Analytic Geometry

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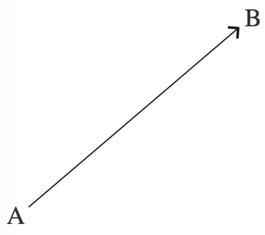
Both knew their vectors pretty well...

- Let $\mathcal{E}$ denote the Euclidean plane $\mathcal{E}_2$ or the Euclidean 3-space $\mathcal{E}_3$. A pair $(A, B) \in \mathcal{E} \times \mathcal{E}$ is called an *ordered pair* of points or a *vector at the point* A. Such a pair is, shortly, denoted by $\overrightarrow{AB}$. The point A is the *original point*, while B is the *terminal point* and the line AB (if $A \neq B$) gives the direction of $\overrightarrow{AB}$. A vector $\overrightarrow{AB}$ at A has the *orientation* from A to B, i.e. from its original to its terminal point.
- The length of the segment $[AB]$ represents the *length* of the vector $\overrightarrow{AB}$ and is denoted by $||\overrightarrow{AB}||$ or by $|\overrightarrow{AB}|$. Usually, the vector $\overrightarrow{AB}$ at A is represented as



An equivalence relation on pairs of points...

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- When the points A, B, C and D are not collinear, this means that $(A, B) \sim (C, D)$ if and only if $ABCD$ is a parallelogram.
- It is not difficult to check that " $\sim$ " is an equivalence relation.
- Let us denote by V_3 the set $(\mathcal{E}_3 \times \mathcal{E}_3)/_\sim$ of equivalence classes and by V_2 the set $(\mathcal{E}_2 \times \mathcal{E}_2)/_\sim$.

- If $\overrightarrow{AB} \in \mathcal{E} \times \mathcal{E}$, its equivalence class is denoted by $\overline{AB}$ and is called a *vector* in $\mathcal{E}$ ($\mathcal{E}_2$ or $\mathcal{E}_3$). In this case, $\overrightarrow{AB}$ is a *representative* of $\overline{AB}$.
- Suppose that $A \neq B$. The line AB defines the *direction* of the vector $\overline{AB}$. The *length* of $\overline{AB}$ is given by

$$||\overline{AB}|| = ||\overrightarrow{AB}|| = AB,$$

the length of the segment $[AB]$. The *orientation* of $\overline{AB}$, from A to B, is given by the orientation of $\overrightarrow{AB}$.

We shall denote the vectors in V_2 or V_3 by small letters: $\overline{a}$, $\overline{b}, \ldots \overline{u}$, $\overline{v}$, $\overline{w}$.

Proposition

Given a vector $\overline{a}$ in V_2 (or V_3) and a fixed point A, there exists a unique representative of $\overline{a}$, having the original point at A.

Proof.

Vector operations

Let $\overline{a}$ and $\overline{b}$ be two vectors in V_3 (or V_2). The *sum* of $\overline{a}$ and $\overline{b}$ is the vector denoted by $\overline{a} + \overline{b}$, so that, if $\overrightarrow{AB} \in \overline{a}$ and $\overrightarrow{BC} \in \overline{b}$, then $\overrightarrow{AC}$ is the representative of $\overline{a} + \overline{b}$.

- If $\overline{v}$ is a vector in V_3 (or V_2), then the *opposite vector* of $\overline{v}$ is denoted by $-\overline{v}$, so that, if $\overrightarrow{AB}$ is a representative of $\overline{v}$, then $\overrightarrow{BA}$ is a representative of $-\overline{v}$.
- The sum $\overline{a} + (-\overline{b})$ will be, shortly, denoted by $\overline{a} - \overline{b}$ and it will be called the *difference* of the vectors $\overline{a}$ and $\overline{b}$.
- Let $\overline{a}$ be a vector in V_3 (or V_2) and k be a real number. The *product* $k \cdot \overline{a}$ is the vector defined as follows:
  1. $\overline{0}$ if $\overline{a} = \overline{0}$ or $k = 0$;
  2. if $k > 0$, then $k \cdot \overline{a}$ has the same direction and orientation as $\overline{a}$ and $||k \cdot \overline{a}|| = k \cdot ||\overline{a}||$;
  3. if $k < 0$, then $k \cdot \overline{a}$ has the same direction as $\overline{a}$, opposite orientation to $\overline{a}$ and $||k \cdot \overline{a}|| = -k \cdot ||\overline{a}||$.

The components of a vector

- Let $\overline{a}$ be a vector in V_2 and xOy be a rectangular coordinates system in $\mathcal{E}_2$. There is a unique point $A \in \mathcal{E}_2$, such that $\overrightarrow{OA} \in \overline{a}$. The coordinates of the point A are called the *components* of the vector $\overline{a}$ and write $\overline{a}(a_1, a_2)$.

