Sequences of Real Numbers - part 1

Ex. 1: a) $x_n = \frac{2^n+7^n}{9^n}$ $\quad x_n \boxed{?} x_{n+1} \iff \frac{2^n+7^n}{9^n} \boxed{?} \frac{2^{n+1}+7^{n+1}}{9^{n+1}} \,\Big|\cdot 9^{n+1}$

$\iff 9(2^n+7^n) \boxed{?} 2^{n+1}+7^{n+1} \iff 9\cdot 2^n + 9\cdot 7^n \boxed{?} 2^{n+1}+7^{n+1} \,\Big| -2^{n+1} - 9\cdot 7^{n+1}$

$\iff 9\cdot 2^n - 2^{n+1} \boxed{?} 7^{n+1} - 9\cdot 7^{n} \iff 2^n(9-2) \boxed{?} 7^n(7-9)$

$\iff 2^n\cdot 7 \boxed{?} 7^n\cdot(-2) \iff 2^n\cdot 7 \boxed{?} -2\cdot 7^n \,\Big| : 7^n$

$\iff \frac{2^n}{7^{n-1}} \boxed{?} -2 \iff x_n > x_{n+1} \Rightarrow (x_n)_{n\in\mathbb{N}}$ is DECREASING

(x_n) dec. $\Rightarrow UB(x_n) = x_0 = \frac{2^0+7^0}{9^0} = \frac{2}{1} = 2$ ①

$LB(x_n) = \lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} \frac{2^n+7^n}{9^n} = \lim\limits_{n\to\infty} \frac{7^n[(\frac{2}{7})^n+1]}{9^n} = \lim\limits_{n\to\infty} (\frac{7}{9})^n \frac{(0+1)}{1} =$

$= \lim\limits_{n\to\infty} (\frac{7}{9})^n\cdot 1 = 0\cdot 1 = 0 \Rightarrow LB(x_n) = 0$ ②

from ① and ② $\Rightarrow (x_n)$ BOUNDED

from ① $\Rightarrow (x_n)$

$\lim\limits_{n\to\infty} x_n = 0 \overset{Te}{\iff} \forall V \in \mathcal{V}(0), \exists n_V \in \mathbb{N}$ s.t. $\forall n \ge n_V, x_n \in V$

$\iff \forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \ge n_\varepsilon, |x_n - 0| < \varepsilon$

choose $\varepsilon > 0$ randomly

$|x_n - 0| < \varepsilon \iff \left|\frac{2^n+7^n}{9^n} - 0\right| < \varepsilon \iff \left|\frac{2^n+7^n}{9^n}\right| < \varepsilon \iff 0 < \frac{2^n+7^n}{9^n} < \varepsilon \,\Big| \ln$

$\iff \ln\frac{2^n+7^n}{9^n} < \ln\varepsilon \iff \ln(2^n+7^n) - \ln 9^n < \ln\varepsilon$

$\iff \ln(2^n+7^n) - n\ln 9 < \ln\varepsilon \quad \ln(2^n+7^n) < \cancel{\ln\varepsilon}\ \ln\varepsilon + n\ln 9$

$\iff n\ln 9 > \ln\frac{2^n+7^n}{\varepsilon} \iff n > \ln\frac{2^n+7^n}{9\varepsilon}$

$\overset{Arch}{\Rightarrow} \exists n_\varepsilon \in \mathbb{N}$ s.t. $n_\varepsilon > \ln\frac{2^n+7^n}{9\varepsilon}$

If $n \ge n_\varepsilon > \ln\frac{2^n+7^n}{9\varepsilon} \iff \ln\frac{2^n+7^n}{9^n} < \ln\varepsilon$ $\quad \varepsilon$ random $\Rightarrow \lim\limits_{n\to\infty} x_n = 0$

b) $x_n = \frac{(-1)^n}{n}$ $\quad x_n \boxed{?} x_{n+1} \iff \frac{(-1)^n}{n} \boxed{?} \frac{(-1)^{n+1}}{n+1} \,\Big|\cdot(n+1)$ $\quad (-1)^n\cdot(n+1) \boxed{?} (-1)^{n+1}$

$\Rightarrow (x_n)_{n\in\mathbb{N}^*}$ is nonmonotonic

$UB(x_n) = x_1 = -1 \quad UB(x_n) = x_2 = \frac{+1}{2}$

$\lim\limits_{n\to\infty} x_n = 0 \iff \forall \varepsilon > 0\ \exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \ge n_\varepsilon\ |x_n - 0| < \varepsilon$

choose $\varepsilon > 0$ random $\quad \left|\frac{(-1)^n}{n} - 0\right| < \varepsilon \iff \frac{1}{n} < \varepsilon$

