Theoretical Mechanics

10. Central forces.

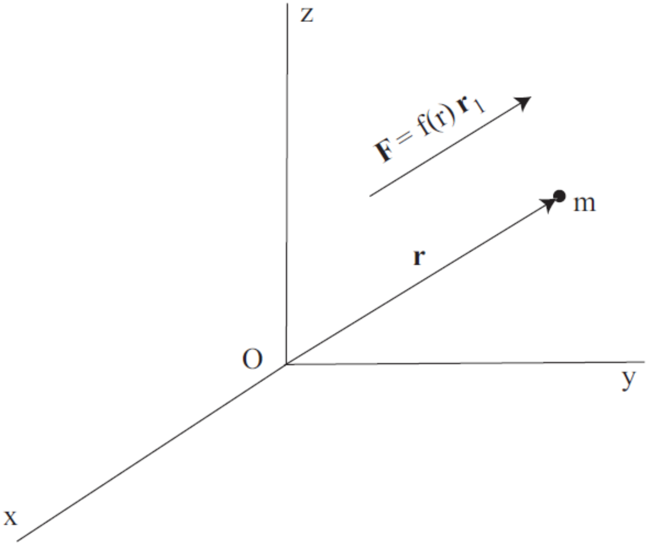
Central forces. Properties

(https://people.maths.bris.ac.uk/~marrk/Mech/15_16_CentralF_notes.pdf)

A central force is a force acting on a particle of mass *m* with the property that

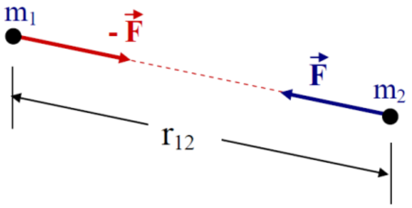
the force is always directed from *m* toward, or away, from a fixed point O.

The particle is said to move in a *central force field*. The point O is referred to as the *centre of force*.



Examples:

Gravitational force



$$\boxed{F_{12} = G\frac{m_1 m_2}{(r_{12})^2}}$$

$$G = 6.67 \text{ x } 10^{-11}\ \mathrm{Nm^2/kg^2}$$

Theoretical Mechanics

Examples:

Sun-Earth system: gravity + perturbations

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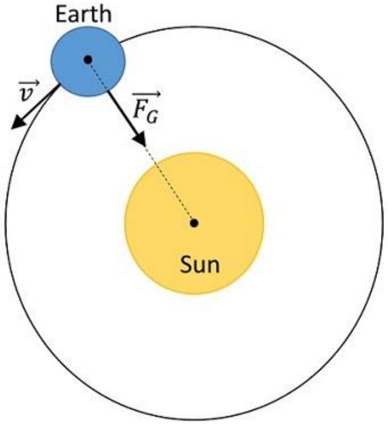
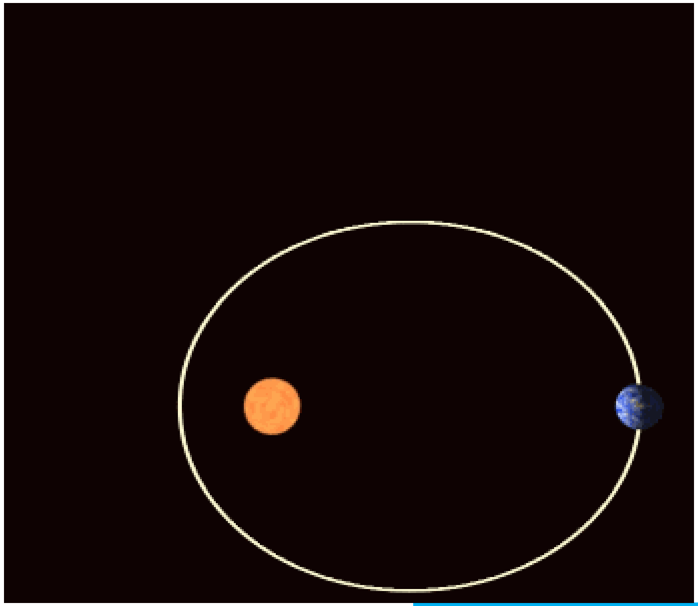
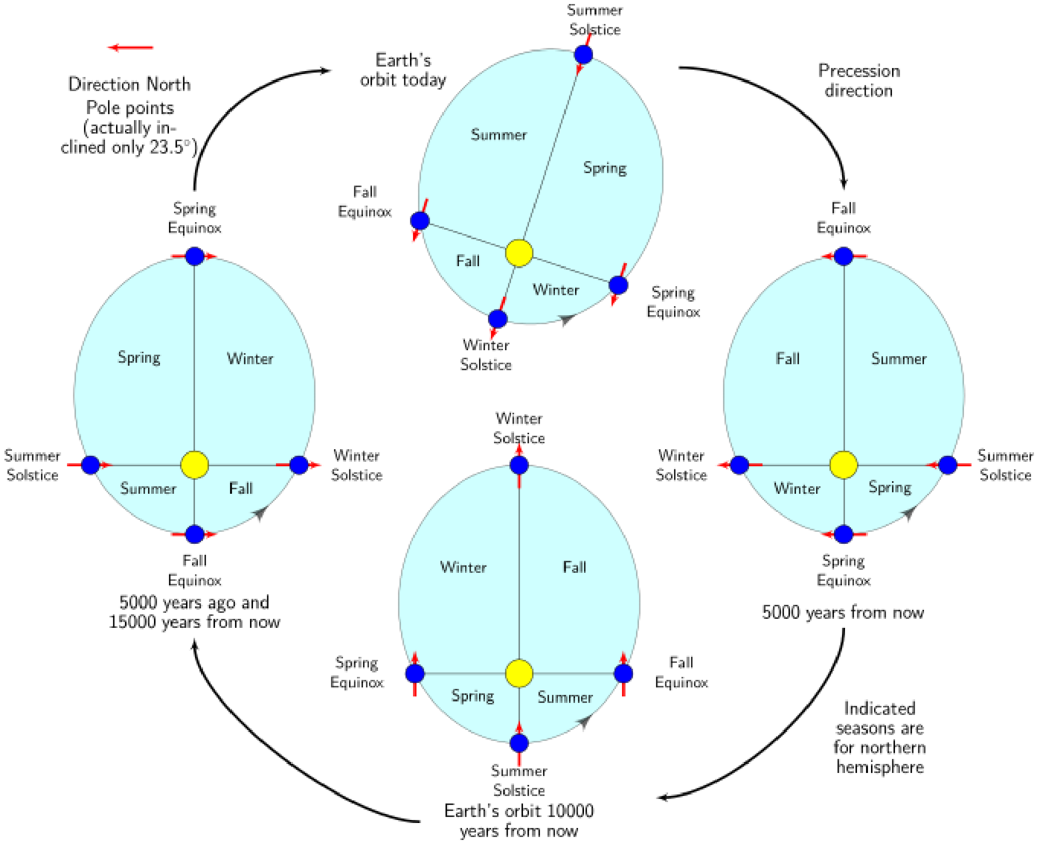




