Tutoring

Saturday, November 18, 2023 5:59 PM

$\overline{a+ib} = a-ib$

$|a+ib| = \sqrt{a^2+b^2}$

$i^2=-1$

1. Let $f:\mathbb{C}\to\mathbb{C}$,

$$f(z) = |z-1|^2(\overline{z}-1) - 2\overline{z}, \quad z\in\mathbb{C}.$$

a) Compute $f(1+i)$ and $f(\frac{1}{i})$.

b) Find the set $S = \{z\in\mathbb{C} : f \text{ is differentiable at } z\}$ and represent it graphically in the complex plane.

c) Compute $f'(z)$ for $z\in S$.

d) Represent graphically in the complex plane the set

$$D = \{z\in S : |f'(z)| < |z+1|^2 - 4\}.$$

$|\zeta|^2 = \zeta\cdot\overline{\zeta}$

$\partial U(z_0, r) = \{z\in\mathbb{C} : |z-z_0| = r\}$

1. a) $f(1+i) = |1+i-1|^2(1-i-1) - 2(1-i) = |i|^2(-i) - 2+2i = 1\cdot(-i) - 2+2i = i-2$

$(i = 0 + i\cdot 1)$

$f(\frac{1}{i}) = f(\frac{i}{-1}) = f(-i) = |-i-1|^2(i-1) - 2i = (\sqrt{2})^2(i-1) - 2i = 2i-2-2i = -2$

b) $f(x+iy) = ((x-1)^2+y^2)(x-1-iy) - 2(x-iy) = u(x,y) + i\,v(x,y)$

$f = u+iv$

$u, v \in C^1(\mathbb{C})$, because they are pol. fct. w.r.t. x, y.

So, f is $\mathbb{R}$-diff.

$C^1(D) = \{$ family of functions $g: D\to\mathbb{R}$ with continuous partial derivatives $\}$

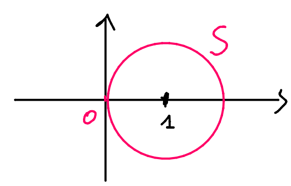
$D\subseteq\mathbb{C}$

$g(x,y) = \dots$

By the Cauchy-Riemann thm., f is diff. at $z\in\mathbb{C} \iff \frac{\partial f}{\partial \overline{z}}(z) = 0$

$\iff \frac{\partial}{\partial\overline{z}}\Big(\underbrace{|z-1|^2}_{(z-1)(\overline{z}-1)}(\overline{z}-1) - 2\overline{z}\Big) = 0 \iff \frac{\partial}{\partial\overline{z}}\Big((z-1)(\overline{z}-1)^2 - 2\overline{z}\Big) = 0 \iff (z-1)\cdot 2(\overline{z}-1)\cdot 1 - 2 = 0$

$\iff 2|z-1|^2 = 2 \iff |z-1| = 1 \iff z\in\partial U(1,1).$



c) $f'(z) = \frac{\partial}{\partial z}\Big(|z-1|^2(\overline{z}-1) - 2\overline{z}\Big) = \frac{\partial}{\partial z}\Big((z-1)\cdot(\overline{z}-1)^2\Big) =$

$z\in S = \partial U(1,1)$

$= (\overline{z}-1)^2, \quad z\in S.$

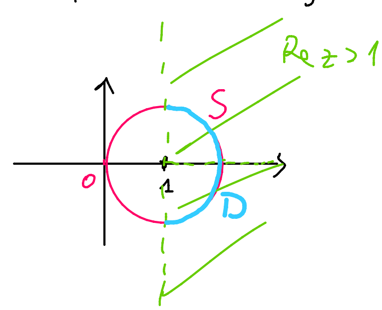
$\boxed{\operatorname{Re}\zeta = \frac{\zeta+\overline{\zeta}}{2}}$

d) $D = \{z\in S : |f'(z)| < |z+1|^2 - 4\}$

$= \{z\in\partial U(1,1) : |(\overline{z}-1)^2| < |z+1|^2 - 4\}$

$= \{z\in\partial U(1,1) : \underbrace{|\overline{z}-1|^2}_{(\overline{z}-1)(z-1)} < \underbrace{|z+1|^2}_{(z+1)(\overline{z}+1)} - 4\} = \{z\in\partial U(1,1) : \cancel{z\overline{z}} - z - \overline{z} + 1 < \cancel{z\overline{z}} + z + \overline{z} + 1 - 4\}$

$= \{z\in\partial U(1,1) : 4 < 2(z+\overline{z})\} = \{z\in\partial U(1,1) : \operatorname{Re} z > 1\}$



2. Let $f:\mathbb{C}\to\mathbb{C}$, $f(z) = e^{-4|z|^2}\cdot\overline{z}$, $z\in\mathbb{C}$. Represent graphically in the complex plane the set of all $z\in\mathbb{C}$ such that f is differentiable at z and compute the derivative at these points.

