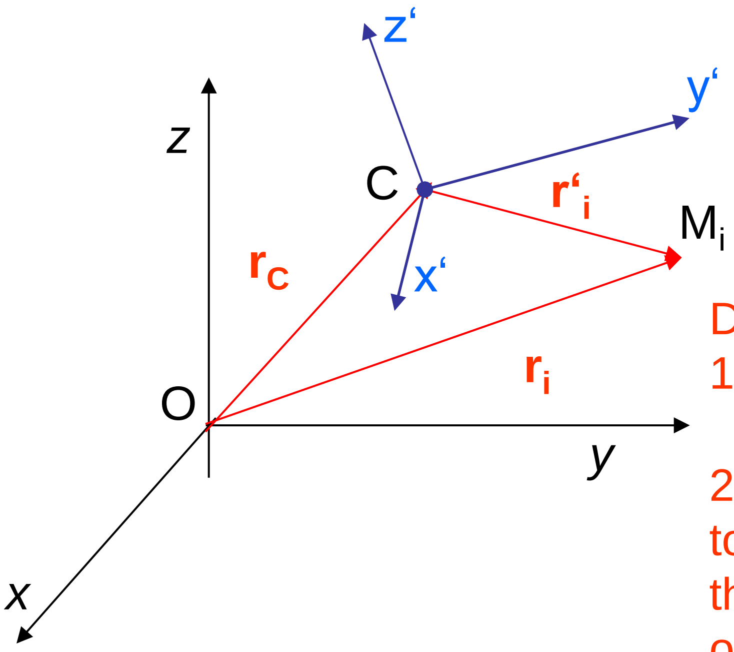
Theoretical Mechanics

15. General theorems of the motion of systems of material points around their mass center

Let be *(S)*: M_i (m_i), $\mathbf{r}_i = \mathbf{OM}_i$, $i = 1,\dots, N$ a moving system of particles moving in an inertial (in particular, fixed) *Oxyz* reference system and consider a Cartesian reference system of coordinates *Cx'y'z'* with origin in the center of the masses *C of* the system *(S)* and invariably oriented in space, *i.e. Cx'y'z'* has only one translation motion with the velocity of the point C relative to the fixed system of coordinates *Oxyz.*



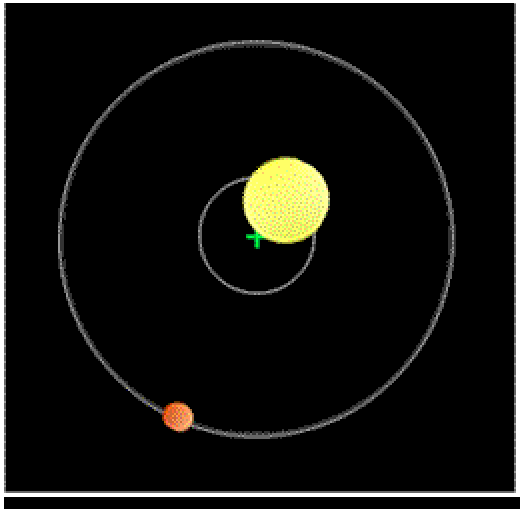
Let be $\mathbf{r'}_i = \mathbf{CM}_i$, $i = 1,\dots, N$. Then we have

$$\vec{r}_i = \vec{r}_C + \vec{r}\,'_i\,, \quad i = 1,\dots, N \tag{15.1}$$

Definition:

1. Cx'y'z' is *a* Koenig frame of reference

2. The motion of the material points system relative to the mobile system of coordinates Cx'y'z' is called the motion of the system around the center of gravity or the motion with respect to a Koenig frame of reference.



Deriving (15.1) we have:

$$\vec{v}_i = \vec{v}_C + \vec{v}'_i \qquad (15.2)$$

Indeed, taking into account that $\vec{r}'_i = x'_i\,\vec{i}\,' + y'_i\,\vec{j}\,' + z'_i\,\vec{k}\,'$ and that Poisson's formulas are

$$\frac{d\vec{i}\,'}{dt} = \vec{\omega}\times\vec{i}\,',\quad \frac{d\vec{j}\,'}{dt} = \vec{\omega}\times\vec{j}\,',\quad \frac{d\vec{k}\,'}{dt} = \vec{\omega}\times\vec{k}\,' \qquad (15.3)$$

Due to the fact that Cx‘y‘z‘ has a motion of translation (**ω** = 0) we have:

$$\frac{d\vec{r}_i}{dt} = \frac{d\vec{r}_C}{dt} + \frac{d\vec{r}\,'_i}{dt} =$$

$$= \frac{d\vec{r}_C}{dt} + \frac{dx'_i}{dt}\vec{i}\,' + \frac{dy'_i}{dt}\vec{j}\,' + \frac{dz'_i}{dt}\vec{k}\,' + \frac{d\vec{i}\,'}{dt}x'_i + \frac{d\vec{j}\,'}{dt}y'_i + \frac{d\vec{k}\,'}{dt}z'_i =$$

$$= \frac{d\vec{r}_C}{dt} + \frac{d\vec{r}\,'_i}{dt} + \vec{\omega}\times\vec{r}\,'_i \underset{\vec{\omega}=0}{=} \frac{d\vec{r}_C}{dt} + \frac{d\vec{r}\,'_i}{dt} = \vec{v}_C + \vec{v}\,'_i$$

But the origin of the mobile frame of reference is the centre of mass, C, and thus we can deduce:

$$0 = \overrightarrow{CC} = \frac{1}{m}\sum_{i=1}^{N} m_i \vec{r}\,'_i, \quad m = \sum_{i=1}^{N} m_i$$

where *m* is the total mass of the system of particles. Therefore, we obtain:

$$\sum_{i=1}^{N} m_i \vec{r}\,'_i = 0 \qquad (15.4)$$

and

$$\sum_{i=1}^{N} m_i \dot{\vec{r}}\,'_i = 0 \qquad (15.5)$$

The moment of momentum of the system of material points with respect to the pole O

$$\vec{K}_O \underset{\text{definition}}{=} \sum_{j=1}^{N} \vec{r}_j \times m_j \vec{v}_j \underset{(1),(2)}{=} \sum_{j=1}^{N} \left(\vec{r}_C + \vec{r}\,'_j\right) \times m_j \left(\vec{v}_C + \vec{v}\,'_j\right) =$$

$$= \vec{r}_C \times m\vec{v}_C + \sum_{j=1}^{N} \vec{r}\,'_j \times m_j \vec{v}\,'_j + \vec{r}_C \times \underbrace{\sum_{j=1}^{N} m_j \vec{v}\,'_j}_{\underset{(14.5)}{=} 0} + \underbrace{\left(\sum_{j=1}^{N} m_j \vec{r}\,'_j\right)}_{\underset{(14.4)}{=} 0} \times \vec{v}_C$$

Thus we have:

$$\vec{K}_O = \vec{r}_C \times m\dot{\vec{r}}_C + \vec{K}\,'_C \qquad (15.6)$$

where

$$\vec{K}\,'_C = \sum_{j=1}^{N} \vec{r}\,'_j \times m_j \dot{\vec{r}}\,'_j \qquad (15.7)$$

Equation (15.7) define the moment of momentum of the of the system with respect to the centre of mass (in the relative motion, with respect to a Koenig frame with the pole at *C.*

Equation (15.6) is the first Koenig's (König) formula.

