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gr. 813

Subject C

1) a) a_1) Define $LB(A)$, $A \subseteq \mathbb{R}$

$LB(A) = \{ x \mid$ ~~$x \in \mathbb{R}$~~, $x \leq a$, $x \in \mathbb{R}$, $a \in A \}$

a_2) Define $\sup(A)$

$\sup(A)$ = the smallest upper bound

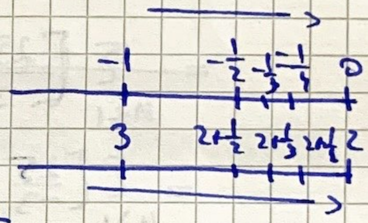
b) b_1) $\sup(A) \in A$ $\quad A = \{1, 2, 3\}$

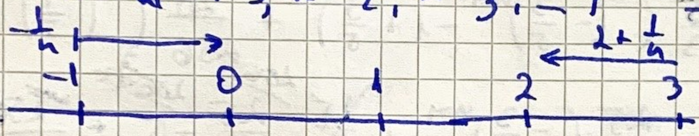
b_2) $\sup B \notin \mathbb{R}$ $\quad B = \mathbb{N}$

c) $M = \bigcap_{n \in \mathbb{N}} \left(-\frac{1}{n}, 2 + \frac{1}{n}\right)$

$-\frac{1}{n} \in \{-1, -\frac{1}{2}, -\frac{1}{3}, \dots\}$

$2 + \frac{1}{n} \in \{3, 2 + \frac{1}{2}, 2 + \frac{1}{3}, \dots\}$





$\Rightarrow LB(M) = (-\infty, -1]$

$UB(M) = [3, \infty)$

$\inf(M) = -1$

$\sup(M) = 3$

$\min M = -1$

$\max M = 3$

2. a) Example of non monotonic seq. of irrational numbers converging to 0: (x_n), $x_n = \frac{(-1)^n}{n}$, $n \in \mathbb{N}$

b) $\lim_{n \to \infty} \frac{1 + \sqrt{2} + \dots + \sqrt{n}}{n\sqrt{n}} = ?$ let $\lim_{n \to \infty} \frac{1 + \sqrt{2} + \dots + \sqrt{n}}{n\sqrt{n}} = \lim_{n \to \infty} x_n$

~~let (a_n) and~~ let $(x_n) = \frac{(a_n)}{(b_n)}$,

$a_n = 1 + \sqrt{2} + \dots + \sqrt{n}$, $a_1 = 1$ $\quad \lim_{n \to \infty} a_n = \infty$ $\quad (a_n) > 0$

$b_n = n\sqrt{n}$, $b_1 = 1$ $\quad \lim_{n \to \infty} b_n = \infty$ $\quad (b_n) > 0$

$\Rightarrow \lim_{n \to \infty} x_n = \lim_{n \to \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} =$ ~~$\lim \frac{}{(n+1)\sqrt{n+1} + n\sqrt{n}}$~~

$= \lim_{n \to \infty} \frac{1 + \sqrt{2} + \dots + \sqrt{n+1} - 1 - \sqrt{2} - \dots - \sqrt{n}}{(n+1)\sqrt{n+1} - n\sqrt{n}} = \lim_{n \to \infty} \frac{\sqrt{n+1}}{(n+1)\sqrt{n+1} - n\sqrt{n}} =$

1)

$$= \lim_{n\to\infty} \frac{(n+1)(n+1) + n\sqrt{n}\cdot\sqrt{n+1}}{(n+1)^2(n+1) - n^2\cdot n} = \lim_{n\to\infty} \frac{(n+1)^2 + n\sqrt{n^2+n}}{(n+1)^3 - n^3} =$$

$$= \lim_{n\to\infty} \frac{n^2+2n+1 + n^2\sqrt{1+\frac{1}{n}}}{n^3+3n^2+3n+1-n^3} = \lim_{n\to\infty} \frac{n^2\left(1+\frac{2}{n}+\frac{1}{n^2}+\sqrt{1+\frac{1}{n}}\right)}{n^2\left(3+\frac{3}{n}+\frac{1}{n^2}\right)} =$$

$$= \frac{1}{3}$$

3) a) $\cancel{\sum_{n\geq 3} \frac{(-3)^{n+2}+4^{n+3}}{5^{n+1}}} = \sum_{n\geq 3} \frac{(-3)^{n-2}+4^{n+3}}{5^{n+1}} =$

$$= \sum_{n\geq 3}\left(\frac{(-3)^{n-2}}{5^{n+1}} + \frac{4^{n+3}}{5^{n+1}}\right) = \sum_{n\geq 3}\left(\frac{5^2}{(-3)}\cdot\frac{(-3)^{n-1}}{5^{n-1}} + \right.$$

$$\left. + \frac{5^2}{4^4}\cdot\frac{4^{n-1}}{5^{n-1}}\right) =$$

$$\cancel{= \sum_{n\geq 1}\left[\frac{25}{-3}\cdot\left(-\frac{3}{5}\right)^{n-1} + \frac{25}{256}\cdot\left(\frac{4}{5}\right)^{n-1}\right] =}$$