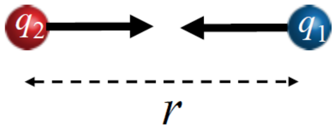


Theoretical Mechanics

Examples:

Coulomb force

The force on a charge due to another charge is proportional to the product of the charges and inversely proportional to the separation squared.



$$F \propto \frac{q_1 q_2}{r^2}$$

Elastic force(spring)

Attractive elastic force

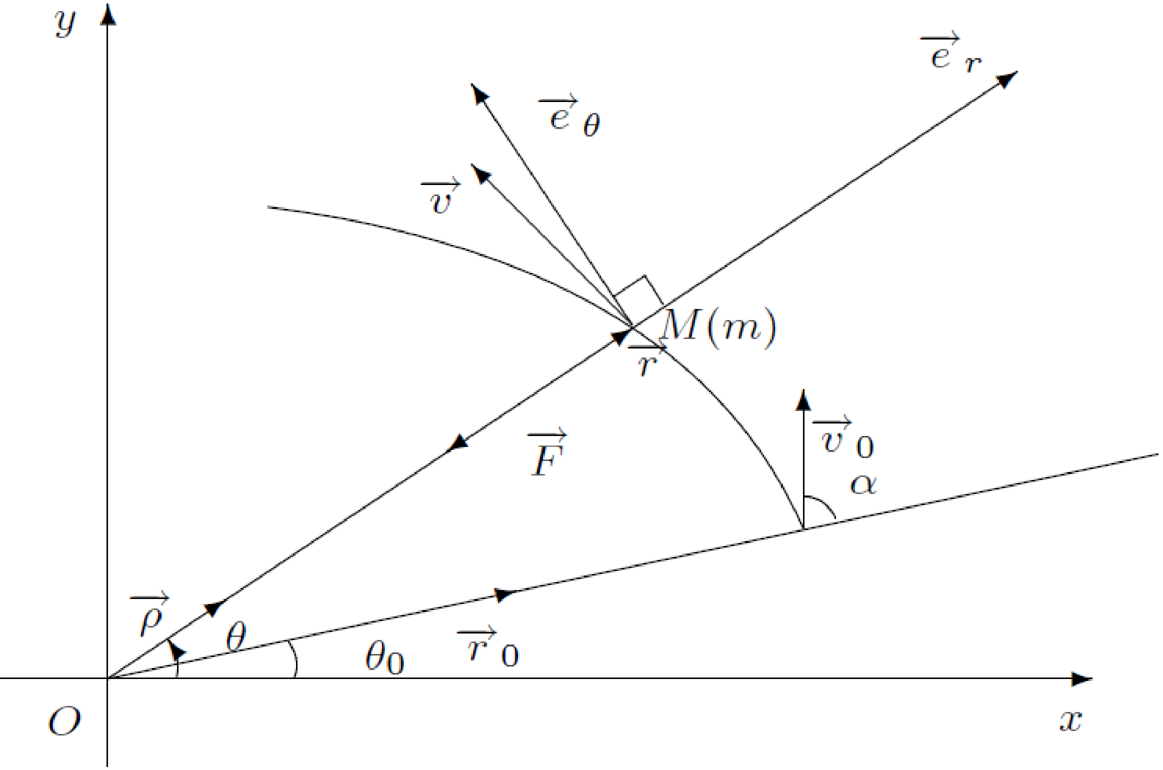
$$\mathbf{F} = -k\mathbf{r} = \operatorname{grad}\left(-\frac{1}{2}kr^2 + \text{const}\right), \quad k > 0\,;$$

