Complements of Geometry

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Optional course for Maths and Computer Science

Week 2

Important lines in a triangle

Although the geometric configuration of a triangle is the simplest (a triangle is defined after-all using just three non-collinear points), its geometry can be extremely rich. This is illustrated by the fact that ETC *Encyclopedia of Triangle Centers* contains as we speak 57030 distinct points.

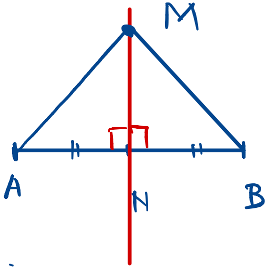
Definition (in Romanian, „mediatoarea")

The perpendicular bisector of a segment $[AB]$ is the line in the plane which is perpendicular on AB and passes through the midpoint of $[AB]$.

Proposition

The perpendicular bisector of $[AB]$ *is the geometric locus of all points* M *in the plane such that* $MA = MB$.

Proof. „⇒" Suppose M is on the perp. bisector of [AB].



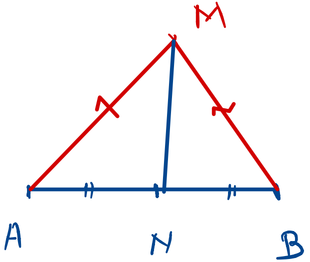
If M = N, we have nothing to prove.

Otherwise draw MA, MB.

$$\left.\begin{cases} \sphericalangle MNA \equiv \sphericalangle MNB \\ NA = NB \\ MN = MN \end{cases}\right\} \underset{S.A.S}{\Rightarrow} \Delta MNA \equiv \Delta MNB$$

$$\Rightarrow \boxed{MA = MB.}$$

"$\Leftarrow$" Suppose $MA = MB$. If M is the mid. of $[AB]$, we are done. Otherwise, we have.

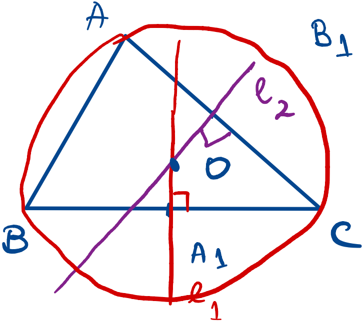


Note that $\Delta MAN \equiv \Delta MBN$ by the S.S.S criterion.

Therefore, $\measuredangle MNA \equiv \measuredangle MNB$. Since $\measuredangle MNA + \measuredangle MNB = \pi$, it follows that $MN \perp AB$.

So M is on the perp. bisector.

Consequence: In a triangle $\triangle ABC$, the perp. bisectors are concurrent. (equivalently, 3 non-colinear points determine a circle.)



Let O be the int. of ℓ_1, ℓ_2, where ℓ_1 and ℓ_2 are the perp. bisect. of [BC] and [CA].

Using the Proposition, $OB = OC$. (1)

Similarly, $OC = OA$ (2)

From (1) and (2) it follows that $OB = OA$, so O is on the perp. bisector of $[AB]$.

O is such that $OA = OB = OC := R > 0$.

So O is the circumcenter of $\triangle ABC$ and R is the circumradius.

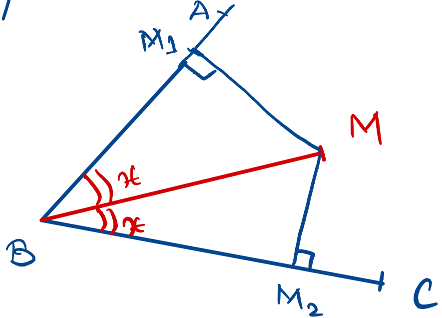
Definition

The bisector of the $\angle ABC$ is the half-line $(BM$, situated in the interior of the angle such that $\angle ABM \equiv \angle MBC$.

Proposition

The angle bisector of $\angle ABC$ is the geometric locus of all points M situated in the interior of $\angle ABC$ such that $d(M, BA) = d(M, BC)$.

„$\Rightarrow$" Suppose M is on the bisector of $\measuredangle ABC$.