Theorem (Koenig) The moment of momentum of a mechanical system with respect to a fixed pole is equal to the sum of the moment of momentum of the same system with respect to the pole of a Koenig frame of reference and the moment of momentum of its mass centre at which the whole mass of the mechanical system is concentrated, with respect to the fixed pole..

Analogously, the kinetic energy is given by:

$$T = \frac{1}{2}\sum_{j=1}^{N} m_j v_j^{\,2} \underset{(2)}{=} \frac{1}{2}\sum_{j=1}^{N} m_j \left(\dot{\vec{r}}_C + \vec{v}'_j\right)^2 =$$

$$=\frac{1}{2}\sum_{j=1}^{N} m_j \vec{v}_C{}^2+\frac{1}{2}\sum_{j=1}^{N} m_j \vec{v}'_j{}^2+\underbrace{\sum_{j=1}^{N} m_j \vec{v}'_j\,\vec{v}_C}_{\underset{(14.5)}{=}\;0} \qquad (15.7)$$

or

$$T=\frac{1}{2}mv_C{}^2+T'_C \qquad (15.8)$$

where

$$T'_C=\frac{1}{2}\sum_{j=1}^{N} m_j \vec{v}'_j{}^2 \qquad (15.9)$$

is the kinetic energy of the system in relative motion (with respect to the mass centre).

Equation (15.8) is the second Koenig's formula.

Theorem (Koenig). The kinetic energy of a mechanical system with respect to a fixed frame of reference is equal to the sum of the kinetic energy of the same system with respect to a Koenig frame of reference and the kinetic energy of the mass centre at which the whole mass of the mechanical system is concentrated, with respect to the fixed frame.

Theorem of the moment of momentum in the motion of a mechanical system around the centre of mass, C

The theorem of the moment of momentum is:

$$\frac{d\vec{K}_O}{dt} = \vec{M}_O \qquad (15.10)$$

where $\vec{M}_O = \sum_{j=1}^{N} \vec{r}_j \times \vec{F}_j$ is the moment of the resultant of the external forces. But,

$$\frac{d\vec{K}_O}{dt} \underset{(14.6)}{=} \frac{d}{dt}\left(\vec{r}_C \times m\vec{v}_C + \vec{K}'_C\right) = \vec{r}_C \times m\vec{a}_C + \dot{\vec{K}}'_C$$

Using the theorem of the centre of mass, m**a**$_C$ = **R**, we have:

$$\frac{d\vec{K}_O}{dt} = \vec{r}_C \times \vec{R} + \dot{\vec{K}}'_C \qquad (15.11)$$

Next, we change the pole in the moment formula

$$\vec{M}_O = \vec{M}_C + \overrightarrow{OC} \times \vec{R} \qquad (15.12)$$

And from (15.11) and (15.12) we obtain :

$$\vec{r}_C \times \vec{R} + \dot{\vec{K}}'_C = \overrightarrow{OC} \times \vec{R} + \vec{M}_C \quad \Rightarrow \quad \frac{d\vec{K}'_C}{dt} = \vec{M}_C \qquad (15.13)$$

Theorem of the moment of momentum in the motion of the system around the center of the mass:

The derivative with respect to time t of the moment of momentum of the system relative to the center of mass is equal to the resulting moment of external forces evaluated with respect to the center of the mass.

Theoretical Mechanics

Theorem of the kinetic energy in the motion of the system around the centre of mass, C

The theorem of the kinetic energy for a system of material points is:

$$dT = \delta L^{(\text{ext})} + \delta L^{(\text{int})} = \sum_{j=1}^{N} \vec{F}_j d\vec{r}_j + \sum_{j=1}^{N} \sum_{k=1}^{j-1} \vec{F}_{jk} d\vec{r}_{jk} \qquad (15.14)$$

Using the second Koenig's formula we have:

$$dT = d\left(\frac{1}{2} m \dot{\vec{r}}_C^{\,2} + T'_C\right) = dT'_C + m\dot{\vec{r}}_C d\dot{\vec{r}}_C \qquad (15.15)$$

The elementary work of the external forces is:

$$\sum_{j=1}^{N} \vec{F}_j d\vec{r}_j \underset{(1)}{=} \sum_{j=1}^{N} \vec{F}_j d\left(\vec{r}_C + \vec{r}\,'_j\right) = \sum_{j=1}^{N} \vec{F}_j d\vec{r}\,'_j + \left(\sum_{j=1}^{N} \vec{F}_j\right) d\vec{r}_C = \sum_{j=1}^{N} \vec{F}_j d\vec{r}\,'_j + \vec{R} d\vec{r}_C$$

(15.16)

Using the theorem of the centre of mass we get:

$$m\frac{d\dot{\vec{r}}_C}{dt}=\vec{R}\Big|\cdot d\vec{r}_C \Rightarrow m\frac{d\dot{\vec{r}}_C}{dt}d\vec{r}_C=\vec{R}\cdot d\vec{r}_C \Rightarrow m d\dot{\vec{r}}_C\cdot\dot{\vec{r}}_C=\vec{R}\cdot d\vec{r}_C$$

(15.17)

Using (15.16) and (15.17) we obtain:

$$\sum_{j=1}^{N}\vec{F}_j d\vec{r}_j=\sum_{j=1}^{N}\vec{F}_j d\vec{r}\,'_j+\vec{R}d\vec{r}=\sum_{j=1}^{N}\vec{F}_j d\vec{r}\,'_j+m d\dot{\vec{r}}_C\dot{\vec{r}}_C \qquad (15.18)$$

and using (15.14), (15.15) and (15.18) we have:

$$dT'=\delta L'^{(\mathrm{ext})}+\delta L'^{(\mathrm{int})}=\sum_{j=1}^{N}\vec{F}_j d\vec{r}\,'_j+\sum_{j,k=1}^{N}\vec{F}_{jk}d\vec{r}\,'_j \qquad (15.19)$$

where $\delta L'^{(ext)}$ is the elementary work of the external forces relative to the system of reference Cx‘y‘z‘, while $\delta L'^{(int)}$ is the elementary work of the inner forces relative to the system of reference Cx‘y‘z‘.

