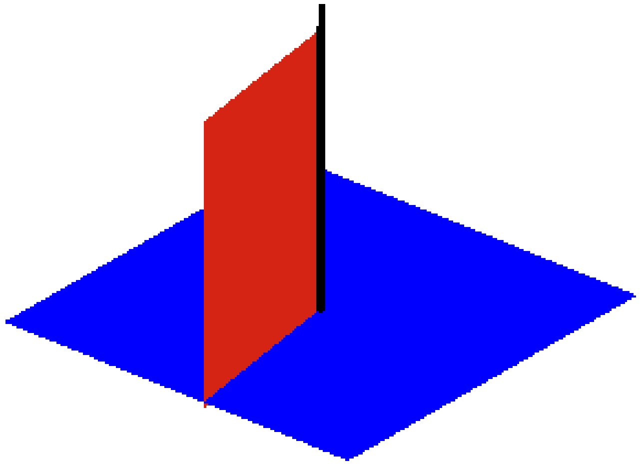
Functional relations and the Jacobian:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z. \tag{2.11}$$

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, z)} = r.$$

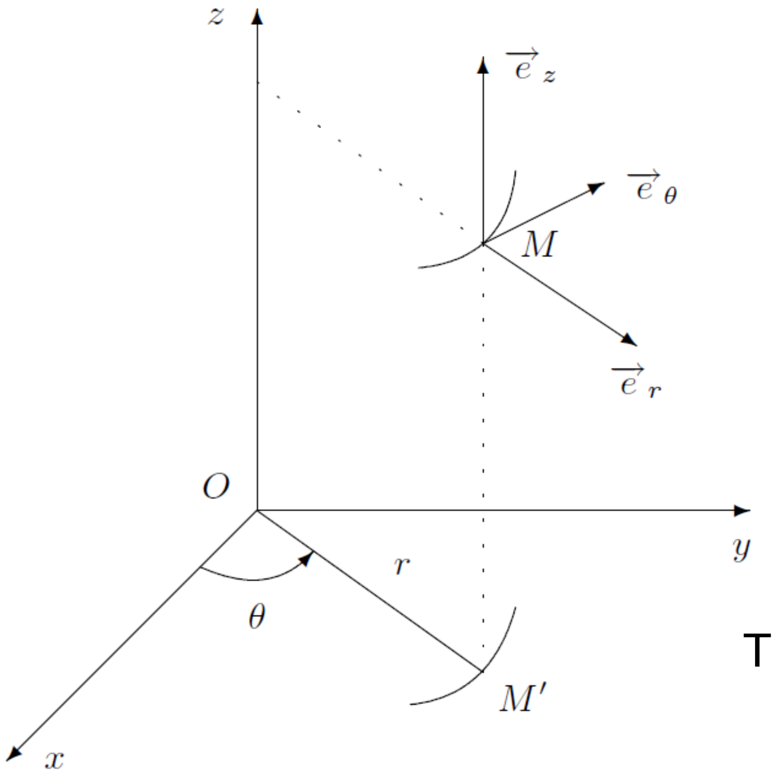
Particular curvilinear coordinates –cylindrical coordinates

(q_1, q_2, q_3) : $r = OM',\ \theta = (\widehat{Ox, OM'}),\ z.$



Cylindrical coordinate surfaces. The three orthogonal components, *r* (green), θ (red), and *z* (blue), each increasing at a constant rate. The point is at the intersection between the three colored surfaces.

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z. \qquad (2.11)$$



The curves:

Γ_r is a line with the versor $\vec{e}_r \Rightarrow H_r = 1$

Γ_θ is a circle with the versor $\vec{e}_\theta \Rightarrow H_\theta = r$

Γ_z is a line with the versor $\vec{e}_z \Rightarrow H_z = 1$

The curves of coordinates (r, θ, z) are orthogonal.

$$\frac{\partial x}{\partial q_i}\frac{\partial x}{\partial q_j} + \frac{\partial y}{\partial q_i}\frac{\partial y}{\partial q_j} + \frac{\partial z}{\partial q_i}\frac{\partial z}{\partial q_j} = 0, \ \ i \neq j.$$ (orthogonality condition)

Displacement:

$$ds_r = H_r dr = dr; \quad ds_\theta = H_\theta d\theta = r d\theta; \quad ds_z = H_z dz = dz$$

$$d\vec{r}_r = ds_r \vec{e}_r = dr\vec{e}_r; \quad d\vec{r}_\theta = ds_\theta \vec{e}_\theta = r d\theta \vec{e}_\theta; \quad d\vec{r}_z = ds_z \vec{e}_r = dz\vec{e}_r; \qquad (2.12)$$

$$d\vec{r} = d\vec{r}_r + d\vec{r}_\theta + d\vec{r}_z = dr\,\vec{e}_r + r d\theta\,\vec{e}_\theta + dz\,\vec{e}_r$$

$$ds = |d\vec{r}| = \sqrt{(dr)^2 + (r d\theta)^2 + (dz)^2}$$

Thus, the velocity and acceleration components are:

$$\overrightarrow{v} = v_r \overrightarrow{e}_r + v_\theta \overrightarrow{e}_\theta + v_z \overrightarrow{e}_z, \quad v_r = \dot{r}, \quad v_\theta = r\dot{\theta}, \quad v_z = \dot{z}, \qquad (2.13)$$