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- Similarly, $\overline{a}$ a vector in V_3 and a rectangular coordinate system $Oxyz$ in $\mathcal{E}_3$, there exists a unique point $A(a_1, a_2, a_3)$, such that $\overrightarrow{OA} \in \overline{a}$. The triple (a_1, a_2, a_3) gives the *components* of $\overline{a}$ and we denote it by $\overline{a}(a_1, a_2, a_3)$.
- Since $\overline{0}(0, 0)$ in V_2 and $\overline{0}(0, 0, 0)$ in V_3, then two vectors are equal if and only if they have the same components.

Theorem

Let $\overline{a}(a_1, a_2)$ and $\overline{b}(b_1, b_2)$ be two vectors in V_2 and $k \in \mathbb{R}$. Then:

(1) *the components of $\overline{a} + \overline{b}$ are $(a_1 + b_1, a_2 + b_2)$;*

(2) *the components of $k \cdot \overline{a}$ are (ka_1, ka_2).*

Proof.

An analogous theorem for 3D

Theorem

Let $\overline{a}(a_1, a_2, a_3)$ *and* $\overline{b}(b_1, b_2, b_3)$ *be two vectors in* V_3 *and* $k \in \mathbb{R}$. *Then:*

(1) *the components of* $\overline{a} + \overline{b}$ *are* $(a_1 + b_1, a_2 + b_2, a_3 + b_3)$;

(2) *the components of* $k \cdot \overline{a}$ *are* (ka_1, ka_2, ka_3).

Theorem

(1) *If* $P_1(x_1, y_1)$ *and* $P_2(x_2, y_2)$ *are two points in* $\mathcal{E}_2$*, then*

$$\overline{P_1 P_2}(x_2 - x_1, y_2 - y_1).$$

(2) *If* $Q_1(x_1, y_1, z_1)$ *and* $Q_2(x_2, y_2, z_2)$ *are two points in* $\mathcal{E}_3$*, then*

$$\overline{Q_1 Q_2}(x_2 - x_1, y_2 - y_2, z_2 - z_1).$$

Proof.

The set of vectors is a very structured one

Theorem (Prop. of the summation)

Let $\overline{a}$, $\overline{b}$ and $\overline{c}$ be vectors in V_3 (or V_2) and $\alpha, \beta \in \mathbb{R}$. Then:

1) $\overline{a} + \overline{b} = \overline{b} + \overline{a}$ *(commutativity);*

2) $(\overline{a} + \overline{b}) + \overline{c} = \overline{a} + (\overline{b} + \overline{c})$ *(associativity);*

3) $\overline{a} + \overline{0} = \overline{0} + \overline{a} = \overline{a}$ *($\overline{0}$ is the neutral element for summation);*

4) $\overline{a} + (-\overline{a}) = (-\overline{a}) + \overline{a} = \overline{0}$ *($-\overline{a}$ is the inverse of $\overline{a}$);*

5) $\alpha(\beta\overline{a}) = (\alpha\beta)\overline{a}$*;*

6) $\alpha \cdot (\overline{a} + \overline{b}) = \alpha \cdot \overline{a} + \beta \cdot \overline{b}$ *(multiplication by real scalars is distributive with respect to the summation of vectors);*

7) $(\alpha + \beta) \cdot \overline{a} = \alpha \cdot \overline{a} + \beta \cdot \overline{a}$ *(multiplication by real scalars is distributive with respect to the summation of scalars);*

8) $1 \cdot \overline{a} = \overline{a}$*.*

Proof.

Proposition

(1) *Let* $\overline{a}(a_1, a_2)$ *be a vector in* V_2. *The length of* $\overline{a}$ *is given by*

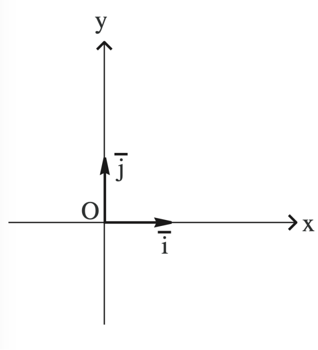
$$||\overline{a}|| = \sqrt{a_1^2 + a_2^2}.$$

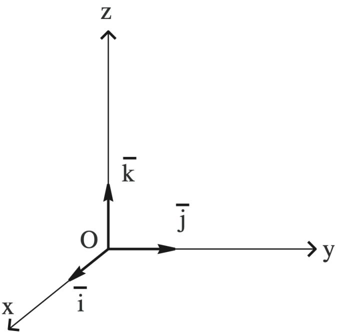
(2) *Let* $\overline{a}(a_1, a_2, a_3)$ *be a vector in* V_3. *The length of* $\overline{a}$ *is given by*

$$||\overline{a}|| = \sqrt{a_1^2 + a_2^2 + a_3^3}.$$

Proof.

