Dynamics of the particle

Newton's law: $m \cdot \frac{d^2\vec{r}}{dt^2} = \vec{F}(X,Y,Z)$

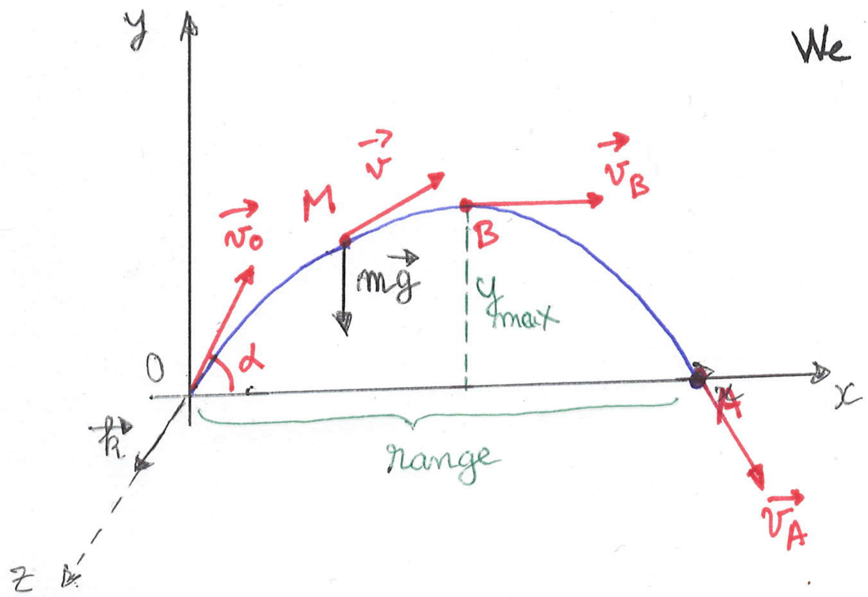
Vectorial form; force

or Cartesian coordinates:
$$\begin{cases} m\ddot{x} = X(x,y,z,\dot{x},\dot{y},\dot{z},t) \\ m\ddot{y} = Y(x,y,z,\dot{x},\dot{y},\dot{z},t) \\ m\ddot{z} = Z(x,y,z,\dot{x},\dot{y},\dot{z},t) \end{cases}$$

Polar coordinates:
$$\begin{cases} m(\ddot{r} - r\dot{\theta}^2) = F_r \\ \frac{m}{r}\cdot\frac{d}{dt}(r^2\dot{\theta}) = F_\theta \end{cases}$$

Frenet:
$$\begin{cases} m\frac{dv}{dt} = F_\tau \\ m\frac{v^2}{R} = F_n \end{cases}$$

① A projectile is launched from the origin O of the frame Oxyz (from the Earth) with the initial velocity $\vec{v}_0$. The angle between the initial velocity and the horizontal is α. Study the motion of the projectile in vacuum under the action of gravity. Find the equation of motion, the trajectory, the maximum height, the impact velocity and the range.



We choose Oxyz such that $\vec{v}_0 \in (Oxy)$

$\vec{k} \perp m\vec{g}$, $\vec{k} \perp \vec{v}_0$ $\Rightarrow$ plane motion in Oxy?

Differential equation of motion:

(1) $m\cdot\vec{a} = m\vec{g} \;|:m$

$\Rightarrow \vec{a} = \vec{g} \;|\cdot\vec{k} \Rightarrow$

$$\Rightarrow \frac{d\vec{v}}{dt}\cdot\vec{m} = 0 \Rightarrow \frac{d}{dt}(\vec{m}\cdot\vec{v}) = 0 \Rightarrow \vec{m}\cdot\vec{v} = \underbrace{\vec{m}\cdot\vec{v_0}}_{\overset{\parallel}{0}} = 0$$

$\vec{m}\perp\vec{v_0}$

$$\Rightarrow \vec{m}\cdot\frac{d\vec{r}}{dt} = 0 \Rightarrow \frac{d}{dt}(\vec{m}\cdot\vec{r}) = 0 \Rightarrow \underline{\vec{r}\cdot\vec{m}} = \vec{r_0}\cdot\vec{m} = \underline{const = 0}$$

Thus, $(x, y, z)\cdot(0, 0, 1) = 0 \Rightarrow \boxed{z=0}$ (the motion takes place in Oxy).

