Analytic Geometry

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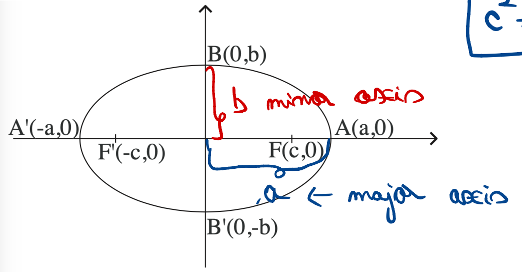
Recap... The ellipse

- An *ellipse* is a plane curve, defined as the geometric locus of the points in the plane, whose distances to two fixed points have a constant sum.

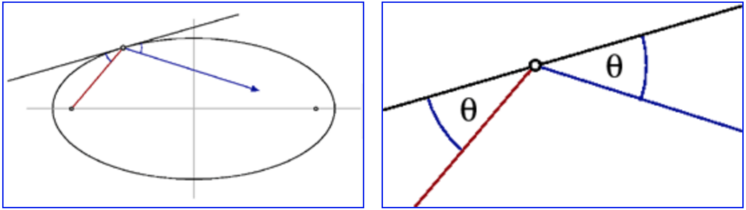
$\boxed{c^2 = a^2 - b^2}$



- The two fixed points are called the *foci* of the ellipse and the distance between the foci is the *focal distance*.

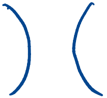
Applications

- Ellipses have an interesting reflective property



- This property affects both light and sound.

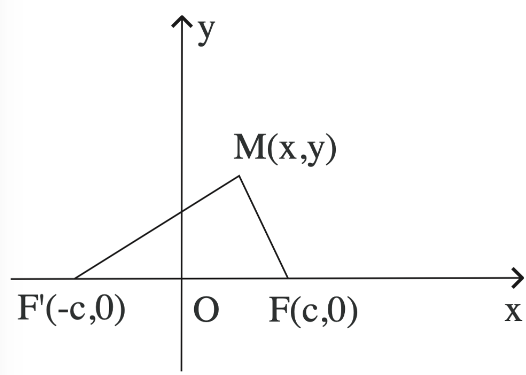
The hyperbola



- A *hyperbola* is a plane curve, defined as the geometric locus of the points in the plane, whose distances to two fixed points have a constant difference.
- The two fixed points are called the *foci* of the hyperbola, and the distance between the foci is the *focal distance*.
- Denote by F and F' the foci of the hyperbola and let $|FF'| = 2c$ be the focal distance. Suppose that the constant in the definition is $2a$. If $M(x, y)$ is an arbitrary point of the hyperbola, then

$$||MF| - |MF'|| = 2a.$$

- Choose a Cartesian system of coordinates, having the center at the midpoint of the segment $[FF']$ and such that $F(c,0)$, $F'(-c,0)$.



- Remark: In the triangle $\Delta MFF'$, $|\underbrace{|MF|-|MF'|}_{2a}| < \underbrace{|FF'|}_{2c}$, so $\boxed{a < c}$.

Say $M(x,y)$ is a point on the hyperbola.

- The metric relation $|MF| - |MF'| = \pm 2a$ becomes

$F(c,0)$

$F'(-c,0)$

$$\sqrt{(x-c)^2+y^2} - \sqrt{(x+c)^2+y^2} = \pm 2a,$$

or

$$\sqrt{(x-c)^2+y^2} = \pm 2a + \sqrt{(x+c)^2+y^2}.$$

- This is

$$x^2 - 2cx + c^2 + y^2 = 4a^2 \pm 4a\sqrt{(x+c)^2+y^2} + x^2 + 2cx + c^2 + y^2 \iff$$

$$\iff cx + a^2 = \pm a\sqrt{(x+c)^2+y^2} \iff$$

$$\iff c^2x^2 + 2a^2cx + a^4 = a^2x^2 + 2a^2cx + a^2c^2 + a^2y^2 \iff$$

$$\iff \underbrace{(c^2-a^2)}_{>0}x^2 - a^2y^2 - a^2(c^2-a^2) = 0. \quad \Big| : a^2(c^2-a^2)$$