$-4|z|^2$

$z \in \mathbb{C}$ such that f is differentiable at z and compute the derivative at these points.

$f(z) = e^{-4|z|^2} \cdot \bar{z}$, $z = x+iy$

$f(x+iy) = e^{-4(x^2+y^2)} \cdot (x-iy)$

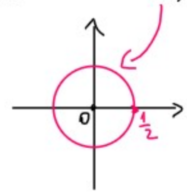
$f = u+iv$; $u, v \in C^1(\mathbb{C}) \Rightarrow f$ is $\mathbb{R}$-diff.

By C.-R. thm., f is diff. at $z \in \mathbb{C} \iff \frac{\partial f}{\partial \bar{z}}(z) = 0$

$$\iff \frac{\partial}{\partial \bar{z}}\left(e^{-4 \cdot z \cdot \bar{z}} \cdot \bar{z}\right) = \left(\frac{\partial}{\partial \bar{z}} e^{-4z\cdot\bar{z}}\right) \cdot \bar{z} + e^{-4z\cdot\bar{z}} \cdot \underbrace{\frac{\partial}{\partial \bar{z}}(\bar{z})}_{=1} =$$

$$= e^{-4z\cdot\bar{z}} \cdot \frac{\partial}{\partial \bar{z}}(-4z\bar{z}) \cdot \bar{z} + e^{-4z\cdot\bar{z}} = \underbrace{e^{-4z\cdot\bar{z}}}_{\neq 0}\left(-4z\bar{z}+1\right) = 0$$

$$\iff 4z\bar{z} = 1 \iff |z|^2 = \frac{1}{4} \iff |z| = \frac{1}{2} \iff z \in \partial U(0, \tfrac{1}{2}).$$



$$f'(z) = \frac{\partial}{\partial z}\left(e^{-4z\bar{z}} \cdot \bar{z}\right) = \bar{z} \cdot e^{-4z\bar{z}} \cdot \underbrace{\frac{\partial}{\partial z}(-4z\bar{z})}_{-4\bar{z}} = -4\bar{z}^2 e^{-4z\bar{z}},$$

$\forall z \in \partial U(0, \frac{1}{2})$.

3. Let $f : \mathbb{C} \setminus \{1\} \to \mathbb{C}$, $f(z) = \dfrac{3z+i-2}{z-1}$, $z \in \mathbb{C} \setminus \{1\}$. Represent graphically in the complex plane the set $A \cap B$, where

$$A = \left\{ z \in \mathbb{C} \setminus \{1\} : |f'(z)| > \sqrt{2} \right\}$$
$$B = \left\{ z \in \mathbb{C} \setminus \{0\} : 0 < \arg z < \frac{\pi}{2} \right\}.$$

$|\zeta_1 \cdot \zeta_2| = |\zeta_1| \cdot |\zeta_2|$

$\left|\frac{\zeta_1}{\zeta_2}\right| = \frac{|\zeta_1|}{|\zeta_2|}$ $\quad |\zeta^2| = |\zeta|^2$

$U(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$

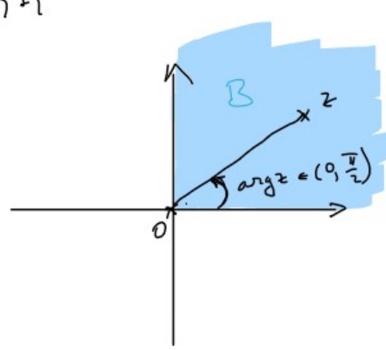
$$f'(z) = \frac{(3z+i-2)'(z-1) - (3z+i-2)\cdot(z-1)'}{(z-1)^2} = \frac{3(z-1) - (3z+i-2)\cdot 1}{(z-1)^2} = \frac{-1-i}{(z-1)^2}, \ z \in \mathbb{C} \setminus \{1\}$$

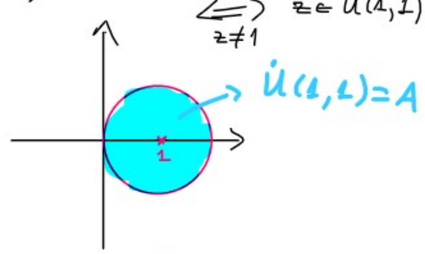
$$|f'(z)| > \sqrt{2} \iff \left|\frac{-1-i}{(z-1)^2}\right| > \sqrt{2} \iff \frac{|-1-i|}{|z-1|^2} > \sqrt{2} \iff \frac{\sqrt{2}}{|z-1|^2} > \sqrt{2} \iff |z-1|^2 < 1$$

