Analytic Geometry

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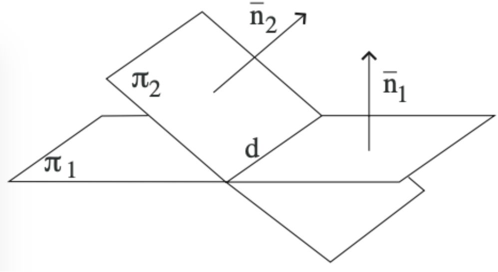
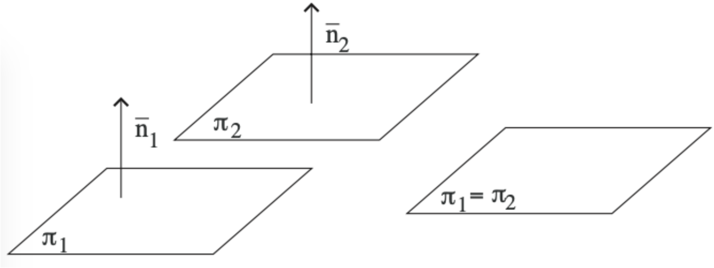
Recap...relative positions of 2 planes

- Let
$$\pi_1 : a_1x + b_1y + c_1z + d_1 = 0, \qquad \overline{n}_1(a_1, b_1, c_1) \neq \overline{0}$$
and
$$\pi_2 : a_2x + b_2y + c_2z + d_2 = 0, \qquad \overline{n}_2(a_2, b_2, c_2) \neq \overline{0}$$
be two planes, having the normal vectors $\overline{n}_1$, respectively $\overline{n}_2$.
- The intersection of these planes is given by the solution of the system of equations
$$\begin{cases} \pi_1 : a_1x + b_1y + c_1z + d_1 = 0 \\ \pi_2 : a_2x + b_2y + c_2z + d_2 = 0 \end{cases} . \tag{1}$$

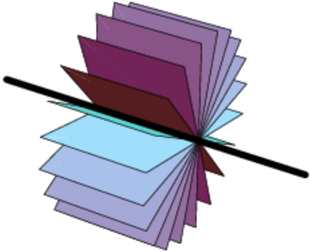

$\bar{n}_2$
π_2
$\bar{n}_1$
d
π_1

- If $\operatorname{rank}\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} = 2$, then the system (1) is compatible and the planes have a line in common. They are *incident*; $\pi_1 \cap \pi_2 = d$.
- If $\operatorname{rank}\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} = 1$, then the rows of the matrix are linearly dependent, which means that the normal vectors of the planes are linearly dependent. There are two possible situations:


$\bar{n}_2$
$\bar{n}_1$
π_2
π_1
$\pi_1 = \pi_2$

- If $\mathrm{rank}(A) = 1 < \mathrm{rank}(\overline{A}) = 2$, then the system (1) is not compatible, and the planes are *parallel*; $\pi_1 \parallel \pi_2$.
- If $\mathrm{rank}(A) = \mathrm{rank}(\overline{A}) = 1$, then the planes are *identical*; $\pi_1 = \pi_2$.

Bundle of planes



Given a line d, the set of all the planes containing the line d is said to be the *bundle* of planes through d.
Let us suppose that d is determined as the intersection of two planes π_1 and π_2, i.e.

$$d : \begin{cases} \pi_1 : a_1x + b_1y + c_1z + d_1 = 0 \\ \pi_2 : a_2x + b_2y + c_2z + d_2 = 0 \end{cases}, \qquad \text{with rank} \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} = 2.$$

- The equation of the bundle is

$$\lambda_1(a_1x + b_1y + c_1z + d_1) + \lambda_2(a_2x + b_2y + c_2z + d_2) = 0,$$

where $(\lambda_1, \lambda_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$.

- The above equation is sometimes written shortly as

$$\lambda_1\pi_1 + \lambda_2\pi_2 = 0, \qquad (\lambda_1, \lambda_2) \in \mathbb{R}^2 \setminus \{(0,0)\}. \tag{2}$$

- We remark that since not both λ_1 and λ_2 are zero at the same time, we may suppose that $\lambda_1 \neq 0$ and divide in (2) by λ_1; This gives the *reduced* equation of the bundle:

$$\pi_1 + \lambda\pi_2 = 0,$$

which contains all the planes through d, except π_2.

