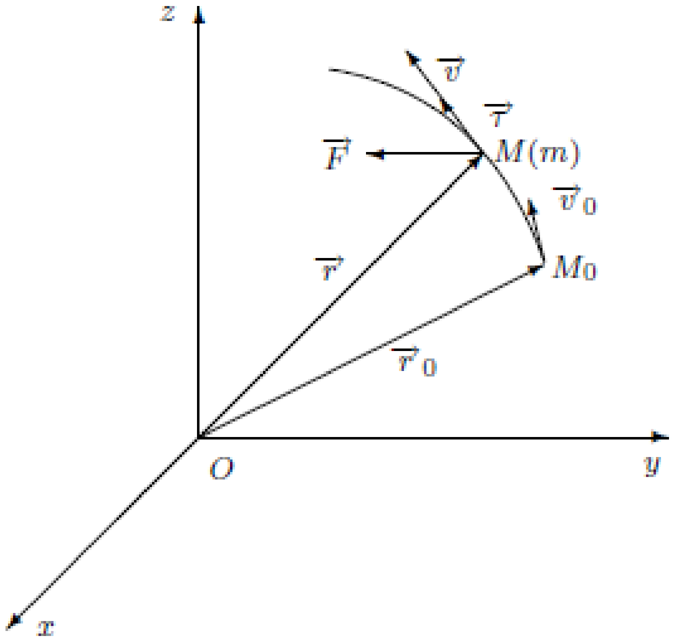
9. Dynamics of the particle. General theorems.

The differential equations of the motion

Consider the material point $M(m)$ moving in the Cartesian frame $Oxyz$ under the action of the force $\vec{F} = \vec{F}(\vec{r}, \vec{v}, t)$ where $\vec{F}(X, Y, Z)$.



The differential equation of the motion is

$$m\frac{d^2\vec{r}}{dt^2} = \vec{F}(\vec{r}, \vec{v}, t), \; t \in (t_0, T] \qquad (9.1)$$

along with the initial conditions

$$\vec{r}(t_0) = \vec{r}_0, \; \vec{v}(t_0) = \vec{v}_0, \qquad (9.2)$$

In the Cartesian frame $Oxyz$, Eqs. (9.1) and (9.2) are

$$m\frac{d^2x}{dt^2} = X, \quad m\frac{d^2y}{dt^2} = Y, \quad m\frac{d^2z}{dt^2} = Z, \tag{9.3}$$

$$\left| \begin{array}{lll} x(t_0) = x_0, & y(t_0) = y_0, & z(t_0) = z_0 \\ \dot{x}(t_0) = \dot{x}_0, & \dot{y}(t_0) = \dot{y}_0, & \dot{z}(t_0) = \dot{z}_0, \end{array} \right. \tag{9.4}$$

where $\overrightarrow{r}_0 = (x_0, y_0, z_0), \ \overrightarrow{v}_0 = (\dot{x}_0, \dot{y}_0, \dot{z}_0)$

A first integral of the differential system of motion (9-3) – (9.4) is a (non-constant) continuously-differentiable function (class C^1)

$$\boxed{\mathcal{F}(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) = c \in \mathbb{R}, \ \forall \ t \geq t_0} \tag{9.5}$$

which reduces to a constant when the functions

$$x = x(t), \quad y = y(t), \quad z = z(t) \tag{9.6}$$

satisfy (9.3).

If we determine $k \leq 6$ first integrals, for which

$$f_j(x_1, x_2, x_3, v_1, v_2, v_3; t) = C_j, \quad C_j = \text{const}, \quad j = 1, 2, \ldots, k,$$

the matrix

$$\mathbf{M} \equiv \left[\frac{\partial(f_1, f_2, \ldots, f_k)}{\partial(x_1, x_2, x_3, v_1, v_2, v_3)} \right]$$

being of rank k, then all the first integrals are functionally independent (for the sake of simplicity, further we say *independent first integrals*) ;

Thus, the problem is reduced to the integration of a system of equations (9.3) with only $6 - k$ unknowns (hence, a smaller number of unknowns). If $k = 6$, then all the first integrals are independent, so that the system (9.5) of first integrals determines all the unknown functions. We notice that for $k > 6$ the first integrals (9.5) are no more independent; we may thus set up at the most six independent first integrals.

