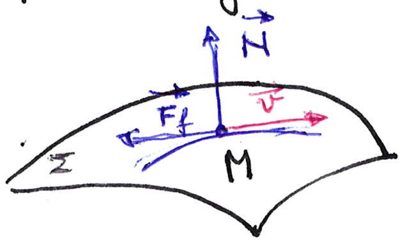
Dynamics of the particle subject to constraints

The dynamics of the particle under constraints reduces to the dynamics of particle (the free particle) where the constraints are replaced by reaction forces.

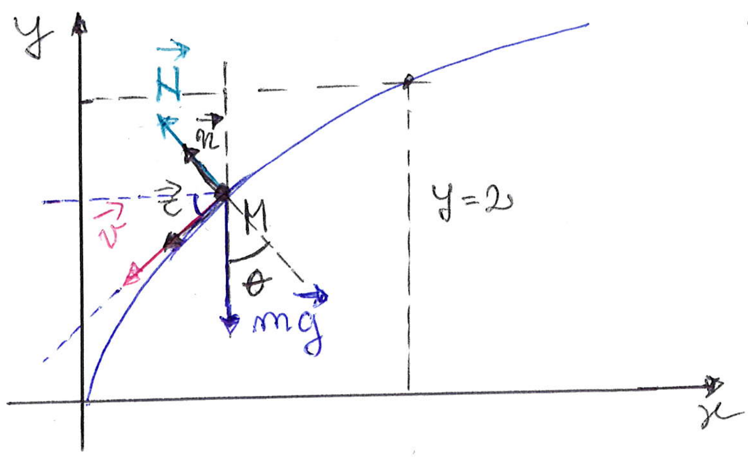


$\vec{N}$ - is the normal reaction
$\vec{F}_f$ - is the tangential reaction (friction force)

$|\vec{F}_f| = f \cdot |\vec{N}|$

f - friction coefficient.

① On the smooth outside of a parabola (with the horizontal axis Ox) of equation $y^2 = 2x$, a heavy small sphere rolls down without slip. The initial velocity of the sphere is zero and its initial ordinate is 2. Find the point where the sphere will leave the parabola. The mass of the sphere is m.



Remark 1. The trajectory of the sphere is known, thus it is suitable to work in the Frénet frame of coordinates.

Frénet $(\vec{\tau}, \vec{n}, \vec{b})$

$$\vec{a}\left(\underbrace{\frac{dv}{dt}}_{a_\tau}, \underbrace{\frac{v^2}{\rho}}_{a_n}, 0\right) = \frac{dv}{dt} \cdot \vec{\tau} + \frac{v^2}{\rho} \cdot \vec{n} \qquad (1)$$

Let be: $\theta = \measuredangle(\vec{n}, Oy) \Rightarrow tg\,\theta = \dfrac{dy}{dx}$ (2)

The differential equation of motion (the Newton's law) is:

$$m \cdot \vec{a} = m\vec{g} + \vec{N} + \underbrace{\vec{F}_f}_{=0} \quad (3)$$

direct applied force

normal reaction (due to the contact between the sphere and the parabola)

tangential reaction (friction force)

! $F_f = 0$ – the parabola is smooth (no friction).

Remark 2: When the sphere leaves the parabola (i.e. there is no contact between ~~the~~ sphere and parabola) we have $\boxed{\vec{N} = 0}$. (4)

On the components of the Frénet frame of coordinates, eq. (3) becomes:

$$\begin{cases} \vec{\tau}: m \cdot \dfrac{dv}{dt} = mg \sin\theta \\ \vec{n}: m \cdot \dfrac{v^2}{\rho} = -mg\cos\theta + N \\ \vec{b}: 0 = 0 \end{cases} \quad (5)$$

We can express N from (5b) and we obtain:

$$N = m\frac{v^2}{\rho} + mg\cos\theta \quad (6).$$

In order to calculate N (and later to impose the condition (4)) we have to find:
- a) the speed (Energy theorem)
- b) the curvature $\left(\dfrac{1}{\rho} = \dfrac{y''}{(1+y'^2)^{3/2}}\right)$
- c) $\cos\theta$ $\left(\text{we use (2)}, tg\,\theta = \dfrac{dy}{dx} = y'\right)$

a) Speed

Energy theorem: $dT = \delta L$ (Kinetic energy, Work)