- Denote $\boxed{c^2 - a^2 = b^2}$ (possible, since $c > a$) and one obtains the equation of the hyperbola

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0.} \tag{1}$$

If $M(x,y) \in \mathcal{H}$, then $M(-x,y) \in \mathcal{H}$

- *Remark*: The equation (1) is equivalent to

$$y = \pm\frac{b}{a}\sqrt{x^2 - a^2}; \qquad x = \pm\frac{a}{b}\sqrt{y^2 + b^2}.$$

Then, the coordinate axes are axes of symmetry for the hyperbola. Their intersection point is the *center* of the hyperbola.

Idea: Think of y as $f(x)$.

- To sketch the graph of the hyperbola, is it enough to represent the function

$$f : (-\infty, -a] \cup [a, \infty) \to \mathbb{R}, \qquad f(x) = \frac{b}{a}\sqrt{x^2 - a^2},$$

by taking into account that the hyperbola is symmetrical with respect to Ox.

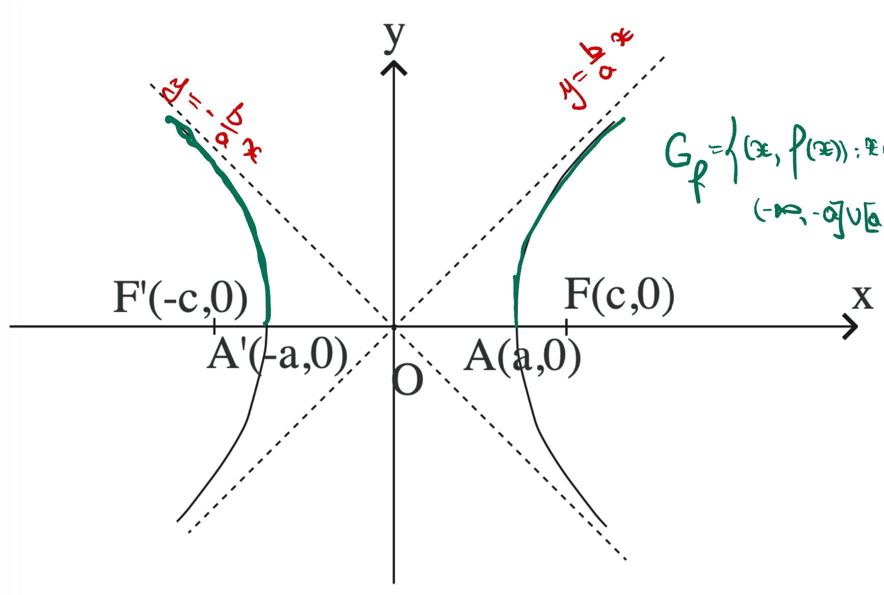
- Since $\lim\limits_{x\to\infty} \dfrac{f(x)}{x} = \dfrac{b}{a}$ and $\lim\limits_{x\to-\infty} \dfrac{f(x)}{x} = -\dfrac{b}{a}$, it follows that $y = \dfrac{b}{a}x$ and $y = -\dfrac{b}{a}x$ are asymptotes of f.

- One has, also,

$$f'(x) = \frac{b}{a}\frac{x}{\sqrt{x^2 - a^2}}, \qquad f''(x) = -\frac{ab}{(x^2 - a^2)\sqrt{x^2 - a^2}}.$$

x	$-\infty$		$-a$		a		∞
$f'(x)$	$-$	$-\ -\ -$	$\vert$	/ / /	$\vert$	$+\ +\ +$	$+$
$f(x)$	∞	$\searrow$	$0\vert$	/ / /	$\vert 0$	$\nearrow$	∞
$f''(x)$	$-$	$-\ -\ -$	$\vert$	/ / /	$\vert$	$-\ -\ -$	$-$

The graph of the hyperbola



$G_f = \{(x, f(x)) : x \in (-\infty, -a] \cup [a, +\infty)\}$

A few remarks

$y = x, \; y = -x.$

- If $a = b$, the equation of the hyperbola becomes $x^2 - y^2 = a^2$. In this case, the asymptotes are the bisectors of the system of coordinates and one deals with an *equilateral* hyperbola.
- As in the case of an ellipse, one can consider the hyperbola having the foci on Oy.
- The number $e = \dfrac{c}{a}$ is called the *eccentricity* of the hyperbola. Since $c > a$, then the eccentricity is always greater than 1.
- Moreover,

$$e^2 = \frac{c^2}{a^2} = 1 + \left(\frac{b}{a}\right)^2,$$

hence e gives informations about the shape of the hyperbola. For e closer to 1, the hyperbola has the branches closer to Ox.

