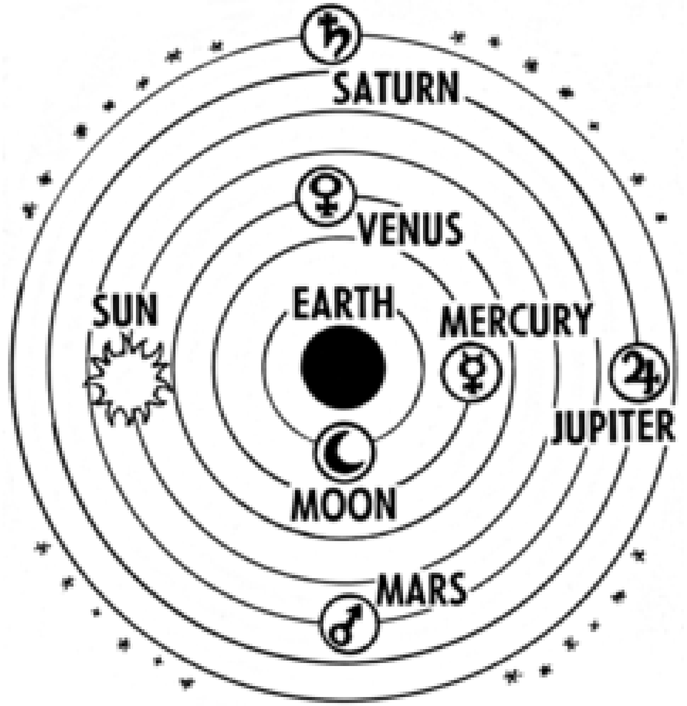
# 11. The universal attraction law. Newton's problem

## The universal attraction law

Historically, in order to obtain the universal attraction law, several steps were done. First, we have to mention Ptolemaeus with his geocentric system

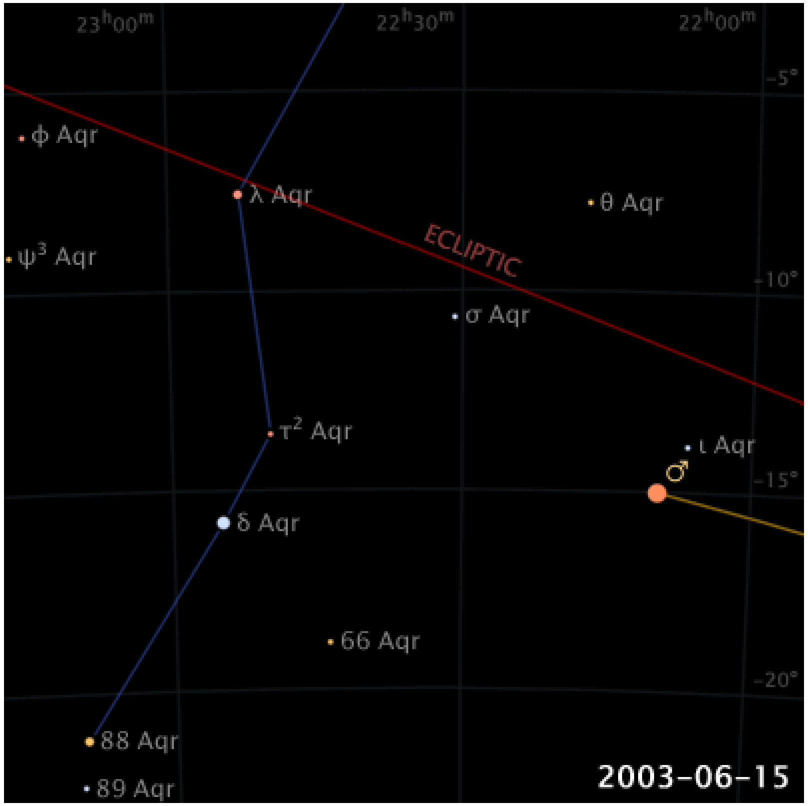
*Claudius Ptolemaeus*
(90 - 168)

