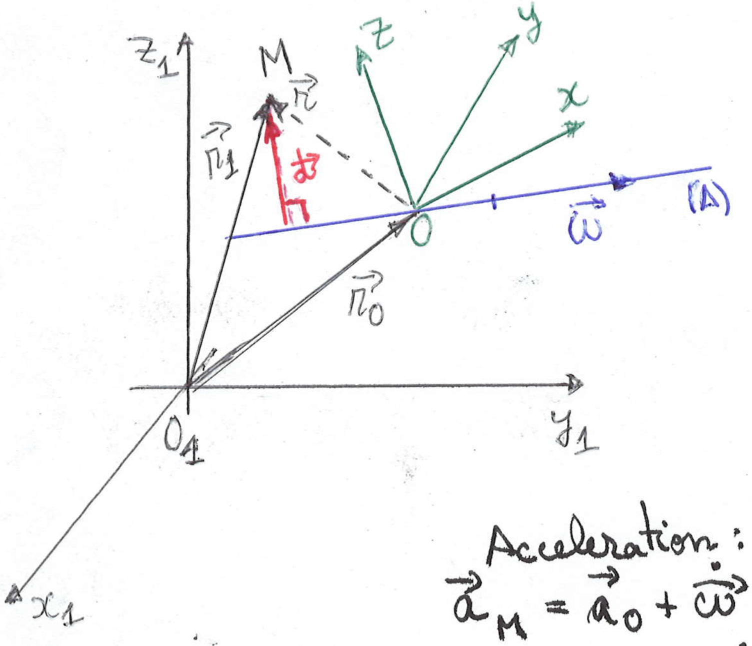
Kinematics of the rigid body



Equations of motion:

$$x_0 = x_0(t);\ y_0 = y_0(t);\ z_0 = z_0(t)$$

$$\varphi = \varphi(t);\ \theta = \theta(t);\ \psi = \psi(t)$$

Velocity:

$$\vec{v}_M = \vec{v}_0 + \vec{\omega} \times \vec{r}$$

$\vec{\omega}$ - instantanous angular velocity

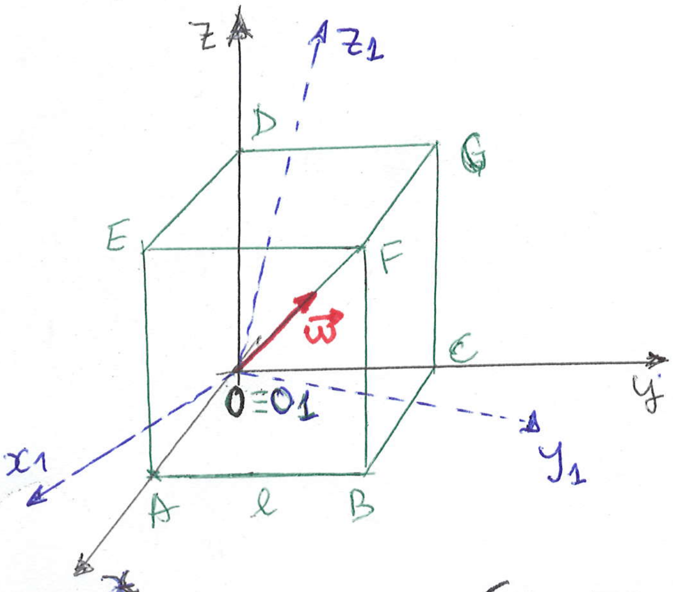
Acceleration:

$$\vec{a}_M = \vec{a}_0 + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) =$$

$$= \underbrace{\vec{a}_0}_{\text{translation}} + \underbrace{\dot{\vec{\omega}} \times \vec{r}}_{\text{angular}} - \underbrace{\omega^2 \vec{d}}_{\text{centripetal acceleration}}$$

Applications:

① A cube OABCDEFG ($OA = \ell$) rotates about the diagonal OF with the angular speed $\omega = \varepsilon_0 \cdot t$, where t is the time and $\varepsilon_0 \in \mathbb{R}$. Find the velocities and accelerations of the points B and D.



