Sequences of real numbers - 2nd part

Ex 1.: $x_n = \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdot \ldots \cdot\left(1-\frac{1}{n^2}\right) =$

$= \frac{2^2-1}{2^2}\cdot\frac{3^2-1}{3^2}\cdot \ldots \cdot \frac{n^2-1}{n^2} = \frac{(2-1)(2+1)}{2^2}\cdot\frac{(3-1)(3+1)}{3^2}\cdot \ldots \cdot\frac{(n-1)(n+1)}{n^2} =$

$= \frac{1}{2}\cdot\frac{n+1}{n} = \frac{n+1}{2n}$

$\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty}\frac{n+1}{2n} = \lim\limits_{n\to\infty}\frac{n(1+\frac{1}{n})}{2n} = \frac{1}{2}\in\mathbb{R} \Rightarrow (x_n)$ convergent

Ex 2.: a) $\lim\limits_{n\to\infty}\frac{5^n}{7^n} = \lim\limits_{n\to\infty}\left(\frac{5}{7}\right)^n = 0$ b) $\lim\limits_{n\to\infty}\frac{4^n+(-4)^n}{6^n} = \lim\limits_{n\to\infty}\frac{4^n(1+(-1))}{6^n} = 0$

c) $\lim\limits_{n\to\infty}\frac{7-3n^3}{5n^2+n} = \lim\limits_{n\to\infty}\frac{n^3(\frac{7}{n^3}-3)}{n^2(5+\frac{1}{n})} = \infty\cdot\frac{-3}{5} = -\infty$

d) $\lim\limits_{n\to\infty}\left(2+\frac{9^n+(-8)^n}{10^n+1}\right)^{25n^3-n^2} = \lim\limits_{n\to\infty}\left(\frac{2\cdot10^n+2+9^n+(-8)^n}{10^n+1}\right)^{25n^3-n^2} =$

$= \lim\limits_{n\to\infty}\frac{10^n\left(2+\frac{2}{10^n}+\left(\frac{9}{10}\right)^n+\left(\frac{-8}{10}\right)^n\right)^{25n^3-n^2}}{10^n\left(1+\frac{1}{10^n}\right)} = \lim\limits_{n\to\infty}2^{25n^3-n^2} = 2^\infty = \infty$

e) $\lim\limits_{n\to\infty}\frac{1+2+\ldots+n}{n^2} = \lim\limits_{n\to\infty}\frac{\frac{n(n+1)}{2}}{n^2} = \lim\limits_{n\to\infty}\frac{n^2+n}{2n^2} = \lim\limits_{n\to\infty}\frac{n^2(1+\frac{1}{n})}{2n^2} = \frac{1}{2}$

f) $\lim\limits_{n\to\infty}\left(\frac{n^3+4n+1}{2n^3+5}\right)^{\frac{-2n^4+1}{n^4+3n+1}} = \lim\limits_{n\to\infty}\left(\frac{n^3(1+\frac{4}{n^2}+\frac{1}{n^3})}{n^3(2+\frac{5}{n^3})}\right)^{\frac{n^4(-2+\frac{1}{n^4})}{n^4(1+\frac{3}{n^3}+\frac{1}{n^4})}} =$

$= \lim\limits_{n\to\infty}\left(\frac{1}{2}\right)^{-2} = \sqrt{2}$

g) $\lim\limits_{n\to\infty}\cos(-2013)^n = \cos(-2013)^\infty = \infty$

h) $\lim\limits_{n\to\infty}\left(\frac{2n^5+3n+1}{7n^5+n^4+3}\right)^{\frac{9n-n^4}{8n^3+1}} = \lim\limits_{n\to\infty}\left(\frac{n^5(2+\frac{3}{n^4}+\frac{1}{n^5})}{n^5(7+\frac{1}{n}+\frac{3}{n^5})}\right)^{\frac{n^4(\frac{9}{n^3}-1)}{n^3(8+\frac{1}{n^3})}} =$