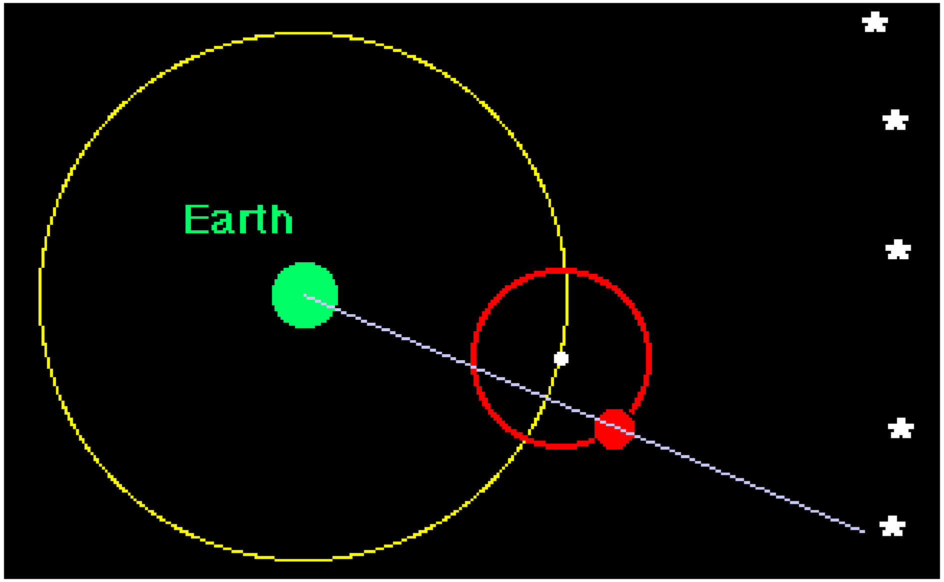
The Univers according to Ptolemaeus



# Theoretical Mechanics

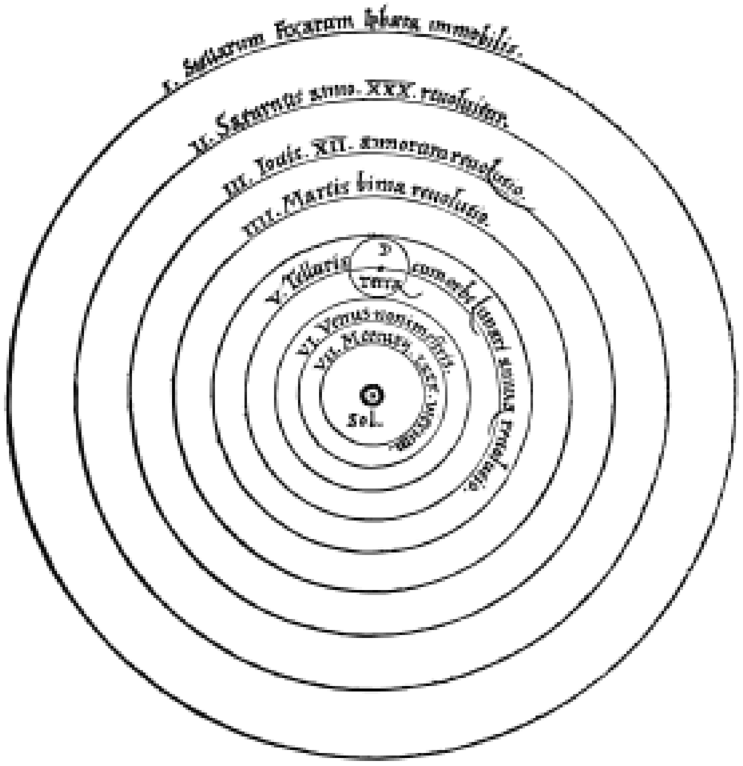
One of the problems of the geocentric system is the retrograde motion. Ptolemaeus explained considered that each planet is moved by a system of two spheres: one called its deferent; the other, its epicycle. A given planet then moves around the epicycle at the same time the epicycle moves along the path marked by the deferent.



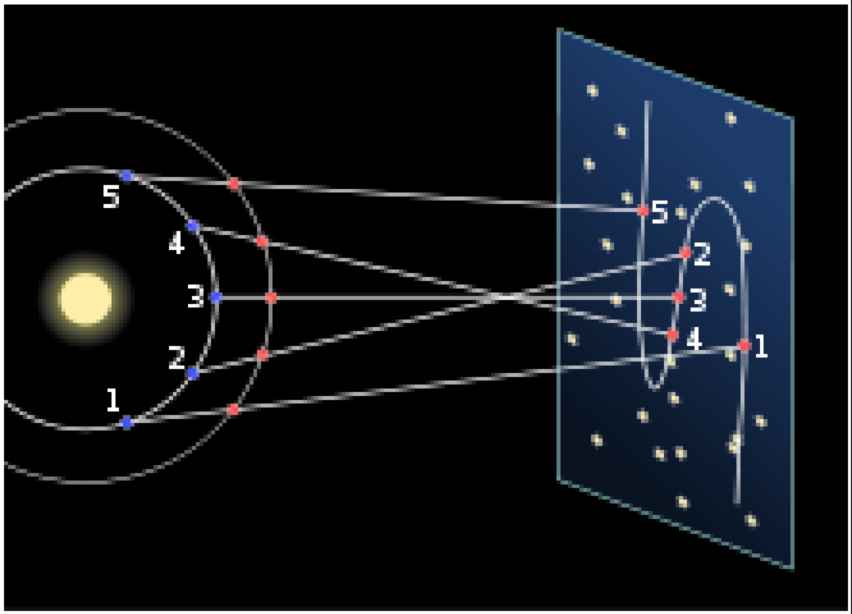
# Theoretical Mechanics

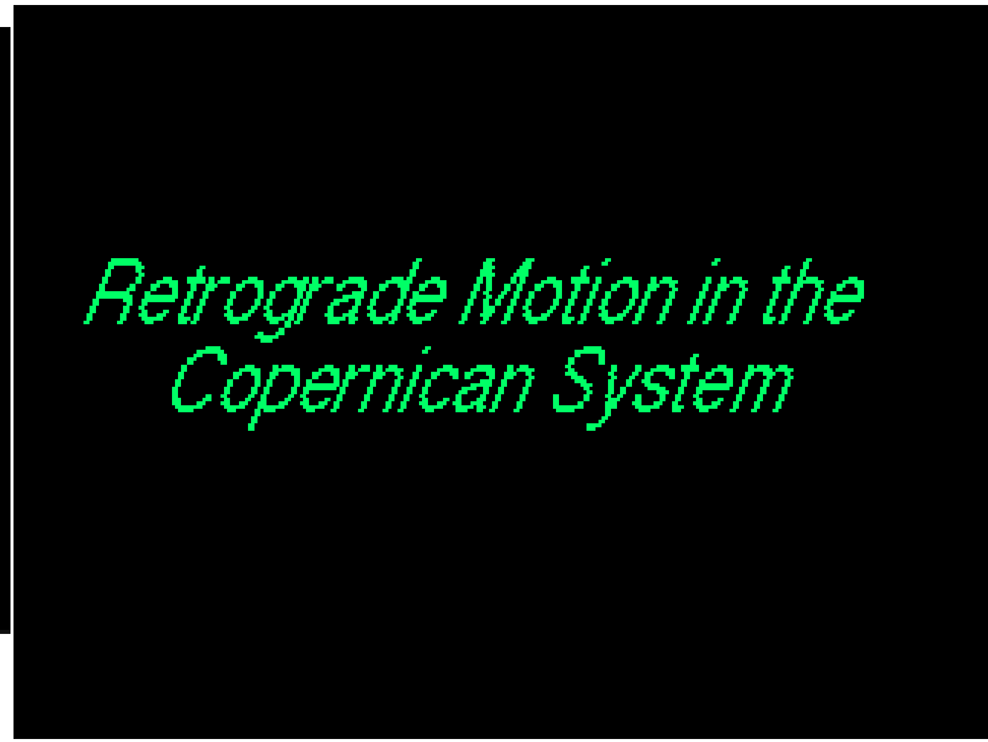
Copernicus (1543) proposed the heliocentric system with the Sun in the center and the planets moving around the Sun on circles. “Psychologically, this is the most important scientific idea in history”.




**Nicolaus Copernicus**
**1473 - 1543**

In heliocentric system the retrograd motion is given by the different speed of the planets moving arround the Sun.

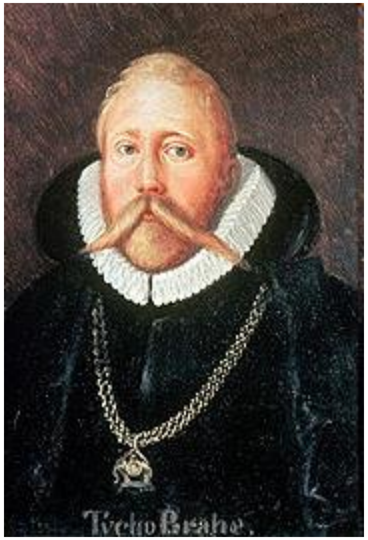


Using the astronomical observations of Tycho Brahe, Kepler (1596, 1609, 1619) formulated the laws describing the planets' motion around the Sun:

1. The motion of a planet around the Sun is an elliptic one, the Sun being at one of the foci.
2. In the motion of a planet around the Sun, the radius vector of it describes equal areas in equal times ( area law)
3. In the motion of planets around the Sun, the ratio of the square of the revolution time to the cube of the semi-major axis is the same for all planets.

