Laboratory 6: Equilibrium points. Stability

In this laboratory there are presented the instructions necessary for the qualitative study of the solutions around the equilibrium points in the case of autonomous scalar equations and the planar systems of autonomous equations.

Scalar autonomous differential equations

An autonomous scalar differential equations has the form:

$$x' = f(x)$$

Constant solutions *x(t)*≡ *x** are called equilibrium solutions, the value *x** is called the equilibrium points.

Let's consider the following equation
$$x' = x\left(1 - x^2\right)$$

```
> with(DEtools): with(plots):
```
we define the function from the right hand side of the equation:
```
> f:=x->x*(1-x^2);
```
$$f := x \to x\left(1 - x^2\right)$$
```
> eqd:=diff(x(t),t)=f(x(t));
```
$$eqd := \frac{\mathrm{d}}{\mathrm{d}t}\, x(t) = x(t)\left(1 - x(t)^2\right)$$

The equilibrium points are real solution of the equation:
$$f(x) = 0$$

```
> equip:=solve(f(x)=0,x);
```
$$equip := 0,\ 1,\ -1$$

For the study of stability there are two methods, either we apply the Stability Theorem in the first approximation, or by the graphical method (analysis of the phase portrait).

THE STABILITY THEOREM IN THE FIRST APPROXIMATION

Suppose x* is an equilibrium point of the differential equation $x' = f(x)$ where f is a continuously differentiable function. Then,
• if $f'(x^*) < 0$, then x^* is locally asymptotically stable (or sink);
• if $f'(x^*) > 0$, then x^* is unstable (or source);

1. Using the stability theorem:

```
> equip[1];
```

$$0$$

```
> D(f)(equip[1]);
```

$$1$$

We notice that $f'(0)=1>0$, so the equilibrium point 0 is **unstable**.

In the same way, we study the stability for the other equilibrium points:

```
> D(f)(equip[2]);
```

$$-2$$

```
> D(f)(equip[3]);
```

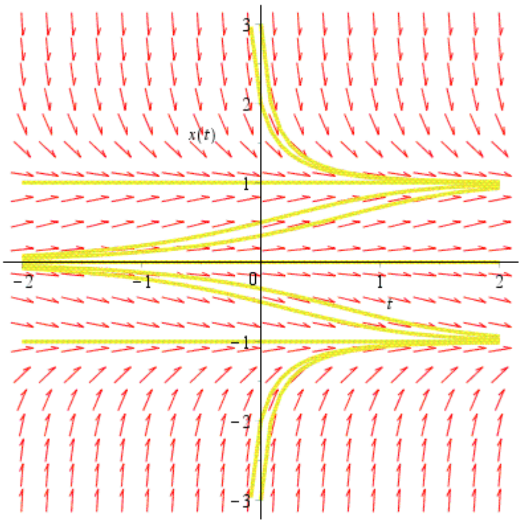
$$-2$$

We obtained that $f'(1)=f'(-1)=2<0$, so the equilibrium points 1 and -1 are **locally asymptotically stable**.

2. The graphic method (analysis of the phase portrait)

In the case of autonomous scalar equations MAPLE does not have a command to generate the phase portrait, the stability of the equilibrium solutions is studied by analyzing the directions field and graphically representing the representative solutions using the **DEplot** command. In the case of our equation, the representative solutions are the solutions that satisfy initial conditions in 0 smaller than -1, solutions that satisfy initial conditions in 0 between -1 and 0, solutions that satisfy initial conditions in 0 between 0 and 1, solutions that satisfy initial conditions in 0 greater than 1:

```
> DEplot(eqd,x(t), t=-2..2, [[x(0)=-3], [x(0)=-2], [x(0)=-1],
[x(0)=-1/2], [x(0)=-1/3], [x(0)=0],[x(0)=1/3], [x(0)=1/2],[x(0)=1],
[x(0)=2],[x(0)=3]]);
```



Linear systems

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = A \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

In the case of linear systems the origin (0,0) is the equilibrium point

Linear systems with real eigenvalues:
Consider a linear system with two nonzero, real, distinct eigenvalues λ_1 and λ_2.

- If $\lambda_1 < 0 < \lambda_2$, then the origin is a **saddle point**. There are two lines in the phase portrait that correspond to straight-line solutions. Solutions along one line tend toward (0, 0) as t increases, and solutions on the other line tend away from (0, 0). All other solutions come from and go to infinity, so, in this case, the origin is **unstable**.
- If $\lambda_1 < \lambda_2 < 0$, then the origin is **a sink node**. All solutions tend to (0, 0) as $t \to \infty$, so the origin is **asymptotically stable**.
- If $0 < \lambda_1 < \lambda_2$, then the origin is **a source node**. All solutions except the equilibrium solution go to infinity as $t \to \infty$, so, the origin is **unstable**.