Repulsive elastic force

$$\mathbf{F} = k\mathbf{r}\,,\ k > 0$$

Theoretical Mechanics

Consider *O the center of the force* $\vec{F}$ acting on the material point *M*(*m*) which has the position vector $\vec{r} = \overrightarrow{OM}$.



Let be

$$\vec{\rho} = \frac{\vec{r}}{r} = \frac{\vec{r}}{|\vec{r}|}$$

the versor of $\vec{r}$. Thus, we have

$$\vec{F} = F\vec{\rho} = F\frac{\vec{r}}{r} \qquad (10.1)$$

where *F* is the algebraic value of $\vec{F}$.

If *F > 0* than $\vec{F}$ is repulsive. Otherwise, if *F < 0* than $\vec{F}$ is attractive.

Theoretical Mechanics

Using the moment of momentum theorem

$$\frac{d\overrightarrow{K}_0}{dt} = \overrightarrow{M}_0(\overrightarrow{F}) = \overrightarrow{r} \times \overrightarrow{F} = 0$$

one obtain the area first integral:

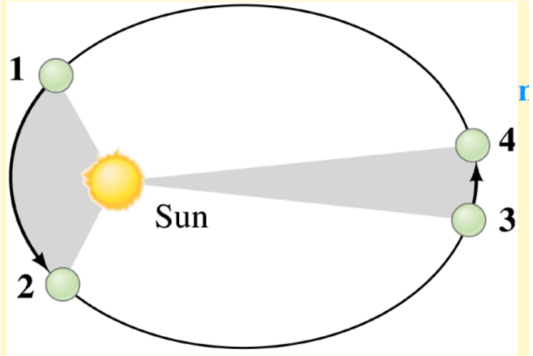
$$\left|\begin{array}{l} \frac{d}{dt}(\overrightarrow{r} \times m\overrightarrow{v}) = 0 \Rightarrow \overrightarrow{r} \times \overrightarrow{v} = \overrightarrow{c} = \overrightarrow{r}_0 \times \overrightarrow{v}_0 \\ \overrightarrow{r}(t_0) = \overrightarrow{r}_0, \ \overrightarrow{v}(t_0) = \overrightarrow{v}_0. \end{array}\right. \qquad (10.2)$$

The areal velocity of *M is*:

$$\frac{d\overrightarrow{A}}{dt} = \frac{1}{2}(\overrightarrow{r} \times \overrightarrow{v}) = \overrightarrow{c}, \ \forall\, t \geq t_0, \qquad (10.3)$$

and, thus, the areal velocity is constant.

Remember: Areal velocity (sector velocity, sectorial velocity) is the rate at which area is swept out by a particle as it moves along a curve.



Theoretical Mechanics

From (10.2) and (10.3) we have:

In the motion under the action of the central force the moment of momentum and the areal velocity are constant vectors at any moment $t \geq t_0$.

We suppose:

$$\overrightarrow{c} = (c_1, c_2, c_3) \neq 0$$

From $\overrightarrow{r} \times \overrightarrow{v} = \overrightarrow{c} = \overrightarrow{r}_0 \times \overrightarrow{v}_0$ we have $\overrightarrow{r} \cdot \overrightarrow{c} = 0$ and then

$$xc_1 + yc_2 + zc_3 = 0 \qquad (10.4)$$

Thus, the motion takes place in a plane determined by $\overrightarrow{r_0}$ and $\vec{v}_0$ (actually the normal to the plane is $\overrightarrow{c} = (c_1, c_2, c_3) \neq 0$).

Theoretical Mechanics

Next, let us consider the motion of the point *M(m)* in the plane Oxy and let be (r, θ) the polar coordinates of the point *M*.

