Seminar 2

1. Consider the English alphabet with 21 consonants and 5 vowels (lowercase letters). In how many ways can we choose 6 letters such that we get 4 distinct consonants and 2 distinct vowels, if: a) the letters are not ordered; b) the letters are ordered? Examples: a) {s,e,a,r,c,h}; b) (a,r,c,h,e,s), (c,a,s,h,e,r), (c,h,a,s,e,r).
A: Applying the basic principle of counting, we get: a) $C_{21}^4 \cdot C_5^2$; b) $A_{21}^4 \cdot A_5^2 \cdot C_6^4$.

2. Two numbers are obtained by rolling two dice. Compute the probabilities of the following events:
a) A: "the numbers are equal".
b) B: "the sum of the numbers is even".
c) C: "the sum of the numbers is at most equal to 10."
A: a) $P(A) = \frac{6}{36} = \frac{1}{6}$; b) $P(B) = \frac{3^2+3^2}{36} = \frac{1}{2}$; c) $P(C) = 1 - \frac{3}{36} = \frac{11}{12}$.

3. For this problem, we assume that the birthday of every person falls equally likely on any month of the year (i.e., the probability that a person was born in a certain month is $\frac{1}{12}$). Compute the probability that
a) in a group of 5 persons there are at least 2 persons that celebrate their birthdays in the same month?
b) in a group of 5 persons all the birthdays fall in at most two months?
A: a) $1 - \frac{A_{12}^5}{12^5} \approx 62\%$. b) $\frac{C_{12}^1 + C_{12}^2(2^5-2)}{12^5}$, where C_{12}^1 is number of cases when all the persons are born in the same month, C_{12}^2 is the number of ways to choose the two months for the cases when all the birthdays fall in exactly 2 months, while $2^5 - 2$ is the number of surjective functions from the persons to chosen 2 months.

