P1 Consider the curve $\mathcal{C}$ parametrised by

$$\mathbf{r}(t) = \left(t/2 + \frac{1}{2}\sin t, \frac{1}{2}\cos t, 2\cos(t/2)\right), \quad t \in \mathbb{R}$$

(a) Prove that the above is an arc length parametrisation of $\mathcal{C}$.

(b) Calculate the unit tangent vector, principle normal vector, binormal vector, and curvature of $\mathcal{C}$ at $\left(\frac{\pi}{2}, -\frac{1}{2}, 0\right)$. Determine the osculating plane at this point.

a) arc - length parametrisation of $\mathcal{C}$ = the distance along the curve you have travelled, from a fixed point

If a curve $r(s)$ is parametrized by its arc length s, then the speed $\boxed{\|r'(s)\| = 1}$ (meaning that if s increases by 1 unit, the length of the curve also increases by 1 unit)

$$\hookrightarrow \int_0^s \|r'_{arc}(t)\|\, dt = s \;\Big|\; \frac{d}{ds} \Rightarrow \|r'_{arc}(s)\| = 1$$

For this problem, we can follow the next steps to prove that $r(t)$ is an arc-length parametrisation of $\mathcal{C}$:

1. compute $r'(t)$
2. compute the magnitude $\|r'(t)\|$
3. check the condition $\|r'(t)\| = 1$

$$r(t) = \left(\frac{t}{2} + \frac{1}{2}\sin(t), \frac{1}{2}\cos(t), 2\cos\left(\frac{t}{2}\right)\right), \quad t \in \mathbb{R}$$

① $r'(t) = \left(\frac{1}{2} + \frac{1}{2}\cos(t), -\frac{1}{2}\sin(t), -\sin\left(\frac{t}{2}\right)\right)$

② $\|r'(t)\| = \sqrt{\left(\frac{1}{2} + \frac{1}{2}\cos(t)\right)^2 + \left(-\frac{1}{2}\sin(t)\right)^2 + \left(-\sin\left(\frac{t}{2}\right)\right)^2} = \sqrt{\left(\frac{1}{2} + \frac{1}{2}\cos(t)\right)^2 + \frac{1}{4}\sin^2(t) + \sin^2\left(\frac{t}{2}\right)} =$

$\cos^2\left(\frac{t}{2}\right) + \sin^2\left(\frac{t}{2}\right) = 1$

$1 + \cos(t) = 2\cos^2\left(\frac{t}{2}\right)$

$$= \sqrt{\left(\frac{1}{2}(1 + \cos(t))\right)^2 + \frac{1}{4}\sin^2(t) + \sin^2\left(\frac{t}{2}\right)} =$$

$\sin^2(t) + \cos^2(t) = 1$

$$= \sqrt{\frac{1}{4}\left(1 + 2\cos(t) + \cancel{\cos^2(t)}\right) + \frac{1}{4}\left(1 - \cancel{\cos^2(t)}\right) + \sin^2\left(\frac{t}{2}\right)} =$$

$$= \sqrt{\frac{1}{4} + \frac{1}{2}\cos(t) + \frac{1}{4} + \sin^2\left(\frac{t}{2}\right)} = \sqrt{\frac{1}{2}(1 + \cos(t)) + \sin^2\left(\frac{t}{2}\right)} =$$

$$= \sqrt{\frac{1}{2}\left(2\cos^2\left(\frac{t}{2}\right)\right) + \sin^2\left(\frac{t}{2}\right)} = \sqrt{\cos^2\left(\frac{t}{2}\right) + \sin^2\left(\frac{t}{2}\right)} = \sqrt{1} = 1$$

③ $\|r'(t)\| = 1 \Rightarrow r(t)$ is an arc-length parametrization

b) The given point is $\left(\frac{\pi}{2}, -\frac{1}{2}, 0\right)$. We need the parameter t corresponding to the given point.

