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Homework 7

Ex1: $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$
choose $a \in \mathbb{R}$ randomly and prove that f is continous at a.
choose a sequence $(b_n) \in A$ with $\lim_{n\to\infty} b_n = a$, randomly
$\cdot \lim_{n\to\infty} f(b_n) = \lim_{n\to\infty} b_n^2 = a^2 = f(a)$ $\Big\} \Rightarrow f$ is continous at a $\Big\} \Rightarrow$
(a_n) random, a-random

$\Rightarrow f$ is c. on $\mathbb{R}$

Ex 2: a) $f(x) = \begin{cases} x: x \in \mathbb{Q} \\ -x: x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

f is c. at $\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in A$ with $|x - a| < \delta$, it holds $|f(x) - f(a)| < \varepsilon$

$\boxed{a=0}$

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in A$ with $|x| < \delta$ it holds $|f(x) - f(0)| < \varepsilon$

$f(x) = \begin{cases} x: x \in \mathbb{Q} \\ -x: x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \Rightarrow \underset{0 \in \mathbb{Q}}{f(0) = 0}$ $\quad |f(x)| < \varepsilon$

① Choose $\varepsilon > 0$ randomly

② $\} \; |f(x)| < \varepsilon$ ⑦ $\quad |f(x)| = \begin{cases} |x|: x \in \mathbb{Q} \\ |-x|: x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} = |x|$

③ $\exists \delta := \varepsilon \; \forall x \in A$ with $|x| < \delta$ it holds $|f(x)| = |x| < \delta = \varepsilon$

for $\varepsilon > 0$, $\exists \delta := \varepsilon$ s.t. $\forall x \in A$ with $|x| < \delta$, it hold $|f(x)| < \varepsilon$ $\Big\} \Rightarrow$
ε-random

$\Rightarrow \forall \varepsilon > 0 \; \overset{\delta = \varepsilon}{\exists} \delta$ f is c. at 0

Conclusion f is c. only at 0 and disc. on $\mathbb{R} \setminus \{0\}$
it is a function with just a single continous point.

b) f is discontinuous on $\mathbb{R}\setminus\{0\}$

choose $b \in \mathbb{R}\setminus\{0\}$ randomly

$\exists (s_n) \subseteq \mathbb{Q}$ s.t. $\lim_{n\to\infty} s_n = b$ $\lim_{n\to\infty} f(s_n) = \lim_{n\to\infty} s_n = b$

$\exists (t_n) \subseteq \mathbb{R}\setminus\mathbb{Q}$ s.t. $\lim_{n\to\infty} t_n = b$, $\lim_{n\to\infty} f(t_n) = \lim_{n\to\infty} -t_n = -b$ $\neq$

? $-b = b \Leftrightarrow 2b = 0 \Leftrightarrow b = 0 \Rightarrow f$ is not c. at b $\}\Rightarrow$ b random

on $\mathbb{R}\setminus\{0\}$ but $b \neq 0$

<u>Ex 3:</u> a) $f: (-\infty, 0] \to \mathbb{R}$, $f(x) = \begin{cases} \sin x : x \in (-\infty, 0) \\ \pi : x = 0 \end{cases}$

is c on $(-\infty, 0)$ not at 0 $\lim_{x\nearrow 0} f(x) = \sin 0 = 0 \neq f(0) = \pi$

b) $f: [-1, 2] \cup \{4\} \to \mathbb{R}$ $f(x) = \begin{cases} 2x+3 : x \in [-1, 2] \\ 0 : x = 4 \end{cases}$

f is c on $[-1, 2]$ c. at 4 as $4 \in \text{Iso} A$

Ex 4: a) $f: \mathbb{R} \to \mathbb{R}$ $f(x) = \begin{cases} \frac{\sin x^2}{|x|} : x \neq 0 & \text{I} \\ 0 : x = 0 & \text{II} \end{cases}$

I c. on $\mathbb{R}\setminus\{0\}$ as an elementary function

II c. at 0 $\lim_{x\to 0} \frac{\sin x^2}{x^2} \cdot \frac{x^2}{|x|} = 0$

b) $f: \mathbb{R} \to \mathbb{R}$ $f(x) = \begin{cases} e^{x-1} : x \in (0, \infty) \\ 0 : x = 0 \\ x^2 + 2x + \sin x : x \in (-\infty, 0) \end{cases}$

c on $\mathbb{R}\setminus\{0\}$ as an elementary function

$\lim_{x\searrow 0} e^{x-1} = \frac{1}{e} \neq 0 = f(0) \Rightarrow$ not c at 0

c)

c) $f: \mathbb{R} \to \mathbb{R}$ $f(x) = \begin{cases} \sin x : x \in \mathbb{Q} \\ \cos x : x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

$\to$ f is c on $\mathcal{C}$ (proof ε and $\delta \to$ Ex2.)

