COURSES 12+13

Bases. Dimension

Let $(K,+,\cdot)$ be a field.

Theorem 1. Let V and V' be vector spaces over K. Then

$$V \simeq V' \Leftrightarrow \dim V = \dim V' .$$

Proof.
$\Rightarrow$. Let $f : V \to V'$ be a K-isomorphism. If $(v_1,\dots,v_n)$ is a basis of V, then $(f(v_1),\dots,f(v_n))$ is a basis of V'. Hence $\dim V = \dim V'$.
$\Leftarrow$. Assume that $\dim V = \dim V' = n$. Let $B = (v_1,\dots,v_n)$ and $B' = (v'_1,\dots,v'_n)$ be bases of V and V' respectively. We know that a K-linear map $f : V \to V'$ is determined by its values on the vectors of the basis B. Define $f(v_i) = v'_i$, for any $i \in \{1,\dots,n\}$. Then it is easy to check that f is a K-isomorphism. □

Corollary 2. Any vector space V over K with $\dim V = n (\in \mathbb{N}^*)$ is isomorphic to the canonical vector space K^n over K.

Remark 3. Corollary 2 is a very important structure result, saying that, up to an isomorphism, any finite dimensional vector space over K is, actually, the canonical vector space K^n over K. Thus, we have an explanation why we have used so often this kind of vector spaces: not only because the operations are very nice and easily defined, but they are, up to an isomorphism, the only types of finite dimensional vector spaces.

We end this section with some important formulas involving vector space dimension.

Theorem 4. Let $f : V \to V'$ be a K-linear map. Then

$$\dim V = \dim(\mathrm{Ker} f) + \dim(\mathrm{Im} f) .$$

Proof. (optional)
Let $(u_1,\dots,u_m)$ be a basis of the subspace $\mathrm{Ker} f$ of V. It can be completed to a basis $B = (u_1,\dots,u_m,v_{m+1},\dots,v_n)$ of V. We are going to prove that $B' = (f(v_{m+1}),\dots,f(v_n))$ is a basis of $\mathrm{Im} f$.

First, we prove that B' is linearly independent in $\mathrm{Im} f$. Let us take $k_{m+1},\dots,k_n \in K$. By the K-linearity of f we have:

$$\sum_{i=m+1}^{n} k_i f(v_i) = 0 \Rightarrow f\Big(\sum_{i=m+1}^{n} k_i v_i\Big) = 0 \Rightarrow \sum_{i=m+1}^{n} k_i v_i \in \mathrm{Ker} f .$$

Since $(u_1,\dots,u_m)$ is a basis of $\mathrm{Ker} f$, there exist $k_1,\dots,k_m \in K$ such that

$$\sum_{i=m+1}^{n} k_i v_i = \sum_{i=1}^{m} k_i u_i ,$$

that is,

$$\sum_{i=1}^{m} k_i u_i - \sum_{i=m+1}^{n} k_i v_i = 0\,.$$

But $B = (u_1, \ldots, u_m, v_{m+1}, \ldots, v_n)$ is a basis of V, hence it follows that $k_i = 0$, for any $i \in \{1, \ldots, n\}$. Therefore, B' is linearly independent in $\mathrm{Im} f$.

Let us now show that B' is a system of generators for $\mathrm{Im} f$. Let $v' \in \mathrm{Im} f$. Then $v' = f(v)$ for some $v \in V$. Since B is a basis of V, there exist $k_1, \ldots, k_n \in K$ such that

$$v = \sum_{i=1}^{m} k_i u_i + \sum_{i=m+1}^{n} k_i v_i\,.$$

By the K-linearity of f and the fact that $u_1, \ldots, u_m \in \mathrm{Ker} f$, it follows that

$$v' = f(v) = f\Big(\sum_{i=1}^{m} k_i u_i + \sum_{i=m+1}^{n} k_i v_i\Big) = \sum_{i=1}^{m} k_i f(u_i) + \sum_{i=m+1}^{n} k_i f(v_i) = \sum_{i=m+1}^{n} k_i f(v_i)\,.$$

Hence B' is a system of generators for $\mathrm{Im} f$.