Intersection of a Hyperbola and a Line

- Let $\mathcal{H} : \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - 1 = 0$ be a hyperbola and $d : y = mx + n$ be a line in $\mathcal{E}_2$. Their intersection is given by the system of equations

$$\begin{cases} \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - 1 = 0 \\ y = mx + n \end{cases}.$$

- By substituting y in the first equation, one obtains

$$(a^2m^2 - b^2)x^2 + 2a^2mnx + a^2(n^2 + b^2) = 0. \tag{2}$$

If $a^2m^2 - b^2 = 0$, (or $m = \pm\frac{b}{a}$), then the equation (2) becomes

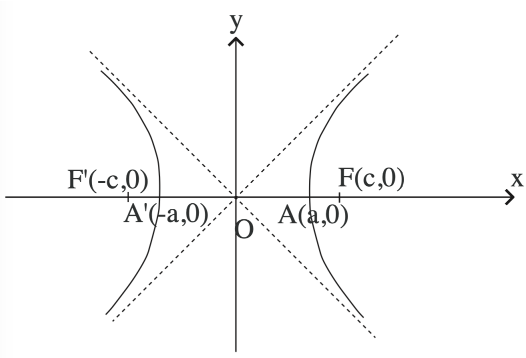
$$\pm 2bnx + a(n^2 + b^2) = 0.$$

- If $n = 0$, there are no solutions (this means, geometrically, that the two asymptotes do not intersect the hyperbola);
- If $n \neq 0$, there exists a unique solution (geometrically, a line d, which is parallel to one of the asymptotes, intersects the hyperbola at exactly one point);

If $a^2m^2 - b^2 \neq 0$, then the discriminant of the equation (2) is

$$\Delta = 4[a^4m^2n^2 - a^2(a^2m^2 - b^2)(n^2 + b^2)].$$

- If $\Delta < 0$, then the line does not intersect the hyperbola;
- If $\Delta = 0$, then the line is *tangent* to the hyperbola (they have a double intersection point);
- If $\Delta > 0$, then the line and the hyperbola have two intersection points.



The tangent to a hyperbola

The line $d : y = mx + n$ is tangent to the hyperbola $\mathcal{H} : \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - 1 = 0$ if the discriminant Δ of the equation

$$(a^2m^2 - b^2)x^2 + 2a^2mnx + a^2(n^2 + b^2) = 0$$

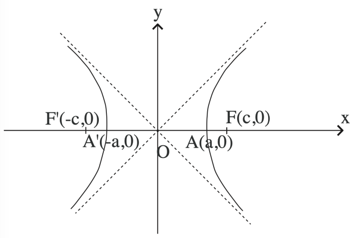
is zero, which is equivalent to $a^2m^2 - n^2 - b^2 = 0.$ $\Leftrightarrow n^2 = a^2m^2 - b^2.$

- If $a^2m^2 - b^2 \geq 0$, i.e. $m \in \left(-\infty, -\frac{b}{a}\right] \cup \left[\frac{b}{a}, \infty\right)$, then $n = \pm\sqrt{a^2m^2 - b^2}$. The equations of the tangent lines to $\mathcal{H}$, having the angular coefficient m are

$$y = mx \pm \sqrt{a^2m^2 - b^2}. \tag{3}$$

- If $a^2m^2 - b^2 < 0$, there are no tangent lines to $\mathcal{H}$, of angular coefficient m.

The Tangent at a Point of the Hyperbola



$P_0(x_0, y_0)$

$y = m(x - x_0) + y_0$.