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2.$$

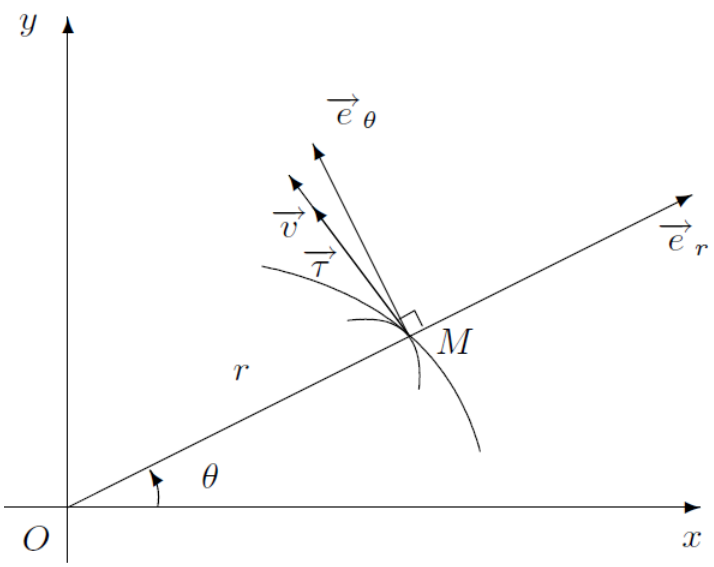
$$a_r := pr_{\overrightarrow{e}_r}\overrightarrow{a} = \frac{1}{2H_r}\left[\frac{d}{dt}\left(\frac{\partial v^2}{\partial \dot{r}}\right) - \frac{\partial v^2}{\partial r}\right]$$

$$= \frac{1}{2}\left[\frac{d}{dt}(2\dot{r}) - 2r\dot{\theta}^2\right] = \ddot{r} - r\dot{\theta}^2 \qquad (2.14)$$

$$a_r = \ddot{r} - r\dot{\theta}^2, \quad a_\theta = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}), \quad a_z = \ddot{z}$$

$$\vec{a} = a_r\vec{e}_r + a_\theta\vec{e}_\theta + a_z\vec{e}_z = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)\vec{e}_\theta + \ddot{z}\vec{e}_z$$

Remark: For z=0, the motion in **polar coordinates** is obtained.



Equations of the motion are:

$$r = r(t),\ \ \theta = \theta(t),\ \ t \in [t_0, T]$$

$$x = r\cos\theta, \quad y = r\sin\theta$$

The velocity and acceleration components are

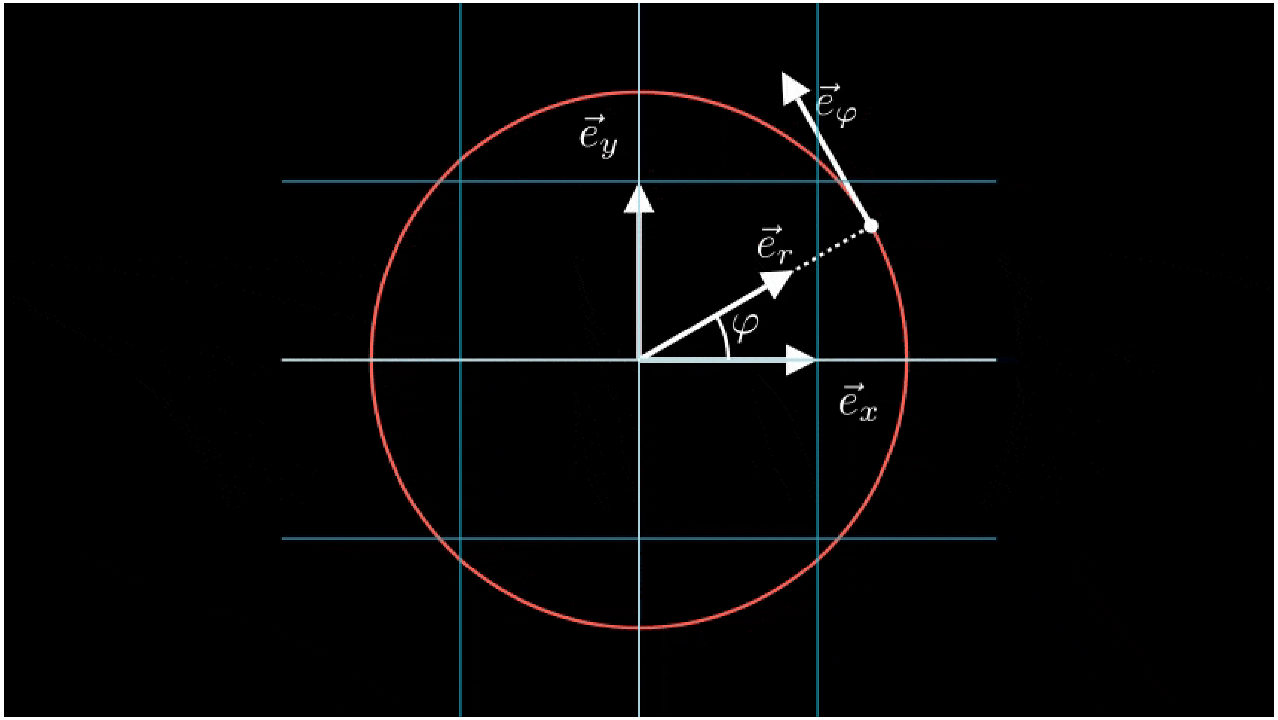
$$\vec{v} = v_r \vec{e}_r + v_\theta \vec{e}_\theta \tag{2.15}$$

$v_r = \dot{r}$ is the radial velocity

$v_\theta = r\dot{\theta}$ is the transversal (circumferential) velocity

$$\vec{a} = a_r \vec{e}_r + a_\theta \vec{e}_\theta; \quad \left| \begin{aligned} a_r &= \ddot{r} - r\dot{\theta}^2 \\ a_\theta &= \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) \end{aligned} \right. \tag{2.16}$$

Using the notation with φ: $v_r = \dot{r}$,$v_\varphi = r\dot{\varphi}$, $\vec{v} = v_r\vec{e}_r + v_\varphi\vec{e}_\varphi = \dot{r}\vec{e}_r + r\dot{\varphi}\,\vec{e}_\varphi$



Remark: It is also possible to calculate directly, using (2.6) the values for the Lame's coefficients in cylindrical coordinates.

