Presentation 2: By Laura Rusanescu

In $\triangle ABC$, let M be a point on AB and N be a point on BC such that

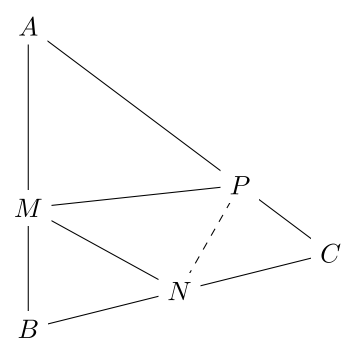
$$\frac{2 \cdot CN}{BC} = \frac{AM}{AB}.$$

Let P be a point on AC. Show that the lines MN and NP are perpendicular if and only if PN is the internal angle bisector of the angle $\angle MPC$.

Given:

1. ABC is a triangle with points M on $\overline{AB}$, N on $\overline{BC}$.
2. $\frac{2 \cdot \overline{CN}}{\overline{BC}} = \frac{\overline{AM}}{\overline{AB}}$.
3. P is a point on $\overline{AC}$.
4. Assume $\overline{MN} \perp \overline{NP}$.

Diagram



Goal:

Show that $\overline{MN} \perp \overline{NP}$ if and only if $\overline{PN}$ bisects $\angle MPC$.

Proof:

Let R be the point symmetric to C with respect to N. From the given equality, we obtain

$$\frac{CR}{BC} = \frac{AM}{AB}.$$

By Thales' Theorem, it follows that $MR \parallel AC$.

Let S denote the intersection of MR and NP.

Since $PC \parallel RS$, we have $\triangle PNC \sim \triangle ASNR$ (AA Similarity while $\angle RNS$ and $\angle CNP$ are opposing angles and $\angle RSN = \angle CPN$). Therefore,

$$\frac{PN}{NS} = \frac{CN}{NR} = 1$$

implying that N is also the midpoint of segment $[PS]$.

Within triangle $\triangle MSP$, we observe that MN is a median.

The desired conclusion, that MN is perpendicular to NP, is equivalent to MN being an altitude in $\triangle AMS$. This holds if and only if $\triangle AMS$ is isosceles.

Isoscelesness of $\triangle AMS$ is equivalent to $\angle MSP = \angle LMS$. However, since $\angle MSP = \angle LSPC$, this translates to requiring $\angle LSPC = \angle LMS$. In other words, we need PN to be the internal angle bisector of $\angle MPC$.