P.P. Teodorescu, Mechanical Systems, Classical Models. Vol. I: Particle Mechanics, Springer, 2007

Momentum of a particle

Newton has introduced the notion of *momentum* (which he called *quantity of motion)* representing the product of the mass by the velocity of a point.

$$\vec{H} := m\vec{v} \qquad (9.7)$$

- **momentum is a vector parallel to the velocity vector**

From (9.1) we have

$$\frac{d}{dt}(m\vec{v}) = \vec{F} \;\Rightarrow\; \frac{d\vec{H}}{dt} = \vec{F},\; \frac{d}{dt}(m\dot{x}) = X, \ldots \qquad \vec{H}(t_0) = \vec{H}_0 \qquad (9.8)$$

Equation (9.8) we have is called the momentum equation. From (9.8) we can formulate the theorem of momentum:

Theorem *(theorem of momentum). The derivative with respect to time of the momentum of a free particle is equal to the resultant of the given forces which act upon it.*

Another usual notation for momentum is: $\vec{P} = m\vec{v}$

Average Force, Momentum, and Impulse

Suppose you are pushing a cart with a force that is non-uniform, but has an average value $\vec{\mathbf{F}}_{\text{ave}}$ during the time interval Δt. We can find the average acceleration according to Newton's Second Law,

$$\vec{\mathbf{F}}_{\text{ave}} = m\,\vec{\mathbf{a}}_{\text{ave}}. \qquad \vec{\mathbf{a}}_{\text{ave}} = \frac{\Delta\vec{\mathbf{v}}}{\Delta t}.$$

Therefore Newton's Second Law can be recast as

$$\vec{\mathbf{F}}_{\text{ave}} = m\,\vec{\mathbf{a}}_{\text{ave}} = \frac{m\,\Delta\vec{\mathbf{v}}}{\Delta t}.$$

The change in momentum is the product of the mass and the change in velocity,

$$\Delta\vec{\mathbf{p}} = m\,\Delta\vec{\mathbf{v}}.$$

Newton's Second Law can be restated as follows: the product of the average force acting on an object and the time interval over which the force acts will produce a change in momentum of the object,

$$\vec{\mathbf{F}}_{\text{ave}}\,\Delta t = \Delta\vec{\mathbf{p}}.$$

This change in momentum is called the *impulse*,

$$\vec{\mathbf{I}} = \vec{\mathbf{F}}_{\text{ave}}\,\Delta t = \Delta\vec{\mathbf{p}}.$$

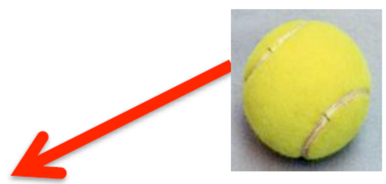
Force is a vector quantity; impulse is obtained by multiplying a vector by a scalar, and so impulse is also a vector quantity. The SI units for impulse are $[\mathrm{N}\cdot\mathrm{s}] = [\mathrm{kg}\cdot\mathrm{m}\cdot\mathrm{s}^{-1}]$, which are the same units as momentum.

https://sites.ualberta.ca/~ygu/courses/phys144/notes/momentum1.pdf

Example

Impulse and momentum:

A tennis ball (mass=0.1 kg) comes in from the right at 40 m/s at an angle 10 deg below horizontal. Then on a bad shot Rafa hits it at 30 deg above the horizontal at a speed of 60 m/s. What is the impulse of the net force and the average net force, assuming the contact (collision) duration is 0.01 sec.



Solution: Take positive X to the Right and positive Y as Up. The velocity components before (subscript 1) and after (2) the tennis ball is hit are:

$$v_{1x} = -40\cos(10) = -39.39 \text{ m/s} \qquad v_{1y} = -40\sin(10) = -6.95 \text{ m/s}$$

$$v_{2x} = 60\cos(30) = 51.96 \text{ m/s} \qquad v_{2y} = 60\sin(30) = 30 \text{ m/s}$$

X-component of impulse is equal to x-component of momentum change

$$J_x = p_{2x} - p_{1x} = m(v_{2x} - v_{1x})$$
$$= 0.1 \times (51.96 - (-39.39)) = 9.14 \text{ kg} \cdot \text{m/s}$$

$$J_y = p_{2y} - p_{1y} = m(v_{2y} - v_{1y})$$
$$= 0.1 \times (30 - (-6.95)) = 3.70 \text{ kg} \cdot \text{m/s}$$

By vector addition

$$J = \sqrt{J_x^{\,2} + J_y^{\,2}} = \sqrt{9.14^2 + 3.70^2} = 9.86 \text{ kg} \cdot \text{m/s}$$

Average net force: $$\overline{F} = \frac{J}{\Delta t} = \frac{9.86}{0.01} = 986 \text{ N}$$

Alternatively, we could use components of J to find components of average net force. The components of average net force are:

$$\overline{F}_x = \frac{J_x}{\Delta t} = \frac{9.14}{0.01} = 914 \text{ N} \qquad \overline{F}_y = \frac{J_y}{\Delta t} = \frac{3.70}{0.01} = 370 \text{ N}$$