Therefore, B' is a basis of $\mathrm{Im} f$ and consequently,

$$\dim V = n = m + (n-m) = \dim(\mathrm{Ker} f) + \dim(\mathrm{Im} f)\,.$$

$\square$

Corollary 5. a) Let V be a K-vector space and let S, T be subspaces of V. Then

$$\dim S + \dim T = \dim(S \cap T) + \dim(S + T).$$

Indeed, $f : S \times T \to S + T$, $f(x, y) = x - y$ is a surjective linear map with the kernel $\mathrm{Ker}\, f = \{(x, x) \mid x \in S \cap T\}$. Hence,

$$\dim(S \times T) = \dim(\mathrm{Ker}\, f) + \dim(S + T).$$

Since $g : S \cap T \to \mathrm{Ker}\, f$, $g(x) = (x, x)$ is an isomorphism, we have

$$\dim(\mathrm{Ker}\, f) = \dim(S \cap T),$$

and by a previous course example we have $\dim(S \times T) = \dim S + \dim T$, which completes the proof of the statement.

b) If V is a K-vector space and $S, T \leq_K V$, then

$$\dim(S + T) = \dim S + \dim T \Leftrightarrow S + T = S \oplus T.$$

c) Let V be a K-vector space and $f \in End_K(V)$. The following statements are equivalent:

(i) f is injective;

(ii) f is surjective;

(iii) f is bijective.

Of course, it is enough to show that (i)$\Leftrightarrow$ (ii).

(i)$\Rightarrow$(ii) If f is injective, then $\mathrm{Ker} f = \{0\}$, hence $\dim(\mathrm{Ker} f) = 0$. By Theorem 4, it follows that $\dim(\mathrm{Im} f) = \dim V$. But $\mathrm{Im} f \leq_K V$, so $\mathrm{Im} f = V$.

(ii)$\Rightarrow$(i) Let us assume that f is surjective. Since $\mathrm{Im} f = V$, it follows by Theorem 4 that $\dim(\mathrm{Ker} f) = 0$, whence $\mathrm{Ker} f = \{0\}$. Thus f is injective.

The matrix of a linear map

First, we define the matrix of a vector in a basis of a vector space. For certain reasons, it is presented as a column-matrix, but it must be said that this is rather a convention than a constraint. But if one changes the convention, the form of the next notions and results must be properly changed.

Let K be a field.

Definition 6. Let V be a K-vector space, $v \in V$ and $B = (v_1, \dots, v_n)$ a basis of V. If

$$v = k_1 v_1 + \dots + k_n v_n \ (k_1, \dots, k_n \in K)$$

is the unique representation of v as a linear combination of the vectors of B, then the **matrix of the vector** v in the basis B is

$$[v]_B = \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix}$$

Definition 7. Let $f : V \to V'$ be a K-linear map, let $B = (v_1, \dots, v_n)$ be a basis of V and let $B' = (v'_1, \dots, v'_m)$ be a basis of V'. Then we can uniquely write the vectors of $f(B)$ as linear combinations of the vectors of B', i.e. there exist $a_{ij} \in K$ ($i \in \{1, \dots, m\}$, $j \in \{1, \dots, n\}$) uniquely determined such that

$$\begin{cases} f(v_1) = a_{11} v'_1 + a_{21} v'_2 + \dots + a_{m1} v'_m \\ f(v_2) = a_{12} v'_1 + a_{22} v'_2 + \dots + a_{m2} v'_m \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ f(v_n) = a_{1n} v'_1 + a_{2n} v'_2 + \dots + a_{mn} v'_m \end{cases}$$

Then the **matrix of the K-linear map** f in the pair of bases (B, B') (or, simply, in the bases B and B') is the matrix whose columns consist of the coordinates of the vectors of $f(B)$ in the basis B', that is,

$$[f]_{BB'} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

For $V = V'$ and $B = B'$, we denote $[f]_B = [f]_{BB}$ and we call it the **matrix of** f in the basis B.

Remarks 8. (1) We complete the matrix of a linear map by columns. This is also a part of the convention we mentioned at the beginning of this section.
(2) As we will see next, the matrix of a linear map depends on the map, on the considered bases, but also by the order of the elements in each basis.

