COURSES 9 + 10

Subspaces. The generated subspace

Let $(K,+,\cdot)$ be a field. Throughout this course this condition on K will always be valid.

We remind that:

• A K-**vector space** is an Abelian group $(V,+)$ with an external operation

$$\cdot : K \times V \to V\,, \quad (k,v) \mapsto k \cdot v\,,$$

satisfying the following axioms: for any $k, k_1, k_2 \in K$ and any $v, v_1, v_2 \in V$,

(L_1) $k \cdot (v_1 + v_2) = k \cdot v_1 + k \cdot v_2$;

(L_2) $(k_1 + k_2) \cdot v = k_1 \cdot v + k_2 \cdot v$;

(L_3) $(k_1 \cdot k_2) \cdot v = k_1 \cdot (k_2 \cdot v)$;

(L_4) $1 \cdot v = v$.

• If V is a vector space over K, a subset $S \subseteq V$ is a **subspace** of V (and we write $S \leq_K V$) if:

(1) S is closed with respect to the addition of V and to the scalar multiplication, that is,

$$\forall x, y \in S\,, \quad x + y \in S\,,$$

$$\forall k \in K\,,\ \forall x \in S\,,\ kx \in S\,.$$

(2) S is a vector space over K with respect to the induced operations of addition and scalar multiplication.

• If $S \leq_K V$ then S contains the zero vector of V, i.e. $0 \in S$.

We have the following **characterization theorem for subspaces**.

Theorem 1. Let V be a vector space over K and let $S \subseteq V$. The following conditions are equivalent:

1) $S \leq_K V$.

2) The following conditions hold for S:

α) $0 \in S$;

β) $\forall x, y \in S\,,\ x + y \in S$;

γ) $\forall k \in K\,,\ \forall x \in S\,,\ kx \in S\,.$

3) The following conditions hold for S:

α) $0 \in S$;

δ) $\forall k_1, k_2 \in K\,,\ \forall x, y \in S\,,\ k_1x + k_2y \in S\,.$

Proof. $\square$

Remark 2. (1) One can replace α) in the previous theorem with $S \neq \emptyset$.

(2) If $S \leq_K V$, $k_1, \ldots, k_n \in K$ and $x_1, \ldots, x_n \in S$ then $k_1x_1 + \cdots + k_nx_n \in S$.

Examples 3. (a) Every non-zero vector space V over K has two subspaces, namely $\{0\}$ and V. They are called the **trivial subspaces**.
(b) Let

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\},$$
$$T = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}.$$

Then S and T are subspaces of the real vector space $\mathbb{R}^3$.

(c) Let $n \in \mathbb{N}$ and let

$$K_n[X] = \{f \in K[X] \mid \deg f \leq n\}.$$

Then $K_n[X]$ is a subspace of the polynomial vector space $K[X]$ over K.
d) Let $I \subseteq \mathbb{R}$ be an interval. The set $\mathbb{R}^I = \{f \mid f : I \to \mathbb{R}\}$ is a $\mathbb{R}$-vector space with respect to the following operations

$$(f + g)(x) = f(x) + g(x), \ (\alpha f)(x) = \alpha f(x)$$

with $f, g \in \mathbb{R}^I$ and $\alpha \in \mathbb{R}$. The subsets

$$C(I, \mathbb{R}) = \{f \in \mathbb{R}^I \mid f \text{ continuous on } I\}, \ D(I, \mathbb{R}) = \{f \in \mathbb{R}^I \mid f \text{ derivable on } I\}$$

are subspaces of $\mathbb{R}^I$ since they are nonempty and

$$\alpha, \beta \in \mathbb{R}, \ f, g \in C(I, \mathbb{R}) \Rightarrow \alpha f + \beta g \in C(I, \mathbb{R});$$

$$\alpha, \beta \in \mathbb{R}, \ f, g \in D(I, \mathbb{R}) \Rightarrow \alpha f + \beta g \in D(I, \mathbb{R}).$$

Theorem 4. Let I be a nonempty set, V be a vector space over K and let $(S_i)_{i \in I}$ be a family of subspaces of V. Then $\bigcap_{i \in I} S_i \leq_K V$.

Proof. □

Remark 5. In general, the union of two subspaces is not a subspace.
For instance, ...

Next, we will see how to complete a subset of a vector space to a subspace in a minimal way.

Definition 6. Let V be a vector space and let $X \subseteq V$. We denote

$$\langle X \rangle = \bigcap \{S \leq_K V \mid X \subseteq S\}$$

and we call it the **subspace generated** (or **spanned**) **by** X. The set X is the **generating set** of $\langle X \rangle$. If $X = \{x_1, \ldots, x_n\}$, we denote $\langle x_1, \ldots, x_n \rangle = \langle \{x_1, \ldots, x_n\} \rangle$.

