min 10 pct

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SEMINAR 1

Propositional Logic →

Symbols parantheses (brackets) "(", ")"
connectives: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$
non/not, and, OR, IMPLIES, IF AND ONLY IF

atomic formulas (atoms): $p, q, r, x_1, x_2, \ldots$

Formulas 1) atoms are formulas
2) If A and B are formulas, then the following are also
$(\neg A)$ - this is a formula
$(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, $(A \leftrightarrow B)$ - formulas
highest priority
3) there are no other formulas

$\underbrace{(p \vee q) \rightarrow (r \leftrightarrow s)}_{\text{formula}}$

$(p \rightarrow q) \vee (\underbrace{r \rightarrow \rightarrow}_{\text{not a formula}} s)$

$p \rightarrow (\neg q)$ - formula

An interpretation means assigning either 0 or 1 to every atom and then deducing the truth value of the whole formula.

Truth tables

p	$\neg p$
0	1
1	0

p	q	$p \wedge q$	$p \vee q$	$p \to q$	$p \leftrightarrow q$
0	0	0	0	1	1
0	1	0	1	1	0
1	0	0	1	0	0
1	1	1	1	1	1

$(p \vee q) \to (\neg p \leftrightarrow q)$

p	q	$p \vee q$	$\neg p$	$\neg p \leftrightarrow q$	$(p \vee q) \to (\neg p \leftrightarrow q)$
0	0	0	1	0	1
0	1	1	1	1	1
1	0	1	0	1	1
1	1	1	0	0	0

if all was 1
↳ then it was true

A formula is a tautology if it's true for any interpretation. ex ~~p~~ $\vee \neg p$ (−)

is a contradiction if it's always false

$p \wedge \neg p$

−〃− ~~a~~ satisfiable if it's not a contradiction
(~~is also a tautology~~)

distributivity

Tautology - Tautologies: $\underbrace{(A \wedge B) \vee C}_{\text{left}} \Leftrightarrow (A \vee C) \wedge (B \vee C)$ the same values as right

A, B, C are formulas

$\Rightarrow$ $A \to B$ if it is always true me me $\Rightarrow$

$\to$

A	B	C	$A\wedge B$	$(A\wedge B)\vee C$	$A\vee C$	$B\vee C$	$(A\vee C)\wedge(B\vee C)$
0	0	0	0	0	0	0	0
1	0	0	0	0	1	0	0
0	1	0	0	0	0	1	0
1	1	0	0	1	1	1	1
0	0	1	1	1	1	1	1
1	0	1	1	1	1	1	1
0	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1

$(A\wedge B)\vee C \Leftrightarrow (A\vee C)\wedge(B\vee C)$

$0 \wedge 1 \Leftrightarrow 0$

$0 \vee 1 \Leftrightarrow 1$

$(A\vee B)\wedge C \Leftrightarrow (A\wedge C)\vee(B\wedge C)$ distrib

$(A\wedge B)\wedge C \Leftrightarrow A\wedge(B\wedge C)$ associativity

$(A\vee B)\vee C \Leftrightarrow A\vee(B\vee C)$

$A\wedge B \Leftrightarrow B\wedge A$ commutativity

$A\vee B \Leftrightarrow B\vee A$

$A\wedge(A\vee B) \Leftrightarrow A$ absorption

$A\vee(A\wedge B) \Leftrightarrow A$

2 negation

$\neg(A\wedge B) \Leftrightarrow (\overline{A\wedge B}) \Leftrightarrow \neg A\vee\neg B$ De Morgan's law

$\neg(A\vee B) \Leftrightarrow (\overline{A\vee B}) \Leftrightarrow \neg A\wedge\neg B$