The relative positions between a line and a plane

Let

$$d : \begin{cases} x = x_1 + pt \\ y = y_1 + qt \\ z = z_1 + rt \end{cases}, \qquad p^2 + q^2 + r^2 > 0$$

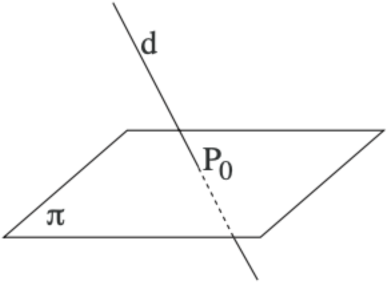
be a line of director vector $\overline{v}(p, q, r)$ and

$$\pi : ax + by + cz + d = 0, \qquad a^2 + b^2 + c^2 > 0$$

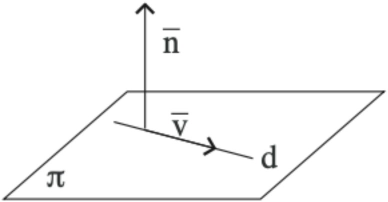
be a plane of normal vector $\overline{n}(a, b, c)$.
The intersection between d and π is given by the solutions of the equation

$$a(x_1 + pt) + b(y_1 + qt) + c(z_1 + rt) + d = 0. \tag{3}$$

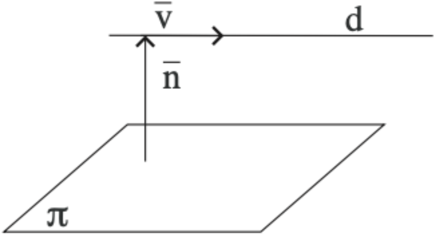
- If (3) has a unique solution t_0, then d and π have one intersection point P_0, corresponding to the parameter t_0. The line and the plane are *incident*; $d \cap \pi = \{P_0\}$.



- If (3) has infinitely many solutions, then d and π have the entire line d in common and d is *contained* into π; $d \subset \pi$. In this case, the normal vector $\overline{n}$ of π is orthogonal on the director vector $\overline{v}$ of d (then $\overline{n} \cdot \overline{v} = 0$, or $ap + bq + cr = 0$) and any point of d is contained into π.

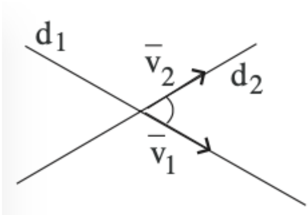


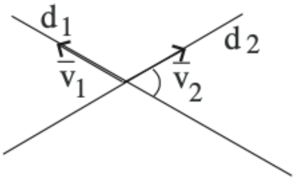
- If the equation $a(x_1+pt)+b(y_1+qt)+c(z_1+rt)+d=0$ has no solutions, then the line d is *parallel* to the plane π; $d \parallel \pi$.



The angle determined by two lines in 3D

Let d_1 and d_2 be two lines on $\mathcal{E}_3$, whose director vectors are $\overline{v}_1$ respectively $\overline{v}_2$. The *angle* determined by d_1 and d_2 is considered to be the acute or right angle formed by d_1 and d_2. It is denoted by $(\widehat{d_1, d_2})$.





It is easy to see that the measure of the angle determined by d_1 and d_2 is given by

$$m(\widehat{d_1, d_2}) = \begin{cases} m(\widehat{\overline{v}_1, \overline{v}_2}), & \text{if} \quad \overline{v}_1 \cdot \overline{v}_2 \geq 0 \\ \pi - m(\widehat{\overline{v}_1, \overline{v}_2}), & \text{if} \quad \overline{v}_1 \cdot \overline{v}_2 < 0 \end{cases} \tag{4}$$

Written with respect to the dot product of two vectors, the relations in (4) become

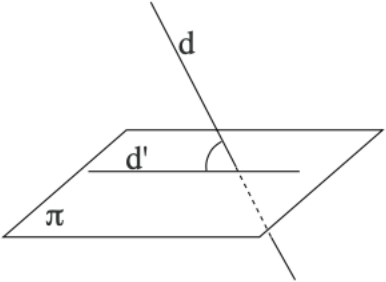
$$m(\widehat{d_1, d_2}) = \begin{cases} \arccos \dfrac{\overline{v}_1 \cdot \overline{v}_2}{|\overline{v}_1||\overline{v}_2|}, & \text{if} \quad \overline{v}_1 \cdot \overline{v}_2 \geq 0 \\ \pi - \arccos \dfrac{\overline{v}_1 \cdot \overline{v}_2}{|\overline{v}_1||\overline{v}_2|}, & \text{if} \quad \overline{v}_1 \cdot \overline{v}_2 < 0 \end{cases} \tag{5}$$

