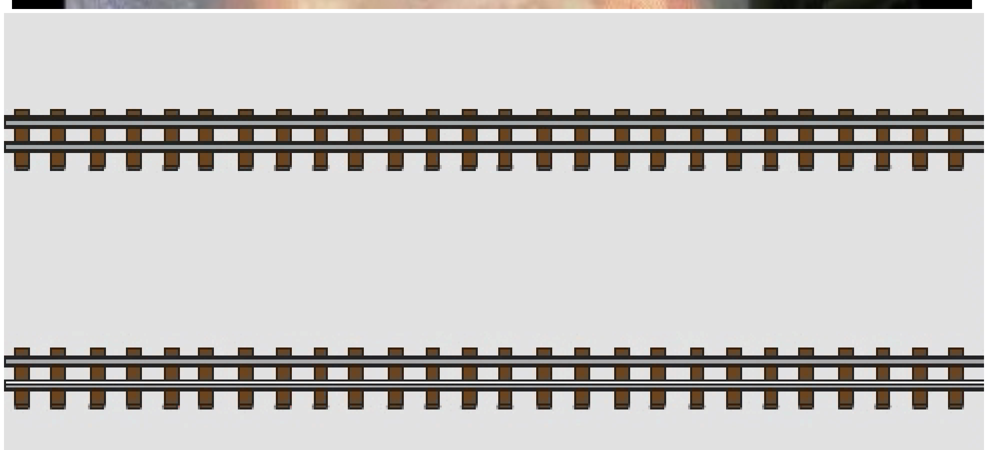
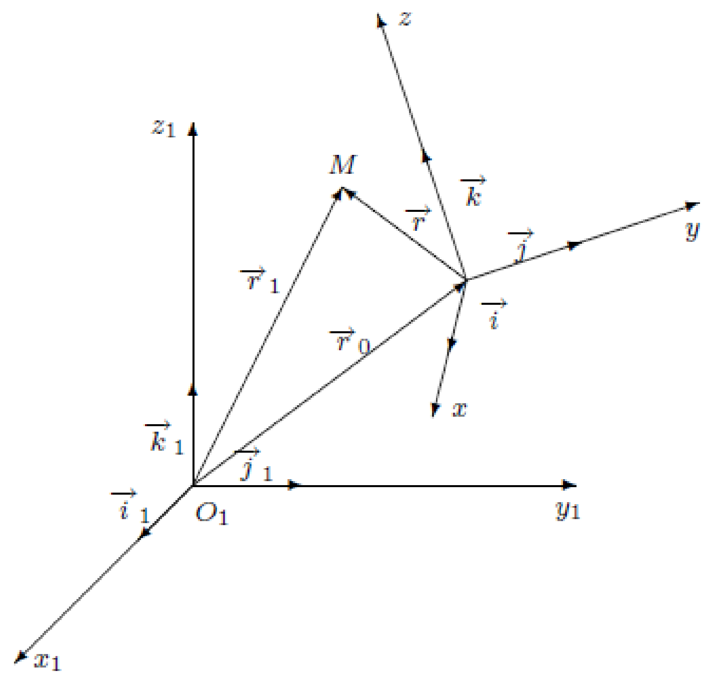
Theoretical Mechanics

13. Dynamics of the relative motion of a particle

(P.P. Teodorescu, Mechanical Systems, Classical Models. Vol. I: Particle Mechanics, Springer, 2007)



Let be an *inertial frame of reference* $O_1x_1y_1z_1$, which is considered to be *fixed*, and a *non-inertial frame* (*movable frame*) *Oxyz* in motion with respect to the fixed one.



In Lecture 5 we have seen that the absolute motion is obtained by the composition of the relative motion and the transportation one (a vector composition for the velocities, while for the accelerations one must add the Coriolis acceleration too).

$$\boxed{\overrightarrow{v}_{rel} := \left(\frac{d\overrightarrow{r}}{dt}\right)_{rel} = \dot{x}\,\overrightarrow{i} + \dot{y}\,\overrightarrow{j} + \dot{z}\,\overrightarrow{k} = \left(\frac{d\overrightarrow{r}}{dt}\right)_{\overrightarrow{i},\overrightarrow{j},\overrightarrow{k}=\overrightarrow{const}}}$$

$$\boxed{\overrightarrow{v}_{tran} = \overrightarrow{v}_0 + \overrightarrow{\omega} \times \overrightarrow{r}}$$

The absolute velocity is the sum of the relative and transport velocity.

$$\overrightarrow{v}_{abs} = \overrightarrow{v}_{rel} + \overrightarrow{v}_{tran} \tag{13.1}$$

$$\vec{a}_{rel} := \left(\frac{d\vec{v}_{rel}}{dt}\right)_{rel} = \ddot{x}\,\vec{i} + \ddot{y}\,\vec{j} + \ddot{z}\,\vec{k} \quad \left(= \left(\frac{d\vec{v}_{rel}}{dt}\right)_{\vec{i},\vec{j},\vec{k}=\overrightarrow{const}} \right)$$

is the relative acceleration,

$$\vec{a}_{tran} = \vec{a}_0 + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

is the transport acceleration, and

$$\vec{a}_c = 2\vec{\omega} \times \vec{v}_{rel}$$

is the **Coriolis acceleration.**

The second Coriolis formula is given by:

$$\vec{a}_{abs} = \vec{a}_{rel} + \vec{a}_{tran} + \vec{a}_c$$

absolute acceleration (13.2)

Starting from the formula (13.2) of composition of the accelerations and multiplying both members by the mass *m* , we can write

$$m\mathbf{a}_a = m\mathbf{a}_t + m\mathbf{a}_r + m\mathbf{a}_C \qquad (13.3)$$

where $\mathbf{a}_t$,$\mathbf{a}_r$,$\mathbf{a}_C$ are the transportation, relative and Coriolis accelerations

Taking into account the equation of motion of a free particle

$$m\,\mathbf{a}_a = \mathbf{F}$$

where **F** is the resultant of the given forces, we get

$$m\mathbf{a}_r = \mathbf{F} + \mathbf{F}_t + \mathbf{F}_C \qquad (13.4)$$

where

$$\begin{aligned} \mathbf{F}_t &= -m\mathbf{a}_t = -m\left[\mathbf{a}_O + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})\right] \\ \mathbf{F}_C &= -m\mathbf{a}_C = -2m\boldsymbol{\omega} \times \mathbf{v}_r \end{aligned} \qquad (13.5)$$

are *complementary forces* (*the transportation force* and *the Coriolis force*, respectively); these forces are added to the given force **F** and allow writing the equation of motion in a non-inertial frame of reference.

