Homework 6

Ex1: a) $\lim\limits_{x\to\infty} x\cos^2\left(\frac{2x^2+2}{3x^2}\right) = \lim\limits_{x\to\infty} x\cos^2\left(\frac{x^2\left(2+\frac{2}{x^2}\right)}{3x^2}\right) = \infty\cos^2\cdot\frac{2}{3} = \infty$

b) $\lim\limits_{x\to 1} \frac{4x^2}{3x^2+5} = \frac{4}{3+5} = \frac{4}{8} = \frac{1}{2}$

c) $\lim\limits_{x\to -\infty} \frac{-2x^2+5}{5x^3} = \lim\limits_{x\to -\infty} \frac{x^2\left(-2+\frac{5}{x^2}\right)}{5x^3} = \lim\limits_{x\to\infty} \frac{-2}{5x}$

d) $\lim\limits_{x\to\infty} \frac{(4x+2)(3x+1)}{7x^2+10x+5} = \lim\limits_{x\to\infty} \frac{x^2\left(4+\frac{2}{x}\right)\left(3+\frac{1}{x}\right)}{x^2\left(7+\frac{10}{x}+\frac{5}{x^2}\right)} = \frac{12}{7}$

e) $\lim\limits_{x\to 1} \frac{x^2-1}{x^3-1} = \lim\limits_{x\to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}(x^2+x+1)} = \lim\limits_{x\to\infty} \frac{x+1}{x^2+x+1} = \frac{2}{3}$

f) $\lim\limits_{x\to 2} \left(\frac{1}{2-x} - \frac{2x}{4-x^2}\right) = \lim\limits_{x\to 2} \frac{2+x-2x}{4-x^2} = \lim\limits_{x\to 2} \frac{\cancel{2-x}}{\cancel{(2-x)}(2+x)} = \frac{1}{4}$

g) $\lim\limits_{x\to 1} \frac{1+x+x^2+\dots+x^n-(n+1)}{x-1} = \lim\limits_{x\to 1} \frac{x-1+x^2-1+\dots+x^n-1}{x-1} =$

$= \lim\limits_{x\to 1} \frac{(x-1)\left[1+(x+1)+\dots+(x^{n-1}+x^{n-2}+\dots+x+1)\right]}{x-1} = 1+2+\dots+n = \frac{n(n+1)}{2}$

h) $\lim\limits_{x\to 1} \frac{x+x^2+\dots+x^n-n}{x+x^2+\dots+x^m-m} = \lim\limits_{x\to 1} \frac{x-1+x^2-1+\dots+x^n-1}{x-1+x^2-1+\dots+x^m-1} =$

$= \lim\limits_{x\to 1} \frac{\cancel{(x-1)}\left[1+(x+1)+\dots+(x^{n-1}+x^{n-2}+\dots+x+1)\right]}{\cancel{(x-1)}\left[1+(x+1)+\dots+(x^{m-1}+x^{m-2}+\dots+x+1)\right]} = \frac{\frac{n(n+1)}{2}}{\frac{m(m+1)}{2}} = \frac{n(n+1)}{m(m+1)}$

i) $\lim\limits_{x\to 27} \frac{x-27}{\sqrt[3]{x}-3} = \lim\limits_{x\to 27} \frac{\cancel{(\sqrt[3]{x}-3)}(\sqrt[3]{x^2}+3\sqrt[3]{x}+3)}{\cancel{\sqrt[3]{x}-3}} = \frac{9+3\cdot 3+3^2}{1} = 21$

j) $\lim\limits_{x\to 1} \frac{\sqrt[4]{x}-1}{\sqrt[3]{x}-1} = \lim\limits_{x\to 1} \frac{(x-1)(\sqrt[4]{x^3}+\sqrt[4]{x^2}+\sqrt[4]{x}+1)}{(x-1)(\sqrt[3]{x^2}+\sqrt[3]{x}+1)} = \frac{4}{3}$

k) $\lim\limits_{x\to\infty} \left(\sqrt[3]{ax^3+x^2+bx+c} - (bx+c)\right) = \lim\limits_{x\to\infty} x\left[\sqrt[3]{a+\frac{1}{x}+\frac{b}{x^2}+\frac{c}{x^3}} - b - \frac{c}{x}\right] =$

$= \infty\cdot(\sqrt[3]{a}-b) = \begin{cases} \infty & : \sqrt[3]{a} > b \\ \infty\cdot 0 & : \sqrt[3]{a} = b \iff a = b^3 \\ -\infty & : \sqrt[3]{a} < b \end{cases}$

case: $a = b^3$ $\lim\limits_{x\to\infty} \left(\sqrt[3]{b^3c^3 + x^2 + bx + c} - (bx+c)\right) =$

$= \lim\limits_{x\to\infty} \left(\sqrt[3]{b^3c^3 + x^2 + bx + c} - \sqrt[3]{(bx+c)^3}\right) = \lim\limits_{x\to\infty} \dfrac{}{\sqrt[3]{(b^3x^3 + x^2 + bx + c)^2} + \sqrt[3]{(b^3x^3 + x^2 + bx + c)(bx+c)^3}}$

$= \lim\limits_{x\to\infty} \dfrac{b^3x^3 + x^2 + bx + c - (bx+c)^3}{\sqrt[3]{(b^3x^3 + x^2 + bx + c)^2} + \sqrt[3]{(b^3x^3 + x^2 + bx + c)(bx+c)^3} + \sqrt[3]{(bx+c)^{3\cdot 2}}} =$

