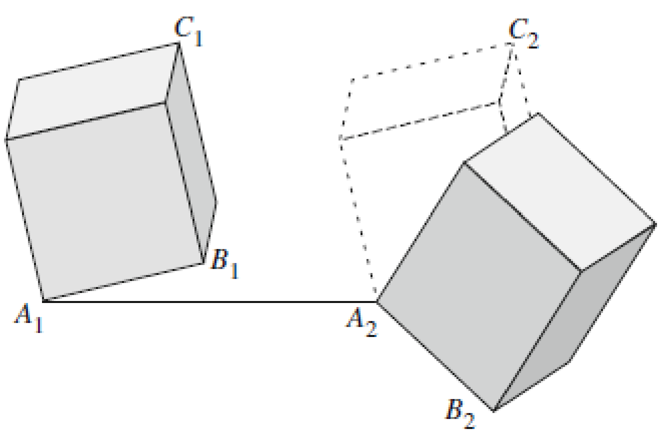
3. Kinematics of the rigid body

Point masses are a good idealization for many natural phenomena (e.g. celestial mechanics).

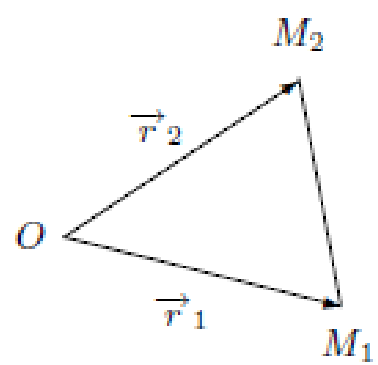
A *rigid body* is a rigid system of an infinite number of points filling continuously in a bounded domain from R^3. In *rigid bodies* the distances between points remain constant in time and this allows a description by ordinary differential equations.



$$(x_{B1}-x_{A1})^2+(y_{B1}-y_{A1})^2+(z_{B1}-z_{A1})^2=\text{const.}$$

Consider $M_1(\vec{r}_1)$ and $M_2(\vec{r}_2)$ two points of the rigid body (S) and an arbitrary point O in space. The rigidity condition is given by:

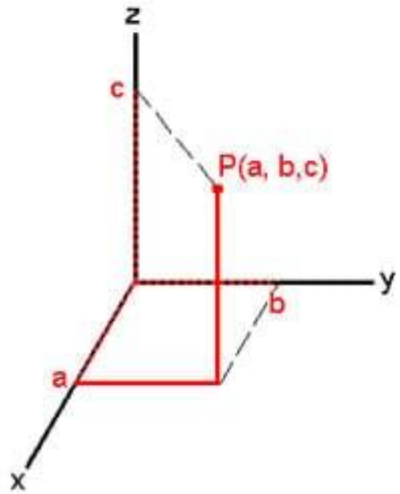
$$\begin{aligned} M_1M_2 &= const., \qquad \text{for} \quad t \geq t_0 \\ |\vec{r}_1-\vec{r}_2| &= const., \qquad \text{for} \quad t \geq t_0 \end{aligned} \tag{3.1}$$



Theoretical Mechanics

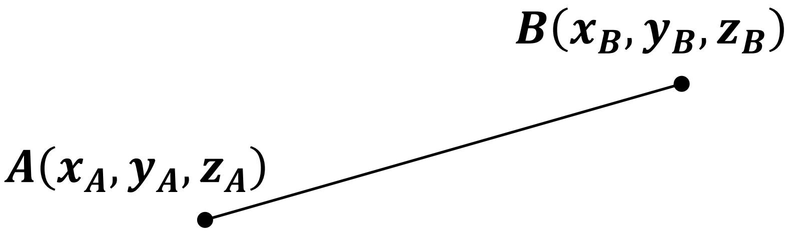
The solid bodies are approximately rigid, their shape does not change if we stretch, compress or torque them. Clearly, this is never rigorously true, because small deformations always take place.
Several dynamical properties of these bodies can be studied considering them as rigid.
The space location (3D) of a rigid body is called its *configuration*, which is the set of the positions of its points.

- The space location of a material point need 3 coordinates (*Degree of Freedom*)



- To define the configuration of a generic system of N points, we need 3 N coordinates.

- ***Five degree of freedom has for a rigid system formed by two points***



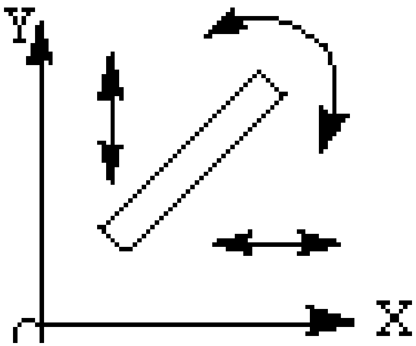
2 x 3 coordinates - 1 rigid condition ($AB = const.$) = 5 DoF

- ***Only six coordinates are necessary for a rigid body* (6 DoF).**

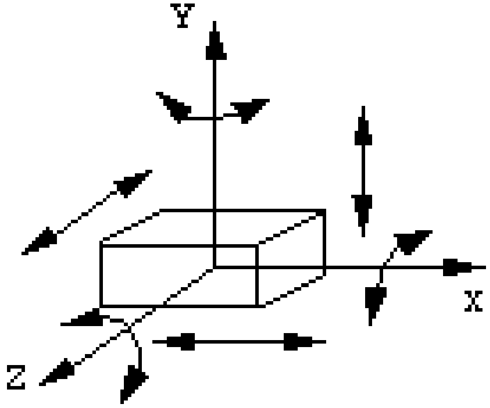
Indeed, for a three points rigid system 3 x 3 coordinates – 3 rigid conditions = 6 coordinates (6 DoF) are necessary.

We suppose that for a N-1 points rigid system 6 coordinates are necessary and we'll prove that this is also true for a N points rigid system. By introducing a new point, we have 9 coordinates. But, taking into account the rigidity condition for three arbitrary non-collinear points the degrees of freedom are reduced to 6. Once you know the distances between mass point m_I and three non-collinear reference mass points (e.g., m1, m2, m3), then you've fixed its position relative to all the other mass points.

- ***Examples:*** The ***degrees of freedom*** (DOF) of a rigid body are also seen as the number of independent movements it has.



To determine the DOF of this body we must consider how many distinct ways the bar can be moved. In a two dimensional plane such as this computer screen, there are 3 DOF. The bar can be *translated* along the *x* axis, translated along the *y* axis, and *rotated* about its centroid.



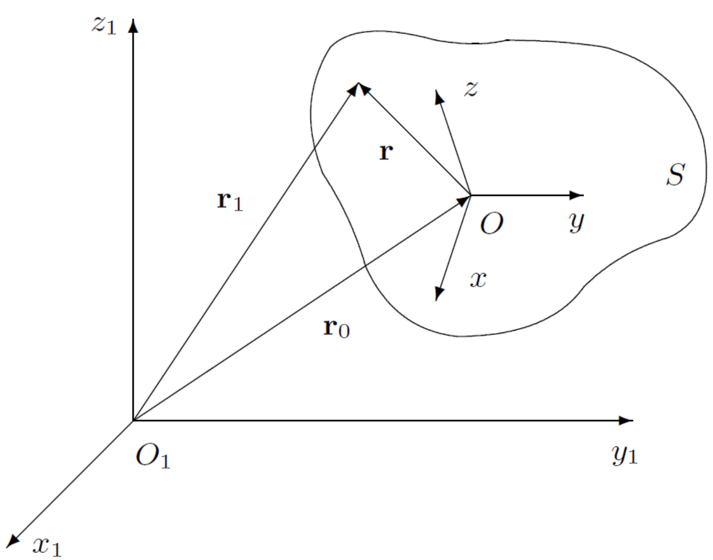
An unrestrained rigid body in space has six degrees of freedom: three translating motions along the *x*, *y* and *z* axes and three rotary motions around the *x*, *y* and *z* axes respectively.

