Seminar 5

1. Let $(S, \mathcal{K}, P)$ is be a probability space and $A, B \in \mathcal{K}$.
(a) Prove that

$$P(A) + P(B) - 1 \leq P(A \cap B) \leq \min\{P(A), P(B)\}$$

$$\max\{P(A), P(B)\} \leq P(A \cup B) \leq P(A) + P(B).$$

(b) It is known that $P(A) = \frac{2}{5}, P(B) = \frac{9}{10}$. Indicate four concrete values $a, b, c, d \in (0, 1]$ such that

$$a \leq P(A \cap B) \leq b < c \leq P(A \cup B) \leq d.$$

Explain why your choice is correct!
(c) It is known that $P(A) = \frac{2}{5}, P(B) = \frac{9}{10}, P(A|B) = \frac{10}{27}$.
(c1) Compute the probabilities: $P(A \cap B), P(A \cup B), P(B \cap \bar{A}), P(B|A)$.
(c2) Are A and B independent events?

A: (a) Using the properties of P we write

- $0 \leq P(A \cup B) \leq 1 \Rightarrow P(A) + P(B) - P(A \cap B) \leq 1 \Rightarrow P(A) + P(B) - 1 \leq P(A \cap B)$;
- $P(A \cap B) \leq P(A) \leq P(A \cup B)$ and $P(A \cap B) \leq P(B) \leq P(A \cup B)$

$\Rightarrow P(A \cap B) \leq \min\{P(A), P(B)\} \leq \max\{P(A), P(B)\} \leq P(A \cup B)$;

- $P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B),$ since $-P(A \cap B) \leq 0$.

(b) We use the above inequalities

$$P(A) + P(B) - 1 \leq P(A \cap B) \leq \min\{P(A), P(B)\} \leq \max\{P(A), P(B)\} \leq P(A \cup B) \leq P(A) + P(B)$$

$$\Rightarrow \frac{2}{5} + \frac{9}{10} - 1 \leq P(A \cap B) \leq \frac{2}{5} < \frac{9}{10} \leq P(A \cup B) \leq \frac{2}{5} + \frac{9}{10}$$

$$\Rightarrow \frac{3}{10} \leq P(A \cap B) \leq \frac{2}{5} < \frac{9}{10} \leq P(A \cup B) \leq \frac{13}{10}$$

$$\Rightarrow a = \frac{3}{10}, b = \frac{2}{5}, c = \frac{9}{10}, d = 1.$$

(c1) From the definition of conditional probability

$$\Rightarrow P(A|B) = \frac{P(A \cap B)}{P(B)} \Rightarrow P(A \cap B) = P(A|B) \cdot P(B) = \frac{10}{27} \cdot \frac{9}{10} = \frac{1}{3};$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{2}{5} + \frac{9}{10} - \frac{1}{3} = \frac{29}{30};$$

$$P(B \cap \bar{A}) = P(B \setminus A) = P(B) - P(A \cap B) = \frac{9}{10} - \frac{1}{3} = \frac{17}{30};$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{3}}{\frac{2}{5}} = \frac{5}{6}.$$

(c2) Are A and B are not independent events, since

$$P(A \cap B) = \frac{1}{3} \neq P(A) \cdot P(B) = \frac{2}{5} \cdot \frac{9}{10} = \frac{9}{25}.$$

2. John has in his pocket 2 red dice and 3 blue dice. He takes randomly a die. If the chosen die is red, he rolls it 3 times. On the other hand, if the chosen die is blue, he rolls it 2 times. Compute the probability of the event E:"the sum of the numbers that show up after the rolls is 10."

A: Let R:"the chosen die is red", B:"the chosen die is blue". Note that $\{R, B\}$ forms a partition. By the formula of total probability, $P(E) = P(E|R)P(R) + P(E|B)P(B) = \frac{27}{6^3} \cdot \frac{2}{5} + \frac{3}{6^2} \cdot \frac{3}{5} = \frac{1}{10}$.

