Complements of Geometry

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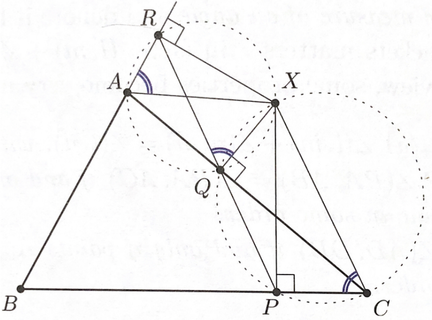
Optional course for Maths and Computer Science

Week 5

The Simson line

Theorem

Let ABC be a triangle and X a point in the plane. If we denote by P, Q, R the perpendicular projections of X on the lines BC, CA and AB respectively, then the points P, Q, R are collinear if and only if X belongs to the circumcircle of the triangle $\triangle ABC$.



Proof

Proposition

(a) *Let $ABCD$ be a convex quadrilateral and let $\{P\} = AC \cap BD$. Then the points A, B, C and D are concyclic if and only if*

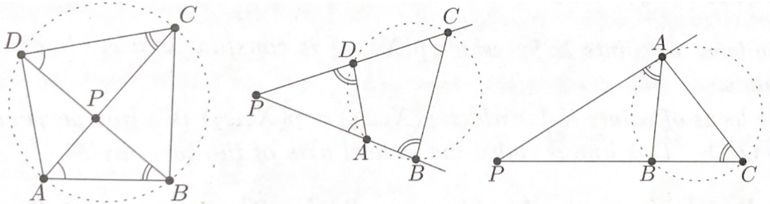
$$PA \cdot PC = PB \cdot PD.$$

(b) *Let $ABCD$ be a convex quadrilateral $\{P\} = AB \cap CD$. Then the points A, B, C, D are concyclic if and only if*

$$PA \cdot PB = PD \cdot PC.$$

(c) *Assume P, B, C are three collinear points in this order and point let A be a point which is not collinear with them. Then PA is tangent to the circumcircle of triangle ABC if and only if*

$$PA^2 = PB \cdot PC.$$


D
C
P
A
B
C
D
P
A
B
A
P
B
C

Continuation of the proof

Power of a point with respect to a circle

Theorem

Given point P and circle ω, let l be an arbitrary line passing through P and intersecting the circle ω at points A and B. Then the value of the product $PA \cdot PB$ does not depend on the choice of l. Also, if P lies outside of ω and PT, $T \in \omega$ is a tangent to ω, then $PA \cdot PB = PT^2$.

If we denote the center of ω by O and its radius by R, then $PA \cdot PB = |OP^2 - R^2|$.

Definition

The quantity $p(P, \omega) = OP^2 - R^2$ is called "the power of the point P with respect to the circle ω".

- $p(P,\omega)$ is negative if and only if P lies in the interior of ω;
- $p(P,\omega) = 0$ if and only if $P \in \omega$;
- $p(P,\omega)$ is positive, if and only if P is in the exterior of ω.

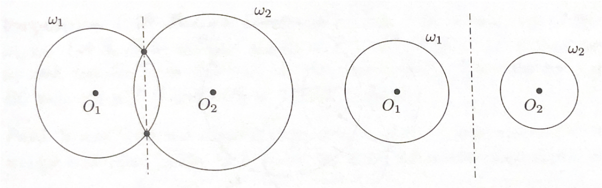
The power of a point is a fundamental concept

Proposition

Let ω_1, ω_2 be two circles with distinct centers O_1, O_2 and radii R_1 and R_2, respectively. Then

(a) *the geometric locus of points X for which $p(X, \omega_1)$ is constant is a circle concentric with ω_1;*

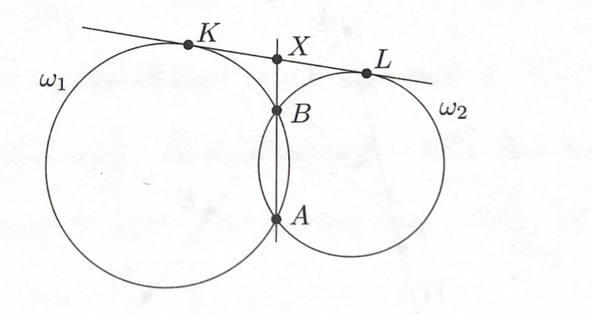
(b) *the geometric locus of points X for which $p(X, \omega_1) = p(X, \omega_2)$ is a line perpendicular to O_1O_2 O_1O_2. This line is called* **the radical axis** *of the two circles.*


ω_1
ω_2
O_1
O_2
ω_1
ω_2
O_1
O_2

The radical axis is a powerful tool in many problems involving intersecting circles since in that case the radical axis is the line joining their intersections, which both have equal (namely zero) power with respect to the two circles.