(https://www.cs.cmu.edu/~rapidproto/mechanisms/chpt4.html)

In order to study the motion of a rigid body two frame of references are required:

- $O_1x_1y_1z_1$ – fixed in space
- $O\,x\,y\,z$ – fixed on the moving rigid body (the frame has the same motion as the rigid body)

The motion of the rigid body is given by the motion of the moving frame (the origin O is an arbitrary point of the rigid body).



The position and the motion of the mobile frame of reference (i.e. the rigid body) is determined by 6 parameters (coordinates).

These can be:

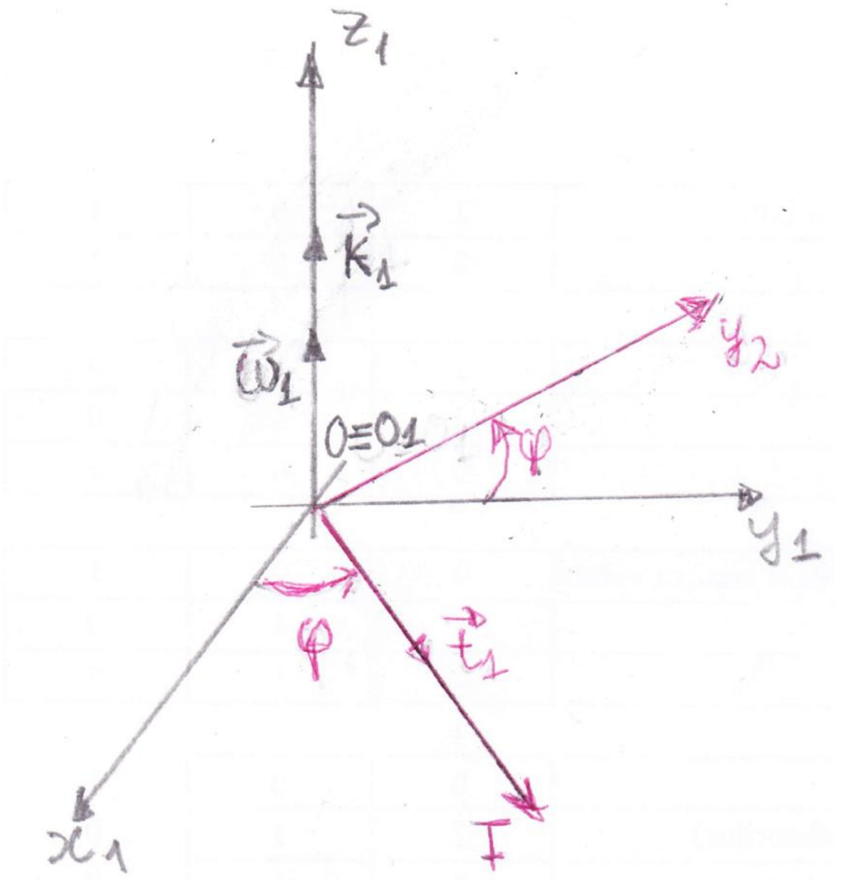
1. The coordinate of 3 arbitrary points (M_1, M_2, M_3) of the rigid body along with the rigidity condition (3.1)

2. The Euler (or **Tait-Bryan**) angles $(\boldsymbol{\phi}, \boldsymbol{\theta}, \boldsymbol{\psi})$ and the coordinates of $\mathrm{O}(\boldsymbol{x_0}, \boldsymbol{y_0}, \boldsymbol{z_0})$ in the fixed frame of reference $\mathrm{O_1x_1y_1z_1}$.
(unit **quaternions** can also be used to represent rotations)
(M. J. Benacquista, J. D. Romano, Classical Mechanics, Springer, 2018)

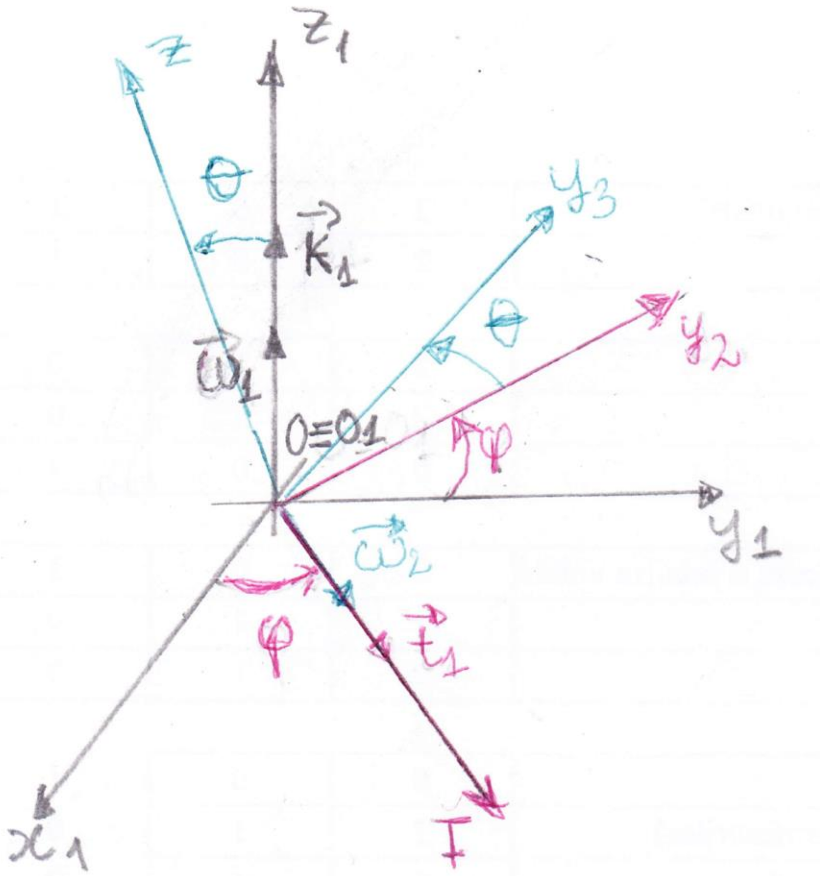
Euler angles

Consider the frame of reference $\mathrm{O\,x_1y_1z_1}$ $(O_1 \equiv O)$. We demonstrate that starting from the frame $\mathrm{Ox_1y_1z_1}$ one can obtain the frame Oxyz by using 3 successive rotations of angles (ϕ, θ, ψ).

Step 1. Rotate the frame $\mathrm{O\,x_1y_1z_1}$ around the $\mathrm{Oz_1}$ axis with the angle ϕ and angular velocity $\vec{\omega}_1 = \dot{\phi}\,\vec{k}_1$ (precession). The obtained frame is $\mathrm{OIy_2z_1}$, where OI is the ***nodes line***.