The average net force is:

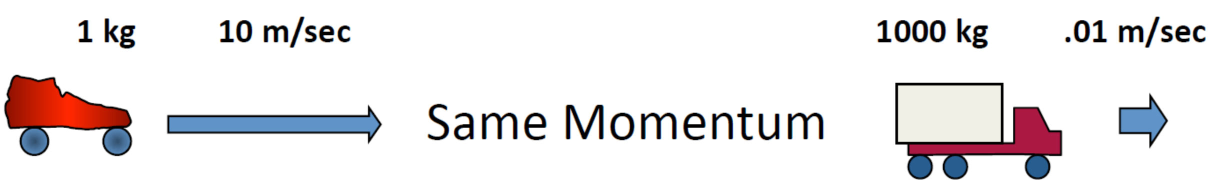
$$\overline{F} = \sqrt{\overline{F}_x^{\,2} + \overline{F}_y^{\,2}} = \sqrt{914^2 + 370^2} = \sqrt{835{,}396 + 136{,}900} = 986.05\ \text{N}$$

Average Force Direction:

$$\theta = \tan^{-1}\left(\frac{\overline{F}_y}{\overline{F}_x}\right) = 22.04°$$

Remarks: **(1) Keep signs straight due to vector use (2) Because the ball was Not at rest initially, average force direction is NOT the same as ball direction (analogy: projectile motion)**

Example



First integrals

- If $\vec{F} = 0$ then $\vec{H} = \overrightarrow{const}$, $\forall\, t \geq t_0$. This first integral is the law of momentum conservation.

$$\mathbf{H} = m\mathbf{v} = \mathbf{C}\,, \quad \mathbf{C} = \overrightarrow{\text{const}}\,, \quad H_i = C_i\,, \quad i = 1,2,3\,;$$

Theorem *(conservation theorem of momentum). The momentum (and the velocity) of a free particle is conserved in time if and only if the resultant of the given forces which act upon it vanishes.*

We notice that the relation $m\mathbf{v} = m\dot{\mathbf{r}} = \mathbf{C}$ leads to

$$m\mathbf{r} = \mathbf{C}t + \mathbf{C}'\,, \quad \mathbf{C}, \mathbf{C}' = \overrightarrow{\text{const}}\,, \quad mx_i = C_i t + C_i'\,, \quad i = 1,2,3\,;$$

Taking into account the initial conditions we obtain:

$$\vec{v} = \vec{v}_0;$$

$$\vec{r}(t) = \vec{v}_0(t - t_0) + \vec{r}_0.$$

The motion of the particle *P* is thus rectilinear and uniform. Besides, this result corresponds to the principle of inertia, which appears thus as a particular case of the principle of action of forces (lex secunda).

- If $\vec{\mathrm{F}} \neq 0,\ \forall\, t \geq t_0$ and exist a fixed direction $\vec{\boldsymbol{u}} = (\alpha, \beta, \gamma)$ such that $\vec{\mathrm{F}} \cdot \vec{u} = 0$, then using this in (9.1) we obtain

$$\frac{d}{dt}(\overrightarrow{H} \cdot \overrightarrow{u}) = \overrightarrow{F} \cdot \overrightarrow{u} = 0 \ \Rightarrow\ \overrightarrow{H} \cdot \overrightarrow{u} = c \in \mathbb{R}.$$