- The vectors $\bar{i}(1,0)$ and $\bar{j}(0,1)$ in V_2 are called the *unit vectors* (or *versors*) of the coordinate axes Ox and Oy.
- The vectors $\bar{i}(1,0,0)$, $\bar{j}(0,1,0)$ and $\bar{k}(0,0,1)$ are called the *unit vectors* (or *versors*) of the coordinate axes Ox, Oy and Oz.
- It is clear that
$$||\bar{i}|| = ||\bar{j}|| = ||\bar{k}|| = 1.$$



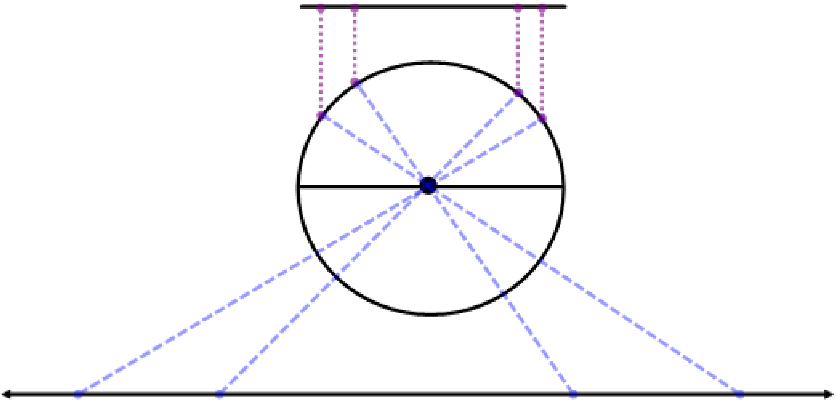


Interlude... not really related to the course

- In general, if we are given an equivalence relation $\sim$ on a set X, then the set of equivalence classes $X/\sim$ is "smaller" than the whole set X.

Interlude... not really related to the course

- In general, if we are given an equivalence relation $\sim$ on a set X, then the set of equivalence classes $X/\sim$ is "smaller" than the whole set X.
- Always smaller?... Take $X = \mathbb{R}$ and say that for $x, y \in \mathbb{R}$ we have $x \sim y$ if and only if $x - y \in \mathbb{Z}$. Then, every real number has a representative in $[0, 1)$, so we can think of $\mathbb{R}/\sim$ as of the interval $[0, 1)$. But is this really "smaller" than $\mathbb{R}$?



So far we have defined the operations

$$+ : V_2 \times V_2 \to V_2, \quad (\overline{a}, \overline{b}) \mapsto \overline{a} + \overline{b}$$

$$\cdot : \mathbb{R} \times V_2 \to V_2, \quad (k, \overline{a}) \mapsto k \cdot \overline{a}$$

and, of course,

$$+ : V_3 \times V_3 \to V_3, \quad (\overline{a}, \overline{b}) \mapsto \overline{a} + \overline{b}$$

$$\cdot : \mathbb{R} \times V_3 \to V_3, \quad (k, \overline{a}) \mapsto k \cdot \overline{a}.$$

V_2 the same thing as $\mathbb{R}^2$, or V_3 the same thing as $\mathbb{R}^3$?

Theorem

1. $(V_2, +)$ *is a vector space over* $\mathbb{R}$*, which is isomorphic to* $(\mathbb{R}^2, +)$*. The set* $\{\bar{i}, \bar{j}\}$ *is a base of* V_2*, therefore* $\dim_{\mathbb{R}} V_2 = 2$*.*
2. $(V_3, +)$ *is a vector space over* $\mathbb{R}$*, which is isomorphic to* $(\mathbb{R}^3, +)$*. The set* $\{\bar{i}, \bar{j}, \bar{k}\}$ *is a base of* V_3*, therefore* $\dim_{\mathbb{R}} V_3 = 3$*.*

Proof.

A few definitions

- Let $\overline{a}$ and $\overline{b}$ be two nonzero vectors in V_3 (or V_2). They are *linearly dependent* if there exist the scalars $\alpha, \beta \in \mathbb{R}^*$ such that $\alpha\overline{a} + \beta\overline{b} = \overline{0}$.
- Let set $\overline{a}$, $\overline{b}$ and $\overline{c}$ be three nonzero vectors in V_3. They are *linearly dependent* if there exist the scalars $\alpha, \beta, \gamma \in \mathbb{R}$, not all equal to zero, such that $\alpha\overline{a} + \beta\overline{b} + \gamma\overline{c} = \overline{0}$.
- The vectors $\overline{a}$ and $\overline{b}$ in V_3 (or V_2), $\overline{a}, \overline{b} \neq \overline{0}$, are *collinear* if they have representatives situated on the same line.
- The vectors $\overline{a}$, $\overline{b}$ and $\overline{c}$ in V_3, $\overline{a}, \overline{b}, \overline{c} \neq \overline{0}$ are *coplanar* if they have representatives situated in the same plane.