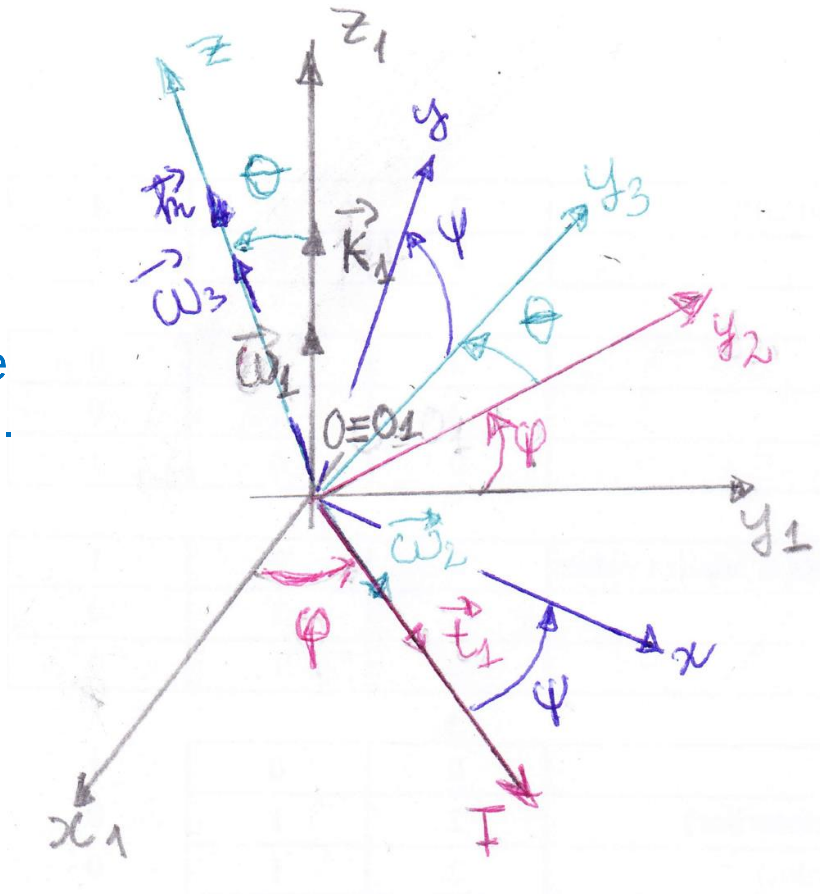


Step 2. Rotate the frame $\mathrm{OIy_2z_1}$ around the OI axis with the angle θ and angular velocity $\vec{\omega}_2 = \dot{\theta}\,\vec{t}_1$ (nutation) and obtain the frame $\mathrm{OIy_3z}$.

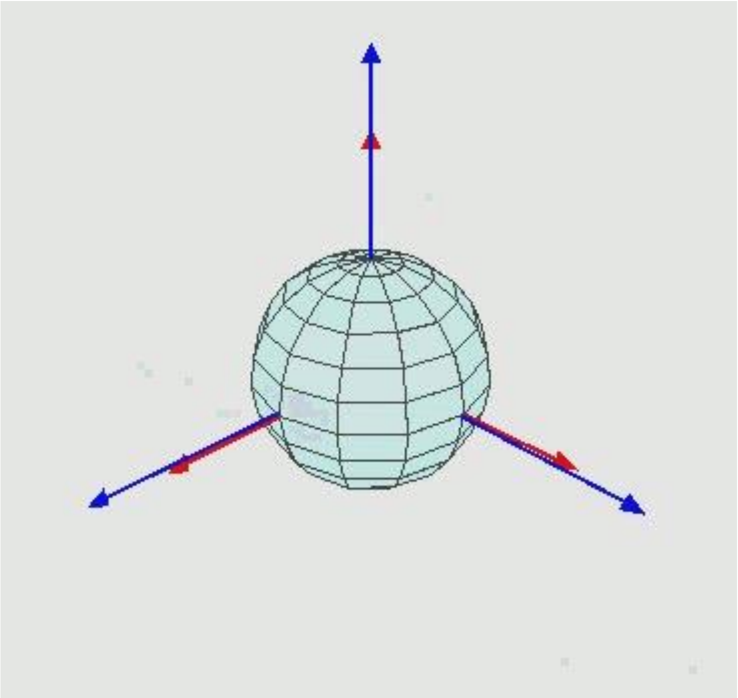


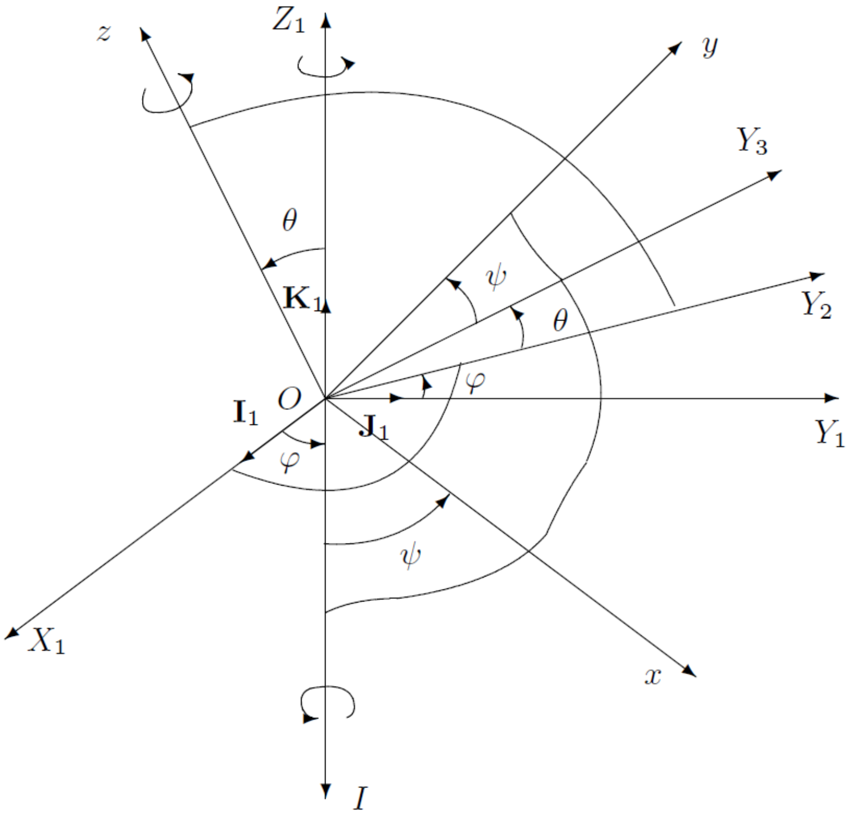
Step 3. Rotate the frame $\mathrm{OIy_3z}$ around the Oz axis with the angle ψ and angular velocity $\vec{\omega}_3 = \dot{\psi}\,\vec{k}$ (intrinsic rotation or spin) and obtain the frame Oxyz.

Remark. Reciprocally, it is possible to obtain the $\mathrm{Ox_1y_1z_1}$ frame from the Oxyz frame using the three rotations.



