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Homework 5

Ex1: a) $\sum_{n\geq 1} \frac{3n^4+7}{\sqrt{n^8+7n^3}} = \sum_{n\geq 1} x_n \quad \lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{3n^4+7}{\sqrt{n^8+7n^3}} = \lim_{n\to\infty} \frac{n^4(3+\frac{7}{n^4})}{n^4\sqrt{1+\frac{7}{n^5}}} =$

$= \lim_{n\to\infty} 3 = 3 \Rightarrow \sum x_n$ is D.

b) $\sum_{n\geq 1} \frac{1}{\sqrt[n]{n}} = \sum_{n\geq 1} x_n \quad \lim_{n\to\infty} x_n = \lim_{n\to\infty} n^{-\frac{1}{n}} = \infty^0 = \lim_{n\to\infty} e^{\ln n^{-\frac{1}{n}}} =$

$= \lim_{n\to\infty} e^{-\frac{1}{n}\ln n} = 1 \Rightarrow \sum x_n$ is D

c) $\sum_{n\geq 1} \frac{1}{\sqrt[n]{n!}} = \sum_{n\geq 1} x_n \quad \lim_{n\to\infty} x_n = \lim_{n\to\infty} (n!)^{-\frac{1}{n}} = \cancel{\lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} n+1 = \infty}$

$= \lim_{n\to\infty} e^{\ln(n!)^{-\frac{1}{n}}} = \lim_{n\to\infty} e^{-\frac{1}{n}\ln n!} = 1 \Rightarrow \sum x_n$ is D

d) $\sum_{n\geq 1} (1+\frac{1}{3n})^{-n} = \sum_{n\geq 1} x_n \quad \lim_{n\to\infty} x_n = \cancel{\lim_{n\to\infty} (\frac{3n+1}{3n})^{-n}} = \lim_{n\to\infty} (1+\frac{1}{3n})^{-n} =$

$= \lim_{n\to\infty} e^{-\frac{1}{3n}\cdot n} = \frac{1}{\sqrt[3]{e}} \Rightarrow \sum x_n$ is D

Ex2: a) $\sum_{n\geq 1} \frac{2^n+3^n}{5^n} = \sum_{n\geq 1} x_n \quad \lim_{n\to\infty} (\frac{2}{5})^n + (\frac{3}{5})^n = 0$

Let $a_n = 2^n+3^n$, ~~$n\in[1,\infty)\cap\mathbb{N}$~~, $b_n = 5^n$, $n\in[1,\infty)\cap\mathbb{N}$

$P(n): a_n \leq b_n$

I E.V. $P(1): a_1 \leq b_1 \Leftrightarrow 2+3 \leq 5$ „True"

$P(2): a_2 \leq b_2 \Leftrightarrow 2^2+3^2 = 4+9 \leq 5^2 = 25 \Leftrightarrow 13 \leq 25$ „True"

II E.D. Pp. that $P(k): a_k \leq b_k \Rightarrow 2^k+3^k \leq 5^k$ „True"

we show that $P(k+1): 2^{k+1}+3^{k+1} \leq 5^{k+1}$

$2^{k+1}+3^{k+1} = 2^k\cdot 2 + 3^k\cdot 3 = 2[2^k + 3^k(1+\frac{1}{2})] = 2(\underbrace{2^k+3^k+\frac{3^k}{2}}_{\leq 5^k}) =$

from P(k): $2^k+3^k \leq 5^k \,|+\frac{3^k}{2} \Leftrightarrow 2^k+3^k+\frac{3^k}{2} \leq 5^k+\frac{3^k}{2} \,|\cdot 2 \Leftrightarrow 2(2^k+3^k+\frac{3^k}{2}) \leq 2\cdot 5^k + 3^k$

~~$2\cdot 5^k + 3^k$~~

$$2\left(2^k+3^k+\frac{3^k}{2}\right) \le 2\cdot 5^k + 3^k \le 5^{k+1} = 5\cdot 5^k \; | -2\cdot 5^k$$

$$3^k \le 3\cdot 5^k \quad 3^{k-1} \le 5^k \text{ "True" } \forall k \in \mathbb{N}$$

$\mathrm{I}, \mathrm{II} \Rightarrow P(n): 2^n+3^n \le 5^n$ True, $\forall n \in \mathbb{N}$

$$x_n = \frac{2^n+3^n}{5^n} \quad x_{n+1} = \frac{2^{n+1}+3^{n+1}}{5^{n+1}} \quad x_{n+1}-x_n = \frac{2^{n+1}+3^{n+1}}{5^{n+1}} - \frac{2^n+3^n}{5^n} =$$

$$= \frac{2\cdot 2^n+3\cdot 3^n-5\cdot 2^n-5\cdot 3^n}{5^{n+1}} = \frac{(-3)\cdot 2^n-2\cdot 3^n}{5^{n+1}} = (-1)\cdot \frac{3\cdot 2^n+2\cdot 3^n}{5^{n+1}} < 0$$

$x_{n+1} < x_n \Rightarrow (x_n)$ decreasing $\Rightarrow$ strictly monotonic $\Big\} \Rightarrow \sum x_n C$