Consider the fixed frame of reference $O_1 x_1 y_1 z_1$ and the moving frame $Oxyz$.

The cube rotates about OF $\Rightarrow$ the angular velocity $\vec{\omega} \parallel OF \Rightarrow$

$$\Rightarrow \boxed{\vec{\omega}\,(p, p, p)} \quad (1)$$

But, $\omega = \varepsilon_0 t \underset{(1)}{\Longrightarrow} \omega^2 = 3p^2 = \varepsilon_0^2 t^2 \Rightarrow \boxed{p = \frac{\varepsilon_0 t \cdot \sqrt{3}}{3}}$ (2)

Thus, $\vec{\omega} = \frac{\sqrt{3}}{3}\varepsilon_0 t(\vec{i}, \vec{j}, \vec{k})$ - in the frame $Oxyz$.

On the other hand we have chosen $O \equiv O_1 \Rightarrow \vec{v}_0 = 0$.

In these conditions the velocity of a point M of the cube has the form:

$$\vec{v}_M = \vec{\omega} \times \vec{r}_M \qquad (3)$$

and the acceleration is:

$$\vec{a}_M = \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \underbrace{(\vec{\omega} \times \vec{r})}_{\vec{v}_M} \qquad (4)$$

For the point B one have:

$$(3) \Rightarrow \quad \vec{v}_B = \vec{\omega} \times \underbrace{\vec{r}_B}_{\vec{OB}} = \frac{\sqrt{3}}{3}\varepsilon_0 t \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ l & l & 0 \end{vmatrix} = \frac{\sqrt{3}}{3} l \varepsilon_0 t(-\vec{i} + \vec{j})$$

$$|\vec{v}_B| = \frac{\sqrt{6}}{3} l \varepsilon_0 t \, (-\vec{i} + \vec{j}).$$

$$(4) \quad \vec{a}_B = \dot{\vec{\omega}} \times \vec{r}_B + \vec{\omega} \times \underbrace{(\vec{\omega} \times \vec{r}_B)}_{\vec{v}_B} =$$

$$\dot{\vec{\omega}} = \frac{\sqrt{3}}{3}\varepsilon_0 (\vec{i}, \vec{j}, \vec{k}) \qquad (5)$$

$$= \frac{\sqrt{3}}{3} l \varepsilon_0 \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} + \frac{1}{3} l \varepsilon_0^2 t^2 \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ -1 & +1 & 0 \end{vmatrix} =$$

$$= \frac{l\varepsilon_0}{3}\left[-\vec{i}(\sqrt{3} + \varepsilon_0 t^2) + \vec{j}(\sqrt{3} - \varepsilon_0 t^2) + \vec{k} \cdot 2\varepsilon_0 t^2\right]$$

For the point D one have:

$$(3) \Rightarrow \vec{v}_D = \vec{\omega} \times \underbrace{\vec{r}_D}_{OD} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 0 & 0 & l \end{vmatrix} \cdot \frac{\varepsilon_0 t \sqrt{3}}{3} = \frac{\sqrt{3}}{3} l \varepsilon_0 t \, (\vec{i} - \vec{j})$$

