① Consider the parabola

$$y = \frac{x^2}{4} - 1$$

Find $\underline{T}$, $\underline{N}$, k, ρ at $(2,0)$. Sketch the osculating circle at this point.

$y = \frac{x^2}{4} - 1$ we can use the parametrization

$$\vec{r}(t) = \left(t, \frac{t^2}{4} - 1\right)$$

For the point $(2,0)$ we have $t = 2$

- First derivative:

$$\vec{r}'(t) = \left(1, \frac{t}{2}\right) \quad \text{at } (2,0) \Rightarrow \vec{r}'(2) = (1,1)$$

- The unit tangent vector T

$$\vec{T}(t) = \frac{r'(t)}{\|r'(t)\|}, \quad \|r'(t)\| = \sqrt{1^2 + \left(\frac{t}{2}\right)^2} = \sqrt{1 + \frac{t^2}{4}} =$$

$$= \sqrt{\frac{4+t^2}{4}} = \frac{\sqrt{4+t^2}}{2}$$

$$\Rightarrow \vec{T}(t) = \left(\frac{1}{\sqrt{1+\frac{t^2}{4}}}, \frac{\frac{t}{2}}{\sqrt{1+\frac{t^2}{4}}}\right) = \left(\frac{2}{\sqrt{4+t^2}}, \frac{t}{\sqrt{4+t^2}}\right)$$

$$\vec{T}(2) = \left(\frac{2}{\sqrt{4+4}}, \frac{2}{\sqrt{4+4}}\right) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

- The normal vector N:

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|T'(t)\|}$$

We need to compute $\vec{T}'(t)$. Let's denote

$\vec{T}(t) = (\vec{T}_1(t), \vec{T}_2(t))$:

$$\vec{T}_1(t) = \frac{2}{\sqrt{4+t^2}}, \quad \vec{T}_2(t) = \frac{t}{\sqrt{4+t^2}}$$

Compute the derivative of $T_1(t)$ and $T_2(t)$

$$\vec{T}_1'(t) = \frac{d}{dt}\left(\frac{2}{\sqrt{4+t^2}}\right) = \frac{2\cdot\left(-\frac{1}{2}\right)\cdot 2t}{(4+t^2)^{\frac{3}{2}}} = -\frac{2t}{(4+t^2)^{\frac{3}{2}}}$$

$$\vec{T}_2'(t) = \frac{d}{dt}\left(\frac{t}{\sqrt{4+t^2}}\right) = \frac{4}{(4+t^2)^{\frac{3}{2}}}$$

$\Rightarrow$

$$\vec{T}'(t) = \left(-\frac{2t}{(4+t^2)^{\frac{3}{2}}}, \frac{4}{(4+t^2)^{\frac{3}{2}}}\right)$$

Compute the magnitude of $\vec{T}'(t)$:

$$\|\vec{T}'(t)\| = \sqrt{\left(-\frac{2t}{(4+t^2)^{\frac{3}{2}}}\right)^2 + \left(\frac{4}{(4+t^2)^{\frac{3}{2}}}\right)^2} = \frac{\sqrt{4t^2+16}}{(4+t^2)^{\frac{3}{2}}} = \frac{2}{4+t^2}$$

So, $\vec{N}(t)$ is:

$$\vec{N}(t) = \cancel{\left(-\frac{2t}{2}, \frac{4}{2}\right)} = \left(-\frac{t}{\sqrt{4+t^2}}, \frac{2}{\sqrt{4+t^2}}\right)$$

$$\vec{N}(2) = \left(-\frac{2}{\sqrt{4+4}}, \frac{2}{\sqrt{4+4}}\right) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

- Curvature K

$$K(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} \text{ ; we have: } \|\vec{T}'(t)\| = \frac{2}{4+t^2}$$

$$\|\vec{r}'(t)\| = \frac{\sqrt{4+t^2}}{2}$$

Thus, $K(t) = \dfrac{2}{(4+t^2)\cdot\frac{\sqrt{4+t^2}}{2}} = \dfrac{4}{(4+t^2)^{\frac{3}{2}}}$

$$K(2) = \frac{4}{(4+4)^{\frac{3}{2}}} = \frac{4}{8\sqrt{8}} = \frac{1}{2\sqrt{8}} = \frac{\sqrt{2}}{\cancel{4}8}$$

- Radius of Curvature ρ

$$\rho(t) = \frac{1}{K(t)} = \frac{(4+t^2)^{\frac{3}{2}}}{4}$$

$$\rho(2) = \frac{(4+4)^{\frac{3}{2}}}{4} = \frac{8\sqrt{8}}{4} = \cancel{4\sqrt{2}}\ 2\sqrt{8}$$

Osculating Circle:

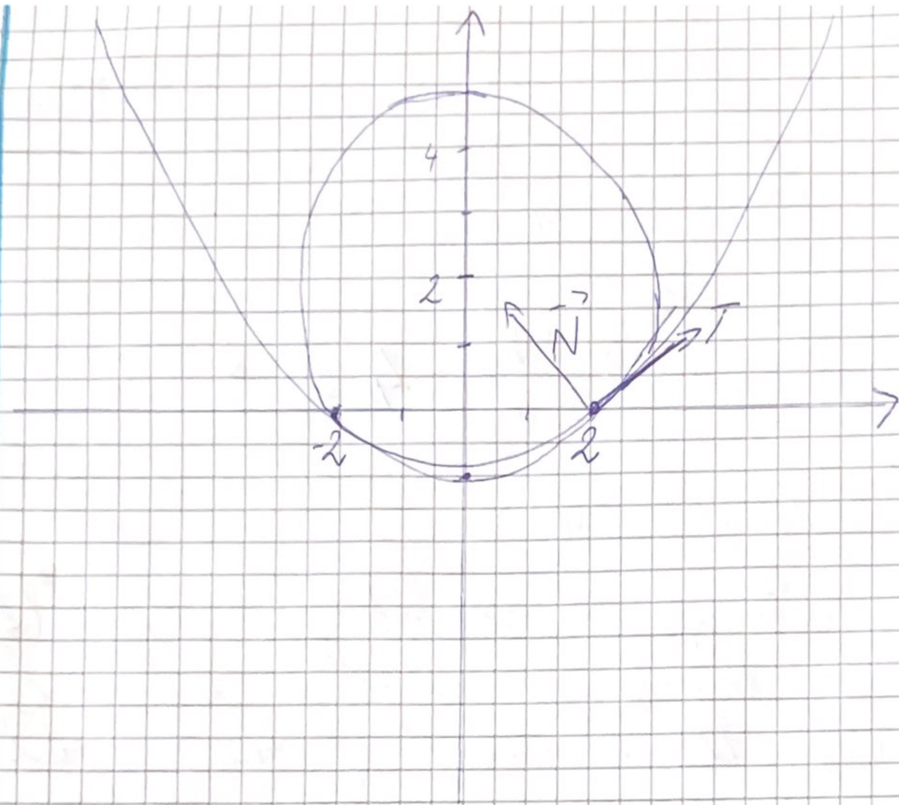
The center of the osculating circle at $(2,0)$ can be found by moving a distance $\rho(2)$ in the direction of the normal vector from the point $(2,0)$

Center of Osc. Circle =

$$= (2,0) + \rho(2) \cdot N(2) = (2,0) + 2\sqrt{2}\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) =$$

$$= (2,0) + (-2,2) = (0,2) \Rightarrow$$ osc circle centered at $\cancel{(0,0)}$ of radius $2\sqrt{2}$

$(0,2)$


4
2
-2
2
$\vec{N}$
$\vec{T}$