Remark: Two (concurrent or skew) lines d_1 and d_2, having the director vectors $\overline{v}_1(p_1, q_1, r_1)$, respectively $\overline{v}_2(p_2, q_2, r_2)$, are orthogonal if their director vectors are orthogonal.

$$d_1 \perp d_2 \iff \overline{v}_1 \cdot \overline{v}_2 = 0 \iff p_1 p_2 + q_1 q_2 + r_1 r_2 = 0. \tag{6}$$

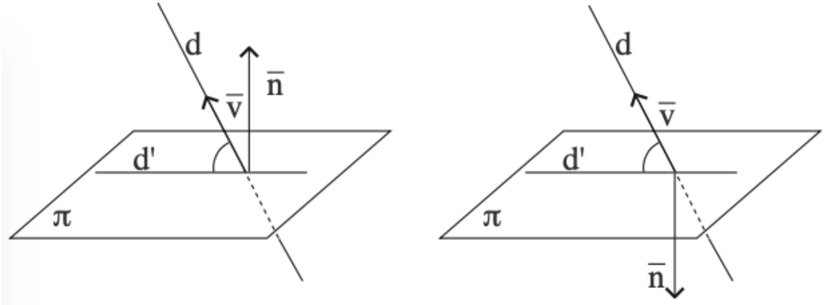
The angle determined by a line and a plane

Let d be a line of director vector $\overline{v}(p, q, r)$ and π be a plane of normal vector $\overline{n}(a, b, c)$. The *angle* determined by d and π, denoted by $(\widehat{d, \pi})$, is the angle determined by d and the orthogonal projection d' of d on π.



The measure of the angle determined by the line d and the plane π is given by

$$m(\widehat{d,\pi}) = \begin{cases} \dfrac{\pi}{2} - m(\widehat{\overline{v},\overline{n}}), & \text{if} \quad \overline{v}\cdot\overline{n} \geq 0 \\ m(\widehat{\overline{v},\overline{n}}) - \dfrac{\pi}{2}, & \text{if} \quad \overline{v}\cdot\overline{n} < 0 \end{cases} \tag{7}$$



Remarks

- The formula (7) has the alternative form

$$m(\widehat{d,\pi}) = \begin{cases} \frac{\pi}{2} - \arccos\frac{\overline{v}\cdot\overline{n}}{|\overline{v}||\overline{n}|}, & \text{if} \quad \overline{v}\cdot\overline{n} \geq 0 \\ \arccos\frac{\overline{v}\cdot\overline{n}}{|\overline{v}||\overline{n}|} - \frac{\pi}{2}, & \text{if} \quad \overline{v}\cdot\overline{n} < 0 \end{cases} \tag{8}$$

- The line d is parallel to the plane π if the vector $\overline{v}$ is orthogonal to $\overline{n}$, hence

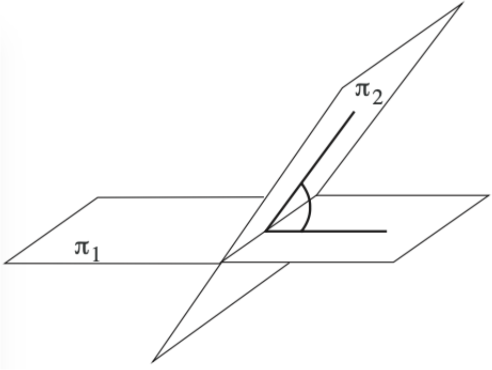
$$d \parallel \pi \iff \overline{v}\cdot\overline{n} = 0 \iff pa + qb + rc = 0. \tag{9}$$

- The line d is orthogonal to the plane π if $\overline{v}$ is parallel to $\overline{n}$. Then

$$d \perp \pi \iff \overline{v} \parallel \overline{n} \iff \exists\, \alpha \in \mathbb{R}^* : \overline{n} = \alpha\overline{v}. \tag{10}$$

The Angle determined by two planes

Let π_1 and π_2 be two planes of normal vectors $\overline{n}_1(a_1, b_1, c_1)$, respectively $\overline{n}_2(a_2, b_2, c_2)$. The *angle* determined by π_1 and π_2, denoted by $(\widehat{\pi_1, \pi_2})$, is the acute or right dihedral angle of π_1 and π_2.