Thus,

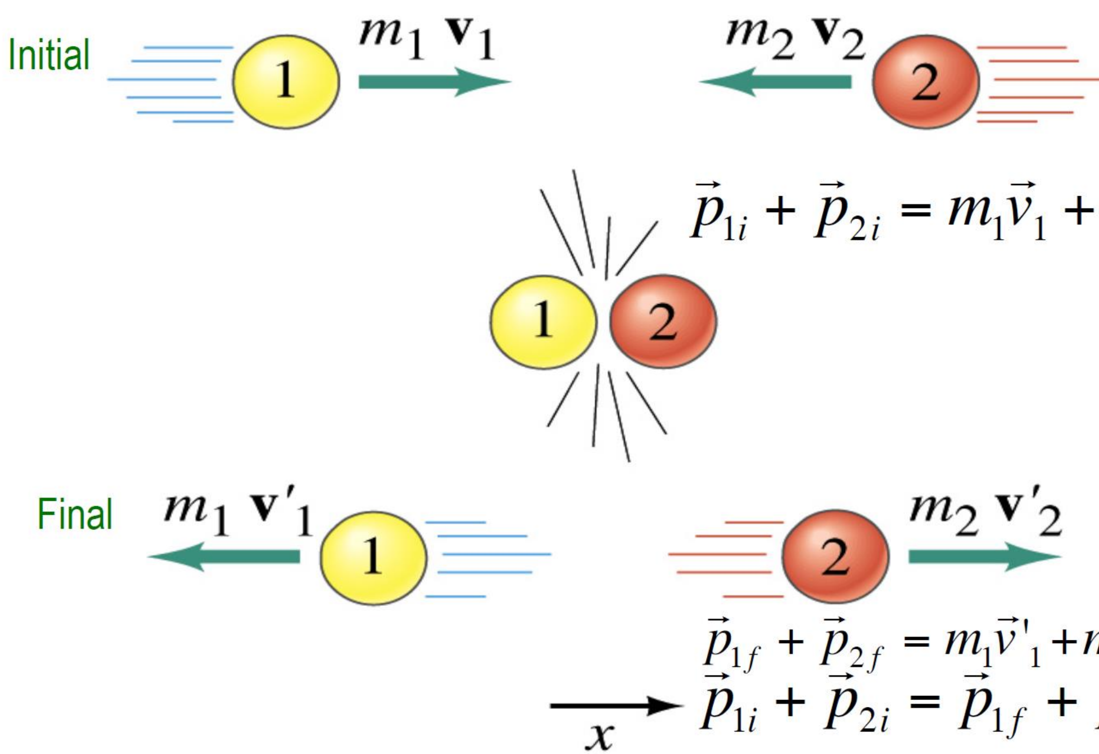
$$\alpha \dot{x} + \beta \dot{y} + \gamma \dot{z} = c,\ \forall\, t \geq t_0, \tag{9.9}$$

is a first integral of the differential equations of motion.

Hence, *if the force* **F** *is parallel to a fixed plane, then the projection of the velocity of the free particle P on the normal to this plane is conserved* (is constant) *in time*

Example

Linear Momentum Conservation (1D)



$$\vec{p}_{1i} + \vec{p}_{2i} = m_1\vec{v}_1 + m_2\vec{v}_2$$

$$\vec{p}_{1f} + \vec{p}_{2f} = m_1\vec{v}'_1 + m_2\vec{v}'_2$$

$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f}$$

$$m_1\vec{v}_1 + m_2\vec{v}_2 = m_1\vec{v}'_1 + m_2\vec{v}'_2$$

Moment of momentum (angular momentum)

The moment of the momentum with respect to the pole *O* (origin of the coordinate system) is called *moment of momentum (angular momentum) of the particle*, with respect to this pole, and is given by

$$\overrightarrow{K}_0 := \overrightarrow{r} \times m\overrightarrow{v} = \overrightarrow{r} \times \overrightarrow{H} \qquad (9.10)$$

$$K_x = m(y\dot{z} - z\dot{y}), \quad K_y = m(z\dot{x} - x\dot{z}), \quad K_z = m(x\dot{y} - y\dot{x}) \qquad (9.11)$$

From the equation of motion (9.1) we have

$$m\frac{d\overrightarrow{v}}{dt} = \overrightarrow{F} \quad | \times \overrightarrow{r}$$

$$\overrightarrow{r} \times \frac{d}{dt}(m\overrightarrow{v}) = \overrightarrow{r} \times \overrightarrow{F} \;\Rightarrow\; \frac{d}{dt}(\overrightarrow{r} \times m\overrightarrow{v}) = \overrightarrow{r} \times \overrightarrow{F}$$

or

$$\frac{d\overrightarrow{K}_0}{dt} = \overrightarrow{M}_0(\overrightarrow{F}) \qquad (9.12)$$

Moment (torque) of force **F** with respect to the pole O: $\vec{M}_O(\vec{F}) = \vec{r} \times \vec{F}$

Theorem (*theorem of angular momentum*) *The derivative with respect to time of the moment of momentum of a particle, with respect to a fixed pole, is equal to the moment of the force which act upon it, with respect to the same pole*.

On projection on the $Oxyz$ axes:

$$m\frac{d}{dt}(y\dot{z} - z\dot{y}) = yZ - zY, \ldots \tag{9.13}$$

First integrals

- If $\overrightarrow{\mathrm{M}}_O(\vec{\mathrm{F}}) = 0$ we deduce the law of the angular momentum conservation.

$$\overrightarrow{K}_0 = \vec{C} \in \mathbb{R}^3, \ \forall\, t \geq t_0 \tag{9.14}$$

Thus,

$$\overrightarrow{K}_0 = \overrightarrow{r} \times m\overrightarrow{v} = \overrightarrow{c}, \ \forall\, t \geq t_0$$

$$\overrightarrow{r} \times \overrightarrow{v} = \overrightarrow{c} = \overrightarrow{r}_0 \times \overrightarrow{v}_0 \tag{9.15}$$

Equation (9.15) is equivalent with

$$\frac{d\overrightarrow{A}}{dt} = \overrightarrow{const} \qquad (9.16)$$

where $\frac{d\overrightarrow{A}}{dt} = \frac{1}{2}(\overrightarrow{r} \times \overrightarrow{v})$ is the areal velocity of the particle.

