Homework10: 10.01.2023

Ex1: a) $f:(-1,1)\to\mathbb{R}$ $f(x)=\frac{1}{\sqrt{1-x^2}}$ $\int f(x)dx=\int\frac{1}{\sqrt{1-x^2}}dx=\arcsin x+C$

$F(x)=\arcsin x$

$\lim_{x\searrow -1}\arcsin x=-\infty$ $\lim_{x\nearrow 1}\arcsin x=\infty \Rightarrow$ divergent

b) $f:[1,\infty)\to\mathbb{R}$, $f(x)=\frac{1}{x(x+1)}$ $\int f(x)dx=\int\frac{1}{x^2+x}dx=\int\frac{1}{x}-\frac{1}{x+1}dx=$

$=\ln|x|-\ln|x+1|+C$ $F(x)=\ln|x|-\ln|x+1|$

$\lim_{x\searrow 1}\ln x-\ln x+1=-\ln 2\Rightarrow$ convergent

$\cancel{\lim_{x\to\infty}\ln x-\ln x+1}$

c) $f:(0,1]\to\mathbb{R}$, $f(x)=\ln x$ $\int f(x)dx=\int\ln x\,dx=x\ln x-x+C$

$F(x)=x\ln x-x$ $\lim_{x\searrow 0}x\ln x-x=0\Rightarrow$ convergent

$\lim_{x\nearrow 1}x\ln x-x=-1\Rightarrow$ convergent

d) $f:[0,1)\to\mathbb{R}$, $f(x)=\frac{\arcsin x}{\sqrt{1-x^2}}$ $\int f(x)dx=\int\frac{\arcsin x}{\sqrt{1-x^2}}dx=\int t\,dt=\frac{t^2}{2}=$

$\arcsin x=t$ $=\frac{\arcsin^2 x}{2}+C$ $F(x)=\frac{\arcsin^2 x}{2}$

$\cancel{dx=}\ \sqrt{1-x^2}=dt$ $\lim_{x\searrow 0}\frac{\arcsin^2 x}{2}=-\infty$

$\lim_{x\nearrow 1}\frac{\arcsin^2 x}{2}=\infty$ $\Rightarrow$ divergent

e) $f:(0,1]\to\mathbb{R}$ $f(x)=\frac{\ln x}{\sqrt{x}}$ $\int f(x)dx=\int\frac{\ln x}{\sqrt{x}}dx=2\sqrt{x}\ln x-\int\frac{2}{\sqrt{x}}dx=$

$=2\sqrt{x}\ln x-4\sqrt{x}+C$ $F(x)=2\sqrt{x}\ln x-4\sqrt{x}$

$\lim_{x\searrow 0}2\sqrt{x}\ln x-4\sqrt{x}=0$ $\lim_{x\nearrow 1}2\sqrt{x}\ln x-4\sqrt{x}=-4\Rightarrow$ convergent

f) $f:[e,\infty)\to\mathbb{R}$, $f(x)=\frac{1}{x(\ln x)^3}$ $\int f(x)dx=\int\frac{1}{x(\ln x)^3}dx=\int\frac{1}{t^3}dt=$

$\ln x=t$ $=-\frac{1}{2t^2}=-\frac{1}{2\ln^2 x}+C$ $F(x)=-\frac{1}{2\ln^2 x}$

$\frac{1}{x}=dt$

$\lim_{x\searrow e}-\frac{1}{2\ln^2 x}=-\frac{1}{2}\Rightarrow$ convergent

g) $f:\left(\frac{1+\sqrt{3}}{2},2\right]\to\mathbb{R}$, $f(x)=\frac{1}{x\sqrt{2x^2-2x-1}}$ $\int f(x)dx=\int\frac{1}{x\sqrt{2x^2-2x-1}}dx=$

$=\int\frac{\sqrt{2x^2-2x-1}}{2x^3-2x^2-x}dx$

h) $f:[0,\infty)\to\mathbb{R}$, $f(x)=\frac{\pi}{2}-\operatorname{arctg}x$ $\int f(x)dx=\int\frac{\pi}{2}-\operatorname{arctg}x\,dx=\frac{\pi}{2}-\int\operatorname{arctg}x\,dx$

$=\frac{\pi}{2}-x\operatorname{arctg}(x)+\int\frac{x}{x^2+1}dx=\frac{\pi}{2}-x\operatorname{arctg}(x)+\frac{1}{2}\ln|x^2+1|+C$

$F(x)=\frac{\pi}{2}-\operatorname{arctg}(x)+\frac{1}{2}\ln|x^2+1|$ $\lim_{x\searrow 0}\frac{\pi}{2}-\operatorname{arctg}x+\frac{1}{2}\ln|x^2+1|=-\infty\Rightarrow$ divergent

i) $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\frac{1}{1+x^2}$ $\int f(x)dx=\int\frac{1}{1+x^2}dx=\operatorname{arctg}x+C$ $F(x)=\operatorname{arctg}x$

j) $f:(\frac{1}{3}, 3] \to \mathbb{R}$ $f(x) = \frac{1}{\sqrt[3]{3x-1}}$ $\int f(x)dx = \int \frac{1}{\sqrt[3]{3x-1}} dx = \int \frac{1}{3\sqrt[3]{t}} dt =$