Theorem of the kinetic energy in the motion of the system around the centre of mass C:

The differential of the kinetic energy of a system moving about the centre of mass C is equal with the sum of the elementary work of the external forces and the elementary work of the inner forces evaluated relative to the system of reference Cx‘y‘z‘.

16. Moments of inertia

Let be (S) a system of material points (discrete – finite number of points M_i (m_i), i = 1,..., N or continuous – rigid body) in an orthogonal frame of reference$Ox_1x_2x_3$ and V a simple manifold in $\mathbf{R}^3$ (point, line or plane)

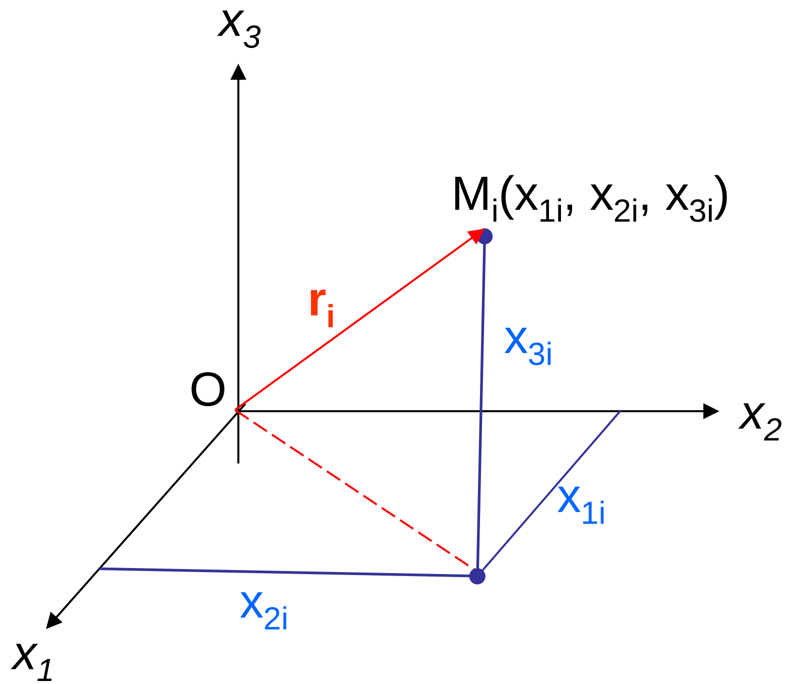
Definition: The scalar value I(V) given by:

(16.1)
$$I(V)=\begin{cases} \sum_{i=1}^{N} m_i d_i^{\,2} & \text{, (S) discrete system (N points)} \\ \int_V d^2 dm = \int_D d^2 \rho \, d\tau & \text{, (S) rigid body (domain D, } d\tau \text{ is the element of volume, area or length)} \end{cases}$$

Is the moment of inertia of the system (S) with respect to the manifold V.
Depending on the manifold the moments can be polar axial or planar.

In equation (16.1)

- d_i is the distance between the point M_i and the manifold V
- d is the distance between the integration point M and the manifold V.



If V = Ox_1 or Ox_2 or Ox_3, we define the moments of inertia about a the axis:

$$I_{11} = I(Ox_1) = \begin{cases} \sum_{i=1}^{N} m_i \left({x_{2i}}^2 + {x_{3i}}^2 \right) & \text{, discrete system} \\ \int_V \left({x_2}^2 + {x_3}^2 \right) dm & \text{, rigid body} \end{cases} \qquad (16.2a)$$

$$I_{22} = I(Ox_2) = \begin{cases} \sum_{i=1}^{N} m_i \left({x_{1i}}^2 + {x_{3i}}^2 \right) & \text{, discrete system} \\ \int_V \left({x_1}^2 + {x_3}^2 \right) dm & \text{, rigid body} \end{cases} \qquad (16.2b)$$

$$I_{33} = I(Ox_3) = \begin{cases} \sum_{i=1}^{N} m_i \left({x_{1i}}^2 + {x_{2i}}^2 \right) & \text{, discrete system} \\ \int_V \left({x_1}^2 + {x_2}^2 \right) dm & \text{, rigid body} \end{cases} \qquad (16.2c)$$

In a similar way we can introduce the product of inertia:

$$I_{12} = \begin{cases} \sum_{i=1}^{N} m_i x_{1i} x_{2i} \\ \int_V x_1 x_2 dm \end{cases} \quad (16.3a)$$

$$I_{23} = \begin{cases} \sum_{i=1}^{N} m_i x_{2i} x_{3i} \\ \int_V x_2 x_3 dm \end{cases} \quad (16.3b)$$

$$I_{31} = \begin{cases} \sum_{i=1}^{N} m_i x_{3i} x_{1i} \\ \int_V x_3 x_1 dm \end{cases} \quad (16.3c)$$

One can observe that $I_{12} = I_{21}$, $I_{23} = I_{32}$, $I_{31} = I_{13}$. Using (16.2) and (16.3) it is possible to build the matrix (tensor) of inertia with respect to the point O

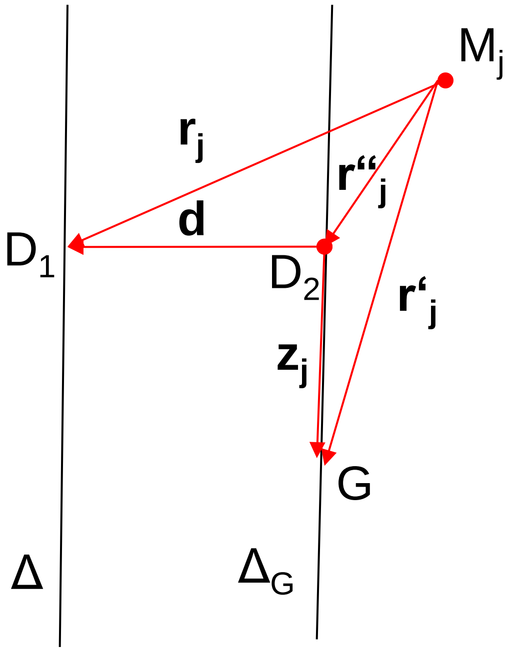
$$I_O = \begin{pmatrix} I_{11} & -I_{12} & -I_{13} \\ -I_{21} & I_{22} & -I_{23} \\ -I_{31} & -I_{32} & I_{33} \end{pmatrix} \quad (16.4)$$

Steiner's theorem:

Let be G the center of massof the material system (S). Consider Δ and Δ_G two parallel axis, such that G∈ Δ_G. We note d = dist (Δ, Δ_G). Then for the moment of inertia of the system with respect to Δ and Δ_G we have:

$$\boxed{I(\Delta) = I(\Delta_G) + md^2} \qquad (16.5)$$

where m is the mass of the system.



