Theoretical Mechanics

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Lecture Notes (in PDF):

Microsoft Teams

Examination (proposal)

1. **Mid Semester Exam (Kinematics, theory and problems) 50%**

2. **Final Exam (Dynamics, theory and problems) 50%**

Theoretical Mechanics

References:

1. Kohr, M., Capitole Speciale de Mecanică, Presa Universitară Clujeană, Cluj-Napoca, 2005.
2. Brădeanu, P., Mecanică Teoretică, vol. 1 şi 2, Litografia Universităţii Babeş-Bolyai, Cluj-Napoca, 1988.
3. Iacob, C., Mecanică Teoretică, Editura Didactică şi Pedagogică, Bucureşti, 1980.
4. Edward A. Desloge, Classical Mechanics, John Wiley & Sons, New York, 1982.
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6. Bose, S., Chattoraj, D., Elementary Analytical Mechanics, Alpha Science International Ltd. 2000.
7. Petre P. Teodorescu, Mechanical Systems, Classical Models, Springer, Dordrecht, 2007.
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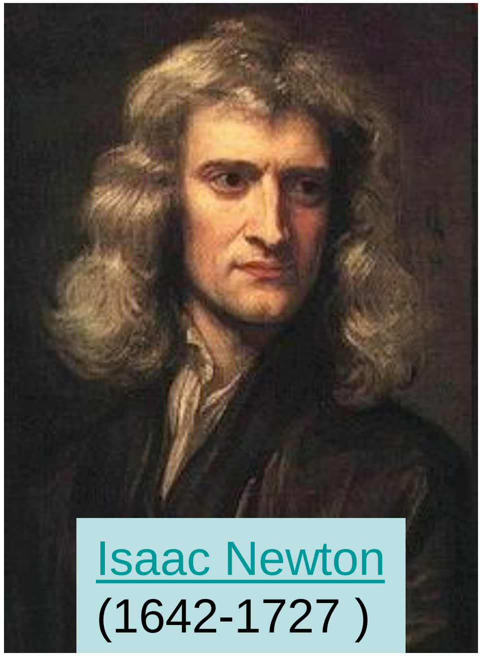
10. Meriam J.L., Kraige, L.G., Dynamics, Wiley, 2020

Theoretical Mechanics

Classical mechanics is the study of the motion of bodies in accordance with the general principles first enunciated by Sir Isaac Newton in his Philosophiae Naturalis Principia Mathematica (1687).

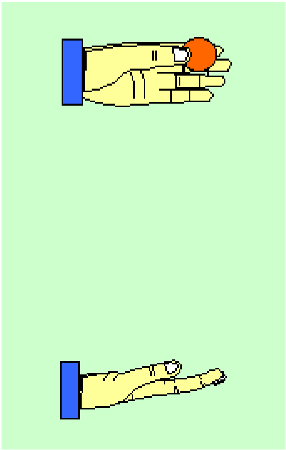
(https://www2.physics.ox.ac.uk/contacts/people/harnew)

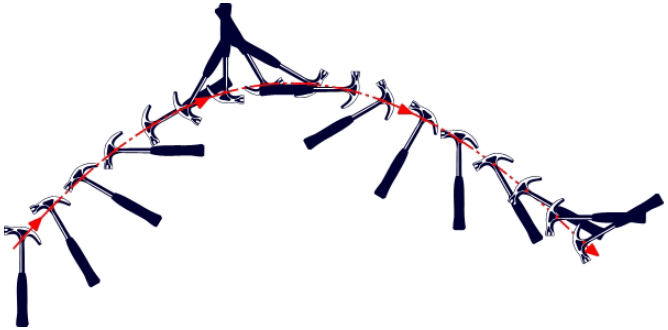


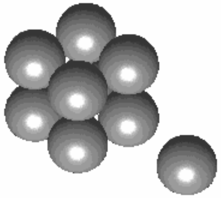
Isaac Newton
(1642-1727)

Classical Mechanics covers:

- The case in which bodies remain at rest
- Translational motion– by which a body shifts from one point in space to another
- More general rotational motion–bodies that are spinning
- Particle collisions



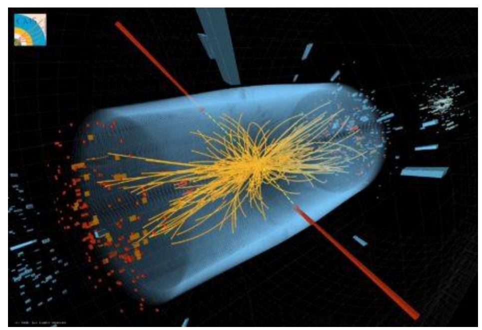


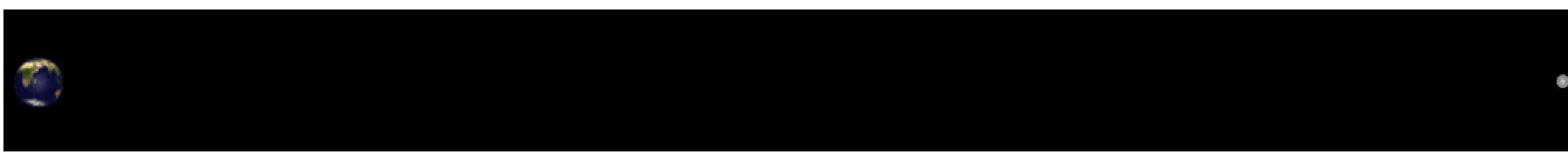


Theoretical Mechanics

Classical Mechanics - valid on scales which are:

- **Not too fast** (e.g.. high energy particle tracks from CERN)
- v << c = **299 792 458 m/s** [speed of light in vacuum]
- If too fast, time is no longer absolute - need special relativity (atomic clocks experiment).





