SEMINAR 10

We will begin by stating and proving **the characterization theorem for subspaces**,

1) Which of the following subsets is a subspace in the space mentioned nearby:
   a) $A = \{(x, y) \in \mathbb{R}^2 \mid ax + by = 0\},\ (a, b \in \mathbb{R}$ are given) in ${}_{\mathbb{R}}\mathbb{R}^2$;
   b) $D = [-1, 1]$ in ${}_{\mathbb{R}}\mathbb{R}$;
   b') $D' = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ in ${}_{\mathbb{R}}\mathbb{R}^2$;
   b") $D'' = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 \leq 1\}$ in ${}_{\mathbb{R}}\mathbb{R}^n$;
   c) $P_n(\mathbb{R}) = \{f \in \mathbb{R}[X] \mid \text{grad} f \leq n\}$ in ${}_{\mathbb{R}}\mathbb{R}[X]$ ($n \in \mathbb{N}$ is given);
   d) $B = \{f \in \mathbb{R}[X] \mid \text{grad} f = n\}$ in ${}_{\mathbb{R}}\mathbb{R}[X]$ ($n \in \mathbb{N}$ is given)?

2) Let V be a K-vector space, $A \leq_K V$ and $C_V A = V \setminus A$.
   i) Is $C_V A$ a subspace in ${}_K V$?
   ii) What about $C_V A \cup \{0\}$?

3) Let V, V' be K-vector spaces, $f : V \to V'$ a linear map, $A \leq_K V$ and $A' \leq_K V'$. Show that:
   a) $f(A) = \{f(a) \in V' \mid a \in A\} \leq_K V'$;
   b) $\overset{-1}{f}(A') = \{x \in V \mid f(x) \in A'\} \leq_K V$.