The complementary forces are called also *inertial forces* because their magnitude is proportional to the inertial mass; these forces are applied to the particle in motion.

The inertial forces are real ones with respect to an observer linked to a noninertial frame of reference; e.g., *the centrifugal force*, which appears in a motion of rotation, is a transportation force.

Theoretical Mechanics

Introducing ***the relative force***

$$\mathbf{F}_r = \mathbf{F} + \mathbf{F}_t + \mathbf{F}_C \qquad (13.6)$$

we may write the equation (13.4) in the form

$$m\mathbf{a}_r = \mathbf{F}_r\,, \quad \mathbf{F}_r = \mathbf{F}_r\left(\mathbf{r}, \mathbf{v}_r; t\right) \qquad (13.7)$$

we put initial conditions of Cauchy type

$$\mathbf{r}(t_0) = \mathbf{r}_0\,, \quad \mathbf{v}_r(t_0) = \mathbf{v}_r^0 \qquad (13.8)$$

Theorem of the relative motion*. The equation of motion of a particle with respect to an inertial frame of reference maintains its form with respect to a noninertial one if the given force is replaced by the force relative to the latter frame.*

Theoretical Mechanics

Theorem of the relative force. *The relative force (with respect to a non-inertial frame of reference) is equal to the sum of the given force and the complementary forces (the force of transportation and the Coriolis force) with respect to an inertial one.*

In case of a particle subjected to bilateral constraints, we use the axiom of liberation of constraints, introducing the constraint force R . The equation of motion (13.4) becomes

$$m\mathbf{a}_r = \mathbf{F} + \mathbf{F}_t + \mathbf{F}_C + \mathbf{R} \tag{13.9}$$

or taking into account (13.6), we may also write

$$m\mathbf{a}_r = \mathbf{F}_r + \mathbf{R} \tag{13.10}$$

Thus, the problem of motion with respect to a non-inertial frame of reference may be reduced to a corresponding problem with respect to an inertial frame, chosen conveniently (with respect to which the Newtonian model of mechanics is verified with a sufficient good approximation).

Particular cases of non-inertial frames of reference

Let us consider first of all the case of a *non-inertial frame of reference in* a *motion of translation* with respect to an inertial one; hence, we assume that ω = **0** . In this case, the equation of motion is ($\boldsymbol{F_t} = -ma_0$, $\boldsymbol{F_c} = 0$)

$$m\boldsymbol{a_r} = \boldsymbol{F} - m\boldsymbol{a_0} \qquad (13.11)$$

We assume, in particular, that the particle P is acted upon by its own weight (**F** = m**g**), so that we can write

$$\boldsymbol{a_r} = \boldsymbol{g} - \boldsymbol{a_0} \qquad (13.12)$$

In the case in which $\boldsymbol{a_0}||\boldsymbol{g}$ we have ***the elevator problem.***

If $\boldsymbol{a_O}$ has the same direction as **g** (it is directed towards the centre of the Earth), then the particle P seems to be lighter ($\boldsymbol{a_r = g - a_O}$), while if $\boldsymbol{a_O}$ has a direction opposite to that of **g** ($\boldsymbol{a_r = g + a_O}$), then the particle seems to be heavier.

In particular, if the elevator is in a free falling ($\boldsymbol{a_O = g}$), then the apparent weight of the particle with respect to it vanishes; we are in ***the case of imponderability***.

Moreover, let us suppose that $\boldsymbol{a_O} = \overrightarrow{\boldsymbol{const}}$. If, for instance, the particle is in free falling ($\boldsymbol{v_0} = 0$ for $t = 0$) from a height h with respect to the floor of the elevator ($x = 0$, the Ox -axis of unit vector **i** along the ascendent vertical), we get, in the movable non-inertial frame,

$$x = -\frac{(g + a_O)t^2}{2} + h$$

The falling time is given by

$$T = \sqrt{\frac{2h}{g + a_O}}$$

assuming that $g + a_O > 0$ ($\mathbf{a}_O = a_O\,\mathbf{i}$, with $a_O > 0$, of an opposite direction to that of the gravitation $\boldsymbol{g} = -\,\mathrm{g}\,\mathbf{i}$, or with $a_O < 0$, of the same direction but with $|\boldsymbol{a_O}| < |\boldsymbol{g}|$).

If $g + a_O = 0$, then the particle is immobile, while if $g + a_O < 0$ ($a_O < 0$ with $|\boldsymbol{a_O}| > |\boldsymbol{g}|$), then the particle goes up along the vertical Ox with respect to an observer linked to the movable frame. If $a_O = 0$, then the movable frame becomes an inertial one, finding again the usual laws of falling.

Another important particular case is that of a *non-inertial frame of reference in motion of rotation* with respect to a fixed Ox_3' -axis ($\mathbf{v}'_O = \mathbf{a}'_O = \mathbf{0}$, $\boldsymbol{\omega} = \omega\mathbf{i}_3$, $\omega = \omega(t)$).

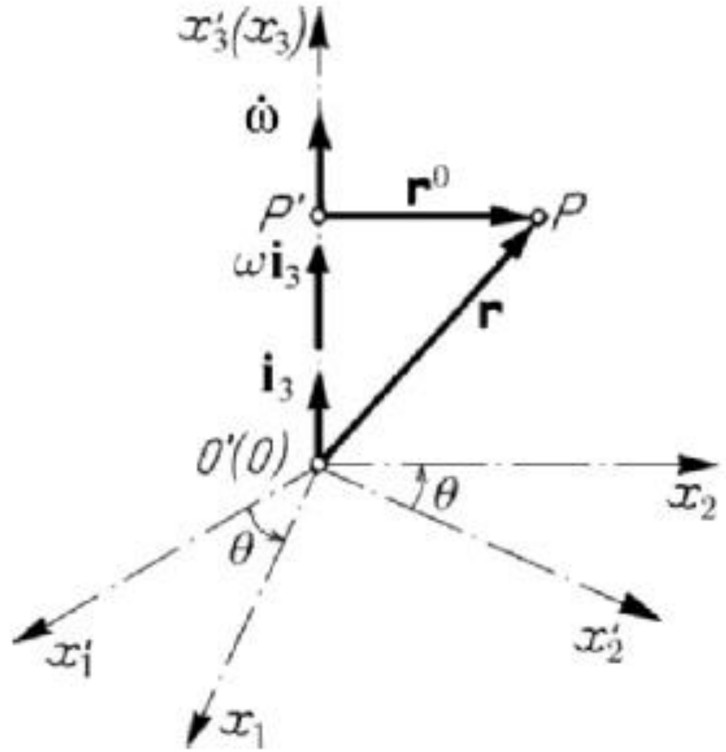
The transportation force is given by

$$\mathbf{F}_t = -m\dot{\boldsymbol{\omega}} \times \mathbf{r} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