We have:

$$\vec{r}_j = \vec{d} + \vec{r}\,''_j \quad \sum m_j \vec{r}\,'_j = 0 \quad \sum m_j \vec{z}_j = 0$$

Indeed: $\qquad 0 = \overrightarrow{GG} = \frac{1}{m}\sum_{i=1}^{N} m_i \vec{r}\,'_i$ and

$$\sum m_j \vec{z}_j = pr_{\Delta_g}\left(\sum m_j \vec{r}\,'_j\right) = 0$$

We obtain:

$$\sum m_j \vec{r}''_j = \sum m_j \left(\vec{r}'_j - \vec{z}_j\right) = 0$$

and the axial moment of inertia with respect to Δ is:

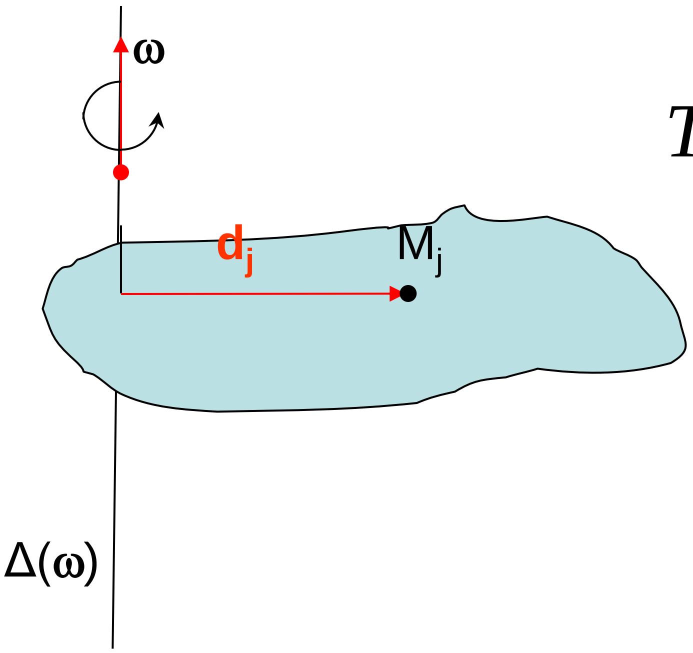
$$I(\Delta) = \sum m_j \vec{r}_j^{\,2} = \sum m_j \left(\vec{d} + \vec{r}''_j\right)^2 =$$

$$= \sum m_j \vec{r}''^{\,2}_j + md^2 + 2\vec{d} \cdot \sum m_j \vec{r}''_j =$$

$$= I(\Delta_G) + md^2$$

The moment of inertia of a mechanical system with respect to an axis Δ is equal to the sum of the moment of inertia of the same system with respect to an axis Δ_G parallel to the first one, passing through the centre of mass, and the moment of inertia of the centre of mass, at which we consider concentrated the mass of the whole mechanical system, with respect to the axis Δ .

Remark: In the case of rotation of rigid system (S) about a fixed axis $\Delta(\omega)$ with the angular velocity ω, the kinetic energy of the system is given by:



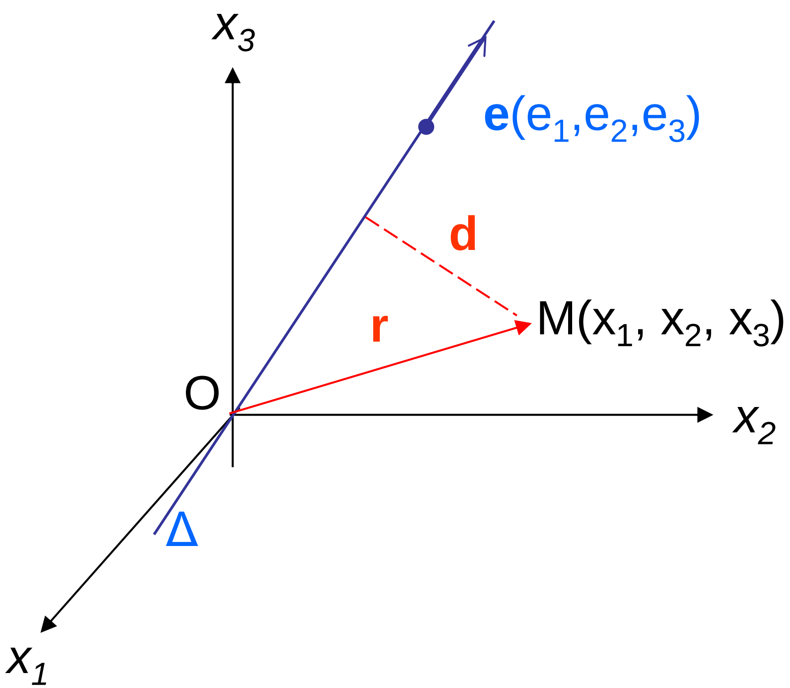
$$T = \frac{1}{2}\sum m_j v_j^{\,2} = \frac{1}{2}\sum m_j \left(\vec{\omega}\times\vec{d}_j\right)^2 =$$

$$= \frac{1}{2}\sum m_j \omega^2 d_j^{\,2} \underbrace{\sin^2\left(\vec{\omega},\vec{d}_j\right)}_{=1} = \frac{1}{2} I(\Delta)\omega^2$$

$$\boxed{T = \frac{1}{2} I(\Delta)\omega^2}$$

Moments of inertia with respect to concurrent axis

Consider $\Delta(\mathbf{e})$ an axis passing through the point O, e = (e_1, e_2, e_3) in the system of reference $Ox_1x_2x_3$, $e_1^2 + e_2^2 + e_3^2 = 1$ ($\mathbf{e}$ is the unit vector of the axis Δ).