Linear systems with complex eigenvalues:
Given a linear system with complex eigenvalues $\lambda = \alpha \pm i\beta$, $\beta > 0$, the solution curves spiral around the origin in the phase plane with a period of $2\pi/\beta$. Moreover:

- If $\alpha < 0$, then the solutions spiral toward the origin. In this case the origin is called a **spiral sink**, origin is **asymptotically stable**.
- If $\alpha > 0$, then the solutions spiral away from the origin. In this case the origin is called a **spiral source**, the origin is **unstable**.
- If $\alpha = 0$, then the solutions are periodic. They return exactly to their initial conditions in the phase plane and repeat the same closed curve over and over. In this case the origin is called a **center** and it is **locally asymptotically stable**.

Let's consider the following linear system $x' = x + y \qquad y' = x - y$
so,

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The given system is introduced by defining the two equations through the `diff` procedure.

```
> restart;
> with(DEtools): with(plots):with(linalg):
>
> eq1:=diff(x(t),t)=x(t)+y(t);
```

$$eq1 := \frac{d}{dt} x(t) = x(t) + y(t)$$

```
> eq2:=diff(y(t),t)=x(t)-y(t);
```

$$eq2 := \frac{d}{dt} y(t) = x(t) - y(t)$$

```
> sist:=eq1,eq2;
```

$$sist := \frac{d}{dt} x(t) = x(t) + y(t), \frac{d}{dt} y(t) = x(t) - y(t)$$

We construct the matrix of the system and we calculate the corresponding eigenvalues:

```
> A:=matrix([[1,1],[1,-1]]);
```

$$A := \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

```
> eigenvals(A);
```

$$\sqrt{2}, -\sqrt{2}$$

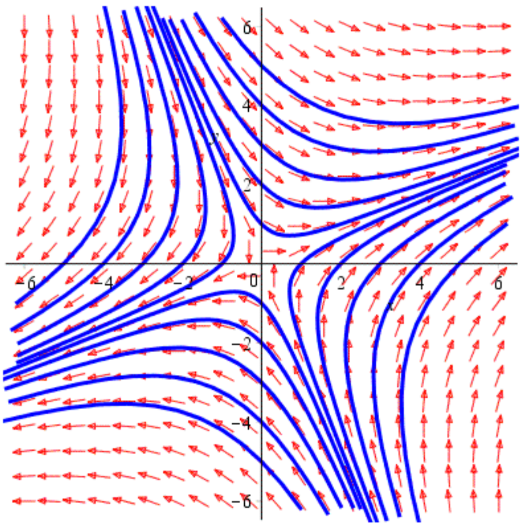
The first eigenvalue is strictly positive and the second is strictly negative then the equilibrium point (0,0) is its unstable equilibrium point (a source node).

To visualize the dynamics generated by this system, the phase portrait is represented along with some orbits using the command **DEplot**

```
> in_cond:=[x(0)=0,y(0)=i]$i=1..5, [x(0)=-i,y(0)=0]$i=1..5,
[x(0)=0,y(0)=-i]$i=1..5,[x(0)=i,y(0)=0]$i=1..5;
```

$$in\_cond := [x(0) = 0, y(0) = 1], [x(0) = 0, y(0) = 2], [x(0) = 0, y(0) = 3], [x(0) = 0, y(0) = 4], [x(0) = 0, y(0) = 5], [x(0) = -1, y(0) = 0], [x(0) = -2, y(0) = 0], [x(0) = -3, y(0) = 0], [x(0) = -4, y(0) = 0], [x(0) = -5, y(0) = 0], [x(0) = 0, y(0) = -1], [x(0) = 0, y(0) = -2], [x(0) = 0, y(0) = -3], [x(0) = 0, y(0) = -4], [x(0) = 0, y(0) = -5], [x(0) = 1, y(0) = 0], [x(0) = 2, y(0) = 0], [x(0) = 3, y(0) = 0], [x(0) = 4, y(0) = 0], [x(0) = 5, y(0) = 0]$$

```
> DEplot([sist],[x(t),y(t)],t=-5..5,x=-6..6,y=-6..6,[in_cond],
arrows=medium, linecolor=blue,stepsize=0.1);
```



Nonlinear systems. Stability of the equilibrium points

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} f_1(x(t), y(t)) \\ f_2(x(t), y(t)) \end{bmatrix}$$

The equilibrium points *X*(x*,y*)* are the real solution of the system

$$\begin{bmatrix} f_1(x, y) \\ f_2(x, y) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The stability of the equilibrium points can be studied using the stability theorem in the first approximation for the system case:

THE STABILITY THEOREM IN THE FIRST APPROXIMATION
Let *X*(x*,y*)* be an equilibrium point for the nonlinear system.