$$= -m\dot{\boldsymbol{\omega}} \times \mathbf{r}^0 - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}^0)$$

$$= -m\dot{\omega}\mathbf{i}_3 \times \mathbf{r}^0 + m\omega^2\mathbf{r}^0$$

$$(\mathbf{r} = \overrightarrow{OP'} + \overrightarrow{P'P} = \overrightarrow{OP'} + \mathbf{r}^0)$$



The Coriolis force is given by

$$\mathbf{F}_C = -2m\boldsymbol{\omega} \times \mathbf{v}_r = -2m\omega\mathbf{i}_3 \times \mathbf{v}_r \,.$$

In case of a uniform rotation ($\dot{\omega} = 0$), we have

$$\mathbf{F}_t = m\omega^2\mathbf{r}^0 \,.$$ (centrifugal force)

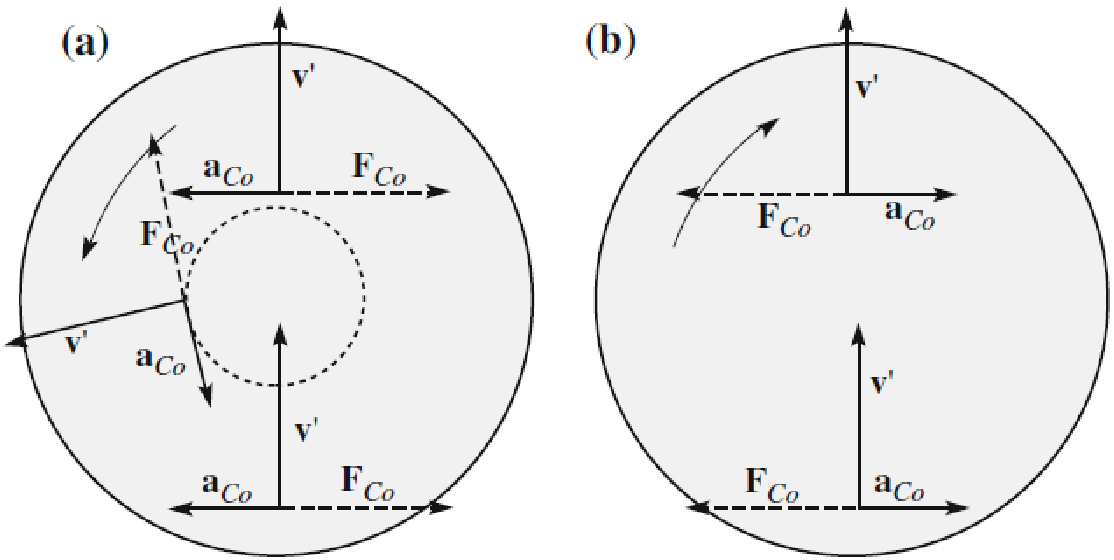
Assuming that the particle P is moving in the Ox_1x_3 -plane (with respect to the noninertial frame), we may write the equation of motion in components, in the form

$$m\ddot{x}_1 = F_1 + m\omega^2 x_1 \,, \quad 0 = F_2 - m\dot{\omega}x_1 - 2m\omega\dot{x}_1 \,, \quad m\ddot{x}_3 = F_3$$

Observing then that $\omega = \dot{\theta}$ and replacing x_1 by r and x_3 by z , we find again the equations of motion of the particle in cylindrical co-ordinates, with respect to a fixed (inertial) frame.

Example (A. Bettini, A Course in Classical Physics 1—Mechanics, Springer, 2014)

Consider the point P lying on the rotating platform. If P does not move relative to the platform, the Coriolis acceleration is null.



Coriolis acceleration and (pseudo)force on a platform rotating. a)Counter-clockwise, b) Clockwise

Let v′ be this velocity, which we assume, for simplicity, to be parallel to the platform. As we have already noticed, the Coriolis acceleration, and consequently the Coriolis force, does not depend on the position of P on the platform and is in any case perpendicular to the relative velocity

If the angular velocity ω is directed out of the plane of the figure, we see the platform turning counter-clockwise. In this case, the Coriolis acceleration is directed towards the left of the motion, and the Coriolis force to the right.

14. Dynamics of systems of material points

Definition: A **system of material points** (S) is a finite set of material points interacting each other. (The movement of each point depends on the movement of the other points).

Material system: - discrete – a finite number of isolated material points

- continuous (rigid) – an infinite number of material points, occupying a domain D in $\mathbf{R}^3$

Material system: - free – particles can be anywhere in space

- with constraints – particles may have geometric or kinetic constraints

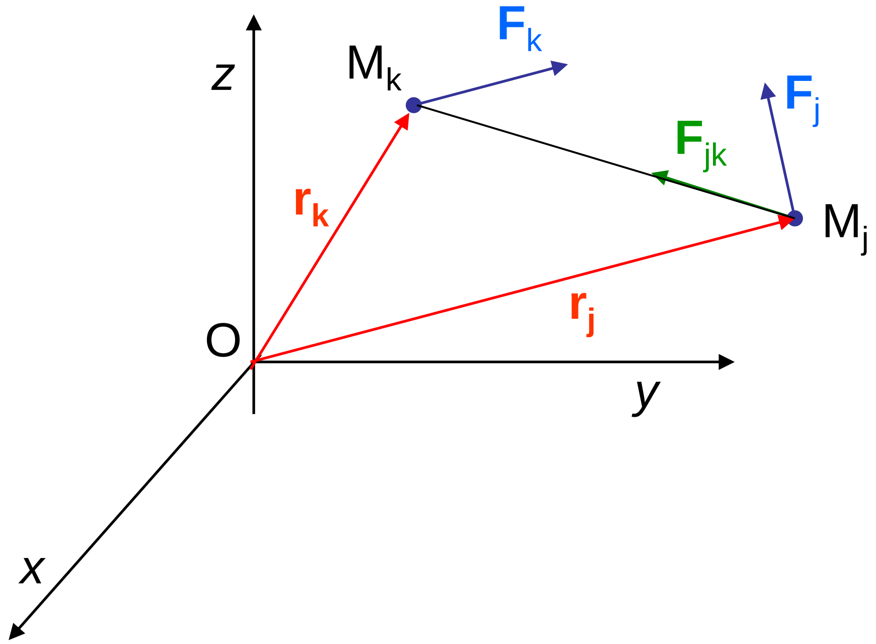
Forces: In a system (S) act **external forces** that come from outside the system (e.g. gravitational attraction) and **inner forces** coming from the points that form the system (S).