We have

$$\overrightarrow{r} \times \overrightarrow{v} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ x & y & 0 \\ \dot{x} & \dot{y} & 0 \end{vmatrix} = (x\dot{y} - y\dot{x})\overrightarrow{k} = r^2\dot{\theta}\,\overrightarrow{k}$$

But, $\overrightarrow{c} = \overrightarrow{r}_0 \times \overrightarrow{v}_0 = r_0^2\dot{\theta}_0\,\overrightarrow{k}$ and we obtain $(\overrightarrow{r}_0 \times \overrightarrow{v}_0 = \overrightarrow{c} = \overrightarrow{r} \times \overrightarrow{v})$:

$$r^2\dot{\theta} = c, \ \forall\ t \geq t_0. \qquad (10.5)$$

Equation (10.5) is the *area integral (*because $\frac{1}{2}\int r^2(\theta)d\theta$ is the area swept by M)

We have

$$c = |\overrightarrow{r}_0 \times \overrightarrow{v}_0| = r_0 v_0 \sin\alpha$$

and it means that

$$c = r_0^2\dot{\theta}_0 = r_0 v_0 \sin\alpha, \quad \alpha = (\widehat{\overrightarrow{r}_0, \overrightarrow{v}_0}). \qquad (10.6)$$

Theoretical Mechanics

Remark. If $\vec{c} = 0$ then the motion of the point *M* is rectilinear.

Next we suppose $\overrightarrow{c} = \overrightarrow{r}_0 \times \overrightarrow{v}_0 \neq 0$

Determination of motion in its plane

Consider the motion of the point *M(m)* in the plane Oxy and let be (r, θ) the polar coordinates of the point *M*. The differential equation of motion is given by

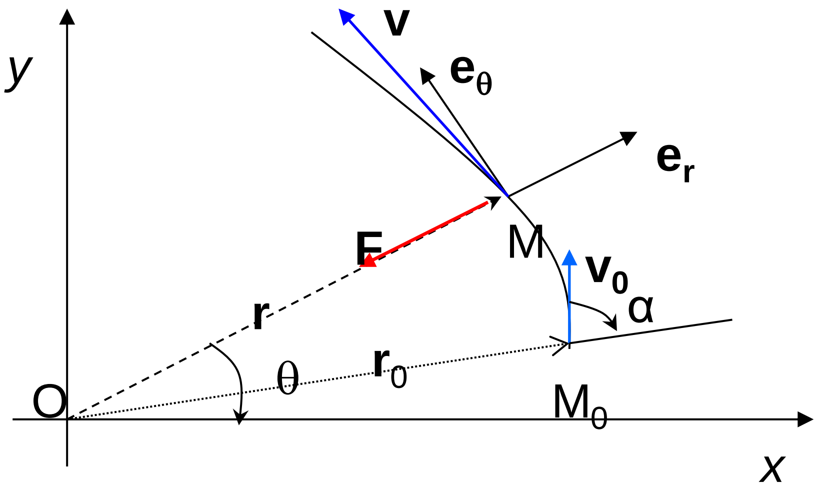
$$m\frac{d^2\overrightarrow{r}}{dt^2} = F\frac{\overrightarrow{r}}{r}, \quad \overrightarrow{r}(t_0) = \overrightarrow{r}_0, \quad \overrightarrow{v}(t_0) = \overrightarrow{v}_0 \tag{10.7}$$

Taking into account the acceleration form in polar coordinate

$$\overrightarrow{a} = \left(\ddot{r} - r\dot{\theta}^2, \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\right),$$

and projecting (10.7) on the polar coordinates versors we obtain:

Theoretical Mechanics



$$\begin{array}{l} \vec{e}_r : \\ \vec{e}_\theta : \end{array} \left\{ \begin{array}{l} m(\ddot{r} - r\dot{\theta}^2) = F \\ \dfrac{m}{r}\dfrac{d}{dt}(r^2\dot{\theta}) = 0 \end{array} \right. \qquad (10.8)$$

From $(10.8)_2$ we obtain the area first integral

$$r^2\dot{\theta} = c, \qquad (10.9)$$

where

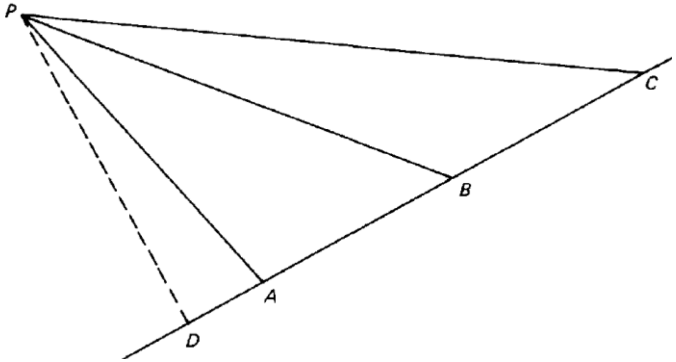
$$c = r_0^2\dot{\theta}_0 = r_0 v_0 \sin\alpha, \quad \alpha = (\widehat{\vec{r}_0, \vec{v}_0}). \qquad (10.10)$$

