ANALYTIC GEOMETRY, PROBLEM SET 2

Warm-up 1. What are the conditions that vectors $\overline{a}$, $\overline{b}$ and $\overline{c}$ should satisfy to ensure the existence of a triangle ABC such that $\overrightarrow{AB} \in a$, $\overrightarrow{BC} \in \overline{b}$ and $\overrightarrow{CA} \in \overline{c}$?

1. On the sides of a triangle ABC, one constructs the parallelograms $ABB'A''$, $BCC'B''$, $CAA'C''$. Show that one can construct a triangle MNP such that $\overrightarrow{MN} \in \overline{A'A''}$, $\overrightarrow{NP} \in \overline{B'B''}$ and $\overrightarrow{PM} \in \overline{C'C''}$.

2. Let M and N be the midpoints of two opposite sides of a quadrilateral $ABCD$ and let P be the midpoint of $[MN]$. Prove that $\overline{PA} + \overline{PB} + \overline{PC} + \overline{PD} = \overline{0}$.

3. In a circle of center O, let M be the intersection point of two perpendicular chords $[AB]$ and $[CD]$. Show that $\overline{OA} + \overline{OB} + \overline{OC} + \overline{OD} = 2\overline{OM}$.

4. Consider, in the 3-dimensional space, the parallelograms $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$. Prove that the midpoints of the segments $[A_1B_1]$, $[A_2B_2]$, $[A_3B_3]$ and $[A_4B_4]$ are the vertices of a new parallelogram.

5. Let ABC be a triangle and a, b, c the lengths of its sides, respectively. If A_1 is the intersection point of the internal bisector of the angle $\angle A$ and BC and M is an arbitrary point, show that

$$\overline{MA_1} = \frac{b}{b+c}\overline{MB} + \frac{c}{b+c}\overline{MC}$$

6. If G is the centroid (center of mass) of a triangle ABC in the plane and O is a given point, then

$$\overline{OG} = \frac{\overline{OA} + \overline{OB} + \overline{OC}}{3}$$

7. Let ABC be a triangle, H its orthocenter, O the circumcenter (center of the circumcircle), G the centroid of the triangle and A' the point on the circumcircle diametrically opposed to A. Then:

(1) $\overline{OA} + \overline{OB} + \overline{OC} = \overline{OH}$; (**Sylvester's formula**)
(2) $\overline{HB} + \overline{HC} = \overline{HA'}$;
(3) $\overline{HA} + \overline{HB} + \overline{HC} = 2\overline{HO}$;
(4) $\overline{HA} + \overline{HB} + \overline{HC} = 3\overline{HG}$;
(5) the points H, G, O are collinear and $2GO = HG$. (the **Euler line**)

Date: October 10, 2022.

8. Let $ABCD$ be a quadrilateral with $AB \cap CD = \{E\}$, $AD \cap BC = \{F\}$ and the points M, N, P the midpoints of $[BD]$, $[AC]$ and $[EF]$, respectively. Then M, N, P are collinear. (the **Newton-Gauss line**)

9. In the plane determined by the triangle ABC, let us consider the points M, N, P, Q such that $\overrightarrow{AM} = \frac{2}{3}\overrightarrow{AB}$, $2\overrightarrow{NA} + \overrightarrow{NC} = \overrightarrow{0}$, $\overrightarrow{AP} = \frac{2}{5}\overrightarrow{AB}$ and $3\overrightarrow{QA} + 2\overrightarrow{QB} + \overrightarrow{QC} = \overrightarrow{0}$.

(1) Find $\alpha \in \mathbb{R}$ such that $\overrightarrow{QN} = \alpha \cdot \overrightarrow{QM}$.
(2) Find $\beta \in \mathbb{R}$ such that $\overrightarrow{CQ} = \beta \cdot \overrightarrow{QP}$.
(3) Find the value of the ratio $\frac{QA}{QR}$, where $AQ \cap BC = \{R\}$.