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

$$H_r = \sqrt{\left(\frac{dx}{dr}\right)^2 + \left(\frac{dy}{dr}\right)^2 + \left(\frac{dz}{dr}\right)^2} = \sqrt{\cos^2\theta + \sin^2\theta + 0^2} = 1$$

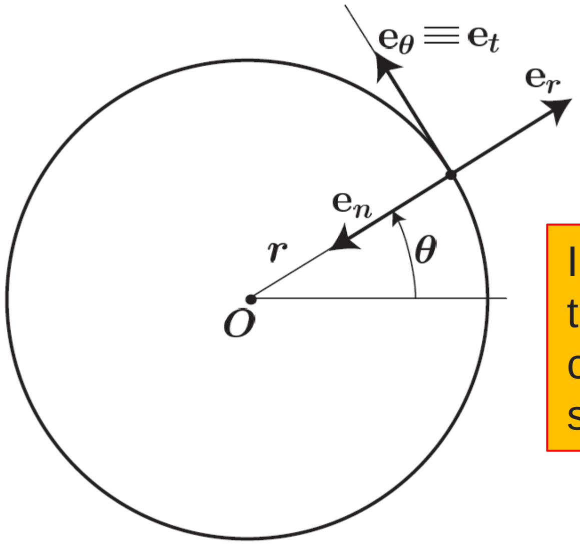
$$H_\theta = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 + \left(\frac{dz}{d\theta}\right)^2} = \sqrt{\mathrm{r}^2\cos^2\theta + r^2\sin^2\theta + 0^2} = r$$

$$H_z = \sqrt{\left(\frac{dx}{dz}\right)^2 + \left(\frac{dy}{dz}\right)^2 + \left(\frac{dz}{dz}\right)^2} = \sqrt{0^2 + 0^2 + 1^2} = 1 \qquad (2.17)$$

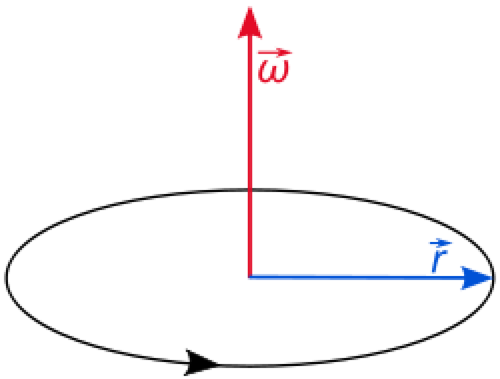
https://ocw.mit.edu/courses/aeronautics-and-astronautics/16-07-dynamics-fall-2009/lecture-notes/MIT16_07F09_Lec05.pdf

Example ***Circular motion***

Consider as an illustration, the motion of a particle in a circular trajectory having angular velocity $\omega = \dot{\theta}$, and angular acceleration $\alpha = \dot{\omega}$.

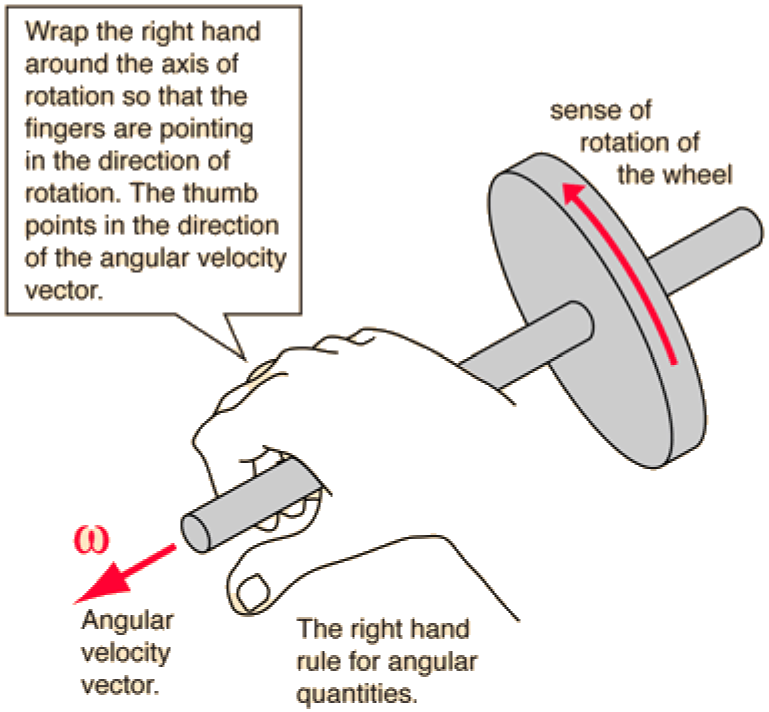


In this case the normal and tangent versors and the polar coordinate versors have the same directions.



The angular velocity describes the rate of change of angular position, and the instantaneous axis of rotation. In this case (counter-clockwise rotation) the vector points up.