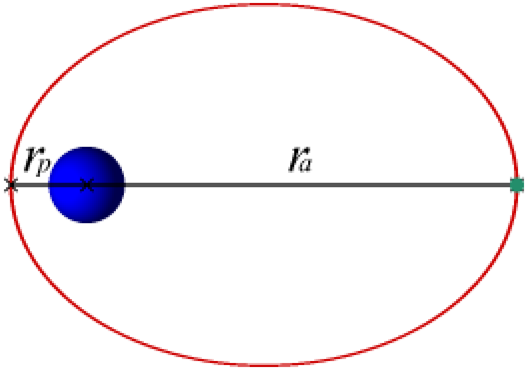
Theoretical Mechanics

From (10.9) we have

$$r^2(\theta)d\theta = cdt \;\Rightarrow\; \int_{\theta_0}^{\theta_1} r^2(\theta)d\theta = c(t_1 - t_0)$$

And this means that the point *M* sweeps equal areas in equal intervals of time (area law).





Thus, the motion of the material the point *M* under the action of a central force respect the area law and takes place in a plane (the plane motion) determined by the initial conditions.

Case I. $F = F(r, \theta, \dot{r}, \dot{\theta})$ (i.e. $\frac{\partial F}{\partial t} = 0$)

One eliminate time t from $(10.8)_1$ by using the area integral (10.9):

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\dot{\theta} = \frac{c}{r^2}\frac{dr}{d\theta} = -c\frac{d}{d\theta}\left(\frac{1}{r}\right)$$

$$\ddot{r} = \frac{d}{dt}\left(-c\frac{d}{d\theta}\left(\frac{1}{r}\right)\right) = \frac{d}{d\theta}\left(-c\frac{d}{d\theta}\left(\frac{1}{r}\right)\right)\underbrace{\dot{\theta}}_{=\frac{c}{r^2}} = -\frac{c^2}{r^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)$$

Thus, Equation $(10.8)_1$ becomes

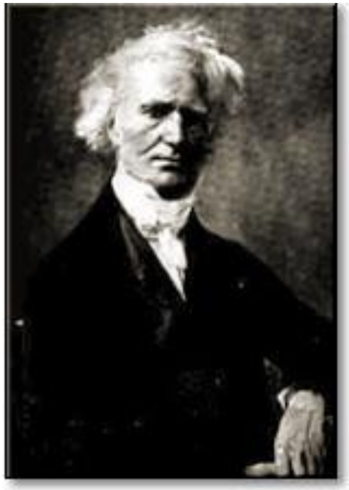
$$-\frac{mc^2}{r^2}\left[\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right] = F\left(r, \theta, -c\frac{d}{d\theta}\left(\frac{1}{r}\right), \frac{c}{r^2}\right) \qquad (10.11)$$

Theoretical Mechanics

When $F = F(r, \theta)$ (i.e. The force depends only by position) the equation is as the Binet‘s equation:

$$-\frac{mc^2}{r^2}\left[\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}\right] = F(r,\theta) \qquad (10.12)$$

By solving eq. (10.12) one obtain the trajectory of the particle in polar coordinates. However, in order to solve (10.12) two initial conditions are necessary. From $\vec{r}(t_0) = \vec{r}_0$ and $\vec{v}(t_0) = \vec{v}_0$ we obtain:



$$r(\theta_0) = r_0, \quad \frac{d}{d\theta}\left(\frac{1}{r}\right)\Big|_{\substack{\theta=\theta_0 \\ (t=t_0)}} = -\frac{1}{r_0}\text{ctg }\alpha. \qquad (10.13)$$

Jacques Philippe Marie Binet
(February 2, 1786 – May 12, 1856)