$$|\vec{v}_D| = \frac{\sqrt{6}}{3} l \varepsilon_0 t$$

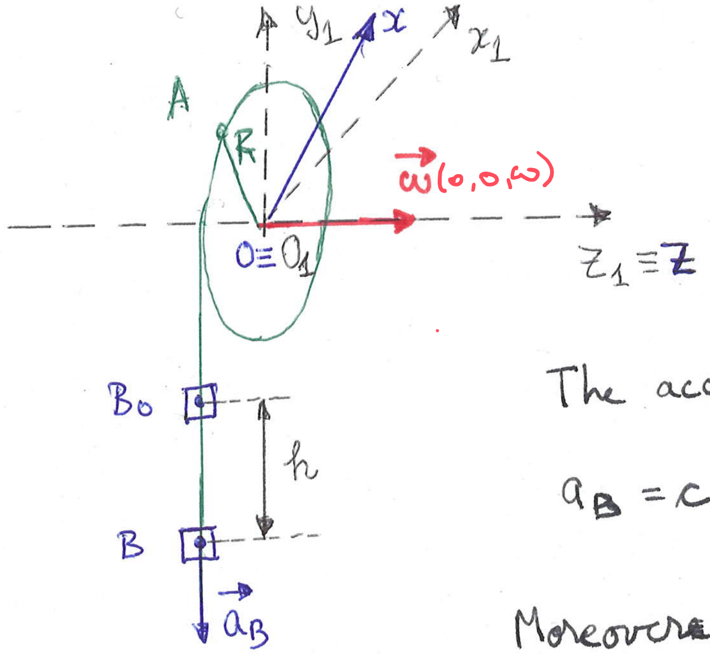
$$(4) \Rightarrow \vec{a}_D = \dot{\vec{\omega}} \times \vec{r}_D + \vec{\omega} \times \underbrace{(\vec{\omega} \times \vec{r}_D)}_{\vec{v}_D} \underset{(5)}{=} \frac{\sqrt{3}}{3} \varepsilon_0 l \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 0 & 0 & l \end{vmatrix} +$$

$$+ \frac{\sqrt{3}}{3} l \varepsilon_0^2 t^2 \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix} =$$

$$= \frac{l \varepsilon_0}{3} \left[\vec{i} \, (\sqrt{3} + \varepsilon_0 t^2) - \vec{j} \cdot (\sqrt{3} - \varepsilon_0 t^2) - \vec{k} \cdot 2 \varepsilon_0 t^2 \right]$$

Remark. $|\vec{v}_B| = |\vec{v}_D|$ and $|\vec{a}_B| = |\vec{a}_D|$ because B and D are situated at the same distance from the diagonal OF (i.e. the axis of rotation).

② Consider a pulley (l.r. scripete) of radius R. A load B descends with a constant acceleration $a_B = c$ (const.). After a time t, B descended with the height h. Find the acceleration of a point A from the pulley.



We choose the frames of reference such that $O \equiv O_1$ and $Oz \equiv Oz_1$.

The acceleration of B is constant:

$$a_B = c \Rightarrow \boxed{v_B = c \cdot t} + \underset{0}{\overset{\parallel}{v_0}} \quad (1)$$

Moreover $\vec{\omega} \perp \vec{R} \Rightarrow$

$$\vec{v}_A = \cancel{\vec{v}_0} + \vec{\omega} \times \vec{R} \Rightarrow \boxed{v_A = \omega \cdot R} \quad (2)$$

$\vec{v}_0 = 0$ $(O \equiv O_1)$

But, B and A are on the same wire, then $\boxed{v_B = v_A}$ (3)

$$(3), (1), (2) \Rightarrow \omega \cdot R = c \cdot t \Rightarrow \boxed{\omega = \frac{c \cdot t}{R}} \quad (4)$$

On the other hand, $v_B = \frac{ds}{dt} = c \cdot t \Rightarrow s = \frac{ct^2}{2}$ $(s_0 = 0)$

$$s = h \Rightarrow h = \frac{ct^2}{2} \Rightarrow \boxed{t^2 = \frac{2h}{c}} \quad (5)$$

How it is possible to find the acceleration:

$$a_A^2 = a_{rot}^2 + a_{ax}^2 = (\dot{\vec{\omega}} \times \vec{R})^2 + (-\omega^2 \vec{R})^2 = \dot{\omega}^2 R^2 + \omega^4 \cdot R^2$$

$\dot{\vec{\omega}} \perp \vec{R}$

$$= \frac{c^2}{R^2} \cdot R^2 + \frac{c^4 t^4}{R^4} \cdot R^2 = c^2 + \frac{4c^2 h^2}{R^2} \Rightarrow a_A = \frac{c}{R}\sqrt{R^2 + 4h^2}.$$

(4) (5)

Remark: $\vec{\omega}(0,0,\omega) \Rightarrow \dot{\vec{\omega}}(0,0,\dot{\omega})$.