Homework 11 16.01.2023

1. $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \frac{\cos nx}{n^\alpha}$, $\alpha > 0$

choose $x \in \mathbb{R}$ random

$\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \frac{\cos nx}{n^\alpha} =$

2. $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = \frac{x(1+n^2)}{n^2}$

choose $x \in [0,1]$ random

$\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \frac{x(1+n^2)}{n^2} = \cancel{x \in [0,1]} \to x \in [0,1]$ $\quad B = [0,1]$

$f : [0,1] \to \mathbb{R}$ $\quad f(x) = x \in [0,1]$ f is cont on $[0,1]$ $\forall n \in \mathbb{N}$ $f_n(x)$ is cont.

choose $n \in \mathbb{N}$ random $a_n = \sup |f_n(x) - f(x)| = \sup\limits_{x\in[0,1]} |f_n(x) - x| =$

$= \sup \left| \frac{x(1+n^2)}{n^2} - x \right| = \sup\limits_{x\in[0,1]} \left| \frac{x + n^2x - n^2x}{n^2} \right| = \sup\limits_{x\in[0,1]} \left| \frac{x}{n^2} \right| = \frac{x}{n^2}$

$g : \mathbb{R} \to \mathbb{R}$ $\quad g(x) = \frac{x}{n^2}$ $\quad g'(x) = \frac{n^2 - 2xn}{n^4}$ $\quad g'(x) = 0$ $\quad \frac{n^2 - 2xn}{n^4} = 0$

$n^2 - 2xn = 0 \mid : n$ $\quad n - 2x = 0$ $\quad x = \frac{n}{2}$ $\quad n = 0$

x		0		$\frac{n}{2}$		1
$n^2 - 2xn$	$+$		$+$	0	$-$	
$\cancel{n^4}$		0				
$g'(x)$	$+$		$+$		$-$	
$g(x)$	$\nearrow$	0	$\nearrow$	0	$\searrow$	$\frac{1}{n^2}$

$g(0) = 0$ $\quad g\left(\frac{n}{2}\right) = 0$ $\quad g(1) = \frac{1}{n^2}$

$g(x) \le 0$, $\forall x \in [0,1]$

$\Rightarrow a_n = \sup\limits_{x\in[0,1]} g(x) = \frac{1}{n^2}$ $\quad a_n = \lim\limits_{n\to\infty} \frac{1}{n^2} = 0 \Rightarrow f_n \rightrightarrows f$

3. $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \frac{x^2}{x^4 + n^2}$ $\quad$ choose $x \in \mathbb{R}$ random

$\cancel{\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \frac{x^2}{x^4+n^2} = 0}$

4. $f_n: [0,\infty) \to \mathbb{R}$, $f_n(x) = \frac{1}{1+nx}$

choose $x \in \mathbb{R}[0,\infty)$ random $\quad \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{1+nx} = 0 \quad \mathcal{C} = [0,\infty)$

$f: [0,\infty) \to \mathbb{R}$, $f_n(x) = 0$, $\forall x \in [0,\infty)$ $\quad f_n \to f$

Choose a random $n \in \mathbb{N}$

$a_n = \sup_{x\in[0,\infty)} |f_n(x) - f(x)| = \sup_{x\in[0,\infty)} |f_n(x)| = \sup_{x\in[0,\infty)} \left|\frac{1}{1+nx}\right|$

We consider the function $g:[0,\infty)\to\mathbb{R}$ $\quad g(x)=\frac{1}{1+nx} \quad g'(x) = \cancel{\frac{-n}{(1+nx)^2}} \; \frac{-n}{(1+nx)^2}$

$g'(x)=0 \quad x = -\frac{1}{n}$

x	0		∞
$1+nx$		$+ + +$	
$g'(x)$		$- - -$	
$g(x)$	1	$\searrow$	0

$g(0)=1 \quad \lim_{x\to\infty}\frac{1}{1+nx}=0$

$g(x) < 1$, $\forall x\in[0,\infty)$

$\Rightarrow a_n = \sup_{x\in[0,\infty)} g(x) = \frac{1}{1+nx}$ a_n

$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{1+nx} = 0 \Rightarrow f_n \rightrightarrows f$

5. $f_n:\mathbb{R}\to\mathbb{R}$, $f_n(x)=\frac{2n^2x}{e^{n^2x^2}}$ choose $x\in\mathbb{R}$ random, $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\frac{2n^2x}{e^{n^2x^2}} = 0$

$\mathcal{C}=\mathbb{R} \quad f:\mathbb{R}\to\mathbb{R} \quad f_n(x)=0$, $\forall x\in\mathbb{R} \quad f_n\to f$

Choose $n\in\mathbb{N}$ random

$a_n = \sup_{x\in\mathbb{R}} |f_n(x)-f(x)| = \sup_{x\in\mathbb{R}}|f_n(x)| = \sup_{x\in\mathbb{R}}\left|\frac{2n^2x}{e^{n^2x^2}}\right|$

We consider the function $g:\mathbb{R}\to\mathbb{R}$, $g(x)=\frac{2n^2x}{e^{n^2x^2}}$

$\cancel{g'(x) = 4nxe^{n^2x^2} + 2n^2e^{n^2x^2} - 2n^2x}$ $\quad g'(x) = \frac{2n^2e^{n^2x^2} - 2n^2xe^{n^2x^2}}{(e^{n^2x^2})^2} =$

$= \frac{2n^2\cancel{e^{n^2x^2}}(1-x)}{(e^{n^2x^2})^2} = \frac{2n^2(1-x)}{e^{n^2x^2}}$