Example: The sun and the planets form a system of material points. Planets and sun are internal points of (S), and other celestial bodies are outer points.

Let be:

- m_k the mass of point M_k,
- $\mathbf{F}_k$ the resultant of the external forces acting on M_k
- $\mathbf{F}_{kj}$ the inner force exerted by the particle M_j on particle M_k,

  $k, j = 1,..., N; \;\; k \neq j$



On the particle M_j acts the resultant force:

$$\vec{F}_j + \sum_{k=1}^{N} \vec{F}_{jk} \qquad (14.1)$$

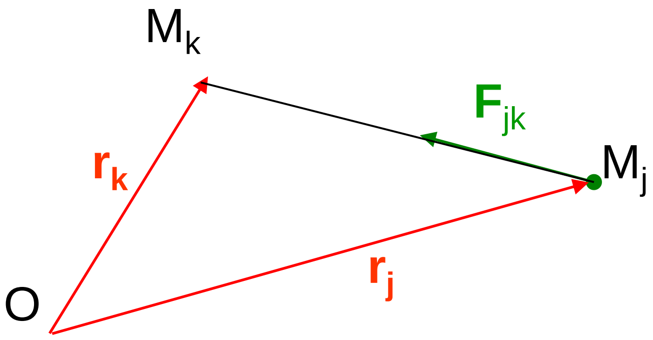
Properties of the inner forces

1. Inner forces respect the action-reaction principle:

$$\vec{F}_{jk} + \vec{F}_{kj} = 0, j, k = 1, \ldots, N$$
$$\vec{F}_{jj} = 0, j = 1, \ldots, N \qquad (14.2)$$

Moreover:

$$\vec{F}_{jk} || \vec{r}_j - \vec{r}_k \Rightarrow$$
$$\Rightarrow \vec{F}_{jk} = F_{jk} \frac{\vec{r}_j - \vec{r}_k}{|\vec{r}_j - \vec{r}_k|} \qquad (14.2')$$



2. The general resultant of the inner forces, **R**$^{(i)}$, and the resultant moment of the inner forces, $M_O^{(i)}$, (O arbitrary in space) are zero:

$$\vec{R}^{(i)} = 0, M_O{}^{(i)} = 0 \tag{14.3}$$

Indeed:

$$\vec{R}^{(i)} = \sum_{j=1}^{N}\left(\sum_{k=1}^{N}\vec{F}_{jk}\right) = \sum_{j,k=1}^{N}\vec{F}_{jk} \underset{\vec{F}_{jk}+\vec{F}_{kj}=0}{=} 0 \tag{14.4}$$

For particles M_k and M_j ($k \neq j$) we have:

$$\underbrace{\vec{r}_j \times \vec{F}_{jk}}_{\vec{M}_j} + \underbrace{\vec{r}_k \times \vec{F}_{kj}}_{\vec{M}_k} \underset{(2)}{=} \left(\vec{r}_j - \vec{r}_k\right) \times \vec{F}_{jk} \underset{(2\prime)}{=} 0$$

Thus:

$$\vec{M}_O{}^{(i)} = \sum_{j=1}^{N}\sum_{k=1}^{N}\vec{r}_j \times \vec{F}_{jk} = 0 \tag{14.5}$$

Differential equations of motion

The differential equations of motion of the system (S) are:

$$m_j \frac{d^2\vec{r}_j}{dt^2} = \vec{F}_j + \sum_{k=1}^{N} \vec{F}_{jk}\, , j = 1, \ldots, N \qquad (14.6)$$

The fundamental problem is to find the motion of the points M_j , i.e. to find the functions

$$\vec{r}_j = \vec{r}_j(t), j = 1, \ldots, N$$

knowing the forces acting on the system ($\mathbf{F}_j$, $\mathbf{F}_{kj}$) and the initial conditions:

$$\vec{r}_j(t_0) = \vec{r}_j^{\,0}, \dot{\vec{r}}_j(t_0) = \dot{\vec{r}}_j^{\,0}\ j = 1, \ldots, N \qquad (14.7)$$

Solving (14.6) along with the initial conditions (14.7) we obtain the equations of motion:

$$\vec{r}_j = \vec{r}_j(t, \vec{r}_1^{\,0}, \ldots, \vec{r}_N^{\,0}, \dot{\vec{r}}_1^{\,0}, \ldots, \dot{\vec{r}}_N^{\,0}), j = 1, \ldots, N \qquad (14.8)$$

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The general theorems of the dynamics of the systems of particles

We consider the discrete systems of particles. Let be:

(S): $M_j(m_j)$, $\mathbf{r}_j$, $\mathbf{F}_j$, $j = 1,\dots, N$

The motion is given by:

$$m_j \ddot{\vec{r}}_j = \vec{F}_j + \sum_{k=1}^{N} \vec{F}_{jk}\,, j = 1, \dots, N \qquad (14.9)$$

1. Theorem of the momentum (quantity of motion)

Definition: The momentum **H** of the system (S) or the quantity of motion is the sum of all momenta of material points:

$$\boxed{\vec{H} = \sum_{j=1}^{N} m_j \dot{\vec{r}}_j = \sum_{j=1}^{N} m_j \vec{v}_j} \qquad (14.10)$$