- From (1) $\Rightarrow \begin{cases} m\cdot\ddot{x} = 0 \\ m\cdot\ddot{y} = -mg \end{cases} \Rightarrow \begin{cases} x(t) = c_1 t + c_2 \\ y(t) = -\frac{gt^2}{2} + c_3 + c_4 \end{cases}$ (2)

Initial condition: $\begin{cases} x(0) = 0;\ y(0) = 0 \\ \dot{x}(0) = v_0\cos\alpha;\ \dot{y}(0) = v_0\sin\alpha \end{cases}$ (3)

Using (3) in (2) we obtain: (the equations of motion) $\begin{cases} x(t) = v_0\cos\alpha\cdot t \\ y(t) = -\frac{gt^2}{2} + v_0\sin\alpha\cdot t \end{cases}$ (4)

- In order to obtain the maximum height we impose:

$$\vec{v_B} \parallel Ox$$

$$\vec{v_B}(\dot{x}_B, \dot{y}_B) \parallel Ox \Rightarrow \dot{y}_B = 0 \Rightarrow \dot{y}_B = -g\cdot t_B + v_0\sin\alpha$$

$$\Rightarrow t_B = \frac{v_0\sin\alpha}{g} \quad (5)$$ (ascending (climb) time).

$$y_B = \frac{-g t_B^2}{2} + v_0\sin\alpha\cdot t_B =$$

$$= -\frac{g}{2g^2}v_0^2\sin^2\alpha + v_0\cdot\frac{v_0\sin\alpha}{g}\cdot\sin\alpha = \frac{v_0^2\sin^2\alpha}{2g}.$$

We have $B\left(x_B = \frac{v_0^2\sin 2\alpha}{2g}, y_B = \frac{v_0^2\sin^2\alpha}{2g}\right)$ – the maximum height point. (6)

- Impact condition: $y_A = 0 \Rightarrow -\frac{gt_A^2}{2} + v_0\sin\alpha\cdot t_A = 0$

$$t_A\left(-\frac{gt_A}{2} + v_0\sin\alpha\right) = 0$$

$\rightarrow t_A = 0$! (not OK)

$\rightarrow t_A = \frac{2v_0\sin\alpha}{g}$ (7)

For $t = t_A$ we obtain the range:

$$x_A(t_A) = \frac{v_0^2 \sin 2\alpha}{g} \qquad (8)$$

The impact velocity is:

$$\underset{\sim}{v_A} = \sqrt{\dot{x}_A^2 + \dot{y}_A^2} = \sqrt{v_0^2 \cos^2\alpha + g^2 t_A^2 + v_0^2 \sin^2\alpha - 2 g t_A v_0 \sin\alpha} =$$

$$= \underset{\sim}{v_0}$$

where $\begin{cases} \dot{x}(t) = v_0 \cos\alpha \\ \dot{y}(t) = -gt + v_0 \sin\alpha \end{cases}$

Remark: The launching speed is equal with the impact speed.

(2) A particle of mass $m = \frac{1}{3}$ kg has the equations of motion in Oxy:

$$\begin{cases} x = 3a \cos 3t \\ y = a \sin 3t \end{cases}, \quad a \in \mathbb{R}$$

Find the force $\vec{F}$ acting on the particle and show that the direction of $\vec{F}$ is $-\vec{r}$.

Solution:

Differential equation of motion: $m\ddot{\vec{r}} = \vec{F}$

But $\vec{r} = (3a \cos 3t, a \sin 3t) \Rightarrow \dot{\vec{r}} = (-9a \cos 3t, 3a \cos 3t)$

$\Rightarrow \ddot{\vec{r}} = (-27a \cos 3t, -9a \sin 3t)$

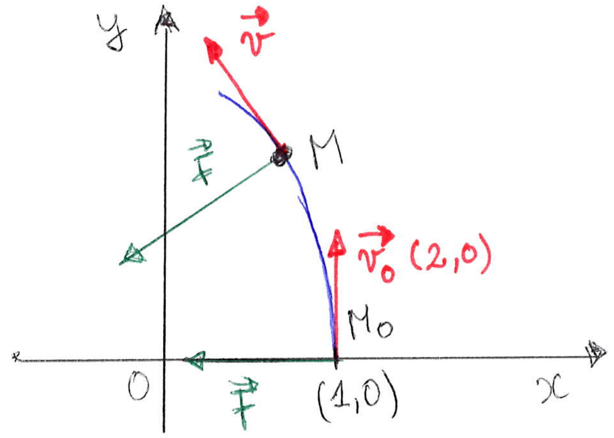
Thus, $\vec{F} = \frac{1}{3}(-27a \cos 3t \cdot \vec{i} - 9a \sin 3t \cdot \vec{j}) =$

$= -9a \cos 3t \cdot \vec{i} - 3a \sin 3t \cdot \vec{j}$

$$\frac{F_x}{x} = \frac{F_y}{y} \Rightarrow \frac{3a \cos 3t}{-9a \cos 3t} = \frac{a \sin 3t}{-3a \sin 3t} = -\frac{1}{3} \Rightarrow$$

$\Rightarrow \vec{F} \parallel -\vec{r}.$

③ A particle ($m=1$) moves in the plane xOy under the action of the force $\vec{F}(X,Y)=(-16x,-4y)$. At the initial moment ($t=0$) we have: $x_0=1$, $y_0=0$, $v_{0x}=0$, $v_{0y}=2$.
Find the trajectory of the particle.



$$\begin{cases} m\cdot\ddot{x}=-16x \\ m\ddot{y}=-4y \end{cases} \quad (1)$$

$m=1$

$\ddot{x}+16x=0 \Rightarrow r^2+16=0 \Rightarrow$
$\Rightarrow r_{1,2}=\pm 4i \quad (2)$

