Analytic Geometry

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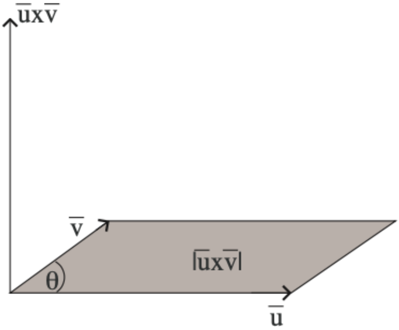
Del capítulo anterior...



- If the vectors $\overline{u}, \overline{v}$ are not collinear, then if $\overrightarrow{OU} \in \overline{u}$ and $\overrightarrow{OV} \in \overline{v}$, then $||\overline{u} \times \overline{v}||$ is the area of the parallelogram formed by $\overrightarrow{OU}$ and $\overrightarrow{OV}$.
- The area of the triangle $\triangle OUV$ can be computed as

$$\mathrm{Area}_{\triangle OUV} = \frac{||\overline{u} \times \overline{v}||}{2}.$$

Algebraically

If $\overline{u} = u_1\overline{i} + u_2\overline{j} + u_3\overline{k}$ and $\overline{v} = v_1\overline{i} + v_2\overline{j} + v_3\overline{k}$ are vectors in V_3, then

$$\overline{u} \times \overline{v} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}. \tag{1}$$

Some observations

The cross product shares a few similarities with the dot product. However, there are some differences which you have to remember:

1. The cross product is not commutative. In fact, it is anti-commutative.
2. The cross product of two vectors is a vector, not a scalar (as it is the case for the result of a dot product). Therefore, it makes sense to consider products with multiple factors. One should be very careful with those, since the cross product is not associative either :)

A closer look at a high-school formula

In high-school, you probably learned how to compute the area of a triangle determined by $A(x_A, y_A)$, $B(x_B, y_B)$ and $C(x_C, y_C)$.

- Let see these in $3D$ and assume WLOG they line in the plane xOy.
- We therefore have $A(x_A, y_A, 0)$, $B(x_B, y_B, 0)$ and $C(x_C, y_C, 0)$. These points determine the vectors $\overline{AB}(x_B - x_A, y_B - y_A, 0)$ and $\overline{AC}(x_C - x_A, y_C - y_A, 0)$.
- Computing, we have

$$\overline{AB} \times \overline{AC} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ x_B - x_A & y_B - y_A & 0 \\ x_C - x_A & y_C - y_A & 0 \end{vmatrix} = \bar{k} \begin{vmatrix} x_B - x_A & y_B - y_A \\ x_C - x_A & y_C - y_A \end{vmatrix},$$

$$\overline{AB} \times \overline{AC} = \bar{k} \begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}.$$

It follows that

$$||\overline{AB} \times \overline{AC}|| = \pm \begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix},$$

hence

$$\mathrm{Area}_{\triangle ABC} = \pm\frac{1}{2} \begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}.$$

Double cross product

Let $\overline{u}, \overline{v}$ and $\overline{w}$ be vectors in V_3. The double cross product of these three vectors is, by definition, the vector $(\overline{u} \times \overline{v}) \times \overline{w}$. The following relation holds

$$\boxed{(\overline{u} \times \overline{v}) \times \overline{w} = (\overline{u} \cdot \overline{w})\overline{v} - (\overline{v} \cdot \overline{w})\overline{u}}.$$

On the other hand,

$$\overline{u} \times (\overline{v} \times \overline{w}) = -(\overline{v} \times \overline{w}) \times \overline{u} = (\overline{v} \cdot \overline{u})\overline{w} - (\overline{w} \cdot \overline{u})\overline{v}$$

Comparing $(\overline{u} \times \overline{v}) \times \overline{w}$ and $\overline{u} \times (\overline{v} \times \overline{w})$ we find that these are equal if

$$-(\overline{v} \cdot \overline{w})\overline{u} + 2(\overline{u} \cdot \overline{w})\overline{v} - (\overline{u} \cdot \overline{v})\overline{w} = 0.$$

We notice that $\overline{u}, \overline{v}$ and $\overline{w}$ being coplanar is a necessary condition for associativity. However, this is not sufficient.