3. Mary studies Probability Theory. She arrives late at the seminar with probability 0.2, if the day is rainy, and with probability 0.1, if the day is sunny. According to the weather forecast, the next day, when Mary has the seminar, is rainy with probability 0.8. Compute the probabilities of the events:
A: "Mary arrives on time at the next seminar."
B: "The next day is rainy, given that Mary arrives on time at the seminar ."

A: Let L: "Mary is late at the next seminar" and S:"the next day is sunny". Note that $\{S, \bar{S}\}$ forms a partition. By the formula of total probability, $P(\bar{L}) = P(\bar{L}|S)P(S) + P(\bar{L}|\bar{S})P(\bar{S}) = 0.9 \cdot 0.2 + 0.8 \cdot 0.8 = 0.82$. By the Bayes formula, $P(\bar{S}|\bar{L}) = \frac{P(\bar{L}|\bar{S})P(\bar{S})}{P(\bar{L})} = \frac{0.8 \cdot 0.8}{0.82} = \frac{32}{41}$.

4. A die is rolled. Let N be the number that is obtained. Next, the die is rolled N times. What is the probability that $N = 3$, given that $N \geq 2$ and the numbers obtained after the N rolls are pairwise **a)** distinct? **b)** equal?

A: Let D: "$N \geq 2$ and the numbers obtained after the N rolls are pairwise distinct" and E: "$N \geq 2$ and the numbers obtained after the N rolls are all equal". Note that $\{\{N = i\} : i \in \{1, \ldots, 6\}\}$ forms a partition. The Bayes formula implies:

a) $P(N = 3|D) = \frac{P(D|N=3)P(N=3)}{\sum_{i=1}^{6} P(D|N=i)P(N=i)} = \frac{\frac{A_6^3}{6^4}}{\sum_{i=2}^{6} \frac{A_6^i}{6^{i+1}}}$; b) $P(N = 3|E) = \frac{P(E|N=3)P(N=3)}{\sum_{i=1}^{6} P(E|N=i)P(N=i)} = \frac{\frac{1}{6^3}}{\sum_{i=2}^{6} \frac{1}{6^i}}$.

5. A pair of dice - one white die and one red die - is rolled two times. Compute the probability that the two pairs of numbers, obtained after the two rolls, are equal. (Example of favorable case: the white die shows number 2 and the red die shows number 4, both after the first roll and the second roll; example of unfavorable case: first roll "2 on white die, 4 on red die", second roll "4 on white die, 2 on red die".)

A: Let (W_k, R_k) be the pair of numbers obtained by the white die, respectively, the red die after the kth roll, $k \in \{1, 2\}$. The desired probability is

$$p = \sum_{i=1}^{6} \sum_{j=1}^{6} P\left(\{W_1 = i\} \cap \{R_1 = j\} \cap \{W_2 = i\} \cap \{R_2 = j\}\right) = \sum_{i=1}^{6} \sum_{j=1}^{6} \frac{1}{6^4} = \frac{6^2}{6^4} = \frac{1}{36},$$

where we use the independence of the events $\{W_1 = i\}, \{R_1 = j\}, \{W_2 = i\}, \{R_2 = j\}, \forall\, i, j \in \{1, \ldots, 6\}$. Alternatively, by the formula of total probability,

$$p = \sum_{i,j=1}^{6} P\left(\{W_2 = i\} \cap \{R_2 = j\} \big| \{W_1 = i\} \cap \{R_1 = j\}\right) P\left(\{W_1 = i\} \cap \{R_1 = j\}\right) = \sum_{i,j=1}^{6} \frac{1}{6^2} \cdot \frac{1}{6^2} = \frac{1}{36}.$$

6. A computer center has three printers A, B, and C, which print at different speeds. Programs are routed to the first available printer. The probability that a program is routed to printers A, B, and C are 0.5, 0.3, and 0.2, respectively. Occasionally a printer will jam and destroy a printout. The probability that printers A, B, and C will jam are 0.02, 0.06 and 0.1, respectively. Your program is destroyed when a printer jams. What is the probability that printer A is involved? Printer B is involved? Printer C is involved?