For rigid body (S) we have:

$$I(\Delta) = \int_S d^2 dm =$$

$$= \int_S \left(\vec{r}^{\,2} - (\vec{r} \cdot \vec{e})^2\right) dm =$$

$$= \int_S \left\{ \left(x_1{}^2 + x_2{}^2 + x_3{}^2 \right)\left(e_1{}^2 + e_2{}^2 + e_3{}^2 \right) - \left(x_1 e_1 + x_2 e_2 + x_3 e_3 \right)^2 \right\} dm =$$

$$= \int_S \left\{ \left(x_2{}^2 + x_3{}^2 \right) e_1{}^2 + \left(x_1{}^2 + x_3{}^2 \right) e_2{}^2 + \left(x_1{}^2 + x_2{}^2 \right) e_3{}^2 - \right.$$

$$\left. - 2 x_1 x_2 e_1 e_2 - 2 x_2 x_3 e_2 e_3 - 2 x_3 x_1 e_3 e_1 \right\} dm$$

Thus

$$I(\Delta) = I_{11} e_1{}^2 + I_{22} e_2{}^2 + I_{33} e_3{}^2 - 2 I_{12} e_1 e_2 - 2 I_{23} e_2 e_3 - 2 I_{31} e_3 e_1$$

(16.6)

or in a matrix (tensor) form:

$$I(\Delta) = \left\{ \vec{e} \right\}^T \left[I_O \right] \left\{ \vec{e} \right\}$$ (16.6‘)

Matrix I_O is symmetric with real element and thus it is diagonalizable (can be reduced to diagonal form)

Property: In the arbitrary point O exist three directions $\Delta_i(\mathbf{e}^{(i)})$, i = 1,2,3 to which the moments of inertia have extreme values. Moreover, these directions are orthogonal ($\mathbf{e}^{(i)} \perp \mathbf{e}^{(j)}$, $i \neq j$) and considering the matrix in the system of reference defined by these directions and the origin in O, the matrix of inertia will have:

$$\begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}$$

where A, B, C are the moments of inertia of the system (S) with respect to the three directions, called principal axis of inertia relative to O.

A, B, C are the eigenvalues of the matrix I_O, while $\mathbf{e}^{(1)}$, $\mathbf{e}^{(2)}$, $\mathbf{e}^{(3)}$ are the eigenvectors.

The kinetic energy can be $(\vec{v} = \vec{\omega} \times \vec{r})$ obtain:

$$T = \frac{1}{2} I(\Delta)\omega^2$$

$$\boxed{T_{\text{rot}} = \frac{1}{2}\boldsymbol{\omega} \cdot \{\mathbf{I}\} \cdot \boldsymbol{\omega}}$$

$$= \frac{1}{2}\omega^2(I_{11}\alpha^2 + I_{22}\beta^2 + I_{33}\gamma^2 - 2I_{12}\alpha\beta - 2I_{23}\beta\gamma - 2I_{31}\gamma\alpha),$$

where $\vec{u} = (\alpha, \beta, \gamma)$ is the versor of $\Delta(\vec{\omega})$

If $Ox_1x_2x_3$ is determined by the principal axis of inertia relative to O ($I_{12} = I_{23} = I_{31} = 0$) we obtain:

$$T = \frac{1}{2}(Ap^2 + Bq^2 + Cr^2)$$

$$\vec{\omega} = (p, q, r)$$

$$\alpha = \frac{p}{\omega}, \quad \beta = \frac{q}{\omega}, \quad \gamma = \frac{r}{\omega}$$

$$\mathbf{I}_0 = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}.$$

For the moment of momentum we obtain:

$$\overrightarrow{K}_0 = \int_S \overrightarrow{r} \times \overrightarrow{v}\, dm = \int_S \overrightarrow{r} \times (\overrightarrow{\omega} \times \overrightarrow{r}) dm$$
$$= \int_S \{r^2 \overrightarrow{\omega} - (\overrightarrow{r} \cdot \overrightarrow{\omega}) \overrightarrow{r}\} dm$$
$$= \int_D \{r^2 \overrightarrow{\omega} - (\overrightarrow{r} \cdot \overrightarrow{\omega}) \overrightarrow{r}\} \rho dv = [\mathbf{I}] \cdot [\omega].$$

Consider: $K_{x_i} = pr_{Ox_i} \overrightarrow{K}_0,\ i = 1, 2, 3.$ we have:

$$K_{x_1} = \int_S \{(x_1^2 + x_2^2 + x_3^2)p - (px_1 + qx_2 + rx_3)x_1\} dm$$
$$= I_{11}p - I_{12}q - I_{13}r$$

Therefore:

$$\Rightarrow \quad K_{x_1} = I_{11}p - I_{12}q - I_{13}r \quad \left(= \frac{\partial T}{\partial p}\right);$$

$$K_{x_2} = -I_{21}p + I_{22}q - I_{23}r \quad \left(= \frac{\partial T}{\partial q}\right);$$

$$K_{x_3} = -I_{31}p - I_{32}q + I_{33}r \quad \left(= \frac{\partial T}{\partial r}\right)$$

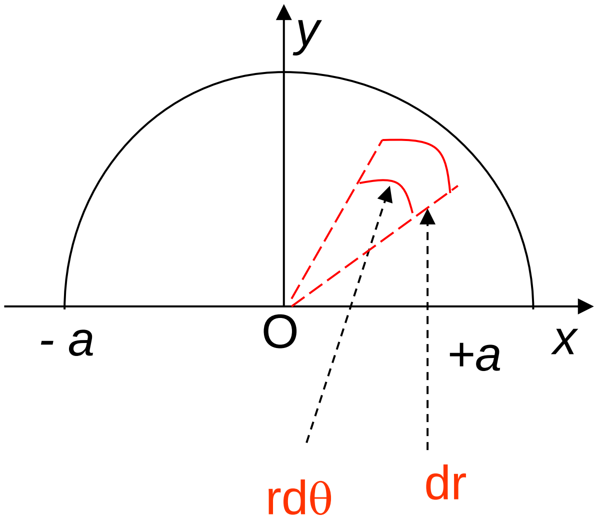
$$\Rightarrow \overrightarrow{K}_0 = grad\ T = \left(\frac{\partial T}{\partial p}, \frac{\partial T}{\partial q}, \frac{\partial T}{\partial r}\right)$$

If $Ox_1x_2x_3$ is determined by the principal axis of inertia relative to O ($I_{12} = I_{23} = I_{31} = 0$) we obtain:

$$\overrightarrow{K}_0 = (Ap, Bq, Cr).$$

Example

1.Calculate the mass M and the coordinates of the centre of mass (x_G, y_G) for a half disk of radius a and density ρ.



$$M = \frac{1}{2} A_{disc} \rho = \frac{\pi a^2 \rho}{2}$$

$$x_G = \frac{1}{M} \iint_D x dm = \frac{\rho}{M} \int_0^a \int_0^\pi r^2 \cos\theta dr d\theta = 0$$

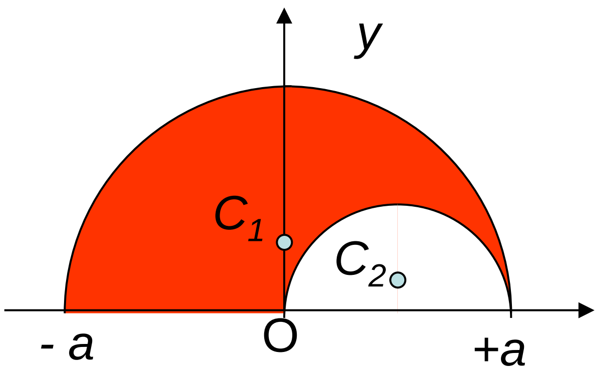
$$dm = \rho dV, \quad dV = dxdy = r dr d\theta$$

