Seminar 6

1. A pair of dice - one white die and one red die - is rolled two times. Compute the probability that the two pairs of numbers, obtained after the two rolls, are equal. (Example of favorable case: the white die shows number 2 and the red die shows number 4, both after the first roll and the second roll; example of unfavorable case: first roll "2 on white die, 4 on red die", second roll "4 on white die, 2 on red die".)

2. A computer center has three printers A, B, and C, which print at different speeds. Programs are routed to the first available printer. The probability that a program is routed to printers A, B, and C are 0.5, 0.3, and 0.2, respectively. Occasionally a printer will jam and destroy a printout. The probability that printers A, B, and C will jam are 0.02, 0.06 and 0.1, respectively. Your program is destroyed when a printer jams. What is the probability that printer A is involved? Printer B is involved? Printer C is involved?

3. a) Let $(S, \mathcal{K}, P)$ be a probability space and $B \in \mathcal{K}$ such that $P(B) > 0$. Prove that $(B, \mathcal{K}_B, P(\cdot|B))$ is a probability space, where $\mathcal{K}_B := \{B \cap A : A \in \mathcal{K}\}$ and $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(A)}{P(B)}, A \in \mathcal{K}_B$.
b) Give examples, by considering a random experiment and its corresponding probability space $(S, \mathcal{K}, P)$, for the probability space $(B, \mathcal{K}_B, P(\cdot|B))$ from **a)**.

Theoretical part

The binomial probabilistic model

Repeated independent trials of an experiment such that there are only two possible outcomes for each trial - which we classify as either *success* or *failure* - and their probabilities remain the same throughout the trials are called **Bernoulli trials**. The binomial model describes the *number of successes* in a series of independent Bernoulli trials:

- *success* appears with probability p, *failure* with probability $1-p$;
- the experiment is repeated n times;
- the probability that success occurs k times in n trials for $k \in \mathbb{N}$, $k \in \{0, \dots, n\}$ is $\boldsymbol{C_n^k p^k (1-p)^{n-k}}$.

▶ $C_n^k p^k (1-p)^{n-k}$ represents the coefficient of x^k in the expansion $(px + 1 - p)^n$ for $k \in \{0, 1, \dots, n\}$.

▶ This model corresponds to the binomial distribution $Bino(n,p), n \in \mathbb{N}^*, p \in (0,1)$.

▶ **Example:** A die is rolled 10 times. The probability that the number 6 shows up 3 times is $C_{10}^3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^7$.

The multinomial probabilistic model

Consider $n \in \mathbb{N}^*$ independent trials such that each trial can have several possible mutually exclusive outcomes $O_1, \dots, O_j$ ($j \in \mathbb{N}^*$) with $P(O_i) = p_i \in (0,1)$, $i \in \{1, \dots, j\}$. Obviously, $p_1 + \cdots + p_j = 1$. The probability that O_i occurs n_i times in n trials for $n_i \in \mathbb{N}$, $i \in \{1, \dots, j\}$ and $n_1 + \cdots + n_j = n$ is
$$\boldsymbol{\frac{n!}{n_1! n_2! \dots n_j!} p_1^{n_1} p_2^{n_2} \dots p_j^{n_j}}.$$

▶ $\frac{n!}{n_1! n_2! \dots n_j!} p_1^{n_1} p_2^{n_2} \dots p_j^{n_j}$ represents the coefficient of $x_1^{n_1} \cdot \ldots \cdot x_j^{n_j}$ in the expansion of $(p_1 x_1 + \ldots + p_j x_j)^n$.

▶ This model corresponds to the multinomial distribution $Multino(n, p_1, \dots, p_j)$, $n \in \mathbb{N}^*$, $p_1, \dots, p_j \in (0,1)$, $p_1 + \ldots + p_j = 1$.

▶ **Example:** Suppose that an urn contains 2 red marbles, 1 yellow marble and 3 blue marbles. 7 marbles are drawn randomly with replacement from the urn (each drawn marble is put back into the urn). The probability that there are drawn 3 red marbles, 2 yellow marbles and 2 blue marble is $\dfrac{7!}{3!2!2!}\left(\dfrac{2}{6}\right)^3\left(\dfrac{1}{6}\right)^2\left(\dfrac{3}{6}\right)^2$.

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4. Let S be the set of all positive integers less or equal than 50, with exactly 2 digits such that one is an even digit and the other is an odd digit. A number is randomly extracted from S. Let X be the sum of its digits. Write the probability distribution of X.

5. The probability that a chipset is defective equals 0.06. A circuit board has 12 such independent chipsets and it's functional if at least 11 chipsets are operating. 4 independent such circuit boards are installed in a computer unit. Compute the probabilities of the following events:
B:"A circuit board is functional."
C:"Exactly two circuit boards are functional in the computer unit."
D:"At least a circuit board is functional in the computer unit."

6. Let (X, Y) be a discrete random vector with the joint probability distribution given by the following contingency table

$X \backslash Y$	-2	1	2
1	0.2	0.1	0.2
2	0.1	0.1	0.3

a) Find the probability distributions of X and Y.
b) Compute the probability that $|X - Y| = 1$, given that $Y > 0$.
c) Are the events $\{X = 2\}$ and $\{Y = 1\}$ independent?
d) Are the random variables X and Y independent?