From (9.15) the following first integrals are obtained

$$y\dot{z} - z\dot{y} = c_1, \ldots \qquad (9.17)$$

In the particular case when $\vec{F} \parallel \vec{r}$ (in this case $\vec{F}$ is called central force) we have

$$\overrightarrow{r} \times \overrightarrow{F} = 0 \Rightarrow \overrightarrow{r} \times \overrightarrow{v} = \overrightarrow{c} = \overrightarrow{r}_0 \times \overrightarrow{v}_0.$$

Notice that:

i) the motion is rectilinear if $\overrightarrow{r}_0 \| \overrightarrow{v}_0 \Rightarrow \overrightarrow{r}_0 \times \overrightarrow{v}_0 = 0 = \overrightarrow{r} \times \overrightarrow{v} \Rightarrow \overrightarrow{r} \| \overrightarrow{v}$,

ii) the motion takes place in a plane if $\overrightarrow{r}_0 \nparallel \overrightarrow{v}_0 \Rightarrow \overrightarrow{r} \cdot (\overrightarrow{r}_0 \times \overrightarrow{v}_0) = 0$

$$\Rightarrow \vec{r} \in (\vec{v}_O, \vec{r}_O)$$

- If $\vec{\mathrm{M}}_O(\vec{\mathrm{F}}) \neq 0, \ \forall\, t \geq t_0$ and exist a fixed direction $\vec{\boldsymbol{u}} = (\alpha, \beta, \gamma)$ such that $\vec{\mathrm{M}}_O(\vec{\mathrm{F}}) \cdot \vec{u} = 0$, then

$$\frac{d}{dt}(\overrightarrow{K}_0 \cdot \overrightarrow{u}) = 0 \ \Rightarrow \ \overrightarrow{K}_0 \cdot \overrightarrow{u} = c \in \mathbb{R}.$$

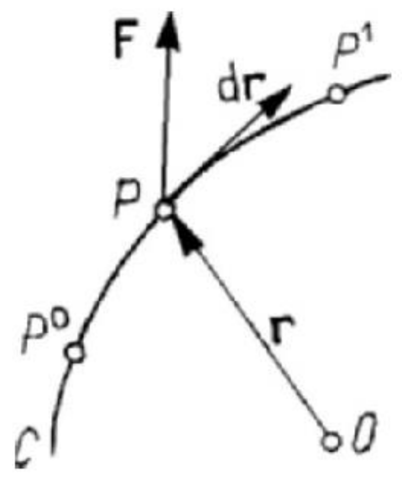
Thus, we obtain the first integral

$$(\overrightarrow{r} \times \overrightarrow{v}) \cdot \overrightarrow{u} = (\overrightarrow{r}_0 \times \overrightarrow{v}_0) \cdot \overrightarrow{u}. \qquad (9.18)$$

Work

Consider the force $\vec{F}$ acting on the point P ($\vec{r} = \overrightarrow{OP}$) which effects a real displacement $\mathrm{d}\vec{r}$. The elementary work of the force $\vec{F}$ is the scalar

$$\delta L = \overrightarrow{F} \cdot d\overrightarrow{r} = xdx + Ydy + Zdz \qquad (9.19)$$



Next, we suppose $\vec{F}$ depends only on $\vec{r}$ ($\vec{F}=\vec{F}(X,Y,Z)$) and exist a function $U=U(x,y,z)$ with the property:

$$X=\frac{\partial U}{\partial x},\quad Y=\frac{\partial U}{\partial y},\quad Z=\frac{\partial U}{\partial z} \tag{9.20}$$

or

$$\mathbf{F}=\operatorname{grad} U=\nabla U=U_{,j}\mathbf{i}_j\,,\quad F_j=U_{,j}\,, \tag{9.21}$$

where $U=U(\mathbf{r})=U(x,y,z)$ is *the force function* (*potential function* or *potential*). Function $V:=-U$ is the *potential energy* and we say that $\vec{F}$ is *conservative*.

Theorem. The force $\vec{F}$ is potential (conservative) if and only if

$$\frac{\partial X}{\partial y}=\frac{\partial Y}{\partial x},\quad \frac{\partial Y}{\partial z}=\frac{\partial Z}{\partial y},\quad \frac{\partial Z}{\partial x}=\frac{\partial X}{\partial z} \tag{9.22}$$

This is equivalent with the fact that

$$\overrightarrow{F}\cdot d\overrightarrow{r}=X\,dx+Y\,dy+Z\,dz \tag{9.23}$$

is an exact differential.

