P1. Sketch the curves parameterised by

(a) $\mathbf{r}(t) = \left(t^2, t^3\right), \quad t \in \mathbb{R};$

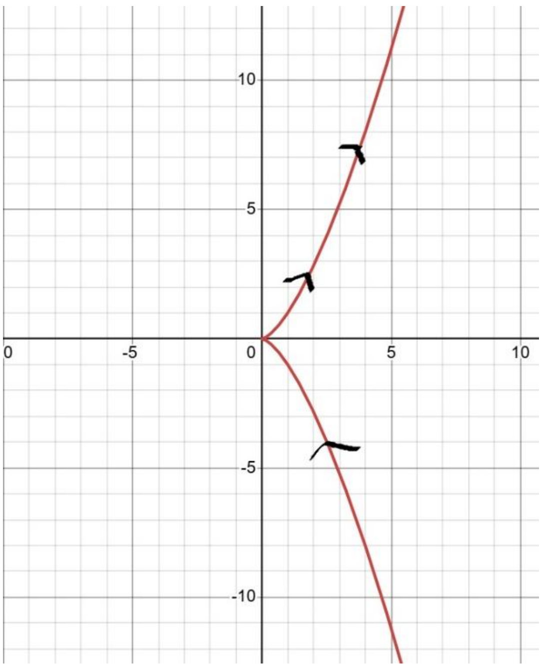
(b) $\mathbf{r}(t) = t\cos t\mathbf{i} + |t|\sin t\mathbf{j}, \quad t \in \mathbb{R};$

(c) $\mathbf{r}(t) = \frac{1+e^{-t}}{1+2e^{-t}}(\cos t, -\sin t), \quad t \in \mathbb{R};$

(d) $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + t^2\mathbf{k}, \quad t \in \mathbb{R}.$

In cases where you cannot show the whole curve, just show a representative portion. Use arrows to indicate the direction in which t increases. These sketches must be accompanied by brief explanations (and a few words/calculations) justifying their shapes; For each curve state, with justification, whether or not it is embedded and whether or not it is closed.

a.) r(t)=(t^2, t^3), t in R;
We can use the graph of the function f:R->R, f(x)=x^3/2 for x>=0.
And the function g:R->R, g(x)=-x^3/2 for x<0.



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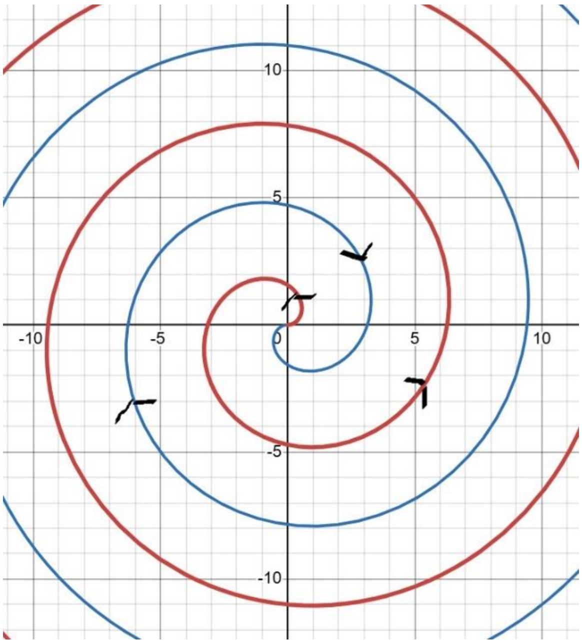
A curve is **embedded** if it doesn't have self-intersecting points. We can also use the fact that r(t) is injective to prove that it is embedded.
Let t1, t2 be real numbers such that t1=/t2. Suppose that r(t1)=r(t2) =>
(t1^2, t1^3)=(t2^2,t2^3) => t1^2=t2^2 and t1^3=t2^3 have to be true at the same time, which can only happen if t1=t2. So we have a contradiction and r(t) is injective, therefore the curve is embedded.
A curve is **closed** if its endpoints meet. In our case the endpoints are -infinity and infinity, which never meet, so the curve is **not closed**.

b.) r(t)=tcos(t)**i**+|t|sin(t)**j,** t in R.
We have x= tcos(t) and y=|t|sin(t). We notice that if we square these two equations and add them together, we get the equation of a circle with radius t, where the radius is changing: x^2+y^2=t^2
So as t increases or decreases, the circles get smaller or larger, and we will have a spiral.