Examples 9. a) For any n-dimensional K-vector space V and any basis B of V, we have

$$[1_V]_B = I_n.$$

b) Consider the $\mathbb{R}$-linear map $f : \mathbb{R}^4 \to \mathbb{R}^3$ defined by

$$f(x, y, z, t) = (x + y + z, y + z + t, z + t + x)\,,\ \forall (x, y, z, t) \in \mathbb{R}^4\,.$$

Let $E=(e_1,e_2,e_3,e_4)$ and $E'=(e'_1,e'_2,e'_3)$ be the standard bases in $\mathbb{R}^4$ and $\mathbb{R}^3$ respectively. Since

$$\begin{cases} f(e_1)=f(1,0,0,0)=(1,0,1)=e'_1+e'_3\\ f(e_2)=f(0,1,0,0)=(1,1,0)=e'_1+e'_2\\ f(e_3)=f(0,0,1,0)=(1,1,1)=e'_1+e'_2+e'_3\\ f(e_4)=f(0,0,0,1)=(0,1,1)=e'_2+e'_3\end{cases}$$

it follows that the matrix of f in the bases E and E' is

$$[f]_{EE'}=\begin{pmatrix}1&1&1&0\\0&1&1&1\\1&0&1&1\end{pmatrix}.$$

c) Let $\mathbb{R}_n[X]$ be the $\mathbb{R}$ - vector space of the polynomials with the degree at most n and real coefficients. The map

$$\varphi:\mathbb{R}_3[X]\to\mathbb{R}_2[X],\ \varphi(a_0+a_1X+a_2X^2+a_3X^3)=a_1+2a_2X+3a_3X^2$$

(which associates a polynomial f its formal derivative f') is a linear map. Let us write the matrix of φ in the pair of basis $B=(1,X,X^2,X^3)$, $B'=(1,X,X^2)$, and then in the pair of basis $B=(1,X,X^2,X^3)$, $B''=(X^2,1,X)$. We have

$$\begin{aligned}\varphi(1)&=0\cdot1+0\cdot X+0\cdot X^2=0\cdot X^2+0\cdot1+0\cdot X\\ \varphi(X)&=1\cdot1+0\cdot X+0\cdot X^2=0\cdot X^2+1\cdot1+0\cdot X\\ \varphi(X^2)&=0\cdot1+2\cdot X+0\cdot X^2=0\cdot X^2+0\cdot1+2\cdot X\\ \varphi(X^3)&=0\cdot1+0\cdot X+3\cdot X^2=3\cdot X^2+0\cdot1+0\cdot X\end{aligned}$$

thus,

$$[\varphi]_{BB'}=\begin{pmatrix}0&1&0&0\\0&0&2&0\\0&0&0&3\end{pmatrix}\ \text{and}\ [\varphi]_{BB''}=\begin{pmatrix}0&0&0&3\\0&1&0&0\\0&0&2&0\end{pmatrix}.$$

d) Let K be a field, $m,n\in\mathbb{N}^*$, $A\in M_{m,n}(K)$, E the standard basis of K^n and E' the standard basis of K^m. Then

$$f_A:K^n\to K^m,\ f_A(x_1,\dots,x_n)=A\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}$$

is a linear map and $[f_A]_{EE'}=A$.

Theorem 10. Let $f:V\to V'$ be a K-linear map, $B=(v_1,\dots,v_n)$ a basis of V, $B'=(v'_1,\dots,v'_m)$ a basis of V' and $v\in V$. Then

$$[f(v)]_{B'}=[f]_{BB'}\cdot[v]_B\,.$$

Proof. Let $[f]_{BB'}=(a_{ij})\in M_{mn}(K)$. Let $v=\sum_{j=1}^n k_jv_j$ and $f(v)=\sum_{i=1}^m k'_iv'_i$ with $k_i,k'_i\in K$. On the other hand, using the definition of the matrix of f in the bases B and B', we have

$$f(v)=f\Big(\sum_{j=1}^n k_jv_j\Big)=\sum_{j=1}^n k_jf(v_j)=\sum_{j=1}^n k_j\Big(\sum_{i=1}^m a_{ij}v'_i\Big)=\sum_{i=1}^m\Big(\sum_{j=1}^n a_{ij}k_j\Big)v'_i\,.$$

