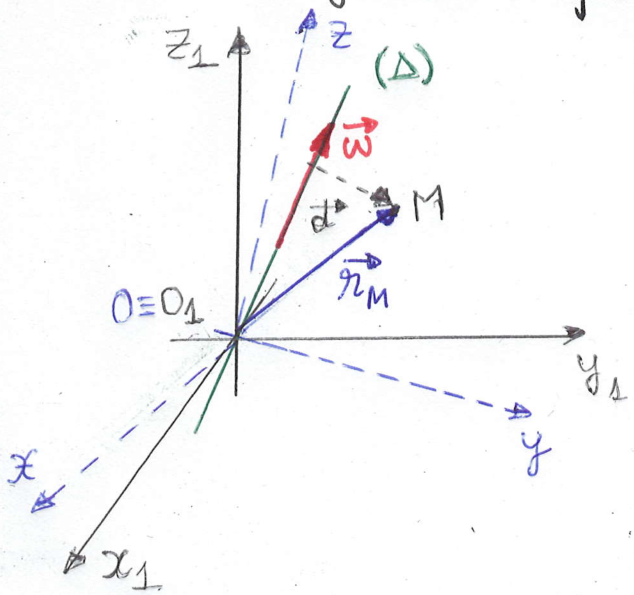
Kinematics of the rigid body

Rigid body with a fixed point:



$$\vec{v}_M = \vec{\omega} \times \vec{r}_M$$

$$\vec{a}_M = \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_M)$$
$$= \dot{\vec{\omega}} \times \vec{r} - \omega^2 \cdot \vec{d}$$

Δ - instantaneous axis of rotation

$$A \in (\Delta) \Rightarrow \boxed{\vec{v}_A = 0}$$

Thus: $\Delta : \dfrac{x}{p(t)} = \dfrac{y}{q(t)} = \dfrac{z}{r(t)}$

where $\vec{\omega} = \vec{\omega}(p, q, t)$.

① Consider a rigid body with a fixed point O. It is known that the velocity of a point $M_1(0,0,2)$ of the rigid is $\vec{v}_{M_1}(1,2,0)$ and the direction's cosines of the velocity of the point $M_2(0,1,2)$ are $\left(-\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$. Find the angular velocity $\vec{\omega}$ and the equation of the instantaneous axis of rotation.

Solution:

Rotation about a fixed point $\Rightarrow \vec{v}_M = \vec{\omega} \times \vec{r}_M$, $\vec{\omega}(p, q, r)$.

$$\underset{(1,2,0)}{\overset{\Vert}{\vec{v}_{M_1}}} = \vec{\omega} \times \vec{r}_{M_1} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ p & q & r \\ 0 & 0 & 2 \end{vmatrix} = 2q \cdot \vec{i} - 2p \cdot \vec{j} \Rightarrow \boxed{\begin{matrix} q = \frac{1}{2} \\ p = -1 \end{matrix}} \quad (1)$$

$$\vec{v}_{M_2} = \vec{\omega} \times \vec{r}_{M_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ p & q & r \\ 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & \frac{1}{2} & r \\ 0 & 1 & 2 \end{vmatrix} = (1-r) \cdot \vec{i} + 2\vec{j} + \vec{k}$$

Direction cosines:

$$\frac{1-r}{\sqrt{(1-r)^2+5}} = -\frac{2}{3}\ ;\quad \frac{2}{\sqrt{(1-r)^2+5}} = \frac{2}{3}\ ;\quad \frac{-1}{\sqrt{(1-r)^2+5}} = -\frac{1}{3} \qquad (2)$$

$$\Downarrow$$

$$9(1-r)^2 = 4(1-r^2)+20$$

$$(1-r)^2 = 4 \Rightarrow r_1 = 3$$

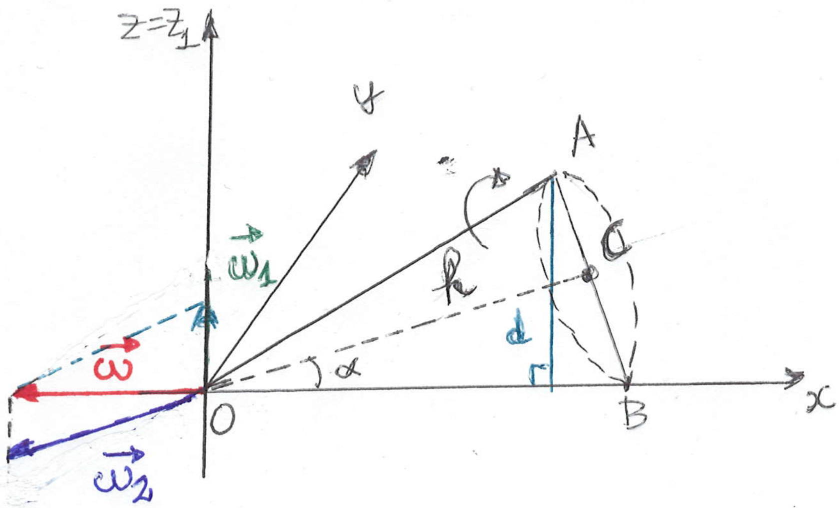
$$r_2 = -1 \text{ (! not OK, does not verify (2 a))}$$

Thus, we have: $\vec{\omega}(-1, \frac{1}{2}, 3)$.