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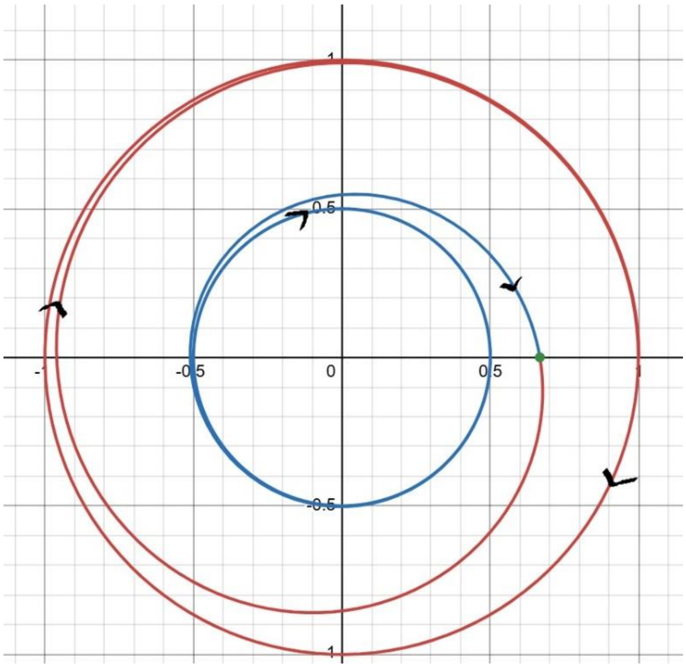
Now similarly to a.), we have to prove that the function r(t)=tcos(t)**i**+|t|sin(t)**j,** which also means that the curve is **embedded**.
Let's assume that r(t1)=r(t2). So we have t1cos(t1)=t2cos(t2) and |t1|sin(t1)= |t2|sin(t2) at the same time.
We get t1=t2cos(t2)/cos(t1), which implies that t1=t2. Thus r(t1)=r(t2) only if t1=t2, which means the function is injective.
Because the spiral keeps growing and approaching infinity, we **do not have a closed curve**.

c.) r(t)=(1+e^-t)/(1+2e^-t)(cost, -sint), t in R.
We have the parametric equations: x=(1+e^-t)/(1+2e^-t)cost and y=(1+e^-t)/(1+2e^-t)(-sint).
Again by squaring them and adding them up we get an equation similar to that of a circle, where the radius R=(1+e^-t)/(1+2e^-t) depends on t.



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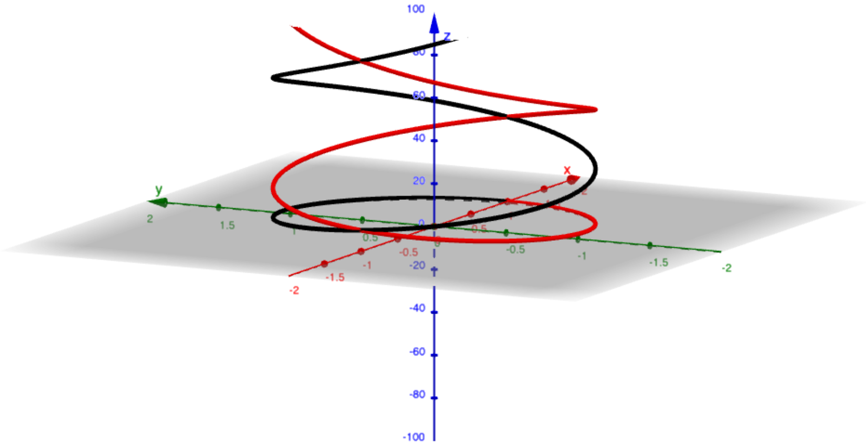
This curve is also **embedded**. Suppose that t1=/t2, but r(t1)=r(t2). This is equivalent to proving that the function f(t)= (1+e^-t)/(1+2e^-t) is injective. So we have (1+e^-t1)/(1+2e^-t1)= (1+e^-t2)/(1+2e^-t2). After some basic computations we get e^-t1+2e^-t2= e^-t2+2e^-t1 => e^-t1=e^-t2, and we know that the exponential function is bijective.

This curve **isn't closed** either as the two endpoints don't meet.

d.) r(t)=cos(t)**i**+sin(t)**j**+t^2**k**, t in R.
We have the parametric equations x=cost, y=sint and z=t^2.



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The curve is **not embedded** because it has self-intersection points which we can find.

Let t1=/t2, we show that r(t1) can be equal to r(t2), with the property that t1^2=t2^2, cos(t1)=cos(t2) and sin(t1)=sin(t2).
From t1^2=t2^2 we have t1=-t2.
Cos(-t2)=cos(t2) is true because it is an even function.
Sin(t1)=sin(t2) is only true if it is equal to 0. So t=kpi, k in Z, are the points where the curve self-intersects.
Just like the rest, this curve also **isn’t closed**, even though it has self-intersection points, it never ‘stops’, but the two endpoints keep approaching infinity along the Oz axis.