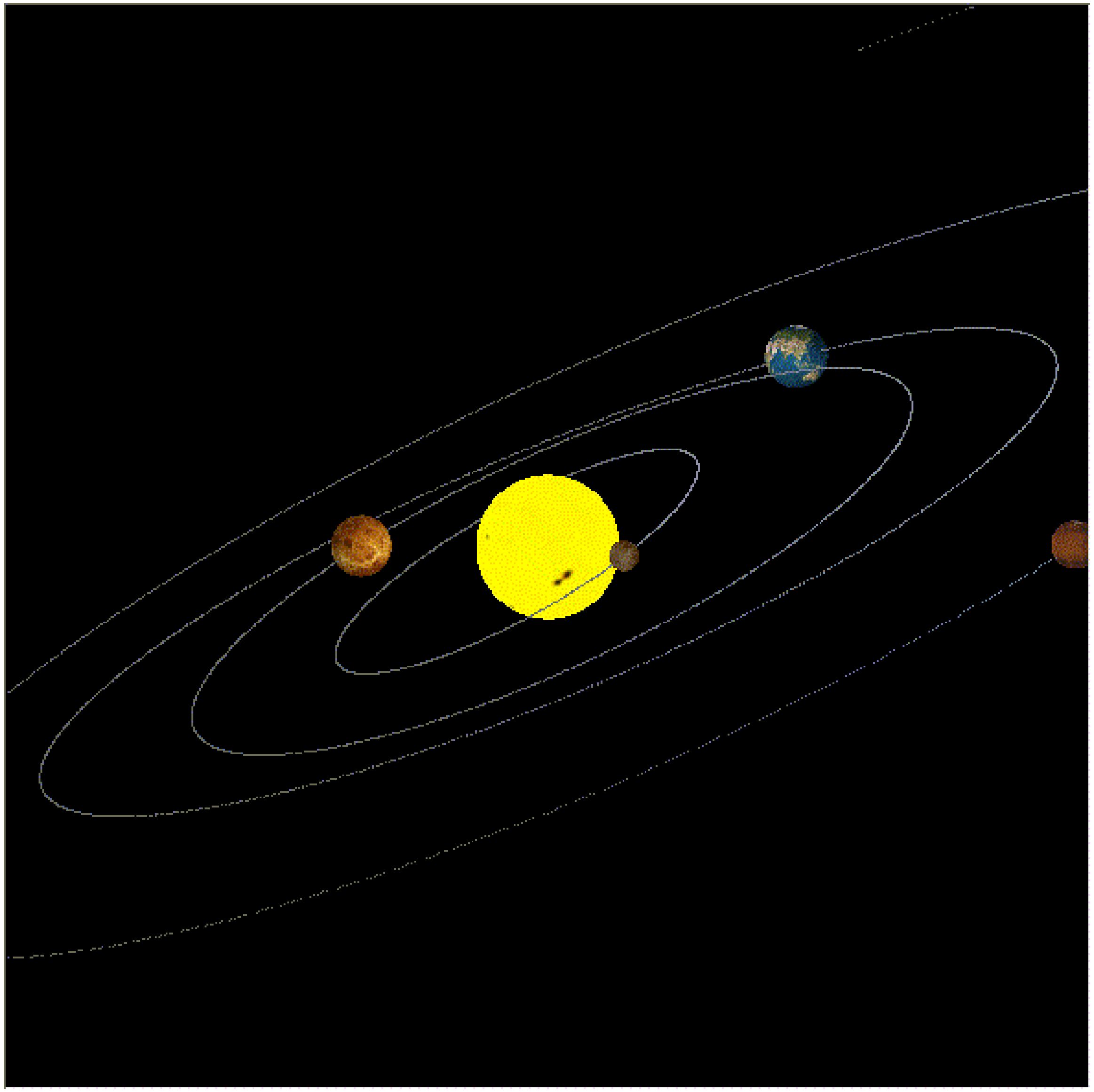
$$a^3/T^2 = \text{constant}$$

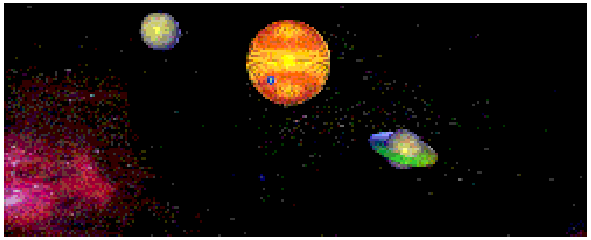



Tycho_Brahe
(1546 - 1601)

Johannes_Kepler
(1571 - 1630)

