Homeworks

Ex1: a) $\sum_{n\geq 3} \frac{7}{9^n} = 7 \cdot \frac{1-\left(\frac{1}{9}\right)^n}{1-\frac{1}{9}} = 7 \cdot \frac{\frac{9^n-1}{9^n}}{\frac{8}{9}} = 7 \cdot \frac{9^n-1}{9^{n+1}} \cdot \frac{9}{8} = \frac{7(9^n-1)}{8 \cdot 9^{n-1}}$

b) $\sum_{n\geq 4} \frac{3^{n-3}+(-4)^{n+3}}{5^n} = \sum_{n\geq 4}\left(\frac{3^{n-3}}{5^n} + \frac{(-4)^{n+3}}{5^n}\right) = \sum_{n\geq 4} \left(\frac{3}{5}\right)^n \cdot \frac{1}{27} +$

$+ \sum_{n\geq 4} \left(\frac{-4}{5}\right)^n \cdot 64 = \frac{1}{27} \cdot \frac{1-\left(\frac{3}{5}\right)^n}{1-\frac{3}{5}} + 64 \cdot \frac{1-\left(-\frac{4}{5}\right)^n}{1+\frac{4}{5}} =$

$= \frac{1}{27} \cdot \frac{5^n-3^n}{5^{n+1}} \cdot \frac{5}{2} + 64 \cdot \frac{5^n-(-4)^n}{5^{n+1}} \cdot \frac{5}{9} = \frac{5^n-3^n}{54 \cdot 5^{n-1}} + \frac{64(5^n-(-4)^n)}{9 \cdot 5^{n-1}}$

c) $\sum_{n\geq 4} e^n = \frac{1-e^n}{1-e}$

d) $\sum_{n\geq 2} \left(-\frac{1}{\pi}\right)^n = \frac{1-\left(-\frac{1}{\pi}\right)^n}{1+\frac{1}{\pi}} = \frac{\frac{\pi^n-(-1)^n}{\pi^n}}{\frac{\pi+1}{\pi}} = [\pi^n-(-1)^n] \cdot \frac{\pi}{\pi+1}$

e) $\sum_{n\geq 3} (-4)^n = \frac{1-(-4)^n}{1+4} = \frac{1-(-4)^n}{5}$

Ex2: a) $\sum_{n\geq 1} \frac{1}{4n^2-1} = \sum_{n\geq 1} \frac{1}{(2n-1)(2n+1)} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \dots + \frac{1}{(2n-1)(2n+1)} =$

$= \frac{1}{2}\left(\frac{1}{1}-\frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3}-\frac{1}{5}\right) + \dots + \frac{1}{2}\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right) =$

$= \frac{1}{2}\cdot\left(1-\frac{1}{2n+1}\right) = \frac{1}{2}\cdot\frac{2n}{2n+1} = \frac{n}{2n+1}$

b) $\sum_{n\geq 1} \frac{1}{\sqrt{n}+\sqrt{n+1}} = \frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \dots + \frac{1}{\sqrt{n}+\sqrt{n+1}} =$

$= \frac{\sqrt{2}-1}{(\sqrt{2}+1)(\sqrt{2}-1)} + \frac{\sqrt{3}-\sqrt{2}}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})} + \dots + \frac{\sqrt{n+1}-\sqrt{n}}{(\sqrt{n}+\sqrt{n+1})(\sqrt{n}-\sqrt{n+1})} =$

$= \frac{\sqrt{2}-1}{1} + \frac{\sqrt{3}-\sqrt{2}}{1} + \dots + \frac{\sqrt{n+1}-\sqrt{n}}{1} = \sqrt{n+1}-1$

c) $\sum\limits_{n\geq 1} \frac{1}{n(n+1)(n+2)} = \frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \ldots + \frac{1}{n(n+1)(n+2)} =$

$= \frac{1}{2}\left(\frac{1}{1\cdot 2} - \frac{1}{2\cdot 3}\right) + \frac{1}{2}\left(\frac{1}{2\cdot 3} - \frac{1}{3\cdot 4}\right) + \ldots + \frac{1}{2}\left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right) =$

$= \frac{1}{2}\left(\frac{1}{1\cdot 2} - \cancel{\frac{1}{2\cdot 3}} + \cancel{\frac{1}{2\cdot 3}} - \cancel{\frac{1}{3\cdot 4}} + \ldots + \frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right) =$

$= \frac{1}{2}\cdot\left(\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right) = \frac{1}{2}\cdot \frac{n^2+3n+2-2}{2(n+1)(n+2)} = \frac{1}{2}\,\frac{n^3+3n}{2(n+1)(n+2)} = \frac{n(n^2+3)}{4(n+1)(n+2)}$

d) $\sum\limits_{n\geq 1} \ln\left(1+\frac{1}{n}\right) = \sum\limits_{n\geq 1} \ln\frac{n+1}{n} = \ln\frac{2}{1} + \ln\frac{3}{2} + \ldots + \ln\frac{n+1}{n} =$

$= \ln(n+1)$

e) $\sum\limits_{n\geq 2} \frac{\ln\left(1+\frac{1}{n}\right)}{\ln\left(n^{\ln(n+1)}\right)} = \sum\limits_{n\geq 2} \frac{\ln\frac{n+1}{n}}{\ln\left(n^{\ln(n+1)}\right)} = \frac{\ln\frac{2}{1}}{\ln\left(1^{\ln 2}\right)} + \frac{\ln\frac{3}{2}}{\ln\left(2^{\ln 3}\right)} + \ldots + \frac{\ln\frac{n+1}{n}}{\ln\left(n^{\ln(n+1)}\right)} =$

$= \frac{\ln 2}{\ln 2\cdot\ln 1} + \frac{\ln\frac{3}{2}}{\ln 3\cdot\ln 2} + \ldots + \frac{\ln\frac{n+1}{n}}{\ln(n+1)\ln n} =$

$= \cancel{\ln 2}\ \frac{1}{\ln 1} - \cancel{\frac{1}{\ln 2}} + \cancel{\frac{1}{\ln 2}} - \cancel{\frac{1}{\ln 3}} + \ldots + \cancel{\frac{1}{\ln n}} - \frac{1}{\ln(n+1)} =$

$= \frac{1}{\ln 1} - \frac{1}{\ln(n+1)} = -\ln(n+1)^{-1}$