General theorems of dynamics (continued)

① A particle of mass $m=1$ moves in a plane along the line $y=x+1$ under the action of the force $\vec{F}(X,Y)$ where
$X = 1-\frac{y^2}{x^2}$, $Y = \frac{2y}{x}$

a) Prove that a function of force U exist and find it.

b) Calculate the work done by $\vec{F}$ when M moves from the point $A(1,2)$ to the point $B(\frac{1}{2},\frac{3}{2})$

Solution:

a) $\frac{\partial X}{\partial y} = -\frac{2y}{x^2}$; $\frac{\partial Y}{\partial x} = -\frac{2y}{x^2}$ $\Longrightarrow \exists\ U(x,y)$ such that

$$\vec{F}(X,Y) = \operatorname{grad} U \iff X = \frac{\partial U}{\partial x} \text{ and } Y = \frac{\partial U}{\partial y}$$

We have $\frac{\partial U}{\partial x} = 1-\frac{y^2}{x^2} \underset{\int}{\Longrightarrow} U = x+\frac{y^2}{x}+\varphi(y)$ (*)

On the other hand:

$$\frac{\partial U}{\partial y} = Y = \frac{2y}{x}$$

(*) $\Longrightarrow$ $\frac{2y}{x}+\varphi'(y) = \frac{2y}{x}$ $\Longrightarrow \varphi'(y)=0 \Rightarrow \varphi(y)=\text{const}$ $(=0)$.

Thus, $U(x,y) = x+\frac{y^2}{x}$

b) $\delta L = dU \Longrightarrow L_{AB} = \int_A^B dU = U(B)-U(A) =$

$= U(\frac{1}{2},\frac{3}{2}) - U(1,2) = 5-5 = 0.$

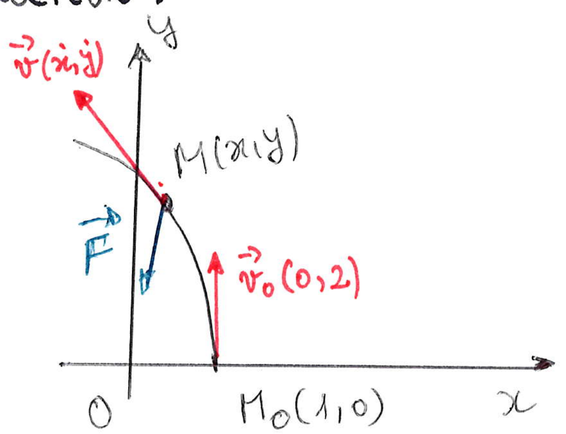
(2) Consider the motion of a particle M $(m=1)$ in the plane xOy under the action of the elastic force, $\vec{F} = -k^2\vec{r}$, $\vec{r} = \overrightarrow{OM}$ $(k>0,\ k=\text{const.})$. At the initial moment, $t=0$, we have: $x(0)=1,\ y(0)=0,\ \dot{x}(0)=0,\ \dot{y}(0)=2$.

i) Prove that the trajectory of the particle is an ellipse.

ii) Prove that $\vec{F}$ is a potential force, find the force function.

iii) Find the work of $\vec{F}$ when M moves from $A\ (t=0)$ to $B\ (t=\frac{\pi}{4k})$.

Solution:



i) $\vec{F}(X,Y) = -k^2(x,y)$

$$\Rightarrow \begin{cases} X = -k^2 x \\ Y = -k^2 y \end{cases}$$

Equations of motion:

$$\begin{cases} m\ddot{x} = -k^2 x \\ m\ddot{y} = -k^2 y \end{cases}$$

The characteristic equation is:

$$u^2 + k^2 = 0 \Rightarrow u_{1,2} = \pm ik$$

$$\Rightarrow \quad x(t) = c_1 \cos kt + c_2 \sin kt$$
$$y(t) = c_3 \cos kt + c_4 \sin kt$$

At $t=0$: $x(0)=1 \Rightarrow \boxed{c_1 = 1}$ $\qquad y(0)=0 \Rightarrow \boxed{c_3=0}$

$\dot{x}(0)=0 \Rightarrow \boxed{c_2=0}$ $\qquad \dot{y}(0)=2 \Rightarrow k c_4 = 2 \Rightarrow \boxed{c_4 = \frac{2}{k}}$

We have $\begin{cases} x = \cos kt \\ y = \frac{2}{k}\sin kt \end{cases} \Rightarrow \boxed{x^2 + \frac{k^2}{4}y^2 = 1}$ – eq. of an ellipse

ii) $X = -k^2 x \; ; \; Y = -k^2 y$

$\dfrac{\partial X}{\partial y} = \dfrac{\partial Y}{\partial x} = 0 \Rightarrow \exists \; U = U(x,y)$ - force function such that