$$= \sum_{n\geq 1}\left[\frac{25}{-3}\left(\left(-\frac{3}{5}\right)^{n-1} - 1 + \frac{3}{5}\right) + \frac{25}{256}\left(\left(\frac{4}{5}\right)^{n-1} - 1 - \frac{4}{5}\right)\right] =$$

$$= \sum_{n\geq 1}\left(\frac{25}{-3}\left(-\frac{3}{5}\right)^{n-1} + \cancel{\frac{25}{3}}\ \frac{25}{3} - 5 + \frac{25}{256}\left(\frac{4}{5}\right)^{n-1} - \frac{25}{256} - \right.$$

$$\left. - \frac{5}{64}\right) =$$

$$= \sum_{n\geq 1}\left[\frac{25}{-3}\left(-\frac{3}{5}\right)^{n-1} + \frac{25}{256}\left(\frac{4}{5}\right)^{n-1} + \frac{6300-3840-75-60}{768}\right]$$

$$= \sum_{n\geq 1}\left[\frac{25}{-3}\left(-\frac{3}{5}\right)^{n-1} + \frac{25}{256}\left(\frac{4}{5}\right)^{n-1} + \frac{6300-3975}{768}\right] =$$

$$= \sum_{n\geq 1}\left[\frac{25}{-3}\left(-\frac{3}{5}\right)^{n-1} + \frac{25}{256}\left(\frac{4}{5}\right)^{n-1} + \frac{2325}{768}\right] = \sum_{n\geq 1} x_n$$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\left[\frac{25}{-3}\left(-\frac{3}{5}\right)^{n-1} + \frac{25}{256}\left(\frac{4}{5}\right)^{n-1} + \frac{2325}{768}\right] =$$

$$= \frac{2325}{768}$$

$$\sum_{n\geq 1} x_n = \frac{2325}{768}$$

~~b)~~ $\cancel{\sum_{n\geq 3}}$

2)

3) a) $\sum_{n\ge3} \frac{1}{n(n-1)} = \sum_{n\ge3} \frac{1}{2\cdot3} + \frac{1}{3\cdot4} + \dots + \frac{1}{n(n-1)} =$

$= \cancel{\sum_{n\ge3}} \frac{1}{3} - \frac{1}{2} + \frac{1}{4} - \frac{1}{3} + \dots + \frac{1}{n} - \frac{1}{n-1} = \cancel{\frac{1}{n} - \frac{1}{2} = \frac{2-n}{2n}}$

$= \sum_{n\ge3} \frac{1}{n} - \frac{1}{2} = \sum_{n\ge3} \frac{2-n}{2n} = \sum_{n\ge1} \left(\frac{2-n}{2n} - \frac{2-1}{2\cdot1} - \frac{2-2}{2\cdot2}\right) =$

$= \sum_{n\ge1} \left(\frac{2-n}{2n} - \frac{1}{2} - 0\right) = \sum_{n\ge1} \left(\frac{2-n}{2n} - \frac{1}{2}\right) = \sum_{n\ge1} x_n$

$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \left(\frac{2-n}{2n} - \frac{1}{2}\right) = \left(-\frac{1}{2}\right) + \lim_{n\to\infty} \frac{2-n}{2n} =$

$= \left(-\frac{1}{2}\right) + \cancel{\lim_{n\to\infty}} \lim_{n\to\infty} \frac{n\left(\frac{2}{n}-1\right)}{2n} = -\frac{1}{2} + \frac{1}{2} = 0$

$\Rightarrow \sum_{n\ge1} x_n = 0$

b) Study the convergence and absolute convergence:

$\sum_{n\ge2} (-1)^{n+1} \cdot \frac{1}{\ln n^4} = \cancel{\sum}$

$= \sum_{n\ge2} \frac{(-1)^{n+1}}{4\ln n} = \cancel{\$} \sum_{n\ge1} \frac{(-1)^{n+1}}{4\ln n} + \frac{1}{4\ln 1} = \sum_{n\ge1} \frac{(-1)^{n+1}}{4\ln n} = \sum_{n\ge1} x_n$

$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{(-1)^{n+2}}{4\ln(n+1)} \cdot \frac{4\ln n}{(-1)^{n+1}} = \lim_{n\to\infty} \frac{(-1)\cdot 4\ln n}{4\ln(n+1)} =$

$= \lim_{n\to\infty} \frac{-4\ln n}{4\ln(n+1)} \cancel{(-1)\lim} = (-1) \lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} = ?$

$\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \sqrt[n]{\frac{(-1)^{n+1}}{\ln n^4}} = \lim_{n\to\infty} \frac{-1}{\sqrt[n]{\ln n^4}} = \lim_{n\to\infty} \frac{-1}{(4\ln n)^{\frac{1}{n}}}$

$\lim_{n\to\infty} \frac{1}{\ln n^4} = 0$

$\lim_{n\to\infty} (-1)^{n+1} = \begin{cases} -1, & n \text{ is even} \\ 1, & n \text{ is odd} \end{cases}$

$\Rightarrow \sum_{n\ge2} (-1)^{n+1} \cdot \frac{1}{\ln n^4}$ is convergent.