$$d\left(\frac{1}{2}mv^2\right) - (m\vec{g}+\vec{N})\cdot d\vec{r} \Rightarrow d\left(\frac{1}{2}mv^2\right) = m\vec{g}\cdot d\vec{r} + \underbrace{\vec{N}\cdot d\vec{r}}_{=0\ (\vec{N}\perp d\vec{r})}$$

$$\left.\begin{aligned} m\vec{g} &= (0,-mg) \\ d\vec{r} &= (dx, dy) \end{aligned}\right\} \Rightarrow m\vec{g}\cdot d\vec{r} = -mg\,dy$$

$$\Rightarrow d\left(\frac{1}{2}mv^2\right) = -mg\,dy \,\Big|\Big|\int \Rightarrow \frac{1}{2}mv^2 - \frac{1}{2}m\underbrace{v_0^2}_{=0\ (v_0=0)} = -mg(y-\underbrace{y_0}_{=2})$$

Initial conditions

$$\Rightarrow \frac{1}{2}mv^2 = -mg(y-2) \Rightarrow \boxed{v^2 = 2g(2-y)} \quad (7)$$

b) Curvature: $\dfrac{1}{\rho} = \dfrac{y''}{(1+y'^2)^{3/2}}$

$$y = \sqrt{2x} = \sqrt{2}\cdot x^{\frac{1}{2}}$$

$$y' = \frac{\sqrt{2}}{2}\cdot x^{-1/2}$$

$$y'' = -\frac{1}{2\sqrt{2}}\cdot x^{-3/2}$$

$$\Rightarrow \frac{1}{\rho} = \frac{-\frac{1}{2\sqrt{2}}\cdot\frac{1}{x^{3/2}}}{\left(1+\frac{1}{2x}\right)^{3/2}}$$

$$\Rightarrow \rho = \frac{-(2x+1)^{3/2}}{\cancel{(2x)^{3/2}}\cdot\left(-\frac{1}{\cancel{2\sqrt{2}}}\right)\cdot\frac{1}{\cancel{x^{3/2}}}} = -(2x+1)^{3/2}$$

$$\Rightarrow \boxed{\rho = -(2x+1)^{3/2}} \quad (8)$$

c) $\cos\theta$: $\operatorname{tg}\theta = \dfrac{dy}{dx} = y' = \dfrac{1}{\sqrt{2x}} \Rightarrow \sqrt{\dfrac{1-\cos^2\theta}{\cos^2\theta}} = \dfrac{1}{\sqrt{2x}} \Rightarrow$

$$\Rightarrow \frac{1-\cos^2\theta}{\cos^2\theta} = \frac{1}{2x} \Rightarrow \cos^2\theta = \frac{2x}{1+2x} \Rightarrow \boxed{\cos\theta = \sqrt{\frac{2x}{1+2x}},\ \theta\in\left[0,\frac{\pi}{2}\right]} \quad (9)$$

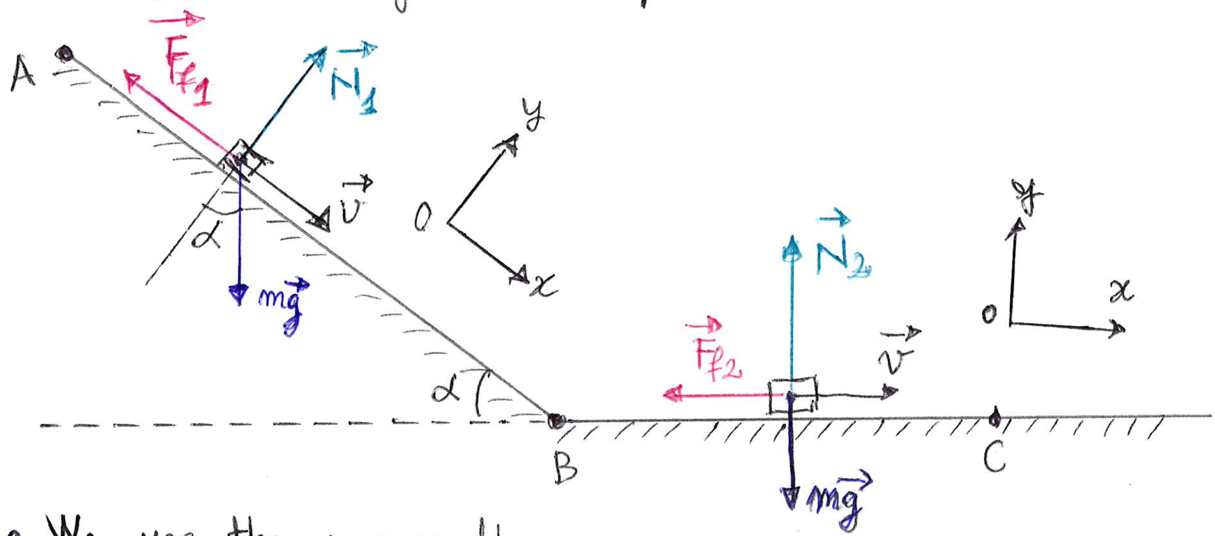
Next, using (6), (7), (8) and (9) we have:

$$N = m\cdot\frac{2g(2-\sqrt{2x})}{-(2x+1)\sqrt{2x+1}} + mg\sqrt{\frac{2x}{1+2x}} \overset{(4)}{=} 0 \;\Big|\; : \frac{mg}{\sqrt{1+2x}} \Rightarrow$$

$$\Rightarrow \frac{4-2\sqrt{2x}}{-(2x+1)} + \sqrt{2x} = 0 \xRightarrow{\sqrt{2x}=u} -\frac{4-2u}{1+u^2} + u = 0 \Rightarrow u^3 + 3u - 4 = 0$$

$$\Rightarrow u = 1 \Rightarrow x = \frac{1}{2} \text{ and } y = 1 \Rightarrow \boxed{D\left(\tfrac{1}{2}, 1\right) \text{ is the detachement point.}}$$

② A vehicle with a weight $G = mg$ starts to move from rest from the point A on the inclined plane $AB = \ell$ and slides horizontally to the point C. The inclined plane has the angle α. Knowing the friction's coefficient f, find the velocity in the point B and the distance BC.



- We use the energy theorem:

$$dT = \delta L \quad (1)$$

- On AB we have: $d\left(\frac{1}{2} m v^2\right) = \left(\vec{F}_{f_1} + \vec{N}_1 + m\vec{g}\right) \cdot d\vec{r} \quad (2)$

But $d\vec{r}\,(dx, 0)$, $m\vec{g}\,(mg \sin\alpha, -mg\cos\alpha)$,
$\vec{N}_1\,(0, N_1)$, $\vec{F}_{f_1}\,(-f \cdot N_1, 0)$

There is no motion on $Oy \Rightarrow N_1 = mg\cos\alpha$

Thus, we obtain:

$$d\left(\frac{1}{2} m v^2\right) = -f\,mg\cos\alpha\, dx + mg\sin\alpha\, dx \Big| \int_{A\,(x=0)}^{B\,(x=\ell)}$$

$$\Rightarrow \frac{1}{2} m v_B^2 - \underbrace{\frac{1}{2} m v_A^2}_{=0} = mg(\sin\alpha - f\cos\alpha) \cdot x \Big|_0^{\ell}$$

$$\Rightarrow \frac{1}{2} v_B^2 = g\ell(\sin\alpha - f\cos\alpha) \Rightarrow v_B = \sqrt{2g\ell(\sin\alpha - f\cos\alpha)} \quad (3)$$

- On BC we have: $d\left(\frac{1}{2} m v^2\right) = \left(\vec{F}_{f_2} + \vec{N}_2 + m\vec{g}\right) d\vec{r}$

But, $d\vec{r}\,(dx, 0)$, $\vec{N}_2\,(0, mg)$, $mg\,(0, -mg)$, $\vec{F}_{f_2}\,(-f \cdot mg, 0)$.

Thus, $d\left(\frac{1}{2} m v^2\right) = -f\,mg\,x \Big| \int_{B(x=0)}^{C(x=x_c)} \Rightarrow$

$$\underbrace{\frac{1}{2} m v_C^2}_{0 \leftarrow \text{Stop condition}} - \frac{1}{2} m v_B^2 = - f m g x \Big|_0^{x_C} \Rightarrow$$

$$\Rightarrow \frac{1}{2} v_B^2 = f g x_C \Rightarrow x_C = \frac{v_B^2}{2 f g} \overset{(3)}{=} \frac{2 g l (\sin\alpha - f \cos\alpha)}{2 f g}$$

$$\Rightarrow \boxed{x_C = \frac{l}{f} (\sin\alpha - f \cos\alpha)}$$