Using the equality

$$\boxed{(\overline{u} \times \overline{v}) \times \overline{w} = (\overline{u} \cdot \overline{w})\overline{v} - (\overline{v} \cdot \overline{w})\overline{u}},$$

one can easily show that the “Jacobi's identity”

$$(\overline{u} \times \overline{v}) \times \overline{w} + (\overline{v} \times \overline{w}) \times \overline{u} + (\overline{w} \times \overline{u}) \times \overline{v} = \overline{0}$$

holds for any $\overline{u}, \overline{v}, \overline{w} \in V_3$.

Triple scalar product

Given three vectors $\overline{a}$, $\overline{b}$ and $\overline{c}$ from V_3, one defines their *triple scalar product* to be the real number $(\overline{a}, \overline{b}, \overline{c}) = \overline{a} \cdot (\overline{b} \times \overline{c})$.
If $\overline{a} = (a_1, a_2, a_3)$, $\overline{b} = (b_1, b_2, b_3)$ and $\overline{c} = (c_1, c_2, c_3)$, then the triple scalar product can be calculated as

$$(\overline{a}, \overline{b}, \overline{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}. \tag{2}$$

Indeed,

$$(\overline{a}, \overline{b}, \overline{c}) = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) =$$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Remark: It can be seen easily that the triple scalar product can be also seen as $(\overline{a}, \overline{b}, \overline{c}) = (\overline{a} \times \overline{b}) \cdot \overline{c}$.

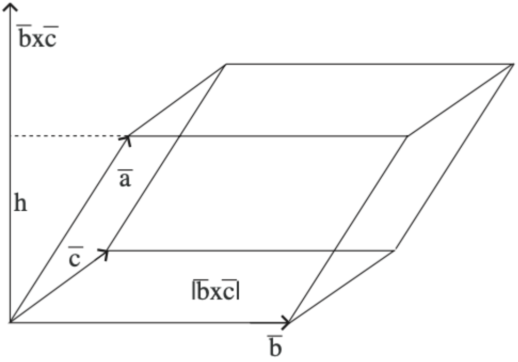
Theorem

If $\overline{a}$, $\overline{b}$ and $\overline{c}$ are vectors in V_3, then:

a) $(\overline{a}, \overline{b}, \overline{c}) = (\overline{c}, \overline{a}, \overline{b}) = (\overline{b}, \overline{c}, \overline{a})$;

b) $(\overline{a}, \overline{b}, \overline{c}) = 0$ *if and only if $\overline{a}$, $\overline{b}$ and $\overline{c}$ are linearly dependent (i.e. they have representatives situated on the same plane).*

The triple scalar product has a geometric meaning. Suppose that the vectors $\overline{a}$, $\overline{b}$ and $\overline{c}$ are linearly independent and choose a representer for each, having the same original point. These form the adjacent sides of a parallelepiped, as below.



Suppose that the base of this parallelepiped is the parallelogram constructed on $\overline{b}$ and $\overline{c}$. The height of the parallelepiped is the length of the orthogonal projection of the vector $\overline{a}$ on the direction of the vector $\overline{b} \times \overline{c}$,

$$h = |\mathrm{pr}_{\overline{b}\times\overline{c}}\overline{a}| = \left|\frac{\overline{a}\cdot(\overline{b}\times\overline{c})}{|\overline{b}\times\overline{c}|}\right| = \frac{|(\overline{a},\overline{b},\overline{c})|}{|\overline{b}\times\overline{c}|}.$$

Then, the volume of the parallelepiped whose adjacent sides are the vectors $\overline{a}$, $\overline{b}$ and $\overline{c}$ is the absolute value of the triple scalar product $(\overline{a},\overline{b},\overline{c})$:

$$V = h \cdot \mathrm{Area}(\overline{b},\overline{c}) = \frac{|(\overline{a},\overline{b},\overline{c})|}{|\overline{b}\times\overline{c}|}|\overline{b}\times\overline{c}| = |(\overline{a},\overline{b},\overline{c})|. \tag{3}$$