Using (14.9) and (14.10) one get:

$$\frac{d\vec{H}}{dt} = \underbrace{\sum_{j=1}^{N} \vec{F}_j}_{=\vec{R}} + \underbrace{\sum_{j=1}^{N} \sum_{k=1}^{N} \vec{F}_{jk}}_{\underset{(4)}{=} 0} \qquad (14.11)$$

where **R** is the resultant of the external forces acting on the points of the system. Thus:

$$\frac{d\vec{H}}{dt} = \vec{R} \qquad (14.12)$$

Equation (14.12) expresses the theorem of the momentum:

„The derivative with respect to time of the momentum of a free discrete mechanical system is equal to the resultant of the given external forces which act upon this system.“

First integrals

A first integral of the differential system of motion (14.6) – (14.7) is a (non-constant) continuously-differentiable function (class C^1)

$$\mathrm{F}(t, \vec{r}_1{}^0, \ldots, \vec{r}_N{}^0, \dot{\vec{r}}_1{}^0, \ldots, \dot{\vec{r}}_N{}^0) = c(\textbf{constant}), \forall t \geq t_0$$

which reduces to a constant when the functions $\boldsymbol{r}_i = \boldsymbol{r}_i(t)$, $i = 1,\ldots N$, satisfy (14.6).

Remark: A first integral can replace an equation in the system (14.9).

Case 1. If:

$$\vec{R} = 0 \Rightarrow \vec{H} = \textbf{constant} \tag{14.13}$$

Case 2. If **R** ≠ 0 is normal to a fixed direction of unit vector **u**, **R · u** = 0 (**R** ⊥ **u**) than using (14.12) we obtain:

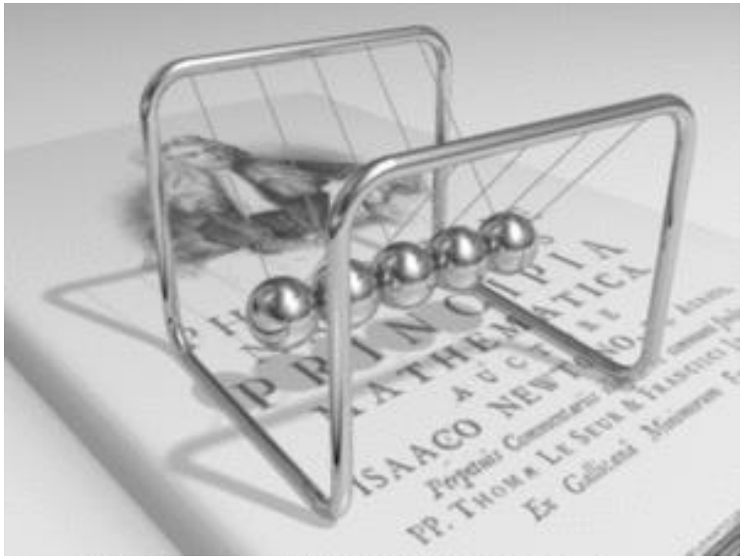
$$\frac{d\vec{H}}{dt} = \vec{R} \Big| \cdot \vec{u} \Rightarrow \frac{d}{dt}\left(\vec{H} \cdot \vec{u}\right) = \vec{R} \cdot \vec{u} = 0 \Rightarrow$$

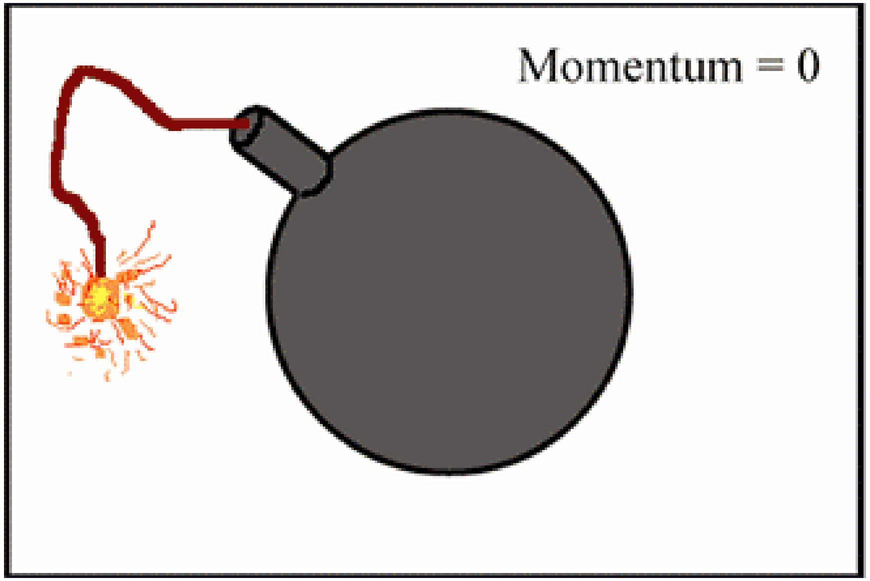
$$\vec{H} \cdot \vec{u} = \textbf{constant,} \forall \mathrm{t} \geq \mathrm{t}_0 \tag{14.14}$$

Equations (14.13) and (14.4) express the principle of the conservation of the system's momentum:

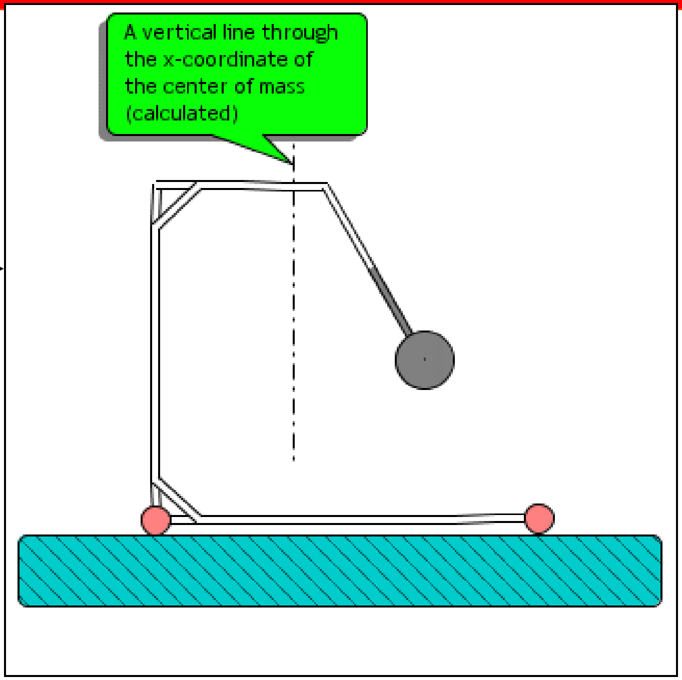
„*If the resultant* **R** *of the given external forces is zero, then the momentum of the free discrete mechanical system S is conserved* (is constant) *in time*."