Earth-Moon: 384 400 km - 1,282 sec

Theoretical Mechanics

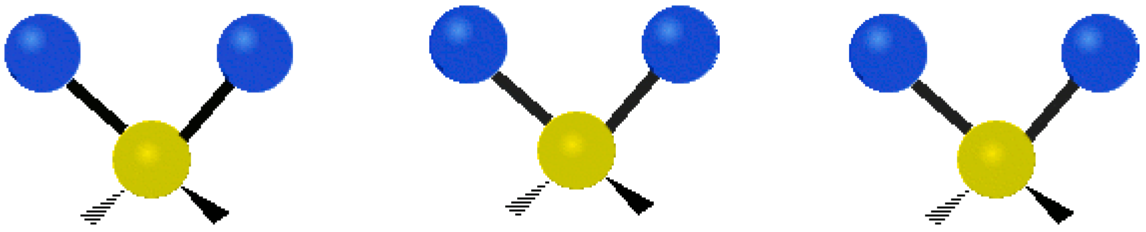
Classical Mechanics - valid on scales which are:

Not too small!

Particles actually have wave-like properties :

(Louis de Broglie postulated that all particles with a specific value of momentum p have a wavelength $\lambda = h/p$, where $h = 6.6 \times 10^{-34}$ J·s is Planck's constant)

Hence for scales >> λ wave properties can be ignored



Molecular vibrations

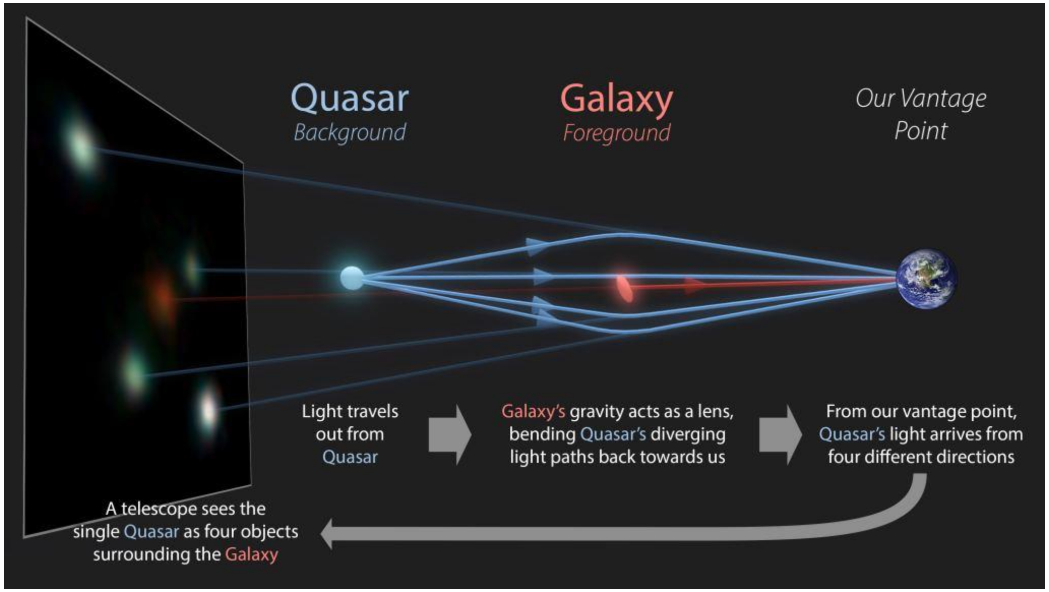
Theoretical Mechanics

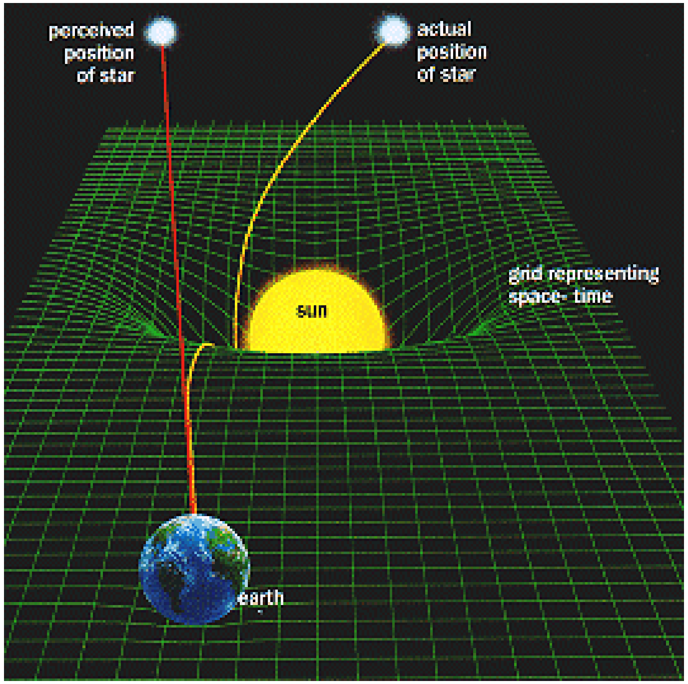
Classical Mechanics valid on scales which are:

- **Not too large**!
- Gravitational lens produced by a cluster of galaxies
- Space is "flat" in classical mechanics - curvature of space is ignored
- Also in Newtonian mechanics, time is absolute

The picture shows a famous cosmic mirage known as the Einstein Cross, and is a direct visual confirmation of the theory of general relativity. It is one of the best examples of the phenomenon of gravitational lensing — the bending of light by gravity as predicted by Einstein in the early 20th century. In this case, the galaxy's powerful gravity acts as a lens that bends and amplifies the light from the quasar behind it, producing four images of the distant object. The quasar, being 11 billion light-years from us in the direction of the constellation of Pegasus, is seen as it was around 11 billion years ago. The galaxy that works as a lens is some ten times closer.

Theoretical Mechanics





0. Preliminaries

(A. Bettini, A Course in Classical Physics 1—Mechanics, Springer, 2016)
(A. Romano, A. Marasco, Classical Mechanics with Mathematica®, 2nd ed., Birkhaeser, 2012)

The **time** is the succession of events. The Newtonian time is linear (has a uniform flow) and irreversible.

