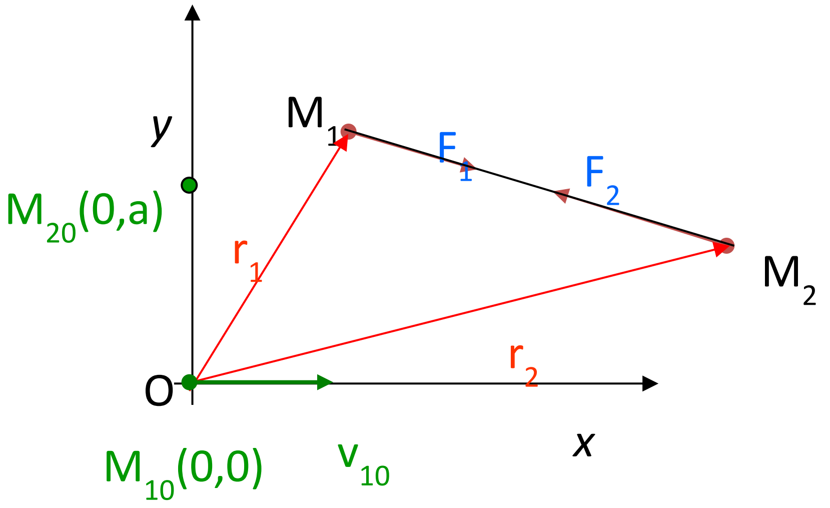
Seminar 12. Dynamics of the systems of material points

1) Two particles M_1 and M_2 with m = 1 attract each other with a force equal to the distance between them the coefficient of proportionality being 1. At t = 0 the point M_1 was in the origin O and had the velocity $v_1 = a\sqrt{2}$ oriented along the Ox axis, while M_2 was on Oy axis having the velocity $v_2 = 0$ and the ordinate *a*. Find the equation of motion of the system formed by M_1 and M_2.



Remark:
The motion takes place in a plane (in Oxy).
Indeed, we can consider the force acting between M1 and M2 as central (for example the center is M2) and then, according to the theory of the central forces, the motion is in a plane.

$$\begin{cases} m_1\ddot{\vec{r}}_1 = \vec{F}_1 = \overrightarrow{M_1M_2} \\ m_2\ddot{\vec{r}}_2 = \vec{F}_2 = \overrightarrow{M_2M_1} = -\overrightarrow{M_1M_2} \end{cases} \tag{1}$$

We preoject equations (1) on the Oxy axes ($m_1 = m_2 = 1$) and we add the initial conditions:

$$\begin{cases} \ddot{x}_1 = x_2 - x_1; \;\; x_1(0) = 0; \;\; \dot{x}_1(0) = a\sqrt{2} \\ \ddot{y}_1 = y_2 - y_1; \;\; y_1(0) = 0; \;\; \dot{y}_1(0) = 0 \\ \ddot{x}_2 = x_1 - x_2; \;\; x_2(0) = 0; \;\; \dot{x}_2(0) = 0 \\ \ddot{y}_2 = y_1 - y_2; \;\; y_2(0) = a; \;\; \dot{y}_2(0) = 0 \end{cases} \tag{2}$$

From (2) we have:

$$\begin{cases} \frac{d^2}{dt^2}(x_1 + x_2) = 0 \\ \frac{d^2}{dt^2}(x_1 - x_2) = -2(x_1 - x_2) \\ \frac{d^2}{dt^2}(y_1 + y_2) = 0 \\ \frac{d^2}{dt^2}(y_1 - y_2) = -2(y_1 - y_2) \end{cases} \tag{3}$$

Integrating (3)we obtain:

$$\begin{cases} x_1 + x_2 = C_1 t + C_2; \quad x_1 - x_2 = C_5 \cos(\sqrt{2}\,t) + C_6 \sin(\sqrt{2}\,t); \\ y_1 + y_2 = C_3 t + C_4; \quad y_1 - y_2 = C_7 \cos(\sqrt{2}\,t) + C_8 \sin(\sqrt{2}\,t); \end{cases} \tag{4}$$