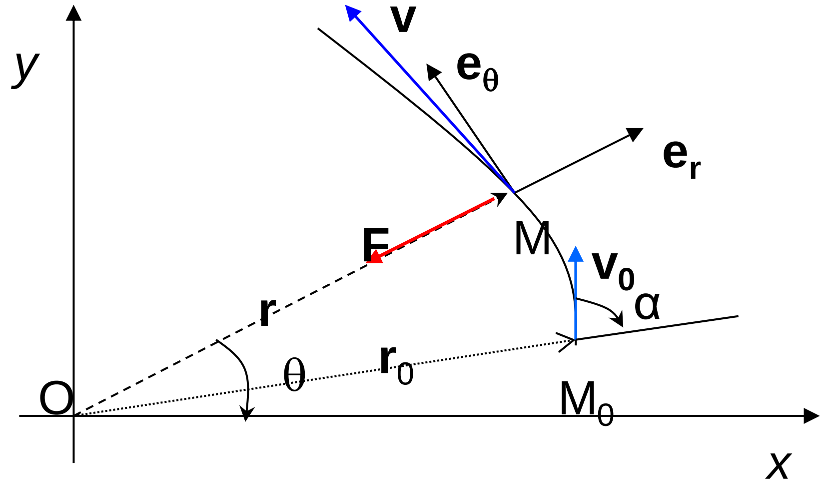
Theoretical Mechanics

Indeed,

$$\frac{d}{d\theta}\left(\frac{1}{r}\right)\Big|_{\substack{\theta=\theta_0\\(t=t_0)}} = -\frac{1}{r_0^2}\frac{dr}{d\theta}\Big|_{t=t_0} = -\frac{1}{r_0^2}\frac{\dot r}{\dot\theta}\Big|_{t=t_0} = -\frac{1}{r_0^2}\frac{\dot r_0}{\dot\theta_0}$$

$$= -\frac{1}{r_0^2}\frac{pr_{\vec{r}_0}\vec{v}_0}{\frac{c}{r_0^2}} = -\frac{1}{c}v_0\cos\alpha = -\frac{1}{r_0 v_0\sin\alpha}v_0\cos\alpha = -\frac{1}{r_0}\mathrm{ctg}\ \alpha,$$

Thus, we have $$\frac{d}{d\theta}\left(\frac{1}{r}\right)\Big|_{\substack{\theta=\theta_0\\(t=t_0)}} = -\frac{1}{r_0}\mathrm{ctg}\ \alpha.$$



Theoretical Mechanics

Solving (10.12) along with the initial conditions (10.13) one obtain the trajectory

$$r = r(\theta). \qquad (10.14)$$

Next, using the area integral $\mathrm{r}^2\,\dot{\theta} = c$ we have $\mathrm{r}^2(\theta)d\theta = cdt$ and we obtain:

$$c(t - t_0) = \int_{\theta_0}^{\theta} r^2(\theta)d\theta \qquad (10.15)$$

and thus

$$\theta = \theta(t) \qquad (10.16)$$

Now, using (10.14) and (10.16) we get $r = r(t)$. In this moment the problem is solved and the motion's equations in polar coordinates are:

$$r = r(t), \quad \theta = \theta(t). \qquad (10.17)$$

Theoretical Mechanics

Case II. $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{r})$

In this case it is possible to apply the general theorems of dynamics. First let us calcuate the elementary work:

$$\delta L = \overrightarrow{F} \cdot d\overrightarrow{r} = F(r)\frac{\overrightarrow{r}}{r} \cdot d\overrightarrow{r} = F(r) \cdot \frac{1}{2r} d(\underbrace{\overrightarrow{r} \cdot \overrightarrow{r}}_{=r^2}) = F(r)dr.$$

Thus, $\delta L = F(r)dr$ is an exact differential

$$\delta L = -dV, \quad V := -\int F(r)dr$$

and the kinetic energy theorem $\mathrm{dT} = \delta L = -dV$ becomes

$$d\left(\frac{mv^2}{2}\right) = F\,dr \quad \Rightarrow \quad \frac{mv^2}{2} - \frac{mv_0{}^2}{2} = \int_{r_0}^{r} F(r)dr \qquad (10.18)$$

Theoretical Mechanics

Thus,

$$v^2 = \frac{2}{m}\int F(r)dr + h \tag{10.19}$$

Energy constant

$$h = \frac{m}{2}v_0^2 + V(r_0)$$

We take into account that:

$$\left.\begin{aligned} v^2 &= \dot{r}^2 + r^2\dot{\theta}^2 \\ \dot{r} &= -c\frac{d}{d\theta}\left(\frac{1}{r}\right) \\ \dot{\theta} &= \frac{c}{r^2} \end{aligned}\right\} \Rightarrow v^2 = c^2\left\{\left[\frac{d}{d\theta}\left(\frac{1}{r}\right)\right]^2 + \frac{1}{r^2}\right\} \tag{10.20}$$

Theoretical Mechanics

Thus,we have to solve

$$c^2\left\{\left[\frac{d}{d\theta}\left(\frac{1}{r}\right)\right]^2+\frac{1}{r^2}\right\}=\frac{2}{m}\int F(r)dr+h \tag{10.21}$$