If the resultant **R** *of the given external forces is parallel to a fixed plane, then the projection of the momentum of the free discrete mechanical system* **S** *on the normal to this plane is conserved* (is constant) *in time.*





The conservation of momentum in the horizontal direction



2. Theorem of motion of the centre of mass

Definition: The point C with the position vector (relative to the reference system Oxyz) given by:

$$\vec{r}_C = \vec{OC} = \sum_{i=1}^{N} m_i \vec{r}_i \Big/ \sum_{i=1}^{N} m_i \qquad (14.15)$$

is the centre of mass (centre of inertia, centre de gravity) of the material system (S).

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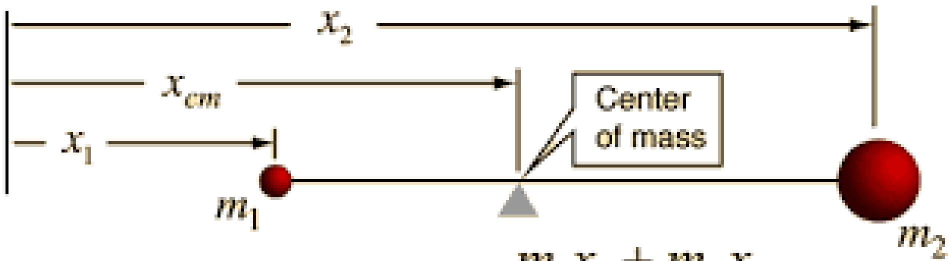
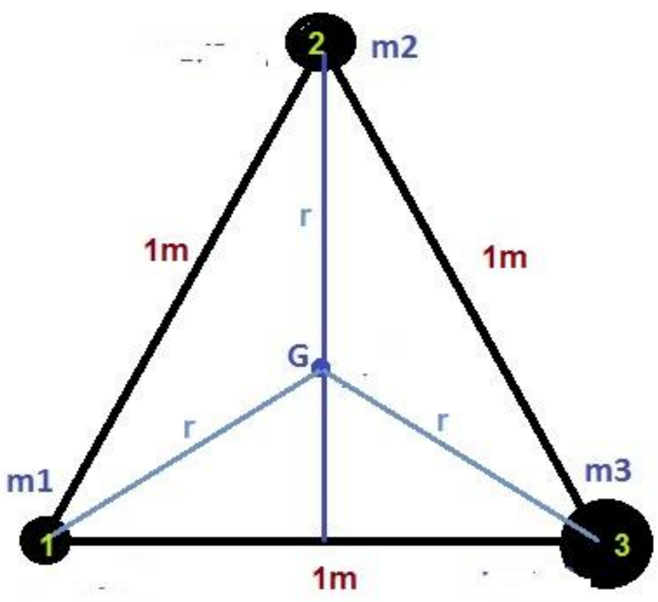
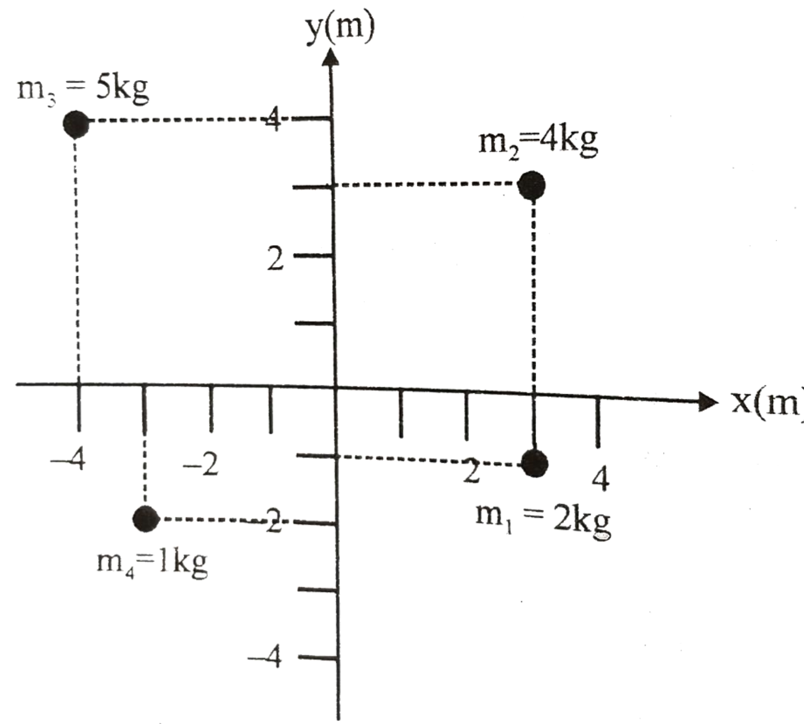


For two masses: $x_{cm} = \dfrac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$





A system of four point masses

$$x_c = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + m_4 x_4}{m_1 + m_2 + m_3 + m_4}$$

$$y_c = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3 + m_4 y_4}{m_1 + m_2 + m_3 + m_4}$$

We mention that $m = \Sigma m_i$ is the total mass of the system (S).
Deriving (14.15) we have:

$$m\dot{\vec{r}}_C = \sum_{i=1}^{N} m_i \dot{\vec{r}}_i = \vec{H} \tag{14.16}$$

And using the momentum theorem (14.12) we obtain:

$$m\ddot{\vec{r}}_C = \vec{R} \tag{14.17}$$

Theorem of the motion of the centre of mass

The centre of mass of a free discrete mechanical system is moving as a free particle at which would be concentrated the whole mass of the system and which would be acted upon by the resultant of the given external forces.

