# Analytic Geometry

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# Last time

We have defined the operations

$$+ : V_2 \times V_2 \to V_2, \quad (\overline{a}, \overline{b}) \mapsto \overline{a} + \overline{b}$$

$$\cdot : \mathbb{R} \times V_2 \to V_2, \quad (k, \overline{a}) \mapsto k \cdot \overline{a}$$

and, of course,

$$+ : V_3 \times V_3 \to V_3, \quad (\overline{a}, \overline{b}) \mapsto \overline{a} + \overline{b}$$

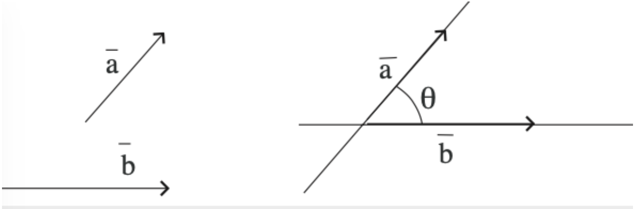
$$\cdot : \mathbb{R} \times V_3 \to V_3, \quad (k, \overline{a}) \mapsto k \cdot \overline{a}.$$

# We also saw that

- The set $\{\overline{a}, \overline{b}\}$ is a basis in $V_2$ if and only if the vectors $\overline{a}$, $\overline{b}$ are not collinear.
- The set $\{\overline{a}, \overline{b}, \overline{c}\}$ is a basis in $V_3$ if and only if the vectors $\overline{a}$, $\overline{b}$, $\overline{c}$ are not coplanar.

# Dot product

The angle between two nonzero vectors $\overline{a}$ and $\overline{b}$ from $V_2$ or $V_3$ is defined as the angle $\theta = \widehat{(\overline{a}, \overline{b})} \in [0, \pi]$ determined by their directions, taking into account their orientations.




Given the vectors $\overline{a}$ and $\overline{b}$ in $V_2$ (or $V_3$), their **dot product** is the real number defined through

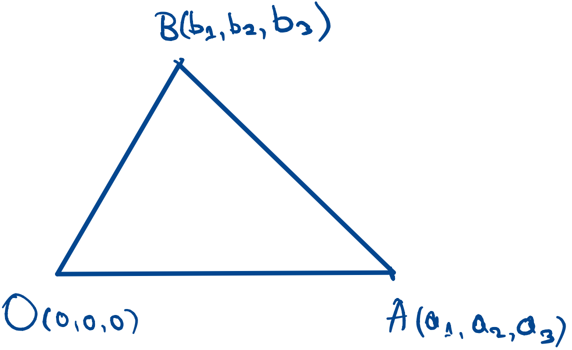
$$\overline{a} \cdot \overline{b} = \begin{cases} |\overline{a}||\overline{b}| \cos\theta, & \text{if } \overline{a} \neq 0 \text{ and } \overline{b} \neq 0 \\ 0, & \text{otherwise} \end{cases}.$$

# Does $\mathbb{R}^3$ (or $\mathbb{R}^n$) "know" any geometry?

**Theorem**

1. *If* $\overline{a}(a_1, a_2)$ *and* $\overline{b}(b_1, b_2)$ *are two vectors in* $V_2$*, then*
$$\overline{a} \cdot \overline{b} = a_1 b_1 + a_2 b_2; \tag{1}$$
2. *If* $\overline{a}(a_1, a_2, a_3)$ *and* $\overline{b}(b_1, b_2, b_3)$ *are two vectors in* $V_3$*, then*
$$\overline{a} \cdot \overline{b} = a_1 b_1 + a_2 b_2 + a_3 b_3. \tag{2}$$

**Proof.** We only prove (2). Choose $\overrightarrow{OA} \in \overline{a}$ and $\overrightarrow{OB} \in \overline{b}$, where $O$ is the origin of the Cartesian system. Then $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$.