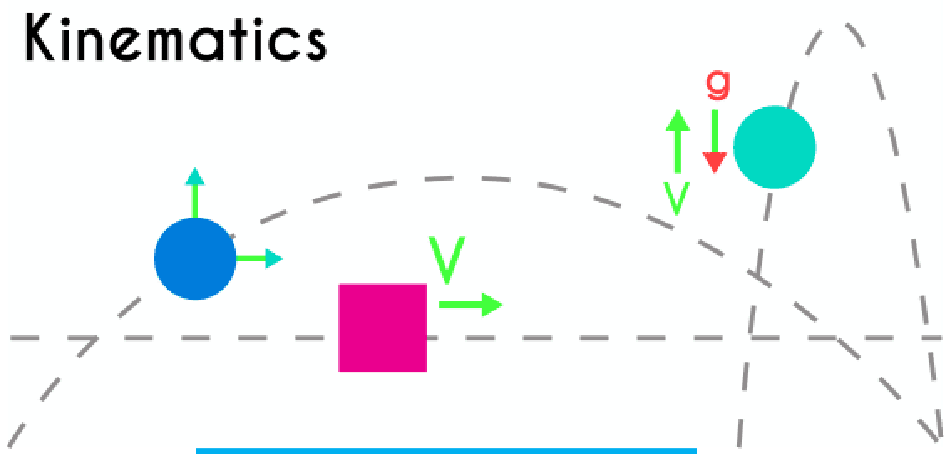
Motion of a body means that its position in space varies in time. The notion of motion is relative: a passenger in a plane sitting in his chair has a fixed position relative to the plane, but moves at, say 800 km/h relative to a person standing on earth. The latter moves at 800 km/h relative to the passenger, in the opposite direction.

To describe the motion we then need a **frame of reference**. Usually, a reference frame is fixed on the earth. The possible choices are still infinite.

Kinematics is the part of classical mechanics that studies the motion of a **body**, ignoring its causes.

A **particle ("material point")** is an object whose size can be ignored in a particular context.
This abstraction allows to simplify the observation of motion, as it gets rid of the complications deriving from the extension of real bodies. Moreover, this abstraction is introductory to the far more complex study of real bodies.

Kinematics of the material point studies the motion of a point-like (particle) object.

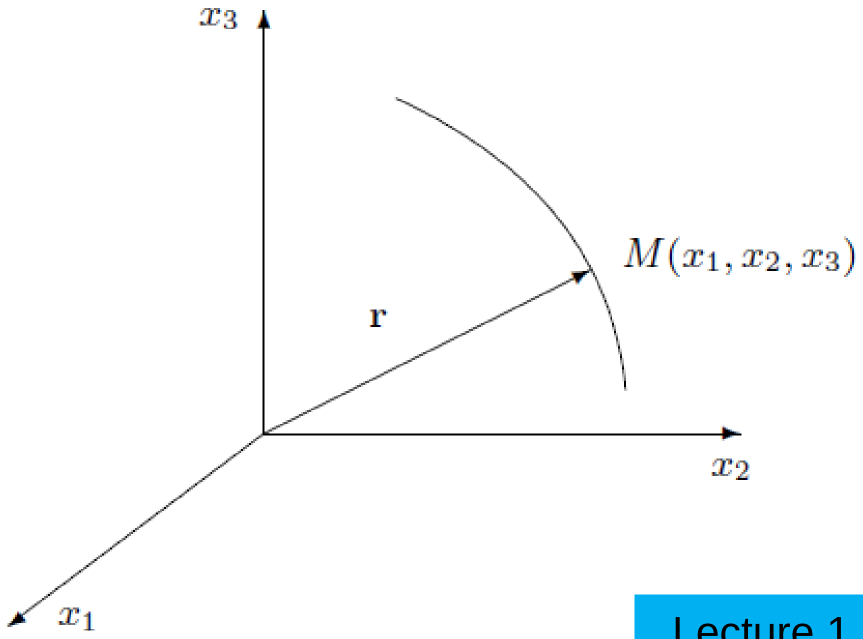


1. Kinematics of the material point

Consider a material point M. The point M is moving relative to an orthonormal frame **$Ox_1x_2x_3$** if his position vector ($\vec{r} = \overrightarrow{OM}$) is variable in time:

$$\vec{r} = \vec{f}(t) \quad \text{or } \vec{r} = \vec{r}(t) \tag{1.1}$$

Equation (1.1) is the ***equation of motion*** of the point M written in the vector form.



In an orthogonal (Cartesian) frame, the scalar equations of the motion are:

$$x_i = x_i(t) \text{ or } x_i = f_i(t), \quad i = 1,2,3 \tag{1.2}$$

If $x_i(t)$ or $f_i(t)$ are known in a time interval $[t_0, T], t_0 > 0; T < \infty,$ then the motion of the material point M in the frame **Ox$_1$x$_2$x$_3$** is known in this time range.

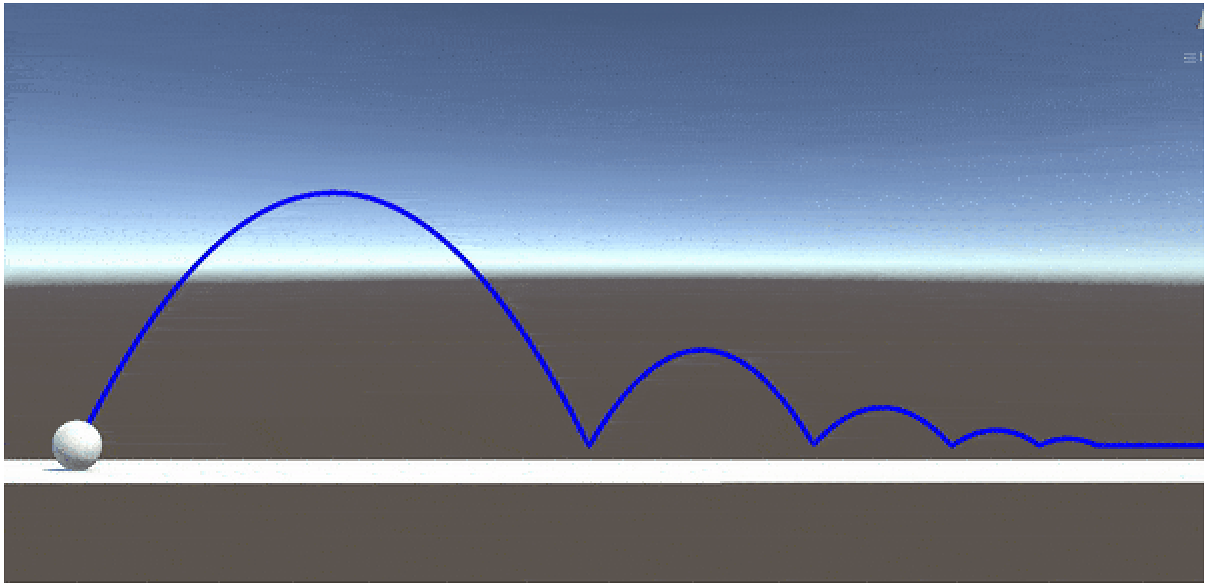
The function

$$\vec{r}: [t_0, T] \rightarrow R^3,\ \vec{r} = \vec{r}(t) \tag{1.3}$$

associates to every moment $t \in [t_0, T]$ a unique position in space for the point M.