$\to$ f is not c. on $\mathbb{R} \setminus \mathcal{C}$

$\sin x = \cos x$

$\mathcal{C} = \{x \in \mathbb{R} : \sin x = \cos x\} = \{\frac{\pi}{4} + k\pi : k \in \mathbb{Z}\} \to$ with the def

$(b_n) \subseteq \mathbb{Q}$ $(t_n) \subseteq \mathbb{R} \setminus \mathbb{Q}$

d) $f: [-2, 1] \cup \{3\} \to \mathbb{R}$

Ex6: $0 < a < b \in \mathbb{R}$ $f: \mathbb{R} \setminus \{0, 1\} \to \mathbb{R}$

$f(x) = \left(\frac{b^x - a^x}{x(b-a)}\right)^{\frac{1}{x-1}}$, $\forall x \in \mathbb{R} \setminus \{0, 1\}$

a) $L = \lim\limits_{x \to \infty} \left(\frac{b^x - a^x}{x(b-a)}\right)^{\frac{1}{x-1}}$ $x \to \infty \Rightarrow \frac{1}{x-1} \to 0$ $0 < a < b$

$\Rightarrow b-a > 0 \to$ constant

$\Rightarrow \lim\limits_{x \to \infty} \left(\frac{1}{b-a}\right)^{\frac{1}{x-1}} = e^0 = 1$

$L = 1 \cdot \lim\limits_{x \to \infty} \left(\frac{b^x - a^x}{x}\right)^{\frac{1}{x-1}}$

$\lim\limits_{x \to \infty} \frac{b^x - a^x}{x} = \lim\limits_{x \to \infty} \frac{b^x}{x} \underbrace{[1 - (\frac{a}{b})^x]}_{1-0=1}$ $0 < a < b \Rightarrow 0 < \frac{a}{b} < 1$

$\to \lim\limits_{x \to \infty} (\frac{a}{b})^x = 0$

$= \lim\limits_{x \to \infty} \frac{b^x}{x} = \begin{cases} \infty : b > 1 \\ 0 : b \leq 1 \end{cases}$ because $\lim\limits_{x \to \infty} b^x = \begin{cases} \infty : b > 1 \\ 1 : b = 1 \\ 0 : |b| < 1 \\ \nexists : b \leq -1 \end{cases}$

$\Rightarrow \lim\limits_{x \to \infty} \left(\frac{b^x - a^x}{x}\right)^{\frac{1}{x-1}} = \infty^0$ or 0^0

$= \lim\limits_{x \to \infty} \left(\frac{b^x}{x}\right)^{\frac{1}{x-1}} \cdot \left(1 - (\frac{a}{b})^x\right)^{\frac{1}{x-1}} = \lim\limits_{x \to \infty} \left(\frac{b^x}{x}\right)^{\frac{1}{x-1}} =$

$\lim\limits_{x \to \infty} \left(\frac{b^x}{x}\right)^{\frac{1}{x-1}} = \lim\limits_{x \to \infty} e^{\ln\left(\frac{b^x}{x}\right) \frac{1}{x-1}} = e^{\lim\limits_{x \to \infty} \ln\left(\frac{b^x}{x}\right) \frac{1}{x-1}}$

$$\lim_{x\to\infty} \ln\left(\frac{b^x}{x}\right)^{\frac{1}{x-1}} = \lim_{x\to\infty} \frac{1}{x-1} \ln\left(\frac{b^x}{x}\right)$$

if $\begin{cases} b>1 \Rightarrow \frac{b^x}{x} \to \infty \Rightarrow \ln \frac{b^x}{x} \to \infty \\ b\le 1 \Rightarrow \frac{b^x}{x} \to 0 \Rightarrow \ln \frac{b^x}{x} \to -\infty \end{cases}$

$$= \lim_{x\to\infty} \frac{\ln(b^x) - \ln x}{x-1} = \lim_{x\to\infty} \frac{x\ln b - \ln x}{x-1} \overset{L'H}{=}$$

$$= \lim_{x\to\infty} \frac{\ln b - \frac{1}{x}}{1} = \lim_{x\to\infty} \left[\ln b - \frac{1}{x}\right] = \ln b$$

$$\lim_{x\to\infty} \left(\frac{b^x}{x}\right)^{\frac{1}{x-1}} = e^{\lim_{x\to\infty} \ln\left(\frac{b^x}{x}\right)^{\frac{1}{x-1}}} = e^{\ln b} = b$$

$c = b \quad \lim_{x\to-\infty} F(x) = a$

EX5. a) $f:[1,3]\to\mathbb{R} \qquad f(x) = \begin{cases} \sqrt{a^2-2ax+x^2}, & x\in[1,2] \\ 3a+2x, & x\in(2,3] \end{cases}$

f continuous on $[1,3] \Leftrightarrow \lim_{x\nearrow 2} f(x) = f(x) = \lim_{x\searrow 2} f(x)$

$$\lim_{x\nearrow 2} f(x) = \lim_{x\to 2} \sqrt{a^2-2ax+x^2} = \sqrt{4-4a+a^2} = \sqrt{(2-a)^2} = (2-a)$$

$$\lim_{x\searrow 2} f(x) = \lim_{x\to 2} 3a+2x = 3a+4$$

$$f(x) = \sqrt{a^2-2a\cdot 2+4} = \sqrt{(a-2)^2} = a-2$$

$a-2 = 3a+4 \qquad 2a = -6 \quad a = -3$

b) $f:(0,\pi)\to\mathbb{R} \quad f(x) = \begin{cases} e^{3x}, & x\in(0,1] \\ a\frac{\sin(x-1)}{x^2-5x+4}, & x\in(1,\pi) \end{cases}$

f continuous on $(0,\pi) \Leftrightarrow \lim_{x\nearrow 1} f(x) = f(1) = \lim_{x\searrow 1} f(x)$

$$\lim_{x\nearrow 1} f(x) = \lim_{x\to 1} e^{3x} = e^3 = f(1)$$

$$\lim_{x\searrow 1} f(x) = \lim_{x\to 1} a\frac{\sin(x-1)}{x^2-5x+4} = a\cdot\frac{\sin 0}{1-5+4} = a\cdot 0 = 0$$

$0 \ne e^3 \Rightarrow f \ne$ continuous