Sequences of functions

Study the pointwise convergence (by specifying the convergence set and the pointwise limit function) and the uniform convergence for the following sequences of functions:

1. $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \dfrac{\cos nx}{n^\alpha}$ unde $\alpha > 0$;
2. $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = \dfrac{x(1+n^2)}{n^2}$;
3. $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \dfrac{x^2}{x^4+n^2}$;
4. $f_n : [0,\infty) \to \mathbb{R}$, $f_n(x) = \dfrac{1}{1+nx}$;
5. $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \dfrac{2n^2x}{e^{n^2x^2}}$;
6. $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \dfrac{nx}{1+n^2x^2}$;
7. $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \sqrt{x^2+\dfrac{1}{n^2}}$;
8. $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = n\left(\sqrt{x+\dfrac{1}{n}} - \sqrt{x}\right)$;
9. $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = \dfrac{nx}{e^{nx^2}}$;
10. $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = \dfrac{x(1+n^2)}{n^2}$;
11. $f_n : [-1,1] \to \mathbb{R}$, $f_n(x) = \dfrac{x}{1+n^2x^2}$;

Theory

Fie $\emptyset \neq D \subseteq \mathbb{R}$. We denote by

$$\mathcal{F}(D) = \{f| \quad f : D \to \mathbb{R}\}$$

the set of all the functions defined on the set D. A **sequence of functions** is each function $x : \mathbb{N}_k \to \mathcal{F}(D)$, which associates uniquely to each natureal number $n \geq k$, a function. Thus

$$x(n) := f_n, \quad \forall n \in \mathbb{N}_k.$$

Recall that $\mathbb{N}_k = \{n \in \mathbb{N} : n \geq k\}$, for a given $k \in \mathbb{N}$.
The usual notations for sequences of functions are

$$(f_n) = (f_n)_{n\in\mathbb{N}_k} = (f_n)_{n\geq k}.$$

We will further use the following framework:

$$(f_n) \subseteq \mathcal{F}(D) \text{ is a sequence of functions defined on } \emptyset \neq D \subseteq \mathbb{R}.$$

A point $x_0 \in D$ is called a (pointwise) **convergence point** if the sequence of the real numbers obtained by applying the sequence of functions to that given point x, is convergent. Namely,

$$\exists \lim_{n\to\infty} f_n(x_0) \in \mathbb{R}.$$

The set of all of the convergence points is called **the convergence set of the sequence of functions** and is denoted by

$$\mathcal{C} = \left\{ x \in D : \lim_{n\to\infty} f_n(x) \in \mathbb{R} \right\}.$$

Whenever the convergence set associated to a sequence of functions is nonempty, to it, we may associated, naturally, a function called the **pointwise limit function**,

$$f : \mathcal{C} \to \mathbb{R},$$

defined by

$$f(x) = \lim_{n\to\infty} f_n(x), \quad \forall x \in \mathcal{C}.$$

The notation for this **pointwise convergence** is:

$$f_n \xrightarrow{p} f \quad sau \quad f_n \to f.$$

By using the ϵ- characterization for the limit of the sequences of real numbers, at each point of the convergence set, we may deduce the following characterization theorem for the pointwise convergence:

Theorem

$$f_n \xrightarrow{p} f \iff \forall x \in \mathcal{C}, \forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}, \quad a.i. \quad \forall n \geq n_\varepsilon, \quad |f_n(x) - f(x)| < \varepsilon.$$

In the following we study another convergence notion for sequences of functions, namely, uniform convergence.

Definition: The sequence of functions (f_n) is said to converge uniformly on the set$D_0 \subseteq D$ if

$$\exists f : D \to \mathbb{R}, \quad a.i. \quad \forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}, \quad a.i. \quad \forall n \geq n_\varepsilon, \forall x \in D_0, \; to\, hold |f_n(x) - f(x)| < \varepsilon.$$

The standard notation for uniform convergence is

$$f_n \overset{u}{\rightrightarrows} f \quad sau \quad f_n \rightrightarrows f.$$

Observaţii:

- $\rightrightarrows \Longrightarrow \longrightarrow$ namely, all uniformly convergent sequences of functions are pointwise convergent as well (having as limit, the limit function defined above), but the converse statement does not hold

- the continuity is inherited through uniform convergence

- In practice, whenever we usually determine explicitly the limit function by computin for each $x \in D$
$$\lim_{n\to\infty} f_n(x).$$
Afterwards we analyze the uniform convergence, usually by applying the Weirstrass theorem:

Weirstrass' theore, Consider a sequence of functions $(f_n) \subseteq \mathcal{F}(D)$ and a sequence of real numbers $(a_n) \subseteq \mathbb{R}$, such that:

a) $\exists n_0 \in \mathbb{N}$ s.t.
$$|f_n(x) - f(x)| < a_n, \quad \forall n \geq n_\varepsilon, \forall x \in \mathcal{C}$$

b) $\lim_{n\to\infty} a_n = 0;$

Then
$$f_n \rightrightarrows f,$$

The continuity inheritance theorem

If $f_n \rightrightarrows f$, and all the functions f_n, $n \in \mathbb{N}$ are continuous, then so is the limit function f as well.