In Cartesian coordinates for $\mathbf{r_C}(x_C, y_C, z_C)$ and $\mathbf{R}(X,Y,Z)$ we have:

$$m\ddot{x}_C = X, m\ddot{y}_C = Y, m\ddot{z}_C = Z \tag{14.18}$$

(simply.science/popups/center-of-mass-motion.html)

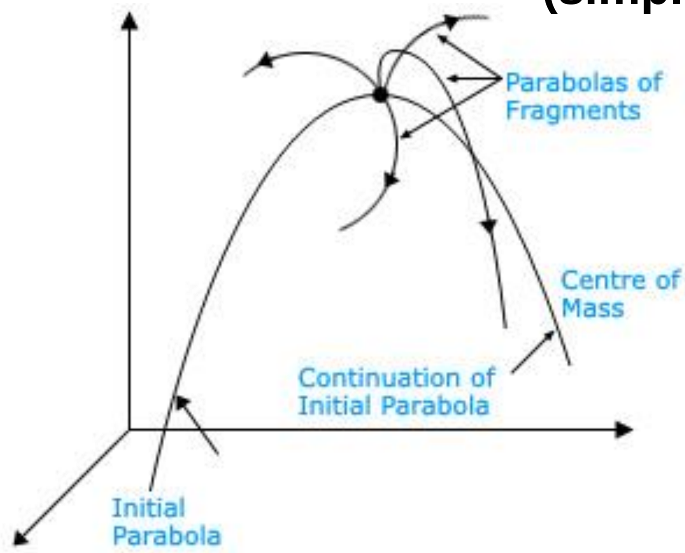


Figure (a)

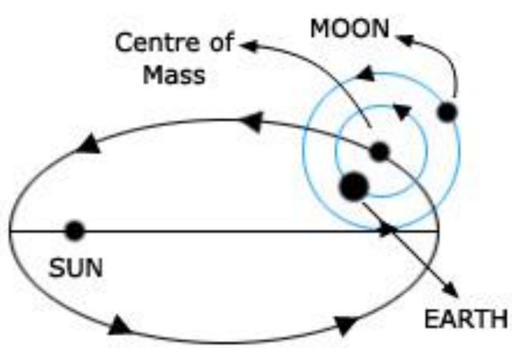


Figure (b)

a. Explosion of a Cracker or a Shell: Initially the cracker is moving along a parabolic path. While in flight it explodes. Each fragment will follow its own parabolic path. But the center of mass of all the fragments will continue to be along the same parabolic path of the shell as before, because there was no external force and explosion is caused by internal force only.

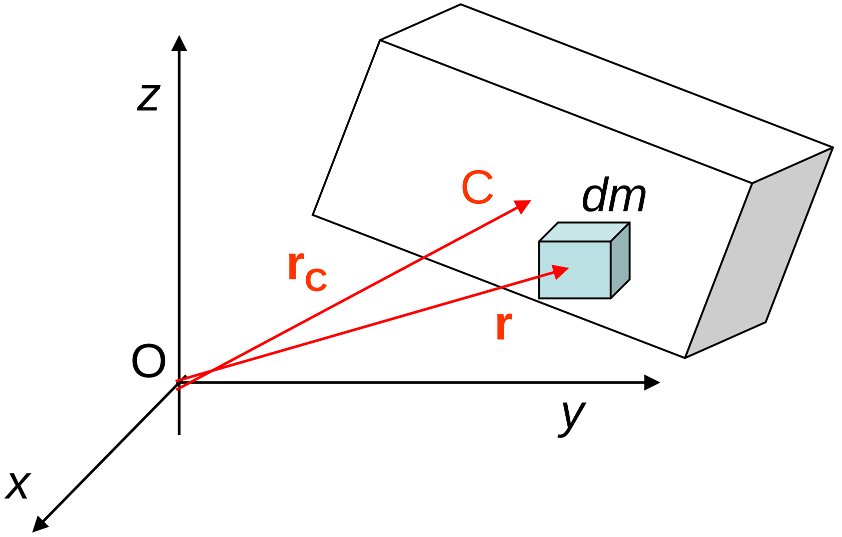
b. Motion of Earth-Moon System: Moon goes round the Sun in a circular orbit. The Earth goes round the Sun in elliptical order. It is more correct to say that the Earth and the Moon both move in circular orbits about their common center of mass describes an elliptical orbit round the Sun. Here Earth and Moon from a system and their gravitational force of attraction is an internal force. But Sun's attraction on both is an external force acting on their center of mass.

Remark: Let be (S) a rigid body of mass m. Then:

$$\vec{r}_C = \frac{1}{m}\int_{(S)} \vec{r}\,dm = \frac{1}{m}\int_{(S)} \rho\vec{r}\,dv$$

is the position vector of the centre of mass of the system (S).

Because internal forces satisfy the principle of action and reaction, then the mass center theorem remains valid for rigid bodies as well. According to this result we assimilate the movement of a rigid body with the movement of a material point, the center of mass.



3. Theorem of the moment of momentum (angular momentum)

Definition: The <u>angular momentum</u> **K_O** of the system (S) with respect to the pole O is the sum of the angular momenta of the points of the system:

$$\vec{K}_O = \sum_{j=1}^{N} \vec{r}_j \times m_j \vec{v}_j \tag{14.19}$$

Deriving **K_O** with respect to time and using (14.9) we obtain:

$$\dot{\vec{K}}_O = \sum_{j=1}^{N} \underbrace{\dot{\vec{r}}_j \times m_j \vec{v}_j}_{=0} + \sum_{j=1}^{N} \vec{r}_j \times m_j \dot{\vec{v}}_j \underset{(9)}{=}$$

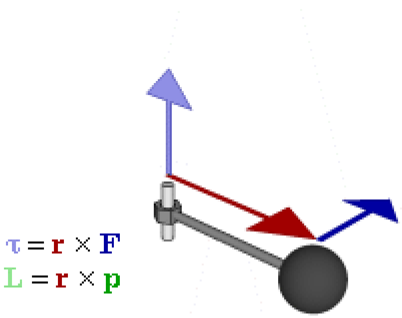
$$= \sum_{j=1}^{N} \vec{r}_j \times \vec{F}_j + \underbrace{\sum_{j=1}^{N} \left(\vec{r}_j \times \sum_{k=1}^{\underline{\ }N} \vec{F}_{jk} \right)}_{\underset{(13.5)}{=} \vec{M}_O^{(i)} = 0} \tag{14.20}$$

We have:

$$\frac{d\vec{K}_O}{dt} = \sum_{j=1}^{N} \vec{r}_j \times \vec{F}_j \underset{\textbf{notation}}{=} \vec{M}_O \qquad (14.21)$$

where M_O is the resultant moment (torque) of the external forces. Equation:

$$\frac{d\vec{K}_O}{dt} = \vec{M}_O \qquad (14.22)$$



is the theorem of the moment of momentum: The derivative with respect to time of the moment of momentum of a continuous mechanical system subjected to constraints, with respect to a fixed pole, is equal to the resultant moment of the given and constraint external forces which act upon that system, with respect to the same pole.