Remarks 7. (1) $\langle X \rangle$ is the smallest subspace of V (with respect to $\subseteq$) which contains X.
(2) Notice that $\langle \emptyset \rangle = \{0\}$.
(3) If V is a K-vector space, then:
(i) If $S \leq_K V$ then $\langle S \rangle = S$.
(ii) If $X \subseteq V$ then $\langle \langle X \rangle \rangle \subseteq \langle X \rangle$.
(iii) If $X \subseteq Y \subseteq V$ then $\langle X \rangle \subseteq \langle Y \rangle$.

Definition 8. A K-vector space V is **finitely generated** if there exist $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in V$ such that $V = \langle x_1, \ldots, x_n \rangle$. The set $\{x_1, \ldots, x_n\}$ is also called **system of generators** for V.

Definition 9. Let V be a K-vector space. A finite sum of the form

$$k_1 x_1 + \cdots + k_n x_n \,,$$

with $k_1, \ldots, k_n \in K$ and $x_1, \ldots, x_n \in V$, is called a **linear combination** of the vectors $x_1, \ldots, x_n$.

Let us show how the elements of a generated subspace look like.

Theorem 10. Let V be a vector space over K and let $\emptyset \neq X \subseteq V$. Then

$$\langle X \rangle = \{k_1 x_1 + \cdots + k_n x_n \mid k_i \in K \,,\ x_i \in X \,, i = 1, \ldots, n \,,\ n \in \mathbb{N}^*\} \,,$$

i.e. $\langle X \rangle$ is the set of all finite linear combinations of vectors of V.

Proof. $\square$

Corollary 11. Let V be a vector space over K and $x_1, \ldots, x_n \in V$. Then

$$\langle x_1, \ldots, x_n \rangle = \{k_1 x_1 + \cdots + k_n x_n \mid k_i \in K \,,\ x_i \in X \,, i = 1, \ldots, n\} \,.$$

Remark 12. Notice that a linear combination of linear combinations is again a linear combination.

Examples 13. (a) Consider the real vector space $\mathbb{R}^3$. Then

$$\langle (1,0,0), (0,1,0), (0,0,1) \rangle = \{k_1(1,0,0) + k_2(0,1,0) + k_3(0,0,1) \mid k_1, k_2, k_3 \in \mathbb{R}\} =$$

$$= \{(k_1,0,0) + (0,k_2,0) + (0,0,k_3) \mid k_1, k_2, k_3 \in \mathbb{R}\} = \{(k_1,k_2,k_3) \mid k_1, k_2, k_3 \in \mathbb{R}\} = \mathbb{R}^3 \,.$$

Hence $\mathbb{R}^3$ is generated by the three vectors $(1,0,0), (0,1,0), (0,0,1)$.
(b) More generally, the subspaces of $\mathbb{R}^3$ are the trivial subspaces, the lines containing the origin and the planes containing the origin.

If $S, T \leq_K V$, the smallest subspace of V which contains the union $S \cup T$ is $\langle S \cup T \rangle$. We will show that this subspace is the sum of the given subspaces.

Definition 14. Let V be a vector space over K and let $S, T \leq_K V$. Then we define the **sum** of the subspaces S and T as the set

$$S + T = \{s + t \mid s \in S\,,\ t \in T\}\,.$$

If $S \cap T = \{0\}$, then $S + T$ is denoted by $S \oplus T$ and is called the **direct sum** of the subspaces S and T.

Remarks 15. a) If V is a K-vector space, $V_1, V_2 \leq_K V$, then $V = V_1 \oplus V_2$ if and only if

$$V = V_1 + V_2 \text{ and } V_1 \cap V_2 = \{0\}.$$

Under these circumstances, we say that V_i $(i = 1, 2)$ is a **direct summand** of V.
b) If $V_1, V_2, V_3 \leq_K V$ and $V = V_1 \oplus V_2 = V_1 \oplus V_3$, we cannot deduce that $V_2 = V_3$.

c) The property of a subspace of being a direct summand is transitive. (during the seminar)

Theorem 16. Let V be a vector space over K and let $S, T \leq_K V$. Then

$$S + T = \langle S \cup T \rangle\,.$$

Proof. $\square$

Remarks 17. (1) Actually, a more general result can be proved: if $S_1, \ldots, S_n$ are subspaces of a K-vector space V then

$$S_1 + \cdots + S_n = \langle S_1 \cup \cdots \cup S_n \rangle.$$

(2) Moreover, if $X_i \subseteq V$ $(i = 1, \ldots, n)$, then $\langle X_1 \cup \cdots \cup X_n \rangle = \langle X_1 \rangle + \cdots + \langle X_n \rangle$.

Linear maps

Definition 18. Let V and V' be vector spaces over K. The map $f : V \to V'$ is called a **(vector space) homomorphism** or a **linear map** (or a **linear transformation**) if

$$f(x + y) = f(x) + f(y)\,,\ \forall x, y \in V\,,$$

$$f(kx) = kf(x),\ \ \forall k \in K,\ \forall x \in V\,.$$

The **(vector space) isomorphism**, **endomorphism** and **automorphism** are defined as usual.

We will mainly use the name *linear map* or *K-linear map*.

