Seminar 12 and 13 - 2024

1. Consider the sequence of independent identically distributed random variables $(X_n)_{n\geq 1}$ such that $X_n \sim Unif[1,3]$ for each $n \geq 1$. Compute the a.s. limit of the sequence which is
i) the arithmetic mean of $X_1, ..., X_n$, as $n \to \infty$;
ii) the geometric mean of $X_1, ..., X_n$, as $n \to \infty$;
iii) the harmonic mean of $X_1, ..., X_n$, as $n \to \infty$.
A: i) the SLLN $\Longrightarrow \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} X_i \xrightarrow{a.s.} E(X_1) = \int_1^3 \frac{x}{2}\, dx = 2.$
ii) the SLLN $\Longrightarrow \sqrt[n]{\displaystyle\prod_{i=1}^{n} X_i} = e^{\frac{1}{n}\sum_{i=1}^{n}\ln X_i} \xrightarrow{a.s.} e^{E(\ln X_1)} = e^{\int_1^3 \frac{\ln x}{2}dx} = \dfrac{3\sqrt{3}}{e} \approx 1,91.$
iii) the SLLN $\Longrightarrow \dfrac{n}{\sum_{i=1}^{n}\frac{1}{X_i}} = \dfrac{1}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_i}} \xrightarrow{a.s.} \dfrac{1}{E(\frac{1}{X_1})} = \dfrac{1}{\int_1^3 \frac{1}{2x}dx} = \dfrac{2}{\ln 3} \approx 1{,}82.$

2. Let $(X_n)_{n\geq 1}$ be a sequence of random variables such that $P(X_n = n^2) = \frac{1}{n}$ and $P(X_n = 0) = 1 - \frac{1}{n}$, for all $n \geq 1$. Prove that:
a) $X_n \xrightarrow{P} 0$.
b) $(X_n)_{n\geq 1}$ does not converge in mean square.
A: a) For every $\varepsilon > 0$,

$$\lim_{n\to\infty} P(|X_n| \geq \varepsilon) = \lim_{n\to\infty} P(X_n = n^2) = \lim_{n\to\infty}\frac{1}{n} = 0.$$

b) We use the proof by contradiction: Assume that $(X_n)_{n\geq 1}$ converges in mean square. Then, by a theorem from the course and a), $X_n \xrightarrow{L^2} 0$. Since $E|X_n|^2 = \frac{n^4}{n} = n^3 \to \infty$, as $n \to \infty$, we get a contradiction. Hence, $(X_n)_{n\geq 1}$ does not converge in mean square.

3. Consider a binary communication channel transmitting codes of n bits each. Assume that the probability of successful transmission of a single bit is $p \in (0,1)$ and that the probability of an error is $1-p$. Assume also that the channel is capable of correcting up to m errors, where $0 < m < n$. If we assume that the transmission of successive bits is independent, compute the probability of successful code transmission.
A: Let X be the number of number of errors in the code. The event A:"the code is transmitted with all errors corrected" is equivalent with $\{X \leq m\}$. Since $X \sim Bino(n, 1-p)$,

$$P(A) = P(X \leq m) = \sum_{k=0}^{m} C_n^k p^{n-k}(1-p)^k.$$

4. The measurement error (in millimeters) of a certain object produced in a factory is a continuous random variable X with the cumulative distribution function $F : \mathbb{R} \to [0,1]$,

$$F(x) = \begin{cases} 0, & x < -1 \\ \frac{1}{4}\left(2 + 3x - x^3\right), & x \in [-1,1] \\ 1, & x > 1. \end{cases}$$

Find: $P(-\frac{1}{2} < X < \frac{1}{2})$, $P(X < \frac{1}{2} | X > -\frac{1}{2})$, $E(X)$.

A: $P(-\frac{1}{2} < X < \frac{1}{2}) = F(\frac{1}{2}) - F(-\frac{1}{2}) = \frac{27}{32} - \frac{5}{32} = \frac{22}{32} = \frac{11}{16}$. $P(X < \frac{1}{2} | X > -\frac{1}{2}) = \frac{P(-\frac{1}{2}<X<\frac{1}{2})}{P(X>-\frac{1}{2})} = \frac{\frac{22}{32}}{1-\frac{5}{32}} = \frac{22}{27}$.
$f(x) = \begin{cases} \frac{3}{4}(1-x^2), & x \in [-1,1] \\ 0, & x \notin [-1,1] \end{cases} \implies E(X) = \int_{-1}^{1} \frac{3}{4}(x - x^3)dx = 0.$

5. A random number generator returns the value Y according to the distribution given by $\begin{pmatrix} -2 & -1 & 0 & 1 \\ \frac{3}{10} & \frac{2}{10} & \frac{1}{10} & \frac{4}{10} \end{pmatrix}$.
Find the mean value and the standard deviation of the generator.
A: $E(Y) = \frac{-6-2+4}{10} = -\frac{4}{10} = -\frac{2}{5}$, $E(Y^2) = \frac{12+2+4}{10} = \frac{18}{10} = \frac{9}{5}$, $\sigma = \sqrt{\frac{9}{5} - \frac{4}{25}} = \frac{\sqrt{41}}{5}$.