How to find the sense of the angular velocity:



In polar coordinates, the equation of the trajectory is

$$r = R = \text{constant}, \qquad \theta = \omega t + \frac{1}{2}\alpha t^2$$

The velocity components are

$$v_r = \dot{r} = 0, \qquad v_\theta = r\dot{\theta} = R(\omega + \alpha t) = v$$

and the acceleration components are

$$a_r = \ddot{r} - r\dot{\theta}^2 = -R(\omega + \alpha t)^2 = -\frac{v^2}{R}$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = R\alpha = a_t$$

where we clearly see that, $a_r \equiv -a_n$, and that $a_\theta \equiv a_t$.

In cartesian coordinates, we have for the trajectory,

$$x = R\cos(\omega t + \frac{1}{2}\alpha t^2), \quad y = R\sin(\omega t + \frac{1}{2}\alpha t^2) \; .$$

For the velocity,

$$v_x = -R(\omega + \alpha t)\sin(\omega t + \frac{1}{2}\alpha t^2), \quad v_y = R(\omega + \alpha t)\cos(\omega t + \frac{1}{2}\alpha t^2) \; ,$$

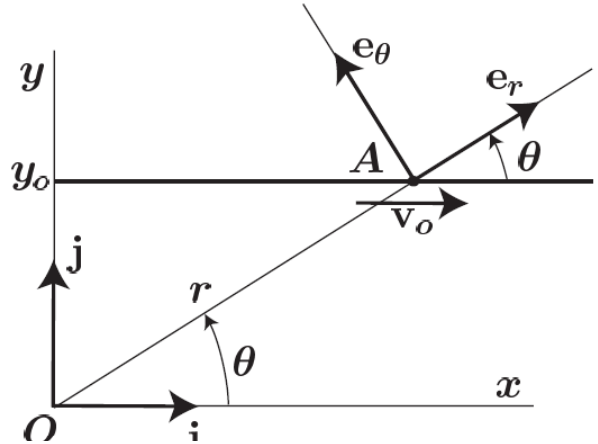
and, for the acceleration,

$$a_x = -R(\omega+\alpha t)^2\cos(\omega t+\frac{1}{2}\alpha t^2) - R\alpha\sin(\omega t+\frac{1}{2}\alpha t^2), \quad a_y = -R(\omega+\alpha t)^2\sin(\omega t+\frac{1}{2}\alpha t^2) + R\alpha\cos(\omega t+\frac{1}{2}\alpha t^2) \; .$$

We observe that, for this problem, the result is much simpler when expressed in polar (or intrinsic) coordinates.

Example ***Motion on a straight line***

Here we consider the problem of a particle moving with constant velocity v_0, along a horizontal line $y = y_0$.



Assuming that at $t = 0$ the particle is at $x = 0$, the trajectory and velocity components in cartesian coordinates are simply,

$$x = v_0 t \qquad y = y_0$$

$$v_x = v_0 \qquad v_y = 0$$

$$a_x = 0 \qquad a_y = 0 \quad .$$

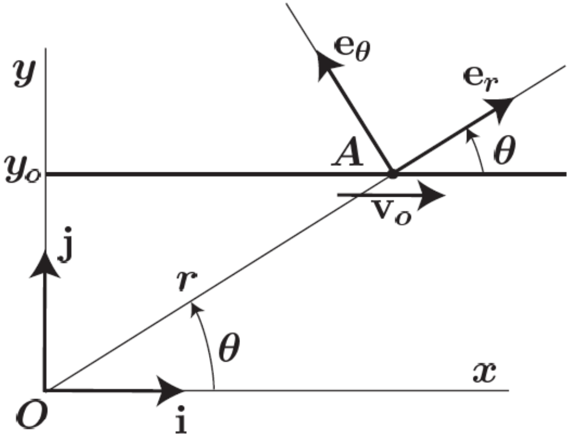
In polar coordinates, we have,

$$r = \sqrt{v_0^2 t^2 + y_0^2} \qquad \theta = \tan^{-1}\left(\frac{y_0}{v_0 t}\right)$$

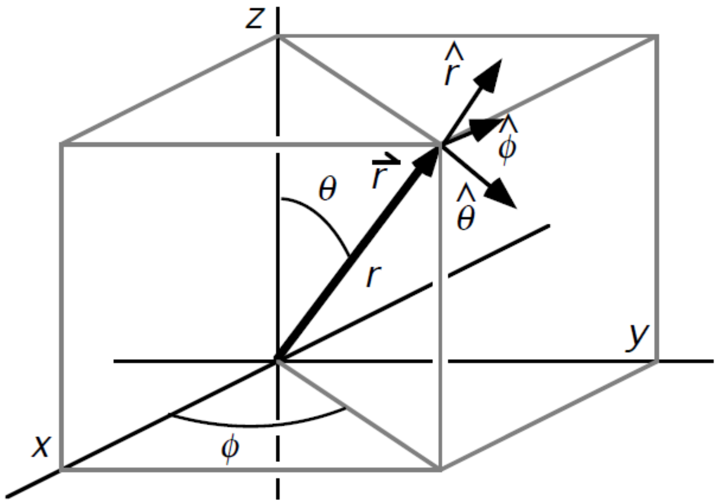
$$v_r = \dot{r} = v_0 \cos\theta \qquad v_\theta = r\dot{\theta} = -v_0 \sin\theta$$

$$a_r = \ddot{r} - r\dot{\theta}^2 = 0 \qquad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \quad .$$

Here, we see that the expressions obtained in cartesian coordinates are simpler than those obtained using polar coordinates. It is also reassuring that the acceleration in both the r and θ direction, calculated from the general two-term expression in polar coordinates, works out to be zero as it must for constant velocity-straight line motion.