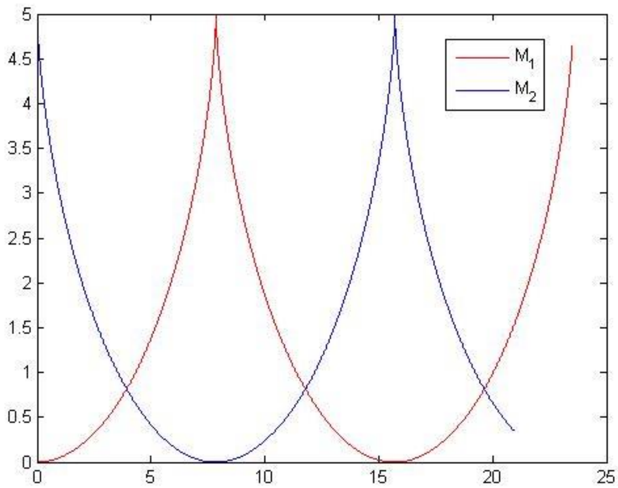
and using the intial conditions (2) one can find the constants of integration C_1,..., C_8:

$$\begin{cases} x_1 + x_2 = a\sqrt{2}\,t; \quad x_1 - x_2 = a \sin(\sqrt{2}\,t); \\ y_1 + y_2 = a; \quad y_1 - y_2 = -a \cos(\sqrt{2}\,t) \end{cases} \tag{5}$$

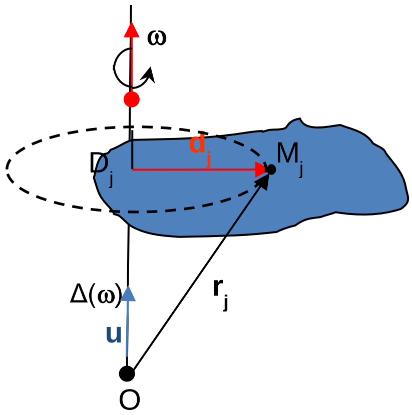
We get:

$$\begin{cases} x_1 = \frac{a}{2}\left[\sqrt{2}\,t + \sin(\sqrt{2}\,t)\right] \\ y_1 = \frac{a}{2}\left[1 - \cos(\sqrt{2}\,t)\right] \end{cases} \qquad \begin{cases} x_2 = \frac{a}{2}\left[\sqrt{2}\,t - \sin(\sqrt{2}\,t)\right] \\ y_2 = \frac{a}{2}\left[1 + \cos(\sqrt{2}\,t)\right] \end{cases} \tag{6}$$

Equations (6) represent two cycloids formed by two diametrically opposed points of a circle that rolls to the line of equation y = a.



2) Find the expression of the kinetic energy of a rotating rigid system (rigid body), (S) about a fixed axis $\Delta(O,\vec{u})$ with an angular velocity $\vec{\omega}$, where $\vec{u}$ is the unit vector of the axis Δ.



Consider the rigid discrete system:

$$(S): M_j(m_j), \vec{r}_j = \overrightarrow{OM}_j, j = 1, \dots, N$$

The velocity $\vec{v}_j$ of the point M_j is given by:

$$\vec{v}_j = \vec{\omega} \times \vec{r}_j \qquad (1)$$

Let be $D_j = pr_\Delta M_j$. Thus, we have

$$\vec{r}_j = \overrightarrow{OD}_j + \vec{d}_j \qquad (2)$$

where $\vec{d}_j = \overrightarrow{D_j M_j}$. Using (1) and (2) we get:

$$\vec{v}_j = \vec{\omega} \times (\overrightarrow{OD}_j + \vec{d}_j) \underbrace{=}_{\vec{\omega} \parallel \overrightarrow{OD}_j} \vec{\omega} \times \vec{d}_j => \vec{v}_j = \vec{\omega} \times \vec{d}_j \qquad (3)$$

Now we can calculate the kinetic energy

$$T = \frac{1}{2}\sum_{j=1}^{N} m_j v_j^2 = \frac{1}{2}\sum_{j=1}^{N} m_j (\vec{\omega} \times \vec{d}_j)^2 = \frac{1}{2}\sum_{j=1}^{N} m_j \omega^2 d_j^2 \underbrace{\sin^2(\vec{\omega}, \vec{d}_j)}_{=1(\vec{\omega} \perp \vec{d}_j)} = \frac{1}{2}\omega^2 \sum_{j=1}^{N} m_j d_j^2 = \frac{1}{2} I(\Delta)\omega^2$$

where

$$I(\Delta) = \sum_{j=1}^{N} m_j d_j^2$$

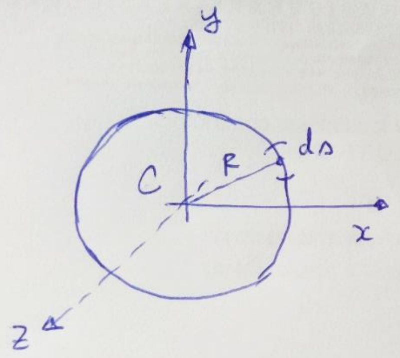
is the moment of inertia of the system (S) with respect to Δ.

