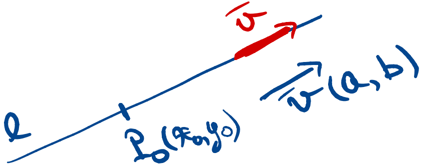
Analytic Geometry

George Țurcaș

Maths & Comp. Sci., UBB Cluj-Napoca

November 13, 2022



P_0 is a point on the line.

We saw the following ways in which one can describe a line in the plane:

- As a vector equation: $\overline{OP} = \overline{OP_0} + t \cdot \overline{v}$, $t \in \mathbb{R}$.
- As two parametric equations:

$$\begin{cases} x = x_0 + t \cdot a \\ y = y_0 + t \cdot b \end{cases}$$

, where $P_0(x_0, y_0)$ and $\overline{v}(a,b)$. $t \in \mathbb{R}$.

- Via a symmetric equation: $\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b}$ for some
- A general equation: $Ax + By + C = 0$, $A, B, C \in \mathbb{R}$.
- A reduced equation:

$y = mx + n$, where $m, n \in \mathbb{R}$.

Intersection of two lines

Let $d_1 : a_1x + b_1y + c_1 = 0$ and $d_2 : a_2x + b_2y + c_2 = 0$ be two lines in $\mathcal{E}_2$. The solution of the system of equation

A

$$\begin{cases} a_1x + b_1y + c_1 = 0 \\ a_2x + b_2y + c_2 = 0 \end{cases} \qquad \begin{pmatrix} a_1 & b_1 & -c_1 \\ a_2 & b_2 & -c_2 \end{pmatrix}$$

$\overline{A}$

will give the set of the intersection points of d_1 and d_2.

1) If $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$, the system has a unique solution (x_0, y_0) and the lines have a unique intersection point $P_0(x_0, y_0)$. They are *secant*.

2) If $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$, the system is not compatible, and the lines have no points in common. They are *parallel*.

3) If $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$, the system has infinitely many solutions, and the lines coincide. They are *identical*.

Suppose $d_i : a_i x + b_i y + c_i = 0$, $i = \overline{1,3}$ are three distinct lines in $\mathcal{E}_2$. Then they are concurrent if and only if

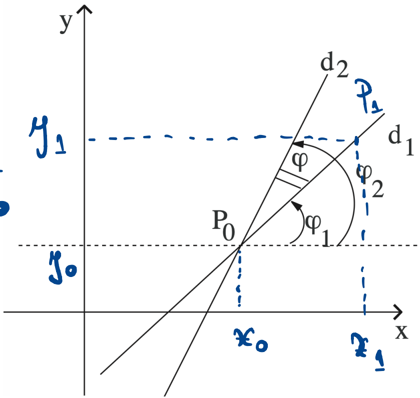
$$\mathrm{Rank}\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} = \mathrm{Rank}\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = 2. \tag{1}$$

The angle between two lines

Let d_1 and d_2 be two concurrent lines, given by their reduced equations:

$$d_1 : y = m_1 x + n_1 \quad \text{and} \quad d_2 : y = m_2 x + n_2.$$

$$\tan \varphi_1 = \frac{y_1 - y_0}{x_1 - x_0} = m_1.$$



$$\tan \varphi_2 = m_2.$$

The angular coefficients of d_1 and d_2 are $m_1 = \tan\varphi_1$ and $m_2 = \tan\varphi_2$.
One may suppose that $\varphi_1 \neq \frac{\pi}{2}$, $\varphi_2 \neq \frac{\pi}{2}$, $\varphi_2 \geq \varphi_1$, such that

$\varphi = \varphi_2 - \varphi_1 \in [0, \pi] \setminus \{\frac{\pi}{2}\}$.
The angle determined by d_1 and d_2 is given by

$$\tan\varphi = \tan(\varphi_2 - \varphi_1) = \frac{\tan\varphi_2 - \tan\varphi_1}{1 + \tan\varphi_1 \tan\varphi_2},$$

hence

$$\tan\varphi = \frac{m_2 - m_1}{1 + m_1 m_2}. \tag{2}$$

1) The lines d_1 and d_2 are parallel if and only if $\tan\varphi = 0$, therefore