# Theoretical Mechanics

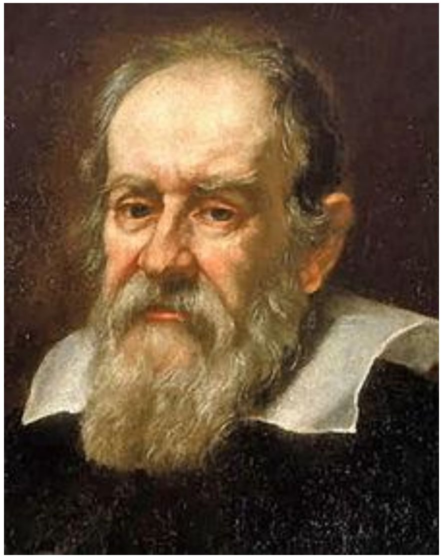




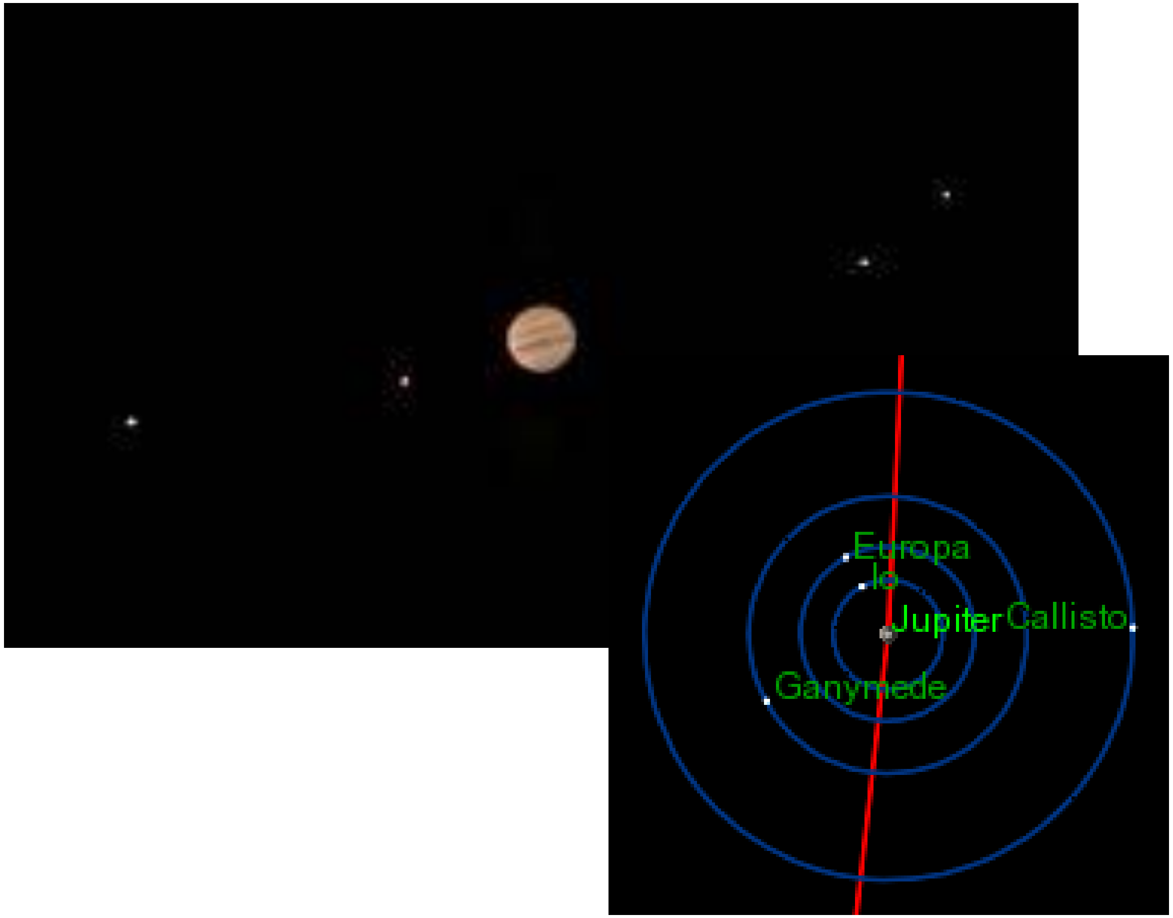
(Solar) planets 1930–2006

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|
| Mercury | Venus | Earth | Mars | Jupiter | Saturn | Uranus | Neptune | Pluto |
| ☿ | ♀ | ⊕ | ♂ | ♃ | ♄ | ♅ | ♆ | ♇ |

Meantime, Galileo Galilei, by using his own design telescope, descovered the four large Jupiters' satellites.  Thus, he proved that other celestial objects (not only the Sun and Earth) have other objects moving around them.



Galileo Galilei
(1564 - 1642)

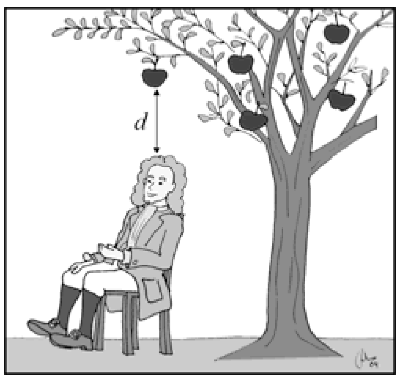


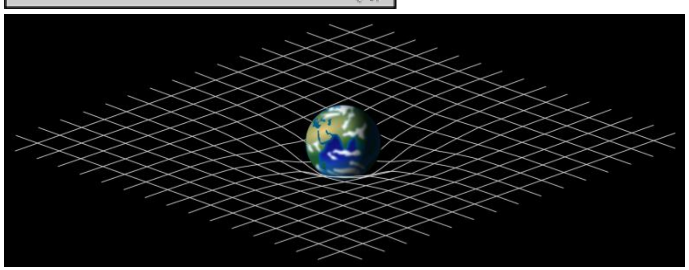
# Theoretical Mechanics

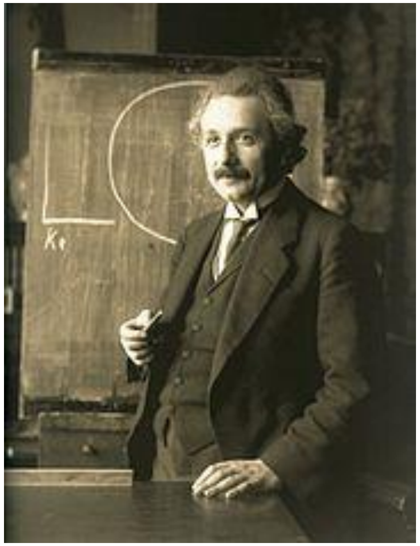
Kepler‘s laws are empirical,  without a theoretical basis. Kepler tried all his life to give a scientific explanation to this phenomena.

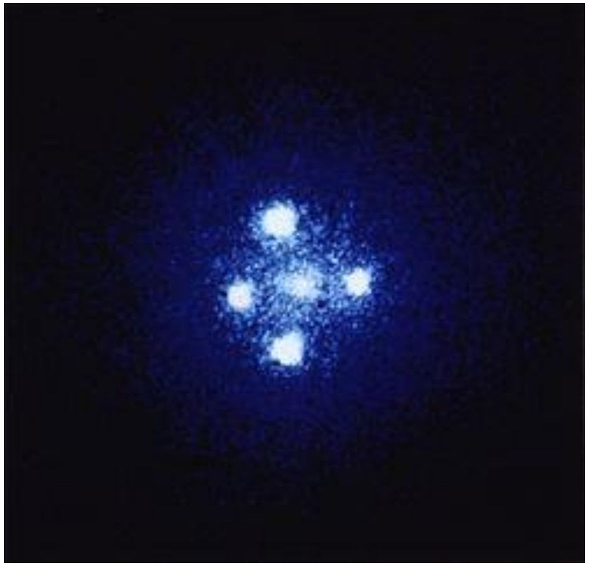
Starting from Kepler’s laws, Isaac Newton (1642-1727) proposed the theory of gravity. He assumed that the force determining apples to fall down to the Earth maintain the Moon in its motion around the Earth and the planets in motion around the Sun.

Later, Albert Einstein (1879-1955) considered gravity a consequence of the spacetime curvature. The Einstein theory does not explain all the physical aspects (dark mass and energy) and have to be further developed.