The particle describes in its motion a curve, which is called the **trajectory.**

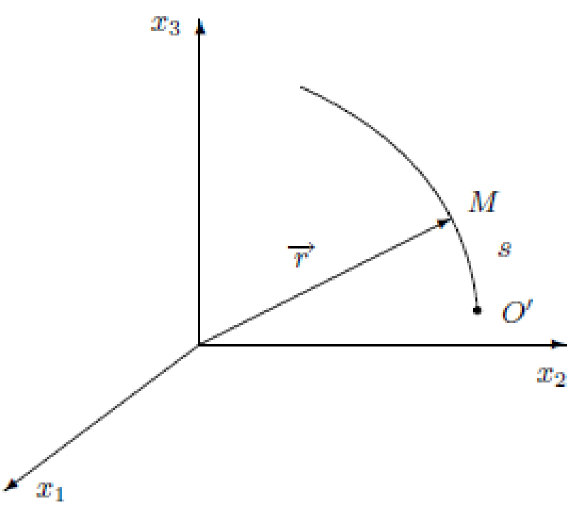
The **trajectory** is the (geometric) locus of the successive positions of the material point in space.



$$x_i = x_i(t) \text{ or } x_i = f_i(t), \quad i = 1,2,3 \qquad (1.2)$$

Equations (1.2) represent the parametric equations of the trajectory (t being the parameter). By eliminating t in these equations the trajectory can be obtained as a intersection of two surfaces: $x_1 = g_1(x_2, x_3), \quad x_2 = g_2(x_3, x_1).$ (1.4)

Further, we suppose that $x_i(t)$ are functions of class C^k $(k \geq 2)$ on the motion interval. Thus, the trajectory is a rectifiable curve* and it is possible to specify the position of the point M on the trajectory by using an intrinsic coordinate, s, the arc length on the trajectory, measured from an initial position O' to the current position of the point M.



Moreover, the parametric representation of the trajectory (depending on the parameter *s*) has the form:

$$\vec{r} = \vec{r}(s), s\epsilon\ [0, S],\ S > 0 \qquad (1.5)$$

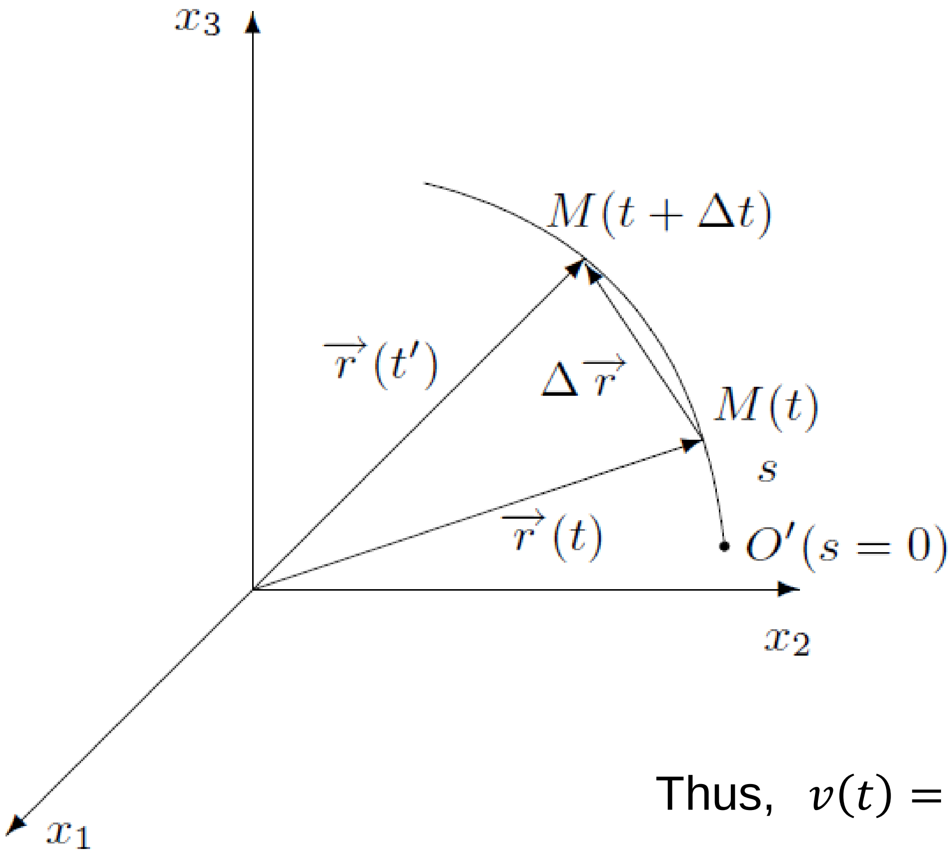
In this case, the equation of the motion of the particle on the trajectory is given by:

$$s = s(t),\ t\epsilon\ [t_0, T] \qquad (1.6)$$

* A rectifiable curve is a curve having finite length.

Velocity

Let us consider the position vector of particle M at the instant of time $t, \vec{r}(t)$ and an immediately following instant $t + \Delta t, \vec{r}(t + \Delta t)$, where Δt is a short time interval.



In this time interval the particle has moved by Δs, which is a step in the space.

The *average speed* of the particle M is defined as follow:

$$v_{\mathrm{m}} = \frac{s(t+\Delta t)-s(t)}{\Delta t}$$

The limit

$$v(t) = \lim_{\Delta t \to 0} \frac{s(t+\Delta t)-s(t)}{\Delta t} \qquad (1.9)$$

is the *speed* of M at the moment *t*.

