Central forces (continued)

The theorem of energy:

When $F = F(r)$ one can use the energy theorem:

$$dT = \delta L \Rightarrow \frac{mv^2}{2} - \frac{mv_0^2}{2} = \int_{r_0}^{r} \pm F(r)\,dr$$

"−" attractive force

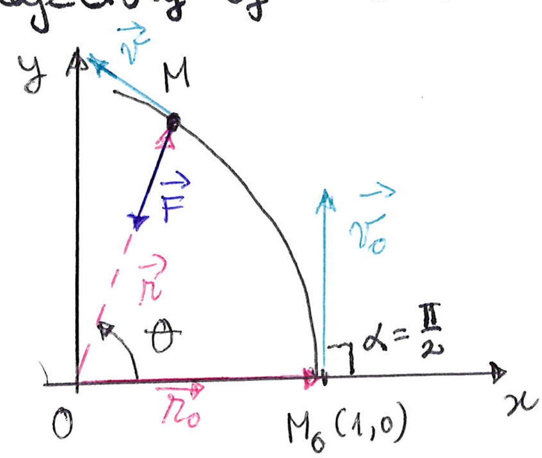
"+" repulsive force

$$v^2 = \frac{2}{m}\int \pm F(r)\,dr + h$$ (h — constant of energy)

In polar coordinates:

$$\underbrace{c^2\left\{\left[\frac{d}{d\theta}\left(\frac{1}{r}\right)\right]^2 + \frac{1}{r^2}\right\}}_{v^2} = \frac{2}{m}\int \pm F(r)\,dr + h \qquad (c = r_0 v_0 \sin\alpha = r^2\dot{\theta})$$

(1) A material point M ($m = 1$) moves under the action of the attractive force, $F = \frac{2}{r^2}$. At the initial moment ($t = 0$) the point was in $M_0(1, 0)$ and its velocity ($v_0 = \sqrt{3}$) was perpendicular on the initial position vector, $\vec{r_0}$. Find the trajectory of M and the equations of motion.



$F = F(r) \Rightarrow$ we can use the theorem of energy.

$$v^2 = \frac{2}{m}\int -F(r)\,dr + h \quad (1)$$

$$v^2 = \frac{2}{m}\int -\frac{2}{r^2}\,dr + h \Rightarrow$$

$$\underset{(m=1)}{\Rightarrow} \boxed{v^2 = \frac{4}{r} + h} \quad (2)$$

Next, we use the initial conditions in eq. (2)

$$\text{At } t=0 \Rightarrow v_0 = \frac{4}{r_0} + h \Rightarrow 3 = 4+h \Rightarrow \boxed{h=-1} \quad (3)$$

On the other hand

$$v^2 = c^2 \left\{ \left[\frac{d}{d\theta}\left(\frac{1}{r}\right)\right]^2 + \frac{1}{r^2} \right\} \quad (4)$$

where $\boxed{c = r_0 v_0 \sin \alpha} = 1 \cdot \sqrt{3} \cdot 1 = \sqrt{3}$

Using (3) and (4) in (2) we get:

$$3 \left\{ \left[\frac{d}{d\theta}\left(\frac{1}{r}\right)\right]^2 + \frac{1}{r^2} \right\} = \frac{4}{r} - 1 \quad (5)$$

We note $u = \frac{1}{r}$ in (5) $\Rightarrow$ $3\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] = 4u - 1 \Rightarrow$

$$\Rightarrow 3\left(\frac{du}{d\theta}\right)^2 = -3u^2 + 4u - 1 \Rightarrow \sqrt{3}\,\frac{du}{d\theta} = \pm\sqrt{-3u^2+4u-1} \quad (6)$$

Remark: In order to choose the sign „+" or „−" in (6) we use

$d\theta > 0$ (θ – increases)

$du < 0$ (r – increases $\Rightarrow u$ – decreases)

Thus, $\frac{du}{d\theta}$ has the sign „−".

$$(6) \Rightarrow \sqrt{3}\,\frac{du}{d\theta} = -\sqrt{-3u^2+4u-1} \Rightarrow -d\theta = \frac{\sqrt{3}}{\sqrt{-3u^2+4u-1}}$$

$$\Rightarrow -\theta + c = \int \frac{\sqrt{3}\,du}{\sqrt{-3u^2+4u-1}} = \int \frac{du}{\sqrt{-u^2+\frac{4}{3}u-\frac{1}{3}}} = \int \frac{du}{\sqrt{\frac{1}{9}-\left(u-\frac{2}{3}\right)^2}} =$$

$$\boxed{\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C}$$

$$= \arcsin \frac{u-\frac{2}{3}}{\frac{1}{3}} = \arcsin(3u-2)$$

We have $\boxed{-\theta + c = \arcsin(3u-2)}$ (7)

At $t=0 \Rightarrow c = \arcsin(1) \Rightarrow c = \frac{\pi}{2}$ (8)

(7), (8) $\Rightarrow$ $-\theta + \frac{\pi}{2} = \arcsin(3u-2)$

$\sin\left(\frac{\pi}{2}-\theta\right) = 3u-2 \Rightarrow \cos\theta = 3u-2 \Rightarrow$

$\boxed{\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta}$ $\qquad u = \frac{2+\cos\theta}{3} \Rightarrow$

$$\Rightarrow \boxed{r = \frac{3}{2+\cos\theta}} \quad (9)$$ – trajectory in polar coordinates.