Remark: Due to the symmetry we notice that x_G =0.

$$y_G = \frac{1}{M} \iint_D y dm = \frac{\rho}{M} \int_0^a \int_0^\pi r^2 \sin\theta dr d\theta = \frac{4a}{3\pi}$$

Example

2.Calculate the mass M and the coordinates of the mass centre (x_G, y_G) for the figure given bellow (density ρ).



$$M = M_1 - M_2 =$$

$$= \frac{\pi a^2 \rho}{2} - \frac{\pi (a/2)^2 \rho}{2} = \frac{3\pi a^2 \rho}{8}$$

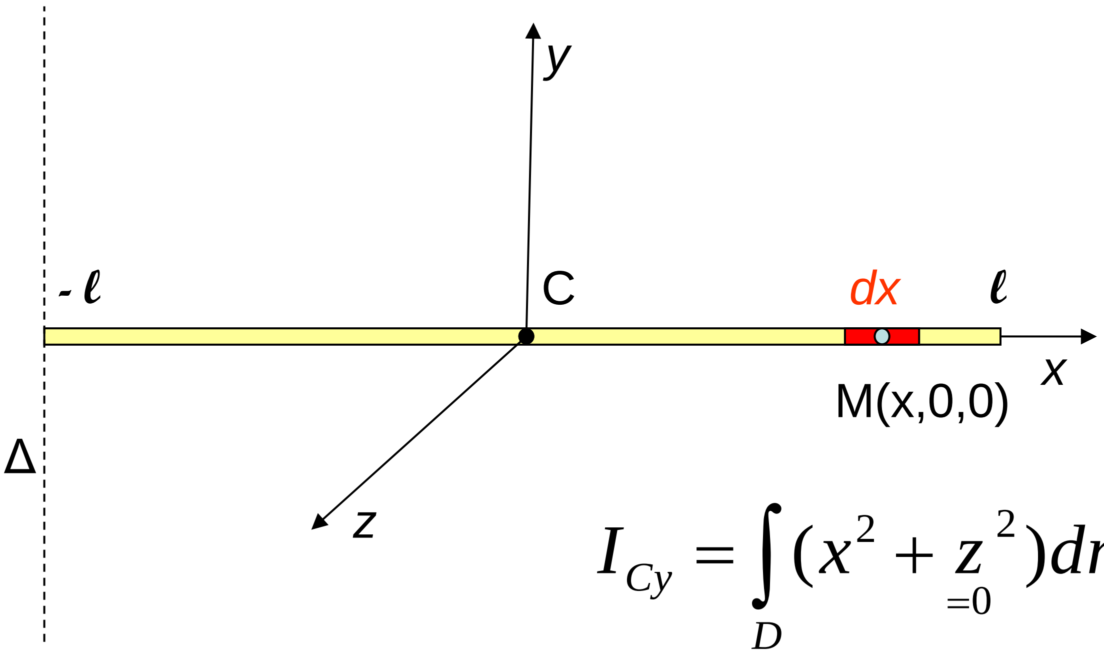
$$x_{G_1} = 0, \quad y_{G_1} = \frac{4a}{3\pi}, \quad x_{G_2} = \frac{a}{2}, \quad y_{G_2} = \frac{2a}{3\pi}$$

Remark: The missing mass can be considered negative.considera masa negativa.

$$x_G = \frac{M_1 x_{G_1} - M_2 x_{G_2}}{M_1 - M_2} = -\frac{a}{6}, \quad y_G = \frac{M_1 y_{G_1} - M_2 y_{G_2}}{M_1 - M_2} = \frac{14a}{9\pi}$$

Example

3. Calculate the moment of inertia relative to the centre of inertia C and relative to the axis Δ for a linear bar of length 2ℓ and density ρ.



$$I_{Cx} = \int_D (\underbrace{y^2}_{=0} + \underbrace{z^2}_{=0}) dm = 0$$

We use:

$$dm = \rho dl = \rho dx$$

$$I_{Cy} = \int_D (x^2 + \underbrace{z^2}_{=0}) dm = \int_D x^2 dm = \rho \int_{-l}^{l} x^2 dx = \frac{ml^2}{3}$$

$$I_{zy} = \int_D yzdm = 0, \ I_{xz} = \int_D xzdm = 0, \ I_{xy} = \int_D xydm = 0$$

Using the Steiner's theorem:

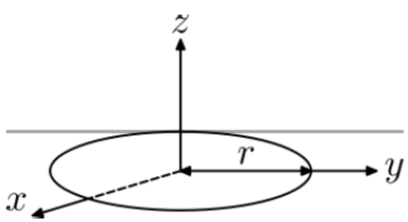
$$I_{\Delta} = I_G + Md^2 = I_{Cy} + ml^2 = \frac{4ml^2}{3}$$

If (S) admits a plane of symmetry π then the centre of mass G, belongs to the plane, two principal axis of inertia belong to the plane π and the third is perpendicular on the plane π.

If Δ is an axis of symmetry of the system (S), then the centre of mass belongs to the axis and Δ is a principal axis of inertia.

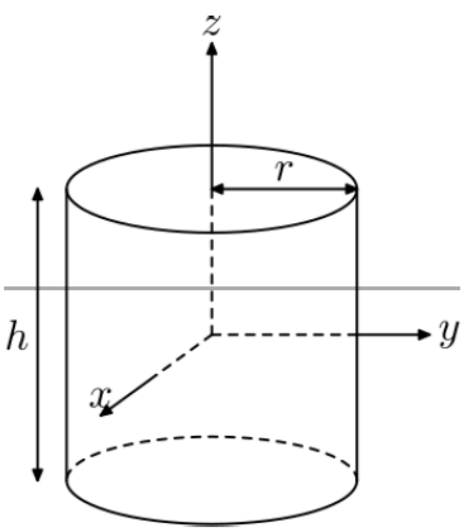
Usual moment of inertia:

Cyrcle



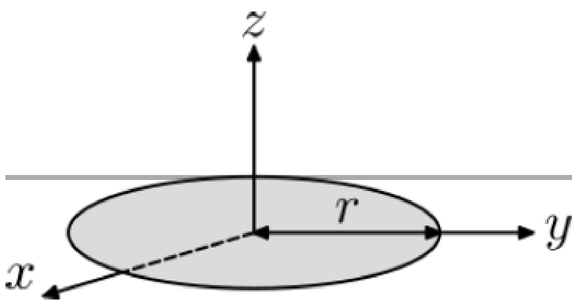
$$I_z = mr^2$$
$$I_x = I_y = \frac{1}{2}mr^2$$