Thus, $v(t) = \frac{ds}{dt}(t) = \dot{s}(t)$ (1.10)

The <u>average velocity</u> in the time interval Δt is the vector obtained by dividing the displacement $\Delta\vec{r}$ by the time interval in which it happens:

$$\vec{v}_{\mathrm{m}} = \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \frac{\Delta\vec{r}}{\Delta t} \tag{1.11}$$

The (instantaneous) <u>*velocity*</u> is the limit for $\Delta t \to 0$ of the average velocity, namely

$$\vec{v}(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} \tag{1.12}$$

Thus, (Cartesian coordinates)

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \dot{\vec{r}}(t) = \dot{x}_1(t)\vec{\iota}_1 + \dot{x}_2(t)\vec{\iota}_2 + \dot{x}_3(t)\vec{\iota}_3 = v_1\vec{\iota}_1 + v_2\vec{\iota}_2 + v_3\vec{\iota}_3 \tag{1.13}$$

However, we have $\vec{r} = \vec{r}(s)$ and $s = s(t)$ and we get

$$\vec{v}(t) = \frac{d\,\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt} = v\,\vec{\tau} \tag{1.14}$$

where $\vec{\tau}$ is the tangent unit vector at the trajectory in the point M.

Remark: The velocity vector is tangent to the trajectory, is oriented in the sense of the displacement and its algebraic magnitude is $v = \dot{s} = \frac{ds}{dt}$.

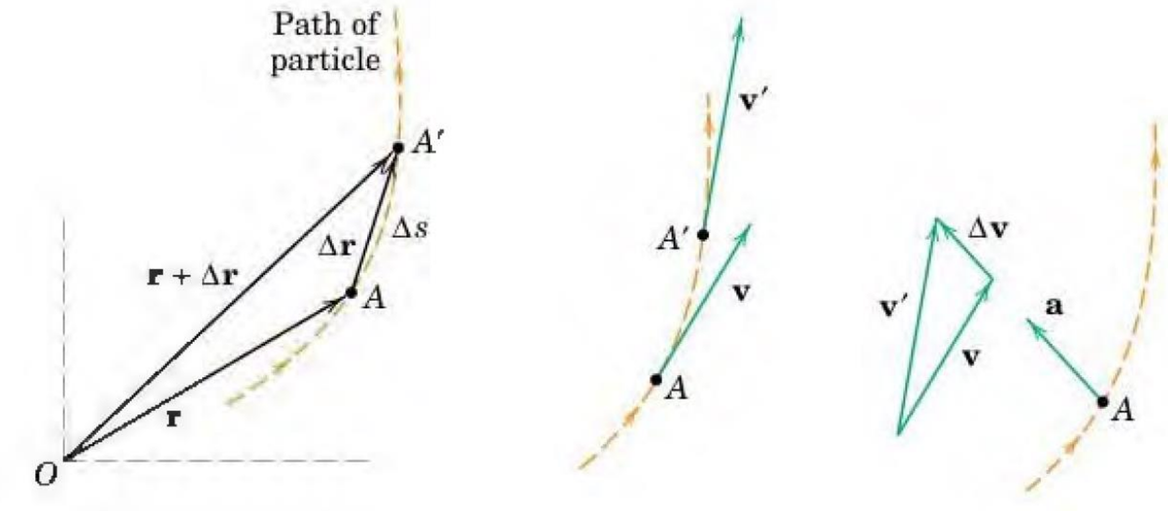
$$\vec{v} = \dot{s}\vec{\tau} \qquad (1.15)$$

Acceleration

The ratio

$$\vec{a}_m(t) = \frac{\vec{v}(t + \Delta t) - \vec{v}(t)}{\Delta t} \qquad (1.16)$$

is the *average acceleration* of the particle M.



The limit

$$\vec{a}(t) = \lim_{\Delta t \to 0} \frac{\vec{v}(t + \Delta t) - \vec{v}(t)}{\Delta t} \quad (1.17)$$

is the <u>*instantaneous acceleration*</u> of the particle M at the moment *t*.

We have (Cartesian coordinates)

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \dot{\vec{v}}(t) = \ddot{x}_1(t)\vec{i}_1 + \ddot{x}_2(t)\vec{i}_2 + \ddot{x}_3(t)\vec{i}_3 = a_1\vec{i}_1 + a_2\vec{i}_2 + a_3\vec{i}_3 \quad (1.18)$$

2. Kinematics of the material point in different frames of coordinates

Cartesian coordinates

Motion equations: $x_i = x_i(t)$**,** $t\epsilon\ [t_0, T]$**,** $i = 1{,}2{,}3$

Trajectory: Eliminate time in the motion equation. The curve is the intersection of two surfaces: $\mathrm{F}_1(x_1, x_2, x_3) = 0$; $\mathrm{F}_2(x_1, x_2, x_3) = 0$

Velocity: $\vec{v}(t) = \frac{d\vec{r}}{dt} = \dot{\vec{r}}(t) = \dot{x}_1(t)\vec{\imath}_1 + \dot{x}_2(t)\vec{\imath}_2 + \dot{x}_3(t)\vec{\imath}_3$ (1.13)

Acceleration: $\vec{a}(t) = \frac{d\vec{v}}{dt} = \dot{\vec{v}}(t) = \ddot{x}_1(t)\vec{\imath}_1 + \ddot{x}_2(t)\vec{\imath}_2 + \ddot{x}_3(t)\vec{\imath}_3$ (1.14)

Theoretical Mechanics

Example (Meriam J.L., Kraige, L.G., Dynamics, Wiley, 2020, page 46)

The curvilinear motion of a particle is defined by $v_x = 50 - 16t$ and $y = 100 - 4t^2$, where v_x is in meters per second, y is in meters, and t is in seconds. It is also known that $x = 0$ when $t = 0$. Plot the path of the particle and determine its velocity and acceleration when the position $y = 0$ is reached.