In order to obtain the equation of motion the area integral is used: $r^2\dot{\theta} = c = \sqrt{3}$ (10)

$$\frac{9}{(2+\cos\theta)^2}\cdot\frac{d\theta}{dt} = \sqrt{3} \Rightarrow \frac{d\theta}{(2+\cos\theta)^2} = \frac{\sqrt{3}}{9}\cdot dt \Rightarrow$$

$$\Rightarrow \frac{\sqrt{3}}{9}t + c = \int\frac{d\theta}{(2+\cos\theta)^2} = \frac{1}{9}\left[4\sqrt{3}\,\mathrm{arctg}\left(\frac{\mathrm{tg}\frac{\theta}{2}}{\sqrt{3}}\right) - \frac{3\sin\theta}{\cos\theta+2}\right]$$

using Wolfram Alpha (wolframalpha.com)
(Integral (1/(2 + Cos[x])^2) = ...)

In order to calculate the constant c we use the initial conditions:

$t = 0$ ($\theta = 0$): $c = 0$

Thus we have: $t = 4\,\mathrm{arctg}\left(\frac{\mathrm{tg}\frac{\theta}{2}}{\sqrt{3}}\right) - \sqrt{3}\frac{\sin\theta}{2+\cos\theta}$

Remark I.1. It is not possible analitically to obtain $\theta = \theta(t)$ and therefore numerical methods are necessary.

2. By giving values to θ we can obtain values for intermadiate values of t, and using (9) and interpolation we finally can obtain:

Matlab:

```
>> theta = 0 : 0.01 : pi;
>> t = 4 * atan (tan (theta/2)/sqrt(3))...
>> r = 3./(2+cos(theta));
>> Theta = interp1 (t, theta, timp)
...
```

timp	θ	r
0	0	1
0.01	0.0173	1.0001
0.02	0.0346	1.0002
0.03	0.0519	1.0005
	...	

Remark. II. 1. The central force $F(r) = \frac{2}{r^2}$ is similar with the Newtonian force.

2. The trajectory (9) : $r = \frac{3}{2+\cos\theta} = \frac{\frac{3}{2}}{1+\frac{1}{2}\cos\theta}$

is an ellipse with the ellipse parameter $p = \frac{3}{2}$ and with the ellipse eccentricity $e = \frac{1}{2}$.

3. It is possible to visualize the trajectory in matlab:

```
>> theta = 0:0.01:2*pi
>> r = 3./(2 + cos(theta))
>> polar(theta, r)
```

(2) Solve the problem from Example 2, page 20, Lecture 8 using the theorem of the energy.

(Homework)

(from Dan B. Marghitu, Mihai Dupac, Advanced Dynamics Analytical and Numerical Calculations with MATLAB, Springer New York, 2012)

Example 4.6. A central force $\mathbf{F}$ attracts a particle P of mass m as shown in Fig. 4.23a. The force $\mathbf{F} = -k^2 m \mathbf{r}$ is on the xy plan of motion. The force $\mathbf{F}$ is proportional to the length of the position vector $\mathbf{r}$ relative to a fixed point O (directed toward the point O), to the mass m of the particle, and to the constant k. The coordinates of the initial position of the particle are $P_0(a, -\frac{g}{k^2})$, and the initial velocity of the particle is $\mathbf{v} = v_0 \mathbf{J}$. Study the motion of the particle. Numerical application: $a = 2$ m, $g = 9.8$ m/s^2, $v_0 = 10$ m/s, and $k = 3$ s^{-1}.

Solution
The Newton's second law of motion of the particle can be written as

$$m\ddot{\mathbf{r}} = m\mathbf{g} + \mathbf{F}.$$

Replacing the expression of the central force $\mathbf{F} = -k^2 m \mathbf{r}$ in the equation of motion, it is obtained