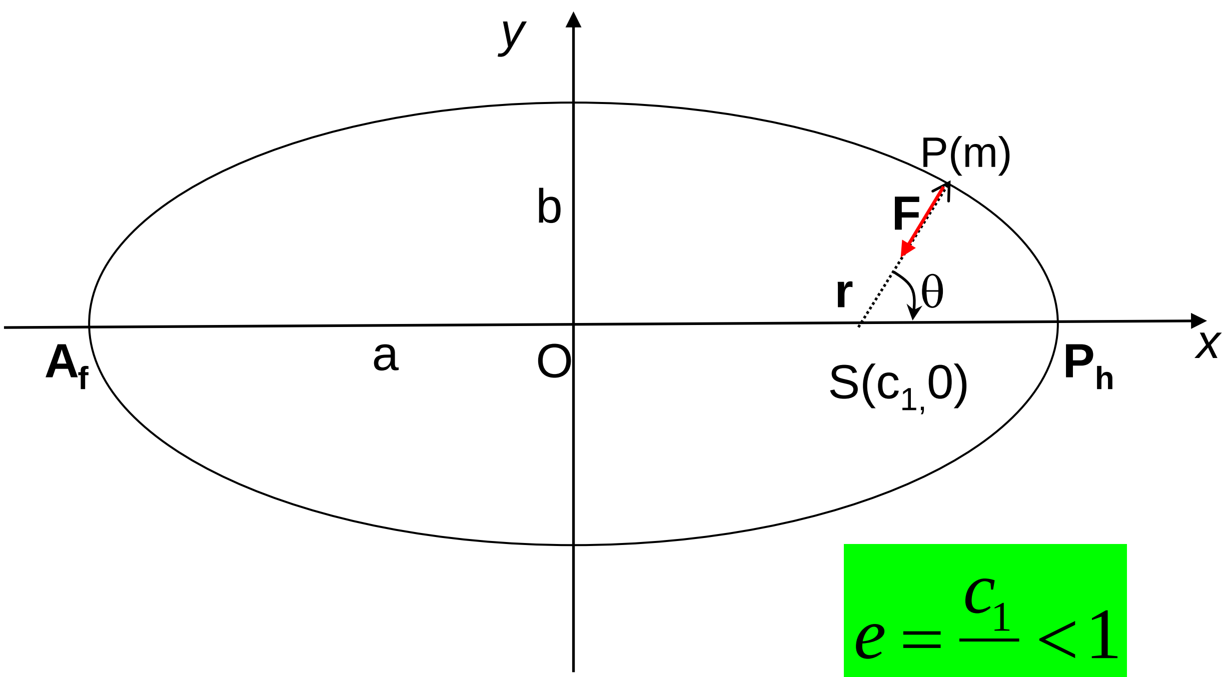







**Next, we will obtain the universal attraction law (given by Newton)**

Consider P(m) a planet moving around the Sun, S(M). The planet and the Sun are considered particles. According to the first law of Kepler we consider that the motion of P(m) is an ellipse from the plane xOy where a is the semi major axis, while b is the semi minor axis.




The trajectory of P(m) in polar coordinates is:

$$r = \frac{p}{1 + e\cos\theta} \tag{11.1}$$

$$e = \frac{c_1}{a} < 1$$

(ellipse eccentricity)

$$p = \frac{b^2}{a}$$

(ellipse parameter)

(https://www.encyclopediaofmath.org/index.php/Ellipse)

Using the second Kepler‘s law we have:

$$r^2\dot{\theta} = c \qquad (11.2)$$

where *c* is the area constant. We deduce that the velocity of P at perihelion $P_h$ (P is nearest to the Sun), is larger than the velocity at aphelion $A_f$ (P is farthest from the Sun)

Using (11.1) we get:

$$\frac{1}{r} = \frac{1+e\cos\theta}{p} \quad \text{thus} \quad \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = \left(\frac{-e\sin\theta}{p}\right)' = \frac{-\cos\theta}{p}$$

(11.3 a,b)

We assume that the force **F** exerted by the sun S on the planet P depends explicitly only by r. Using the moment of momentum theorem

$$\frac{d\vec{K}_o}{dt} = \vec{M}_o\left(\vec{F}\right)$$

we prove that **F** is a central force:

$$\frac{d}{dt}\left(r^2\dot{\theta}\,\vec{k}\right) = \vec{r}\times\vec{F} \quad \Leftrightarrow \quad \underbrace{\frac{d}{dt}\left(r^2\dot{\theta}\right)}_{\underset{(11.2)}{=}\;0}\vec{k} = \vec{r}\times\vec{F} \quad \Leftrightarrow \quad \vec{r}\times\vec{F} = 0$$

Thus, **F** is always directed from P toward the fixed point S (i.e. **F** is a central force).

Taking into account that **F** is a central force one can use the Binet's equation:

$$-\frac{mc^2}{r^2}\left[\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}\right]=F(r) \tag{11.4}$$

Using (11.3b) in (11.4) we have:

$$-\frac{mc^2}{r^2}\left[\frac{-e\cos\theta}{p}+\frac{1}{r}\right]=F(r) \tag{11.5}$$

and using (11.3a) in (11.5) one get:

$$F(r)=-\frac{mc^2}{p}\frac{1}{r^2} \tag{11.6}$$

According to Kepler's third law

$$\frac{a^3}{T^2} = \text{constant} \Rightarrow 4\pi \frac{a^3}{T^2} = \text{constant} \underset{\text{notation}}{=} \mu \in \Re \tag{11.7}$$

Taking into account that T is the rotation period of the planet around the Sun, integrating (11.2) we have:

$$r^2\dot{\theta} = c \Rightarrow \underbrace{\int_0^{2\pi} r^2(\theta)d\theta}_{=2\times\text{ellipse area}=2\pi\,\mathrm{a\,b}} = \int_{t_0}^{t_0+T} c\,dt \Rightarrow cT = 2\pi\,\mathrm{a\,b}$$

One get:

$$c = \frac{2\pi\,a\,b}{T} \tag{11.8}$$

From (11.6) and (11.8) we have:

$$F(r) = -\frac{m}{p}\frac{4\pi^2 a^2 b^2}{T^2}\frac{1}{r^2} \tag{11.9}$$

Taking into account (11.7), equation (11.9) becomes:

$$F(r) = -m\underbrace{\frac{4\pi^2 a^3}{T^2}}_{\underset{(11.7)}{=}\mu}\frac{b^2}{ap}\frac{1}{r^2} \underset{p=\frac{b^2}{a}}{=} -\frac{\mu m}{r^2} \tag{11.10}$$

Equation (11.10) is the algebraic magnitude of the force exerted by the sun S on the planet P.