By definition, $\overline{a} \cdot \overline{b} = |\overline{a}| \cdot |\overline{b}| \cdot \cos(\measuredangle \overline{a}, \overline{b})$

$$= OA \cdot OB \cdot \cos(\measuredangle AOB).$$

Using Cosine Thm in $\Delta AOB$, we have

$$OA \cdot OB \cdot \cos(\measuredangle AOB) = \frac{OA^2 + OB^2 - AB^2}{2}.$$

Using the distance formula,

$$OA \cdot OB \cdot \cos(\measuredangle AOB) = \frac{1}{2}\Big[a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2$$
$$- (a_1 - b_1)^2 - (a_2 - b_2)^2 - (a_3 - b_3)^2\Big]$$
$$= a_1 b_1 + a_2 b_2 + a_3 b_3 \quad \blacksquare$$

Since $\cos\theta = \dfrac{\overline{a}\cdot\overline{b}}{|\overline{a}||\overline{b}|}$, then, for two nonzero vectors $\overline{a}$ and $\overline{b}$, one has

$$\cos\widehat{(\overline{a},\overline{b})} = \frac{a_1b_1 + a_2b_2}{\sqrt{a_1^2+a_2^2}\cdot\sqrt{b_1^2+b_2^2}},\ \text{for } \overline{a},\overline{b}\in V_2; \tag{3}$$

$$\cos\widehat{(\overline{a},\overline{b})} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\cdot\sqrt{b_1^2+b_2^2+b_3^2}},\ \text{for } \overline{a},\overline{b}\in V_3. \tag{4}$$

**Theorem**

*If $\overline{u}$ and $\overline{v}$ are nonzero vectors in $V_2$ (or $V_3$) and $\theta$ is the angle between them, then*

**a)** *$\theta$ is acute if and only if $\overline{u} \cdot \overline{v} > 0$;*

**b)** *$\theta$ is obtuse if and only if $\overline{u} \cdot \overline{v} < 0$;*

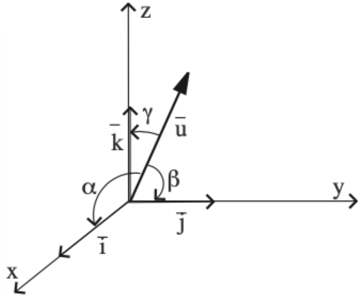
**c)** *$\theta = \dfrac{\pi}{2}$ if and only if $\overline{u} \cdot \overline{v} = 0$.*

**Proof**. The sign of the cosine of $\theta$ coincides with the sign of the dot product $\overline{a} \cdot \overline{b}$. The assertions follow trivially. $\square$

The notions of "acute", "obtuse" or orthogonal (perpendicular) can be generalized to vectors with more than 3 components using the algebraic form of the dot product, even if there's no obvious "geometrical" interpretation.

# The 3 axes determine 3 angles

Given an arbitrary vector $\overline{u} \in V_3$ and an associated Cartesian system of coordinates, one defines the *director angles* of $\overline{u}$ to be the three angles determined by $\overline{u}$ with the versors of the system of coordinates $\alpha = \widehat{(\overline{u}, \overline{i})}$, $\beta = \widehat{(\overline{u}, \overline{j})}$ and $\gamma = \widehat{(\overline{u}, \overline{k})}$, respectively.




The values $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are sometimes called *director cosines* of the vector $\overline{u}$.

**Theorem**

*The director cosines of a vector* $\overline{u}(u_1, u_2, u_3) \in V_3$, $\overline{u} \neq \overline{0}$, *are*

$$\cos\alpha = \frac{u_1}{|\overline{u}|},\ \cos\beta = \frac{u_2}{|\overline{u}|},\ \cos\gamma = \frac{u_3}{|\overline{u}|}. \tag{5}$$

**Proof.**

$\alpha = \sphericalangle(\overline{u}, \overline{i})$

$\overline{i}(1, 0, 0)$

$\overline{u} \cdot \overline{i} = |\overline{u}| \cdot |\overline{i}| \cdot \cos\alpha$

$u_1 \cdot 1 + u_2 \cdot 0 + u_3 \cdot 0 = |\overline{u}| \cos\alpha$

$\Rightarrow \cos\alpha = \dfrac{u_1}{|\overline{u}|}$ $\boxtimes$

For any nonzero vector $\overline{u} \in V_3$, $\dfrac{\overline{u}}{|\overline{u}|}$ is a unit vector, called the *versor of* $\overline{u}$. Moreover, it is easy to see that

$$\frac{\overline{u}}{|\overline{u}|} = \cos\alpha \cdot \bar{i} + \cos\beta \cdot \bar{j} + \cos\gamma \cdot \overline{k}, \text{ with } (\cos\alpha)^2 + (\cos\beta)^2 + (\cos\gamma)^2 = 1.$$