$\Rightarrow \exists n_\varepsilon \in \mathbb{N}^*$ s.t. $\forall n \ge n_\varepsilon\ \frac{1}{n} < \varepsilon \iff \left|\frac{(-1)^n}{n} - 0\right| < \varepsilon \iff \lim\limits_{n\to\infty} a_n = 0$

$\Rightarrow (x_n)$ convergent

c) $x_n = \frac{5^n}{n!}$ $\quad x_n ? x_{n+1}$ $\quad \frac{5^n}{n!} ? \frac{5^{n+1}}{(n+1)!} \quad |\cdot(n+1)!$

$5^n(n+1) ? 5^{n+1} \quad |:5^n \quad n+1 ? 5 \quad |-1 \quad n ? 4$

$\Rightarrow$ for $n<4 \Rightarrow (x_n)$ increasing

$\Rightarrow$ for $n \geq 4 \Rightarrow (x_n)$ decreasing

~~LB(x_n) = ...~~ $UB(x_n) = x_4 = \frac{5^4}{4!} = \frac{625}{24}$

$\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} \frac{5^n}{n!} = 0 \Rightarrow$

?

d) $x_n = \frac{3n}{4n^2+1}$ $\quad x_n ? x_{n+1} \Leftrightarrow \frac{3n}{4n^2+1} ? \frac{3n+3}{4n^2+8n+5} \quad |(4n^2+1)(4n^2+8n+5)$

$3n\cdot(4n^2+8n+5) ? (3n+3)(4n^2+1) \Leftrightarrow 12n^3+24n^2+15n ? 12n^3+3n+12n^2+3$

$\cancel{12n^3+39n ? 12n^3+4n^2}$ $\quad 12n^3+24n^2+15n ? 12n^3+12n^2+3n+3$

$20n^2+12n-1 ? 0 \quad \Delta = 144+80 = 224 > 0$

$\Rightarrow (x_n)$ is decreasing

$UB(x_n) = x_1 = \frac{3}{4+1} = \frac{3}{5}$

$LB(x_n) = \lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} \frac{3n}{4n^2+1} = 0 \Leftrightarrow \forall V \in \mathcal{V}(0), \exists n_0 \in \mathbb{N}$ s.t.

$\forall n \geq n_0, x_n \in V \Leftrightarrow \forall \varepsilon>0, \exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\varepsilon, |x_n - 0| < \varepsilon$

choose randomly $\varepsilon>0$

$\left|\frac{3n}{4n^2+1} - 0\right| < \varepsilon \Leftrightarrow \frac{3n}{4n^2+1} < \frac{3n}{4n^2} = \frac{3}{4n} = \frac{3}{4}\cdot\frac{1}{n} < \frac{1}{n}$

$\Rightarrow \exists n_\varepsilon \in \mathbb{N}$ s.t. $\frac{1}{n_\varepsilon} < \varepsilon \Rightarrow \forall n \geq n_\varepsilon, |x_n - 0| < \varepsilon$

ε-random

$\Rightarrow \lim\limits_{n\to\infty} x_n = 0 \Rightarrow (x_n)$ convergent.

Ex2: a) $\lim_{n\to\infty} \frac{3n}{4n^2+1} = 0 \iff \forall V \in \mathcal{V}(0), \exists n_V \in \mathbb{N}$, s.t. $\forall n \geq n_V, X_n \in V$

$\iff \forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\varepsilon, |X_n - 0| < \varepsilon$

choose randomly $\varepsilon > 0$ $\left\{ \left|\frac{3n}{4n^2+1} - 0\right| < \varepsilon \iff \frac{3n}{4n^2+1} < \frac{3n}{4n^2} = \frac{3}{4n} = \frac{3}{4}\cdot\frac{1}{n} < \frac{1}{n} \right.$

$\Rightarrow \exists n_\varepsilon \in \mathbb{N}$ s.t. $\frac{1}{n_\varepsilon} < \varepsilon \Rightarrow \forall n \geq n_\varepsilon, |X_n - 0| < \varepsilon \}$ $\Rightarrow$ true

ε random

b) $\lim_{n\to\infty} \frac{5n^2}{-2n+4} = -\infty \iff \forall V \in \mathcal{V}(-\infty), \exists n_V \in \mathbb{N}$, st. $\forall n \geq n_V, X_n \in V$

