Seminars 10 and 11 - 2024

1. A program returns a value according to a random variable X with $E(X) = m \in \mathbb{R}$ and $V(X) = \sigma^2$, $\sigma > 0$. Prove that X takes values in the interval $(m - 3\sigma, m + 3\sigma)$ with more than 88% probability.

2. The number of items produced in a factory during a day is a random variable with mean 50. If we consider a day, which event is more likely: E_1:"the production is more than 100 items in this day" or E_2:"the production is at most 100 items in this day"?

3. Let $(X_n)_{n\geq 1}$ be a sequence of independent random variables with $Unif[a, b]$ distribution, where $a < b$. Define for each $n \in \mathbb{N}^*$

$$Y_n = \max\{X_1, \ldots, X_n\} \text{ and } Z_n = \min\{X_1, \ldots, X_n\}.$$

Prove that $Y_n \xrightarrow{P} b$ and $Z_n \xrightarrow{P} a$.

▶ The density function for $Unif[a, b]$ is $f(t) = \begin{cases} \frac{1}{b-a}, & \text{if } t \in [a, b] \\ 0, & \text{if } t \notin [a, b]\,. \end{cases}$

4. Let $(X_n)_{n\geq 1}$ be a sequence of Bernoulli random variables. Prove that $X_n \xrightarrow{P} 0$ if and only if $X_n \xrightarrow{L^2} 0$.

5. Let $\lambda > 0$. A calling center has the following property, for every $n \in \mathbb{N}$, $n \geq 100$, during an hour interval $(0, 1]$: the calls arrive independently with at most one call in each time subinterval $\left(\frac{i}{n}, \frac{i+1}{n}\right]$, one call has probability $\frac{\lambda}{n}$ to occur, $i = \overline{0, n-1}$. Let's denote by X_n the corresponding total number of calls. Prove that $X_n \xrightarrow{d} X$, where $X \sim Poiss(\lambda)$.

▶ $X \sim Poiss(\lambda) \Rightarrow P(X = k) = \dfrac{\lambda^k}{k!} e^{-\lambda}$, for $k \in \{0, 1, 2, ...\}$.

6. Let $(X_n)_{n\geq 1}$ be a sequence of independent random variables with $Unif[0, 1]$ distribution. Define for each $n \in \mathbb{N}^*$

$$Y_n = \max\{X_1, \ldots, X_n\} \text{ and } Z_n = \min\{X_1, \ldots, X_n\}.$$

Prove that $Y_n \xrightarrow{L^2} 1$ and $Z_n \xrightarrow{L^2} 0$.

7. Consider a sequence of distinct coins such that the probability of getting a head with the nth coin is $\dfrac{1}{n}$, $n \in \mathbb{N}^*$. Let X_n be 1, if the toss of the nth coin shows a head, and 0, otherwise. Do we have $X_n \xrightarrow{a.s.} 0$?