Remarks 19. (1) When defining a linear map, we consider vector spaces over the same field K.
(2) If $f : V \to V'$ is a K-linear map, then the first condition from its definition tells us that f is a group homomorphism between $(V, +)$ and $(V', +)$. Thus we have

$$f(0) = 0' \text{ and } f(-x) = -f(x),\ \forall x \in V.$$

We denote by $V \simeq V'$ the fact that two vector spaces V and V' are isomorphic and

$$Hom_K(V, V') = \{f : V \to V' \mid f \text{ is a } K\text{-linear map}\},$$

$$End_K(V) = \{f : V \to V \mid f \text{ is a } K\text{-linear map}\},$$

$$Aut_K(V) = \{f : V \to V \mid f \text{ is a } K\text{-isomorphism}\}.$$

Theorem 20. Let V, V' be K-vector spaces. Then $f : V \to V'$ is a linear map if and only if

$$f(k_1v_1 + k_2v_2) = k_1f(v_1) + k_2f(v_2),\ \forall k_1, k_2 \in K,\ \forall v_1, v_2 \in V.$$

Proof. □

One can easily prove by way of induction the following:

Corollary 21. If $f : V \to V'$ is a linear map, then

$$f(k_1v_1 + \cdots + k_nv_n) = k_1f(v_1) + \cdots + k_nf(v_n),\ \forall v_1, \ldots, v_n \in V,\ \forall k_1, \ldots, k_n \in K.$$

Examples 22. (a) Let V and V' be K-vector spaces and let $f : V \to V'$ be defined by $f(x) = 0'$, for any $x \in V$. Then f is a K-linear map, called the **trivial linear map**.
(b) Let V be a vector space over K. Then the identity map $1_V : V \to V$ is an automorphism of V.
(c) Let V be a vector space and $S \leq_K V$. Define $i : S \to V$ by $i(x) = x$, for any $x \in S$. Then i is a K-linear map, called the **inclusion linear map**.
(d) Let us consider $\varphi \in \mathbb{R}$. The map

$$f : \mathbb{R}^2 \to \mathbb{R}^2,\ f(x, y) = (x\cos\varphi - y\sin\varphi, x\sin\varphi + y\cos\varphi),$$

i.e. the plane rotation with the rotation angle φ, is a linear map.
(e) If $a, b \in \mathbb{R}$, $a < b$, $I = [a, b]$, and $C(I, \mathbb{R}) = \{f : I \to \mathbb{R} \mid f \text{ continuous on } I\}$, then

$$F : C(I, \mathbb{R}) \to \mathbb{R},\ F(f) = \int_a^b f(x)dx$$

is a linear map.

As in the case of group homomorphisms, we have the following:

Theorem 23. Let V, V', V'' be K-vector spaces.
(i) If $f : V \to V'$ and $g : V' \to V''$ are K-linear maps (isomorphisms) then $g \circ f : V \to V''$ is a K-linear map (isomorphism).
(ii) If $f : V \to V'$ is an isomorphism of K-vector spaces then $f^{-1} : V' \to V$ is again an isomorphism of K-vector spaces.

Proof. □

Definition 24. Let $f : V \to V'$ be a K-linear map. Then the set

$$\mathrm{Ker} f = \{x \in V \mid f(x) = 0'\}$$

is called the **kernel** of the K-linear map f and the set

$$\mathrm{Im} f = \{f(x) \mid x \in V\}$$

is called the **image** of the K-linear map f.

Theorem 25. Let $f : V \to V'$ be a K-linear map. Then we have
1) $\mathrm{Ker} f \leq_K V$ and $\mathrm{Im} f \leq_K V'$.
2) f is injective if and only if $\mathrm{Ker} f = \{0\}$.

Proof. □

Theorem 26. Let $f : V \to V'$ be a K-linear map and let $X \subseteq V$. Then

$$f(\langle X \rangle) = \langle f(X) \rangle \, .$$

Proof. □

Theorem 27. Let V and V' be vector spaces over K. For any $f, g \in Hom_K(V, V')$ and for any $k \in K$, we consider $f + g, k \cdot f \in Hom_K(V, V')$,

$$(f + g)(x) = f(x) + g(x), \ \forall x \in V \, ,$$

$$(kf)(x) = kf(x), \ \forall x \in V \, .$$

The above equalities define an addition and a sclar multiplication on $Hom_K(V, V')$ and $Hom_K(V, V')$ is a vector space over K.

Proof. □

Corollary 28. If V is a K-vector space, then $End_K(V)$ is a vector space over K.

Remarks 29. a) Let V be a K-vector space. From Theorem 23 one deduces that $End_K(V)$ is a subgroupoid of $(V^V, \circ)$ and from Example 22 (b) it follows that $(End_K(V), \circ)$ is a monoid. Moreover, the endomorphism composition $\circ$ is distributive with respect to endomorphism addition $+$, thus $End_K(V)$ also has a unitary ring structure, $(End_K(V), +, \circ)$.

b) The set $Aut_K(V)$ is the group of the units of $(End_K(V), \circ)$.