$\iff \forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\varepsilon, X_n < -\varepsilon$

choose randomly $\varepsilon > 0$ $\left\{ \frac{5n^2}{-2n+4} < -\varepsilon \iff \frac{5n^2}{2n-4} > \varepsilon \iff \right.$ $\cancel{5n^2}$

$\iff \frac{5n^2}{2n-4} > \frac{5n^2}{2n} = \frac{5n}{2} > n > \varepsilon \Rightarrow \exists n_\varepsilon \in \mathbb{N}$ s.t. $n_\varepsilon > \varepsilon \Rightarrow \forall n \geq n_\varepsilon, X_n < -\varepsilon$

Ex3: a) $\lim_{n\to\infty} \frac{4^n+1}{9^n+1} = \lim_{n\to\infty} \frac{4^n(1+\frac{1}{4^n})}{9^n(1+\frac{1}{9^n})} = \lim_{n\to\infty} \frac{4^n}{9^n}\frac{(1+0)}{(1+0)} = \lim_{n\to\infty} 0\cdot 1 = 0$

b) $\lim_{n\to\infty} \frac{8^n+(-4)^n}{8^{n-1}+4^n} = \lim_{n\to\infty} \frac{8^n\left[1+\left(-\frac{1}{2}\right)^n\right]}{8^{n-1}\left(1+\frac{4^n}{8^{n-1}}\right)} = \lim_{n\to\infty} 8\cdot\frac{1-0}{1+0} = 8$

c) $\lim_{n\to\infty}\left(\sin\frac{5n}{109}\right)^n = \lim_{n\to\infty}\left(\sin\frac{5n}{105+n}\right)^n = \cancel{\lim}$ $\Big\} \Rightarrow \lim_{n\to\infty}\left(\sin\frac{5n}{108}\right)^n = 0$

$\frac{5n}{109} < \frac{\pi}{6}$ ~~(109~~ ~~30n < 109n~~ $\Rightarrow \frac{5n}{109} \in (0,1)$

d) $\lim_{n\to\infty}\left(\sqrt{9n^2+2n+1} - 3n\right) = \lim_{n\to\infty}\left(\sqrt{9n^2\left(1+\frac{2}{9n}+\frac{1}{9n^2}\right)} - 3n\right) = 0$

e) $\lim_{n\to\infty}\left(9+\frac{1-5n^3}{6n^4+2}\right)^2 = \lim_{n\to\infty}\left(\frac{6n^4+2-5n^3+1}{6n^4+2}\right)^2 =$

$= \lim_{n\to\infty}\left(\frac{6n^4-5n^3+3}{6n^4+2}\right)^2 = \lim_{n\to\infty}\left\{\frac{6n^4\left(1-\frac{5}{6n}+\frac{3}{6n^4}\right)}{6n^4\left(1+\frac{2}{6n^4}\right)}\right\}^2 =$

$= 1^2 = 1$

f) $\lim_{n\to\infty}\left(\sqrt[3]{n^3+n+3} - \sqrt[3]{n^3+1}\right) = \lim_{n\to\infty}\left(\sqrt[3]{n^3\left(1+\frac{1}{n^2}+\frac{3}{n^3}\right)} - \sqrt[3]{n^3\left(1+\frac{1}{n^3}\right)}\right) =$

$= \lim_{n\to\infty}\left[n\left(\sqrt[3]{1+\frac{1}{n^2}+\frac{3}{n^3}} - \sqrt[3]{1+\frac{1}{n^3}}\right)\right] = \lim_{n\to\infty}\left(n\cdot(1-1)\right) = \infty\cdot 0 = 0$

g) $\lim_{n\to\infty}\left(\frac{n^3+5n+1}{n^2-1}\right)^{\frac{1-7n^4}{9n^4+1}} = \lim_{n\to\infty}\left[\frac{n^3\left(1+\frac{5}{n^2}+\frac{1}{n^3}\right)}{n^2\left(1-\frac{1}{n^2}\right)}\right]^{\lim_{n\to\infty}\frac{7n^4\left(\frac{1}{7n^4}-1\right)}{9n^4\left(1+\frac{1}{9n^4}\right)}} =$

$= \lim_{n\to\infty}(n\cdot 1)^{\frac{7}{9}} = \lim_{n\to\infty} n^{\frac{7}{9}} = \infty$

h) $\lim_{n\to\infty}\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdot\ldots\cdot\left(1-\frac{1}{n}\right) = \lim_{n\to\infty}\frac{1}{2}\cdot\frac{2}{3}\cdot\ldots\cdot\frac{n-1}{n} = \lim_{n\to\infty}\frac{1}{n} = 0$

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__Ex 4__: a) b) By the density property of the rationals $\Leftrightarrow$ Archimedian property

$\Rightarrow$ ~~ε~~ Let $t \in \mathbb{R}$, $\forall n \in \mathbb{N}$, $t - \frac{1}{n} < t + \frac{1}{n}$ s.t. $\exists r \in \mathbb{R}$ s.t.

