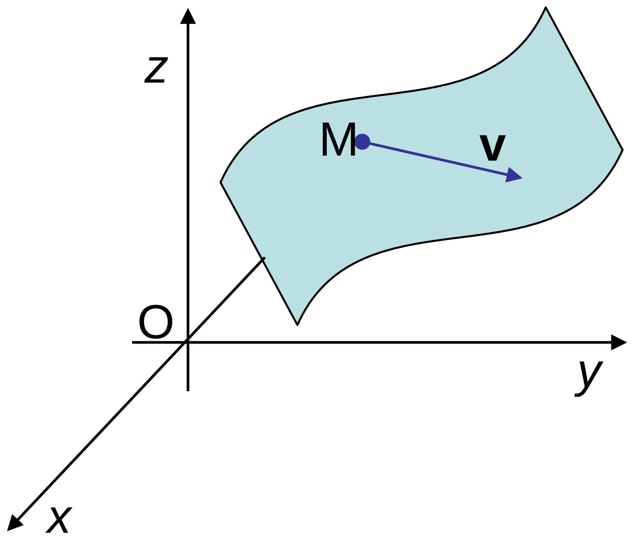
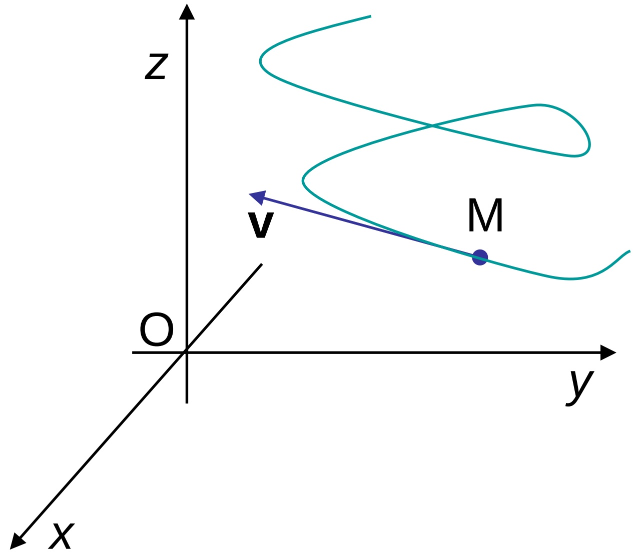
12. Dynamics of material point subject to constraints

Until now, we have studied the motion of the free particle, we have considered only the directly applied external forces. However, there are cases when the particle is forced to move on a specified manifold **(curve** or **surface).**





I. On a surface. The coordinates of the particle satisfy the **equation of the surface.**

$$h\left(t,\vec{r}\right)=0 \tag{12.1}$$

II. On a curve. The coordinates of the particle satisfy the **equation of the curve** (given as a intersection of two surfaces).

$$\begin{cases} h_1(t, \vec{r}) = 0 \\ h_2(t, \vec{r}) = 0 \end{cases} \tag{12.2}$$

In both cases the particle has geometrical constraints

Definition:

A **constraint** is a restriction of geometric kind or kinematic kind imposed to the moving particle.

A **geometric constraint** is a restriction on the particle‘s position, a relation between the particle‘s coordinates and time.

A **kinematic constraint** is a restriction on the particle‘s velocity, a relation between the coordinates of the point and the components of the velocity and time.

The motion of a particle M(m) on a tridimensional manifold Σ (curve or surface) has to take into account the interaction between the particle and the manifold, the last one acting on the particle that has the tendency to leave it.

The Cauchy‘s postulate: There is a force **R** whose action on the particle is perfectly equivalent to the action of the manifold Σ on the particle. This, allows us to replace these constraints by constraint forces (reactions); in this case, the particle *P* may be considered to be a free particle, subjected to the action of the given as well as of the constraint forces, so that one can use the considerations of the previous subsection.

(the axiom of liberation from constraints, the axiom of constraint forces)

Using the axiom of liberation from constraints, we introduce the constraint force **R** so that, the equation of motion becomes:

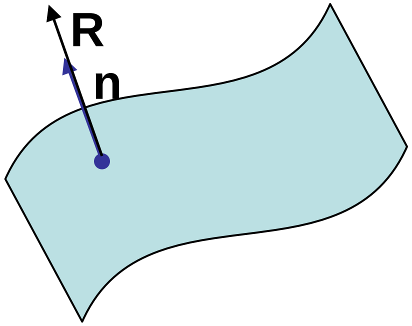
$$m\vec{a} = \vec{F} + \vec{R} \tag{12.3}$$

$\vec{F}$ - is the resultant of the all directly applied forces (it is given)

$\vec{R}$ - it is an unknown force; it has to be determined and it is called the reaction or constraint force. The reaction (constraint) force depends on the manifold.