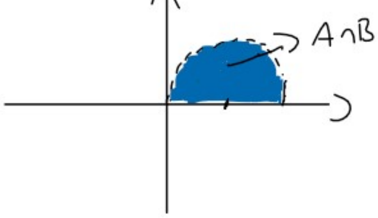
$z \in \mathbb{C} \setminus \{1\}$ $\qquad (|-1-i| = \sqrt{2})$

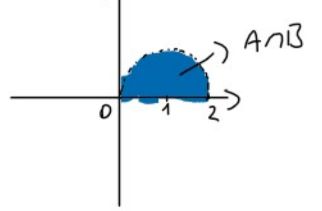
$\iff |z-1| < 1$

$\underset{z \neq 1}{\iff} z \in \dot{U}(1,1)$.









$\left|\ \lim |z_n| = 0 \iff \lim z_n = 0\ \right|$

$\boxed{\begin{array}{l}\lim_{n\to\infty}|z_n| = 0 \iff \lim_{n\to\infty} z_n = 0\\ \lim_{n\to\infty}|z_n| = \infty \iff \lim_{n\to\infty} z_n = \infty\end{array}}$

4, Find $\lim_{n\to\infty} z_n$, where the sequence $(z_n)_{n\in\mathbb{N}^*}$ is given by

$$z_n = \left(\frac{1}{2}+\frac{i\pi}{n}\right)^n + \left(1+\frac{\pi}{in}\right)^n,\ n\in\mathbb{N}^*.$$

$\boxed{\begin{array}{c}\lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n = e^z\\ e^{x+iy} = e^x(\cos y + i\sin y)\\ z=x+iy\end{array}}$

$|\zeta^n| = |\zeta|^n$

5. Find $\lim_{n\to\infty} z_n$, where the sequence $(z_n)_{n\in\mathbb{N}^*}$ is given by

$$z_n = n^2\cdot\left(\frac{1}{2}+\frac{2i}{3}\right)^n + \left(1+\frac{\pi}{2in}\right)^{3n},\ n\in\mathbb{N}^*.$$

4. $$\lim_{n\to\infty}\left|\left(\frac12+\frac{i\pi}{n}\right)^n\right| = \lim_{n\to\infty}\left|\frac12+i\frac{\pi}{n}\right|^n = \lim_{n\to\infty}\left(\sqrt{\frac14+\frac{\pi^2}{n^2}}\right)^n = \lim_{n\to\infty}\left(\frac14+\frac{\pi^2}{n^2}\right)^{\frac n2} = \left(\frac14\right)^\infty = 0$$

$$\frac14+\frac{\pi^2}{n^2}\to\frac14$$

$$\Downarrow$$

$$\lim_{n\to\infty}\left(\frac12+\frac{i\pi}{n}\right)^n = 0$$

$$\lim_{n\to\infty}\left(1+\frac{\pi}{in}\right)^n = \lim_{n\to\infty}\left(1+\frac{\frac{\pi}{i}}{n}\right)^n \overset{z=\frac{\pi}{i}}{=} \lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n = e^z = e^{\frac{\pi}{i}} = e^{-\pi i} = \cos(-\pi)+i\sin(-\pi) = -1.$$

So, $\lim z_n = -1$.

5. $$\lim_{n\to\infty}\left|n^2\cdot\left(\frac12+\frac{2i}{3}\right)^n\right| = \lim_{n\to\infty} n^2\cdot\left|\frac12+\frac{2i}{3}\right|^n = \lim_{n\to\infty} n^2\left(\sqrt{\frac14+\frac49}\right)^n = \lim_{n\to\infty} n^2\left(\sqrt{\frac{25}{36}}\right)^n = \lim_{n\to\infty} n^2\left(\frac56\right)^n = 0$$

$$\lim_{x\to\infty} x^2\cdot\left(\frac56\right)^x = \lim_{x\to\infty}\frac{x^2}{\left(\frac65\right)^x} \overset{l'H}{=} \lim_{x\to\infty}\frac{2x}{\left(\frac65\right)^x\cdot\ln\frac65} \overset{l'H}{=} \lim_{x\to\infty}\frac{2}{\left(\frac65\right)^x\cdot\left(\ln\frac65\right)^2} = 0$$

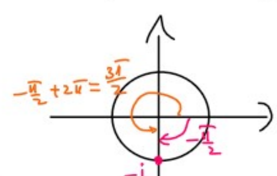
$$\Downarrow$$

$$\lim_{n\to\infty} n^2\left(\frac12+\frac{2i}{3}\right)^n = 0$$

$$\lim_{n\to\infty}\left[\left(1+\frac{\frac{\pi}{2i}}{n}\right)^n\right]^3 \overset{z=\frac{\pi}{2i}}{=} \left[e^z\right]^3 = \left(e^{\frac{\pi}{2i}}\right)^3 = \left(e^{-\frac{\pi}{2}i}\right)^3 = \left(\underbrace{\cos\left(-\frac{\pi}{2}\right)}_{=0} + i\underbrace{\sin\left(-\frac{\pi}{2}\right)}_{=-1}\right)^3 = (-i)^3 = i$$

$(i^2=-1)$

So, $\lim_{n\to\infty} z_n = i$.