The energy constant can be calculated from

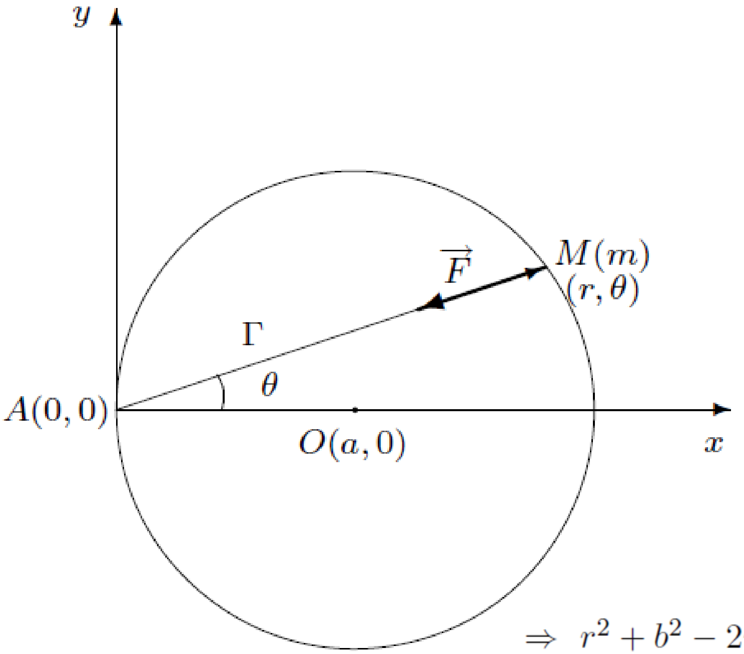
$$v_0{}^2=\frac{2}{m}\int F(r)dr+h \tag{10.22}$$

Remark. It is also possible to use the Binet‘s equation for the case F=F(r).

Theoretical Mechanics

Example 1.

A particle *M(m)* moves on a circle of radius *b* being attracted by a fixed point *A* of the circle. Find the attractive force and the velocity of the particle as functions of *r*, the distance between M and A.



Solution

Consider the equation of the circle

$$(x-b)^2+y^2=b^2$$

In polar coordinates we have

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Rightarrow (r\cos\theta - b)^2 + r^2\sin^2\theta = b^2$$

$$\Rightarrow r^2+b^2-2rb\cos\theta=b^2 \Rightarrow \left.\begin{array}{l} r(r-2b\cos\theta)=0 \\ r\neq 0 \text{ (punctul descrie cercul)} \end{array}\right\} \Rightarrow r=2b\cos\theta$$

Theoretical Mechanics

Consider the Binet‘s equation

$$-\frac{mc^2}{r^2}\left[\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}\right]=\pm F(r,\theta)$$

sign „+“ for repulsive force
sign „–“ for attractive force

Next, in order to obtain the force, we calculate the left hand term of the equation in order to obtain the force.

$$r = 2b\cos\theta \;\Rightarrow\; \frac{d}{d\theta}\left(\frac{1}{r}\right) = \frac{d}{d\theta}\left(\frac{1}{2b\cos\theta}\right) = \frac{1}{2b}\frac{\sin\theta}{\cos^2\theta}$$

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = \frac{d}{d\theta}\left(\frac{1}{2b}\frac{\sin\theta}{\cos^2\theta}\right) = \frac{1}{2b}\left[\frac{\cos\theta}{\cos^2\theta} + \frac{2\sin^2\theta}{\cos^3\theta}\right] = \frac{1}{2b}\left(\frac{\cos^2\theta + 2\sin^2\theta}{\cos^3\theta}\right)$$

Theoretical Mechanics

$$F=\frac{mc^2}{r^2}\left[\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}\right]=\frac{mc^2}{r^2}\left[\frac{1}{2a}\frac{\cos^2\theta+2\sin^2\theta}{\cos^3\theta}+\frac{1}{r}\right]$$

$$=\frac{mc^2}{r^2}\left[\frac{1}{2a}\left(\frac{1}{\cos\theta}+\frac{2-2\cos^2\theta}{\cos^3\theta}\right)+\frac{1}{r}\right]=\frac{mc^2}{r^2}\left[\frac{1}{2a}\left(\frac{2}{\cos^3\theta}-\frac{1}{\cos\theta}\right)+\frac{1}{r}\right]$$

$$=\frac{mc^2}{r^2}\left[\frac{1}{2a}\left(\frac{2}{\frac{r^3}{8a^3}}-\frac{1}{\frac{r}{2a}}\right)+\frac{1}{r}\right]=\frac{mc^2}{r^2}\left[\frac{8a^2}{r^3}-\frac{1}{r}=\frac{1}{r}\right]=\frac{8mc^2a^2}{r^5}$$

$$\Rightarrow\quad F=\frac{8mc^2a^2}{r^5}$$

$$v^2=\dot{r}^2+r^2\dot{\theta}^2=c^2\left[\left(\frac{d}{d\theta}\left(\frac{1}{r}\right)\right)^2+\frac{1}{r^2}\right]=c^2\left[\frac{1}{4b^2}\frac{1-\cos^2\theta}{\cos^4\theta}+\frac{1}{r^2}\right]$$

$$=c^2\left[\frac{1}{4b^2}\left(\frac{1}{r^4/(16b^4)}-\frac{1}{r^2/(4b^2)}\right)+\frac{1}{r^2}\right]=\frac{4b^2c^2}{r^4}\qquad\Rightarrow\quad v=\frac{2bc}{r^2}$$

