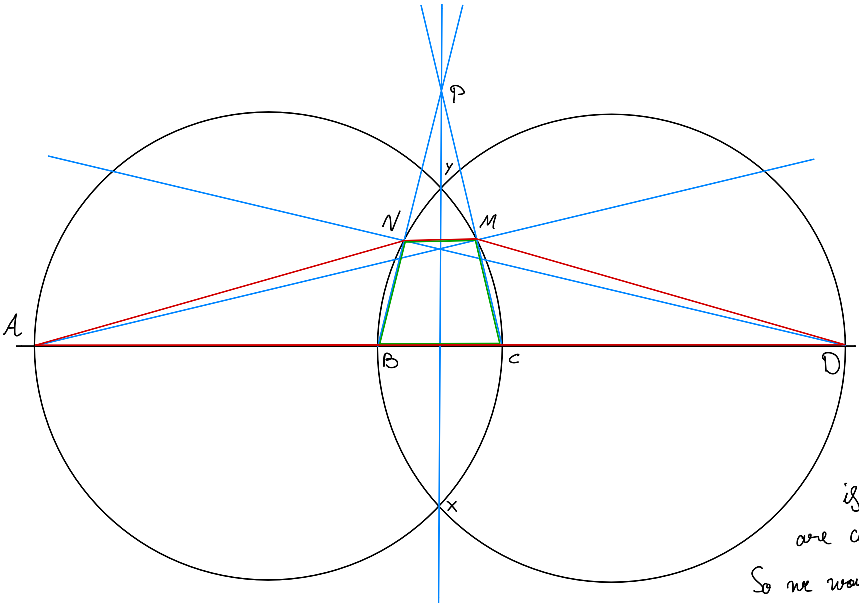
Presentation 1. Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y. Let P be a point on the line XY such that $P \notin BC$. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN and XY are concurrent.

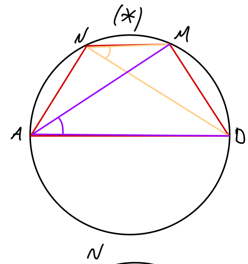


We assume P lies outside the two circles (the other case can be treated similarly)

We must proove AM, DN, XY are concurrent
Using the Radial Lemma (Lecture 5), it is sufficient to show that $ADMN$ is a cyclic quadrilateral:

Isolating the quadrilateral in (*), we notice that if the points were concyclic, the angles $\widehat{DNM}$ and $\widehat{DAM}$ are congruent (and equal to the angle that inscribes the arc DM)
So we would like to show $\widehat{DNM} \equiv \widehat{DAM}$

(*)



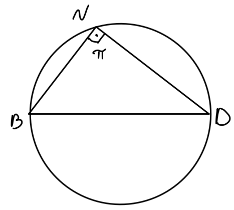
From the other side of the problem, from the original construction, XY is the radial axis and BN, CM, XY are concurrent.
Using the Radial Lemma again, this implies that B, C, M, N are concyclic.

To bring our two halves of the proof together, we look for relations between $\underbrace{\widehat{DNM}, \widehat{DAM}}_{\text{must prove congruency}}$ and $\underbrace{BCMN}_{\text{we know is cyclic}}$.

(1) $\widehat{DAM} \overset{(i)}{=} \widehat{CAM} \overset{(ii)}{=} \pi - \widehat{ACM} - \widehat{AMC} = \pi - \widehat{BCM} - \widehat{AMC}$

(2) $\widehat{DNM} = \widehat{BNM} - \widehat{BND}$

(i) A, B, C, D colinear from the problem statement
(ii) CAM Triangle



The angles $\widehat{BND}$ and $\widehat{AMC}$ inscribe the diameter $[BD]$ and $[AC]$ respectively, so the measure π radians

$\Rightarrow \widehat{DAM} = \widehat{DNM} \Leftrightarrow \pi - \widehat{BCM} - \pi = \widehat{BNM} - \pi \Leftrightarrow \widehat{BNM} + \widehat{BCM} = 0$ which is true, since $BCMN$ is cyclic.

In summary, $\begin{cases} XY \text{ radial axis} \\ BN, CM, XY, \text{ concurrent} \end{cases} \Rightarrow B, C, M, N$ concyclic $\Rightarrow \widehat{DNM} = \widehat{BNM} - \frac{\pi}{2} = \frac{\pi}{2} - \widehat{BCM} = \frac{\pi}{2} - \left(\frac{\pi}{2} - \widehat{DAM}\right) = \widehat{DAM} \Rightarrow ADMN$ cyclic $\Rightarrow BN, CM, XY$ concurrent