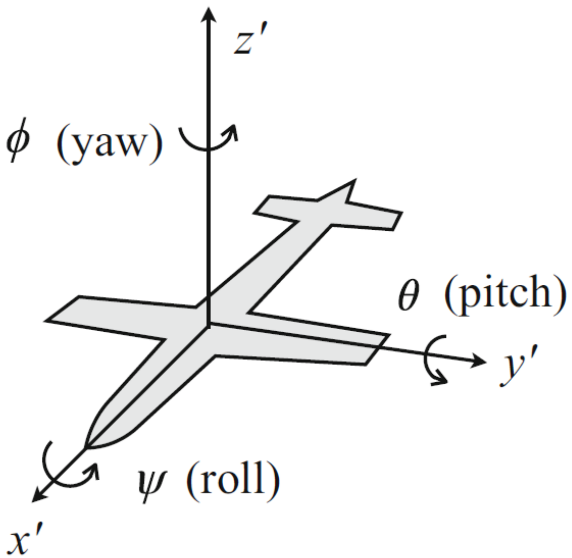
Definition. The angles $\boldsymbol{\phi}, \boldsymbol{\theta}, \boldsymbol{\psi}$ are the Euler angles.





Remark

If the angles (ϕ, θ, ψ) describing the rotations are associated with three *different* axes (e.g., z, y, x instead of z, y, z), then the angles are called **Tait-Bryan angles**



The Tait-Bryan angles *(φ, θ, ψ)* are most often used to describe the orientation of an aircraft (or similar object), and in such a context go by the names **yaw**, **pitch**, and **roll**, or **heading**, **elevation**, and **bank**.

Yaw or heading specifies the left-right direction of the aircraft;

Pitch or elevation specifies whether the aircraft is ascending or descending;

Roll or bank specifies if the aircraft has any rotational motion (spin) about its direction of motion.

The equations of motion for the rigid body are:

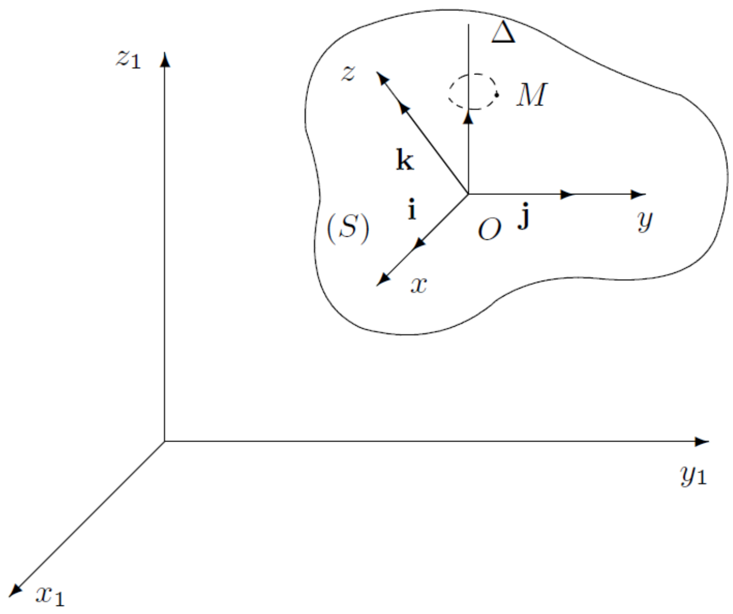
$$\begin{cases} x_0 = x_0(t), & y_0 = y_0(t), & z_0 = z_0(t) \\ \varphi = \varphi(t), & \theta = \theta(t), & \psi = \psi(t), \end{cases} \qquad t \in [t_0, T]. \tag{3.2}$$

The motion determined by 6 independent parameters (6 DoF) is the most general motion of the rigid body.

Poisson formulas

Consider the motion of a rigid body S in a fixed frame $O\,x_1y_1z_1$ and a mobile frame $Oxyz$ (linked to the rigid body in the arbitrary point *O*). Let be $\vec{i}, \vec{j}, \vec{k}$ the unit vectors of the mobile frame. During the motion the unit vectors vary in time (they change their position). Thus, it mean we have:

$$\overrightarrow{i} = \overrightarrow{i}(t), \overrightarrow{j} = \overrightarrow{j}(t), \overrightarrow{k} = \overrightarrow{k}(t), \quad \overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k} \in C^k[t_0, T], \; k \geq 2).$$

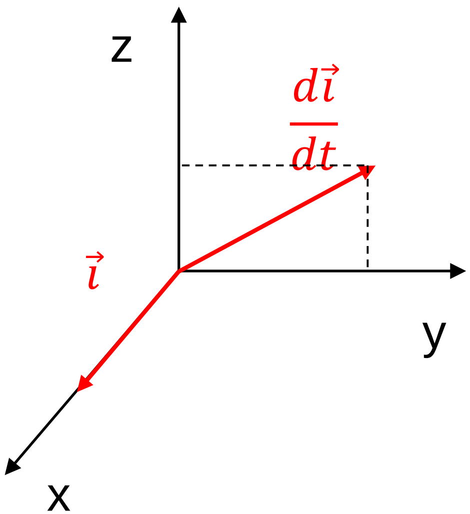


We have to find

$$\frac{d\overrightarrow{i}}{dt} = ?, \quad \frac{d\overrightarrow{j}}{dt} = ?, \quad \frac{d\overrightarrow{k}}{dt} = ?$$

From $\vec{\imath} \cdot \vec{\imath} = 1 \;\Rightarrow \vec{\imath}\,\frac{d\vec{\imath}}{dt} = 0$ and we have

$$\begin{matrix} \frac{d\vec{\imath}}{dt} \in (\vec{\jmath}, \vec{k}) \\ \frac{d\vec{\jmath}}{dt} \in (\vec{k}, \vec{\imath}) \\ \frac{d\vec{k}}{dt} \in (\vec{\imath}, \vec{\jmath}) \end{matrix} \quad \Rightarrow \quad \begin{matrix} \frac{d\overrightarrow{i}}{dt} = \lambda_{12}\overrightarrow{j} + \lambda_{13}\overrightarrow{k} \\ \frac{d\overrightarrow{j}}{dt} = \lambda_{23}\overrightarrow{k} + \lambda_{21}\overrightarrow{i} \\ \frac{d\overrightarrow{k}}{dt} = \lambda_{31}\overrightarrow{i} + \lambda_{32}\overrightarrow{j}, \end{matrix} \qquad \lambda_{\ell r} \in \mathbb{R}. \tag{3.3}$$



On the other hand the frame $Oxyz$ is orthonormal and we have

$$\overrightarrow{i} \cdot \overrightarrow{j} = 0, \; \overrightarrow{j} \cdot \overrightarrow{k} = 0, \; \overrightarrow{k} \cdot \overrightarrow{i} = 0$$

from where

$$\frac{d\overrightarrow{i}}{dt} \cdot \overrightarrow{j} + \overrightarrow{i} \cdot \frac{d\overrightarrow{j}}{dt} = 0.$$

Thus, taking into account (3.3), we can use the following notations

$$\begin{cases} \lambda_{12} = -\lambda_{21} := r \\ \lambda_{23} = -\lambda_{32} := p \\ \lambda_{31} = -\lambda_{13} := q \end{cases} \qquad (3.4)$$

and we have

$$\frac{d\vec{i}}{dt} = r\vec{j} - q\vec{k} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ p & q & r \\ 1 & 0 & 0 \end{vmatrix} = \vec{\omega} \times \vec{i}$$

$$\frac{d\vec{j}}{dt} = p\vec{k} - r\vec{i} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ p & q & r \\ 0 & 1 & 0 \end{vmatrix} = \vec{\omega} \times \vec{j}$$

$$\frac{d\vec{k}}{dt} = q\vec{i} - p\vec{j} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ p & q & r \\ 0 & 0 & 1 \end{vmatrix} = \vec{\omega} \times \vec{k}.$$

These relations are the Poisson's formulas

$$\frac{d\overrightarrow{i}}{dt} = \overrightarrow{\omega} \times \overrightarrow{i}\,, \quad \frac{d\overrightarrow{j}}{dt} = \overrightarrow{\omega} \times \overrightarrow{j}\,, \quad \frac{d\overrightarrow{k}}{dt} = \overrightarrow{\omega} \times \overrightarrow{k} \tag{3.5}$$

The vector $\vec{\omega}(p,q,r)$ is a free vector (it is not bound). Let be Δ its direction (support) at the moment t.