- if Re(λ)<0 for all the eigenvalues of J_f (*X**) then *X** is locally asymptotically stable
- if there exists an eigenvalue of J_f (*X**) with Re(λ)>0 then *X** is unstable

where J_f (*X*) is the jacobian of the vectorial function $f = (f_1, f_2)$

Let's consider the system

$$x' = x\left(1 - \frac{x}{2} - y\right)$$
$$y' = y\left(x - 1 - \frac{y}{2}\right)$$

```
> f1:=(x,y)->x*(1-x/2-y);
```

$$f1 := (x, y) \to x\left(1 - \frac{1}{2}x - y\right)$$

```
> f2:=(x,y)->y*(x-1-y/2);
```

$$f2 := (x, y) \to y\left(x - 1 - \frac{1}{2}y\right)$$

```
> eq1:=diff(x(t),t)=f1(x(t),y(t));
```

$$eq1 := \frac{d}{dt}x(t) = x(t)\left(1 - \frac{1}{2}x(t) - y(t)\right)$$

```
> eq2:=diff(y(t),t)=f2(x(t),y(t));
```

$$eq2 := \frac{d}{dt}y(t) = y(t)\left(x(t) - 1 - \frac{1}{2}y(t)\right)$$

```
> sist2:=eq1,eq2;
```

$$sist2 := \frac{d}{dt}x(t) = x(t)\left(1 - \frac{1}{2}x(t) - y(t)\right), \frac{d}{dt}y(t) = y(t)\left(x(t) - 1 - \frac{1}{2}y(t)\right)$$

```
> EquiP:=solve({f1(x,y)=0,f2(x,y)=0},{x,y});
```

$$EquiP := \{x = 0, y = 0\}, \{x = 0, y = -2\}, \{x = 2, y = 0\}, \left\{x = \frac{6}{5}, y = \frac{2}{5}\right\}$$

Notice that the system has four equilibrium points.

```
> EquiP[1,1];EquiP[1,2];
```

$$x=0$$

$$y=0$$

First we construct the jacobian of the vectorial function $f=(f_1,f_2)$ and then we calculate the eigenvalues of J_f (0,0)

```
> J:=jacobian([f1(x,y),f2(x,y)],[x,y]);
```

$$J:=\begin{bmatrix} 1-x-y & -x \\ y & x-1-y \end{bmatrix}$$

```
>
> A1:=subs(EquiP[1,1],EquiP[1,2],y=0,eval(J));
```

$$A1:=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

```
> eigenvals(A1);
```

$$1, -1$$

So, in this case the equilibrium point (0,0) is unstable (saddle point) since the first eigenvalue is positive.

We do the same for the other equilibrium points.

```
> EquiP[2,1];EquiP[2,2];
```

$$x=0$$

$$y=-2$$

```
> A2:=subs(EquiP[2,1],EquiP[2,2],eval(J));
```

$$A2:=\begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix}$$

```
> eigenvals(A2);
```

$$3, 1$$

In this case the equilibrium point (0, -2) is an unstable equilibrium point of the source node type.

```
> EquiP[3,1];EquiP[3,2];
```

$$x=2$$

$$y=0$$

```
> A3:=subs(EquiP[3,1],EquiP[3,2],eval(J));
```

$$A3:=\begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}$$

```
> eigenvals(A3);
```

$$-1, 1$$

Also, in this case the equilibrium point (2, 0) is an unstable equilibrium point of the saddle type.

```
> EquiP[4,1];EquiP[4,2];
```

$$x=\frac{6}{5}$$

$$y=\frac{2}{5}$$

```
> A4:=subs(EquiP[4,1],EquiP[4,2],y=0,eval(J));
```

$$A4 := \begin{bmatrix} -\frac{3}{5} & -\frac{6}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix}$$

```
> eigenvals(A4);
```

$$-\frac{2}{5} + \frac{1}{5} I \sqrt{11}, -\frac{2}{5} - \frac{1}{5} I \sqrt{11}$$

In this case we have complex conjugate eigenvalues whose real part is -2/5, so the equilibrium point (6 / 5,2 / 5) is a locally asymptotically stable equilibrium point of the sink spiral type.

In order to represent the phase portrait, a window containing all the equilibrium points must be chosen, in this case we can choose x = -3..3, y = -3..3, and then set a list of conditions to represent some orbits.

```
> condin:=[x(0)=-1,y(0)=1], [x(0)=-0.5,y(0)=1], [x(0)=1,y(0)=1],
[x(0)=1,y(0)=3], [x(0)=2,y(0)=0.5], [x(0)=-1,y(0)=-1],
[x(0)=-0.5,y(0)=-1], [x(0)=-1,y(0)=-2.5], [x(0)=1,y(0)=-1],
[x(0)=1.5,y(0)=-1], [x(0)=1,y(0)=-2.5];
```

$$condin := [x(0) = -1, y(0) = 1], [x(0) = -0.5, y(0) = 1], [x(0) = 1, y(0) = 1], [x(0) = 1, y(0) = 3], [x(0) = 2, y(0) = 0.5], [x(0) = -1, y(0) = -1], [x(0) = -0.5, y(0) = -1], [x(0) = -1, y(0) = -2.5], [x(0) = 1, y(0) = -1], [x(0) = 1.5, y(0) = -1], [x(0) = 1, y(0) = -2.5]$$

```
> DEplot([sist2],[x(t),y(t)],t=-10..10,x=-3..3,y=-3..3,
[condin],linecolor=blue,stepsize=0.1);
```