Cylinder



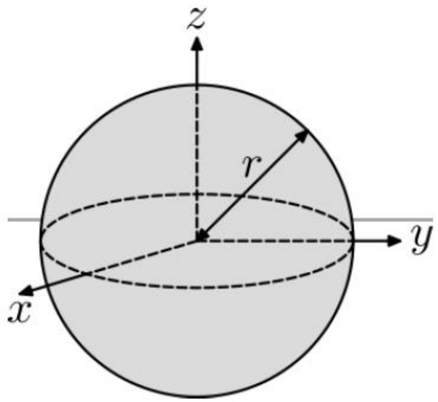
$$I_z = \frac{1}{2}mr^2$$ [1]
$$I_x = I_y = \frac{1}{12}m\left(3r^2 + h^2\right)$$

-disk



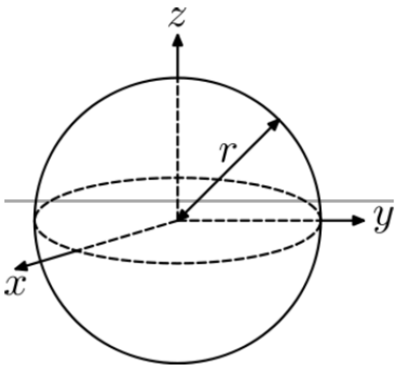
$$I_z = \frac{1}{2}mr^2$$
$$I_x = I_y = \frac{1}{4}mr^2$$

-sphere

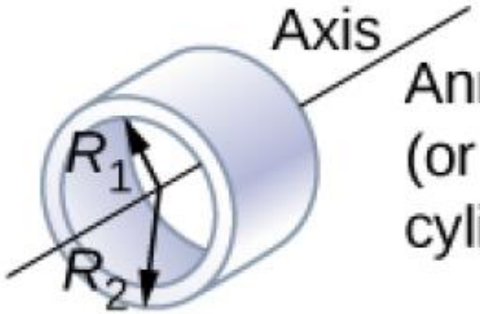


$$I = \frac{2}{5}mr^2$$

- hollow sphere

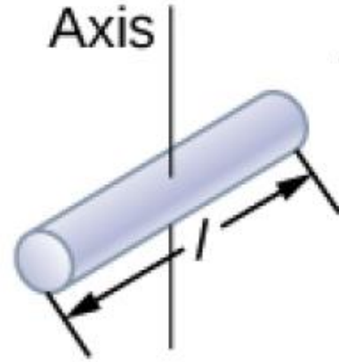


$$I = \frac{2}{3}mr^2$$



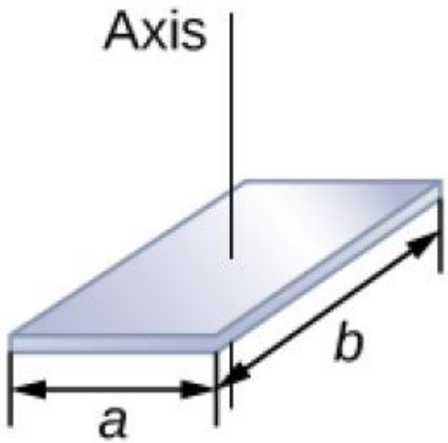
Annular cylinder (or ring) about cylinder axis

$$I = \frac{M}{2}\left(R_1^2 + R_2^2\right)$$



Thin rod about axis through center ⊥ to length

$$I = \frac{Ml^2}{12}$$



Slab about ⊥ axis through center

$$I = \frac{M(a^2 + b^2)}{12}$$

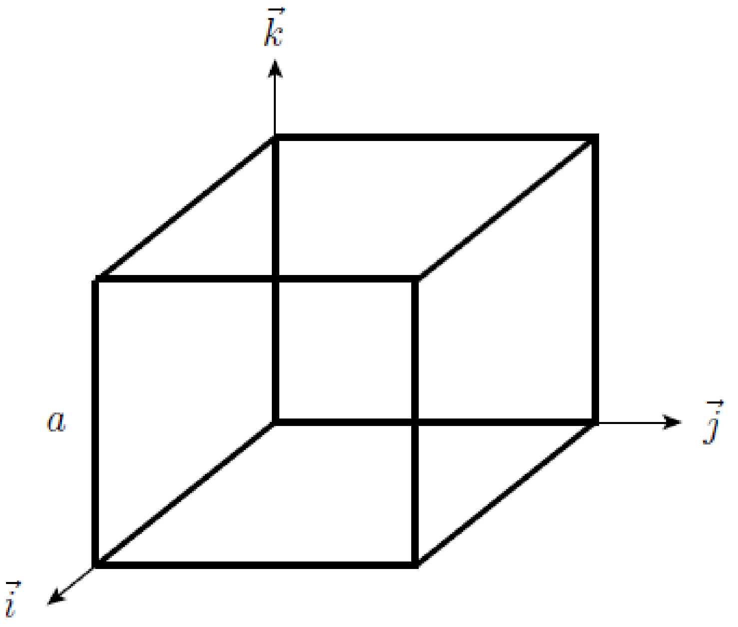
If (S) admits a plane of symmetry π then the centre of mass G, belongs to the plane, two principal axis of inertia belong to the plane π and the third is perpendicular on the plane π.

If Δ is an axis of symmetry of the system (S), then the centre of mass belongs to the axis and Δ is a principal axis of inertia.

Example. Calculate the inertia tensor with respect to the coordinate axes {**i**, **j**, **k**} for a cube of uniform mass density and side *a.*

$$I_{11} = \rho \int_V dV \left(\delta_{11}(x^2 + y^2 + z^2) - x^2 \right)$$

$$= \rho \int_V dV (y^2 + z^2) .$$



Introducing the integration limits

$$\begin{aligned} I_{11} &= \rho \int_0^a \int_0^a \int_0^a dx\,dy\,dz\,(y^2+z^2) \\ &= \rho\,a \int_0^a \int_0^a dy\,dz\,(y^2+z^2) \\ &= \rho\,a \left(a\frac{1}{3}a^3 + a\frac{1}{3}a^3 \right) \\ &= \frac{2}{3}\rho\,a^5\,, \end{aligned}$$

or in terms of the mass

$$I_{11} = \frac{2}{3} M a^2\,.$$

One can straightforwardly check that $I_{11} = I_{22} = I_{33}$. One can also observe from the definition of the inertia tensor that it is, in general, symmetric i.e., $I_{ij} = I_{ji}$. Let us now calculate the remaining components. We obtain

$$I_{12} = I_{21} = -\rho \int_0^a \int_0^a \int_0^a dx\, dy\, dz\, x\, y = -\frac{1}{4}\rho\, a^5 = -\frac{1}{4} M a^2 .$$

