Improper Integrals

Exercise 1: Study (with the help of the definiton and by using the Leibniz-Newton formula) the imporper integrability of the following functions, and, in case of convergence, determine the value of the improper integral.

For the examples within this exercise, the following steps should be followed:

Step 1: Compute the nondeterminate integral of f.

Pasul 2: Choose an antiderivative of f (usually the function having the constant $c = 0$ is chosen).

Pasul 3: Compute the limit towards a from the antiderivative. If the limit exists and is finite, than we are in a convergence case.

a)

$$f : (-1, 1) \to \mathbb{R} \quad f(x) = \frac{1}{\sqrt{1 - x^2}}$$

b)

$$f : [1, \infty) \to \mathbb{R} \quad f(x) = \frac{1}{x(x + 1)}.$$

c)

$$f : (0, 1] \to \mathbb{R} \quad f(x) = \ln x.$$

d)

$$f : [0, 1) \to \mathbb{R} \quad f(x) = \frac{\arcsin x}{\sqrt{1 - x^2}}.$$

e)

$$f : (0, 1] \to \mathbb{R} \quad f(x) = \frac{\ln x}{\sqrt{x}}.$$

f)

$$f : [e, \infty) \to \mathbb{R} \quad f(x) = \frac{1}{x \cdot (\ln x)^3}.$$

g)

$$f : \left(\frac{1 + \sqrt{3}}{2}, 2 \right] \to \mathbb{R} \quad f(x) = \frac{1}{x\sqrt{2x^2 - 2x - 1}}.$$

h)

$$f : [0, \infty) \to \mathbb{R}, f(x) = \frac{\pi}{2} - \mathrm{arctg} x.$$

i)

$$f\colon \mathbb{R} \to \mathbb{R}, \quad f(x) = \frac{1}{1+x^2},$$

j)

$$f\colon \left(\frac{1}{3}, 3\right] \to \mathbb{R} \quad f(x) = \frac{1}{\sqrt[3]{3x-1}}$$

k)

$$f\colon [1,\infty) \to \mathbb{R} \quad f(x) = \frac{x}{(1+x^2)^2}.$$

Exercise 2: Study the improper integrability (with the help of the comparison criteria) for the following functions. (Notice that in case of convergence, this criteria does not provide us the directly with the value of the improper integral, since the criteria mainly assures the nature).

For the examples within this exercise, the following steps should be followed:

Step 1: Determine the problematic boundary points of the domain,

Step 2: Compute

$$\lim_{x \uparrow b} (b-x)^p f(x)$$

and set p such that the value of the limit to belong to $\in (0,\infty)$, for $f : [a,b) \to [0,\infty)$.

If the domain is open at în a, then we compute

$$L = \lim_{x \downarrow a} (x-a)^p f(x)$$

and if the domain is upper unbounde and we compute (deci $[a,\infty)$)

$$L = \lim_{x \to \infty} x^p f(x).$$

Step 3: For the first two cases, if $p < 1$ we have convergent improper integrability, while in the third one, improper integrability is convergent if $p > 1$.

a)

$$f : [1,\infty) \to \mathbb{R} \quad f(x) = \frac{1}{s\sqrt{1+x^2}}$$

b)

$$f : [0, \frac{\pi}{2}) \to \mathbb{R} \quad f(x) = \frac{1}{\cos x}$$

c)

$$f : (0,\infty) \to \mathbb{R} \quad f(x) = \left(\frac{arctgx}{x}\right)^2$$

d)

$$f : (1,\infty) \to \mathbb{R} \quad f(x) = \left(\frac{\ln x}{x\sqrt{x^2-1}}\right)^2$$

e)

$$f:(0,1)\to\mathbb{R}\quad f(x)=\left(\frac{\ln x}{x\sqrt{1-x^2}}\right)^2$$

Power Series

Let $(a_n)_{n\geq 0}\subseteq\mathbb{R}$ be a sequence of real numbers. **A power series** is series of functions of the from:

$$\sum_{n\geq 0} a_n x^n,$$

with the convention that the first function in this series, is the constant function a_0. Thus, for a given $x_0\in\mathbb{R}$, we get the following series of real numbers

$$\sum_{n\geq 0} a_n {x_0}^n = a_0 + a_1 x_0 + a_2 x_0^2 + \ldots + a_n x_0^n + \ldots.$$