Particular curvilinear coordinates –spherical coordinates



Transforms

The forward and reverse coordinate transformations are

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \arctan\left(\sqrt{x^2 + y^2}, z\right)$$

$$\phi = \arctan(y, x)$$

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta$$

$$\left|\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)}\right| = r^2 \sin\phi.$$

The curves:

Γ_r is a line with the versor $\vec{e}_r(\hat{r}) \Rightarrow H_r = 1$

Γ_θ is a circle of radius r with the versor $\vec{e}_\theta(\hat{\theta}) \Rightarrow H_\theta = r$

Γ_ϕ is a circle of radius $r \sin\theta$ with the versor $\vec{e}_\phi(\hat{\phi}) \Rightarrow H_\phi = r \sin\theta$

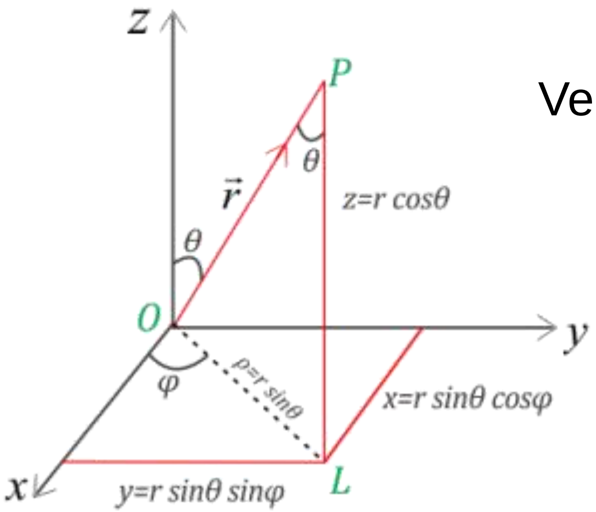
Displacement:

$$ds_r = H_r dr = dr; \quad ds_\theta = H_\theta d\theta = r d\theta; \quad ds_\phi = H_\phi d\phi = r\sin\theta\, d\phi$$

$$d\vec{r}_r = dr\vec{e}_r; \quad d\vec{r}_\theta = rd\theta\vec{e}_\theta; \quad d\vec{r}_\phi = r\sin\theta\ \ d\phi\vec{e}_\phi; \qquad (2.18)$$

$$d\vec{r} = d\vec{r}_r + d\vec{r}_\theta + d\vec{r}_\phi = dr\ \vec{e}_r + rd\theta\ \vec{e}_\theta + r\sin\theta\ \ d\phi\vec{e}_\phi$$

$$ds = |d\vec{r}| = \sqrt{(dr)^2 + (rd\theta)^2 + (r\sin\theta\ \ d\phi)^2}$$



Velocity:

$$\vec{v} = v_r\vec{e}_r + v_\theta\vec{e}_\theta + v_\phi\vec{e}_\phi$$

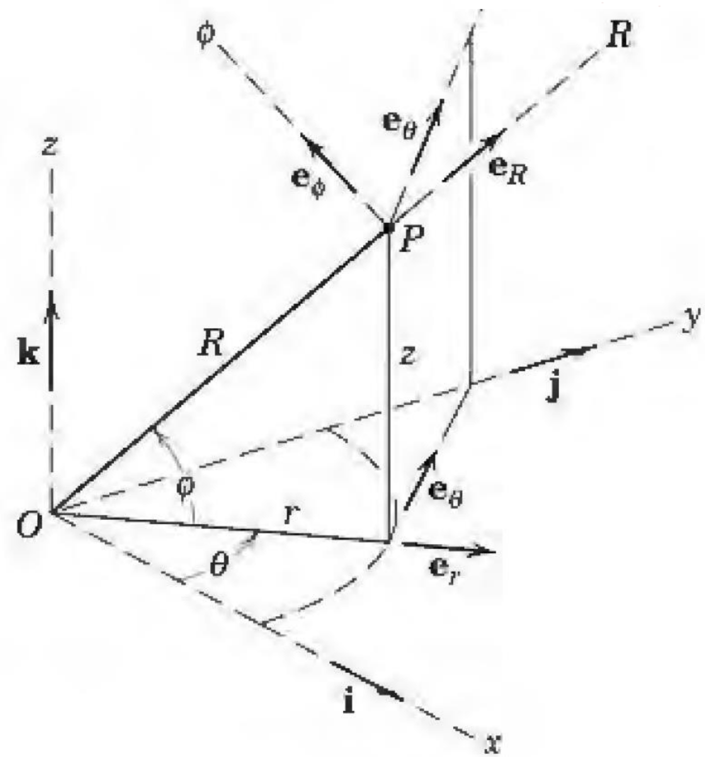
$$v_r = \dot{r}$$

$$v_\theta = r\,\dot{\theta} \qquad (2.19)$$

$$v_\phi = r\sin\theta\,\dot{\phi}$$

Acceleration:

$$\mathbf{a} = \left(\ddot{r} - r\,\dot{\theta}^2 - r\,\dot{\varphi}^2 \sin^2\theta\right)\hat{\mathbf{r}} + \left(r\,\ddot{\theta} + 2\dot{r}\,\dot{\theta} - r\,\dot{\varphi}^2 \sin\theta\cos\theta\right)\hat{\boldsymbol{\theta}} + \left(r\ddot{\varphi}\,\sin\theta + 2\dot{r}\,\dot{\varphi}\,\sin\theta + 2r\,\dot{\theta}\,\dot{\varphi}\,\cos\theta\right)\hat{\boldsymbol{\varphi}}. \quad (2.20)$$



