

## Limits of Functions

Recall the following:

$$\lim_{x \rightarrow \infty} q^x = \begin{cases} +\infty & : q > 1 \\ 1 & : q = 1 \\ 0 & : |q| < 1 \\ \varnothing & : q \leq -1 \end{cases}$$

$$\lim_{x \rightarrow x_0} q^x = q^{x_0}, \forall q \in (0, \infty) \quad \text{and} \quad x_0 \in \mathbb{R}$$

$$\lim_{x \rightarrow x_0} \log_a x = \log_a x_0, \forall a \in (0, \infty) \setminus \{1\}, x_0 > 0.$$

$$\lim_{x \rightarrow \infty} \log_a x = \begin{cases} +\infty & : a > 1 \\ -\infty & : 0 < a < 1 \end{cases}$$

$$\lim_{x \rightarrow 0} \frac{q^x - 1}{x} = \ln q, \forall q > 0 \quad \text{and} \quad \lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1.$$

**Exercise 1:** Compute the limits of the following functions at the specified points:

$$a) \lim_{x \rightarrow \infty} x \cos^2 \frac{2x^2 + 2}{3x^2} \quad b) \lim_{x \rightarrow 1} \frac{4x^2}{3x^3 + 5} \quad c) \lim_{x \rightarrow -\infty} \frac{-2x^2 + 5}{5x^3} \quad d) \lim_{x \rightarrow \infty} \frac{(4x + 2)(3x + 1)}{7x^2 + 10x + 5}$$

$$e) \lim_{x \rightarrow 1} \frac{x^2 - 1}{x^3 - 1} \quad f) \lim_{x \rightarrow 2} \left( \frac{1}{2 - x} - \frac{2x}{4 - x^2} \right)$$

$$g) \lim_{x \rightarrow 1} \frac{1 + x + x^2 + \dots + x^n - (n + 1)}{x - 1}, n \in \mathbb{N} \quad h) \lim_{x \rightarrow 1} \frac{x + x^2 + \dots + x^n - n}{x + x^2 + \dots + x^m - m}, \forall m, n \in \mathbb{N}.$$

$$i) \lim_{x \rightarrow 27} \frac{x - 27}{\sqrt[3]{x} - 3} \quad j) \lim_{x \rightarrow 1} \frac{\sqrt[3]{x} - 1}{\sqrt[4]{x} - 1}$$

$$k) \lim_{x \rightarrow \infty} \left( \sqrt[3]{ax^3 + x^2 + bx + c} - (bx + c) \right) \forall a, b, c > 0.$$

**Exercise 2:** Compute the limits of the following functions at the specified points:

$$a) \lim_{x \rightarrow \infty} \left( \frac{1}{3x} \right)^{\frac{7x+1}{4x+4}} \quad b) \lim_{x \rightarrow 0} \left( \frac{5 \sin x - 2 \tan x}{x} \right)^{\frac{4 \sin x + 9x}{x}}$$

$$c) \lim_{x \rightarrow 0} (1 + \cos x)^{\frac{1}{x^2}} \quad d) \lim_{x \rightarrow 0} (e^x - x + 1)^{\frac{1}{1 - \cos x}}$$

$$e) \lim_{x \rightarrow 0} (1 + \sin x)^{\frac{1}{x}} \quad f) \lim_{x \rightarrow \infty} \left( \frac{x+7}{x} \right)^x$$

**Exercise 3:**

$$a) \lim_{n \rightarrow \infty} \left[ \lim_{x \rightarrow 0} (1 + \sin^2 x + \sin^2 2x + \dots + \sin^2 nx)^{\frac{1}{n^3 x^2}} \right]$$

$$a) \lim_{n \rightarrow \infty} \left[ \lim_{x \rightarrow 0} (1 + \ln(1+x) + \ln(1+2x) + \dots + \ln(1+nx))^{\frac{1}{n^2 x}} \right]$$

**Exercise 4:** Compute the following limits:

$$a) \lim_{x \rightarrow 0} \frac{e^{2x} - 1}{3x}; \quad b) \lim_{x \rightarrow 0} \frac{e^x - \cos x}{3x}.$$