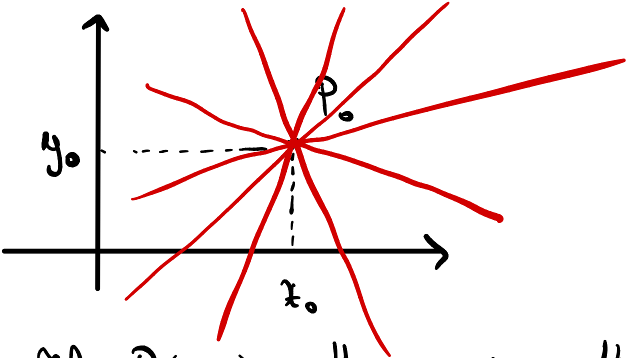
$$d_1 \parallel d_2 \iff m_1 = m_2. \tag{3}$$

2) The lines d_1 and d_2 are orthogonal if and only if they determine an angle of $\frac{\pi}{2}$, hence

$$d_1 \perp d_2 \iff m_1 m_2 + 1 = 0. \tag{4}$$

A bundle of lines

The set of all the lines passing through a given point P_0 is said to be a *bundle* of lines. The point P_0 is called the *vertex* of the bundle.



Example: If $P(0,0)$ is the origin. How do lines that pass through P look like?

$$\{Ax + By = 0 \mid A, B \in \mathbb{R},\ A^2 + B^2 > 0\}.$$

In general, given $P(x_0, y_0)$.

$$\{A \cdot (x - x_0) + B \cdot (y - y_0) = 0 \mid A, B \in \mathbb{R} \text{ and } A^2 + B^2 > 0\}.$$ consists of all lines that pass through P_0.

<u>Remark</u>: If $B \neq 0$, then

$\frac{A}{B}$

$$y - y_0 = s \cdot (x - x_0), \text{ where } s \in \mathbb{R}.$$

The bundle is $\{x = x_0\} \cup \{y - y_0 = s(x - x_0) \mid s \in \mathbb{R}\}$.

If the point P_0 is given as the intersection of two lines, then its coordinates are the solution of the system

$$\begin{cases} d_1 : a_1 x + b_1 y + c_1 = 0 \\ d_2 : a_2 x + b_2 y + c_2 = 0 \end{cases},$$

supposed to be compatible. The equation of the bundle of lines through P_0 is

$$r(a_1 x + b_1 y + c_1) + s(a_2 x + b_2 y + c_2) = 0, \qquad (r, s) \in \mathbb{R}^2 \setminus \{(0,0)\}. \quad (5)$$

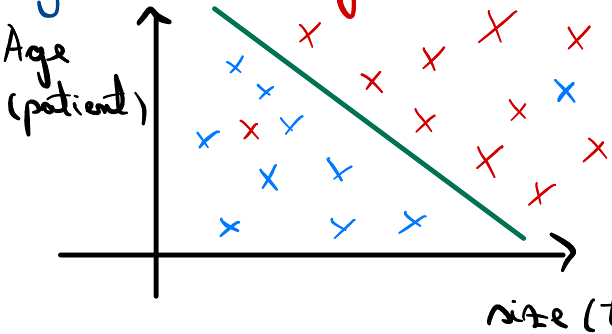
Remark: As before, if $r \neq 0$ (or $s \neq 0$), one obtains the reduced equation of the bundle, containing all the lines through P_0, except d_1 (or, respectively except d_2).

$t = \frac{s}{r}$.

$\{a_2 x + b_2 y + c_2 = 0\} \cup \{a_1 x + b_1 y + c_1 + t \cdot (a_2 x + b_2 y + c_2) = 0 \mid t \in \mathbb{R}\}.$

An interlude, if time permits...