According to the principle of action and reaction, we deduce that planet P attracts the sun S with an $\mathbf{F_P}$ force equal in size, but of the opposite direction:

$$F_P = \frac{\mu_P M}{r^2} \tag{11.11}$$

In the above relationships the variables are:
M = the mass of the Sun
m = mass of the planet
$\mu_P$ = coefficient specific to the planet's attractive center
$\mu$ = specific factor of the sun's attractive center (the same for all planets orbiting the Sun)

Therefore:

$$F = F_P \Leftrightarrow \frac{\mu m}{r^2} = \frac{\mu_P M}{r^2} \Leftrightarrow \mu m = \mu_P M \Leftrightarrow$$

$$\Leftrightarrow \frac{\mu}{M} = \frac{\mu_P}{m} \underset{notation}{=} f = \text{constant}$$

same for all planets

The constant $f$ is called the constant of the attraction.

$$f = \frac{\mu}{M} = \frac{\mu_P}{m} \approx 6.674 \times 10^{-11}\,\mathrm{N\,(m/kg)^2}. \qquad (11.12)$$

**(Henry Cavendish, 1798)**

Using (11.12) in (11.10) we obtain **the attraction law:**

$$F = -f\,\frac{m\,M}{r^2}$$

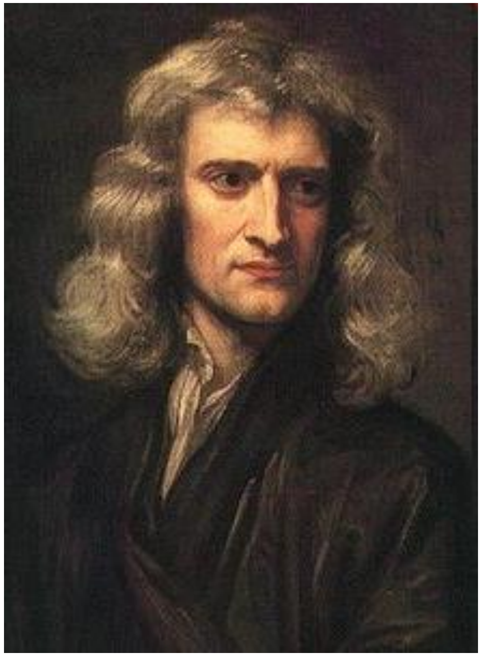
or in vector form

$$\vec{F} = -f\,\frac{m\,M}{r^2}\,\frac{\vec{r}}{r} \qquad (11.13)$$

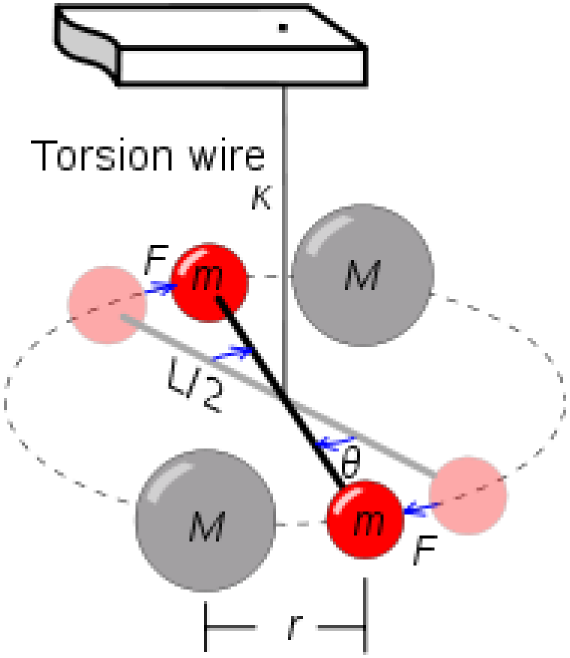
**Law of attraction:** The force of attraction exercised by the Sun in the movement of a planet around it is directly proportional to the mass of the planet and the mass of the Sun and inversely proportional to the square of the distance between the Sun and the planet.

It has been proven that this law is valid not only for the Solar System but for all the bodies of the Universe. That is why the law is **called the universal attraction law**.

A force F given by (11.10) is called **Newtonian force.**



Isaac Newton
(1642-1727 )




Cavendish’s experiment

## Newton‘s problem

Find the motion of a particle P(m) under the action of Newtonian type force:
(the inverse problem of the attraction law)

$$F(r) = -\frac{\mu m}{r^2} \tag{11.14}$$

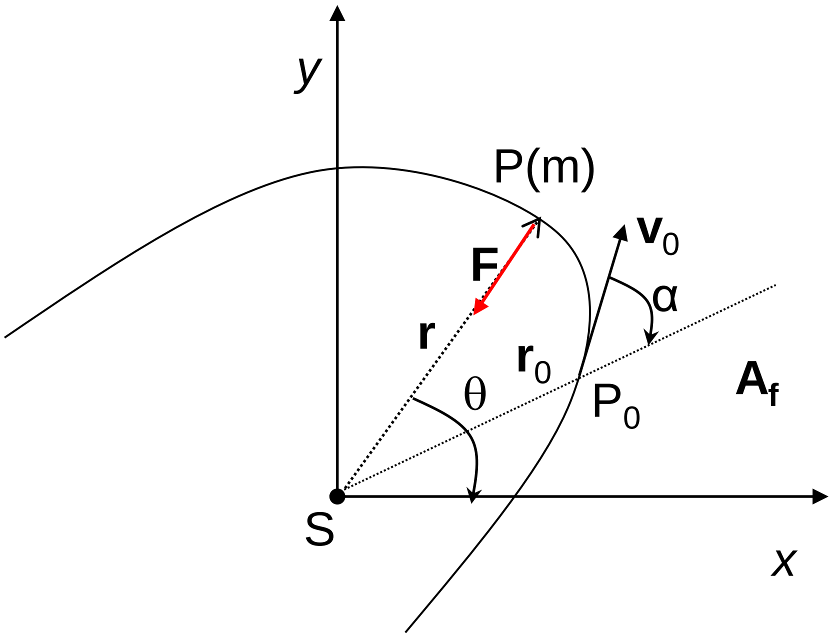
where $\mu > 0$ is a constant and *r* is the distance from P to the attractive cener S.

Therefore, the force F is a central fore and thus the motion of particle P(m) takes place in a plane and the area law is respected:

$$r^2\dot{\theta} = c \tag{11.15}$$

Remark. This problem has application in celestial mechanics.

Next, in orer to deduce the equation of motion of the particle P around the attractive center S we use the equation of Binet.