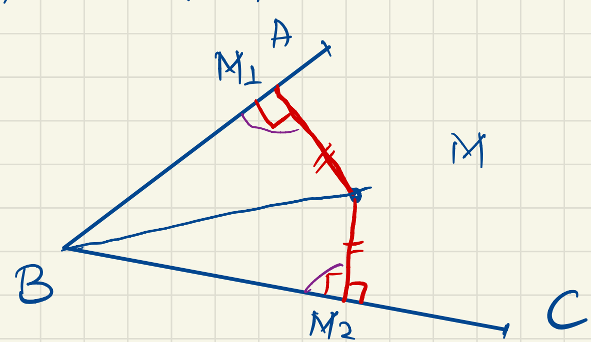


$$\begin{cases} \sphericalangle M_1BM \equiv \sphericalangle M_2BM & (BM - \text{bisector}) \\ \sphericalangle BMM_1 \equiv \sphericalangle BMM_2 & (\text{both equal to } \frac{\pi}{2} - x) \\ MB = MB \end{cases}$$

$$\underset{S.A.S}{\Rightarrow} \quad \Delta MBM_1 \equiv \Delta MBM_2$$

$$\Rightarrow MM_1 = MM_2 \iff d(M, AB) = d(M, AC).$$

" $\Leftarrow$ " Suppose $M \in \text{Int}(\sphericalangle ABC)$ s.t. $d(M, AB) = d(M, AC)$.



$$\begin{cases} MM_1 = MM_2 & \text{(by hypothesis)} \\ MB = MB & \\ \underline{\underline{M_1B}}^2 = BM^2 - MM_1^2 = BM^2 - MM_2^2 = \underline{\underline{M_2B^2}} & \end{cases}$$

S.S.S

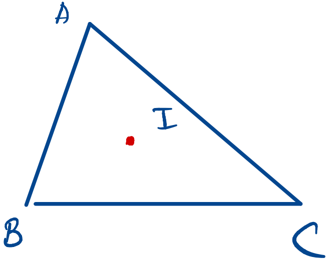
$$\Rightarrow \triangle MBM_1 \equiv \triangle MBM_2$$

$$\Rightarrow \measuredangle MBA \equiv \measuredangle MBC$$

$\boxtimes$

Proposition

In a $\triangle ABC$, the internal angle bisectors are concurrent.



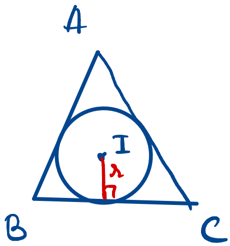
Proof. Suppose I is on the bisectors of $\measuredangle B$ and $\measuredangle C$.

By the perp. prop.,

$d(I, BC) = d(I, BA)$ and $d(I, CB) = d(I, CA)$.

$\Rightarrow d(I, AB) = d(I, AC)$, so I is on the bisector of $\measuredangle A$.

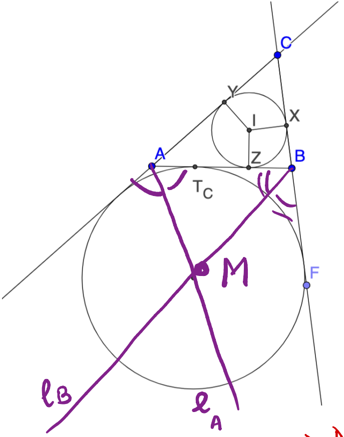
Let $r := d(I, AB)$. Then $\mathcal{C}(I, r)$ is the incircle of ΔABC. I - incenter and r - inradius.



The external bisectors of a triangle are the bisectors of the external angles.

Proposition

In every triangle $\triangle ABC$, two external bisectors and one internal bisector are concurrent.



Let $\{M\} = \ell_B \cap \ell_A$, the int. of two ext. $\measuredangle$ bisectors.

$d(M, AC) = d(M, AB)$

$d(M, AB) = d(M, CB)$

$\Rightarrow d(M, CA) = d(M, CB)$

$\Rightarrow M$ is on the internal $\measuredangle$ bisector of $\measuredangle C$.

Definition

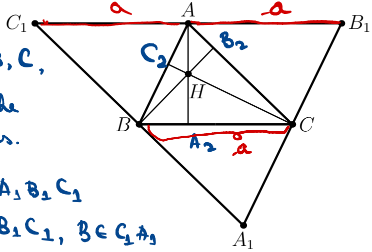
The *height* (or *altitude*) of a triangle is a line segment determined by a vertex and the perpendicular projection of the vertex on the opposite side.

Proposition

The altitudes of a triangle are concurrent. Their meeting point is usually denoted by H and is called the orthocenter of the triangle.

Through $A, B, C,$ draw $\parallel$ to the opposite sides.