But there is only one way to write $f(v)$ as a linear combination of the vectors of the basis B', hence we have

$$k'_i = \sum_{j=1}^{n} a_{ij} k_j, \forall i \in \{1, \dots, m\}.$$

Therefore, $[f(v)]_{B'} = [f]_{BB'} \cdot [v]_B$. □

For a K-linear map $f : V \to V'$ the dimension $\dim(\mathrm{Im} f)$ is also called **the rank of** f. We denote it by $\mathrm{rank}(f)$. The rank of a linear map and the rank of its matrix in a pair of bases are strongly connected.

Theorem 11. Let $f : V \to V'$ be a K-linear map. Then

$$\mathrm{rank}(f) = \mathrm{rank}[f]_{BB'} ,$$

where B and B' are arbitrary bases of V and V' respectively.

Proof. Let $B = (v_1, \dots, v_n)$ and $[f]_{BB'} = A$. We have

$$\mathrm{rank}(f) = \dim(\mathrm{Im} f) = \dim f(V) = \dim f(\langle v_1, \dots, v_n \rangle) = \dim\langle f(v_1), \dots, f(v_n)\rangle =$$

$$= \mathrm{rank}(\ {}^tA) = \mathrm{rank}(A) = \mathrm{rank}[f]_{BB'} .$$

□

Remark 12. The matrices of a linear map in different pairs of bases have the same rank.

We continue this section by presenting one of the key results in Linear Algebra, connecting linear maps and matrices.

Theorem 13. Let V, V' and V'' be vector spaces over K with $\dim V = n$, $\dim V' = m$ and $\dim V'' = p$ and let B, B' and B'' be bases of V, V' and V'' respectively. If $f, g \in Hom_K(V, V')$, $h \in Hom_K(V', V'')$ and $k \in K$, then

$$[f + g]_{BB'} = [f]_{BB'} + [g]_{BB'} ,\ [kf]_{BB'} = k \cdot [f]_{BB'} ,$$

$$[h \circ f]_{BB''} = [h]_{B'B''} \cdot [f]_{BB'} .$$

Proof. If $[f]_{BB'} = (a_{ij}) \in M_{mn}(K)$, $[g]_{BB'} = (b_{ij}) \in M_{mn}(K)$ and $[h]_{B'B''} = (c_{ki}) \in M_{pm}(K)$ then

$$f(v_j) = \sum_{i=1}^{m} a_{ij} v'_i , \quad g(v_j) = \sum_{i=1}^{m} b_{ij} v'_i , \quad h(v'_i) = \sum_{k=1}^{p} c_{ki} v''_k$$

for any $j \in \{1, \dots, n\}$ and for any $i \in \{1, \dots, m\}$.

Then for any $k \in K$ and for any $j \in \{1, \dots, n\}$ we have

$$(f + g)(v_j) = f(v_j) + g(v_j) = \sum_{i=1}^{m} a_{ij} v'_i + \sum_{i=1}^{m} b_{ij} v'_i = \sum_{i=1}^{m} (a_{ij} + b_{ij}) v'_i ,$$

$$(kf)(v_j) = kf(v_j) = k \cdot \left(\sum_{i=1}^{m} a_{ij} v'_i \right) = \sum_{i=1}^{m} (ka_{ij}) v'_i ,$$

hence $[f + g]_{BB'} = [f]_{BB'} + [g]_{BB'}$ and $[kf]_{BB'} = k \cdot [f]_{BB'}$.

Finally, for any $j \in \{1, \dots, n\}$ we have

$$(h \circ f)(v_j) = h(f(v_j)) = h\left(\sum_{i=1}^{m} a_{ij} v'_i\right) = \sum_{i=1}^{m} a_{ij} h(v'_i) = \sum_{i=1}^{m} a_{ij}\left(\sum_{k=1}^{p} c_{ki} v''_k\right) =$$

$$= \sum_{k=1}^{p}\sum_{i=1}^{m} (c_{ki} a_{ij}) v''_k = \sum_{k=1}^{p}\left(\sum_{i=1}^{m} c_{ki} a_{ij}\right) v''_k \,,$$

hence $[h \circ f]_{BB''} = [h]_{B'B''} \cdot [f]_{BB'}$. □

Theorem 14. Let V and V' be vector spaces over K with $\dim V = n$ and $\dim V' = m$ and let B and B' be bases of V and V' respectively. Then the map $\varphi : Hom_K(V, V') \to M_{mn}(K)$ defined by

$$\varphi(f) = [f]_{BB'} \,,\ \forall f \in Hom_K(V, V')$$

is an isomorphism of vector spaces.