Instantaneous axis of rotation:

$$(\Delta):\ \frac{x}{p} = \frac{y}{q} = \frac{z}{r} \Rightarrow (\Delta):\ \frac{x}{-1} = \frac{y}{\frac{1}{2}} = \frac{z}{3}.$$

② A circular cone of height h and with the vertex angle of 2α is rolling without slipping on a plane wich rotates around a fixed axis Oz_1 perpendicular on the plane with the angular velocity ω_1. Find the rotation and centripetal acceleration of a point A from the cone's base.



The rolling of the cone on the plane consists in a rotation about OC with the angular velocity $\vec{\omega}_2$ and a rotation about Oz_1 with the angular velocity $\vec{\omega}_1$.

On the other hand, the points from OB <u>have not rotation</u>. It means that the two rotations $\vec{\omega}_1$ and $\vec{\omega}_2$ combine in a unique rotation $\vec{\omega}$, oriented on the <u>instantaneous axis of rotation</u> OB.

In order to find $\vec{\omega}$ we calculate the velocity of the point C in two ways:

1. C rotates about $Oz_1 \Rightarrow v_C = \omega_1 \cdot d(C, Oz_1) = \omega_1 \cdot h \cdot \cos\alpha$
2. C rotates about $OB \Rightarrow v_C = \omega \cdot d(C, OB) = \omega \cdot h \cdot \sin\alpha$

$$\Rightarrow \omega = \omega_1 \cdot \mathrm{ctg}\,\alpha \Rightarrow \boxed{\vec{\omega} = -\omega_1 \cdot \mathrm{ctg}\,\alpha \cdot \vec{i}} \quad (1)$$

We have $\vec{\omega} = \vec{\omega}_1 + \vec{\omega}_2 \Rightarrow \vec{v}_C = \underline{\underline{\vec{\omega} \times \vec{OC}}} = (\vec{\omega}_1 + \vec{\omega}_2) \times \vec{OC} =$

$$= \underbrace{\underline{\underline{\vec{\omega}_1 \times \vec{OC}}}}_{\vec{OC} \parallel \omega_2}$$

Rotation acceleration of point A:

$$\vec{a}_{rot} = \dot{\vec{\omega}} \times \vec{r}_A = \dot{\vec{\omega}} \times \vec{OA}$$

$$\dot{\vec{\omega}} = \frac{d\vec{\omega}}{dt} \underset{(1)}{=} -\omega_1 \cdot \mathrm{ctg}\,\alpha \cdot \frac{d\vec{i}}{dt} = -\omega_1 \cdot \mathrm{ctg}\,\alpha \cdot \omega_1 \cdot \underbrace{\vec{k} \times \vec{i}}_{\vec{j}} = -\omega_1^2 \mathrm{ctg}\,\alpha \cdot \vec{j}$$

Poisson formulas: $\frac{d\vec{i}}{dt} = \vec{\omega}_1 \times \vec{i}$

angular velocity in the rolling rigid $(Oxyz)$.

$$\Delta OAC \Rightarrow OA = \frac{h}{\cos\alpha} \Rightarrow$$

$$\Rightarrow \vec{OA}\left(\frac{h}{\cos\alpha} \cdot \cos 2\alpha,\ 0,\ \frac{h \sin 2\alpha}{\cos\alpha}\right)$$

$$\vec{a}_{rot} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -\omega_1 \mathrm{ctg}\,\alpha & 0 \\ \frac{h}{\cos\alpha}\cos 2\alpha & 0 & \frac{h}{\cos\alpha}\sin 2\alpha \end{vmatrix} = -\frac{h\omega_1^2 \mathrm{ctg}\,\alpha}{\cos\alpha} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ \cos 2\alpha & 0 & \sin 2\alpha \end{vmatrix} =$$

$$= -\frac{h\omega_1^2}{\sin\alpha}\left(\sin 2\alpha\, \vec{i} - \cos 2\alpha\, \vec{k}\right) \Rightarrow |\vec{a}_{rot}| = \frac{h\omega_1^2}{\sin\alpha}$$

Centripetal acceleration of point A:
(axipetal)

$$\left.\begin{aligned} \vec{a}_{ax} &= -\omega^2 \cdot \vec{d} = -\omega^2 \cdot d \cdot \vec{r} \\ d &= OA \cdot \sin 2\alpha = \frac{h \sin 2\alpha}{\cos\alpha} = 2h\sin\alpha \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \vec{a}_{ax} \underset{(1)}{=} -\omega_1^2 \, ctg^2\alpha \cdot 2h\sin\alpha \cdot \vec{r} = -\frac{2h\omega_1^2 \cos^2\alpha}{\sin\alpha} \cdot \vec{r}$$

$$|\vec{a}_{ax}| = \frac{2h\omega_1^2\cos^2\alpha}{\sin\alpha} = 2h\omega_1^2\cos\alpha \, ctg\,\alpha .$$