$t - \frac{1}{n} < r_n < t + \frac{1}{n} \Leftrightarrow |r_n - t| < \frac{1}{n} \Rightarrow r_n$ converges to t

~~Ex 5. $a > 0$ $x_0 \in \mathbb{R}$ $0 < x_0 < \frac{1}{a}$ $(x_n)_{n \in \mathbb{N}}$ sequence of real numbers~~

$\cancel{x_{n+1} = 2x_n - a x_n^2, \forall n \in \mathbb{N}}$

$\cancel{x_1 = 2x_0 - a x_0^2}$

$\cancel{x_2 = 2x_1 - a x_1^2 = 2(2x_0 - a x_0^2) - a(2x_0 - a x_0^2)^2 =}$

$\cancel{= 4x_0 - 2a x_0^2 - a(4x_0^2 - 4a x_0^3 + a^2 x_0^4) =}$

$\cancel{= -a^3 x_0^4 + 4a^2 x_0^3 + 2x_0^2 + 4x_0}$

$\cancel{x_n = x_{n+1} = 2x_n - a x_n^2} \quad \cancel{x_n = \frac{x_{n+1} + a x_n^2}{2}}$

$\cancel{x_1 = 2x_0 - a x_0^2}$

$\cancel{x_2 = 2x_1 - a x_1^2}$

$\cancel{x_3 = 2x_2 - a x_2^2}$

$\cancel{x_4 = 2x_3 - a x_3^2}$

$\vdots$

$\cancel{x_{n-1} = 2x_{n-2} - a x_{n-2}^2}$

$\cancel{x_n = 2x_{n-1} - a x_{n-1}^2}$

$\oplus$

$\cancel{x_1 + x_2 + \dots + x_n = 2(x_0 + x_1 + \dots + x_{n-1}) - a(x_0^2 + x_1^2 + \dots + x_{n-1}^2)}$

$\cancel{a(x_0^2 + x_1^2 + \dots + x_{n-1}^2) = 2x_0 + x_1 + x_2 + \dots + x_{n-1} - x_n}$

$\cancel{2x_0 = a x_0^2 + x_1 + x_2 + \dots + x_{n-1} - x_n + a(x_1^2 + \dots + x_{n-1}^2)}$

$\cancel{2x_1 + x_2 + \dots + x_{n-1} - x_n = a(x_1^2 + \dots + x_{n-1}^2)}$

$\cancel{2x_2 + \dots + x_{n-1} - x_n = a(x_2^2 + \dots + x_{n-1}^2)}$

$\cancel{0 < x_0 < \frac{1}{a} \mid \cdot 2 \quad 0 < 2x_0 < \frac{2}{a}}$

$\cancel{0 < x_0 < \frac{1}{a} \mid (\cdot)^2 \quad a > 0 \quad 0 < x_0^2 < \frac{1}{a^2} \mid (-a) < 0 \quad 0 > -a x_0^2 > -\frac{1}{a}}$

EX I: $a>0$ $x_0 \in \mathbb{R}$ $0<x_0<\frac{1}{a}$

$x_{n+1} = 2x_n - a x_n^2$, $\forall n \in \mathbb{N}$

a) $x_{n+1} = 2x_n - a x_n^2$ $\qquad$ $x_{n+1} = x_n(2 - a x_n)$

$\cancel{x_n} = 2\cancel{x_{n-1}} - a x_{n-1}^2 \mid \cdot 2$ $\qquad$ $\cancel{x_n} = \cancel{x_{n-1}}(2 - a x_{n-1})$

$\cancel{x_{n-1}} = 2\cancel{x_{n-2}} - a x_{n-2}^2 \mid \cdot 2^2$

$\cancel{x_{n-2}} = 2\cancel{x_{n-3}} - a x_{n-3}^2 \mid \cdot 2^3$

$\cancel{x_2} = \cancel{x_1}(2 - a x_1)$

$\cancel{x_1} = x_0(2 - a x_0)$

$x_{n+1} = (2 - a x_n)(2 - a x_{n-1}) \cdot \ldots \cdot (2 - a x_0)$

$\cancel{x_2} = 2 \cdot x_1 - a x_1^2 \mid \cdot 2^{n-2}$

$\cancel{x_1} = 2 \cdot x_0 - a x_0^2 \mid \cdot 2^{n-1}$

$x_{n+1} = -a(x_n^2 + 2x_{n-1}^2 + 2^3 x_{n-2}^3 + \ldots + 2^{n-2} \cdot x_1^2 + 2^{n-1} \cdot x_0^2) + 2x_0$