$3x-1 = t$ $\quad = \frac{1}{3} \cdot \frac{3}{2} \cancel{(3x-1)}^{t} {}^{\frac{2}{3}} = \frac{1}{2} \cdot (3x-1)^{\frac{2}{3}} + C$

$3dx = dt$ $\quad F(x) = \frac{1}{2}(3x-1)^{\frac{2}{3}}$ $\lim\limits_{x \searrow \frac{1}{3}} \frac{1}{2}(3x-1)^{\frac{2}{3}} = 0$ $\lim\limits_{x \nearrow 3} \frac{1}{2}(3x-1)^{\frac{2}{3}} = 2$

$\Rightarrow$ convergent

k) $f: [1, \infty) \to \mathbb{R}$, $f(x) = \frac{x}{(1+x^2)^2}$ $\int f(x) dx = \int \frac{x}{(1+x^2)^2} dx = \int \frac{1}{2t^2} dt = \frac{1}{2}\left(-\frac{1}{1+x^2}\right) + C$

$1+x^2 = t$ $\quad F(x) = -\frac{1}{2x^2+2}$ $\lim\limits_{x \searrow 1} -\frac{1}{2x^2+2} = -\frac{1}{4} \Rightarrow$ convergent

$2x dx = dt$

$x dx = \frac{1}{2} dt$

Ex2: a) $f: [1, \infty) \to \mathbb{R}$, $f(x) = \frac{1}{x\sqrt{1+x^2}}$ $\int_1^{\infty} f(x) dx = \cancel{\int_1^t \lim}$

$= \lim\limits_{t \to \infty} \int_1^t \frac{1}{x\sqrt{1+x^2}} dx = \lim\limits_{t \to \infty} \int_1^t \frac{1}{u^2-1} du = -\lim\limits_{t \to \infty} \int_1^t \frac{1}{1-u^2} du =$

$= -\lim\limits_{t \to \infty} \frac{\ln|u+1|}{2} - \frac{\ln|u-1|}{2} = \lim\limits_{t \to \infty} \left(\frac{\ln|\sqrt{1+x^2}+1|}{2} - \frac{\ln|\sqrt{1+x^2}-1|}{2}\right)\Big|_1^t =$

$\cancel{= \lim\limits_{t \to \infty} \frac{1}{2} \ln|\sqrt{1+x^2}-1|} = \lim\limits_{t \to \infty} \left(\frac{1}{2} \ln|\sqrt{1+x^2}+1| - \frac{1}{2} \ln|\sqrt{1+x^2}+1|\right)\Big|_1^t$

$= \lim\limits_{t \to \infty} \frac{1}{2} \ln|\sqrt{1+t^2}-1| - \frac{1}{2} \ln|\sqrt{1+t^2}+1| - \frac{1}{2} \ln|\sqrt{2}-1| + \frac{1}{2} \ln|\sqrt{2}+1| =$

$= \lim\limits_{t \to \infty} \infty - \infty = 0 \Rightarrow$ convergent

b) $f: [0, \frac{\pi}{2}) \to \mathbb{R}$, $f(x) = \frac{1}{\cos x}$ $\int_0^{\frac{\pi}{2}} f(x) dx = \lim\limits_{t \to \frac{\pi}{2}} \int_0^t \frac{1}{\cos x} dx =$

$= \lim\limits_{t \to \frac{\pi}{2}} \ln|tg(x)|\Big|_0^t = \infty \Rightarrow$ divergent

c) $f: (0, \infty) \to \mathbb{R}$, $f(x) = \left(\frac{\operatorname{arctg} x}{x}\right)^2$ $\int_0^{\infty} f(x) dx = \lim\limits_{t \to \infty} \int_0^t \left(\frac{\operatorname{arctg} x}{x}\right)^2 dx =$

$= \cancel{\lim\limits_{t \to \infty}}$

d) $f: (1, \infty) \to \mathbb{R}$, $f(x) = \left(\frac{\ln x}{x\sqrt{x^2-1}}\right)^2 \cancel{dx}$ $\int_1^{\infty} f(x) dx = \lim\limits_{t \to \infty} \int_1^t \left(\frac{\ln x}{x\sqrt{x^2-1}}\right)^2 dx =$

$=$

e) $f: (0, 1) \to \mathbb{R}$, $f(x) = \left(\frac{\ln x}{x\sqrt{1-x^2}}\right)^2$ $\int_0^1 f(x) dx = \lim\limits_{t \to 1} \int_0^t \left(\frac{\ln x}{x\sqrt{1+x^2}}\right)^2 dx$