6. A random value X is generated according to the density function $f_X : \mathbb{R} \to \mathbb{R}$, $f(x) = \dfrac{1}{2}e^{-|x|}$, for all $x \in \mathbb{R}$.
Compute:
a) the cumulative distribution function of X;
b) the cumulative distribution function of the random value X^2;
c) $P(X^2 \geq 1)$;
d) the mean value and the variance of X.
A: a) $F_X(x) = \begin{cases} \frac{1}{2}\int_{-\infty}^{x} e^t dt, & x < 0 \\ \frac{1}{2}\int_{-\infty}^{0} e^{-t}dt + \frac{1}{2}\int_0^x e^{-t}dt, & x \geq 0 \end{cases} = \begin{cases} \frac{e^x}{2}, & x < 0 \\ \frac{1}{2} + \frac{1-e^{-x}}{2}, & x \geq 0 \end{cases} = \begin{cases} \frac{e^x}{2}, & x < 0 \\ 1 - \frac{e^{-x}}{2}, & x \geq 0 \end{cases}$.
b) $F_{X^2}(x) = P(X^2 \leq x) = \begin{cases} 0, & x < 0 \\ F(\sqrt{x}) - F(-\sqrt{x}), & x \geq 0 \end{cases} = \begin{cases} 0, & x < 0 \\ 1 - \frac{e^{-\sqrt{x}}}{2} - \frac{e^{-\sqrt{x}}}{2}, & x \geq 0 \end{cases} = \begin{cases} 0, & x < 0 \\ 1 - e^{-\sqrt{x}}, & x \geq 0 \end{cases}$.
c) $P(X^2 \geq 1) = 1 - F_{X^2}(1) = \frac{1}{e}$. d) $E(X) = \frac{1}{2}\int_{-\infty}^{\infty} xe^{-|x|}dx = 0$ (we integrate an odd function on a symmetric interval), $V(X) = E(X^2) - E^2(X) = \frac{1}{2}\int_{-\infty}^{\infty} x^2 e^{-|x|}dx = \int_0^{\infty} x^2 e^{-x}dx = -x^2e^{-x}\Big|_0^{\infty} - 2xe^{-x}\Big|_0^{\infty} - 2e^{-x}\Big|_0^{\infty} = 2$.
Another solution is to find first a density function for X^2: $f_{X^2}(y) = \begin{cases} 0, & y < 0 \\ (1 - e^{-\sqrt{y}})', & y \geq 0 \end{cases} = \begin{cases} 0, & y < 0 \\ \frac{e^{-\sqrt{y}}}{2\sqrt{y}}, & y \geq 0 \end{cases}$.
Then, if $Y = X^2$, $E(Y) = \int_0^{\infty} y\frac{e^{-\sqrt{y}}}{2\sqrt{y}}dy \overset{y=x^2}{=} \int_0^{\infty} x^2e^{-x}dx = 2$.

7. For each $n \in \mathbb{N}, n \geq 2$, consider

$$X_n \sim \begin{pmatrix} -1 & 1 \\ \frac{1}{n} & 1 - \frac{1}{n} \end{pmatrix}$$

such that $(X_n)_{n\geq 2}$ is a sequence of pairwise independent random variables.

(a) Does $(X_n)_{n\geq 2}$ obey the weak law of large numbers?

(b) Compute $\lim\limits_{n\to\infty} V\left(\frac{1}{2}(X_{n-1} + X_n)\right)$.

A: (a)
$V(X_n) = E(X_n^2) - (E(X_n))^2 = 1 - \left(1 - \frac{2}{n}\right)^2 = \frac{4}{n} - \frac{4}{n^2} \leq 4$.
$(X_n)_{n\geq 2}$ is a sequence of pairwise independent random variables, we use Theorem 36 from the lecture to deduce that $(X_n)_{n\geq 2}$ obeys the weak law of large numbers.

(b) By the independence property $\Rightarrow V\left(\frac{1}{2}(X_{n-1} + X_n)\right) = \frac{1}{4}(V(X_{n-1}) + V(X_n)) = \frac{1}{n-1} - \frac{1}{(n-1)^2} + \frac{1}{n} - \frac{1}{n^2}$.
Therefore, $\lim\limits_{n\to\infty} V\left(\frac{1}{2}(X_{n-1} + X_n)\right) = 0$.