$A\to B \Leftrightarrow \neg A\vee B$

$\underset{0}{1\to 0} \Leftrightarrow \underset{0}{0\vee 0}$

<u>Normal forms</u>

Elementary conjunction is a conjunction of atoms and/or their negations

ex. $p\wedge\neg q$, $p\wedge\neg q\wedge r\wedge\neg p$

$p\wedge(p\to q)$ not elementary conjunction

$r\wedge\neg p\wedge\neg t$ el. conj

Elementary disjunction is a disjunction of atoms and/or their negations

disj $p \vee \neg r \vee q$

$(p \wedge q) \vee \neg r$ - neither

- Conjunctive normal form (CNF) is a conjunction of elementary disjunction $(r \vee p) \wedge (\neg s \vee t \vee r)$

$(p \vee q) \wedge \neg r$ CNF

single atom

and between parentheses that have OR

- Disjunctive normal form (DNF) is a ~~disj~~ disjunction of elementary conjunction

$p \vee q \vee \underbrace{(\neg s \wedge r)}_{\text{el conj}}$ DNF

. For an arbitrary formula we do the following steps:

1) Transform the implication using $\neg, \wedge, \vee$

$p \to q \Leftrightarrow \neg p \vee q$

2) We get rid of the negations using De Morgan's Law $\overline{p \wedge q} \Leftrightarrow (\neg p \vee \neg q)$

3) We obtain CNF or DNF using the other tautologies

$\underline{\neg x \to (x \wedge y)} \Leftrightarrow (\underset{abs}{\neg\neg} x) \vee (x \wedge y)$

$\Leftrightarrow^{A} x \vee (x \wedge y) \Leftrightarrow x$

A is satisfiable but not a tautology

Exercises

1) $\left((x \to y) \to (z \to \neg x)\right) \to (\neg y \to \neg z) \Leftrightarrow$

$\Leftrightarrow \left((\neg x \vee y) \to (\neg z \vee \neg x)\right) \to (y \vee \neg z) \Leftrightarrow$

$\Leftrightarrow \overline{\left(\overline{(\neg x \vee y)} \vee (\neg z \vee \neg x)\right)} \to (y \vee \neg z) \Leftrightarrow$

$\Leftrightarrow \left((x \wedge y) \vee (\neg z \vee \neg x)\right) \vee y \vee \neg z \Leftrightarrow$

$\Leftrightarrow \left(\neg(x \wedge \neg y) \wedge \neg(\neg z \vee \neg x)\right) \vee y \vee \neg z \Leftrightarrow$

$\Leftrightarrow \left((\neg x \vee y) \wedge (z \wedge x)\right) \vee y \vee \neg z \Leftrightarrow$

$\Leftrightarrow \left((\neg x \vee y) \wedge z \wedge x\right) \vee y \vee \neg z \Leftrightarrow$

$\Leftrightarrow \underbrace{(\neg x \wedge z \wedge x)}_{0} \vee (y \wedge z \wedge x) \vee y \vee \neg z$

$\Leftrightarrow (y \wedge z \wedge x) \vee y \vee \neg z$ DNF

$\left.\begin{array}{l} v(x)=1 \\ v(y)=1 \\ v(z)=1 \end{array}\right\} \Rightarrow v(A)=1 \Rightarrow A$ is satisfiable

$\left.\begin{array}{l} v(x)=0 \\ v(y)=0 \\ v(z)=1 \end{array}\right\} \Rightarrow v(A)=0 \Rightarrow A$ is not a tautology

$A \Leftrightarrow (\overbrace{y \vee y}^{y} \vee \neg z) \wedge (\overbrace{z \vee y \vee \neg z}^{1}) \wedge (x \vee y \vee \neg z)$

$\Leftrightarrow (y \vee \neg z) \wedge (x \vee y \vee \neg z)$ CNF

2. $\dfrac{A \to B}{\neg B \to \neg A}$

$(A \to B) \to (\neg B \to \neg A)$

$\Leftrightarrow (\neg A \vee B) \to (\neg\neg B \vee \neg A)$

$\Leftrightarrow \neg(\neg A \vee B) \vee (B \vee \neg A)$

$(A \wedge \neg B) \vee B \vee \neg A \Leftrightarrow$

$(\underbrace{A \vee B \vee \neg A}_{1}) \wedge (\underbrace{\neg B \vee B \vee B \neg A}_{1}) = 1$

tautology