We say that a manifold (curve sau surface) is ideal or smooth if the reactive force acting on a material point during motion is orthogonal to the constraint (there is no friction). The constraint force **R** has only the orthogonal component (**R** II **n**) and the tangential one is zero.

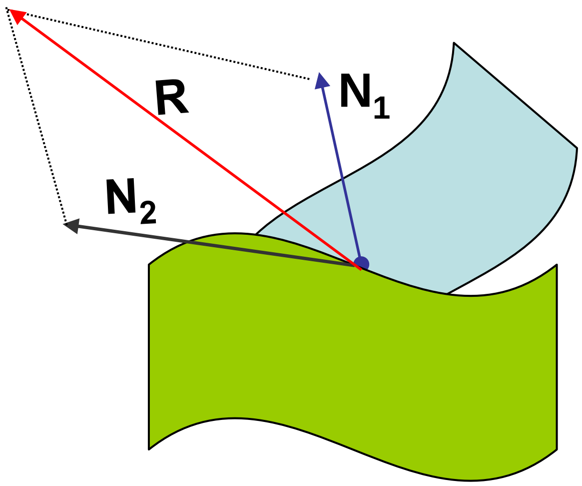
In this case the constraint is an ideal one.



In this casei, tahing into account (12.1) we obtain:

$$\vec{R} = \lambda\, grad\, h \tag{12.4}$$

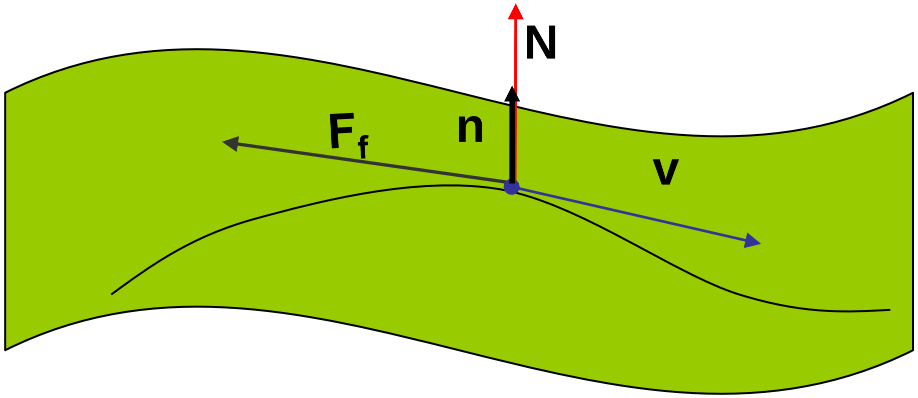
If the motion takes place on a curve given by equations (12.2) the normal reaction force **R** belongs to the normal plane to the curve *C* , hence it is of the form:



$$\vec{R} = \vec{N}_1 + \vec{N}_2 = \lambda_1 \, grad\, h_1 + \lambda_2 \, grad\, h_2 \qquad (12.5)$$

The motion of the material point on a smooth manifold is calld <u>frictionless motion</u>.

However, in real world, friction is present. A constraint with friction implies a constraint force tangent to the surface *S* or to the curve *C* , which is determined by a supplementary modelling of the mechanical phenomenon .



In this case the constriction force is:

$$\vec{R} = \vec{N} + \vec{F}_f \qquad (12.6)$$

where **N** is the normal constraint reaction (parallel with the normal to the manifold, **n**), and $\mathbf{F_f}$ is the tangential constraint reaction (friction force)
Therefore, the equation of motion of the material point is:

$$m\vec{a} = \vec{F} + \vec{N}$$

(frictionless motion)

$$m\vec{a} = \vec{F} + \vec{N} + \vec{F}_f$$

(motion with friction)

The motion of the material point on a fixed curve of class C^1

Consider the material point M(m) forced to move on the curve:

$$(C):\begin{cases} h_1(x,y,z)=0 \\ h_2(x,y,z)=0 \end{cases} \tag{12.7}$$

The differential equation of motion is given by:

$$m\frac{d^2\vec{r}}{dt^2}=\vec{F}+\vec{N}+\vec{F}_f=\vec{F}+\vec{N}_1+\vec{N}_2+\vec{F}_f \tag{12.8}$$