Proof. Let us prove first that φ is bijective.

Let $f, g \in Hom_K(V, V')$ such that $\varphi(f) = \varphi(g)$. Then $[f]_{BB'} = [g]_{BB'} = (a_{ij})$ and

$$f(v_j) = a_{1j} v'_1 + a_{2j} v'_2 + \dots + a_{mj} v'_m = g(v_j) \,,\ \forall j \in \{1, \dots, n\}.$$

Then $f = g$ by the universal property of vector spaces. Thus, φ is injective.

Now let $A = (a_{ij}) \in M_{mn}(K)$, seen as a list of column-vectors $(a^1, \dots, a^n)$, where $a^j = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix}$.
Consider $B = (v_1, \dots, v_n)$ and $B' = (v'_1, \dots, v'_m)$ and consider the K-linear map $f : V \to V'$ defined on the basis B of V by

$$f(v_j) = a_{1j} v'_1 + \dots + a_{mj} v'_m \,,\ \forall j \in \{1, \dots, n\}.$$

Then

$$\varphi(f) = [f]_{BB'} = (a_{ij}) = A \,.$$

Thus, φ is surjective.

The proof is completed by Theorem 13. □

Remark 15. The extremely important isomorphism given in Theorem 14 allows us to work with matrices instead of linear maps, which is much simpler from a computational point of view.

As we previously saw, $(End_K(V), +, \circ)$ is a unitary ring.

Theorem 16. Let V be a vector space over K with $\dim V = n$ and let B be a basis of V. Then the map $\varphi : End_K(V) \to M_n(K)$ defined by

$$\varphi(f) = [f]_B \,,\ \forall f \in End_K(V)$$

is an isomorphism of vector spaces and of rings.

Proof. It follows by Theorem 13 and Theorem 14. □

Corollary 17. Let V be a K-vector space, B an arbitrary basis of V and $f \in End_K(V)$. Then

$$f \in Aut_K(V) \Leftrightarrow \det[f]_B \neq 0\,.$$

Indeed, $f \in Aut_K(V)$ (i.e. f is a unit in the ring $(End_K(V), +, \circ)$) if and only if $[f]_B$ is a unit in $(M_n(K), +, \cdot)$ which means that $\det[f]_B \neq 0$.

Definition 18. Let $f \in End_K(V)$ and let $B = (v_1, \ldots, v_n)$ and $B' = (v_1', \ldots, v_n')$ be bases of V. Then we can write

$$\begin{cases} v_1' = t_{11}v_1 + t_{21}v_2 + \cdots + t_{n1}v_n \\ v_2' = t_{12}v_1 + t_{22}v_2 + \cdots + t_{n2}v_n \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ v_n' = t_{1n}v_1 + t_{2n}v_2 + \cdots + t_{nn}v_n \end{cases}$$

for some $t_{ij} \in K$. Then the matrix $(t_{ij}) \in M_n(K)$, having as columns the coordinates of the vectors of the basis B' in the basis B, is called the **transition matrix from** B **to** B' and is denoted by $T_{BB'}$.

Remarks 19. 1) Sometimes the basis B is referred to as the "old" basis and the basis B' is referred to as the "new" basis.
2) The j-th column of $T_{BB'}$ $(j = 1, \cdots, n)$ consists of the coordinates of $v_j' = 1_V(v_j')$ in the basis B, hence $T_{BB'} = [1_V]_{B'B}$.

Theorem 20. Let $B = (v_1, \ldots, v_n)$ and $B' = (v_1', \ldots, v_n')$ be bases of V. Then the transition matrix $T_{BB'}$ is invertible and its inverse is the transition matrix $T_{B'B}$.

