

Analytic Geometry

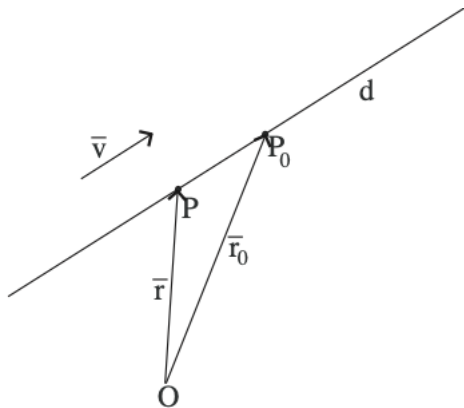
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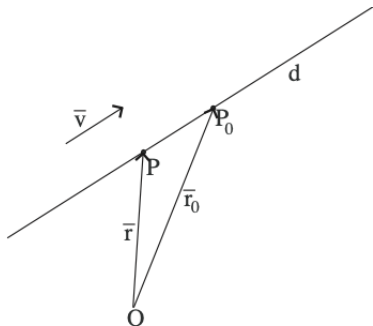
The line in the plane. Several forms of its equation

The vector language can be used to “describe” a line in a short form. Let d be a line passing through a fixed point P_0 and parallel to the fixed vector \bar{v} (**director vector**). Fixing an arbitrary point O in the plane, one can characterize any point P by its *position vector*, i.e. the vector having the original point O and the terminal point P .



The point P belongs to the line d if and only if the vectors $\overline{P_0P}$ and \bar{v} are linearly dependent. This means that there exists $t \in \mathbb{R}$, such that $\overline{P_0P} = t\bar{v}$. But $\overline{P_0P} = \overline{OP} - \overline{OP_0} = \bar{r} - \bar{r}_0$, hence $t\bar{v} = \bar{r} - \bar{r}_0$, and the *vector equation* of the line passing through P_0 and of director vector \bar{v} is

$$\bar{r} = \bar{r}_0 + t\bar{v}. \quad (1)$$

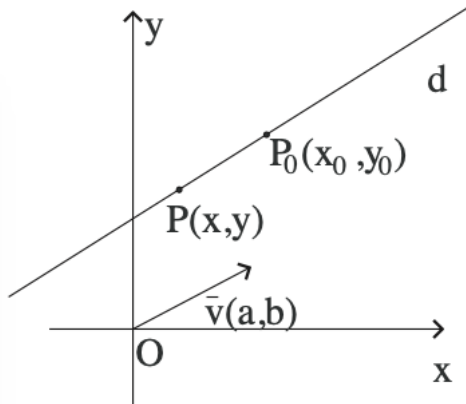


A glimpse in the future. The line as an affine space

Another glimpse in the future. The plane as an affine space

Parametric equation of a line

A line d can be determined by specifying a point $P_0(x_0, y_0)$ on the line and a nonzero vector $\bar{v}(a, b)$, parallel to the line (the direction of the line).



In the diagram above, we assume that $O(0,0)$ is the origin. Let us write the vector equation of the line d .

The line d in a 2-space, passing through the point $P_0(x_0, y_0)$ and parallel to the nonzero vector $\bar{v}(a, b)$ has the *parametric equations*

$$d : \begin{cases} x = x_0 + at \\ y = y_0 + bt \end{cases} \quad t \in \mathbb{R}. \quad (2)$$

The symmetric equation of a line

Starting with the parametric equations,

$$d : \begin{cases} x = x_0 + at \\ y = y_0 + bt \end{cases} \quad t \in \mathbb{R}, \quad (3)$$

if $a, b \neq 0$ then, by expressing t from both equations, we see that

$$d : \frac{x - x_0}{a} = \frac{y - y_0}{b}. \quad (4)$$

This is called the “symmetric equation” of a line and $\vec{v}(a, b)$ is the director vector of the line d .

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What happens if a or b are zero?