Form the $\Delta A_1B_1C_1$ with $A \in B_1C_1$, $B \in C_1A_1$ and $C \in A_1B_1$.



Proof

Note that $AA_2 \perp BC \Rightarrow AA_2 \perp B_1C_1$.

Similarly, $BB_2 \perp A_1C_1$ and $CC_2 \perp A_1B_1$.

Since BCB_1A is parallelogram, $BC = AB_1$.

Similarly, $BCAC_1$ is a parallelogram, $AC_1 = CB$

So $AB_1 = AC_1$, hence AA_2 is a perp. bisector of $[B_1C_1]$.

Analogously, $[BB_2]$ and $[CC_2]$ are perp. bisectors of $[A_1C_1]$ and $[A_1B_1]$.

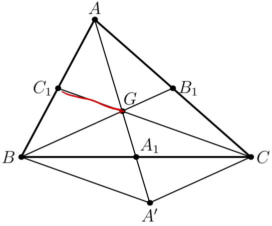
Now, AA_2, BB_2 and CC_2 are concurrent.

Definition

The line segment connecting a vertex of a triangle with the midpoint of the opposite side is called a *median*.

Proposition

The three medians in a triangle are concurrent. Their meeting point, usually denoted by G, is called the centroid (or the center of gravity) of the triangle.



Proof

- Let B_1, C_1 be the midpoints of ~~$[BC]$~~, $[CA]$ and $[AB]$, respectively.
- Consider $\{G\} = BB_1 \cap CC_1$. We will show that AG is actually situated on the median starting from A, namely that $\{A_1\} := AG \cap BC$ is the midpoint of $[BC]$.
- On the line AG construct the point A' such that $\boxed{AG = GA'}$. We have that $[GC_1]$ is a midline in triangle ABA' so $CG \| BA'$. Similarly, GB_1 is a midline in triangle ACA' so $BG \| CA'$.
- It follows that $BGCA'$ is a parallelogram, so its diagonals half each other. This proves that AG passes through the midpoint of $[BC]$.

Collinearity

Menelaus of Alexandria was a Greek mathematician and astronomer who lived in the first century AD. He is best known for his work in geometry and one of his eponymous theorem.

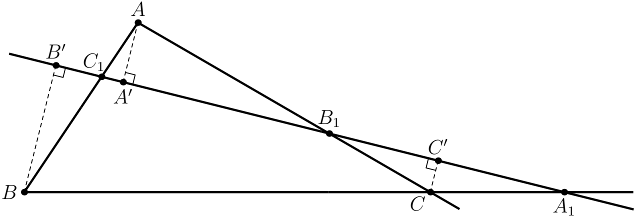
Menelaus also wrote a book on spherical trigonometry called "Sphaerica," which was highly regarded by mathematicians in ancient Greece and Rome. Unfortunately, only a few fragments of his work have survived to the present day.

Theorem (Menelaus's theorem)

Let l be a line which intersects the sides (or their extensions) of the triangle ABC in the points A_1, B_1, C_1. Then the following relation holds:

$$\frac{A_1B}{A_1C} \cdot \frac{B_1C}{B_1A} \cdot \frac{C_1A}{C_1B} = 1. \qquad (*)$$

Proof. Project the vertices of the triangle on the line l to obtain the points A', B', C'.



- Using similar triangles, we obtain

$$\triangle A_1C'C \sim \triangle A_1B'B \Rightarrow \frac{A_1B}{A_1C} = \frac{BB'}{CC'}$$

$$\triangle B_1C'C \sim \triangle B_1A'A \Rightarrow \frac{B_1C}{B_1A} = \frac{CC'}{AA'}$$

$$\triangle C_1A'A \sim \triangle C_1B'B \Rightarrow \frac{C_1A}{C_1B} = \frac{AA'}{BB'}.$$

- Multiplying the relations above gives

$$\frac{A_1B}{A_1C} \cdot \frac{B_1C}{B_1A} \cdot \frac{C_1A}{C_1B} = 1.$$

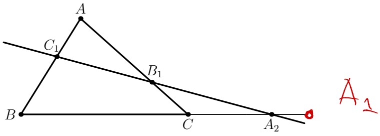
Theorem (Reciprocal version of Menelaus')

Let ABC be a triangle and consider the points A_1, B_1, C_1 situated on BC, CA, AB, such that either two of them are on the sides and one is on an extension or all three of them are on the extensions of the sides. Suppose that

$$\frac{A_1B}{A_1C} \cdot \frac{B_1C}{B_1A} \cdot \frac{C_1A}{C_1B} = 1. \qquad (*)$$

Then the points A_1, B_1, C_1 are collinear.