where

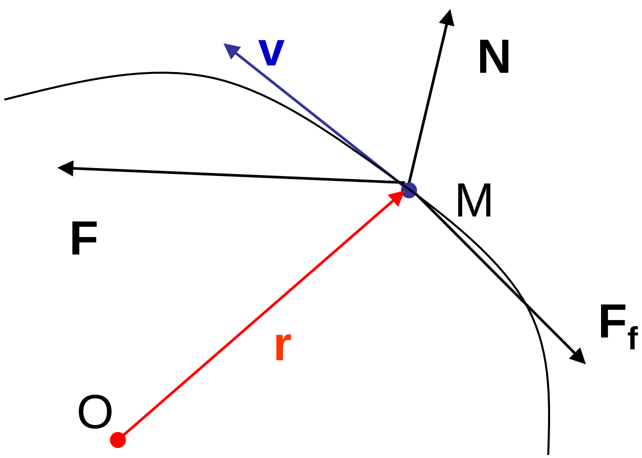
$\vec{F}$ is the given force

$\vec{N}=\vec{N}_1+\vec{N}_2=\lambda_1\, grad\, h_1+\lambda_2\, grad\, h_2$ is the normal constrict reaction at the curve, while λ_1, λ_2 are unknown real constants.

$\vec{F}_f$ is the friction force

The friction force is parallel with the tangent to the curve in the point M and it is opposing to the motion (it has the opposite sign of the velocity of the point M) :

$$\vec{F}_f = -F_f \frac{\vec{v}}{v} \qquad (12.9)$$



The magnitude of the friction force is given by the Coulomb's law:

$$F_f = f\,N \qquad (12.10)$$

where $f > 0$ is the friction coefficient depending on the nature of the curve (or surface), while N is the magnitude of the normal reaction **N**.

The numerical coefficient *f* , which depends only on the nature and the state (dry or wet) of the rough surface, and does not depend on the velocity **v** of the particle if it begins to move, is called *coefficient of friction* (coefficient of *static* friction, unlike a *dynamic* one, which is of the form *f* = *f* (*v*)).

wood on wood, dry …………	0.25-0.50	leather on metals, dry ………..	0.56
wood on wood, soapy ………	0.20	leather on metals, wet ………..	0.36
metals on oak, dry …………..	0.50-0.60	leather on metals, greasy …….	0.23
metals on oak, wet …………..	0.24-0.26	leather on metals, oily ………	0.15
metals on oak, soapy ……….	0.20	steel on agate, dry ……………	0.20
metals on elm, dry …………..	0.20-0.25	steel on agate, oily …………...	0.107
hemp on oak, dry ……………	0.53	iron on stone …………………	0.30-0.70
hemp on oak, wet ……………	0.33	wood on stone ……………….	0.40
leather on oak ………………..	0.27-0.38	earth on earth ………………...	0.25-1.00
metals on metals, dry ……….	0.30	earth on earth, wet clay ……..	0.31
metals on metals, wet ……….	0.15-0.20	metals on ice …………………	0.01-0.03
smooth surfaces, best results ...	0.03-0.036	smooth surfaces, occasionally greased ………………………	0.07-0.08

v (km/h)	0	10.93	21.08	43.5	65.8	87.6	96.48
f	0.242	0.088	0.072	0.07	0.057	0.038	0.027

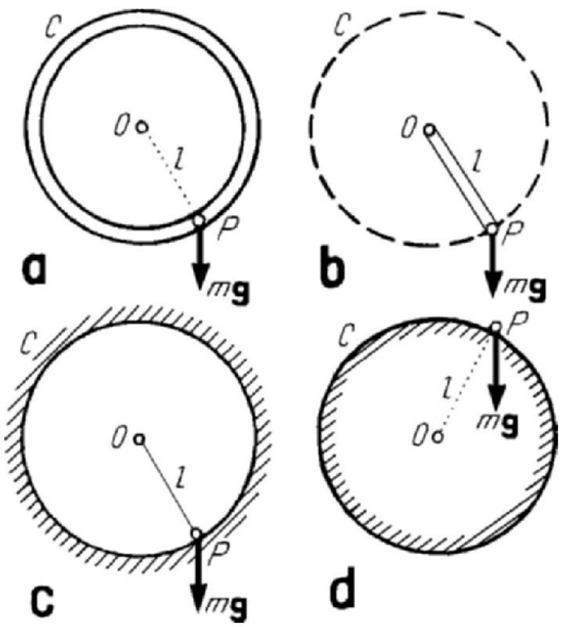
In order to solve (12.8) we need the initial conditions:

$$\vec{r}(t_0) = \vec{r}_0, \ \vec{v}(t_0) = \vec{v}_0$$

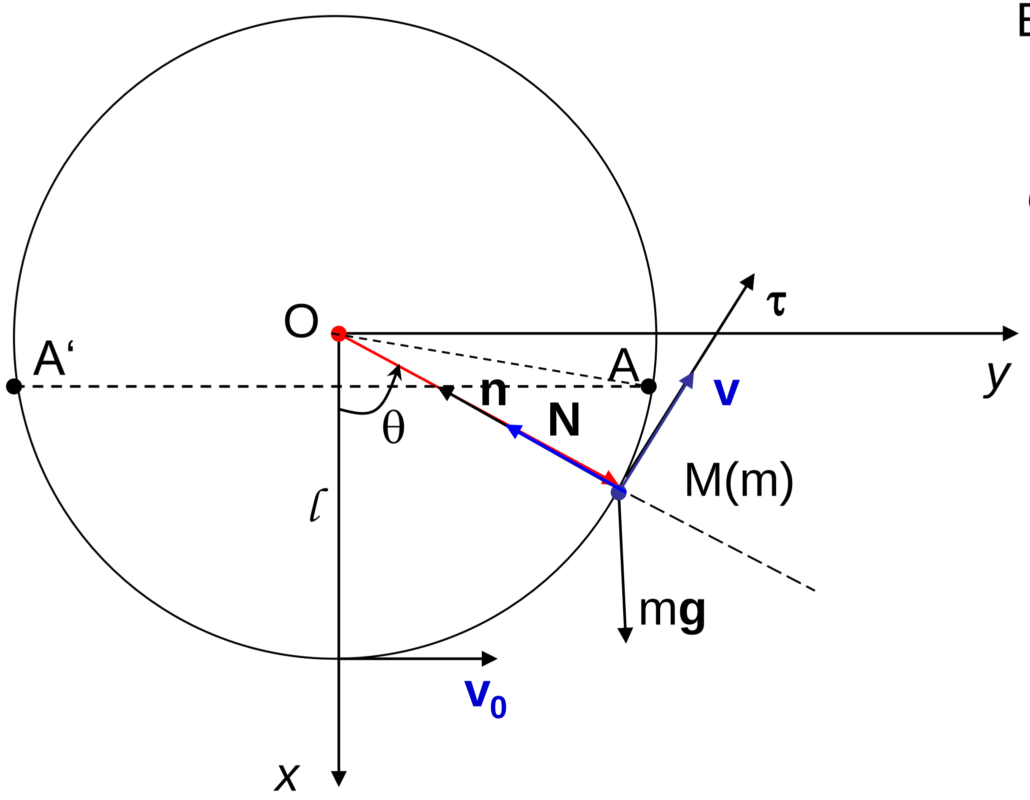
and the constriction‘s equations (12.7).

Mathematical pendulum

A *simple pendulum* (or *mathematical pendulum*) is a heavy particle which moves without friction on a circle *C* of radius *l* , situated in a vertical plane.



The constraint may be bilateral (e.g., a ball modelled as a particle constrained to move in the interior of a circular tube (Fig.a) or a ball linked to the centre *O* of the circle by an inextensible and incompressible bar *OP* , of negligible mass with respect to that of the particle (Fig.b)) or *unilateral* (e.g., a ball linked to the centre *O* by an inextensible and perfectly flexible thread (Fig.c) or a ball constrained to move on a whole cylinder, which has a horizontal axis (Fig.d)).



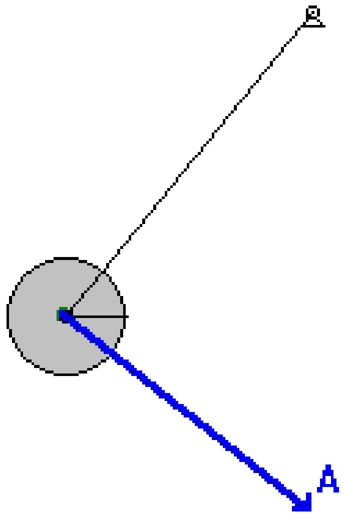
Ecuation of motion:

$$\theta = \theta(t),\ \ t \in [0, T] \qquad (12.11)$$

The motion depends on only one parameter, the angle θ.

The differential equation of motion is:

$$m\vec{a} = \vec{F} + \vec{N} \qquad (12.12)$$

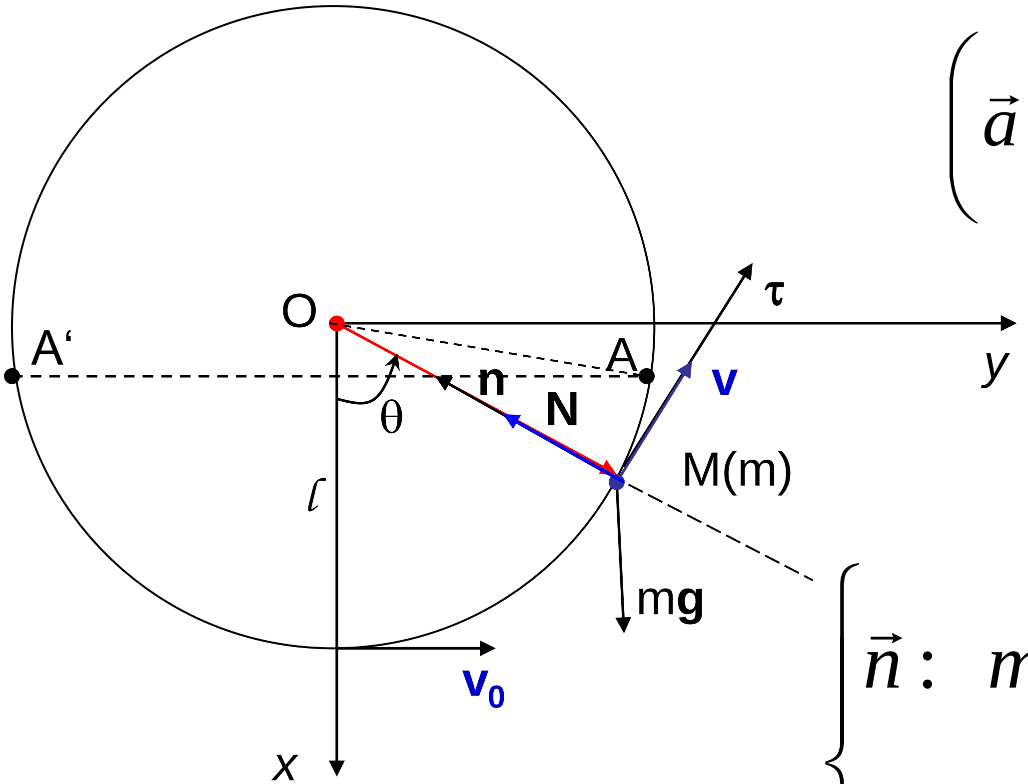


We add the initial condition. We assume the motion starting from the Ox axis with the velocity $\mathbf{v}_0$ perpendicular on the circle's radius. Thus:

$$\theta(0) = 0 \tag{12.13}$$

$$v(0) = v_0 \quad (\vec{v}_o \perp Ox)$$

Projecting ecuation (12.12) on the tangent and on the principal normal we get:



$$\left(\vec{a} = a_n \vec{n} + a_\tau \vec{\tau} = \frac{v^2}{l} \vec{n} + \frac{dv}{dt} \vec{\tau} \right)$$

$$\begin{cases} \vec{n}: \quad m\dfrac{v^2}{l} = N - mg\cos\theta & (12.14) \\ \vec{\tau}: \quad m\dfrac{dv}{dt} = -mg\sin\theta & (12.15) \end{cases}$$

From (12.14) we obtain:

$$N = m\frac{v^2}{l} + mg\cos\theta \tag{12.16}$$

We take into account that in the circular motion $v = l\dot{\theta}$ and (12.15) becomes:

$$\frac{d(l\dot{\theta})}{dt} = -mg\sin\theta$$

or

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0 \tag{12.17}$$

along with the boundary condition

$$\theta(0) = 0$$
$$\dot{\theta}(0) = \frac{v_0}{l} \tag{12.18}$$

Case I. If the oscilations are small we use the approximation: $\sin\theta \approx \theta$ and equation (12.17) becomes:

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \tag{12.19}$$

Equation (12.19) is a second order homogenous differential equation. The solution is of the form:

$$\theta(t) = C_1 \cos\sqrt{\frac{g}{l}}t + C_2 \sin\sqrt{\frac{g}{l}}t \tag{12.20}$$

where the constants C_1 and C_2 can be obtained from the initial conditions. One notice that the motion is periodic and the period is:

$$T = 2\pi\sqrt{\frac{l}{g}} \tag{12.21}$$

Case II. For large oscillations we take into account that the force m**g** is conservative, i.e. a function V exists such that:

$$m\vec{g} = -gradV \;\Rightarrow\; mg = -\frac{dV}{dx} \;\Rightarrow V = -mgx$$

The elementary work is:

$$\delta L = m\vec{g}\cdot d\vec{r} + \underbrace{\vec{N}\cdot d\vec{r}}_{=0\ (\text{because } \vec{N}\perp d\vec{r})} \underset{\substack{m\vec{g}\,(mg,0)\\ d\vec{r}\,(dx,dy)}}{=} mgdx = -dV$$

Therefore

$$dT = \delta L = -dV \;\Rightarrow\; d(T+V) = 0 \;\Rightarrow\; T+V = h$$

We get:

$$\frac{1}{2}mv^2 - mgx = h$$

where h is the integration‘s constant.