4. For this problem, we assume that the birthday of every person falls equally likely on any day of the year 2003 (i.e., the probability that a person was born in a certain day is $\frac{1}{365}$). Which is the minimum number n such that the probability of the event "at least two persons share a birthday in an arbitrary group of n persons born in 2003" is at least 50%?
A: The probability that at least two out of n share a birthday is $1 - \frac{A_{365}^n}{365^n} = 1 - \frac{365}{365} \cdot \frac{365-1}{365} \cdot \ldots \cdot \frac{365-(n-1)}{365}$, if $2 \leq n \leq 365$. The minimum number n such that $1 - \frac{364}{365} \cdot \ldots \cdot \frac{365-(n-1)}{365} \geq 0.5$ can be found by running the following script in Octave:

```
n=2; Q=364/365;
while Q>=0.5
  n++;
  Q=Q*(365-(n-1))/365;
endwhile
disp(n)
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The answer is 23.

5. a) How many solutions $(x_1, \ldots, x_n) \in \mathbb{N}^* \times \cdots \times \mathbb{N}^*$ does the equation $x_1 + \ldots + x_n = k$ ($k, n \in \mathbb{N}^*$, $k \geq n$) have?
b) How many solutions $(x_1, \ldots, x_n) \in \mathbb{N} \times \cdots \times \mathbb{N}$ does the equation $x_1 + \ldots + x_n = k$ ($k, n \in \mathbb{N}^*$) have?
a) A: C_{k-1}^{n-1}.
Consider k sticks with $k-1$ spaces between them: | | | $\cdots$ | | (the first stick has no space before it and the last stick has no space after it). If we put $n-1$ symbols + on $n-1$ spaces between the k sticks, then the k sticks are divided into n groups (each group has at least one stick). Let x_i = number of sticks in the ith group , $i = \overline{1,n}$. We have $x_i \in \mathbb{N}^*$, $i = \overline{1,n}$, and $x_1 + \ldots + x_n = k$. There are C_{k-1}^{n-1} ways to choose the spaces for the $n-1$ symbols +, so this is the number of desired solutions.
Example: $k = 6, n = 3$, | | | | | |, + +. In | | + | | | + | we have 3 groups, so $x_1 = 2, x_2 = 3, x_3 = 1$ and $x_1 + x_2 + x_3 = 6$. So, there are $C_{6-1}^{3-1} = C_5^2 = 10$ solutions $(x_1, x_2, x_3) \in \mathbb{N}^* \times \mathbb{N}^* \times \mathbb{N}^*$ of the equation $x_1 + x_2 + x_3 = 6$.
b) A: C_{n+k-1}^{n-1}, which is equal to C_{n+k-1}^k.
Every solution $(x_1, \ldots, x_n) \in \mathbb{N} \times \cdots \times \mathbb{N}$ of the equation $x_1 + \ldots + x_n = k$ corresponds uniquely to a solution $(y_1, \ldots, y_n) \in \mathbb{N}^* \times \cdots \times \mathbb{N}^*$ of the equation $y_1 + \ldots + y_n = n + k$ and vice versa, by choosing $y_i = x_i + 1$, $i = \overline{1,n}$. According to a), there are $C_{n+k-1}^{n-1} = C_{n+k-1}^k$ such solutions.

6. How many *multisubsets* with $k \in \mathbb{N}$ elements does a set S with $n \in \mathbb{N}^*$ distinct elements have?
A *multisubset*, called also a *combination with repetitions*, is a collection of elements *not necessarily distinct* and *not ordered.* Example: $\{1, 1, 3, 4, 4\}$ is a *multisubset* of the set $\{1, 2, 3, 4\}$.
A: Let $S = \{e_1, \ldots, e_n\}$. Every multisubset with k elements corresponds uniquely to a solution $(x_1, \ldots, x_n) \in \mathbb{N} \times \cdots \times \mathbb{N}$ of the equation $x_1 + \ldots + x_n = k$ and vice versa. Applying the result of problem 5 b), there are $C^{n-1}_{n+k-1} = C^k_{n+k-1}$ multisubsets with k elements of the set S. In the above example $k = 5, n = 4$, $x_1 = 2, x_2 = 0, x_3 = 1, x_4 = 2$. There are $C^{4-1}_{4+5-1} = C^5_{4+5-1} = \frac{8!}{5!3!} = 56$ multisubsets with 5 elements of the set $\{1, 2, 3, 4\}$.
7. A person sends 10 memes by choosing for each meme a recipient from a list of 20 friends. Compute the probability that the first friend in the list receives exactly 5 memes?
A: $\frac{C^5_{10} 19^5}{20^{10}}$, where 20^{10} is the number of possible cases, which is the number of functions from the memes to friends, C^5_{10} is the number of ways to choose the 5 emails for the first friend and 19^5 is the number of functions from the remaining memes to the remaining friends.
8. 5 balls numbered from 1 to 5 are randomly placed on a line. Compute the probability that:
a) the first and the last balls have even numbers;
b) the first two balls have odd numbers;
c) the balls with even numbers are next to each other;
d) at least two balls that are placed next to each other have the same parity.
A: a) $\frac{2! \cdot 3!}{5!} = \frac{1}{10}$. b) $\frac{A^2_3 \cdot 3!}{5!} = \frac{3}{10}$. c) $\frac{2 \cdot 4 \cdot 3!}{5!} = \frac{2}{5}$. d) $1 - \frac{2! \cdot 3!}{5!} = \frac{9}{10}$.
9. In how many ways can we split the following marbles among 3 persons:
a) 1 red marble, 1 blue marble, 1 green marble, 1 yellow marble and 1 orange marble?
b) 5 red marbles;
c) 5 red marbles and 3 blue marbles;
d) 5 red marbles, 3 blue marbles and 4 green marbles.
A: a) 3^5 (number of functions form the set of 5 marbles to the set of 3 persons); we apply the result of Problem 5 b) and the basic principle of counting to get: b) $C^{3-1}_{5+3-1} = C^2_7$; c) $C^{3-1}_{3+5-1} \cdot C^{3-1}_{3+3-1} = C^2_7 \cdot C^2_5$; d) $C^{3-1}_{3+5-1} \cdot C^{3-1}_{3+3-1} \cdot C^{3-1}_{3+4-1} = C^2_7 \cdot C^2_5 \cdot C^2_6$.