$\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$

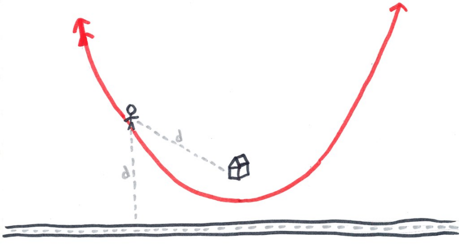
- One can prove, as in the case of the ellipse that, if $\mathcal{H} : \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$ is a hyperbola, and $P_0(x_0, y_0)$ is a point of $\mathcal{H}$, then the equation of the tangent to $\mathcal{H}$ at P_0 is

$$\frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} - 1 = 0. \tag{4}$$

$\frac{x \cdot x}{a^2} - \frac{y \cdot y}{b^2} - 1 = 0.$

The parabola

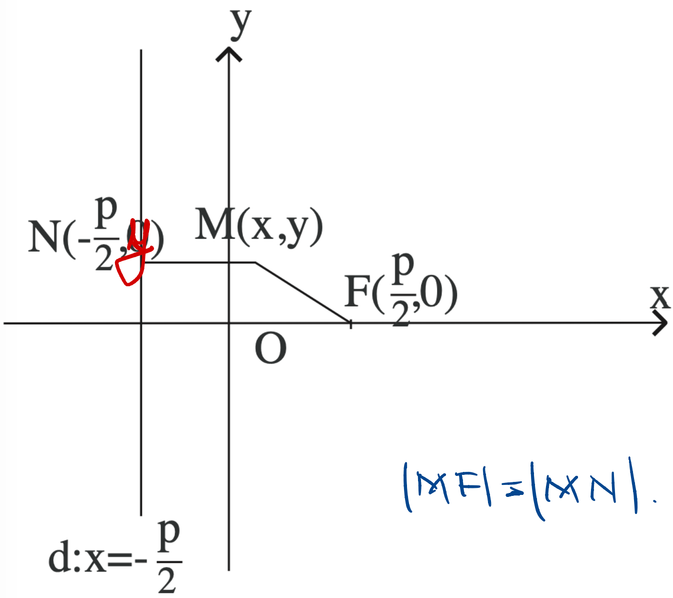
The *parabola* is a plane curve defined to be the geometric locus of the points in the plane, whose distance to a fixed line d is equal to its distance to a fixed point F.



- The line d is the *director line* and the point F is the *focus*. The distance between the focus and the director line is denoted by p and represents the *parameter* of the parabola.
- Consider a Cartesian system of coordinates xOy, in which $F\left(\frac{p}{2}, 0\right)$ and $d : x = -\frac{p}{2}$. If $M(x, y)$ is an arbitrary point of the parabola, then it verifies

$$|MN| = |MF|,$$

where N is the orthogonal projection of M on d.



$|MF| = |MN|$.

Thus, the coordinates of a point of the parabola verify

$$\sqrt{\left(x+\frac{p}{2}\right)^2+0}=\sqrt{\left(x-\frac{p}{2}\right)^2+y^2}\Leftrightarrow$$

$y^2=-10x$

$$\Leftrightarrow\left(x+\frac{p}{2}\right)^2=\left(x-\frac{p}{2}\right)^2=y^2\Leftrightarrow$$