Binet‘s equation:

$$-\frac{\cancel{m}c^2}{\cancel{r}^2}\left[\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}\right]=F\underset{(11.14)}{=}-\frac{\mu\,\cancel{m}}{\cancel{r}^2}$$

We obtain:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}=\frac{\mu}{c^2} \tag{11.16}$$

Equation (11.16) is a linear nonhomogeneous second order differential equation. The general solution of equation (11.16) is:

$$\frac{1}{r}=C_1\cos(\theta-\theta_1)+\frac{\mu}{c^2}, \qquad C_1,\ \theta_1\in\Re \tag{11.17}$$

where the integrating constants $C_1$ and $\theta_1$ can be found from the initial conditions:

$$r(\theta_0)=r_0; \qquad \frac{d}{d\theta}\left(\frac{1}{r}\right)\Bigg|_{\substack{t=0\\(\theta=\theta_0)}}=-\frac{1}{r_0}ctg\alpha \tag{11.18}$$

Using in (11.17) the following:

$$\frac{\mu}{c^2} \underset{notation}{=} \frac{1}{p} \Leftrightarrow p = \frac{c^2}{\mu} \quad \text{and} \quad C_1 \underset{notation}{=} \frac{e}{p} \tag{11.19}$$

we get:

$$\frac{1}{r} = C_1 \cos(\theta - \theta_1) + \frac{\mu}{c^2} = \frac{e}{p}\cos(\theta - \theta_1) + \frac{1}{p} = \frac{1 + e\cos(\theta - \theta_1)}{p}$$

or

$$\boxed{r = \frac{p}{1 + e\cos(\theta - \theta_1)}} \tag{11.20}$$

Equation (11.20) is the equation of a conic with the **focus** in S, and the angle between the **focal axis** and the **polar axis** is $\theta_1$. Parameter p is the **parameter of the conic**, while e is the **eccentricity**.

Using (11.20) and the initial conditions (11.18) we get:

$$\begin{cases} r(\theta_0) = r_0 \\ \dfrac{d}{d\theta}\left(\dfrac{1}{r}\right)\Bigg|_{\substack{t=0 \\ (\theta=\theta_0)}} = -\dfrac{1}{r_0} ctg\alpha \end{cases} \underset{(11.20)}{\Leftrightarrow}$$

$$\Leftrightarrow \begin{cases} e\cos(\theta_0 - \theta_1) = \dfrac{p}{r_0} - 1 \\ \dfrac{d}{d\theta}\left(\dfrac{1+e\cos(\theta-\theta_1)}{p}\right)\Bigg|_{\theta=\theta_0} = -\dfrac{e}{p}\sin(\theta_0-\theta_1) = -\dfrac{1}{r_0} ctg\alpha \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} e\cos(\theta_0 - \theta_1) = \dfrac{p}{r_0} - 1 \\ e\sin(\theta_0 - \theta_1) = \dfrac{p}{r_0} ctg\alpha \end{cases} \qquad (11.21a,b)$$

Using (26a) and (26b) at power two and adding them we get:

$$e^2 = \frac{p^2}{r_0^{\ 2}} + 1 - \frac{2p}{r_0} + \frac{p^2}{r_0^{\ 2}} ctg^2\alpha = \frac{p^2}{r_0^{\ 2}} \frac{1}{\sin^2\alpha} - \frac{2p}{r_0} + 1$$

Thus

$$e^2 = 1 + \frac{p}{r_0}\left(\frac{p}{r_0}\frac{1}{\sin^2\alpha} - 2\right) \qquad (11.22)$$

Next, considering $F(r) = -\dfrac{\mu m}{r^2}$ we prove that the force is conservative, i.e. a potential function V=V(r) exists, such that $F(r) = -\dfrac{dV}{dr}$.

We obtain:

$$V = -\frac{\mu m}{r} \qquad (11.23)$$

The elementary work is given by:

$$\delta L = \vec{F} \cdot d\vec{r} = F(r)dr = -\frac{dV}{dr}dr = -dV$$

and from the kinetic energy theorem we have:

$$dT = \delta L = -dV \;\Rightarrow\; d(T+V) = 0$$

Integrating we get the first integral of the energy

$$T + V = h', \quad h' \in \Re, \; \forall\, t \geq t_0 \tag{11.24}$$

Therefore,

$$\frac{1}{2}mv^2 - \frac{\mu m}{r} = \frac{1}{2}\cancel{m}v_0{}^2 - \frac{\mu \cancel{m}}{r_0} = h' \underset{notation}{=} \cancel{m}h$$

$$\boxed{\frac{1}{2}v_0{}^2 - \frac{\mu}{r_0} = h} \tag{11.25}$$

In (11.22) we have:

$$e^2 = 1 + \frac{p}{r_0}\left(\frac{p}{r_0}\frac{1}{\sin^2\alpha} - 2\right) \underset{\substack{p=\frac{c^2}{\mu},\\ c=r_0 v_0 \sin\alpha}}{=} 1 + \frac{c^2}{\mu r_0}\left(\frac{c^2}{\mu r_0}\frac{1}{\sin^2\alpha} - 2\right) =$$

$$= 1 + \frac{c^2}{\mu r_0}\left(\frac{r_0^{\ 2} v_0^{\ 2}\sin^2\alpha}{\mu r_0}\frac{1}{\sin^2\alpha} - 2\right) = 1 + \frac{c^2}{\mu r_0}\left(\frac{r_0 v_0^{\ 2}}{\mu} - 2\right) =$$

$$= 1 + \frac{c^2}{\mu r_0}\left[\frac{2r_0}{\mu}\left(\frac{v_0^{\ 2}}{2} - \frac{\mu}{r_0}\right)\right]$$

Taking into account (11.25) one get:

$$e^2 = 1 + \frac{c^2}{\mu r_0}\left[\frac{2r_0}{\mu}\left(\frac{v_0^{\;2}}{2} - \frac{\mu}{r_0}\right)\right] \underset{\frac{v_0^{\;2}}{2} - \frac{\mu}{r_0} = h}{=} 1 + \frac{2c^2}{\mu^2} h \qquad (11.26)$$

