SEMINAR 11

1) Let V be a K-vector space, $S \leq_K V$ and $x, y \in V$. We denote $\langle S, x\rangle = \langle S \cup \{x\}\rangle$. Show that if $x \in V \setminus S$ and $x \in \langle S, y\rangle$ then $y \in \langle S, x\rangle$.

2) Let V be a K-vector space and $\alpha, \beta, \gamma \in K$, $x, y, z \in V$ such that $\alpha\gamma \neq 0$ and $\alpha x + \beta y + \gamma z = 0$. Show that $\langle x, y\rangle = \langle y, z\rangle$.

3) Can you find an $\mathbb{R}$-linear map $f : \mathbb{R}^3 \to \mathbb{R}^2$ such that

$$f(1, 0, 3) = (1, 1) \text{ şi } f(-2, 0, -6) = (2, 1)\,?$$

4) Let us consider:
a) $f_1 : \mathbb{R}^2 \to \mathbb{R}^2$, $f_1(x, y) = (-x, y)$ (the symmetry with respect to Oy);
b) $f_2 : \mathbb{R}^2 \to \mathbb{R}^2$, $f_2(x, y) = (x, -y)$ (the symmetry with respect to Ox);
c) $f_3 : \mathbb{R}^2 \to \mathbb{R}^2$, $f_3(x, y) = (x\cos\varphi - y\sin\varphi, x\sin\varphi + y\cos\varphi)$, $\varphi \in \mathbb{R}$, (the plane rotation of angle φ);
d) $f_4 : \mathbb{R}^2 \to \mathbb{R}^3$, $f_4(x, y) = (x + y, 2x - y, 3x + 2y)$.
Show that f_1, f_2, f_3, f_4 are $\mathbb{R}$-linear maps. Are they isomorphisms? Are they endomorphisms? Are they automorphisms?

5) Let V, V_1, V_2 be K-vector spaces, $f : V \to V_1$, $g : V \to V_2$ and

$$h : V \to V_1 \times V_2,\ h(x) = (f(x), g(x)).$$

Show that h is a linear map if and only if f and g are linear maps. Generalize this statement.

6) a) Let $m \in \mathbb{N}^*$ and $f : \mathbb{R}^m \to \mathbb{R}$. Show that f is an $\mathbb{R}$-linear map if and only if there exist $a_1, \ldots, a_m \in \mathbb{R}$, uniquely determined, such that

$$f(x_1, \ldots, x_m) = a_1x_1 + \cdots + a_mx_m,\ \forall (x_1, \ldots, x_m) \in \mathbb{R}^m.$$

b) Determine the $\mathbb{R}$-linear maps $f : \mathbb{R}^m \to \mathbb{R}^n$ $(m, n \in \mathbb{N}^*)$.