$d: x=-\frac{5}{2}.$

$$\Leftrightarrow x^2+px+\frac{p^2}{4}=x^2-px+\frac{p^2}{4}+y^2,$$

and the equation of the parabola is

$y^2=10x$

$\Rightarrow p=5$

$$y^2=2px. \tag{5}$$

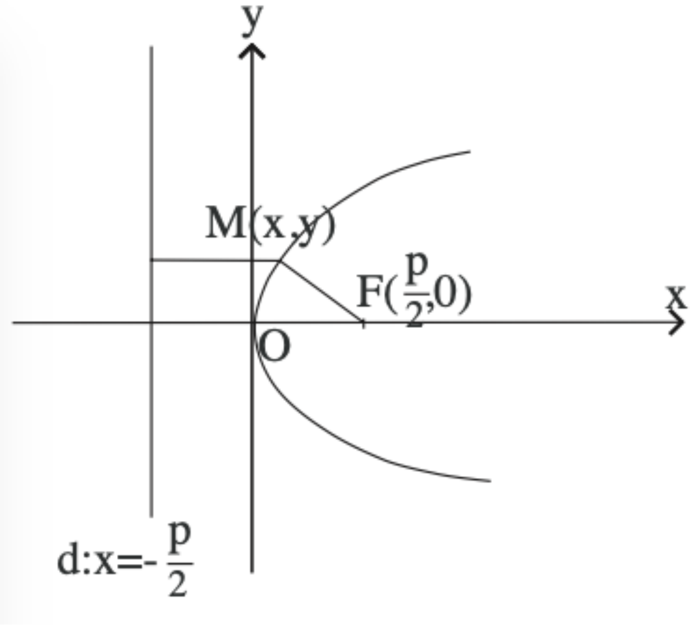
$F\left(\frac{5}{2},0\right)$

Remark: The equation (5) is equivalent to $y=\pm\sqrt{2px}$, so that the parabola is symmetrical with respect to Ox.

Representing the graph of the function $f : [0, \infty) \to [0, \infty)$ and using the symmetry of the curve with respect to Ox, one obtains the graph of the parabola. One has

$$f'(x) = \frac{p}{\sqrt{2px}}; \qquad f''(x) = -\frac{p}{2x\sqrt{2x}}.$$

x	0		∞
$f'(x)$	$\vert$	$+ \ + \ +$	$+$
$f(x)$	0	$\nearrow$	∞
$f''(x)$	$-$	$- \ - \ -$	$-$


y
M(x,y)
F(p/2,0)
x
O
d:x=-p/2

Intersection of a Parabola and a Line

Let $\mathcal{P} : y^2 = 2px$ be a parabola, $d : y = mx + n$ $(m \neq 0)$ be a line and

$$\begin{cases} y^2 = 2px \\ y = mx + n \end{cases}$$

be the system determined by their equations.

This leads to a second degree equation

$$m^2x^2 + 2(mn - p)x + n^2 = 0,$$

having the discriminant

$$\Delta = 4p(2mn - p) \tag{6}$$

- If $\Delta < 0$, then the line does not intersect the parabola;
- If $\Delta > 0$, then there are two intersection points between the line and the parabola;
- If $\Delta = 0$, then the line is *tangent* to the parabola and they have a unique intersection point.

The tangent to a parabola with a given direction

A line $d : y = mx + n$ (with $m \neq 0$) is tangent to the parabola $\mathcal{P} : y^2 = 2px$ if the discriminant Δ which appears in (6) is zero, i.e. $2mn = p$. Then, the equation of the tangent line to $\mathcal{P}$, having the angular coefficient m, is

$$y = mx + \frac{p}{2m}. \tag{7}$$

The tangent to a parabola with a given point

Let $\mathcal{P} : y^2 = 2px$ be a parabola and $P_0(x_0, y_0)$ be a point of $\mathcal{P}$. Suppose that $y_0 > 0$, so that the point P_0 belongs to the graph of the function $f : [0, \infty) \to [0, \infty)$, $f(x) = \sqrt{2px}$. The angular coefficient of the tangent at P_0 to the curve is

$$f'(x_0) = \frac{p}{\sqrt{2px_0}} = \frac{p}{y_0}.$$

A similar computation leads to the angular coefficient of the tangent for $y_0 < 0$, which is still $\dfrac{p}{y_0}$.

The equation of the tangent at P_0 to $\mathcal{P}$ is

$$y - y_0 = f'(x_0)(x - x_0),$$

or, replacing $f'(x_0)$,

$$y - y_0 = \frac{p}{y_0}(x - x_0) \Leftrightarrow$$

$$\Leftrightarrow yy_0 - y_0^2 = p(x - x_0) \Leftrightarrow$$

$$yy_0 - 2px_0 = p(x - x_0),$$

hence the equation of the tangent is

$$yy_0 = p(x + x_0). \tag{8}$$

$$y^2 = 2px = p(x + x)$$

Thank you very much for your attention!