<u>Problem</u>: Decide whether a tumor is benign or malignant.

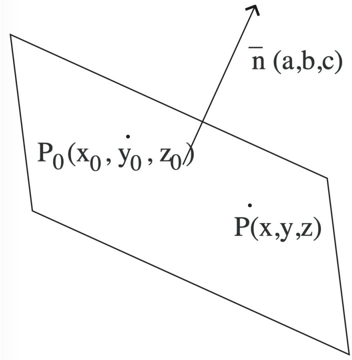


$ax + by = c$

New patient: $a \cdot x_p + b \cdot y_p \underset{<?}{\overset{?}{>}} c$

The analytic representation of planes in space

- Recall that if we endow the 3-dimensional Euclidean space $\mathcal{E}_3$ with a rectangular system of coordinates $Oxyz$, a point $P \in \mathcal{E}_3$ is characterized by three real numbers, the coordinates of the point, $P(x, y, z)$



- A plane π in the 3-dimensional space can be uniquely determined by specifying a point $P_0(x_0, y_0, z_0)$ in the plane and a nonzero vector $\overline{n}(a, b, c)$, orthogonal to the plane. $\overline{n}$ is called the *normal vector* to the plane π.
- An arbitrary point $P(x, y, z)$ is contained into the plane π if and only if

$$\overline{n} \perp \overline{P_0 P},$$

or

$$\overline{n} \cdot \overline{P_0 P} = 0.$$

- But $\overline{P_0 P}(x - x_0, y - y_0, z - z_0)$ and one obtains the *normal* equation of the plane π containing the point $P_0(x_0, y_0, z_0)$ and of normal vector $\overline{n}(a, b, c)$.

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0. \tag{6}$$

Remark: The equation (6) can be written in the form $ax + by + cz + d = 0$.

Theorem

Given $a, b, c, d \in \mathbb{R}$, with $a^2 + b^2 + c^2 > 0$, the equation

$$\pi : \ ax + by + cz + d = 0 \tag{7}$$

describes a plane in $\mathcal{E}_3$. This plane has $\overline{n}(a, b, c)$ as a normal vector.

Short proof: Let's take $P_0, P_1, P_2 \in \pi$.

$$\overline{P_0P_1} \cdot \overline{n} = \overline{P_0P_2} \cdot \overline{n} = 0$$

$$\left(a(x_1 - x_0) + b \cdot (y_1 - y_0) + c(z_1 - z_0) = 0 \right)$$

- The equation $ax + by + cz + d = 0$ with $(a, b, c) \neq (0, 0, 0)$ is sometimes referred to as the "general equation" of the plane.
- Given a fixed point O in the 3-space, any point P is characterized by its position vector $\overline{r}_P = \overline{OP}$.

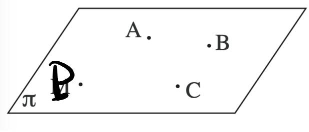
Theorem

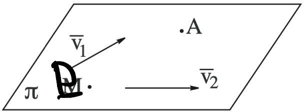
a) The vector equation of the plane π, determined by three noncollinear points A, B and C, is

$$\underset{P}{\overline{r}} = (1 - \alpha - \beta)\overline{r}_A + \alpha\overline{r}_B + \beta\overline{r}_C, \qquad \alpha, \beta \in \mathbb{R}. \tag{8}$$

b) The vector equation of the plane π, determined by a point A and two nonparallel directions $\overline{v}_1$ and $\overline{v}_2$ contained into the plane, is

$$\underset{P}{\overline{r}} = \overline{r}_A + \alpha\overline{v}_1 + \beta\overline{v}_2, \qquad \alpha, \beta \in \mathbb{R}. \tag{9}$$





a) $P \in \pi$ iff. $\overline{AP}$, $\overline{AB}$ and $\overline{AC}$ are linearly dependent. So $\exists\, \alpha, \beta \in \mathbb{R}$ s.t.