Solution. The x-coordinate is obtained by integrating the expression for v_x, and the x-component of the acceleration is obtained by differentiating v_x. Thus,

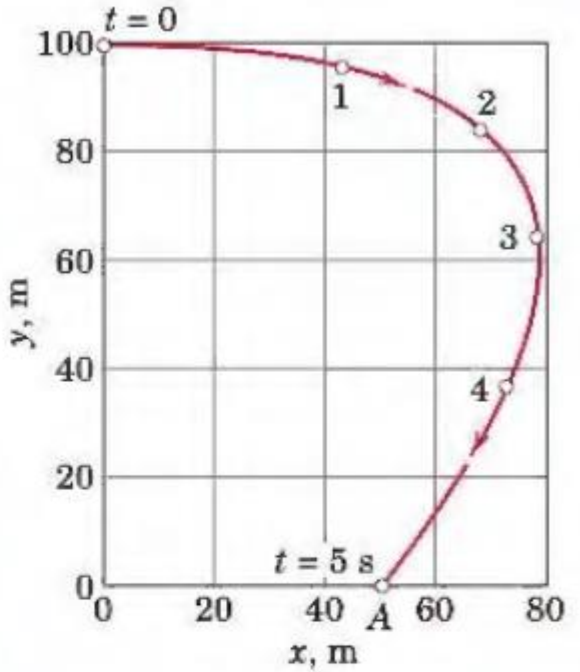
$$\left[\int dx = \int v_x\,dt\right] \qquad \int_0^x dx = \int_0^t (50 - 16t)\,dt \qquad x = 50t - 8t^2 \text{ m}$$

$$[a_x = \dot{v}_x] \qquad a_x = \frac{d}{dt}(50 - 16t) \qquad a_x = -16 \text{ m/s}^2$$

The y-components of velocity and acceleration are

$$[v_y = \dot{y}] \qquad v_y = \frac{d}{dt}(100 - 4t^2) \qquad v_y = -8t \text{ m/s}$$

$$[a_y = \dot{v}_y] \qquad a_y = \frac{d}{dt}(-8t) \qquad a_y = -8 \text{ m/s}^2$$



We now calculate corresponding values of x and y for various values of t and plot x against y to obtain the path as shown.

When $y = 0$, $0 = 100 - 4t^2$, so $t = 5$ s. For this value of the time, we have

$$v_x = 50 - 16(5) = -30 \text{ m/s}$$

$$v_y = -8(5) = -40 \text{ m/s}$$

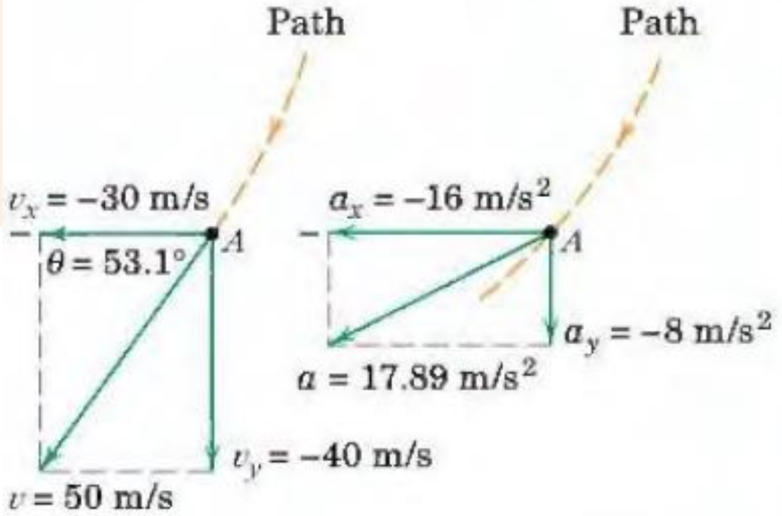
$$v = \sqrt{(-30)^2 + (-40)^2} = 50 \text{ m/s}$$

$$a = \sqrt{(-16)^2 + (-8)^2} = 17.89 \text{ m/s}^2$$

The velocity and acceleration components and their resultants are shown on the separate diagrams for point A, where $y = 0$. Thus, for this condition we may write

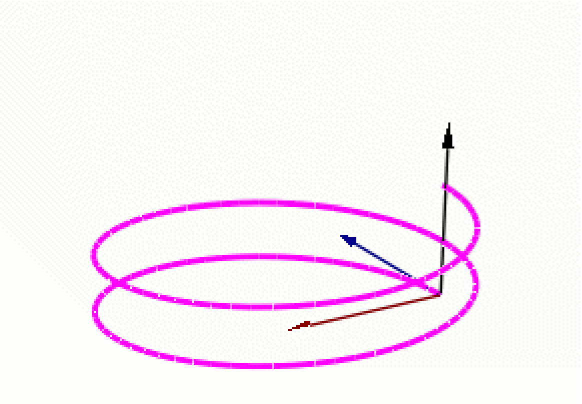
$$\mathbf{v} = -30\mathbf{i} - 40\mathbf{j} \text{ m/s}$$

$$\mathbf{a} = -16\mathbf{i} - 8\mathbf{j} \text{ m/s}^2$$



Frenet – Serrat frame of coordinates ($\vec{\tau}, \vec{n}, \vec{b}$) (or (T, N, B))

$\vec{\tau}$ - is the unit vector tangent to the curve, pointing in the direction of motion
$\vec{n}$ **-** is the normal unit vector
$\vec{b}$ **-** is the binormal unit vector, the cross product of $\vec{\tau}$ and $\vec{n}$.
Remark: The Frenet frame of coordinates has the origin in the moving particle and it is moving along with the particle.



