Symbols 1) "(", ")" - parantheses **FIRST ORDER LOGIC**

2) $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ - connections

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Seminar 2

3) $\forall$, $\exists$ - quantifiers

4) "=" - equality sign

5) $x, y, z, \dots$ - variables

6) $a, b, c, \dots$ - constants

7) $f, g, h, \dots$ - functions $(x, f(x))$

8) P, Q, R - predicates

M-set

- A predicate P is an open expression $P(x_1, x_2, \dots, x_n)$ in which if we replace $x_1, \dots, x_n$ with $a_1, \dots, a_n$ etc we get a formula (proposition)

$M = \mathbb{R}$ "$x+y=z$" predicate   ~~binary~~ $f(x,y)$ binary

$\mathbb{R}^3$ $1+1=2$ formula (true) $f(x)$ unary

$2+1=1$ formula (false)   n-ary

$\mathbb{R}^2$ "$x<y$" predicate

(x, y) $x<y$

subset $= \{ (a_1, a_2, \dots, a_n) \in M^n \mid P(a_1, \dots, a_n) \text{ true} \}$

$P(1,2) =$ "$1<2$" true

Terms (expressions): - any variable

- any constant
- If f is an ~~n-ary~~ n-ary function (n variables) and $t_1, \dots, t_n$ are terms then $f(t_1, \dots, t_n)$ is a term
- there are no other terms

2 - constant

2 term

f(2) term

Formulas: - If P is ~~a predicate~~ an n-ary predicate and $t_1, \dots, t_n$ are terms then $P(t_1, \dots, t_n)$ is a formula

- If t_1, t_2 are terms then $(t_1 = t_2)$ is a formula $\underbrace{x+y}_{t_1} = \underbrace{z}_{t_2}$

$x+y$ ~~(formula)~~ $\to$ term

- If A and B are formulas then $(\neg A)$, $(A \to B)$, $(A \land B)$, $(A \lor B)$, $A \leftrightarrow B$ are formulas
- If A is a formula, x is a variable then $(\forall) x A$, $\exists x A$ are formulas
- there are no other formulas

Bound variable has quantifier in front $\forall x$, otherwise it's free

14) f, g, h are functions of 1, 2, 3 variables respect. and P, Q ~~R~~ are predicates with 1, resp. 3 variables

a) Are the following terms?

$\underbrace{f(g(x,y))}_{f(t_1)}$ - term

$g(\underbrace{f(z)}_{t_1}, \underbrace{h(x,y,z)}_{t_2})$ - term

$f(\overset{\text{not a term}}{g(x)}, h(x,y,z))$ - not a term

f must be unary

b) Are the following formulas

$Q(x, f(x), \underbrace{h(y,z,z)})$ - formula

$\underbrace{P(x)}_{\text{formula}} \to (\forall y)(\underbrace{Q(x,y,z)}_{\text{formula}} \wedge \underbrace{P(g(x,y))}_{\text{formula}})$ - formula

$Q(\underbrace{P(x)}_{\substack{\text{not term} \\ \downarrow \\ \text{has to be var}}}, f(y), z)$ - not a formula

$f(h(x,y,z))$ - term but not a formula

$g(x,y) = f(h(x,y,z))$ - formula

List of tautologies:

p.17 2.3.9

1) $\forall x \forall y A \Leftrightarrow \forall y \forall x A$

$\exists x \exists y A \Leftrightarrow \exists y \exists x A$

2) $\exists x \forall y A \Rightarrow \forall y \exists x A$

$\forall x A \Rightarrow \exists x A$

$\ulcorner \underbrace{\forall y \in \mathbb{R}\ \exists x \in \mathbb{R}\ (y-x=0)}_{1} \xrightarrow{1 \to 0} \underbrace{\exists x \in \mathbb{R}\ \forall y \in \mathbb{R} (y-x=0)}_{0}$

false

5) $\forall x A \vee \forall x B \Rightarrow \forall x (A \vee B)$

$\ulcorner \forall x (A \vee B) \to (\forall x A) \vee (\forall x B)$

$\underbrace{(\forall) x \in \mathbb{N} (x>0) \vee (x=0)}_{1} \to \underbrace{(\forall) x \in \mathbb{N} (x>0)}_{0}$