$$\left.\begin{array}{l} r(t) = \left(\frac{t}{2} + \frac{1}{2}\sin(t),\ \frac{1}{2}\cos(t),\ 2\cos\left(\frac{t}{2}\right)\right) \\ r(t) = \left(\frac{\pi}{2},\ -\frac{1}{2},\ 0\right) \end{array}\right\} \Rightarrow \begin{array}{l} \frac{t}{2} + \frac{1}{2}\sin(t) = \frac{\pi}{2} \\ \frac{1}{2}\cos(t) = -\frac{1}{2} \Rightarrow \cos(t) = -1 \\ 2\cos\left(\frac{t}{2}\right) = 0 \Rightarrow \cos\left(\frac{t}{2}\right) = 0 \end{array}$$

$$\left.\begin{array}{l} \cos\left(\frac{t}{2}\right) = 0 \Rightarrow \arccos(0) = \frac{t}{2} \Rightarrow \frac{\pi}{2} = \frac{t}{2} \Rightarrow t = \pi \\ \cos(t) = -1 \Rightarrow \arccos(-1) = t \Rightarrow \pi = t \end{array}\right\} \Rightarrow \boxed{t = \pi}$$

① Unit tangent vector $T(t)$: $\quad T = \frac{r'}{\underbrace{\|r'\|}_{=1}} \xrightarrow[\text{param. ✓}]{\text{arc-length}} T = r'$

$$r'(\pi) = \left(\frac{1}{2} + \frac{1}{2}\cos(\pi),\ -\frac{1}{2}\sin(\pi),\ -\sin\left(\frac{\pi}{2}\right)\right) = \left(\frac{1}{2} + (-1)\frac{1}{2},\ 0,\ -1\right) = (0, 0, -1)$$

$$\boxed{T(\pi) = (0, 0, -1)}$$

② Principal Normal Vector $N(t)$: $\quad N = \frac{T'}{\|T'\|}$

$$T' = r'' \Rightarrow \left(-\frac{1}{2}\sin(t),\ -\frac{1}{2}\cos(t),\ -\frac{1}{2}\cos\left(\frac{t}{2}\right)\right)$$

$$r''(\pi) = \left(-\frac{1}{2}\sin(\pi),\ -\frac{1}{2}\cos(\pi),\ -\frac{1}{2}\cos\left(\frac{\pi}{2}\right)\right) = \left(0, \frac{1}{2}, 0\right)$$

$$\|T'\| = \sqrt{0^2 + \left(\frac{1}{2}\right)^2 + 0^2} = \frac{1}{2}$$

$$N = \frac{\left(0, \frac{1}{2}, 0\right)}{\frac{1}{2}} = (0, 1, 0) \qquad \boxed{N(\pi) = (0, 1, 0)}$$

③ Binormal vector $B(t)$: $\quad B(t) = T(t) \times N(t)$

$$B(\pi) = (0, 0, -1) \times (0, 1, 0) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix} = \vec{i}\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} - \vec{j}\begin{vmatrix} 0 & -1 \\ 0 & 0 \end{vmatrix} + \vec{k}\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} =$$

$$= \vec{i}(0 + 1) - \vec{j}(0 - (-1)\cdot 0) + \vec{k}(0\cdot 1 - 0\cdot 0) = \vec{i} \Rightarrow (1, 0, 0)$$

$$\boxed{B(\pi) = (1, 0, 0)}$$

④ Curvature: $\quad \kappa = \left\|\frac{dT}{ds}\right\|$ or $\kappa = \frac{\|T'\|}{\|r'\|}$

$$\left.\begin{array}{l} \|T'\| = \frac{1}{2} \\ \|r'\| = 1 \end{array}\right. \Rightarrow \boxed{\kappa = \frac{1}{2}}$$

5. Osculating plane of a curve at a particular point is the plane that best approximates the behavior of the curve at that point.

It is spanned by T and N. Its normal vector is the binormal vect. $B(t)$.

$$B_x(x - x_0) + B_y(y - y_0) + B_z(z - z_0) = 0 \quad , \text{ where } \begin{array}{l} r(t_0) = (x_0, y_0, z_0) \\ B(t) = (B_x, B_y, B_z) \end{array}$$

$B(\pi) = (1, 0, 0)$ at $\left(\frac{\pi}{2}, -\frac{1}{2}, 0\right)$ $\rightarrow 1\left(x - \frac{\pi}{2}\right) + \underbrace{0\left(y + \frac{1}{2}\right)}_{0} + \underbrace{0(z - 0)}_{0} = 0$

$$\Downarrow$$

$\boxed{x = \frac{\pi}{2}}$ $\rightarrow$ osculating plane equation

Useful link:

https://youtu.be/VIqA8U9ozIA?si=B4OnVSAZCOVutoZt