Motion equations: $s = s(t)$**,** $t\epsilon\ [t_0, T]$

Velocity: $\vec{v}(t) = v\vec{\tau} = (v, 0,0) = \left(\frac{ds}{dt}, 0,0\right)$

Acceleration: $\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d}{dt}(v\vec{\tau}) =$

$$= \frac{dv}{dt}\vec{\tau} + v\frac{d\vec{\tau}}{ds}\frac{ds}{dt}$$

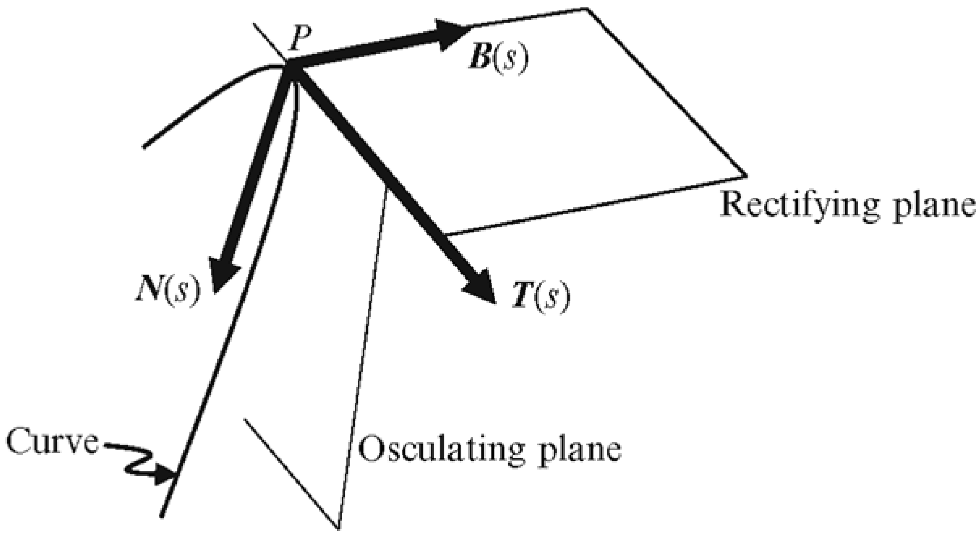
$$=\dot{v}\vec{\tau} + \frac{v^2}{R}\vec{n}$$

R is the radius of curvature
$\frac{d\vec{\tau}}{ds} = \frac{\vec{n}}{R}$ is the Frenet's formula

Thus, we have

$$\vec{a}(t) = \left(\dot{v}, \frac{v^2}{R}, 0\right) = a_\tau \vec{\tau} + a_n \vec{n}$$

where $a_\tau = \dot{v}$ is the tangential acceleration and $a_n = \frac{v^2}{R}$ is the normal acceleration.



In 2D the curvature $\rho = \frac{1}{R}$ is:

$$x = x(t), \quad y = y(t),$$

$$\rho = \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}.$$

or for $y = f(x),$

$$\rho = \frac{|y''|}{(1 + f'^2)^{3/2}}$$

Theoretical Mechanics

Example: (https://math.libretexts.org/Bookshelves/Calculus/)

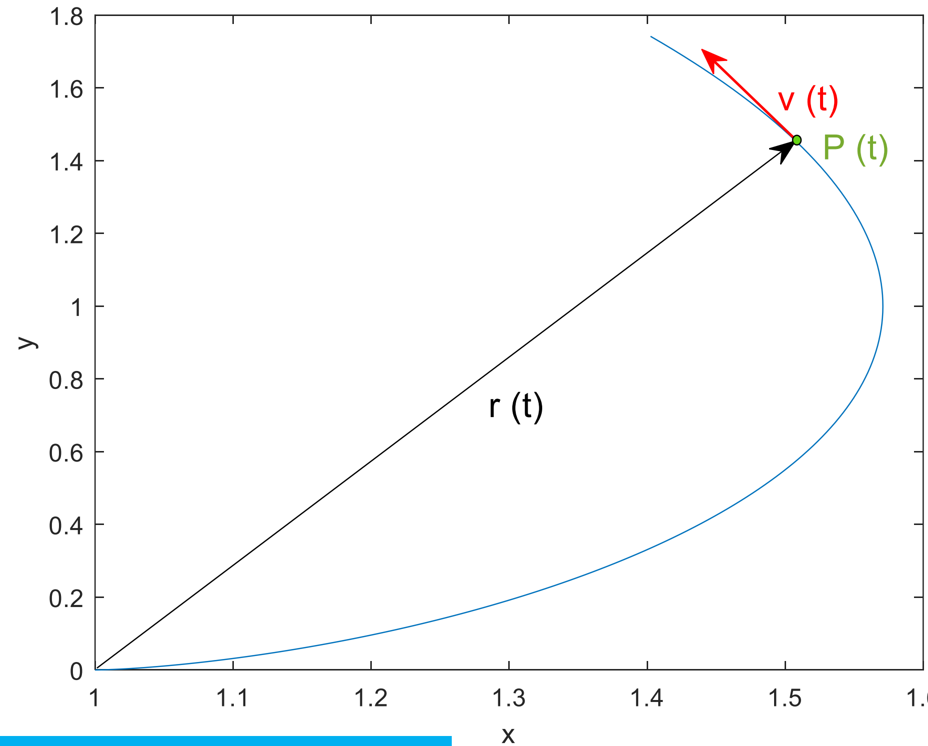
Without finding **T** and **N,** write the acceleration of the motion

$$\mathbf{r}(t) = (\cos t + t\sin t)\,\hat{\mathbf{i}} + (\sin t - t\cos t)\,\hat{\mathbf{j}}$$

for $t > 0.$

(Matlab)

```
t=0:0.01:2;
x=cos(t)+t.*sin(t);
y=sin(t)-t.*cos(t);
plot(x,y)
xlabel('x'); ylabel('y');
```



To solve this problem, we must first find the particle's velocity.

$$\mathbf{r}(t) = (\cos t + t\sin t)\,\hat{\mathbf{i}} + (\sin t - t\cos t)\,\hat{\mathbf{j}}$$

for $t > 0$.

$$\begin{aligned}\mathbf{v} &= \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \\ &= (-\sin t + \sin t + t\cos t)\,\hat{\mathbf{i}} + (\cos t - \cos t + t\sin t)\,\hat{\mathbf{j}} \\ &= (t\cos t)\,\hat{\mathbf{i}} + (t\sin t)\,\hat{\mathbf{j}}\end{aligned}$$

Next find the speed.

$$|v| = \sqrt{t^2\cos^2 t + t^2\sin^2 t} = \sqrt{t^2} = |t|$$

When $t > 0$, $|t|$ simply becomes t.

We know that $a_T = \dfrac{\mathrm{d}}{\mathrm{d}t}|v|$, which we can use to find that $\dfrac{\mathrm{d}}{\mathrm{d}t}(t) = 1$.