Proof. Let $B_1C_1 \cap BC = \{A_2\}$. We need to show that $A_2 = A_1$.



From Menelaus's theorem, aplied in the triangle ABC with the transversal $C_1 - B_1 - A_2$, we get

$$\frac{A_2B}{A_2C} \cdot \frac{B_1C}{B_1A} \cdot \frac{C_1A}{C_1B} = 1.$$

But using (*) we find that

$$\frac{A_2B}{A_2C} = \frac{A_1B}{A_1C},$$

and hence $A_2 = A_1$, from where the conclusion follows.

Concurrent lines in a triangle

Definition

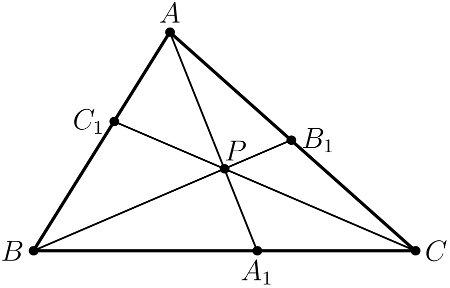
A *cevian* is a line segment that connects a vertex of a triangle to the opposite side, or an extension of that line segment.

The name "cevian" comes from the Italian mathematician Giovanni Ceva (1647-1734), who made significant contributions to geometry, including the theorem that bears his name. We now present this theorem, which was originally published in the work "De lineis rectis" in 1678.

Theorem (Ceva)

Let AA_1, BB_1, CC_1, be concurrent cevians in $\triangle ABC$. Then we have the relation:

$$\frac{A_1B}{A_1C} \cdot \frac{B_1C}{B_1A} \cdot \frac{C_1A}{C_1B} = 1.$$



Proof

By Menelaus Thm in $\triangle ABA_1$ for the transversal $C_1 - P - C$, we get:

$$\frac{AC_1}{C_1B} \cdot \frac{CB}{CA_1} \cdot \frac{PA_1}{PA} = 1. \quad (1)$$

Similarly, by Menelaus Thm in $\triangle ACA_1$ for the transversal $B - P - B_1$ we have:

$$\frac{AP}{PA_1} \cdot \frac{BA_1}{BC} \cdot \frac{CB_1}{B_1A} = 1 \quad (2)$$

Proof

Multiplying (1) and (2), we get:

$$\frac{AC_1}{C_1B} \cdot \frac{\cancel{CB}}{CA_1} \cdot \frac{\cancel{PA_1}}{\cancel{PA}} \cdot \frac{\cancel{AP}}{\cancel{PA_1}} \cdot \frac{BA_1}{\cancel{BC}} \cdot \frac{CB_1}{B_1A} = 1.$$

$$\frac{AC_1}{C_1B} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} = 1.$$

$\boxtimes$

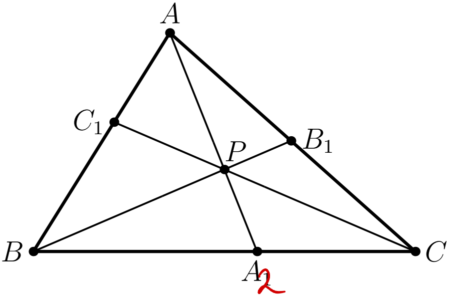
Theorem

Let ABC be a triangle and A_1, B_1, C_1 situated on its sides. such that the following relation holds:

$$\frac{A_1B}{A_1C} \cdot \frac{B_1C}{B_1A} \cdot \frac{C_1A}{C_1B} = 1.$$

Then the cevians AA_1, BB_1, CC_1 are concurrent.

Proof. Let $BB_1 \cap CC_1 = \{P\}$ and $AP \cap BC = \{A_2\}$.



From Ceva's theorem applied to AA_2, BB_1, CC_1, it follows that

$$\frac{A_2B}{A_2C} \cdot \frac{B_1C}{B_1A} \cdot \frac{C_1A}{C_1B} = 1.$$

Using the hypothesis we get that

$$\frac{A_1B}{A_1C} = \frac{A_2B}{A_2C}.$$

One shows that $A_1 = A_2$, hence the cevians AA_1, BB_1, CC_1 are concurrent. $\square$

Thank you very much for your attention!