$\lim\limits_{n\to\infty} x_n = 0$

b) $\sum\limits_{n\ge 1} \frac{2^n}{3^n+5^n}$ $\quad \lim\limits_{n\to\infty} x_n = \frac{2^n\cdot 5^n - 2^n\cdot 3^n}{5^{2n}-3^{2n}} = \lim\limits_{n\to\infty} \frac{10^n-6^n}{25^n-9^n} =$

$$\sum_{n\ge 1}\frac{2^n}{3^n+5^n} = \sum_{n\ge 1} x_n \qquad x_n \le \left(\frac{2}{5}\right)^n$$

$$\sum y_n = \sum\left(\frac{2}{5}\right)^n \quad \lim_{n\to\infty} y_n = \lim_{n\to\infty}\left(\frac{2}{5}\right)^n = 0$$

$$\frac{y_{n+1}}{y_n} \; ? \; 1 \qquad \frac{\left(\frac{2}{5}\right)^{n+1}}{\left(\frac{2}{5}\right)^n} = \frac{2^{n+1}}{5^{n+1}}\cdot\frac{5^n}{2^n} = \frac{2}{5} < 1 \Rightarrow y_{n+1} < y_n$$

$\Rightarrow (x_n)$ decreasing $\Rightarrow$ strictly monotonic $\Big\} \Rightarrow \sum y_n C.$

$\lim\limits_{n\to\infty} y_n = 0$

$\xrightarrow{C_1C} \sum x_n C.$

Ex 2: a) $\sum\limits_{n\ge 2} \frac{1}{3n-4}$ $\quad \lim\limits_{n\to\infty} \frac{1}{3n-4} = 0$

Let $\sum\limits_{n\ge 2}\frac{1}{3n-4} = \sum\limits_{n\ge 2} x_n$ ~~let $y_n = \frac{1}{3n}$~~ ~~$x_n < y_n \Big\} \xrightarrow{C_1C} \sum y_n D.$~~

we set $y_n = \frac{1}{n^\alpha}$

$$\lim_{n\to\infty}\frac{x_n}{y_n} = \lim_{n\to\infty}\frac{\frac{1}{3n-4}}{\frac{1}{n^\alpha}} = \lim_{n\to\infty}\frac{n^\alpha}{3n-4} = \frac{1}{3} \in (0,\infty)$$

$\alpha = 1$

$\xrightarrow{C_2C} \sum x_n \sim \sum \frac{1}{n^1} \Big\} \Rightarrow \sum x_n$ is D

D.

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b) $\sum_{n\geq 1} \frac{1}{(4n-1)^3} = \sum_{n\geq 1} x_n$ we set $y_n = \frac{1}{n^\alpha}$ $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{(4n-1)^3} = 0$

$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{n^\alpha}{(4n-1)^3} \overset{\alpha=3}{=} \frac{1}{64} \in (0,\infty) \Rightarrow \sum x_n \sim \sum \frac{1}{n^3} \Rightarrow C.$

c) $\sum_{n\geq 1} \frac{1}{\sqrt{4n^2-1}}$ we set $y_n = \frac{1}{n^\alpha}$ and compute $\lim_{n\to\infty} \frac{x_n}{y_n} =$

$= \lim_{n\to\infty} \frac{n^\alpha}{\sqrt{4n^2-1}} \overset{\alpha=1}{=} \frac{1}{2} \in (0,\infty) \Rightarrow \sum x_n \sim \sum \frac{1}{n^1} \Big\} \Rightarrow \sum x_n\ D.$

D.

d) $\sum_{n\geq 1} \frac{\sqrt{n^2+n}}{\sqrt[3]{n^5-n}} = \sum_{n\geq 1} x_n$ $y_n = \frac{1}{n^\alpha}$

$\alpha+1 = \frac{5}{3} \Rightarrow \alpha = \frac{2}{3}$

$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{\sqrt{n^2+n} \cdot n^\alpha}{\sqrt[3]{n^5-n}} = \lim_{n\to\infty} \frac{n^{\alpha+1}\sqrt{1+\frac{1}{n}}}{n^{\frac{5}{3}}\sqrt{1-\frac{1}{n^4}}} = 1 \in (0,\infty) \Rightarrow$

$\overset{C_2C}{\Longrightarrow} \sum x_n \sim \sum \frac{1}{n^{2/3}} \Big\} \Rightarrow \sum x_n\ D.$

$\frac{2}{3} < 1 \Rightarrow D.$

Ex 4.: a) $\sum_{n\geq 1} \frac{100^n}{n!} = \sum_{n\geq 1} x_n$

$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{100^{n+1}}{(n+1)!} \cdot \frac{n!}{100^n} = \lim_{n\to\infty} \frac{100}{n+1} = 0 < 1 \overset{\text{D'Alembert}}{\Longrightarrow} \sum x_n\ C$

b) $\sum_{n\geq 1} \frac{2^n \cdot n!}{n^n} = \sum_{n\geq 1} x_n$

$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{2^{n+1} \cdot (n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{2^n \cdot n!} = 2 \cdot \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = 2 \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n =$