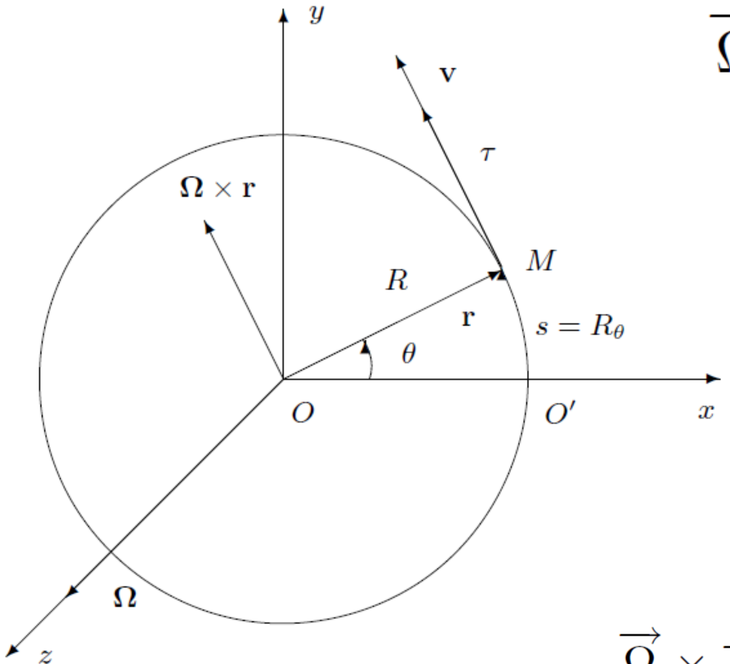
Geometrical interpretation

Consider the motion of a point M on a circle of radius R.

Equation of motion: $\theta = \theta(t), \ \theta \in [t_0, T].$

Velocity of *M*: $M\colon \overrightarrow{v} = v\overrightarrow{\tau} = \dot{s}\overrightarrow{\tau} = R\dot{\theta}\overrightarrow{\tau}\,.$

Consider $\overrightarrow{\Omega} \parallel Oz$ such that $\Omega = \dot{\theta}$.



$$\overrightarrow{\Omega} \times \overrightarrow{r} \parallel \overrightarrow{v} \ (\overrightarrow{v} \perp \overrightarrow{r}; \ \overrightarrow{v} \perp \overrightarrow{\Omega} \ \Rightarrow$$

$$\overrightarrow{v} \perp (\overrightarrow{\Omega}, \overrightarrow{r}) \ \Rightarrow \ \overrightarrow{v} \parallel \overrightarrow{\Omega} \times \overrightarrow{r})$$

$$|\overrightarrow{\Omega} \times \overrightarrow{r}| = \underbrace{|\overrightarrow{\Omega}|}_{=|\Omega|} \, \underbrace{|\overrightarrow{r}|}_{=R} \underbrace{\sin(\underbrace{\overrightarrow{\Omega}, \overrightarrow{r}}_{=\pi/2})}_{=1} = R|\dot{\theta}| = |\overrightarrow{v}|$$

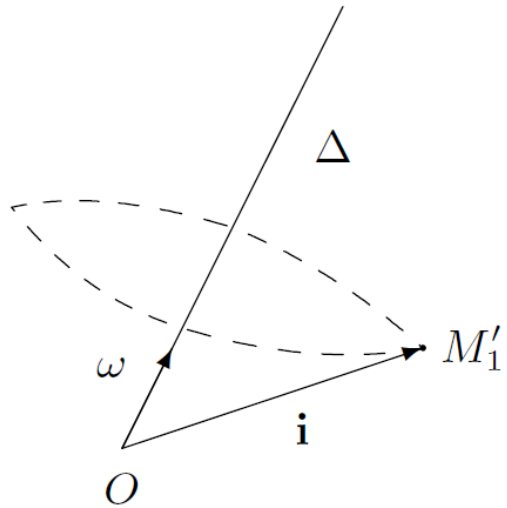
$$\Rightarrow \left. \begin{array}{l} \overrightarrow{\Omega} \times \overrightarrow{r} \parallel \overrightarrow{v} \\ |\overrightarrow{\Omega} \times \overrightarrow{r}| = |\overrightarrow{v}| \end{array} \right\} \Rightarrow \ \overrightarrow{\Omega} \times \overrightarrow{r} = \overrightarrow{v}$$

Thus
$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{\Omega} \times \vec{r}. \tag{3.6}$$

Comparing the formulas

$$\frac{d\,\vec{i}}{dt} = \vec{\omega} \times \vec{i}\,, \quad \frac{d\,\vec{r}}{dt} = \vec{\Omega} \times \vec{r}\,, \tag{3.7}$$

we can deduce that the extremity of versor $\vec{\imath}$ has a circular motion around the axis Δ with the velocity $\vec{\omega}$.



The motion of the rigid is given by the motion of 3 arbitrary non collinear points (e.g. the extremities of the versors $\vec{\imath}, \vec{\jmath}, \vec{k}$). Thus, at the moment t the rigid body has a circular motion around the axis Δ with the velocity $\vec{\omega}$.
Generally,

