ANALYTIC GEOMETRY, PROBLEM SET 4

1. Given the vectors $\overline{a}(3,-1,-2)$ and $\overline{b}(1,2,-1)$. Compute
$$\overline{a}\times\overline{b}, (2\overline{a}+\overline{b})\times\overline{b} \text{ and } (2\overline{a}+\overline{b})\times(2\overline{a}-\overline{b}).$$

2. Given the vectors $\overline{a}(2,-3,1)$, $\overline{b}(-3,1,2)$ and $\overline{c}(1,2,3)$, compute $(\overline{a}\times\overline{b})\times\overline{c}$ and $\overline{a}\times(\overline{b}\times\overline{c})$. Is the cross product associative?

3. Find the distances between the opposite sides of the parallelogram constructed on $\overrightarrow{AB}(6,0,2)$ şi $\overrightarrow{AC}(1.5,2,1)$.

4. Given the points $A(1,-1,2)$, $B(5,-6,2)$ and $C(1,3,-1)$, find the length of the altitude from the vertex B in the triangle $\triangle ABC$.

5. Find the vector $\overline{p}$, knowing that $\overline{p}$ is perpendicular on $\overline{a}(2,3,-1)$ and $\overline{b}(1,-1,3)$ and its dot product with $\overline{c}(2,-3,4)$ is equal to 51.

6. Let $ABCD$ be a convex quadrilateral. Show that if the diagonal AC passes through the midpoint of the diagonal BD, then the triangles ACB and ACD have equal areas.

7. Prove that the points $A(1,2,-1)$, $B(0,1,5)$, $C(-1,2,1)$ and $D(2,1,3)$ are situated in the same plane.

8. Find the volume of the tetrahedron which has $A(2,-1,1)$, $B(5,5,4)$, $C(3,2,1)$ and $D(4,1,3)$ as vertices.

9. Let $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ be coplanar representatives of vectors with modulus 1 and such that A, B, C are on the same side of a line that passes through O. Show that $||\overline{OA}+\overline{OB}+\overline{OC}|| \geq 1$.

10. Let $ABCD$ be a quadrilateral and E, F the midpoints of $[AB]$ and $[CD]$. Denote by K, L, M and N the midpoints of the segments $[AF], [CE], [BF]$ and $[DE]$, respectively. Prove that $KLMN$ is a parallelogram.

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I expect you are able to prove equalities as the ones below. Have a go at them!

12. Let $\overline{a}, \overline{b}, \overline{c}$ be vectors in $\mathcal{V}_3$. Prove the following formulae:

1. $\overline{a} \times (\overline{b} \times \overline{c}) = (\overline{a} \cdot \overline{c}) \cdot \overline{b} - (\overline{a} \cdot \overline{b}) \cdot \overline{c} = \begin{vmatrix} \overline{b} & \overline{c} \\ \overline{a} \cdot \overline{b} & \overline{a} \cdot \overline{c} \end{vmatrix}$;
2. $(\overline{a} \times \overline{b}) \times \overline{c} = (\overline{a} \cdot \overline{c}) \cdot \overline{b} - (\overline{b} \cdot \overline{c}) \cdot \overline{a} = \begin{vmatrix} \overline{b} & \overline{a} \\ \overline{b} \cdot \overline{c} & \overline{a} \cdot \overline{c} \end{vmatrix}$.
3. $(\overline{a} \times \overline{b}) \cdot (\overline{c} \times \overline{d}) = \begin{vmatrix} \overline{a} \cdot \overline{c} & \overline{a} \cdot \overline{d} \\ \overline{b} \cdot \overline{c} & \overline{b} \cdot \overline{d} \end{vmatrix}$;
4. $(\overline{a} \times \overline{b}) \times (\overline{c} \times \overline{d}) = (\overline{a}, \overline{c}, \overline{d}) \cdot \overline{b} - (\overline{b}, \overline{c}, \overline{d}) \cdot \overline{a} = (\overline{a}, \overline{b}, \overline{d}) \cdot \overline{c} - (\overline{a}, \overline{b}, \overline{c}) \cdot \overline{d}$;
5. $(\overline{a} \times \overline{b}, \overline{b} \times \overline{c}, \overline{c} \times \overline{a}) = (\overline{a}, \overline{b}, \overline{c})^2$