$= 2 \lim_{n\to\infty} \left(1 + \frac{-1}{n+1}\right)^n = 2 e^{\lim_{n\to\infty} \frac{-1}{n+1} \cdot n} = 2e^{-1} = 2\frac{1}{e}$

$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \frac{2}{e} < 1 \underset{\text{D'Alembert}}{\Longrightarrow} \sum x_n$ is C.

c) $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \frac{3}{e} > 1 \Rightarrow \sum x_n$ is D.

d) $\sum_{n\geq 1} \frac{(n!)^2}{2^{n^2}} = \sum x_n \quad \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{[(n+1)!]^2}{2^{(n+1)^2}} \cdot \frac{2^{n^2}}{(n!)^2} =$

$= \lim_{n\to\infty} \frac{(n+1)^2}{2^{2n+1}} = 0 < 1 \Rightarrow 2 \leq (n+1)^2 \Rightarrow 0 \leq \frac{(n+1)^2}{2^{2n+1}} \leq \frac{1}{2^{2n-2}}$

$\Rightarrow \sum x_n$ is C

e) $\sum_{n\geq 1} \frac{n^2}{(2+\frac{1}{n})^n} = \sum_{n\geq 1} x_n \quad \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{(n+1)^2}{(2+\frac{1}{n+1})^{n+1}} \cdot \frac{(2+\frac{1}{n})^n}{n^2} =$

$= \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \cdot \frac{\left(\frac{2n+1}{n}\right)^n}{\left(\frac{2n+3}{n+1}\right)^{n+1}} = \lim_{n\to\infty} \frac{(n+1)^3}{(2n+3)n^2} \cdot \left(\frac{(2n+1)(n+1)}{(2n+3)n}\right)^n =$

$= \frac{1}{2} \lim_{n\to\infty} \left(\frac{2n+1}{2n+3}\right)^n \cdot \underbrace{\left(\frac{n+1}{n}\right)^n}_{(1+\frac{1}{n})^n = e} = \frac{e}{2} \lim_{n\to\infty} \left(\frac{2n+1}{2n+3}\right)^n = \cancel{\frac{e}{2} \lim}$

$= \frac{e}{2} \lim_{n\to\infty} \left(1 + \frac{-2}{2n+3}\right)^n = \cancel{\frac{e}{2} \lim_{n\to\infty}} \frac{e}{2} \cdot e^{\lim_{n\to\infty} \frac{-2n}{2n+3}} =$

$= \frac{e}{2} \cdot e^{-1} = \frac{1}{2} < 1 \xrightarrow{\text{D'Alembert}} \sum x_n$ is C

Ex5. $a > 0 \quad \sum_{n\geq 0} \frac{a^n}{n^n} = \sum_{n\geq 0} x_n$

$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{a^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{a^n} = a \cdot \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1} = \frac{a}{e} \lim_{n\to\infty} \frac{1}{n+1} =$

$= \frac{a}{e} \cdot 0 = 0 < 1 \Rightarrow \sum x_n$ C.

b) $\sum_{n\geq 1} \left(\frac{n^2+n+1}{n^2} a\right)^n = \sum x_n$ $\qquad a<1$ C, $a \geq 1$ D

$\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n^2+n+1}{n^2} a\right)^n} = \lim_{n\to\infty} \frac{n^2+n+1}{n^2} \cdot a = a$ <1 C, >1 D.

$a=1 \quad x_n = \left(\frac{n^2+n+1}{n^2}\right)^n \quad \lim_{n\to\infty} x_n = \lim_{n\to\infty} \left(1 + \frac{n+1}{n^2}\right)^n = e^{\lim_{n\to\infty} n\frac{(n+1)}{n^2}} = e \neq 0$

$\Rightarrow \sum x_n$ D

c) $\sum_{n\geq 1} \frac{3^n}{2^n+a^n} = \sum x_n \quad x_n \leq \left(\frac{3}{a}\right)^n = y_n$

$a>3 \Rightarrow \frac{3}{a} \in (0,1) \Rightarrow \sum y_n$ is geometric C. $\left.\begin{matrix} \sum y_n\, C \\ x_n \leq y_n \end{matrix}\right\} \Rightarrow \sum x_n\, C$

$a=3 \Rightarrow x_n = \frac{3^n}{2^n+3^n} \quad \lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{3^n}{2^n+3^n} = \lim_{n\to\infty} \frac{1}{\left(\frac{2}{3}\right)^n+1} = 1 \neq 0$

$\Rightarrow \sum x_n\, D$

$a<3 \quad x_n = \frac{3^n}{2^n+a^n} \quad \lim_{n\to\infty} x_n = \begin{cases} \lim_{n\to\infty} \left(\frac{3}{2}\right)^n \cdot \frac{1}{1+\left(\frac{a}{2}\right)^n} = \infty \neq 0 \\ \lim_{n\to\infty} \left(\frac{3}{a}\right)^n \cdot \frac{1}{\left(\frac{2}{a}\right)^n+1} = \infty \neq 0 \end{cases}$

$\Rightarrow \sum x_n\, D$

conclusion $\sum x_n = \begin{cases} C & a>3 \\ D & a\leq 3 \end{cases}$