$$\vec{\omega} = \vec{\omega}(t), \quad \Delta = \Delta(t).$$

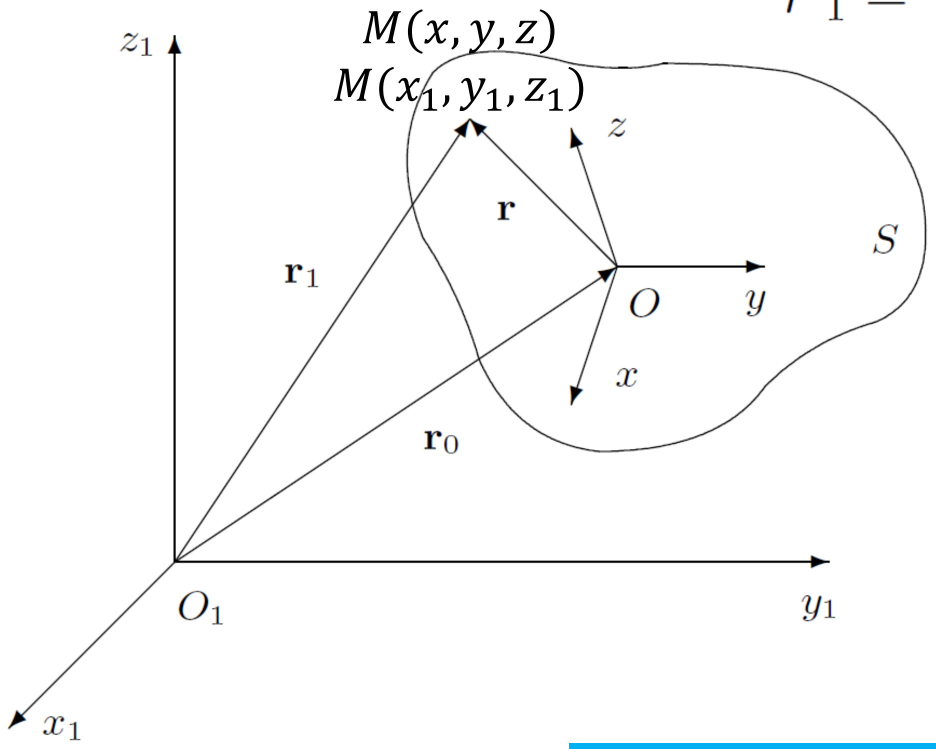
4. Velocity and acceleration of the rigid body. Particular motions

Velocity

Consider the motion of a solid rigid S and $M \in S$ an arbitrary point. The position vector of M in $O_1x_1y_1z_1$is:

$$\overrightarrow{O_1M} = \overrightarrow{O_1O} + \overrightarrow{OM}$$

$$\overrightarrow{r}_1 = \overrightarrow{r}_0 + \overrightarrow{r} = \overrightarrow{r}_0 + x\overrightarrow{i} + y\overrightarrow{j} + z\overrightarrow{k}$$



Velocity is given by

$$\begin{aligned}
\vec{v}_M &:= \frac{d\vec{r}_1}{dt} \\
&= \frac{d\vec{r}_0}{dt} + (\dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k}) + \Big(x \underbrace{\frac{d\vec{i}}{dt}}_{=\vec{\omega}\times\vec{i}} + y \underbrace{\frac{d\vec{j}}{dt}}_{=\vec{\omega}\times\vec{j}} + z \underbrace{\frac{d\vec{k}}{dt}}_{=\vec{\omega}\times\vec{k}} \Big) \\
&= \frac{d\vec{r}_0}{dt} + \vec{\omega}\times\vec{r} = \vec{v}_0 + \vec{\omega}\times\vec{r}.
\end{aligned} \tag{4.1}$$

where

$$\dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k} = 0,$$

because M is fixed in $Oxyz$.

Thus,

$$\vec{v}_M = \vec{v}_0 + \vec{\omega}\times\vec{r} \tag{4.2}$$

where $\vec{v}_0 = \dfrac{d\vec{r}_0}{dt}$ is the velocity of O in $O_1x_1y_1z_1$.

Consider $\vec{\omega}(p, q, r,)$ in $Oxyz$. We have

$$\overrightarrow{\omega} \times \overrightarrow{r} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ p & q & r \\ x & y & z \end{vmatrix} \tag{4.3}$$

$$v_{M_x} = v_{Ox} + qz - ry, \quad v_{M_y} = v_{Oy} + rx - pz, \quad v_{M_z} = v_{Oz} + py - qx$$

Remark. Equation (4.2) gives the distribution of velocities in the rigid body. The characteristics of the velocity are:

$$\overrightarrow{v}_0 = \overrightarrow{v}_0(t), \quad \overrightarrow{\omega} = \overrightarrow{\omega}(t).$$

Acceleration

$$
\begin{aligned}
\vec{a}_M &:= \frac{d\vec{v}_M}{dt} = \frac{d}{dt}(\vec{v}_0 + \overline{\omega} \times \vec{r}) \\
&= \vec{a}_0 + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt} \\
&= \vec{a}_0 + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \underbrace{\left(\underbrace{\frac{d\vec{r}_1}{dt}}_{=\vec{v}_M} - \underbrace{\frac{d\vec{r}_0}{dt}}_{=v_0} \right)}_{=\vec{\omega}\times\vec{r}} \\
&= \vec{a}_0 + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})
\end{aligned}
$$

Thus,

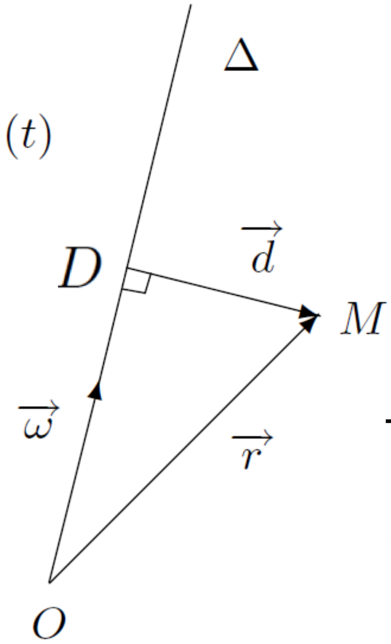
$$
\vec{a}_M = \vec{a}_0 + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}), \tag{4.4}
$$

where $\vec{a}_0 := \frac{d\vec{v}_0}{dt}$ is the acceleration of O in $O_1x_1y_1z_1$.

Consider

$$\vec{d} = \overrightarrow{DM} \perp \Delta \;\Rightarrow\; \vec{r} = \overrightarrow{OD} + \vec{d} \;\Rightarrow$$

$$\vec{\omega} \times \vec{r} = \underbrace{\vec{\omega} \times \overrightarrow{OD}}_{=0} + \vec{\omega} \times \vec{d} = \vec{\omega} \times \vec{d} \;\Rightarrow$$



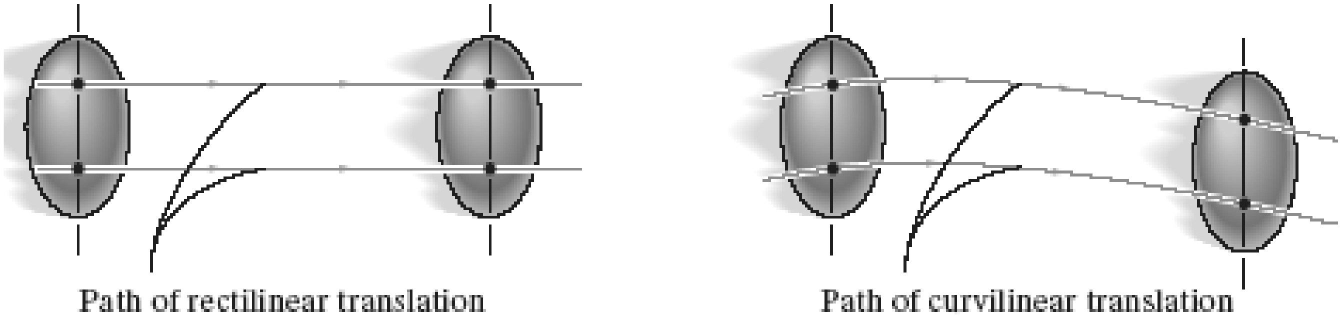
$$\vec{\omega} \times (\vec{\omega} \times \vec{r}) = \vec{\omega} \times (\vec{\omega} \times \vec{d}\,) = (\underbrace{\vec{\omega} \cdot \vec{d}}_{=0})\vec{\omega} - \omega^2 \vec{d} = -\omega^2 \vec{d}$$

$$\Rightarrow\; \vec{a}_M = \vec{a}_0 + \dot{\vec{\omega}} \times \vec{r} - \omega^2 \vec{d}\,. \qquad (4.5)$$