The measure of the angle determined by π_1 and π_2 is given by

$$m(\widehat{\pi_1, \pi_2}) = \begin{cases} m(\widehat{\overline{n}_1, \overline{n}_2}), & \text{if} \quad \overline{n}_1 \cdot \overline{n}_2 \geq o \\ \pi - m(\widehat{\overline{n}_1, \overline{n}_2}), & \text{if} \quad \overline{n}_1 \cdot \overline{n}_2 < o \end{cases} \tag{11}$$

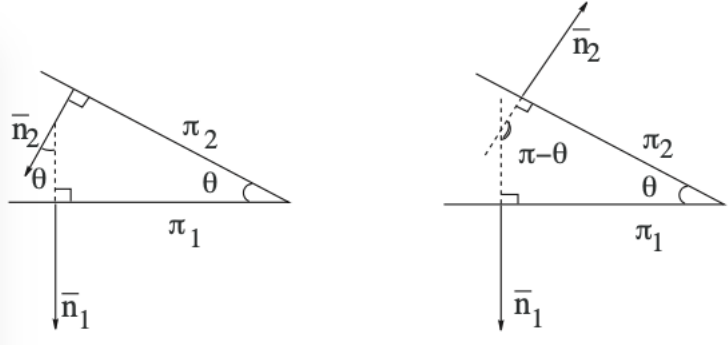
The formula (11) can be written in the form

$$m(\widehat{\pi_1, \pi_2}) = \begin{cases} \arccos \dfrac{\overline{n}_1 \cdot \overline{n}_2}{|\overline{n}_1||\overline{n}_2|}, & \text{if} \quad \overline{n}_1 \cdot \overline{n}_2 \geq o \\ \pi - \arccos \dfrac{\overline{n}_1 \cdot \overline{n}_2}{|\overline{n}_1||\overline{n}_2|}, & \text{if} \quad \overline{n}_1 \cdot \overline{n}_2 < o \end{cases} \tag{12}$$

Remark: The planes π_1 and π_2 are orthogonal if and only if their normal vectors are orthogonal, hence

$$\pi_1 \perp \pi_2 \iff \overline{n}_1 \cdot \overline{n}_2 = 0 \iff a_1 a_2 + b_1 b_2 + c_1 c_2 = 0. \tag{13}$$

A diagram explaining the previous slide



Metric problems concerning distances. The distance between a point and a plane

Let $P_0(x_0, y_0, z_0)$ be a point and $\pi : ax + by + cz + d = 0$ (with $a^2 + b^2 + c^2 > 0$) be a plane in $\mathcal{E}_3$.

Theorem

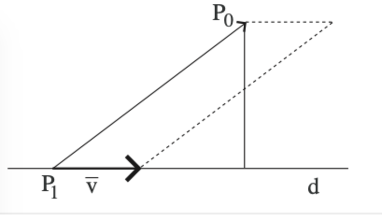
The distance from the point $P_0(x_0, y_0, z_0)$ to the plane $\pi : ax + by + cz + d = 0$ is given by

$$d(P_0, \pi) = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}. \tag{14}$$

The distance between a point and a line

Given a point $P_0(x_0, y_0, z_0)$ and a line $d : \begin{cases} x = x_1 + pt \\ y = y_1 + qt \\ z = z_1 + rt \end{cases}$, $t \in \mathbb{R}$, with $p^2 + q^2 + r^2 > 0$, we present two ways to find the distance from P_0 to d.

- Let $\overline{v}(p, q, r)$ be the director vector of d and $P_1(x_1, y_1, z_2)$ be an arbitrary point on d. The distance from P_0 to d is the altitude of the parallelogram determined by $\overline{v}$ and $\overline{P_1P_0}$.



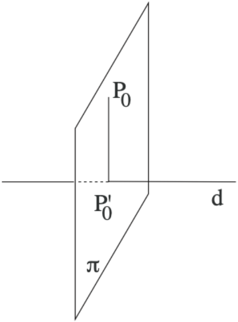
This altitude can be expressed using the area of the parallelogram and one has

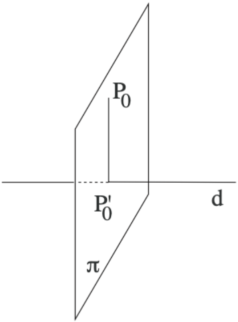
$$d(P_0, d) = \frac{|\overline{v} \times \overline{P_1P_0}|}{|\overline{v}|}. \tag{15}$$

- Let π be the plane passing through P_0 and orthogonal on d. Its equation is

$$\pi : p(x - x_0) + q(y - y_0) + r(z - z_0) = 0.$$

Let P_0' be the intersection point of π and d; $\{P_0'\} = d \cap \pi$.