If

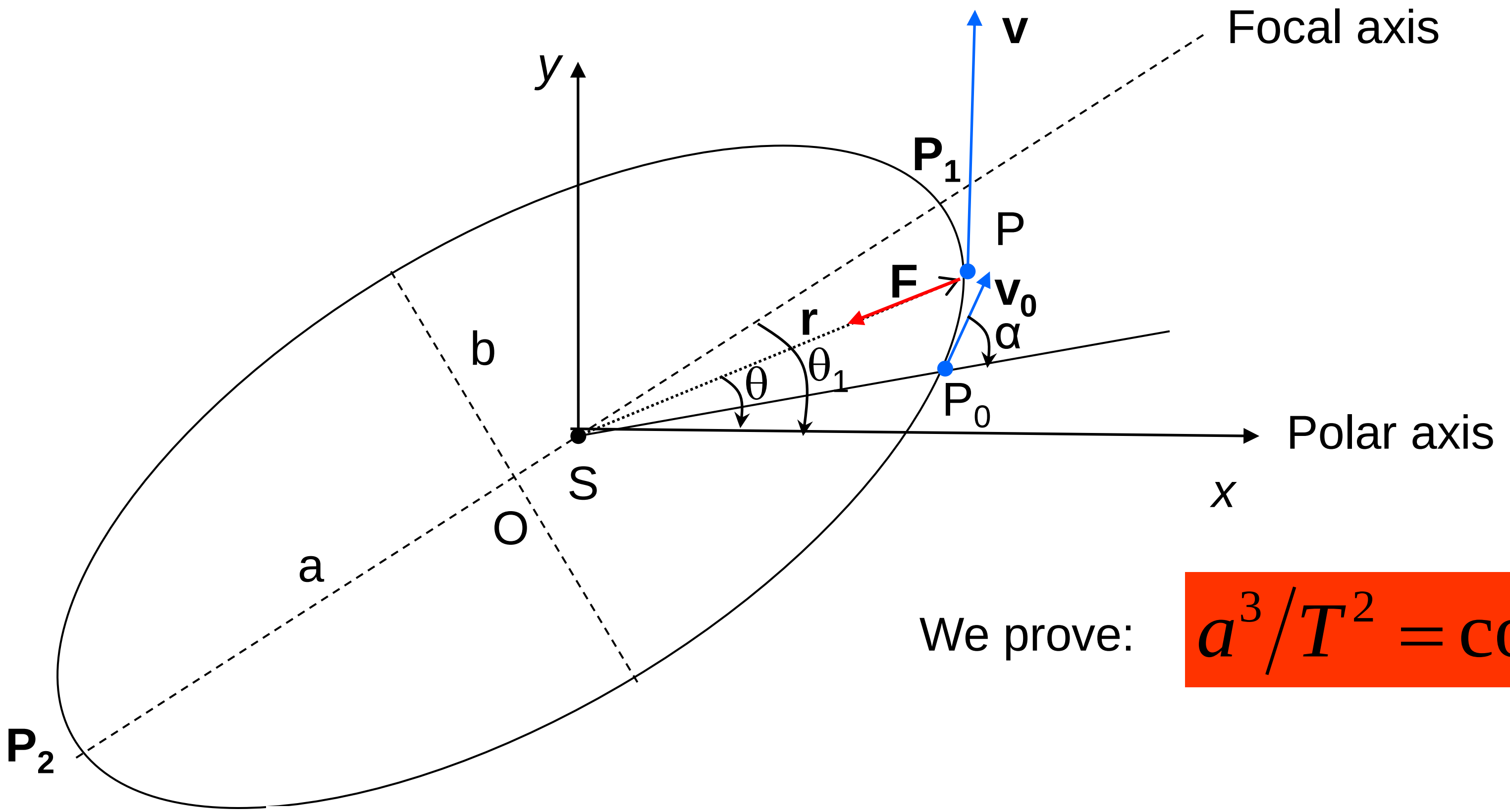
- **e < 1 (h < 0)** then the conic is **an ellipse**
- **e > 1 (h > 0)** then the conic is **a hyperbola**
- **e = 1 (h = 0)** then the conic is a **parabola**

Thus, for h < 0, the conic is an ellipse and the **Kepler's first law** is fulfilled.

**The Kepler's second law** results from the central force's properties (area law)

However, we still have to obtain the Kepler's third law.

We consider that the motion of P takes place on an ellipse:




We prove: $\boxed{a^3/T^2 = \text{constant}}$

We obtain: $$a = \frac{1}{2}(SP_1 + SP_2) = \frac{1}{2}\left(\frac{p}{1+e\cos(\theta_1-\theta_1)} + \frac{p}{1+e\cos(\theta_1+\pi-\theta_1)}\right)$$

We have:

$$a = \frac{p}{1-e^2} \tag{11.27}$$

But

$$e^2 = \frac{OS^2}{a^2} = \frac{a^2 - b^2}{a^2} = 1 - \frac{b^2}{a^2}$$

$$b^2 = a^2(1-e^2) \Rightarrow b = a\sqrt{1-e^2} \underset{(11.27)}{=} \frac{p}{\sqrt{1-e^2}} \tag{11.28}$$

Taking into account (11.8) $c = \dfrac{2\pi a b}{T}$

we get:

$$T = \frac{2\pi a b}{c} \tag{11.29}$$

Using (11.27), (11.29) and (11.28) we calculate the ratio

$$\frac{a^3}{T^2} = \frac{a^3c^2}{4\pi^2a^2b^2} = \frac{ac^2}{4\pi^2b^2} \underset{(11.28)}{=} \frac{ac^2}{4\pi^2a^2(1-e^2)} =$$

$$= \frac{c^2}{4\pi^2a(1-e^2)} \underset{p=c^2/\mu}{=} \frac{p\mu}{4\pi^2a(1-e^2)} \underset{(11.27)}{=}$$

$$= \frac{p\mu}{4\pi^2 p} = \frac{\mu}{4\pi^2} = \text{constant} \qquad (35)$$

Thus, we obtained $\boxed{a^3/T^2 = \text{constant}}$ and this is **the Kepler‘s third law.**

Remark. We proved that the motion of a particle P under the action of a central force of Newtonian type respect the Kepler‘s laws.

## Accelaration due to the gravity

Let us consider the motion of the Moon around the Earth; a force of attraction of the form

$$F = -\mu \frac{m}{r^2}, \quad F = -f \frac{mM}{r^2}$$

for which $\mu = 4\pi^2 a^3 / T^2$

A particle of mass equal to unity at the Earth surface is attracted to that one by a force

equal to $1 \cdot g = 1 \cdot \mu / R^2$, being a constant which depends on the mass of the Earth of radius

$R$ . We obtain thus the relation

$$g = 4\pi^2 \frac{a^3}{R^2 T^2},$$

which allows to calculate the gravity acceleration. The trajectory of the Moon is quasi-circular, with $a \cong 60R \cong 384\ 000\ \mathrm{km}$; it results $g \cong 4\pi^2 60^3 R/T^2$. Because $R \cong 6370\ \mathrm{km}$ and $2\pi R \cong 40\ 000\ \mathrm{km}$, the time of revolution of the Moon being 27 days 7 hours 43 minutes $= 39\ 343 \cdot 60\ \mathrm{s}$, Newton has obtained $g \cong 9.8\ \mathrm{m/s^2}$, in a good concordance with the result previously obtained by Galileo.

At the Earth surface we have

$$\boldsymbol{G} = \boldsymbol{F}, \quad \boldsymbol{mg} = \boldsymbol{f}\frac{\boldsymbol{mM}}{\boldsymbol{R^2}}, \qquad \boldsymbol{M} = \frac{\boldsymbol{gR^2}}{\boldsymbol{f}}$$

But, $M = \frac{4\pi\rho R^3}{3}$ and from here we obtain: $\mu = \dfrac{3}{4\pi}\dfrac{g}{fR}$.

We get thus $\rho$ = 5.51 g/cm$^3$. This density is much greater than that of the superior spherical strata of the Earth; we may thus conclude that the density is much greater towards the centre of the Earth.