First integral

If $\mathbf{M}_O = 0$ then from (14.22) we have :

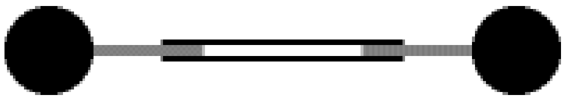
$$\vec{K}_O = \textbf{constant} \qquad (14.23)$$

Equation (14.23) expresses the conservation of the moment of momentum.

$\vec{K}_O = \textbf{constant}$

(http://www.cleonis.nl/physics/phys256/angular_momentum.php)

Two weights connected to pistons. Hydraulic machinery pulls the weights closer to the center of rotation, causing angular acceleration



4. Theorem of the kinetic energy

Definition: The kinetic energy of the system (S) is the scalar value:

$$T = \frac{1}{2}\sum_{j=1}^{N} m_j {v_j}^2 \tag{14.24}$$

Let be $\delta L^{(ext)}$ and $\delta L^{(int)}$ the elementary work of the external forces, and of the inner forces, respectively. Thus:

$$\delta L^{(\mathbf{ext})} = \sum_{j=1}^{N} \vec{F}_j d\vec{r}_j\,; \quad \delta L^{(\mathrm{int})} = \sum_{j=1}^{N}\sum_{k=1}^{N} \vec{F}_{jk}\, d\vec{r}_j \tag{14.25}$$

Let be $M_j\,(m_j)$ a point of the system. Using its kinetic energy and the theorem of the kinetic energy we have:

$$dT_j = d\left(\frac{1}{2} m_j {v_j}^2\right) = \vec{F}_j d\vec{r} + \left(\sum_{k=1}^{N} \vec{F}_{jk}\right) d\vec{r}_j, \forall j = 1,\ldots,N \tag{14.26}$$

Summing (14.26) with respect to *j* we obtain the theorem of the kinetic energy:

$$dT = \delta L^{(\mathbf{ext})} + \delta L^{(\mathrm{int})} \tag{14.27}$$

Theorem of kinetic energy (Daniel Bernoulli)
"The differential of the kinetic energy of a free discrete mechanical system is equal to the elementary work of the given external and internal forces which act upon this system. "

5. Theorem of mechanical energy conservation

If a state function

$$V^{(\mathrm{int})} = V^{(\mathrm{int})}(x_1, y_1, z_1, \ldots, x_N, y_N, z_N)$$

called the internal potential energy of the system, such that,

$$\delta L^{(\mathrm{int})} = -dV^{(\mathrm{int})} \tag{14.28}$$

exist, then (14.27) becomes:

$$d\left(T + V^{(\mathrm{int})}\right) = \delta L^{(\mathbf{ext})} \tag{14.29}$$

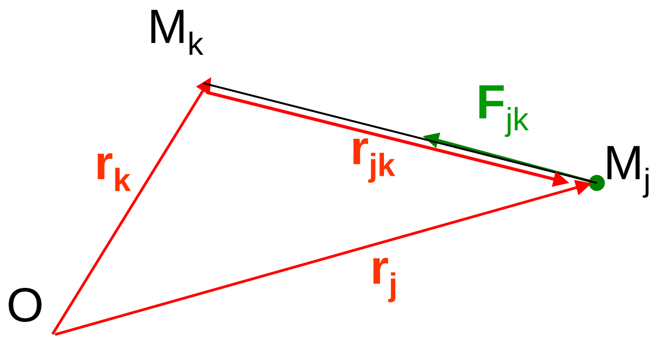
If the internal forces depend only on position, i.e. $\mathbf{F}_{jk} = \mathbf{F}_{jk}(\mathbf{r}_{jk})$ then

$$F_{jk} \cdot dr_{jk} = -dV_{jk}{}^{(\text{int})}, \Rightarrow V_{jk}{}^{(\text{int})} = -\int F_{jk} dr_{jk} \tag{14.30}$$

But,

$$\delta L_{jk}{}^{(\text{int})} = \vec{F}_{jk} \cdot d\vec{r}_j + \vec{F}_{kj} \cdot d\vec{r}_k \underset{\vec{F}_{jk}=-\vec{F}_{kj}}{=} \vec{F}_{jk} \cdot d(\vec{r}_j - \vec{r}_k) =$$

$$= \vec{F}_{jk} \cdot d\vec{r}_{jk} = F_{jk}\vec{u} \cdot d(r_{jk}\vec{u}) = F_{jk}\frac{1}{2}d(r_{jk}\vec{u} \cdot \vec{u}) = \frac{1}{2}F_{jk}dr_{jk}$$



Then

$$\delta L^{(\mathrm{int})} = \sum_{j,k=1}^{N} \vec{F}_{jk} \cdot d\vec{r}_j = \frac{1}{2} \sum_{\substack{j,k=1 \\ j \neq k}}^{N} F_{jk} dr_{jk} = -dV^{(\mathrm{int})} \qquad (14.30‘)$$

where

$$V^{(\mathrm{int})} = \frac{1}{2} \sum_{\substack{j,k=1 \\ j \neq k}}^{N} V_{jk}{}^{(\mathrm{int})} \qquad (14.31)$$

If a state function

$$V^{(\mathbf{ext})} = V^{(\mathbf{ext})}(x_1, y_1, z_1, \ldots, x_N, y_N, z_N)$$

called external potential energy of the system, such that,

$$\delta L^{(\mathbf{ext})} = -dV^{(\mathbf{ext})} \qquad (14.32)$$

exist, then (14.29) become:

$$d\left(T + V^{(\mathbf{ext})} + V^{(\mathrm{int})}\right) = 0$$

Thus

$$T + V^{(\mathbf{ext})} + V^{(\mathrm{int})} = h(\mathbf{constant}) \tag{14.33}$$

Equation (14.33) is the theorem of the conservation of the mechanical energy:

„In the motion of a system of material points in a conservative field (inernal and external forces are potential) the mechanical energy $E = T + V^{(ext)} + V^{(int)}$ is conserved in time. “