On the other hand

$$\mathbf{a} = (\cos t - t\sin t)\,\hat{\mathbf{i}} + (\sin t + t\cos t)\hat{\mathbf{j}}$$

$$|\mathbf{a}|^2 = t^2 + 1$$

$$a_N = \sqrt{(t^2+1) - (1)} = t$$

Thus,

$$|\mathbf{a}| = (1)\mathbf{T} + (t)\mathbf{N} = \mathbf{T} + t\mathbf{N}$$

A similar problem

Write a in the form $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$ without finding **T** or **N**.

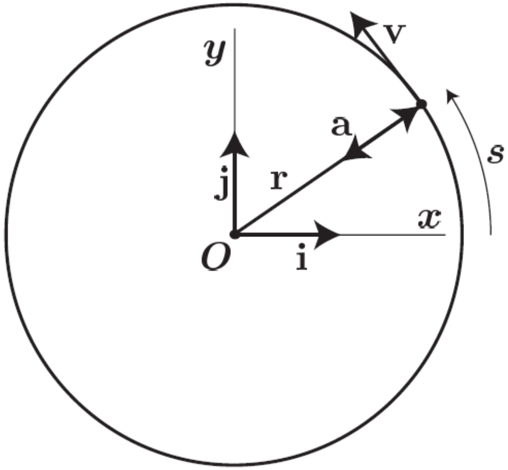
$$\mathbf{r}(t) = (t+1)\,\hat{\mathbf{i}} + 2t\hat{\mathbf{j}} + t^2\hat{\mathbf{k}}$$

(*https://ocw.mit.edu/courses/aeronautics-and-astronautics/*)

Example ***Circular Motion***

We consider motion of a particle along a circle of radius R at a constant speed v_0. The parametrization of a circle in terms of the arc length is

$$\boldsymbol{r}(s) = R\cos(\frac{s}{R})\boldsymbol{i} + R\sin(\frac{s}{R})\boldsymbol{j} \; .$$



Since we have a constant speed v_0, we have $s = v_0 t$. Thus,

$$\boldsymbol{r}(t) = R\cos(\frac{v_0 t}{R})\boldsymbol{i} + R\sin(\frac{v_0 t}{R})\boldsymbol{j} \; .$$

The velocity is

$$\boldsymbol{v}(t) = \frac{d\boldsymbol{r}(t)}{dt} = -v_0\sin(\frac{v_0 t}{R})\boldsymbol{i} + v_0\cos(\frac{v_0 t}{R})\boldsymbol{j} \; .$$

which, clearly, has a constant magnitude $|\boldsymbol{v}| = v_0$. The acceleration is,

$$\boldsymbol{a}(t) = \frac{d\boldsymbol{r}(t)}{dt} = -\frac{v_0^2}{R}\cos(\frac{v_0 t}{R})\boldsymbol{i} - \frac{v_0^2}{R}\sin(\frac{v_0 t}{R})\boldsymbol{j} \; .$$

Remark

Note that, the acceleration is perpendicular to the path (in this case it is parallel to $\boldsymbol{r}$), since the velocity vector changes direction, but not magnitude.

We can also verify that, from $\boldsymbol{r}(s)$, the unit tangent vector, $\boldsymbol{e}_t$, could be computed directly as

$$\boldsymbol{e}_t = \frac{d\boldsymbol{r}(s)}{ds} = -\sin(\frac{s}{R})\boldsymbol{i} + \cos(\frac{s}{R}) = -\sin(\frac{v_0 t}{R})\boldsymbol{i} + \cos(\frac{v_0 t}{R})\boldsymbol{j} \ .$$

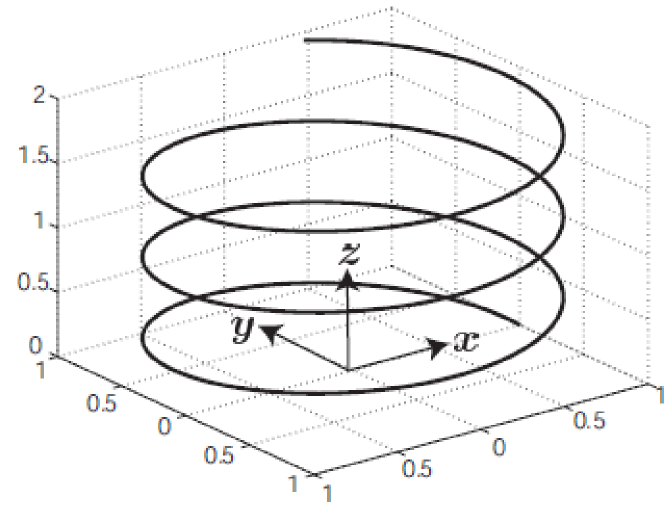
Example *Motion along a helix*

The equation $\boldsymbol{r}(t) = R\cos t\boldsymbol{i} + R\sin t\boldsymbol{j} + ht\boldsymbol{k}$, defines the motion of a particle moving on a helix of radius R, and pitch $2\pi h$, at a constant speed. The velocity vector is given by

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = -R\sin t\boldsymbol{i} + R\cos t\boldsymbol{j} + h\boldsymbol{k} \ ,$$

and the acceleration vector is given by,

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = -R\cos t\boldsymbol{i} + -R\sin t\boldsymbol{j} \ .$$



In order to determine the speed at which the particle moves we simply compute the modulus of the velocity vector,

$$v = |\boldsymbol{v}| = \sqrt{R^2\sin^2 t + R^2\cos^2 t + h^2} = \sqrt{R^2 + h^2} \ .$$

If we want to obtain the equation of the path in terms of the arc-length coordinate we simply write,

$$ds = |d\boldsymbol{r}| = v dt = \sqrt{R^2 + h^2}\, dt\ .$$

Integrating, we obtain $s = s_0 + \sqrt{R^2 + h^2}\ t$, where s_0 corresponds to the path coordinate of the particle at time zero. Substituting t in terms of s, we obtain the expression for the position vector in terms of the arc-length coordinate. In this case, $\boldsymbol{r}(s) = R\cos(s/\sqrt{R^2+h^2})\boldsymbol{i} + R\sin(s/\sqrt{R^2+h^2})\boldsymbol{j} + hs/\sqrt{R^2+h^2}\boldsymbol{k}$. The figure below shows the particle trajectory for $R = 1$ and $h = 0.1$.