A: Let $A_i,\ i=\overline{1,3}$ denote the events that the program was routed to printers A, B and C, respectively, and let E denote the event that the program was destroyed. Then $\{A_1, A_2, A_3\}$ form a partition and we have

$$P(A_1) = 0.5,\ P(A_2) = 0.3,\ P(A_3) = 0.2,$$

and

$$P(E|A_1) = 0.02,\ P(E|A_2) = 0.06,\ P(E|A_3) = 0.1.$$

By the formula of total probability we have

$$P(E) = P(E|A_1)\, P(A_1) + (E|A_2)\, P(A_2) + (E|A_3)\, P(A_3) = 0.5 \cdot 0.02 + 0.3 \cdot 0.06 + 0.2 \cdot 0.1 = 0.048.$$

By Bayes' formula, we get

$$P(A_1|E) = \frac{0.5 \cdot 0.02}{0.048} \approx 0.2083; P(A_2|E) = \frac{0.3 \cdot 0.06}{0.048} = 0.375; P(A_3|E) = \frac{0.2 \cdot 0.1}{0.048} \approx 0.4166.$$

7. a) Let $(S, \mathcal{K}, P)$ be a probability space and $B \in \mathcal{K}$ such that $P(B) > 0$. Prove that $(B, \mathcal{K}_B, P(\cdot|B))$ is a probability space, where $\mathcal{K}_B := \{B \cap A : A \in \mathcal{K}\}$ and $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(A)}{P(B)}, A \in \mathcal{K}_B$.
b) Give examples, by considering a random experiment and its corresponding probability space $(S, \mathcal{K}, P)$, for the probability space $(B, \mathcal{K}_B, P(\cdot|B))$ from **a)**.
A: a) Since $(S, \mathcal{K})$ is a measurable space, we deduce that $(B, \mathcal{K}_B)$ is a measurable space (note that $\mathcal{K}_B$ has the properties of a σ-field):
(i) $\mathcal{K}_B = B \cap \mathcal{K} \neq \emptyset$, because $B \neq \emptyset$ and $\mathcal{K} \neq \emptyset$;
(ii) if $A \in \mathcal{K}_B$, then $\bar{A} \in \mathcal{K}_B$, because there is $C \in \mathcal{K}$ such that $A = B \cap C$ and thus $\bar{A} = B \setminus A = B \setminus (B \cap C) = B \setminus C = B \cap \bar{C} \in B \cap \mathcal{K}$;
(iii) if $A_n \in \mathcal{K}_B$, $n \in \mathbb{N}^*$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{K}_B$, because there are $C_n \in \mathcal{K}$, $n \in \mathbb{N}^*$, such that $A_n = B \cap C_n$, so $\bigcup\limits_{n=1}^{\infty} A_n = \bigcup\limits_{n=1}^{\infty}(B \cap C_n) = B \cap \left(\bigcup\limits_{n=1}^{\infty} C_n\right) \in \mathcal{K}_B$.
Since P is a probability, we deduce that $P(\cdot|B)$ is a probability:
(1) $P(B|B) = \frac{P(B)}{P(B)} = 1$;
(2) $P(A|B) = \frac{P(A)}{P(B)} \geq 0$, for $A \in \mathcal{K}_B$;
(3) if $(A_n)_{n\geq 1}$ is a sequence of pairwise disjoint events from $\mathcal{K}_B$, then

$$P(\bigcup_{n=1}^{\infty} A_n|B) = \frac{P(\bigcup\limits_{n=1}^{\infty} A_n)}{P(B)} = \frac{\sum\limits_{n=1}^{\infty} P(A_n)}{P(B)} = \sum_{n=1}^{\infty} P(A_n|B).$$

b) Consider the experiment of rolling a die. Then we can choose $S = \{1, 2, 3, 4, 5, 6\}, \mathcal{K} = \mathcal{P}(S), P(A) = \frac{\#A}{6}, A \in \mathcal{K}$. Next, let $B = \{2, 4, 6\}$ (i.e., an even number shows up on the die). Then $\mathcal{K}_B = B \cap \mathcal{K} = \mathcal{P}(\{2, 4, 6\})$ and $P(A|B) = \frac{P(A)}{P(B)} = \frac{\#A}{3}, A \in \mathcal{K}_B$.