The coordinates of the point P_0' correspond to the parameter t_0, solution of the equation

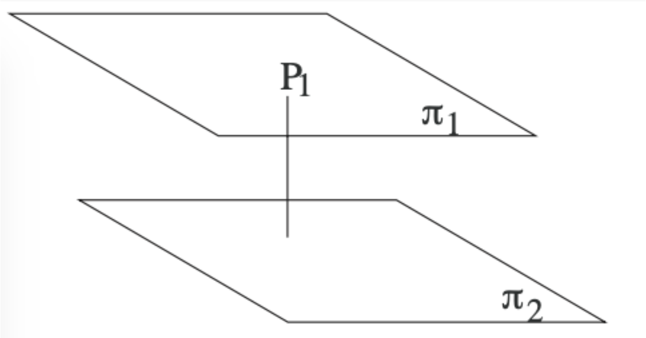
$$p(x_1 + pt - x_0) + q(y_1 + qt - y_0) + r(z_1 + rt - z_0) = 0.$$

Finally, $d(P_0, d) = d(P_0, P_0')$.

The distance between two parallel planes

Let π_1 and π_2 be two parallel planes. Choose an arbitrary point $P_1 \in \pi_1$. Then

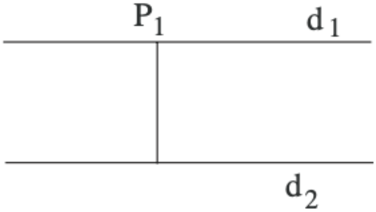
$$d(\pi_1, \pi_2) = d(P_1, \pi_2).$$



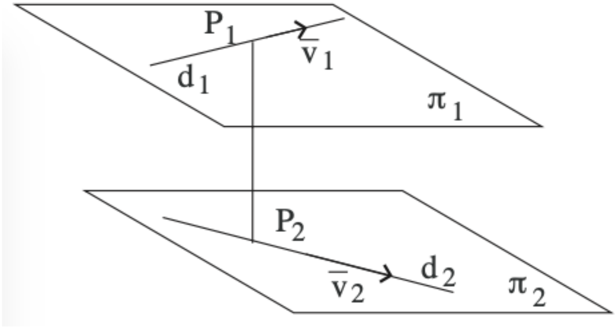
The distance between two lines

Let d_1 and d_2 be two lines in the 3-space.

- If the lines are identical or concurrent, then $d(d_1, d_2) = 0$.
- If the lines are parallel, it is enough to choose an arbitrary point $P_1 \in d_1$ and $d(d_1, d_2) = d(P_1, d_2)$.



- If d_1 and d_2 are skew, there exists a unique line which is orthogonal on both d_1 and d_2 and intersects both d_1 and d_2. The length of the segment determined by these intersection points is the distance between the skew lines.
  Suppose that
  $$d_1 : \begin{cases} x = x_1 + p_1 t \\ y = y_1 + q_1 t \\ z = z_1 + r_1 t \end{cases}, t \in \mathbb{R} \text{ and } d_2 : \begin{cases} x = x_2 + p_2 s \\ y = y_2 + q_2 s \\ z = z_2 + r_2 s \end{cases}, s \in \mathbb{R}$$
  are, respectively, the parametric equations of the lines of director vectors $\overline{v}_1(p_1, q_1, r_1) \neq \overline{0}$, respectively $\overline{v}_2(p_2, q_2, r_2) \neq \overline{0}$.
  One can determine the equations of two parallel planes $\pi_1 \parallel \pi_2$, such that $d_1 \subset \pi_1$ and $d_2 \subset \pi_2$. The normal vector $\overline{n}$ of these planes has to be orthogonal on both $\overline{v}_1$ and $\overline{v}_2$, hence $\overline{n} = \overline{v}_1 \times \overline{v}_2$.


P_1
$\bar{v}_1$
d_1
π_1
P_2
d_2
$\bar{v}_2$
π_2

Then $\overline{n}(A, B, C)$, with $A = \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix}$, $B = \begin{vmatrix} r_1 & p_1 \\ r_2 & p_2 \end{vmatrix}$ and $C = \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix}$.
The equations of the planes π_1 and π_2 are:

$$\pi_1 : A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$$

$$\pi_2 : A(x - x_2) + B(y - y_2) + C(z - z_2) = 0.$$

Now, the distance between d_1 and d_2 is the distance between the parallel planes π_1 and π_2; $d(d_1, d_2) = d(\pi_1, \pi_2)$, and one has the following theorem.

Theorem

The distance between two skew lines d_1 and d_2 is given by

$$d(d_1, d_2) = \frac{|A(x_1 - x_2) + B(y_1 - y_2) + C(z_1 - z_2)|}{\sqrt{A^2 + B^2 + C^2}}. \tag{16}$$

Thank you very much for your attention!