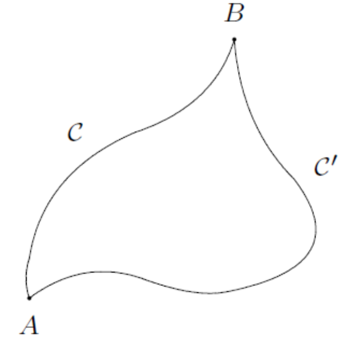
A force field $\vec{F}: D \subset R^3 \rightarrow R^3$ is a conservative force field if $\exists U: D \rightarrow R$ with the property:

$$\overrightarrow{F} = \mathrm{grad} U = -\mathrm{grad} V$$

Theorem. Consider $D \subset R^3$ a simply connected domain.

A force field $\vec{F}: D \rightarrow R^3$ is potential (conservative) if and only if one of the following equivalent conditions takes place:

- $$\mathrm{rot}\,\overrightarrow{F} = \nabla \times \overrightarrow{F} = 0$$
- $$\int_\Gamma \underbrace{\overrightarrow{F} \cdot d\overrightarrow{r}}_{=X dx + Y dy + Z dz} = 0, \quad \forall\ \gamma \text{ closed curve}$$
- $$\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = \int_{C'} \overrightarrow{F} \cdot d\overrightarrow{r},$$



$\forall\ C, C' \in D$ arbitrary curved from *A* to *B*.

- the differential form

$$\overrightarrow{F} \cdot d\overrightarrow{r} = X dx + Y dy + Z dz$$

is an exact differential, i.e.

$$\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x}, \quad \frac{\partial Y}{\partial z} = \frac{\partial Z}{\partial y}, \quad \frac{\partial Z}{\partial x} = \frac{\partial X}{\partial z}$$

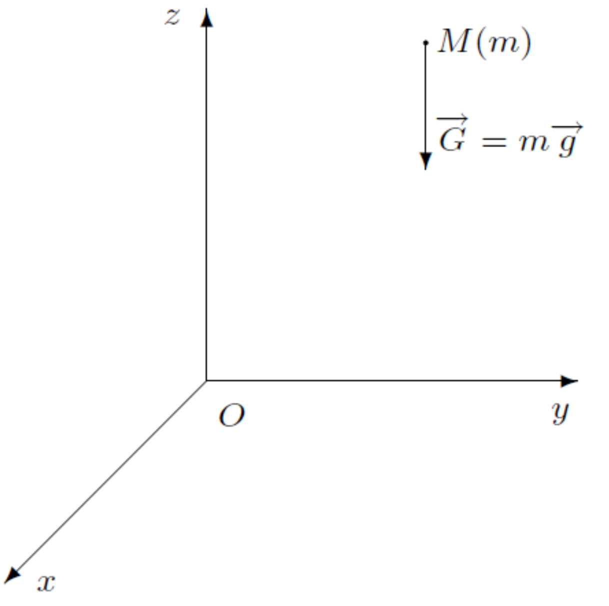
Example

$$\overrightarrow{F} = m\overrightarrow{g} = -\mathrm{grad} V \;\Leftrightarrow\; \frac{\partial V}{\partial x} = 0,\; \frac{\partial V}{\partial y} = 0,\; \frac{\partial V}{\partial z} = mg$$

$$\Rightarrow\; dV = d(mgz)$$

$$\Rightarrow\; V = mgz + c, \quad U = -mgz$$

$\Rightarrow\; \overrightarrow{G} = m\overrightarrow{g}$ is a potential (conservative) force



Force function in the plane

Consider $\overrightarrow{F}:D\to\mathbb{R}^2,\ \overrightarrow{F}=(X,Y),\ X=X(x,y),\ Y=Y(x,y)$

such that

$$\frac{\partial X}{\partial y}=\frac{\partial Y}{\partial x} \tag{9.24}$$

Thus, $\exists\ U:D\to\mathbb{R}$ such that $\overrightarrow{F}=\mathrm{grad}U\ \Leftrightarrow\ X=\dfrac{\partial U}{\partial x},\quad Y=\dfrac{\partial U}{\partial y}$ (9.25)