Therefore, the kinetic energy of the rigid body rotating about the axis Δ is given by:

$$\boxed{T = \frac{1}{2} I(\Delta)\omega^2}$$

Moments of inertia.

3) Calculating the moment of inertia of a thin ring about the central axis



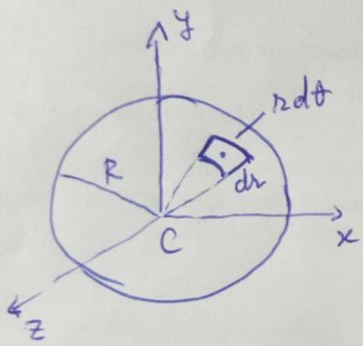
$$I_c = \int_{(\mathcal{C})} R^2 \, dm =$$

$$dm = \rho \, ds = \rho R \, d\theta$$

$$= \rho R^3 \int_0^{2\pi} d\theta = 2\pi \rho R^3 = \underbrace{2\pi R \rho}_{m} \cdot R^2 =$$

$$= mR^2$$

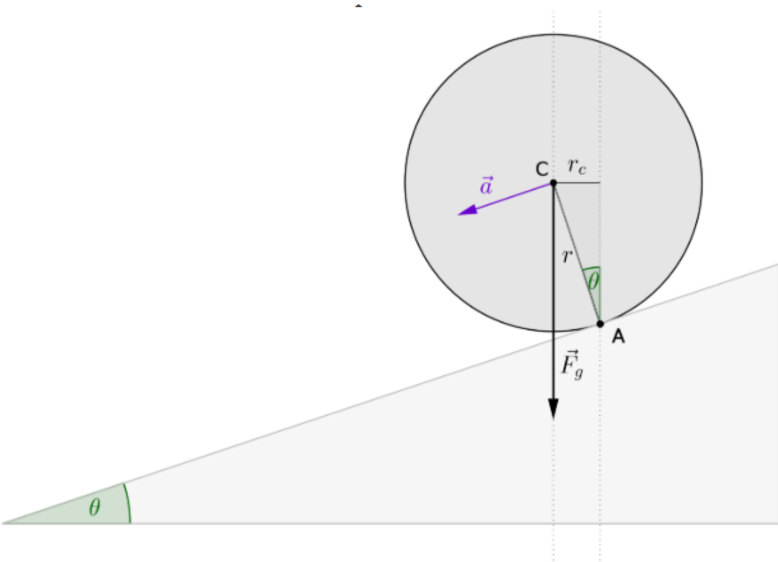
4) Calculating the moment of inertia of a disk about the central axis



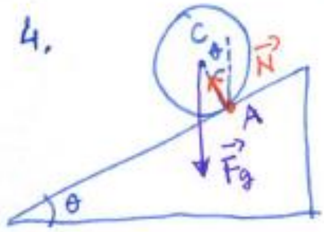
$$I_c = \int_D r^2 \, dm = \rho \int_0^R \int_0^{2\pi} r^3 \, dr \, d\theta = \rho \pi R^2 \cdot \frac{R^2}{2} = \frac{mR^2}{2}$$

$$dm = \rho \cdot d\sigma = \rho r \, dr \, d\theta$$

5. A sphere of mass m and radius r is rolling down a slope of inclination θ without slipping under the action of its own weight $\vec{F}_g = m\vec{g}$. At what rate does the sphere accelerate down the slope?



4.



$$\vec{M}_A = \vec{AC} \times \vec{F_g} + \underbrace{\vec{AA}}_{=0} \times \vec{N}$$

$$M_A = r\,mg \sin\theta$$

$$\frac{d\vec{K}_A}{dt} = \vec{M}_A \quad (*)$$

$$\vec{K}_A = I_A \cdot \vec{\omega} \underset{\text{T. Steiner}}{=} (I_c + mr^2)\vec{\omega} = \left(\frac{2}{5}mr^2 + mr^2\right)\vec{\omega} =$$

$$= \frac{7m}{5} r^2 \vec{\omega}$$

$$(*) \Rightarrow \frac{7}{5} \cancel{m} r^2 \dot{\omega} = r \cancel{m} g \sin\theta \Rightarrow \dot{\omega} = \frac{5}{7}\frac{g}{r}\sin\theta$$

$$\Rightarrow \vec{a}_c = \dot{\vec{\omega}} \times \vec{r} \Rightarrow |\vec{a}_c| = \dot{\omega} r = \frac{5}{7} g \sin\theta$$