Proof. Since $T = T_{BB'}$ is the transition matrix from the basis B to the basis B' we have

$$v_j' = \sum_{i=1}^{n} t_{ij}v_i\,,\ \forall j \in \{1, \ldots, n\}.$$

Denote $S = (s_{ij}) \in M_{mn}(K)$ the transition matrix from the basis B' to the basis B. Then

$$v_i = \sum_{k=1}^{n} s_{ki}v_k'\,,\ \forall i \in \{1, \ldots, n\}.$$

It follows that

$$v_j' = \sum_{i=1}^{n} t_{ij}\left(\sum_{k=1}^{n} s_{ki}v_k'\right) = \sum_{k=1}^{n}\left(\sum_{i=1}^{n} s_{ki}t_{ij}\right)v_k'\,.$$

By the uniqueness of writing of each v_j' as linear combination of the vectors of the basis B', it follows that

$$\sum_{i=1}^{n} s_{ki}t_{ij} = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases},$$

that is, $S \cdot T = I_n$.

Similarly, one can show that $T \cdot S = I_n$. Thus, T is invertible and its inverse is S. □

Theorem 21. Let $B = (v_1, \ldots, v_n)$ and $B' = (v_1', \ldots, v_n')$ be bases of V and let $v \in V$. Then

$$[v]_B = T_{BB'} \cdot [v]_{B'}\,.$$

Proof. Let $v \in V$ and let us write v in the two bases B and B'. Then

$$v = \sum_{i=1}^{n} k_i v_i \text{ and } v = \sum_{j=1}^{n} k'_j v'_j$$

for some $k_i, k'_j \in K$. Since $T_{BB'} = (t_{ij}) \in M_n(K)$, we have

$$v'_j = \sum_{i=1}^{n} t_{ij} v_i \,,\ \forall j \in \{1, \ldots, n\}.$$

It follows that

$$v = \sum_{j=1}^{n} k'_j \left(\sum_{i=1}^{n} t_{ij} v_i \right) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} t_{ij} k'_j \right) v_i \,.$$

By the uniqueness of writing of v as a linear combination of the vectors of B, it follows that

$$k_i = \sum_{j=1}^{n} t_{ij} k'_j \,,$$

hence $[v]_B = T_{BB'} \cdot [v]_{B'}$. □

Remark 22. Usually, we are interested in computing the coordinates of a vector v in the new basis B', knowing the coordinates of the same vector v in the old basis B and the transition matrix from B to B'. Then by Theorem 21, we have

$$[v]_{B'} = T_{BB'}^{-1} \cdot [v]_B = T_{B'B} \cdot [v]_B \,.$$

Theorem 23. Let $f \in Hom_K(V, V')$, let B_1 and B_2 be bases of V and let B'_1 and B'_2 be bases of V'. Then

$$[f]_{B_2 B'_2} = T_{B'_1 B'_2}^{-1} \cdot [f]_{B_1 B'_1} \cdot T_{B_1 B_2} \,.$$

Proof. We have $T_{B_1 B_2} = [1_V]_{B_2 B_1}$ and $T_{B'_1 B'_2} = [1_{V'}]_{B'_2 B'_1}$ (see Remark 19 2)). Of course, we also vave $T_{B'_1 B'_2}^{-1} = [1_{V'}]_{B'_1 B'_2}$ and by applying Theorem 13 to the equality $f = 1_{V'} \circ f \circ 1_V$ we get

$$[f]_{B_2 B'_2} = [1_{V'}]_{B'_1 B'_2} \cdot [f]_{B_1 B'_1} \cdot [1_V]_{B_2 B_1} = T_{B'_1 B'_2}^{-1} \cdot [f]_{B_1 B'_1} \cdot T_{B_1 B_2} \,,$$

hence the expected conclusion. □

If we take $V' = V$, $B_1 = B'_1$ and $B_2 = B'_2$, we deduce:

Corollary 24. Let $f \in End_K(V)$ and let B_1 and B_2 be bases of V. Then

$$[f]_{B_2} = T_{B_1 B_2}^{-1} \cdot [f]_{B_1} \cdot T_{B_1 B_2} \,.$$