Let be A the point of maximum height ($\mathbf{v}_A = 0$):

$$\frac{1}{2}mv^2 - mgx = \frac{1}{2}m\underbrace{v_A{}^2}_{v_A=0} - mgx_A \quad \Rightarrow \quad v^2 = 2g(x - x_A) \quad (12.22)$$

At t = 0 we have:

$$v_0{}^2 = 2g(l - x_A) \Rightarrow x_A = l - \frac{v_0{}^2}{2g}$$

In order to keep an oscillatory motion we impose the condition:

$$-l < x_A < l \Rightarrow -l < l - \frac{v_0{}^2}{2g} < l \Rightarrow -2l < -\frac{v_0{}^2}{2g} < 0$$

$$\Rightarrow v_0{}^2 < 4l\,g \Rightarrow v_0 < 2\sqrt{\text{lg}}$$

Consider α the angle when the point reach the point x_A. Thus:

$$x_A = l\cos\alpha$$
$$x = l\cos\theta \qquad (12.23)$$

but $v = l\dot{\theta}$ and from (12.22) and (12.23) we get:

$$l^2\dot{\theta}^2 = 2gl(\cos\theta - \cos\alpha) \Rightarrow l\frac{d\theta}{dt} = \pm\sqrt{2g(\cos\theta - \cos\alpha)} \qquad (12.24)$$

where the „+“ sign corresponds to the case when the point ascends while the „-“ sign corresponds to the case when the point descends. By integrating (12.24) one obtin:

$$t - t_0 = \sqrt{\frac{l}{2g}}\int_{\theta_0}^{\theta}\frac{d\theta}{\pm\sqrt{(\cos\theta - \cos\alpha)}} \qquad (12.25)$$

We consider $t_0 = 0$ and $\theta_0 = 0$ and the movement to take place from the point of minimum to the maximum point (point A). Then (12.25) becomes:

$$t = \sqrt{\frac{l}{2g}} \int_0^{\theta} \frac{d\theta}{\sqrt{(\cos\theta - \cos\alpha)}} \qquad (12.26)$$

Taking into account $\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$ and (12.26) we get:

$$t = \sqrt{\frac{l}{2g}} \int_0^{\theta} \frac{d\theta}{\sqrt{2\left(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}\right)}} = \frac{1}{2}\sqrt{\frac{l}{g}} \int_0^{\theta} \frac{d\left(\frac{\theta}{2}\right)}{\sqrt{\left(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}\right)}} \qquad (12.27)$$

Theoretical Mechanics

We use in (12.27) the transformation

$$\sin\varphi = \frac{\sin(\theta/2)}{\sin(\alpha/2)} \underset{notatie}{=} \frac{1}{k}\sin(\theta/2) \Rightarrow \cos\varphi\, d\varphi = \frac{1}{k}\cos\frac{\theta}{2}\, d\left(\frac{\theta}{2}\right)$$

and (12.26) becomes:

$$t = \sqrt{\frac{l}{g}}\int_0^{\varphi}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} \underset{notation}{=} \sqrt{\frac{l}{g}}\underbrace{F(k,\varphi)}_{\substack{\text{elliptic integral}\\ \text{of first kind}}} \qquad (12.28)$$

For a quarter of period θ varies from 0 to α, and therefore φ varies from 0 to $\pi/2$. Thus:

$$T = 4\sqrt{\frac{l}{g}}\int_0^{\pi/2}\frac{d\varphi}{\sqrt{(1-k^2\sin^2\varphi)}} = 4\sqrt{\frac{l}{g}}F\left(k,\frac{\pi}{2}\right) \qquad (12.29)$$

IN order to calculate te period T we can use an expansion:

$$\frac{1}{\sqrt{1-k^2\sin^2\varphi}}=\left(1-k^2\sin^2\varphi\right)^{-1/2}=1+\frac{k^2}{2}\sin^2\varphi+\ldots+\frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{2\cdot 4\cdot\ldots\cdot 2n}k^{2n}\sin^{2n}\varphi+\ldots$$

