Seminars 12 and 13 - 2024

1. Consider the sequence of independent identically distributed random variables $(X_n)_{n\geq 1}$ such that $X_n \sim Unif[1,3]$ for each $n \geq 1$. Compute the a.s. limit of the sequence which is
i) the arithmetic mean of $X_1, ..., X_n$, as $n \to \infty$;
ii) the geometric mean of $X_1, ..., X_n$, as $n \to \infty$;
iii) the harmonic mean of $X_1, ..., X_n$, as $n \to \infty$.

2. Let $(X_n)_{n\geq 1}$ be a sequence of random variables such that $P(X_n = n^2) = \frac{1}{n}$ and $P(X_n = 0) = 1 - \frac{1}{n}$, for all $n \geq 1$. Prove that:
a) $X_n \xrightarrow{P} 0$.
b) $(X_n)_{n\geq 1}$ does not converge in mean square.

3. Consider a binary communication channel transmitting codes of n bits each. Assume that the probability of successful transmission of a single bit is $p \in (0,1)$ and that the probability of an error is $1-p$. Assume also that the channel is capable of correcting up to m errors, where $0 < m < n$. If we assume that the transmission of successive bits is independent, compute the probability of successful code transmission.

4. The measurement error (in millimeters) of a certain object produced in a factory is a continuous random variable X with the cumulative distribution function $F : \mathbb{R} \to [0,1]$,

$$F(x) = \begin{cases} 0, & x < -1 \\ \frac{1}{4}\left(2 + 3x - x^3\right), & x \in [-1,1] \\ 1, & x > 1. \end{cases}$$

Find: $P(-\frac{1}{2} < X < \frac{1}{2})$, $P(X < \frac{1}{2} | X > -\frac{1}{2})$, $E(X)$.

5. A random number generator returns the value Y according to the distribution given by $\begin{pmatrix} -2 & -1 & 0 & 1 \\ \frac{3}{10} & \frac{2}{10} & \frac{1}{10} & \frac{4}{10} \end{pmatrix}$. Find the mean value and the standard deviation of the generator.

6. A random value X is generated according to the density function $f_X : \mathbb{R} \to \mathbb{R}$, $f(x) = \dfrac{1}{2} e^{-|x|}$, for all $x \in \mathbb{R}$. Compute:
a) the cumulative distribution function of X;
b) the cumulative distribution function of the random value X^2;
c) $P(X^2 \geq 1)$;
d) the mean value and the variance of X.

7. For each $n \in \mathbb{N}, n \geq 2$, consider

$$X_n \sim \begin{pmatrix} -1 & 1 \\ \frac{1}{n} & 1 - \frac{1}{n} \end{pmatrix}$$

such that $(X_n)_{n\geq 2}$ is a sequence of pairwise independent random variables.

(a) Does $(X_n)_{n\geq 2}$ obey the weak law of large numbers?

(b) Compute $\lim\limits_{n\to\infty} V\left(\frac{1}{2}(X_{n-1} + X_n)\right)$.