Thus,

$$\vec{a}_M = \vec{a}_0 + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) = \vec{a}_0 + \dot{\vec{\omega}} \times \vec{r} - \omega^2 \vec{d}\,. \qquad (4.6)$$

Translation of the rigid body



In every moment of the motion we have

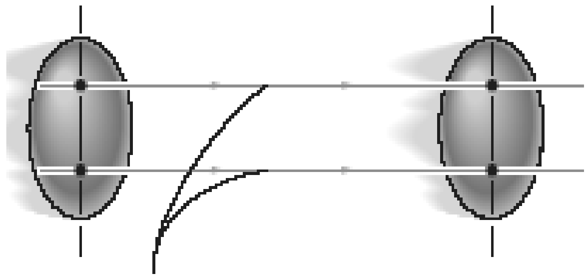
$$\vec{v}_0(t) \neq 0, \quad \vec{\omega}(t) = 0,\ t \in [t_0, T] \tag{4.7}$$

Every vector $\overrightarrow{OM}$ is equipollent (parallel) with itself.
The equations of motion are:

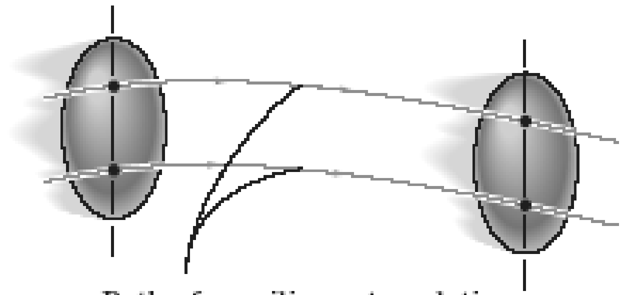
$$x_0 = x_0(t), \quad y_0 = y_0(t), \quad z_0 = z_0(t), \quad t \in [t_0, T] \tag{4.8}$$

$$\vec{r}_0 = \vec{r}_0(t),\ t \in [t_0, T] \tag{4.9}$$

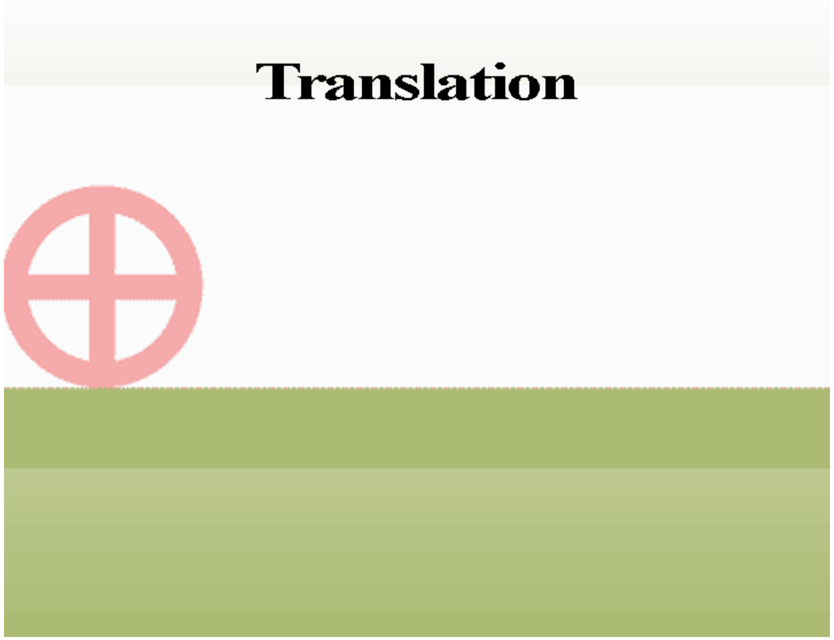
Translation of the rigid body

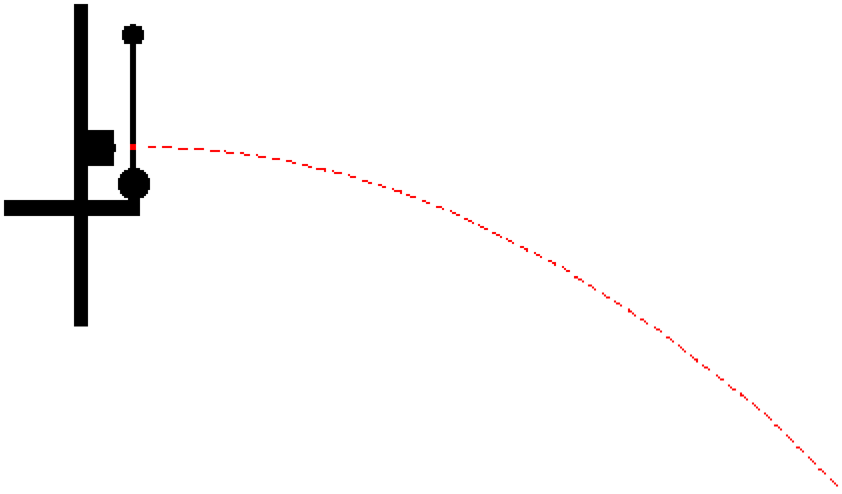


Path of rectilinear translation



Path of curvilinear translation





Velocity and acceleration:

$$\overrightarrow{v}_M = \overrightarrow{v}_0, \quad \overrightarrow{a}_M = \overrightarrow{a}_0 \tag{4.10}$$

Remark. The motion of translation of a rigid body reduces to the translation of the point O. All the points of the rigid body have the same velocity and acceleration.

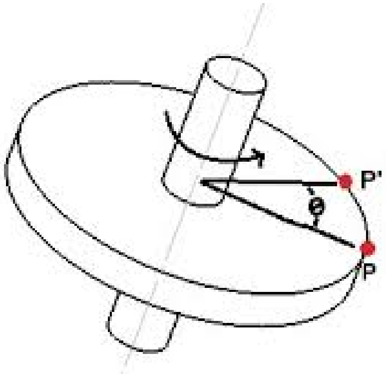
Definition. If at a moment *t* we have

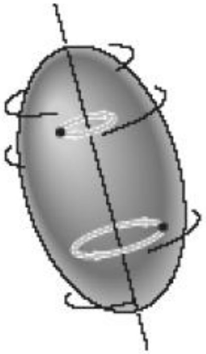
$$\overrightarrow{v}_0(t) \neq 0, \quad \overrightarrow{\omega}(t) = 0,$$

then the motion of the rigid body is an instantaneous translation.

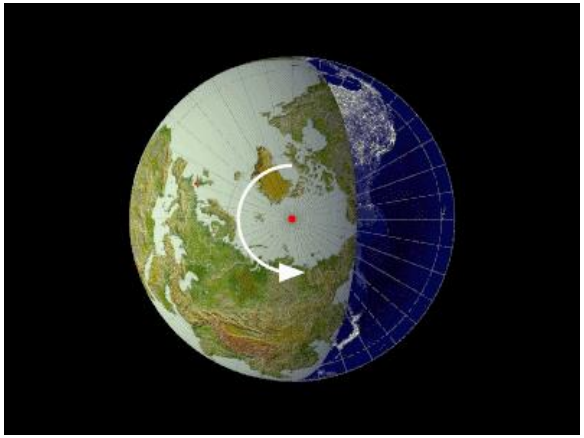
Remark. According to the eq. (4.7) and the previous definition the translation of a rigid body is a succession of instantaneous translation.