# Algebraic properties of the dot product

Given $\overline{a}$, $\overline{b}$, $\overline{c} \in V_3$ (or $V_2$) and $\lambda \in \mathbb{R}$, one has:

**1)** $\overline{a} \cdot \overline{b} = \overline{b} \cdot \overline{a}$ (commutativity of the dot product);

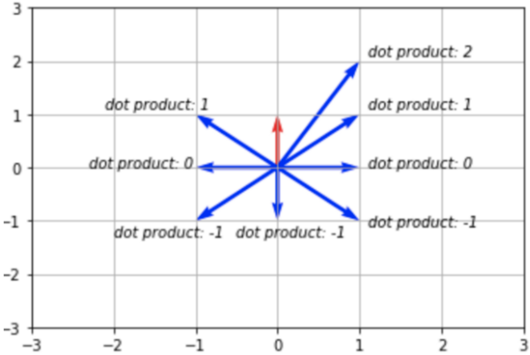
**2)** $\overline{a} \cdot (\overline{b} + \overline{c}) = \overline{a} \cdot \overline{b} + \overline{a} \cdot \overline{c}$ (distributivity of the dot product with respect to the summation of vectors);

**3)** $\lambda(\overline{a} \cdot \overline{b}) = (\lambda \overline{a}) \cdot \overline{b} = \overline{a} \cdot (\lambda \overline{b})$;

**4)** $\overline{a} \cdot \overline{a} = |\overline{a}|^2$.

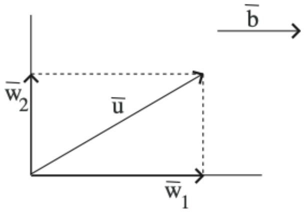
1)-3) can all be proved by comparing the components of the vectors on both sides.

# Exercise




- Dot product is larger when the direction of the blue vectors is similar to the direction of the red one.
- Dot product is larger when the magnitude of the blue vector is larger.

Sometimes it is useful to decompose a vector into a sum of two terms, one of them having a given direction and the other being orthogonal on this direction.




Let $\overline{u}$ and $\overline{b}$ be two nonzero vectors and project (orthogonally) a representative of the vector $\overline{u}$ on a line passing through the original point of this representative and parallel to the direction of $\overline{b}$. One gets the vector $\overline{w}_1$, having the direction of $\overline{b}$ and, by making the difference $\overline{u} - \overline{w}_1$, another vector $\overline{w}_2$, orthogonal on the direction of $\overline{b}$; $\overline{u} = \overline{w}_1 + \overline{w}_2$.

$$pr_{\overline{b}}(\overline{u}) = \overline{w}_1$$

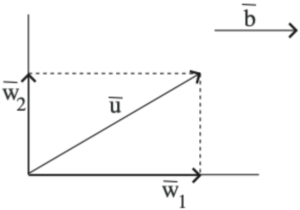
# Projections

- The vector $\overline{w}_1$ is called the *orthogonal projection of* $\overline{u}$ *on* $\overline{b}$ and it is denoted by $\mathrm{pr}_{\overline{b}}\overline{u}$.
- The vector $\overline{w}_2$ is called the *vector component of* $\overline{u}$ *orthogonal to* $\overline{b}$ and $\overline{w}_2 = \overline{u} - \mathrm{pr}_{\overline{b}}\overline{u}$.

## Theorem

*If $\overline{u}$ and $\overline{b}$ are vectors in $V_2$ or $V_3$ and $\overline{b} \neq 0$, then*

- *the orthogonal projection of $\overline{u}$ on $\overline{b}$ is* $pr_{\overline{b}}\overline{u} = \dfrac{\overline{u} \cdot \overline{b}}{|\overline{b}|^2} \cdot \overline{b}$;
- *the vector component of $\overline{u}$ orthogonal to $\overline{b}$ is* $\overline{u}\text{-}pr_{\overline{b}}\overline{u} = \overline{u} - \dfrac{\overline{u} \cdot \overline{b}}{|\overline{b}|^2} \cdot \overline{b}$.




Proof: Let $\overline{u} = \overline{w}_1 + \overline{w}_2$

where $\overline{w}_1 = pr_{\overline{b}}(\overline{u})$ and

$\overline{w}_2 \perp \overline{b}$.