$-i = \cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)$

6. Solve in $\mathbb{C}$ the equation $\sin z = i$.

$$\Downarrow$$

$\boxed{\sin z = \frac{e^{iz}-e^{-iz}}{2i}}$

$$\frac{e^{iz}-e^{-iz}}{2i} = i \iff e^{iz}-e^{-iz} = -2 \underset{t=e^{iz}\neq 0}{\iff} t-\frac{1}{t} = -2 \quad |\cdot t$$

$$\iff t^2+2t-1=0 \qquad t_{1,2} = \frac{-2\pm2\sqrt2}{2} \begin{cases} -1-\sqrt2 \\ -1+\sqrt2\end{cases}$$

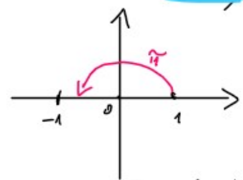
$$\Delta = 4+4 = 8$$

$$\text{I} \quad t_1 = -1-\sqrt2 \iff e^{iz} = -1-\sqrt2 \underset{z=x+iy}{\iff} e^{ix-y} = -1-\sqrt2$$

I $t_1 = -1-\sqrt{2} \iff e^{iz} = -1-\sqrt{2} \underset{z=x+iy}{\iff} e^{ix-y} = -1-\sqrt{2}$

$\iff e^{-y}(\cos x + i\sin x) = (1+\sqrt{2})\cdot(-1) = (1+\sqrt{2})(\cos\pi + i\sin\pi)$

$\iff \begin{cases} e^{-y}=1+\sqrt{2} \\ x=\pi \pmod{2\pi}\end{cases}$



$\iff \begin{cases} y=-\ln(1+\sqrt{2})\\ x\in\{\pi+2k\pi : k\in\mathbb{Z}\}\end{cases} \iff z_k = \pi(2k+1) - i\ln(1+\sqrt{2}),\ k\in\mathbb{Z}.$

II $t_2=-1+\sqrt{2} = \underbrace{\sqrt{2}-1}_{>0} \iff e^{iz}=e^{-y+ix} = (\underbrace{\sqrt{2}-1}_{>0})\cdot 1 = (\sqrt{2}-1)\cdot(\cos 0+i\sin 0)$

$\iff \begin{cases} e^{-y}=\sqrt{2}-1\\ x=0 \pmod{2\pi}\end{cases} \iff z_k = 2k\pi - i\ln(\sqrt{2}-1),\ k\in\mathbb{Z}.$

$\boxed{\cos z = \frac{e^{iz}+e^{-iz}}{2}}$ $\boxed{e^{-y}=\frac{1}{e^{y}}}$

$\cancel{e^{z_1\cdot z_2} = (e^{z_1})^{z_2}}$ $\cancel{i}$

6. Solve in $\mathbb{C}$ the equation: $3\cos z - 5i\sin z = 4i$.

$\iff 3\cdot\frac{e^{iz}+e^{-iz}}{2} - 5i\cdot\frac{e^{iz}-e^{-iz}}{2i} = 4i \quad |\cdot 2i$

$\iff 3i\cdot(e^{iz}+e^{-iz}) - 5i(e^{iz}-e^{-iz}) = -8$

$\iff -2i e^{iz}+8i e^{-iz} = -8 \quad |:2i \iff -e^{iz}+4e^{-iz}=4i \underset{t=e^{iz}}{\iff} -t+4\cdot\frac{1}{t} = 4i \quad |\cdot t$

$\iff t^2+4it-4=0 \qquad \Delta = -16+16=0$

$t=\frac{-4i\pm 0}{2} = -2i$

$t=-2i \iff e^{iz}=-2i \underset{z=x+iy}{\iff} e^{-y+ix} = 2\cdot(\underbrace{\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)}_{-i})$

$\iff e^{-y}(\cos x+i\sin x) = 2\left(\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right)$

$\iff \begin{cases}-y=\ln 2\\ x\in\{-\frac{\pi}{2}+2k\pi : k\in\mathbb{Z}\}\end{cases} \iff z_k = -\frac{\pi}{2}+2k\pi - i\ln 2,\ k\in\mathbb{Z}.$

8. Find all $f\in\mathcal{H}(\mathbb{C})$ such that

$$|f(z)|=e^{xy}, \forall z=x+iy\in\mathbb{C}.$$

T1, C.5