Other choice :

$$\boxed{\mathbf{v} = v_R\mathbf{e}_R + v_\theta\mathbf{e}_\theta + v_\phi\mathbf{e}_\phi}$$

$$v_R = \dot{R}$$

$$v_\theta = R\,\dot{\theta}\,\cos\phi$$

$$v_\phi = R\,\dot{\phi}$$

(2/16)

$$\boxed{\mathbf{a} = a_R\mathbf{e}_R + a_\theta\mathbf{e}_\theta + a_\phi\mathbf{e}_\phi}$$

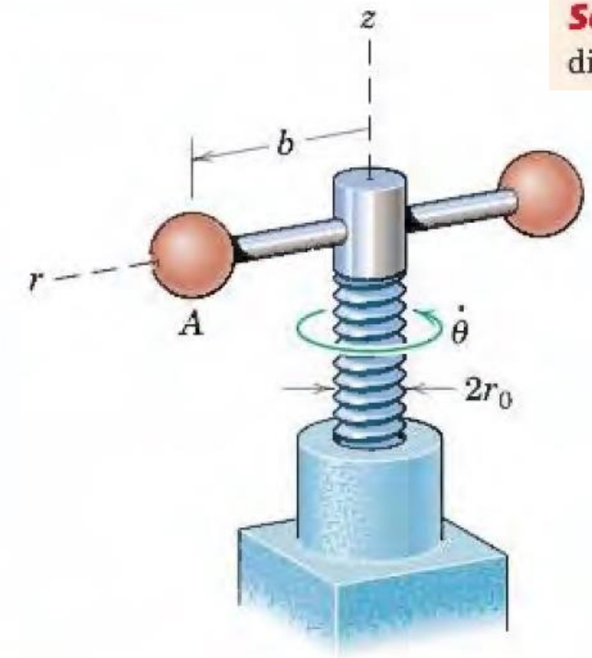
$$a_R = \ddot{R} - R\dot{\phi}^2 - R\,\dot{\theta}^2\cos^2\phi$$

$$a_\theta = \frac{\cos\phi}{R}\,\frac{d}{dt}(R^2\dot{\theta}) - 2R\,\dot{\theta}\,\dot{\phi}\,\sin\phi$$

$$a_\phi = \frac{1}{R}\,\frac{d}{dt}(R^2\dot{\phi}) + R\,\dot{\theta}^2\sin\phi\,\cos\phi$$

Example: J. L. Meriam, L. G. Kraige, J. N. Bolton, Engineering Mechanics: Dynamics, 8th Ed, Wiley, 2015

The power screw starts from rest and is given a rotational speed $\dot{\theta}$ which increases uniformly with time t according to $\dot{\theta} = kt$, where k is a constant. Determine the expressions for the velocity v and acceleration a of the center of ball A when the screw has turned through one complete revolution from rest. The lead of the screw (advancement per revolution) is L.



Solution. The center of ball A moves in a helix on the cylindrical surface of radius b, and the cylindrical coordinates r, θ, z are clearly indicated.

Integrating the given relation for $\dot{\theta}$ gives $\theta = \Delta\theta = \int \dot{\theta}\, dt = \frac{1}{2}kt^2$. For one revolution from rest we have

$$2\pi = \frac{1}{2}kt^2$$

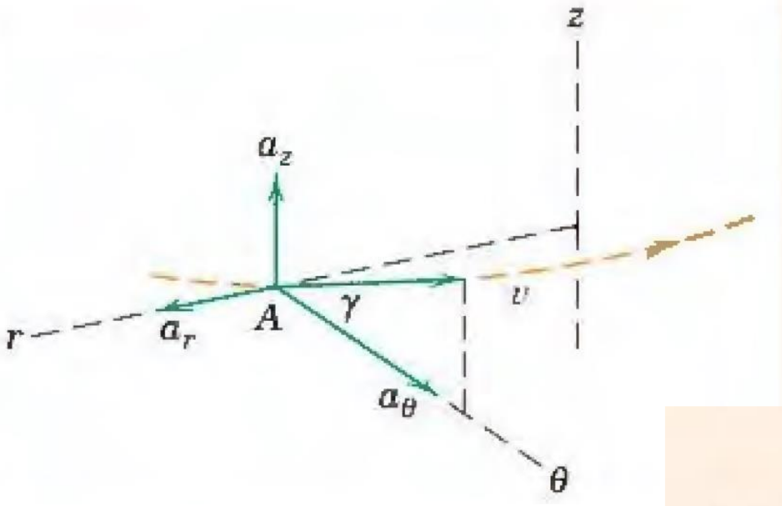
giving

$$t = 2\sqrt{\pi/k}$$

Thus, the angular rate at one revolution is

$$\dot{\theta} = kt = k(2\sqrt{\pi/k}) = 2\sqrt{\pi k}$$

① The helix angle γ of the path followed by the center of the ball governs the relation between the θ- and z-components of velocity and is given by $\tan\gamma = L/(2\pi b)$. Now from the figure we see that $v_\theta = v\cos\gamma$. Substituting $v_\theta = r\dot{\theta} = b\dot{\theta}$
② from Eq. 2/16 gives $v = v_\theta/\cos\gamma = b\dot{\theta}/\cos\gamma$. With $\cos\gamma$ obtained from $\tan\gamma$ and with $\dot{\theta} = 2\sqrt{\pi k}$, we have for the one-revolution position



$$v = 2b\sqrt{\pi k}\,\frac{\sqrt{L^2 + 4\pi^2 b^2}}{2\pi b} = \sqrt{\frac{k}{\pi}}\sqrt{L^2 + 4\pi^2 b^2}$$