Since $\overline{w}_1 \parallel \overline{b}$, we know that

$\overline{w}_1 = k \cdot \overline{b}$, for some $k \in \mathbb{R}^*$.

$$\overline{u} = \overline{w}_1 + w_2 \iff$$

$$\overline{u} = k \cdot \overline{b} + \overline{w}_2 \quad \Big| \cdot \overline{b}$$

$\therefore \quad \overline{u} \cdot \overline{b} = k \cdot |\overline{b}|^2$, since

$\overline{w}_2 \cdot \overline{b} = 0$.

$$\therefore \quad k = \frac{\overline{u} \cdot \overline{b}}{|\overline{b}|^2}.$$

$$pr_{\overline{b}}(\overline{u}) = \frac{\overline{u} \cdot \overline{b}}{|\overline{b}|^2} \cdot \overline{b}$$

$\boxed{Z}$

The length of the orthogonal projection of the vector $\overline{u}$ on $\overline{b}$ can be obtained as following:

$$|\mathrm{pr}_{\overline{b}}\overline{u}| = \left|\frac{\overline{u}\cdot\overline{b}}{|\overline{b}|^2}\cdot\overline{b}\right| = \left|\frac{\overline{u}\cdot\overline{b}}{|\overline{b}|^2}\right||\overline{b}|,$$

which yields

$$|\mathrm{pr}_{\overline{b}}\overline{u}| = \frac{|\overline{u}\cdot\overline{b}|}{|\overline{b}|},$$

and if $\theta$ is the angle between $\overline{u}$ and $\overline{b}$, then

$$|\mathrm{pr}_{\overline{b}}\overline{u}| = |\overline{u}||\cos\theta|.$$

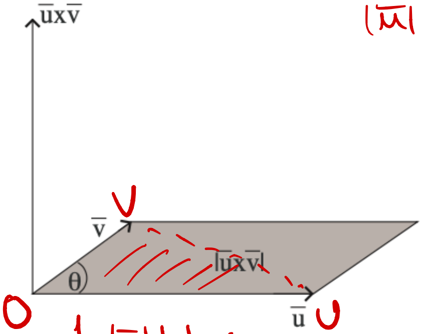
# The cross product

**Definition**

The cross product of two vectors $\overline{u}$ and $\overline{v}$ is another vector $\overline{u} \times \overline{v}$, which can be determined by the following conditions:

- If $\overline{u}$ and $\overline{v}$ are colinear, then $\overline{u} \times \overline{v} := \overline{0}$;
- Else, let $0 < \theta < \pi$ be the angle between them. The vector $\overline{u} \times \overline{v}$ is such that:
  1. $||\overline{u} \times \overline{v}|| = ||\overline{u}|| \cdot ||\overline{v}|| \cdot \sin(\theta)$;
  2. $\overline{u} \times \overline{v}$ is perpendicular on $\overline{u}$ and on $\overline{v}$;
  3. the orientation of $\overline{u} \times \overline{v}$ is given by the right-hand rule.

$|\overline{u}| \cdot |\overline{v}| \cdot$




$\frac{1}{2} \cdot |\overline{u}| \cdot |v| \cdot \sin\theta$

- If the vectors $\overline{u}, \overline{v}$ are not collinear, then if $\overrightarrow{OU} \in \overline{u}$ and $\overrightarrow{OV} \in \overline{v}$, then $||\overline{u} \times \overline{v}||$ is the area of the parallelogram formed by $\overrightarrow{OU}$ and $\overrightarrow{OV}$.
- The area of the triangle $\triangle O\cancel{AB}UV$ can be computed as

$$\text{Area}_{\triangle O\cancel{AB}UV} = \frac{||\overline{u} \times \overline{v}||}{2}.$$

# The algebraic form of the cross product

If $\overline{u} = u_1\overline{i} + u_2\overline{j} + u_3\overline{k}$ and $\overline{v} = v_1\overline{i} + v_2\overline{j} + v_3\overline{k}$ are vectors in $V_3$, then their *cross product* $\overline{u} \times \overline{v}$ is the vector

$$\overline{u} \times \overline{v} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \overline{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \overline{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \overline{k}, \tag{6}$$

or, shortly,

$$\overline{u} \times \overline{v} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}. \tag{7}$$

# Did we defined the same thing?

Let $\overline{u}(u_1, u_2, u_3)$ and $\overline{v}(v_1, v_2, v_3)$. Using the algebraic definition, we get $\overline{u} \times \overline{v}(u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$.