$$m\ddot{\mathbf{r}} = m\mathbf{g} - k^2 m \mathbf{r}.$$

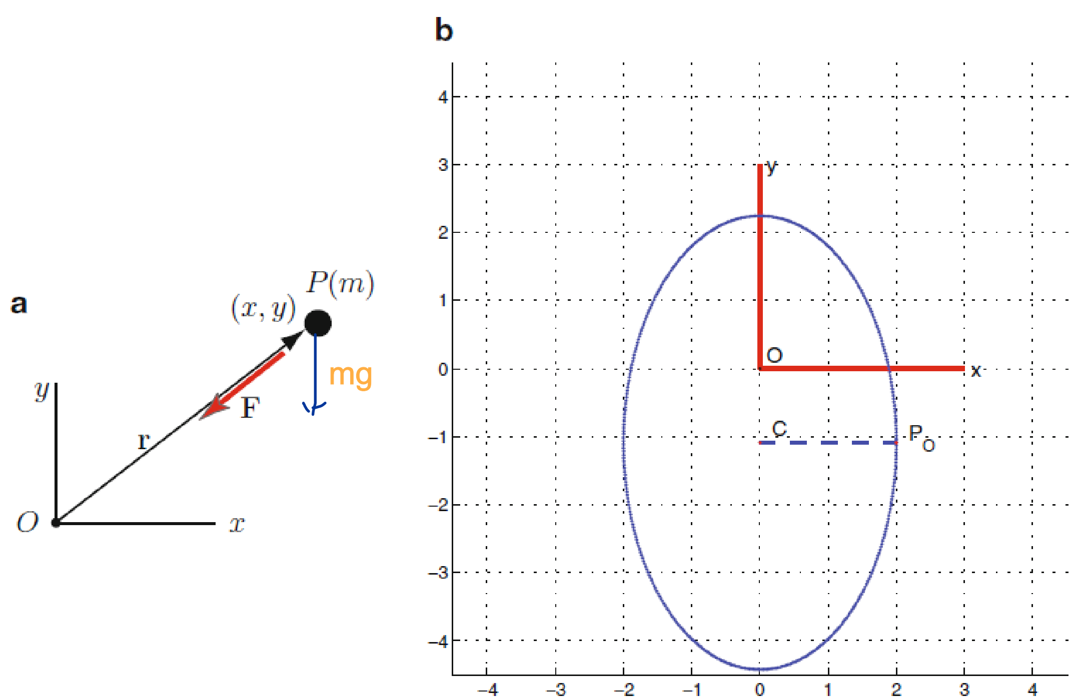


Fig. 4.23 Example 4.6

Or the two equations of motion are

$$\begin{aligned}\ddot{x}+k^2x&=0,\\ \ddot{y}+k^2y&=-g.\end{aligned}\tag{4.62}$$

Solving (4.62), it results

$$\begin{aligned}x&=c_1\cos(kt)+c_2\sin(kt),\\ y&=c_3\cos(kt)+c_4\sin(kt)-\frac{g}{k^2},\end{aligned}$$

and

$$\begin{aligned}\dot{x}&=-kc_1\sin(kt)+kc_2\cos(kt),\\ \dot{y}&=-kc_3\sin(kt)+kc_4\cos(kt).\end{aligned}$$

Using the initial conditions at $t=0$

$$x(0)=a,\ y(0)=-\frac{g}{k^2},\ \dot{x}(0)=0,\ \dot{y}(0)=v_0,$$

the integration constants c_1, c_2, c_3, and c_4 are calculated as

$$\begin{aligned}c_1&=a,\\ c_2&=0,\\ c_3&=0,\\ c_4&=\frac{v_0}{k}.\end{aligned}$$

The parametric equations for the trajectory of the particle are

$$\begin{aligned}x&=a\cos(kt),\\ y&=\frac{v_0}{k}\sin(kt)-\frac{g}{k^2}.\end{aligned}$$

The set of parametric equations is converted to a single equation, eliminating the variable t from the simultaneous equations

$$\cos^2(kt) = \frac{x^2}{a^2},$$
$$\sin^2(kt) = \frac{\left(y + \frac{g}{k^2}\right)^2}{\left(\frac{v_0}{k}\right)^2},$$

and using the Pythagorean trigonometric identity

$$\cos^2(kt) + \sin^2(kt) = \frac{x^2}{a^2} + \frac{\left(y + \frac{g}{k^2}\right)^2}{\left(\frac{v_0}{k}\right)^2} = 1.$$

The trajectory of the particle

$$\frac{x^2}{a^2} + \frac{\left(y + \frac{g}{k^2}\right)^2}{\left(\frac{v_0}{k}\right)^2} = 1$$

is an elliptical curve with the center of the ellipse located at $(0, -\frac{g}{k^2})$ and the semi-axes a and v_0/k. Since the trigonometric functions $\sin(kt)$ and $\cos(kt)$ are periodic functions, the motion of the particle is also periodic. The period for the motion T of the particle is calculated from

$$\sin(kt) = \sin[k(t+T)] = \sin(kt + 2n\pi),$$

or

$$kt + kT = kt + 2n\pi,$$

and it results

$$T = \frac{2n\pi}{k} \quad \left(\text{for } n = 1 \implies T = \frac{2\pi}{k}\right).$$