The volume of a tethrahedron

Suppose we have a tetrahedron $OABC$ such that $\overrightarrow{OA} \in \overline{a}$, $\overrightarrow{OB} \in \overline{b}$ and $\overrightarrow{OC} \in \overline{c}$. Then, the volume of the tetrahedron can be computed as

$$\mathrm{Vol}_{OABC} = \frac{1}{3}\mathrm{d}(A, OBC) \cdot \mathrm{Area}_{\triangle OBC}$$

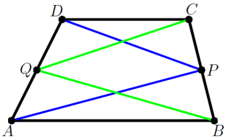
$$\mathrm{Vol}_{OABC} = \frac{1}{6}|\mathrm{pr}_{\overline{b}\times\overline{c}}(\overline{a})| \cdot |\overline{b} \times \overline{c}| = \frac{1}{6}\frac{|(\overline{a}, \overline{b}, \overline{c})|}{|\overline{b} \times \overline{c}|}|\overline{b} \times \overline{c}| = \frac{1}{6}|(\overline{a}, \overline{b}, \overline{c})|$$

An easy application

We are given a tetrahedron $ABCD$ of volume 5 with three of its vertices $A(2,1,-1)$, $B(3,0,1)$ and $C(2,-1,3)$. Its fourth vertex D is situated on somewhere on the Oy axis. Find the coordinates of the point D.

An application of the cross product

Consider a trapezoid $ABCD$ with $AB \parallel CD$. Let P and Q be the midpoints of $[BC]$ and $[DA]$. Prove that the triangles APD and CQB have the same area.



The dot and triple scalar products in play..

The vectors $\overline{a}(8,4,1)$, $\overline{b}(2,2,1)$ and $\overline{c}(1,1,1)$ are given. Determine the vector $\overline{d}$ such that:

1. $\angle(\overline{d},\overline{a}) = \angle(\overline{d},\overline{b})$;
2. $\overline{d} \perp \overline{c}$;
3. $||\overline{d}|| = 1$;
4. The triples $\{\overline{a},\overline{b},\overline{c}\}$ and $\{\overline{a},\overline{b},\overline{d}\}$ have the same orientation.

Digression: The Diffie-Hellman key exchange protocol

- Diffie–Hellman key exchange establishes a shared secret between two parties that can be used for secret communication for exchanging data over a public network.
- The simplest implementation of the protocol uses the multiplicative group of integers modulo p, where p is prime, and g is a primitive root modulo p. These two values are chosen in this way to ensure that the resulting shared secret can take on any value from 1 to $p-1$. Here is an example of the protocol, with non-secret values in blue, and secret values in red.

- Alice and Bob publicly agree to use a modulus $p = 23$ and base $g = 5$ (which is a primitive root modulo 23).
- Alice chooses a secret integer $a = 4$, then sends $A = g^a \mod p$ to Bob.

$$A = 5^5 \pmod{23} = 4$$

- Bob chooses a secret integer $b = 3$, then sends $B = g^b \mod p$ to Alice.
- Alice computes $s = B^a \pmod{p}$

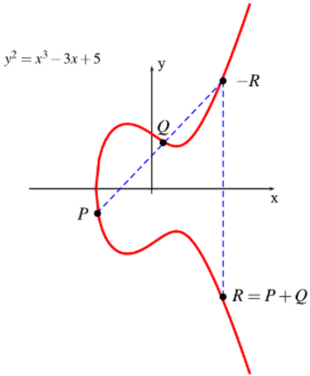
$$s = 10^4 \pmod{23} = 18.$$

- Bob computes $s = A^b \pmod{p}$, this being

$$s = 4^3 \pmod{23} = 18.$$

Now both Alice and Bob share the secret key s.

Get inspiration from analytic geometry



The problem set for this week has already been posted. Ideally you would think about the problems before the seminar.

Thank you very much for your attention!