$= \lim\limits_{x\to\infty} \dfrac{b^3x^3 + x^2 + bx + c - (b^3x^3 + 3b^2x^2c + 3bxc^2 + c^3)}{x^2\left(\sqrt[3]{b^6 + \underset{\downarrow}{\ldots}} + \sqrt[3]{b^6 + \underset{\downarrow}{\ldots}} + \sqrt[3]{b^6 + \underset{\downarrow}{\ldots}}\right)} = \lim\limits_{x\to\infty} \dfrac{x^2(1 - 3b^2c) + \ldots}{x^2 \cdot 3b^2 \underset{\downarrow}{-} }$

$0 \quad 0 \quad 0 \quad 0$

$= \begin{cases} \dfrac{1 - 3b^2c}{3b^2} : 1 - 3b^2c = 0 \\ 0 : c = \dfrac{1}{3b^2} \end{cases}$

~~Ex 2: a) $\lim\limits_{x\to\infty} \left(\frac{1}{3x}\right)^{\frac{7x+1}{4x+4}} = 0^{\frac{7}{4}}$~~ $= \lim\limits_{x\to\infty} f(x) = \begin{cases} \infty : a > b^3 \\ \dfrac{1 - 3b^2c}{3b^2} : a = b^3 \text{ and } c \neq \dfrac{1}{3b^2} \\ 0 : a = b^3 \text{ and } c = \dfrac{1}{3b^2} \\ -\infty : a < b^3 \end{cases}$

Ex 2: a) $\lim\limits_{x\to\infty} \left(\frac{1}{3x}\right)^{\frac{7x+1}{4x+4}} = 0^{\frac{7}{4}} = 0$

b) $\lim\limits_{x\to 0} \left(\dfrac{5\sin x - 2\tan x}{x}\right)^{\frac{4\sin x + 9x}{x}} = \lim\limits_{x\to 0} \left(\dfrac{5\sin x}{x} - \dfrac{2\tan x}{x}\right)^{\frac{4\sin x}{x} + \frac{9x}{x}} =$

$= \lim\limits_{x\to 0} (5-2)^{4+9} = 3^{13}$

c) $\lim\limits_{x\to 0} (1 + \cos x)^{\frac{1}{x^2}} = 2^{\frac{1}{0^+}} = 2^{\infty} = \infty$

d) $\lim\limits_{x\to 0} (e^x - x + 1)^{\frac{1}{1-\cos x}} = 2^{\infty} = \infty$

e) $\lim\limits_{x\to 0} (1 + \sin x)^{\frac{1}{x}} \overset{1^\infty}{=} e^{\lim\limits_{x\to 0} \frac{\sin x}{x}} = e^1 = e$

f) $\lim\limits_{x\to\infty} \left(\dfrac{x+7}{x}\right)^x = e^7$

Ex 3: a) $\lim\limits_{n\to\infty} \left[\lim\limits_{x\to 0} (1 + \sin^2 x + \sin^2 2x + \ldots + \sin^2 nx)^{\frac{1}{n^3x^2}}\right]$

$\lim\limits_{x\to 0} (1 + \sin^2 x + \ldots + \sin^2 nx)^{\frac{1}{n^3x^2}} = \lim e^{\lim\limits_{x\to 0} \frac{\sin^2 x + \ldots + \sin^2 nx}{n^3x^2}} = e^{\frac{1}{n^3} \lim\limits_{x\to 0} \left[\frac{\sin^2 x}{x^2} + \ldots + \frac{\sin^2 nx}{x^2}\right]} =$

$= e^{\frac{1}{n^3} \cdot (1 + 2^2 + \ldots + n^2)} = e^{\frac{n(n+1)(2n+1)}{6}} = e^{\frac{(n+1)(2n+1)}{6n^2}}$

$\lim\limits_{n\to\infty} e^{\frac{(n+1)(2n+1)}{6n^2}} = e^{\frac{2}{6}} = e^{\frac{1}{3}} = \sqrt[3]{e}$

b) $\lim_{n\to\infty} [\lim_{x\to 0} (1+\ln(1+x)+\ln(1+2x)+\dots+\ln(1+nx))^{\frac{1}{n^2 x}}] =$

$\lim_{x\to 0} (1+\ln(1+x)+\ln(1+2x)+\dots+\ln(1+nx))^{\frac{1}{n^2 x}} = \frac{1}{n^2}$

$= e^{\frac{1}{n^2}\lim_{x\to 0}\left(\frac{\ln(1+x)}{x}+\dots+\frac{\ln(1+nx)}{x}\right)} = \lim_{x\to 0}\frac{\ln(1+kx)}{x} =$

$= \lim_{x\to 0}\frac{\ln(1+kx)}{kx}\cdot\frac{kx}{x} = k =$

$= e^{\frac{1}{n^2}(1+2+\dots+n)} = e^{\frac{n(n+1)}{2n^2}} = e^{\frac{n+1}{2n}}$

$\lim_{n\to\infty} e^{\frac{n+1}{2n}} = e^{\frac{1}{2}} = \sqrt{e}$

Ex5: a) $\lim_{x\to 0}\frac{e^{2x}-1}{3x} \overset{L'H}{=} \lim_{x\to 0}\frac{2}{3} = \frac{2}{3}$

b) $\lim_{x\to 0}\frac{e^{x}-\cos x}{3x} \overset{L'H}{=} \frac{1}{3}$