- $\overline{u} \cdot (\overline{u} \times \overline{v}) = 0$, so $\overline{u} \times \overline{v}$ is orthogonal on $\overline{u}$;

$$u_1(u_2v_3 - u_3v_2) + u_2(u_3v_1 - u_1v_3)$$
$$+ u_3(u_1v_2 - u_2v_1) = 0$$

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Let $\overline{u}(u_1, u_2, u_3)$ and $\overline{v}(v_1, v_2, v_3)$. Using the algebraic definition, we get $\overline{u} \times \overline{v}(u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1)$.

- $\overline{u} \cdot (\overline{u} \times \overline{v}) = 0$, so $\overline{u} \times \overline{v}$ is orthogonal on $\overline{u}$; Indeed, notice that

$$\overline{u} \cdot (\overline{u} \times \overline{v}) = u_1(u_2 v_3 - u_3 v_2) + u_2(u_3 v_1 - u_1 v_3) + u_3(u_1 v_2 - u_2 v_1) = 0.$$

- Similarly, $\overline{v} \cdot (\overline{u} \times \overline{v}) = 0$.

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- $\overline{u} \cdot (\overline{u} \times \overline{v}) = 0$, so $\overline{u} \times \overline{v}$ is orthogonal on $\overline{u}$; Indeed, notice that

$$\overline{u} \cdot (\overline{u} \times \overline{v}) = u_1(u_2v_3 - u_3v_2) + u_2(u_3v_1 - u_1v_3) + u_3(u_1v_2 - u_2v_1) = 0.$$

- Similarly, $\overline{v} \cdot (\overline{u} \times \overline{v}) = 0$.
- We have that

$$\underline{|\overline{u} \times \overline{v}|}^2 = |\overline{u}|^2|\overline{v}|^2 - (\overline{u} \cdot \overline{v})^2$$

(Lagrange's identity).

$$= |\overline{u}|^2 \cdot |\overline{v}|^2 - (|\overline{u}| \cdot |\overline{v}| \cdot \cos\theta)^2$$

$$= |\overline{u}|^2 \cdot |\overline{v}|^2 (1 - \cos^2\theta)$$

To prove Lagrange's identity, one just has to open the brackets and check that

$$|\overline{u} \times \overline{v}|^2 = (u_2 v_3 - u_3 v_2)^2 + (u_3 v_1 - u_1 v_3)^2 + (u_1 v_2 - u_2 v_1)^2$$

equals to

$$|\overline{u}|^2 |\overline{v}|^2 - (\overline{u} \cdot \overline{v})^2 = (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1 v_1 + u_2 v_2 + u_3 v_3)^2.$$

Using Lagrange's identity,

$$|\overline{u} \times \overline{v}|^2 = |\overline{u}|^2 |\overline{v}|^2 - (\overline{u} \cdot \overline{v})^2 = |\overline{u}|^2 |\overline{v}|^2 - |\overline{u}|^2 |\overline{v}|^2 \cos^2 \theta = |\overline{u}|^2 |\overline{v}|^2 \sin^2 \theta.$$

Are you convinced that the cross product defined geometrically and the cross product defined algebraically are one and the same?

$\boxed{\overline{u} \times \overline{v} = -\overline{v} \times \overline{u}}$ ;

An immediate consequence of the Lagrange's identity is that $|\overline{u}|^2|\overline{v}|^2 - (\overline{u} \cdot \overline{v})^2 \geq 0$, or $|\overline{u} \cdot \overline{v}| \leq |\overline{u}||\overline{v}|$, which leads, after replacing the components of the vectors, to the Cauchy-Schwartz inequality. The equality $|\overline{u} \cdot \overline{v}| = |\overline{u}||\overline{v}|$ holds if and only if the vector $\overline{u} \times \overline{v}$ is the zero vector, i.e. its components are all zero, which happens if and only if $\dfrac{v_1}{u_1} = \dfrac{v_2}{u_2} = \dfrac{v_3}{u_3} = \lambda$, or $\overline{v} = \lambda \overline{u}$, $\lambda \in \mathbb{R}^*$. In summary, one has:

**Theorem**

*If $\overline{u}$ and $\overline{v}$ are nonzero vectors in $V_3$, then $\overline{u} \times \overline{v} = \overline{0}$ if and only if $\overline{u}$ and $\overline{v}$ are parallel.*

The problem set for this week has been posted. Please have a look at it before attending the seminar.

Thank you very much for your attention!