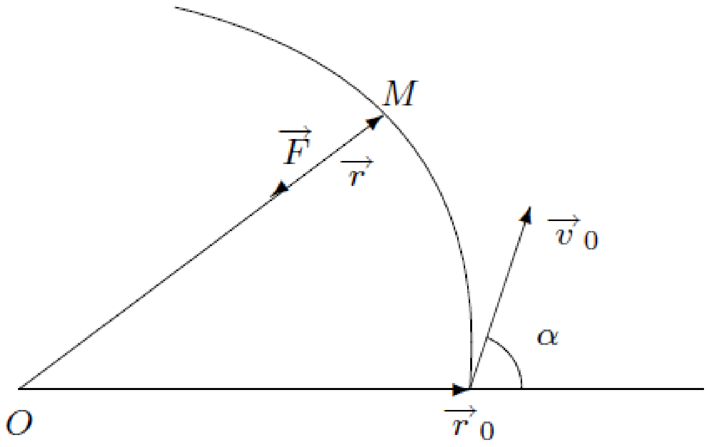
Theoretical Mechanics

Example 2.

Find the motion of a particle *M(m=1)* that moves under the action of an attractive force $F(r) = \frac{1}{r^3}$. At the initial moment we have:

$$t = 0: \quad \theta_0 = 0, \qquad r_0 = 2, \qquad v_0 = \frac{1}{2}, \qquad \alpha = \widehat{(\vec{r}_0, \vec{v}_0)} = \frac{\pi}{4}$$

Solution



$$\left.\begin{aligned} &-\frac{mc^2}{r^2}\left(\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}\right) = -\frac{1}{r^3} \\ &c = r_0 v_0 \sin\alpha = 2\frac{1}{2}\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2} \\ &m = 1 \end{aligned}\right\} \Rightarrow$$

$$\frac{1}{2}\left(\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}\right) = \frac{1}{r}$$

Theoretical Mechanics

$$\Rightarrow \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) - \frac{1}{r} = 0 \Rightarrow \frac{1}{r} = C_1 e^{\theta} + C_2 e^{-\theta}$$

$$\left.\begin{array}{l} r(0) = 2 \\ \theta(0) = 0 \end{array}\right\} \Rightarrow \frac{1}{2} = C_1 + C_2$$

$$\left.\begin{array}{l} \frac{d}{d\theta}\left(\frac{1}{r}\right)\Big|_{t=0} = -\frac{1}{r_0}\mathrm{ctg}\ \alpha = -\frac{1}{2} \\ \frac{d}{d\theta}\left(\frac{1}{r}\right)\Big|_{t=0} = C_1 - C_2 \\ \theta(0) = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} C_1 - C_2 = -\frac{1}{2} \\ C_1 + C_2 = \frac{1}{2} \end{array}\right. \Rightarrow 2C_1 = 0 \Rightarrow C_1 = 0$$

$$C_1 = 0 \Rightarrow C_2 = \frac{1}{2} \Rightarrow \frac{1}{r} = \frac{1}{2}e^{-\theta} \Rightarrow r = 2e^{\theta} \longleftarrow$$

(equation of the trajectory in polar coordinates, i.e. a logarithmic spiral)

$$\boxed{r = ae^{b\theta}}$$

Using the area law

$$r^2\dot{\theta} = c = \frac{\sqrt{2}}{2} \;\Rightarrow\; 4e^{2\theta}\dot{\theta} = \frac{\sqrt{2}}{2} \;\Rightarrow\; 4e^{2\theta}d\theta = \frac{\sqrt{2}}{2}dt$$

$$\Rightarrow\; 2d(e^{2\theta}) = \frac{\sqrt{2}}{2}dt \;\Rightarrow\; 2e^{2\theta}\Big|_0^{\theta} = \frac{\sqrt{2}}{2}t \;\Rightarrow\; e^{2\theta} - 1 = \frac{\sqrt{2}}{4}t$$

$$\Rightarrow\; \frac{r^2}{4} - 1 = \frac{\sqrt{2}}{2}t \;\Rightarrow\; r^2 = \sqrt{2}t + 4$$

$$\Rightarrow\; 2\theta = \ln\left(1 + \frac{\sqrt{2}}{4}t\right) \;\Rightarrow\; \begin{cases} \theta = \dfrac{1}{2}\ln\left(\dfrac{\sqrt{2}}{4}t + 1\right) \\ r = (\sqrt{2}t + 4)^{1/2} \end{cases}$$

(equations of motion)

Theoretical Mechanics

One can use ***matlab*** for visualization

```
t=0:0.01:100;
r=(sqrt(2)*t+4).^(1/2);
theta=(1/2)*log(sqrt(2)/4*t+1);
polar(theta,r)
```

$$\begin{cases} \theta = \frac{1}{2}\ln\left(\frac{\sqrt{2}}{4}t + 1\right) \\ r = (\sqrt{2}t + 4)^{1/2} \end{cases}$$