$\ddot{y}+4y=0 \Rightarrow r^2+4=0 \Rightarrow$
$\Rightarrow r_{1,2}=\pm 2i \quad (3)$

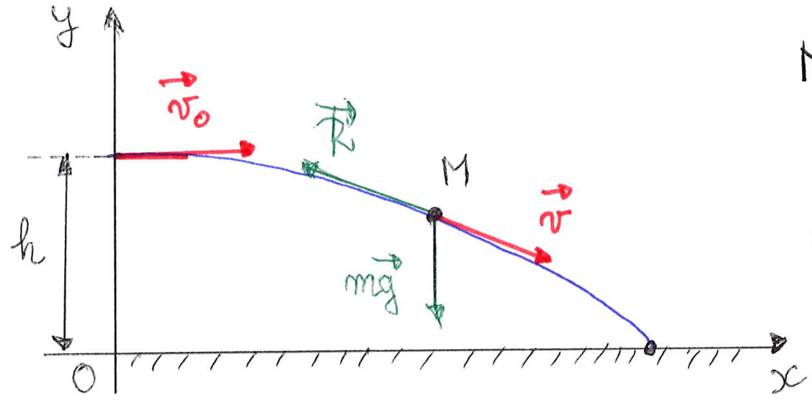
$$(2),(3) \Rightarrow \begin{cases} x(t)=A\cos 4t+B\sin 4t \\ y(t)=C\cos 2t+D\sin 2t \end{cases}$$

$$\begin{cases} \dot{x}(t)=-4A\sin 4t+4B\cos 4t \\ \dot{y}(t)=-2C\sin 2t+2D\cos 2t \end{cases}$$

At $t=0$: $\begin{cases} A=1 \quad ; \quad B=0 \\ C=0 \quad ; \quad D=1 \end{cases} \Rightarrow \begin{cases} x=\cos 4t \\ y=\sin 2t \end{cases}$

But $2\sin^2 2t=1-\cos 4t \Rightarrow 2y^2=1-x \Rightarrow \boxed{x=1-2y^2}$
(parabola).

(4) A particle of mass m is launched from the height h from the Earth's surface in the horizontal direction with the speed v_0. The drag force (due to the air resistance) is $\vec{R} = -k m \vec{v}$, $k>0$. Find the equations of motion.



Newton's laws:

$$m\vec{a} = m\vec{g} + \vec{R} \quad (1)$$

where $\vec{a}(\ddot{x}, \ddot{y})$, $m\vec{g}(0, -mg)$

$\vec{R}(-km\dot{x}, -km\dot{y})$.

$$(1) \Rightarrow \begin{cases} m\ddot{x} = -km\dot{x} \\ m\ddot{y} = -km\dot{y} - m\cdot g \end{cases} \quad (2) \Rightarrow \begin{cases} \ddot{x} + k\dot{x} = 0;\ x(0)=0;\ \dot{x}(0)=v_0 & (3a) \\ \ddot{y} + k\dot{y} = -g;\ y(0)=h;\ \dot{y}(0)=0 & (3b) \end{cases}$$

$$(3a) \Rightarrow \ddot{x} + k\dot{x} = 0 \Rightarrow r^2 + kr = 0 \Rightarrow \begin{cases} r_1 = 0 \\ r_2 = -k \end{cases}$$

$$\Rightarrow x(t) = C_1 + C_2 e^{-kt}$$

$$\left.\begin{aligned} x(0) = 0 &\Rightarrow C_1 + C_2 = 0 \\ \dot{x}(0) = v_0 &\Rightarrow -kC_2 = v_0 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} C_1 &= \frac{v_0}{k} \\ C_2 &= -\frac{v_0}{k} \end{aligned}\right\} \Rightarrow \boxed{x(t) = \frac{v_0}{k}\left(1 - e^{-kt}\right)} \quad (4)$$

$$(3b) \Rightarrow \ddot{y} + k\dot{y} = -g \quad (5) \quad (r^2 + kr = 0)$$

Attach the homogenous eq. $\ddot{y} + k\dot{y} = 0 \Rightarrow y_0 = C_1 + C_2 e^{-kt}$ (6)

Search the particular solution: $\boxed{y_p(t) = A\cdot t}$ (7) ($r_1 = 0$ is the root of the characteristic equation).

Use (7) in (5) $\Rightarrow$

$$kA = -g \Rightarrow A = -\frac{g}{k}$$

Thus, $y(t) = y_0(t) + y_p(t) = -\frac{g}{k}t + C_1 + C_2 e^{-kt}$

$$\left.\begin{aligned} \dot{y}(0) = 0 &\Rightarrow C_2 = -\frac{g}{k^2} \\ y(0) = h &\Rightarrow C_1 = h + \frac{g}{k^2} \end{aligned}\right\} \Rightarrow \boxed{y(t) = h - \frac{g}{k}t + \frac{g}{k^2}\left(1 - e^{-kt}\right)} \quad (8)$$

The eq. of motion are (4) and (8).