③ $[a_r = \ddot{r} - r\dot{\theta}^2]$ $\quad a_r = 0 - b(2\sqrt{\pi k})^2 = -4b\pi k$

$[a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}\,]$ $\quad a_\theta = bk + 2(0)(2\sqrt{\pi k}) = bk$

$[a_z = \ddot{z} = \dot{v}_z]$ $\quad a_z = \dfrac{d}{dt}(v_z) = \dfrac{d}{dt}(v_\theta \tan\gamma) = \dfrac{d}{dt}(b\dot{\theta}\tan\gamma)$

$$= (b\tan\gamma)\ddot{\theta} = b\frac{L}{2\pi b}k = \frac{kL}{2\pi}$$

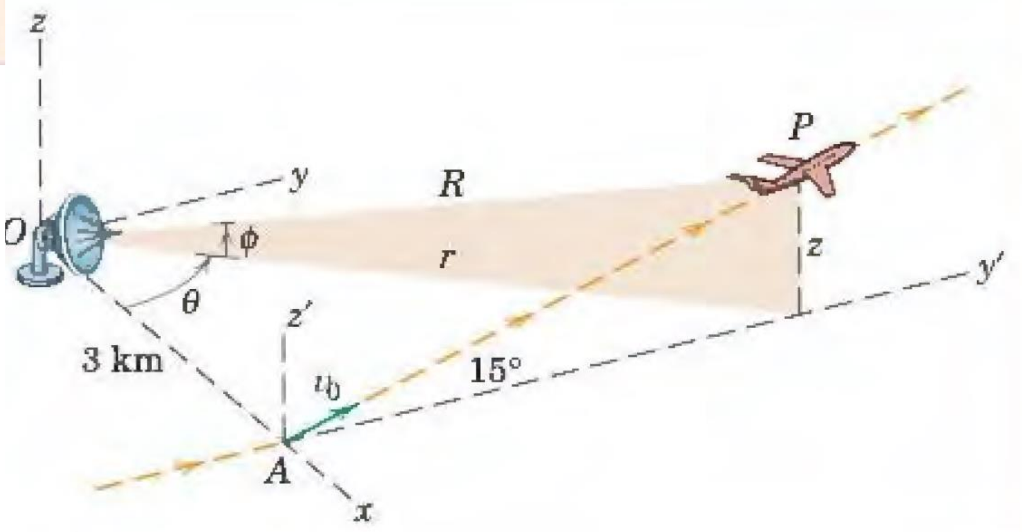
Now we combine the components to give the magnitude of the total acceleration, which becomes

$$a = \sqrt{(-4b\pi k)^2 + (bk)^2 + \left(\frac{kL}{2\pi}\right)^2}$$

$$= bk\sqrt{(1 + 16\pi^2) + L^2/(4\pi^2 b^2)} \qquad \textit{Ans.}$$

Example: J. L. Meriam, L. G. Kraige, J. N. Bolton, Engineering Mechanics: Dynamics, 8th Ed, Wiley, 2015

An aircraft P takes off at A with a velocity v_0 of 250 km/h and climbs in the vertical y'-z' plane at the constant 15° angle with an acceleration along its flight path of 0.8 m/s^2. Flight progress is monitored by radar at point O. (a) Resolve the velocity of P into cylindrical-coordinate components 60 seconds after takeoff and find $\dot{r}$, $\dot{\theta}$, and $\dot{z}$ for that instant. (b) Resolve the velocity of the aircraft P into spherical-coordinate components 60 seconds after takeoff and find $\dot{R}$, $\dot{\theta}$, and $\dot{\phi}$ for that instant.



Solution. (a) The accompanying figure shows the velocity and acceleration vectors in the y'-z' plane. The takeoff speed is

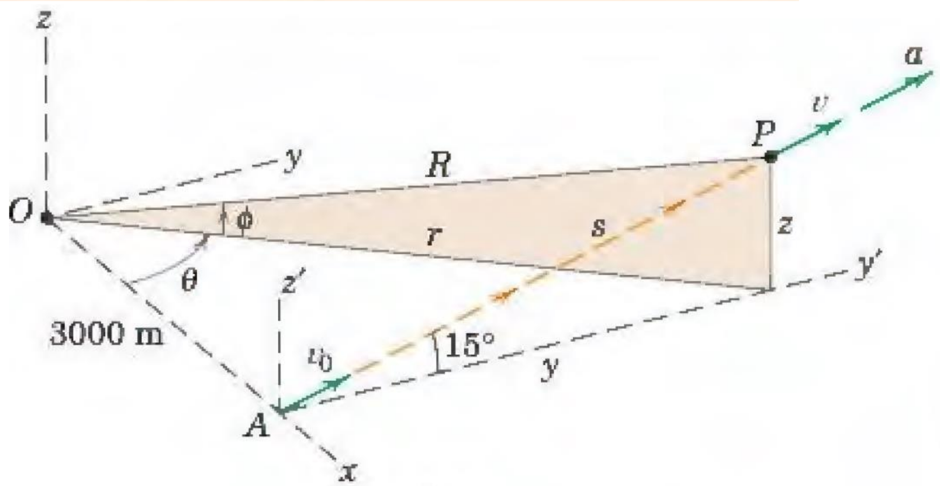
$$v_0 = \frac{250}{3.6} = 69.4 \text{ m/s}$$

and the speed after 60 seconds is

$$v = v_0 + at = 69.4 + 0.8(60) = 117.4 \text{ m/s}$$

The distance s traveled after takeoff is

$$s = s_0 + v_0 t + \frac{1}{2}at^2 = 0 + 69.4(60) + \frac{1}{2}(0.8)(60)^2 = 5610 \text{ m}$$



