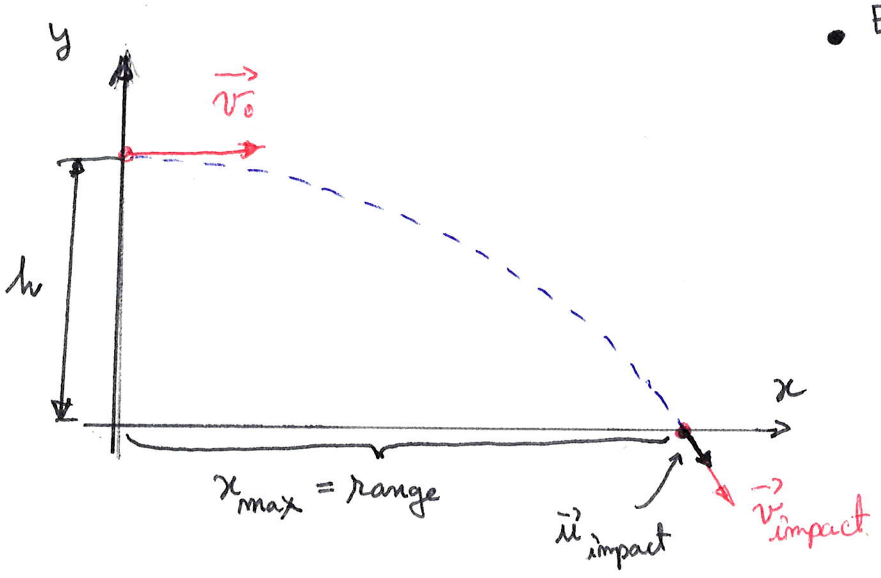
Kinematics of a material point

① The 2D motion of a projectile, launched from a plane in the horizontal direction, is given by:

$$x = v_0 t$$
$$y = h - \frac{gt^2}{2}$$

where v_0 is the initial speed, g is the acceleration due to the gravity and h is the elevation. Find the trajectory, the ground impact speed, the velocity direction at the impact and the range.



- Eqs. of motion:

$$\begin{cases} x = v_0 \cdot t & (1) \\ y = h - \frac{gt^2}{2} & \end{cases}$$

- Trajectory (eliminate time in eq. (2))

$$t = \frac{x}{v_0}$$

$$\boxed{y = h - \frac{g \cdot \frac{x^2}{v_0^2}}{2} = h - \frac{g}{2v_0^2} \cdot x^2} \quad (3)$$

a parabola.

- Velocity: $\vec{v}\,(\dot{x}, \dot{y}) = (v_0, -g \cdot t)$

$\underset{\uparrow \text{ notation}}{\dot{x}} = \frac{dx}{dt}$ (x-dot / overdot); $\ddot{x} = \frac{d^2x}{dt}$ (x-double dot).

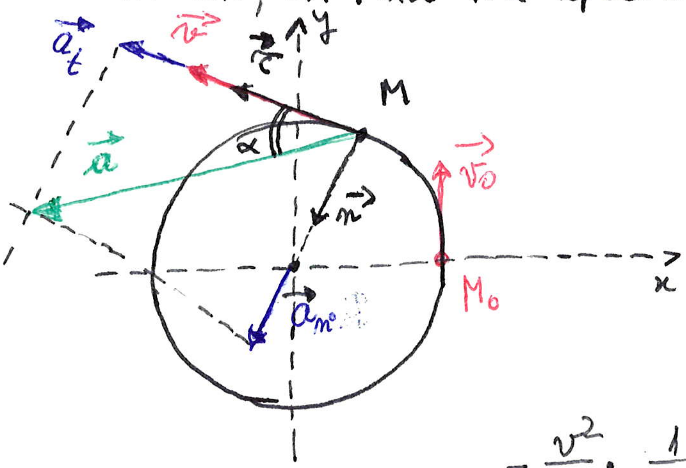
- Ground impact: $\underbrace{\boxed{y = 0}}_{\text{impact condition}}$ $\Rightarrow h - \frac{gt^2}{2} = 0 \Rightarrow t_{impact} = \sqrt{\frac{2h}{g}}$ (descending time)

- Impact velocity & speed: $\vec{v}_{impact} = (v_0, -g \cdot t_{impact}) = (v_0, -\sqrt{2gh})$, $v_{impact} = \sqrt{v_0^2 + 2gh}$.

- Impact direction: $\underset{\text{versor}}{\vec{u}_{impact}} = \frac{\vec{v}_{impact}}{v_{impact}} = \left(\frac{v_0}{\sqrt{v_0^2 + 2gh}}, \frac{-\sqrt{2gh}}{\sqrt{v_0^2 + 2gh}} \right)$

- Range: We use $y = 0$ in (3) (trajectory) $\Rightarrow 0 = h - \frac{g}{2v_0^2} \cdot x^2 \Rightarrow x_{max} = v_0 \sqrt{\frac{2h}{g}}$.

③ A particle moves on a circle of radius R with the initial speed v_0. The angle between the acceleration and the velocity of the particle is constant, α. Find the speed of the particle as a function of time.