A simple computation shows that (4) can be written in the form

$$Ax + By + C = 0, \quad \text{with } A^2 + B^2 \neq 0, \quad (5)$$

meaning that any line from the 2-space is characterized by a first degree equation. Suppose WLOG that $A \neq 0$. Then conversely, such of an equation represents a line, since the formula (5) is equivalent to

$$\frac{x + \frac{C}{A}}{-\frac{B}{A}} = \frac{y}{1} \text{ and this is the symmetric equation of the line passing through } P_0 \left(-\frac{C}{A}, 0 \right) \text{ and parallel to } \bar{v} \left(-\frac{B}{A}, 1 \right).$$

The equation (5) is called *general equation* of the line.

Given points $P_1, P_2 \in d$, how do we write the general equation of d ?

Reduced equation of lines

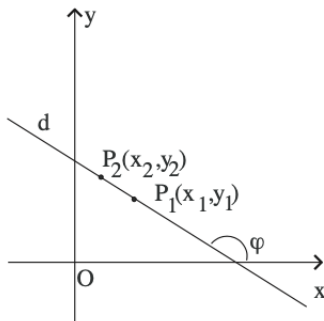
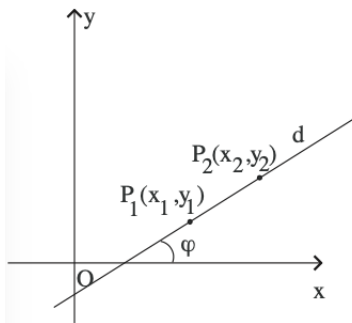
Consider a line given by its general equation $Ax + By + C = 0$, where A or B is nonzero. One may suppose that $B \neq 0$, so that the equation can be divided by B . One obtains

$$y = mx + n \quad (6)$$

which is said to be the *reduced equation* of the line.

Remark: If $B = 0$, then the general equation is $Ax + C = 0$, or $x = -\frac{C}{A}$, a line parallel to Oy . (In the same way, if $A = 0$, one obtains the equation of a line parallel to Ox).

Let d be a line of equation $y = mx + n$ in a Cartesian system of coordinates and suppose that the line is not parallel to Oy . Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two different points on d and φ be the angle determined by d and Ox ; $\varphi \in [0, \pi] \setminus \{\frac{\pi}{2}\}$.



The points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ belong to d , hence

$\begin{cases} y_1 = mx_1 + n \\ y_2 = mx_2 + n \end{cases}$, and $x_2 \neq x_1$, since d is not parallel to Oy . Then,

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \tan \varphi. \quad (7)$$

The number $m = \tan \varphi$ is called the *angular coefficient* (or slope) of the line d .

It is immediate that the equation of the line passing through the point $P_0(x_0, y_0)$ and of the given angular coefficient m is

$$y - y_0 = m(x - x_0). \quad (8)$$

Line determined by two points

A line can be uniquely determined by two distinct points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ on the line. The line can be seen to be the line passing through the point $P_1(x_1, y_1)$ and having $\overline{P_1P_2}(x_2 - x_1, y_2 - y_1)$ as director vector, therefore its equation is

$$d : \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}. \quad (9)$$

The equation (9) can be put in the form

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0. \quad (10)$$

Given three points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$, they are collinear if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

Take home!

We saw the following ways in which one can describe a line in the plane:

- As a vector equation:
- As two parametric equations:
- Via a symmetric equation:
- A general equation:
- A reduced equation:

Intersection of two lines

Let $d_1 : a_1x + b_1y + c_1 = 0$ and $d_2 : a_2x + b_2y + c_2 = 0$ be two lines in \mathcal{E}_2 . The solution of the system of equation

$$\begin{cases} a_1x + b_1y + c_1 = 0 \\ a_2x + b_2y + c_2 = 0 \end{cases}$$

will give the set of the intersection points of d_1 and d_2 .

- 1) If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, the system has a unique solution (x_0, y_0) and the lines have a unique intersection point $P_0(x_0, y_0)$. They are *secant*.
- 2) If $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$, the system is not compatible, and the lines have no points in common. They are *parallel*.
- 3) If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, the system has infinitely many solutions, and the lines coincide. They are *identical*.

Suppose $d_i : a_i x + b_i y + c_i = 0$, $i = \overline{1, 3}$ are three distinct lines in \mathcal{E}_2 . Then they are concurrent if and only if

$$\text{Rank} \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} = \text{Rank} \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = 2. \quad (11)$$

Thank you very much for your attention!