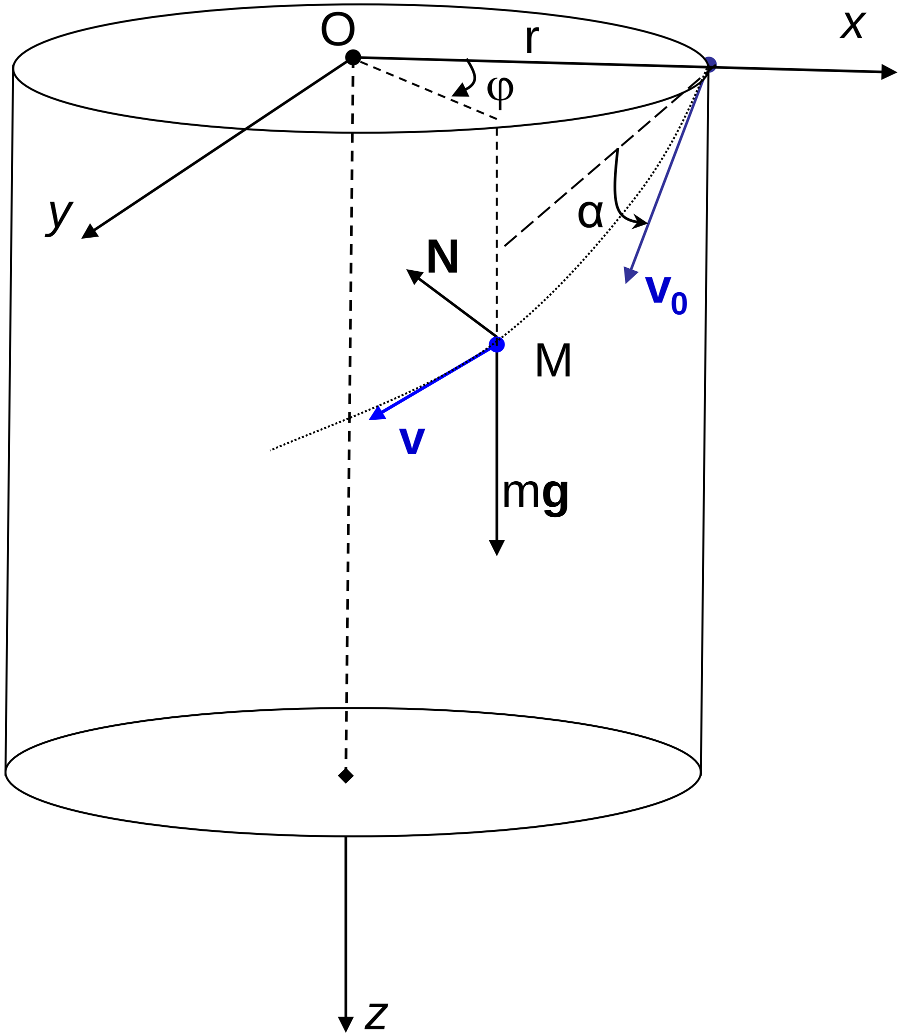
In the above serie we keep only the first two terms, and the period becomes:

$$T=4\sqrt{\frac{l}{g}}\int_0^{\pi/2}(1+k^2\sin^2\varphi)d\varphi=4\sqrt{\frac{l}{g}}\left(\frac{\pi}{2}+\frac{k^2}{2}\frac{\pi}{4}\right) \qquad (12.30)$$

$$\int_0^{\pi/2}\sin^2\varphi\, d\varphi=\int_0^{\pi/2}\frac{1-\cos 2\varphi}{2}d\varphi=\frac{\pi}{4}-\sin 2\varphi\Big|_0^{\pi/2}=\frac{\pi}{4}$$

Using the aproximation: $\sin^2\frac{\alpha}{2}\approx\frac{\alpha^2}{4}$ we obtain:

$$T=2\pi\sqrt{\frac{l}{g}}\left(1+\frac{\alpha^2}{16}\right) \qquad (12.31)$$



Example:

A material point M(m) moves on the inner surface of a circular radius of radius r. Considering the surface of the cylinder absolutely smooth, the axis of the vertical cylinder Oz, and taking into account the weight force, determine the motion of the point and the pressure it is exercise on the cylinder. At the initial moment the velocity of the point on the Ox axis is v_0 and makes the angle α with the horizontal plane.

Cylinder equation: $$f(x,y,z)=x^2+y^2-r^2=0 \qquad (12.32)$$

The equation of motion:

$$m\vec{a}=m\vec{g}+\vec{N} \qquad (12.33)$$

$$\vec{N}=\lambda\, grad\ f=\lambda\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right)=\lambda(2x,2y,0)$$

Equation of motion in Cartesian coordinates and the initial conditions:

$$\begin{cases} Ox:\ m\ddot{x}=2\lambda x\ , \quad x(0)=r;\ \ \dot{x}(0)=0 \\ Oy:\ m\ddot{y}=2\lambda y\ , \quad y(0)=0;\ \ \dot{y}(0)=v_0\cos\alpha \\ Oz:\ m\ddot{z}=mg\ , \quad z(0)=0;\ \ \dot{z}(0)=v_0\sin\alpha \end{cases} \qquad (12.35)$$