Frenel formulas: $\vec{a}_t \parallel \vec{v}$

$$\vec{a} = a_t \cdot \vec{\tau} + a_n \cdot \vec{n}$$

$$tg(\alpha) = (\text{const.}) = \frac{a_n}{a_t} = \frac{\frac{v^2}{R}}{\frac{dv}{dt}} =$$

$$= \frac{v^2}{R} \cdot \frac{1}{\frac{dv}{dt}} \Rightarrow \frac{dv}{dt} = ctg\alpha \cdot \frac{v^2}{R} \Rightarrow$$

$$\Rightarrow \frac{dv}{v^2} = \frac{ctg\alpha}{R} \cdot dt \xrightarrow{\int_0^t} -\left(\frac{1}{v} - \frac{1}{v_0}\right) = \frac{ctg\alpha}{R} \cdot (t - \underset{0}{\underset{\parallel}{t_0}}) \Rightarrow v = \frac{v_0 R}{R - v_0 ctg\alpha \cdot t}.$$

④ Write $\vec{a}$ in the form $\vec{a} = a_\tau \cdot \vec{\tau} + a_n \cdot \vec{n}$ without finding $\vec{\tau}$ and $\vec{n}$. The equation of motion of the particle is:

$$\vec{r}(t) = (t+1) \cdot \vec{i} + 2t \cdot \vec{j} + t^2 \cdot \vec{k}.$$

We have: $\vec{v} = v \cdot \vec{\tau} = \left(\frac{ds}{dt}, 0, 0\right)$; $\vec{a} = \left(\frac{dv}{dt}, \frac{v^2}{R}, 0\right)$; $a^2 = a_\tau^2 + a_n^2$

$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{i} + 2\vec{j} + 2t \cdot \vec{k} \Rightarrow \boxed{v = \sqrt{1+4+4t^2} = \sqrt{5+4t^2}} \quad (1)$$

$$\vec{a} = \frac{d\vec{v}}{dt} = 2\vec{k} \Rightarrow \boxed{a = 2} \quad (2)$$

$$a_\tau = \frac{dv}{dt} = \left(\sqrt{5+4t^2}\right)' = \frac{1}{2} \cdot \frac{8t}{\sqrt{5+4t^2}} = \frac{4t}{\sqrt{5+4t^2}}.$$

$$a_n^2 = a^2 - a_\tau^2 = 4 - \frac{16t^2}{5+4t^2} = \frac{20 + \cancel{16t^2} - \cancel{16t^2}}{5+4t^2} = \frac{20}{5+4t^2}$$

$$a_n = 2\sqrt{\frac{5}{5+4t^2}}$$

Thus, $\vec{a} = 2\sqrt{\frac{5}{5+4t^2}} \cdot \vec{n} + \frac{4t}{\sqrt{5+4t^2}} \cdot \vec{\tau}.$

(2) Find the trajectory of a particle M moving in a plane with the velocity $\vec{v}(k(x+y), k(x-y))$. At the time $t=0$ the particle M was in the position $A(1,-1)$. (k - constant).

Sol.: $\vec{v}(\dot{x}, \dot{y})$

$$\begin{cases} \dot{x} = k(x+y) & (1) \\ \dot{y} = k(x-y) & (2) \end{cases} \overset{\frac{d}{dt}}{\iff} \begin{cases} \ddot{x} = k(\dot{x}+\dot{y}) \\ \dot{y} = k(x-y) \end{cases} \iff \ddot{x} = k(kx+ky+kx-ky)$$

$$\Rightarrow \boxed{\ddot{x} - 2k^2 x = 0} \quad (3)$$ – differential equation, linear, constant coefficients.

The characteristic equation is: $r^2 - 2k^2 = 0 \Rightarrow r_{1,2} = \pm k\sqrt{2}$ (4)

The solution has the form: $x(t) = C_1 e^{k\sqrt{2}t} + C_2 e^{-k\sqrt{2}t}$ (5). $C_1, C_2 \in \mathbb{R}$

Using (1) and (5) we obtain:

$$\underline{y(t)} = \frac{1}{k}\cdot\dot{x}(t) - x(t) = \frac{1}{\cancel{k}}\cdot \cancel{k}\sqrt{2}\left(C_1 e^{k\sqrt{2}t} - C_2 e^{-k\sqrt{2}t}\right) - C_1 e^{k\sqrt{2}t} - C_2 e^{-k\sqrt{2}t} = \underline{(\sqrt{2}-1)C_1 e^{k\sqrt{2}t} - (\sqrt{2}+1)C_2 e^{-k\sqrt{2}t}} \quad (6)$$

In order to find the constants C_1 and C_2 we use the initial cond's.

(7) $t=0: \begin{cases} x(0) = 1 \\ y(0) = -1 \end{cases} \Rightarrow \begin{cases} x(0) = C_1 + C_2 = 1 \\ y(0) = (\sqrt{2}-1)C_1 - (\sqrt{2}+1)C_2 = -1 \end{cases} +$

$$\Rightarrow \begin{cases} C_1 + C_2 = 1 \\ \sqrt{2}(C_1 - C_2) = 0 \end{cases} \Rightarrow C_1 = C_2 = \frac{1}{2} \quad (8)$$

hyperbolic cosine

Thus, we have: $\boxed{x(t)} = \frac{1}{2}\left(e^{\sqrt{2}kt} + e^{-\sqrt{2}kt}\right) = \underline{ch(\sqrt{2}kt)}$ (9)

$\boxed{y(t)} = \frac{\sqrt{2}}{2}\left(e^{\sqrt{2}kt} - e^{-\sqrt{2}kt}\right) - \frac{1}{2}\left(e^{\sqrt{2}kt} + e^{-\sqrt{2}kt}\right) = \underline{\sqrt{2}\, sh(\sqrt{2}kt) - ch(\sqrt{2}kt)}$ (10)

$x(t)$.

• Trajectory (eliminate time t in (9) and 10).

$$\begin{aligned} x(t) &= ch(\sqrt{2}kt) \\ \frac{y(t)+x(t)}{\sqrt{2}} &= sh(\sqrt{2}kt) \end{aligned} \Rightarrow x^2 - \frac{(x+y)^2}{2} = ch^2(\sqrt{2}kt) - sh^2(\sqrt{2}kt) = 1$$

$$\Downarrow$$

$$\boxed{x^2 - y^2 - 2xy = 2}$$ – equilateral hyperbola.