Theorem

1. *The vectors $\overline{a}$ and $\overline{b}$ are linearly dependent if and only if they are collinear.*
2. *The vectors $\overline{a}$, $\overline{b}$ and $\overline{c}$ are linearly dependent in V_3 if and only if they are coplanar.*

Proof.

1. If the vectors $\overline{a}$ and $\overline{b}$ are collinear, then there exists a scalar $\alpha \in \mathbb{R}^*$ such that $\overline{a} = \alpha \cdot \overline{b}$, i.e.

$$1 \cdot \overline{a} + (-\alpha) \cdot \overline{b} = \overline{0},$$

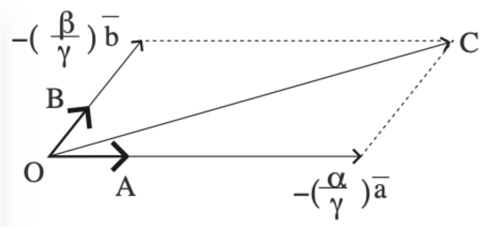
so, by definition, $\overline{a}$ and $\overline{b}$ are linearly dependent.

Conversely, if $\alpha\overline{a} + \beta\overline{b} = \overline{0}$ for some scalars $\alpha, \beta \in \mathbb{R}^*$, then we can write $\overline{a} = \left(-\dfrac{\beta}{\alpha}\right)\overline{b}$. By definition, $\overline{a}$ and $\overline{b}$ are collinear.

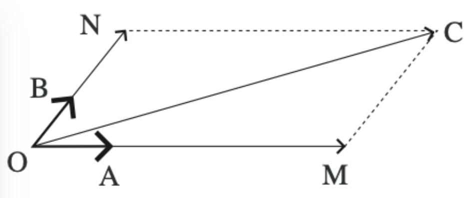
2. Suppose that the vectors $\overline{a}$, $\overline{b}$ and $\overline{c}$ are linearly dependent. Then, there exist $\alpha, \beta, \gamma \in \mathbb{R}$ not all zero, such that $\alpha\overline{a} + \beta\overline{b} + \gamma\overline{c} = \overline{0}$. Suppose that $\gamma \neq 0$. One obtains

$$\overline{c} = \left(-\frac{\alpha}{\gamma}\right)\overline{a} + \left(-\frac{\beta}{\gamma}\right)\overline{b}.$$

If $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are representative of $\overline{a}$ respectively $\overline{b}$, then the representative $\overrightarrow{OC}$ of $\overline{c}$, constructed as below, is coplanar with $\overrightarrow{OA}$ and $\overrightarrow{OB}$.



Conversely, if $\overline{a}$, $\overline{b}$ and $\overline{c}$ are coplanar, let us consider the representatives $\overrightarrow{OA} \in \overline{a}$, $\overrightarrow{OB} \in \overline{b}$ and $\overrightarrow{OC} \in \overline{c}$, situated in the same plane. In the diagram below, $OMCN$ is a parallelogram.



Then, there exist $\alpha, \beta \in \mathbb{R}$ such that $\overrightarrow{OM} = \alpha \cdot \overrightarrow{OA}$ and $\overrightarrow{ON} = \beta \cdot \overrightarrow{OB}$. Hence $\overline{OC} = \overline{OM} + \overline{ON} = \alpha \cdot \overline{OA} + \beta \cdot \overline{OB}$ and $\overline{c} = \alpha \cdot \overline{a} + \beta \cdot \overline{b}$, so that $\alpha \cdot \overline{a} + \beta \cdot \overline{b} + (-1) \cdot \overline{c} = \overline{0}$ and the vectors $\overline{a}$, $\overline{b}$ and $\overline{c}$ are linearly dependent.

$\square$

To keep in mind...

- The set $\{\overline{a}, \overline{b}\}$ is a base in V_2 if and only if the vectors $\overline{a}$, $\overline{b}$ are not collinear.
- The set $\{\overline{a}, \overline{b}, \overline{c}\}$ is a base in V_3 if and only if the vectors $\overline{a}$, $\overline{b}$, $\overline{c}$ are not coplanar.

Thank you very much for your attention!