Rotation around a fixed axis





Rotation about a fixed axis



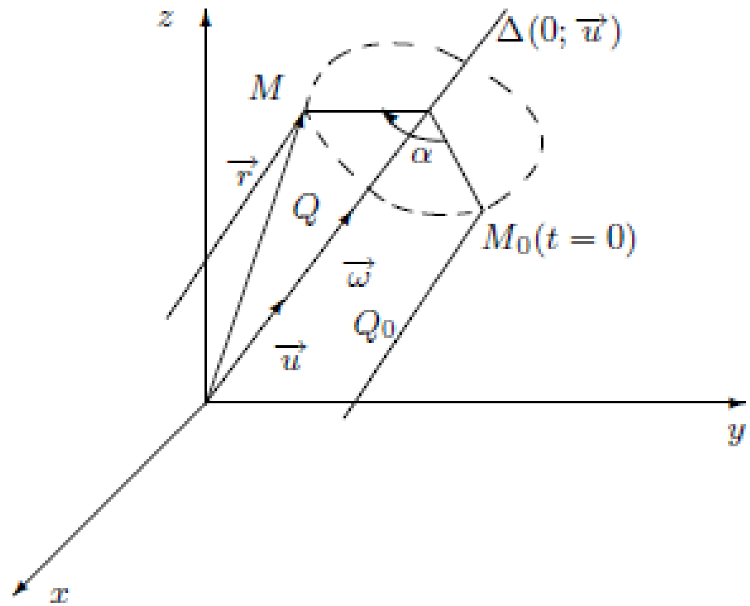
In every moment of the motion we have

$$\vec{v}_0(t) = 0, \quad \vec{\omega}(t) = \omega(t)\,\vec{u} \qquad t \in [t_0, T] \qquad (4.11)$$

where $\vec{u}$ is a fixed versor ($\dot{\vec{u}} = 0$).

The rigid body executes a rotation motion around a fixed axis $\Delta(O;\vec{u})$ called ***rotation axis***. All the points of the rigid body describe arcs of circle in planes perpendicular on Δ.

Consider the moving frame $Oxyz$ and α the rotation angle around Δ (determined by an initial fix plane Q_0 and the current plane $Q = (M, \Delta)$).



We have

$$\vec{\omega} = \omega \vec{u} = p\,\vec{i} + q\,\vec{j} + z\,\vec{k}$$

Equation of the motion:

$$\alpha = \alpha(t), \ \ t \in [t_0, T]. \qquad (4.12)$$

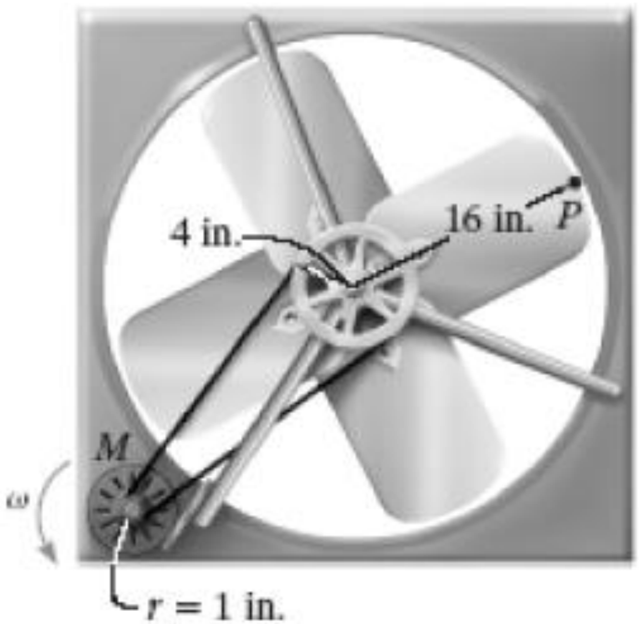
Velocity and acceleration:

$$\vec{v}_M = \vec{\omega} \times \vec{r}, \quad \vec{a}_M = \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) = \dot{\vec{\omega}} \times \vec{r} - \omega^2 \vec{d} \qquad (4.13)$$

Equations of Δ (in $Oxyz$):

$$A \in \Delta \ \Leftrightarrow \ \vec{r}_A = \overline{OA} \parallel \vec{\omega} \ \Leftrightarrow \ \vec{r}_A \times \vec{\omega} = 0 \ \Leftrightarrow \ \frac{x}{p} = \frac{y}{q} = \frac{z}{r} \qquad (4.14)$$

Example



Given: The motor M begins rotating at $\omega = 4(1 - e^{-t})$ rad/s, where t is in seconds. The radii of the motor, fan pulleys, and fan blades are 1 in, 4 in, and 16 in, respectively.

Find: The magnitudes of the velocity and acceleration at point P on the fan blade when t = 0.5 s.

(4.14)

Plan: 1) Determine the angular velocity and acceleration of the motor using kinematics of angular motion.

2) Assuming the belt does not slip, the angular velocity and acceleration of the fan are related to the motor's values by the belt.

3) The magnitudes of the velocity and acceleration of point P can be determined from the scalar equations of motion for a point on a rotating body.

1) Since the angular velocity is given as a function of time, $\omega_m = 4(1 - e^{-t})$, the angular acceleration can be found by differentiation.

$$\alpha_m = d\omega_m/dt = 4e^{-t} \text{ rad/s}^2$$

When t = 0.5 s,

$\omega_m = 4(1 - e^{-0.5}) = 1.5739$ rad/s, $\alpha_m = 4e^{-0.5} = 2.4261$ rad/s^2

2) Since the belt does not slip (and is assumed inextensible), it must have the same speed and tangential component of acceleration at all points. Thus the pulleys must have the same speed and tangential acceleration at their contact points with the belt. Therefore, the angular velocities of the motor (ω_m) and fan (ω_f) are related as

$$v = \omega_m r_m = \omega_f r_f \Rightarrow (1.5739)(1) = \omega_f(4) \Rightarrow \omega_f = 0.3935 \text{ rad/s}$$

3) Similarly, the tangential accelerations are related as

$$a_t = \alpha_m r_m = \alpha_f r_f \Rightarrow (2.4261)(1) = \alpha_f(4) \Rightarrow \alpha_f = 0.6065 \text{ rad/s}^2$$

4) The speed of point P on the the fan, at a radius of 16 in, is now determined as

$$v_P = \omega_f r_P = (0.3935)(16) = 6.30 \text{ in/s}$$

The normal and tangential components of acceleration of point P are calculated as

$$a_n = (\omega_f)^2 r_P = (0.3935)^2 (16) = 2.477 \text{ in/s}^2$$
$$a_t = \alpha_f r_P = (0.6065)(16) = 9.704 \text{ in/s}^2$$

The magnitude of the acceleration of P can be determined by

$$a_P = \sqrt{(a_n)^2 + (a_t)^2} = \sqrt{(2.477)^2 + (9.704)^2} = 10.0 \text{ in/s}^2$$