$= \lim\limits_{n\to\infty}\left(\frac{2}{7}\right)^{\infty\cdot(-\frac{1}{8})} = \left(\frac{2}{7}\right)^{-\infty} = \infty$

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Ex3: a) $\lim_{n\to\infty}\left(1+\frac{1}{-5n^3+3n}\right)^{n^2-5n^3}=\lim_{n\to\infty}\left(\frac{-5n^3+3n+1}{-5n^3+3n}\right)^{n^2-5n^3}=$

$=\lim_{n\to\infty} e^{\frac{1}{-5n^3+3n}(n^2-5n^3)}=e^{\lim_{n\to\infty}\frac{n^2-5n^3}{3n-5n^3}}=e^1=e$

b) $\lim_{n\to\infty}(9n^2+5)\ln\left(1+\frac{1}{4n^2}\right)=\lim_{n\to\infty}\ln\left(1+\frac{1}{4n^2}\right)^{9n^2+5}=\ln\lim_{n\to\infty} e^{\frac{1}{4n^2}\cdot(9n^2+5)}=$

$=\ln\lim_{n\to\infty} e^{\frac{9}{4}}=\ln e^{\frac{9}{4}}=\frac{9}{4}$

c) $\lim_{n\to\infty}\frac{n^n}{1^1+2^2+\dots+n^n}=\lim_{n\to\infty}\frac{n^n}{n^n\left(\frac{1}{n^n}+\frac{2^2}{n^n}+\dots+1\right)}=1$

d) $\lim_{n\to\infty}\frac{x_1+2x_2+\dots+nx_n}{n^2}=\lim_{n\to\infty}\frac{n\left(\frac{x_1}{n}+\frac{2x_2}{n}+\dots+x_n\right)}{n^2}=0$

Ex4: a) $x_n=\frac{a^n-a^{-n}}{a^n+a^{-n}},\ a\neq 0$ $\lim_{n\to\infty}x_n=\lim_{n\to\infty}\frac{a^n-a^{-n}}{a^n+a^{-n}}=\lim_{n\to\infty}\frac{a^n(1-a^{-2n})}{a^n(1+a^{-2n})}=1$

b) $y_n=\frac{a^n+b^n}{a^{n+1}+b^{n+1}},\ a\neq -b$

I $a<b \Rightarrow \lim_{n\to\infty}y_n=\lim_{n\to\infty}\frac{b^n\left[\left(\frac{a}{b}\right)^n+1\right]}{b^{n+1}\left[\left(\frac{a}{b}\right)^{n+1}+1\right]}=\frac{1}{b}$

II $a>b \Rightarrow \lim_{n\to\infty}y_n=\lim_{n\to\infty}\frac{a^n\left[1+\left(\frac{b}{a}\right)^n\right]}{a^{n+1}\left[1+\left(\frac{b}{a}\right)^{n+1}\right]}=\frac{1}{a}$

III $a=b \Rightarrow \lim_{n\to\infty}y_n=\lim_{n\to\infty}\frac{a^n(1+1)}{a^{n+1}(1+1)}=\frac{1}{a}=\frac{1}{b}$

c) $z_n=\frac{1+a+\dots+a^n}{1+b+\dots+b^n},\ a,b>0$ $\lim_{n\to\infty}z_n=\lim_{n\to\infty}\frac{1+a+\dots+a^n}{1+b+\dots+b^n}=\lim_{n\to\infty}\frac{a^n\left(\frac{1}{a^n}+\frac{1}{a^{n-1}}+\dots+1\right)}{b^n\left(\frac{1}{b^n}+\frac{1}{b^{n-1}}+\dots+1\right)}=$

$=\lim_{n\to\infty}\left(\frac{a}{b}\right)^n=\begin{cases}\infty, & a>b\\ 0, & a<b\\ 1, & a=b\end{cases}$