Proposition

Let the circles ω_1, ω_2 intersect at points A, B. Denote by K and L the points of tangency of a common external tangent with circles ω_1, ω_2, respectively. Then, the line AB passes through the midpoint of the segment $[KL]$.



Proof

The radical center

Lemma

Let ω_1, ω_2 and ω_3 with pairwise distinct centers. Show that their pairwise radical axes are either parallel or concurrent. The point of concurrence is called the radical center of the three circles

The Radical Lemma

Lemma

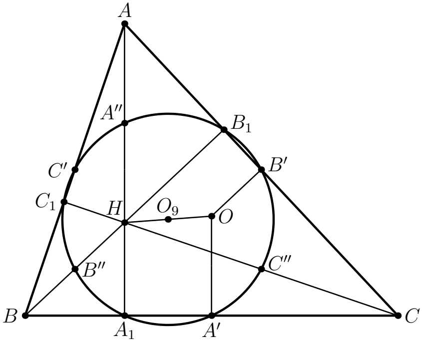
Let l be the radical axis of the circles ω_1 and ω_2. Let A, D be distinct points on ω_1 and B, C be distinct points on ω_2, such that the lines AD and BC are not parallel. Then the lines AD, BC and l are concurrent if and only if $ABCD$ is a cyclic quadrilateral.

The nine-point circle

We saw already that given a triangle ABC, we can associate to it a few circles, namely: the circumcircle (with center O and radius R), the inscribed circle (with center I and radius r), and the ex-circles (with centers I_a, I_b, I_c and radii r_a, r_b, r_c). Now we present another circle which plays an important role in the geometry of the triangle.

Theorem (Euler's 9-point circle)

The midpoints of the sides, the feet of the altitudes and the midpoints of the segments AH, BH, CH belong to a circle. The center of this circle is denoted by O_9 and coincides with the midpoint of $[OH]$. The radius of this circle is $\dfrac{R}{2}$.


A
A''
B_1
B'
C'
C_1
H
O_9
O
B''
C''
B
A_1
A'
C

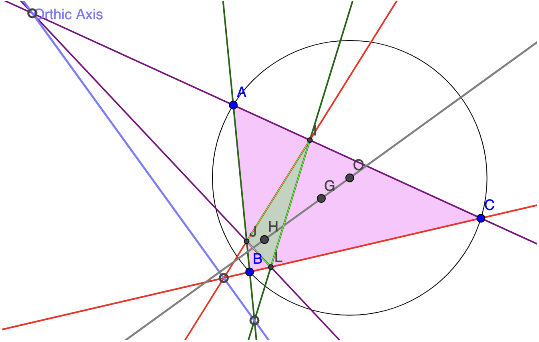
Remark

The segments $[A'A'']$, $[B'B'']$, $[C'C'']$ are diameters on the Euler's circle, so the lines A_1A'', B_1B'', C_1C'' are concurrent in the midpoint of OH, the center O_9 of Euler's circle.

The orthic line

Theorem (orthic line)

Suppose ABC is not a right triangle and that is not isosceles. Denote by A_1, B_1, C_1 the feet of the altitudes from A, B, and C, respectively and by A_2, B_2, C_2 the intersection of the lines B_1C_1, A_1C_1, B_1C_1 with the sides of the triangle ABC. Then the points A_2, B_2, C_2 are collinear and the line they form is called the orthic line of the triangle ABC.



Proof

Let $\mathcal{C}$ and $\mathcal{C}_1$ be the circumcircles of ABC and $A_1B_1C_1$. The powers of the point A_2 with respect to the two circles are $A_2B \cdot A_2C$ and $A_2B_1 \cdot A_2C_1$. Since BCB_1C_1 is a cyclic quadrilateral, it follows that the two powers are equal, hence A_2 is situated on the radical axis of the two circles.
We proceed similarly with B_2 and C_2. $\square$

Remark

The circumscribed circle of $A_1B_1C_1$ is the Euler's circle for the triangle ABC. Because the radical axis is perpendicular on the line determined by the two centers it follows that the orthic line is perpendicular on OH.

Thank you very much for your attention!