$$\overline{AP} = \alpha \cdot \overline{AB} + \beta \cdot \overline{AC}. \iff$$

$$\overline{r}_P - \overline{r}_A = \alpha \cdot (\overline{r}_B - \overline{r}_A) + \beta(\overline{r}_C - \overline{r}_A)$$

$\boxed{3}$

b) $P \in \pi$ iff. $\overline{AP}, \overline{v}_1, \overline{v}_2$ are linearly dependent. $\exists\, \alpha, \beta \in \mathbb{R}$ s.t.

$$\overline{AP} = \alpha \cdot \overline{v}_1 + \beta \cdot \overline{v}_2$$

$$\overline{r}_P = \overline{r}_A + \alpha \cdot \overline{v}_1 + \beta \cdot \overline{v}_2$$

$\boxtimes$

If the points A, B and C which determine the plane π are of coordinates $A(x_A, y_A, z_A)$, $B(x_B, y_B, z_B)$ and $C(x_C, y_C, z_C)$ and an arbitrary point of π is $\cancel{M}P(x, y, z)$, then the equation (8) decomposes into three linear equations:

$$\begin{cases} x = (1 - \alpha - \beta)x_A + \alpha x_B + \beta x_C \\ y = (1 - \alpha - \beta)y_A + \alpha y_B + \beta y_C \\ z = (1 - \alpha - \beta)z_A + \alpha z_B + \beta z_C \end{cases}.$$

This system must have solutions (α, β), so that

$$\begin{vmatrix} x & y & z & 1 \\ x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \end{vmatrix} = 0, \tag{10}$$

which is the analytic equation of the plane determined by three noncollinear points.

The points A, B, C and D are coplanar if and only if:

$$\begin{vmatrix} x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \\ x_D & y_D & z_D & 1 \end{vmatrix} = 0. \tag{11}$$

Replacing now, in (9), the vectors $\overline{v}_1(p_1, q_1, r_1)$ and $\overline{v}_2(p_2, q_2, r_2)$ and the points $A(x_A, y_A, z_A)$ and $M(x, y, z)$, the equation (9) becomes

$$\begin{cases} x = x_A + \alpha p_1 + \beta p_2 \\ y = y_A + \alpha q_1 + \beta q_2 \\ z = z_A + \alpha r_1 + \beta r_2 \end{cases}, \qquad \alpha, \beta \in \mathbb{R}, \tag{12}$$

and these are the parametric equations of the plane. Again, this system must have solutions (α, β), so that

$$\begin{vmatrix} x - x_A & y - y_A & z - z_A \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} = 0, \tag{13}$$

which is the analytic equation of the plane determined by a point and two nonparallel directions.

The points A, B, C and D are coplanar if and only if:

$$\begin{vmatrix} x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \\ x_D & y_D & z_D & 1 \end{vmatrix} = 0. \tag{11}$$

Replacing now, in (9), the vectors $\overline{v}_1(p_1, q_1, r_1)$ and $\overline{v}_2(p_2, q_2, r_2)$ and the points $A(x_A, y_A, z_A)$ and $P(x, y, z)$, the equation (9) becomes

$$\begin{cases} x = x_A + \alpha p_1 + \beta p_2 \\ y = y_A + \alpha q_1 + \beta q_2 \\ z = z_A + \alpha r_1 + \beta r_2 \end{cases}, \qquad \alpha, \beta \in \mathbb{R}, \tag{12}$$

and these are the parametric equations of the plane. Again, this system must have solutions (α, β), so that

$$\begin{vmatrix} x - x_A & y - y_A & z - z_A \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} = 0, \tag{13}$$

which is the analytic equation of the plane determined by a point and two nonparallel directions.

The 6th problem set has been posted. Ideally you would think about it before next weeks' seminar.

Thank you very much for your attention!