We integrate $(9.25)_1$ with respect to *x*

$$U(x,y)=\int_{x_0}^{x}X(s,y)ds+\varphi(y), \tag{9.26}$$

where $\phi(y)$ is determined from $(9.25)_2$:

$$\int_{x_0}^{x}\frac{\partial U(s,y)}{\partial y}ds+\varphi'(y)=Y(x,y)$$

Taking into account (9.24) we obtain

$$\int_{x_0}^{x} \frac{\partial Y(s,y)}{\partial s} ds + \varphi'(y) = Y(x,y)$$

$$Y(x,y) - Y(x_0,y) + \varphi'(y) = Y(x,y)$$

$$\varphi'(y) = Y(x_0,y),$$

or

$$\varphi(y) = \int_{y_0}^{y} Y(x_0,u)du + const. \tag{9.27}$$

Using (9.26) and (9.27) we have

$$U(x,y) = \int_{x_0}^{x} X(s,y)ds + \int_{y_0}^{y} Y(x_0,u)du. \tag{9.28}$$

Next, we suppose that the force field $\vec{\mathrm{F}} = \vec{F}(\vec{r}) = \vec{F}(x, y, z)$ is conservative (potential), i,.e $\exists \mathrm{U}: \mathrm{D} \rightarrow \mathrm{R}$ with the property:

$$\overrightarrow{F} = \mathrm{grad} U = -\mathrm{grad} V$$

or

$$X = \frac{\partial U}{\partial x}, \quad Y = \frac{\partial U}{\partial y}, \quad Z = \frac{\partial U}{\partial z}$$

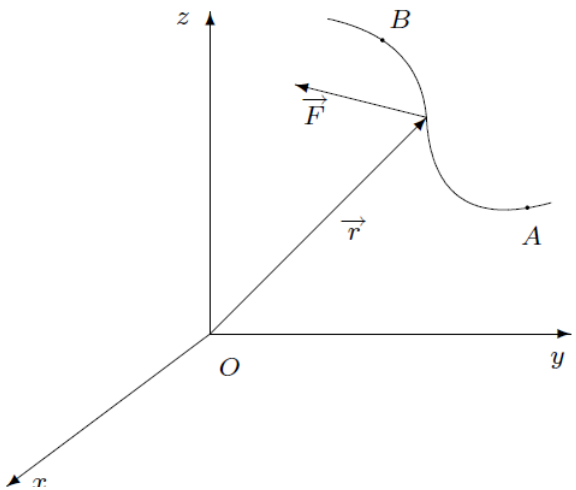
Thus, the elementary work is an exact differential

$$\delta L = X dx + \ldots = dU = -dV$$

Total work along an arc AB of a curve

Consider AB an arc of a curve defined by the equations:

$$x = x(q), \quad y = y\ (q), \quad z = z(q), \ q \in [q_0, q_1]. \tag{9.29}$$



The curvilinear integral

$$L_{AB} = L := \int_{AB} \overrightarrow{F} \cdot d\overrightarrow{r} = \int_{AB} X dx + Y dy + Z dz = \int_{AB} (X(x(z), \ldots) dx(q)$$

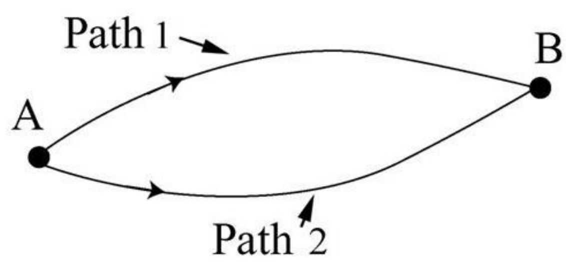
$$= \int_{q_0}^{q_1} (X(x(q), y(q), z(q)) x'(q) + \ldots) dq \tag{9.30}$$

is the total work of the force $\vec{F}$ along the arc AB.

If the force is conservative, $\overrightarrow{F} = \mathrm{grad}U$, then

$$L = \int_{AB} \overrightarrow{F} \cdot d\overrightarrow{r} = \int_{AB} dU = U(B) - U(A) \qquad (9.31)$$

and the work is independent of the path from A to B.



If the force is not conservative we use:

$$L = \int_{AB} \overrightarrow{F} \cdot d\overrightarrow{r} = \int_{q_0}^{q_1} [X(x(q), y(q), z(q))x'(q) + \ldots]dq \qquad (9.32)$$

and if the force is a function of velocity and time we can use:

$$L = \int_{AB} Xdx + Ydy + Zdz = \int_{t_0}^{t_1} X(x(t), \ldots)\dot{x}(t) + \ldots \qquad (9.33)$$

Kinetic energy

The scalar quantity

$$\boxed{T = \frac{1}{2} m v^2} \qquad (9.34)$$

is the *kinetic energy of the particle P* ; this quantity depends on the mass and the velocity of the particle.