The remaining components are equal to the previously calculated one. Hence, we can write the inertia tensor in matrix form as

$$I = M a^2 \begin{pmatrix} \frac{2}{3} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{pmatrix} .$$

Examples:

Rotation about x axis ($\boldsymbol{\omega} = (\omega, 0, 0)$): K (not)**=L= $\mathbf{I}\boldsymbol{\omega} = Ma^2/12\ (8\omega, -3\omega, -3\omega)$**

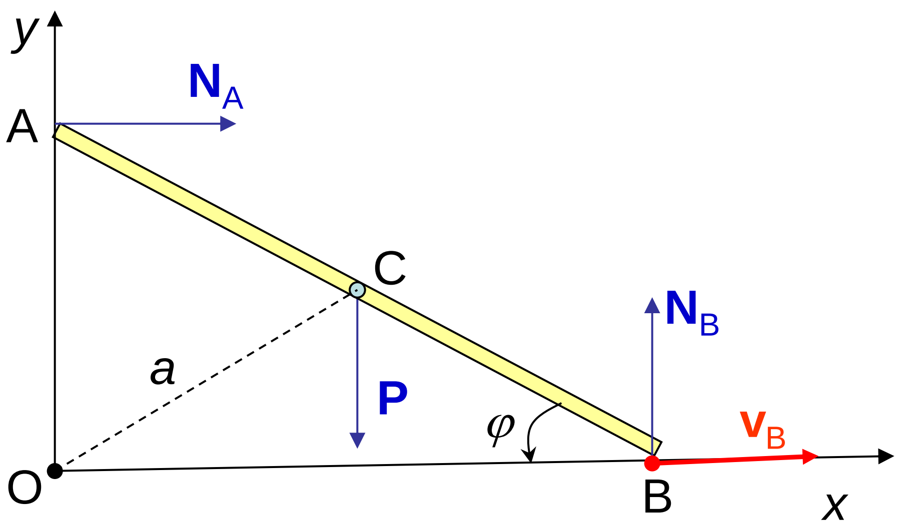
$= Ma^2\omega\ (2/3, -1/4, -1/4)$.

Rotation about diagonal through O ($\boldsymbol{\omega} = \omega / \sqrt{3}\ (1, 1, 1)$):

$$\mathbf{L} = \mathbf{I}\boldsymbol{\omega} = \frac{Ma^2}{12} \frac{\omega}{\sqrt{3}} \begin{bmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{Ma^2}{12} \frac{\omega}{\sqrt{3}} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \frac{Ma^2}{6} \boldsymbol{\omega}.$$

Example

4. A homogenous bar AB of length 2a and weight **P** moves under the action of its own weight, sliding with the ends A and B on the smooth vertical wall Oy and on the smooth horizontal floor Ox, respectively. Find the angular velocity $\boldsymbol{\omega}$ of the bar and the pressures $\mathbf{N}_A$ and $\mathbf{N}_B$ exerted by the wall and the floor as functions of the angle φ between the bar and Ox. At t=0 the angle was $\varphi = \varphi_0$. Find the value of φ for which the bar breaks off the wall.



The weight **P** is a potential force and thus we can use the theorem of the energy conservation. The work due to $\mathbf{N}_A$ and $\mathbf{N}_B$ is zero, the reactions are perpendicular on the displacements.

We can use the Koenig's theorem :

$$T = \frac{1}{2} m v_C{}^2 + T'_C \qquad \text{(i)}$$

where T'$_C$ is the kinetic energy of the system in motion around the centre of mass (inertia).
In the motion of the rigid body about the fixed axis Δ with the angular velocity ω the kinetic energy is given by:

$$T = \frac{1}{2} I_\Delta \omega^2 \qquad \text{(ii)}$$

Therefore, we have:

$$T = \frac{1}{2} m v_C{}^2 + \frac{1}{2} I_{Cz} \dot{\varphi}^2, \quad I_{Cz} = \frac{ml^2}{3} \qquad \text{(iii)}$$

Consider: $P = mg, \; x_C = a\cos\varphi, \; y_C = a\sin\varphi$ and we get:

$$T = \frac{2}{3} m a^2 \dot{\varphi}^2 \qquad \text{(iv)}$$

The elementary work is, $\delta L^{(ext)} = -dV$:

$$\delta L^{(ext)} = \vec{P} \cdot d\overrightarrow{OC} + \underbrace{\vec{N}_A \cdot d\overrightarrow{OA}}_{=0} + \underbrace{\vec{N}_B \cdot d\overrightarrow{OB}}_{=0} = -mgdy_C$$

$$\vec{P} = -gradV \;\Rightarrow -mg = \frac{dV}{dy_C} \;\Rightarrow V = mgy_C = mga\sin\varphi$$

Using the theorem of energy conservation , T+V=h, we obtain:

$$T + V = \frac{2}{3}ma^2\dot{\varphi}^2 + mga\sin\varphi = h$$

The constant h can be obtained using the initial conditions and we have:

$$\frac{2}{3}ma^2\dot{\varphi}^2 + mga(\sin\varphi - \sin\varphi_0) = 0 \qquad \text{(v)}$$

Thus:

$$\dot{\varphi}^2 = \frac{3}{2}\frac{g}{a}\left(\sin\varphi_0 - \sin\varphi\right) \qquad \text{(vi)}$$

And deriving (vi) we obtain the differential equation for the angular velocity ω =dφ/dt.

$$\ddot{\varphi} = -\frac{3g}{4a}\cos\varphi, \;\; \varphi(0) = \varphi_0, \;\; \dot{\varphi}(0) = 0 \qquad \text{(vii)}$$

In order to obtain the reactions we use the equation of the centre of mass:

$$m\vec{a}_C = \vec{N}_A + \vec{N}_B + \vec{P} \qquad \text{(viii)}$$

Eq. (viii) in Cartesian coordinates:

$$\begin{cases} m\ddot{x}_C = N_A \\ m\ddot{y}_C = N_B - mg \end{cases} \qquad \text{(ix)}$$

Using (iii), (vi) and (vii) in (ix) we obtain:

$$N_A = \frac{3}{4} P \cos\varphi \left(3\sin\varphi - 2\sin\varphi_0\right) \qquad \text{(x)}$$

$$N_B = \frac{mg}{4}\left(1 + 9\sin^2\varphi - 6\sin\varphi\sin\varphi_0\right) \qquad \text{(xi)}$$

The bar break off the wall when $N_A = 0$ and using (x) we obtain:

$$\varphi_{\text{desprindere}} = \arcsin\left(\frac{2}{3}\sin\varphi_0\right) \qquad \text{(xii)}$$