A point $x_0\in\mathbb{R}$ is called a **convergence point** of the series of real numbers $\sum\limits_{n\geq 0} a_n x_0^n$, is convergent, thus

$$\sum_{n=0}^{\infty} a_n x_0^n \in \mathbb{R}\,.$$

The set of all convergence points is said to be **the convergence set of the power series** and is denoted by

$$\mathcal{C}=\left\{x_0\in\mathbb{R}:\sum_{n=0}^{\infty}a_n x_0^n\in\mathbb{R}\right\}.$$

Recall that for each power sereies

$$0\in\mathcal{C},$$

due to the fact that

$$\sum_{n=0}^{\infty} a_n 0 = a_0 \in\mathbb{R}\,.$$

The convergence radius of the power series is

$$R=\frac{1}{\lambda}\quad unde\quad \lambda=\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\sqrt[n]{|a_n|}.$$

According to Cauchy-Hadamard' s theorem

$$(-R,R)\subseteq\mathcal{C}\subseteq[-R,R].$$

The particular cases

$$x=-R\quad and\quad x=R$$

must be analyzed separately in order to determine exactly $\mathcal{C}$.

Determine the convergence radius and the convergence set for the following power series

Exemple 1:

$$\sum_{n\geq 0} x^n.$$

Solution:

$$\lambda = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{1}{1} = 1 \Longrightarrow R = 1.$$

For $x = 1$, the series of real numbers $\sum_{n=0}^{\infty} 1 = \infty$, is divergent, thus $1 \notin \mathcal{C}$.

For $x = -1$,the series of real numbers $\sum_{n\geq 0}(-1)^n$ does not have a sum, due to the fact that the sequence of patial sums is constantly osscilating between 1 and 0, therefore $\nexists \lim_{n\to\infty}(-1)^n$. Thus $-1 \notin \mathcal{C}$. We conclude that

$$\mathcal{C} = (-1, 1).$$

Remark: The power serires may be written as developed around another point in $\mathbb{R}$, case in which they are sated as

$$\sum_{n\geq 0} a_n(x - x_0)^n.$$

In such a case the convergence radus is computeted accordingly tot he algorithm above.The convergece radius is computed exactly like in the case above, the only difference relies in the formulation of the convergence set, namely:

$$(x_0 - R, x_0 + R) \subseteq \mathcal{C} \subseteq [x_0 - R, x_0 + R].$$

The cases when $x = x_0 - R$ and $x = x_0 + R$ must be analyzed separately in order to specify completely the convergence set.

Exemple 2 :

$$\sum_{n\geq 1} \frac{(-1)^n}{n(2n+1)}(x+2)^n.$$

Solution: The power series is expanded about the point $x_0 = -2$, and the generating sequence is $(a_n)_{n\geq 1}$, having the general term

$$a_n = \frac{(-1)^n}{n(2n+1)}, \quad \forall n \in \mathbb{N}.$$

We compute

$$\lambda = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left|\frac{(n+1)(2n+3)}{n(2n+1)}\right| = 1 \Longrightarrow R = \frac{1}{1} = 1.$$

Hence

$$(-2-1, -2+1) = (-3, -1) \subseteq \mathcal{C}.$$

Next we check the boundaries:

For $x = -3$, the series of real numbers

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n(2n+1)} \cdot (-1)^n = \sum_{n=1}^{\infty} \frac{1}{n(2n+1)} \quad \sim \sum_{n\geq 1} \frac{1}{n^2},$$

is convergent, therefore $-3 \in \mathcal{C}$.

For $x = -1$, the series of real numbers $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n(2n+1)} \cdot (1)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n(2n+1)}$ is alternating. Due to the fact that the sequence of real numbers $\left(\frac{1}{n(2n+1)}\right)_{n\geq 1}$ is decreasing and has the limit 0, according to Liebniz' criterion, we obtain convergence, hence $-1 \in \mathcal{C}$.

In conclusion

$$\mathcal{C} = [-3, -1].$$

Exercises: Determine both the convergence radius and the convergence set for the following power series:

$$a) \sum_{n\geq 0} (n+1)^n x^n \quad b) \sum_{n\geq 0} \frac{1}{n} x^n \quad c) \sum_{n\geq 0} \frac{(-1)^n}{n} x^n \quad d) \sum_{n\geq 0} \frac{1}{n(n+1)} x^n \quad e) \sum_{n\geq 0} \frac{1}{n!} x^n \quad f) \sum_{n\geq 0} n! x^n.$$