From $m\frac{d\overrightarrow{v}}{dt} = \overrightarrow{F} \mid \cdot d\overrightarrow{r}$ we deduce

$$m \underbrace{\frac{d\overrightarrow{r}}{dt}}_{=\overrightarrow{v}} \cdot \underbrace{\frac{d\overrightarrow{v}}{dt} dt}_{=d\overrightarrow{v}} = \overrightarrow{F} \cdot d\overrightarrow{r} \;\Rightarrow\; m\overrightarrow{v} \cdot d\overrightarrow{v} = \overrightarrow{F} \cdot d\overrightarrow{r} \;\Rightarrow\; dT = \delta L \qquad (9.35)$$

Equation (9.35) expresses the following theorem

Theorem (*theorem of kinetic energy*). *The differential of the kinetic energy of a free particle is equal to the elementary work of the resultant of the given forces which act upon it.*

Theoretical Mechanics

If the motion of the particle takes place in a conservative field of forces, $\overrightarrow{F} = \overrightarrow{F}(\overrightarrow{r})$, then exist $V = V(\overrightarrow{r})$ such that $\overrightarrow{F} = -\mathrm{grad}V$.

Thus, $\delta L = -dV$ and Eq. (9.35) becomes

$$d(T+V) = 0 \;\Rightarrow\; T+V = h, \;\forall\, t \geq t_0. \tag{9.36}$$

The first integral (9.36) is the *energy integral*, *V* is the *potential energy*, while *h* is the energy constant and has to be calculated from the initial conditions.

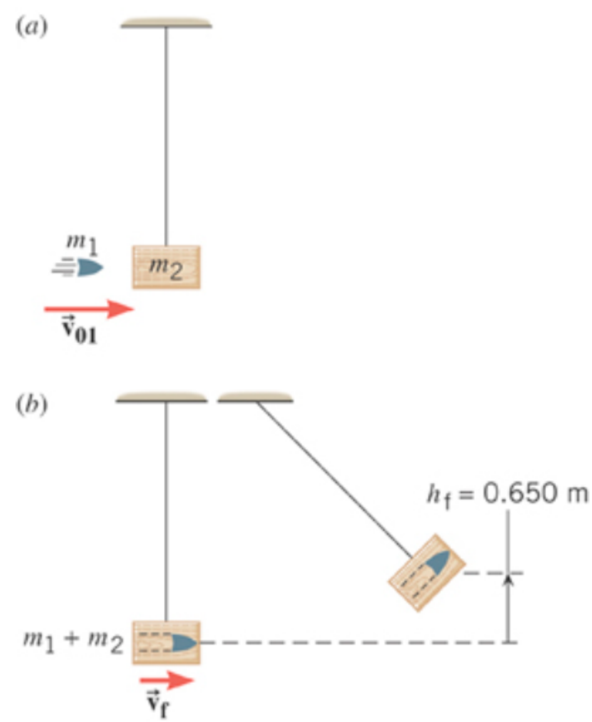
$E = T + V$ total (mechanical) energy

Theorem (*mechanical energy conservation theorem*). *The mechanical energy of a free particle is conserved in time if and only if the resultant of the given forces which act upon it is conservative.*

https://sites.ualberta.ca/~ygu/courses/phys144/notes/momentum2.pdf

Ex. A Ballistic Pendulum

The mass of the block of wood is 2.50-kg and the mass of the bullet is 0.0100-kg. The block swings to a maximum height of 0.650 m above the initial position. Find the initial speed of the bullet.



What kind of collision? **Perfectly inelastic**

No net external force → momentum conserved

$$m_1 v_{f1} + m_2 v_{f2} = m_1 v_{o1} + m_2 v_{o2}$$

$$(m_1 + m_2)\, v_f = m_1 v_{o1}$$

Solve for V_{01}

$$v_{o1} = \frac{(m_1 + m_2) v_f}{m_1}$$

What do we not know? The final speed!!

How can we get it? Using the mechanical energy conservation!

Now using the mechanical energy conservation

$$\tfrac{1}{2}mv^2 = mgh$$

$$\cancel{(m_1 + m_2)}gh_f = \tfrac{1}{2}\cancel{(m_1 + m_2)}v_f^2$$

$$gh_f = \tfrac{1}{2}v_f^2$$

Solve for V_f

$$v_f = \sqrt{2gh_f} = \sqrt{2\left(9.80\,\mathrm{m/s^2}\right)(0.650\ \mathrm{m})}$$

Using the solution obtained previously, we obtain

$$v_{o1} = \frac{(m_1 + m_2)v_f}{m_1} = \frac{(m_1 + m_2)\sqrt{2gh_f}}{m_1}$$

$$= \left(\frac{0.0100\ \mathrm{kg} + 2.50\ \mathrm{kg}}{0.0100\ \mathrm{kg}}\right)\sqrt{2\left(9.80\,\mathrm{m/s^2}\right)(0.650\ \mathrm{m})}$$

$$= +896\,\mathrm{m/s}$$