$$\vec{F} = \text{grad}\, U$$

Thus, $\dfrac{\partial U}{\partial x} = X = -k^2 x$

$\dfrac{\partial U}{\partial y} = Y = -k^2 y$

$\Rightarrow dU = -\dfrac{k^2}{2} d(x^2+y^2) \Rightarrow$

$\Rightarrow \boxed{U = -\dfrac{k^2}{2}(x^2+y^2)}$

iii) $L_{AB} = \int_{AB} \vec{F} d\vec{r} = U(B) - U(A) =$

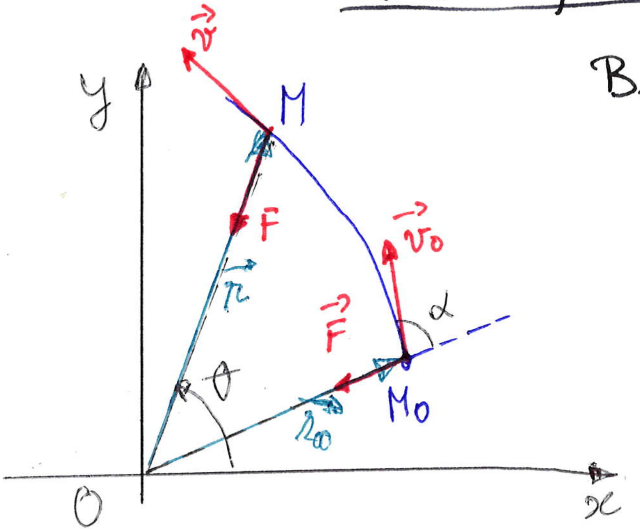
$$= -\frac{k^2}{2}(x^2+y^2)\Big|_A^B =$$

$$= -\frac{k^2}{2}\left\{\cos^2 kt + \frac{4}{k^2}\sin^2 kt\right\}\Big|_0^{\frac{\pi}{4k}} =$$

$$= -\frac{k^2}{2}\left\{\cos^2\frac{\pi}{4} + \frac{4}{k^2}\sin^2\frac{\pi}{4}\right\} + \frac{k^2}{2} = -\frac{k^2}{2}\left(\frac{1}{2} + \frac{4}{k^2}\cdot\frac{1}{2}\right) +$$

$$+ \frac{k^2}{2} = -\frac{k^2}{4} - 1 + \frac{k^2}{2} = \frac{k^2}{4} - 1.$$

Central forces



Binet's equation:

$$-\frac{mc^2}{r^2}\left[\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}\right]=\pm F(r,\theta)$$

"−" attractive force

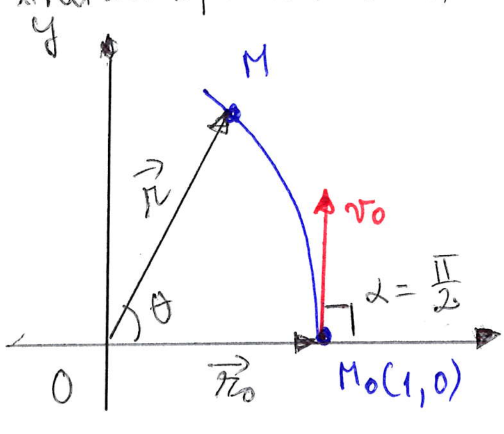
"+" repulsive force

$c = r_0 v_0 \sin\alpha = r^2\dot{\theta}$

$r(\theta_0) = r_0$

$$\left.\frac{d}{d\theta}\left(\frac{1}{r}\right)\right|_{\substack{\theta=\theta_0\\(t=t_0)}} = -\frac{1}{r_0}\cdot ctg\,\alpha$$

① Find the trajectory of a particle M moving under the action of the attractive force $F=\frac{2}{r^2}$. At the initial moment $(t=0)$ the particle was in $M_0(1,0)$ and its initial speed $v_0=\sqrt{3}$ was perpendicular on $\vec{r}_0$.



$c = r_0\cdot v_0\cdot \sin\alpha = 1\cdot\sqrt{3}\cdot 1=\sqrt{3}$

Binet's equation:

$$-\frac{3}{r^2}\left[\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}\right]=-\frac{2}{r^2}$$

or $$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}=\frac{2}{3} \quad (*)$$

– nonhomogenous linear differential equation.

initial conditions: $t=0 \Rightarrow r_0=1$

$$\left.\frac{d}{d\theta}\left(\frac{1}{r}\right)\right|_{t=0} = -\frac{1}{1}\cdot ctg\frac{\pi}{2}=0$$

We have to solve eq (*).

We attach the homogenous equation: $\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}=0 \quad (**)$

The characteristic equation of (**) is

$$\rho^2 + 1 = 0 \Rightarrow \rho_{1,2} = \pm i$$

Thus, the homogenous solution is:

$$\frac{1}{r_\theta} = C_1 \cos\theta + C_2 \sin\theta \quad (***)$$

The right hand part of (*) is a constant and then we consider a particular solution of the form:

$$\frac{1}{r_p} = A \text{ (const.)}. \quad (****)$$

In order to find A we use (****) in (*) $\Rightarrow A = \frac{2}{3}$.

Thus the solution of (*) is

$$\frac{1}{r} = \frac{1}{r_p} + \frac{1}{r_0} = \frac{2}{3} + C_1 \cos\theta + C_2 \sin\theta.$$

We use the initial conditions to find C_1 and C_2:

$$\left.\begin{array}{l} r_0 = 1 \Rightarrow \frac{2}{3} + C_1 = 1 \Rightarrow C_1 = \frac{1}{3} \\ \left.\frac{d}{d\theta}\left(\frac{1}{r}\right)\right|_{t=0} = 0 \Rightarrow C_1 = 0 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \frac{1}{r} = \frac{2}{3} + \frac{1}{3}\cos\theta = \frac{2+\cos\theta}{3}$$

$$\Rightarrow \boxed{r = \frac{3}{2+\cos\theta}}$$ – the trajectory in polar coordinates.