I. E.V. $P(1)$: $x_1 < \frac{1}{a}$ $\qquad$ $x_1 = 2x_0 - a x_0^2$

$0 < x_0 < \frac{1}{a} \mid \cdot 2$ $\quad$ $2x_0 < \frac{2}{a}$

$0 < x_0 < \frac{1}{a} \mid \cdot ()^2$ $\quad$ $x_0^2 < \frac{1}{a^2} \mid \cdot a$ $\quad$ $a x_0^2 < \frac{1}{a}$

$\overset{\ominus}{\Rightarrow} 2x_0 - a x_0^2 < \frac{2}{a} - \frac{1}{a} = \frac{1}{a}$

$\Rightarrow x_1 < \frac{1}{a}$

II E.V. Pp that $P(k)$: $x_k < \frac{1}{a}$ A

show that $P(k+1)$: $x_{k+1} < \frac{1}{a}$ A

$x_{k+1} = 2x_k - a x_k^2$

$P(k) \Rightarrow x_k < \frac{1}{a} \mid \cdot 2 \Rightarrow 2x_k < \frac{2}{a}$

$P(k) \Rightarrow x_k < \frac{1}{a} \mid \cdot ()^2 \Rightarrow x_k^2 < \frac{1}{a^2} \mid \cdot a$ $\quad$ $a x_k^2 < \frac{1}{a}$ $\quad \overset{\ominus}{\Rightarrow}$

$\rightarrow 2x_k - a x_k^2 < \frac{2}{a} - \frac{1}{a} = \frac{1}{a} \Leftrightarrow x_{k+1} < \frac{1}{a}$, $\forall k \in \mathbb{N}$

from I, II $\Rightarrow P(n)$: $x_n < \frac{1}{a}$, A, $\forall n \in \mathbb{N}$

b) $P(n)$: $x_n > 0$

I E.V. $P(1)$: $x_1 > 0$ $\qquad$ $x_1 = 2x_0 - a x_0^2$

$x_0 > 0 \mid \cdot 2$ $\quad$ $2x_0 > 0$

$x_0 > 0 \mid ()^2$ $\quad$ $x_0^2 > 0 \mid \cdot a$ $\quad$ $a x_0^2 > 0$

$\overset{\ominus}{\Rightarrow} 2x_0 - a x_0^2 > 0 \Leftrightarrow x_1 > 0$

II E.D. Pp that $P(k)$: $x_k > 0$, A

show that $P(k+1)$: $x_{k+1} > 0$, A $\qquad$ $x_{k+1} = 2x_k - a x_k^2$

$x_k > 0 \mid \cdot 2$ $\quad$ $2x_k > 0$

$x_k > 0 \mid ()^2$ $\quad$ $x_k^2 > 0 \mid \cdot a$ $\quad$ $a x_k^2 > 0$

$\overset{\ominus}{\Rightarrow} 2x_k - a x_k^2 > 0 \Leftrightarrow x_{k+1} > 0 \Rightarrow P(k+1)$, A from I, II $\Rightarrow 0 < x_n, \forall n \in \mathbb{N}$ A*

c) x_n is increasing $\Leftrightarrow$ $\cancel{x_{n+1} < x_n}$ $x_n < x_{n+1}$ $\quad x_n < 2x_n - ax_n^2$

$\cancel{x_{n+1} - x_n < 0 \quad 2x_n - ax_n^2 - x_n < 0 \quad x_n - ax_n^2 < 0 \mid + ax_n^2}$

~~true from a) $\Rightarrow$~~ $\cancel{0 < x_n < ax_n^2 \mid : x_n \quad 1 < ax_n \mid : a, \quad x_n > \frac{1}{a}}$

$0 < x_n - ax_n^2 \quad ax_n^2 < x_n \mid : x_n \quad ax_n < 1 \quad x_n < \frac{1}{a}$

true from a)

$0 < x_n < \frac{1}{a}$

true from b)

d) $\cancel{\lim_{n\to\infty}} x_{n+1}$ $\quad 0 < x_n < \frac{1}{a} \mid \lim_{n\to\infty}$ $\quad 0 < \lim_{n\to\infty} x_n < \lim_{n\to\infty} \frac{1}{a} = 0$

0

$\xRightarrow{S.T.} \lim_{n\to\infty} x_n = 0$