$\vee \underbrace{(\forall) x \in \mathbb{N} (x=0)}_{0}$

6) $\exists x (A \wedge B) \Rightarrow (\exists x A) \wedge (\exists x B)$

$(\exists x A) \wedge (\exists x B) \to \exists x (A \wedge B)$

$\underbrace{\exists x \in \mathbb{N} (x=1) \wedge \exists x \in \mathbb{N} (x=2)}_{1} \to \underbrace{\exists x \in \mathbb{N} (x=1 \wedge x=2)}_{0}$

18) $M = \mathbb{N}$

S, P are predicates with 3 variables

$S(x, y, z)$ true $\Leftrightarrow x + y = z$ $\quad S(1,1,2)$ $S(1,4,2)$

true false

$P(x, y, z)$ true $\Leftrightarrow xy = z$ $\quad P(1,1,1)$ $P(1,0,2)$

true false

Write a formula with free variable x (free) true if and only if

a) $x = 0 \Leftrightarrow \cancel{P(x,1,0)}\ P(x, 1, 0)$ $\quad x \cdot 1 = 0$

$x = 0 \Leftrightarrow \underbrace{P(x, 1, 0)}_{x \cdot 1 = 0} \Leftrightarrow \underbrace{S(x, 1, 1)}_{x + 1 = 1}$

b) $x = 1 \Leftrightarrow \underbrace{S(x, 0, 1)}_{x + 0 = 1} \Leftrightarrow P(x, 1, 1)$

$\Leftrightarrow \underbrace{P(1, 1, x)}_{1 \cdot 1 = x}$

c) $x = 2 \Leftrightarrow S(1, 1, x) \Leftrightarrow P(1, 2, x)$

d) x is even $\Leftrightarrow \exists y\ \cancel{\dots} \ P(2, y, x)$ $\quad x = 2/.2 \quad 2y = x$

$\Leftrightarrow \exists y\, S(y, y, x)$

x is free

e) x is odd $\Leftrightarrow \exists y\, S(2y, 1, x)$ $\quad 2y = 1 + x$

$\Leftrightarrow \underbrace{\neg \exists y\, \underbrace{P(2, y, x)}_{\text{even}}}_{\text{odd}}$ $\quad \Leftrightarrow \neg \exists y\, S(y, y, x)$

f) x is prime $\quad$ 7 prime if $y / x \Rightarrow y = 1 \vee y =$

~~$x \neq 0, x \neq 1$~~

$(x \neq 0) \wedge (x \neq 1) \wedge$ if $(y / x) \wedge (y = 1) \vee (y = x)$

x is prime $\iff (\neg P(x,1,0)) \land (\neg S(x,0,1)) \land (\forall y \exists z P(y,z,x)$
$\to (S(y,0,1) \lor P(x,1,x)))$

$y \mid x \iff \exists z P(y,z,x)$
$x = y \cdot z$

Two free variables x and y

2) a) $x = y \iff P(x,1,y)$ $\quad x \cdot 1 = y$

b) $x \leq y \iff \exists z \, S(x,z,y)$

c) $x < y \iff \exists z \, (S(x,z,y) \land (\neg P(x,1,y)))$

3) Three free variables x, y, z

a) z is the greatest common divisor of x and y

$z = \gcd(x,y)$

$z = \gcd(x,y) \iff (z \mid x) \land (z \mid y) \land \text{if} ((t \mid x) \land$
$t \mid y)) \to t \mid z$

$z = \gcd(x,y) \iff \exists p \, P(z,p,x) \land \exists x P(z,t,y)$
$\land (\forall t) \, \Delta \, (\exists t_2 \, P(t_2, \Delta, x) \land \exists t_3 \, P(t_3, \Delta, y)$
$\to \exists t_4 \, P(t_4, \Delta, z)) \, P(t_2, t_4, z) \land \exists$