From equation (12.35c) we obtain:

$$\begin{cases} z(t) = \dfrac{gt^2}{2} + C_1 t + C_2 \\ z(0) = 0; \;\; \dot{z}(0) = v \end{cases} \Rightarrow \quad z(t) = \frac{gt^2}{2} + (v_0 \sin\alpha)t \qquad (12.36)$$

Next, we eliminate λ between equations (12.35a) and (12.35b):

$$\left.\begin{cases} m\ddot{x} = 2\lambda x \big| \cdot y \\ m\ddot{y} = 2\lambda y \big| \cdot x \end{cases}\right| - \;\Rightarrow\; m(\ddot{x}y - x\ddot{y}) = 0 \;\Rightarrow\; \frac{d}{dt}(\dot{x}y - x\dot{y}) = 0$$

$$\Rightarrow \;\; \dot{x}y - x\dot{y} = C_3$$

At t = 0 we have: $C_3 = \dot{x}(0)y(0) - x(0)\dot{y}(0) \underset{(12.35)}{=} -rv_0 \sin\alpha$

Therefore: $\dot{x}y - x\dot{y} = -rv_0 \sin\alpha$ (12.37)

We use the cylindrical coordinates (we use the constraint equation (12.32)).

$$x = r\cos\varphi;\ \ y = r\sin\varphi;\ \ z = z \tag{12.38}$$

Using (12.38) in (12.37) we get:

$$-r\dot{\varphi}\sin\varphi \cdot r\sin\varphi - r\cos\varphi \cdot r\dot{\varphi}\cos\varphi = -rv_0\sin\alpha$$

$$r\dot{\varphi} = v_0\sin\alpha \ \Rightarrow \left.\begin{matrix} \varphi = \left(\dfrac{v_0}{r}\cos\alpha\right)t + C_4 \\ t = 0:\ \varphi = 0 \end{matrix}\right\} \Rightarrow \varphi = \left(\dfrac{v_0}{r}\cos\alpha\right)t \tag{12.39}$$

From (12.38) and (12.39) we obtain the equation of motion in Cartesian coordinates, (along with (12.36)):

$$x = r\cos\left(\frac{v_0 t}{r}\cos\alpha\right); \quad y = r\sin\left(\frac{v_0 t}{r}\cos\alpha\right) \tag{12.40}$$

In order to find the normal reaction force N parameter λ have to be calculated. from (12.40a) and (12.35a) we have:

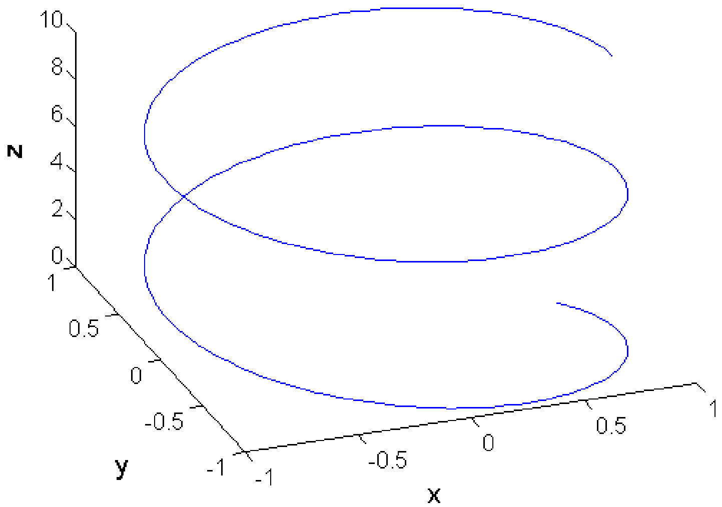
$$-mr\frac{v_0^{\,2}\cos^2\alpha}{r^2}\cos\left(\frac{v_0 t}{r}\cos\alpha\right) = 2\lambda\cos\left(\frac{v_0 t}{r}\cos\alpha\right)$$

$$\lambda = -\frac{mv_0^{\,2}\cos^2\alpha}{2r^2} \tag{12.41}$$

Thus,

$$\vec{N} = -\frac{mv_0{}^2 \cos^2\alpha}{2r^2}(2x, 2y, 0) \qquad (12.42)$$

And the magnitude of the normal reaction force is:



$$N = \frac{mv_0{}^2 \cos^2\alpha}{r^2}\underbrace{\sqrt{x^2 + y^2}}_{r} \Rightarrow$$

$$\Rightarrow N = \frac{mv_0{}^2 \cos^2\alpha}{r} \qquad (12.43)$$