(a)

The y-coordinate and associated angle θ are

$$y = 5610 \cos 15° = 5420 \text{ m}$$

$$\theta = \tan^{-1} \frac{5420}{3000} = 61.0°$$

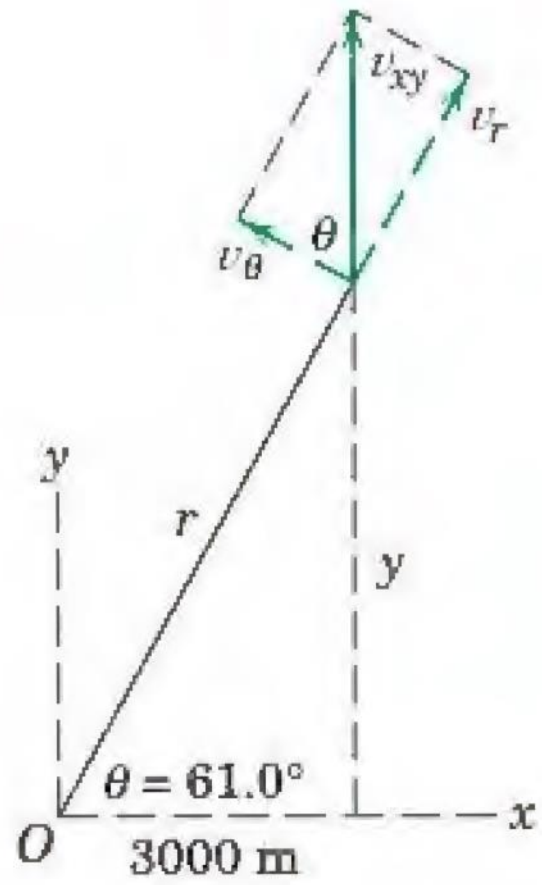
From the figure (b) of x-y projections, we have

$$r = \sqrt{3000^2 + 5420^2} = 6190 \text{ m}$$

$$v_{xy} = v \cos 15° = 117.4 \cos 15° = 113.4 \text{ m/s}$$

$$v_r = \dot{r} = v_{xy} \sin \theta = 113.4 \sin 61.0° = 99.2 \text{ m/s}$$

$$v_\theta = r\dot{\theta} = v_{xy} \cos \theta = 113.4 \cos 61.0° = 55.0 \text{ m/s}$$



So $$\dot{\theta} = \frac{55.0}{6190} = 8.88(10^{-3}) \text{ rad/s}$$

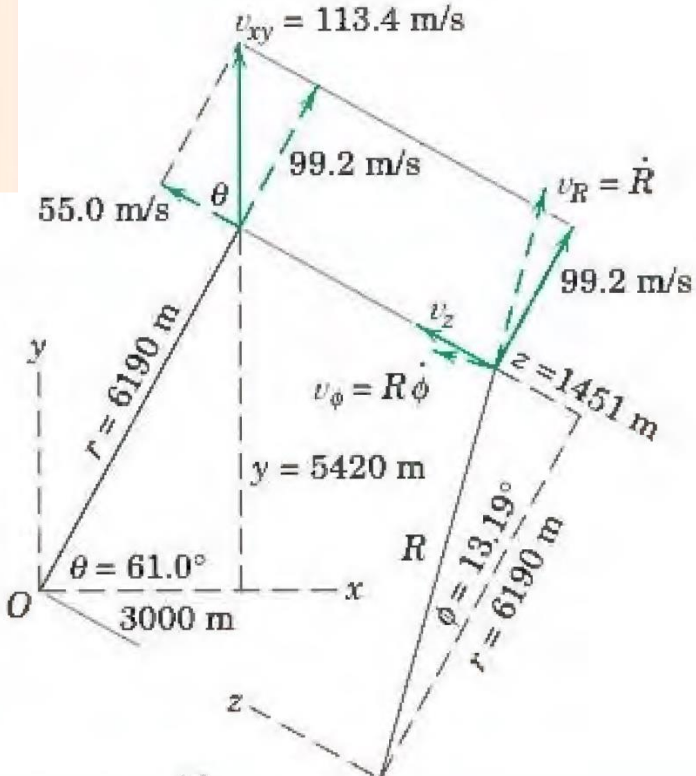
Finally $$\dot{z} = v_z = v \sin 15° = 117.4 \sin 15° = 30.4 \text{ m/s}$$

(b) Refer to the accompanying figure (c), which shows the x-y plane and various velocity components projected into the vertical plane containing r and R. Note that

$$z = y \tan 15° = 5420 \tan 15° = 1451 \text{ m}$$

$$\phi = \tan^{-1}\frac{z}{r} = \tan^{-1}\frac{1451}{6190} = 13.19°$$

$$R = \sqrt{r^2 + z^2} = \sqrt{6190^2 + 1451^2} = 6360 \text{ m}$$



(c)

From the figure,

$$v_R = \dot{R} = 99.2 \cos 13.19° + 30.4 \sin 13.19° = 103.6 \text{ m/s}$$

$$\dot{\theta} = 8.88(10^{-3}) \text{ rad/s, as in part } (a)$$

$$v_\phi = R\dot{\phi} = 30.4 \cos 13.19° - 99.2 \sin 13.19° = 6.95 \text{ m/s}$$

$$\dot{\phi} = \frac{6.95}{6360} = 1.093(10^{-3}) \text{ rad/s}$$