Complements of Geometry

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Optional course for Maths and Computer Science

Week 1

Organisation

- The entire comunication is done through MS Teams;
- The components of your final grade are: 60% Test (last week of the term) 40% Seminar presentations and activity

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- Passing equirements: ≥ 5 in the final exam or ≥ 4 in the final exam, if one has good activity.

We plan to cover the following chapters

- The geometry of the triangle (2 weeks)
- The circle. The power of a point w.r.t. to a circle (2 weeks)
- Quadrilaterals (2 weeks)
- Parameterization of curves and surfaces with applications (3 weeks)
- Elliptic curves with applications (2 weeks)

Main characters

C. F. Gauss

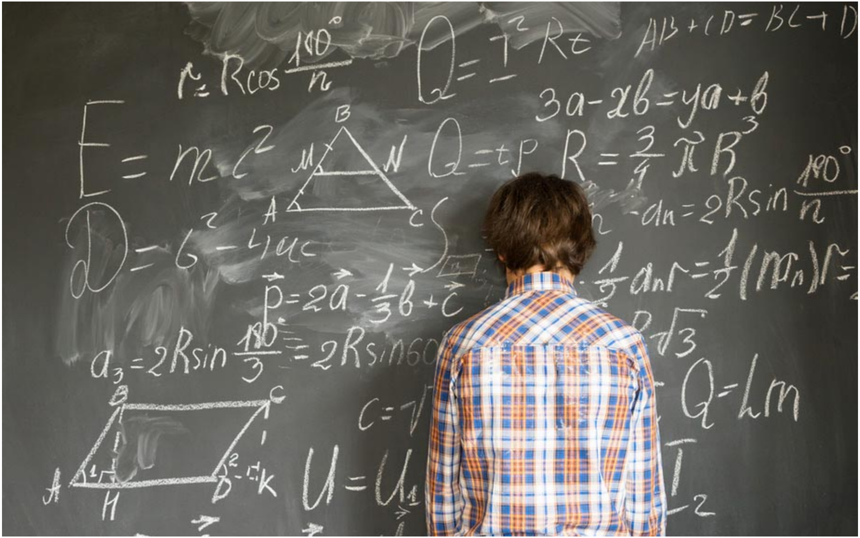
J. Bolyai

N. Lobachevsky

- C. F. Gauss (1777–1855)
- N. Lobachevsky (1792–1856)
- J. Bolyai (1802–1860)
- E. Beltrami (1835–1900)
- F. Klein (1849–1922)
- H. Poincaré (1854 – 1912)

- Philosophers, scientists and humankind in general, thinking about the physical space we inhabit – the shape of the universe as a whole – influenced and were influenced by the mathematical constructs that become known as "geometry".
- Euclid (300 B.C.) postulates:
  1. A straight line may be drawn from any one point to any other point.
  2. A terminated line can be produced indefinitely.
  3. A circle can be drawn with any centre and any radius.
  4. All right angles are equal to one another.
  5.

The fifth postulate is a little bit more complicated



Euclid's fifth postulate

That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

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Or, equivalently

For every line l and for every point P not lying on l, there exists a unique line l' passing through P and parallel to l.

The use made by Newton of the formalism of Euclidean geometry, together with the overwhelming success of his approach to physics and astronomy based on the model of Euclidean space, enshrined Euclidean geometry and cloaked it in a certainty and inevitability that made questioning it appear to be a sign of mental instability.

But what if ...

Jacob's Dream, Jusepe de Ribera, 1639

Earlier researchers knew that ...

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- Sum of angles in a triangle is 180°;
- There are similar triangles which are not congruent;

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Bolyai and Lobacevsky (1830) published this new "Non-Euclidean Geometry".

- The story is really a sad one, since their were ignored by their contemporaries.
- The young J. Bolyai was really dismayed by that.

Gauss and the Bolyai's

- The story of Gauss and the Bolyais starts back with the close friendship between Gauss and János's father, Farkas (also known by his German name of Wolfgang), dating back to their student days together in Göttingen.

- The family obsession with Euclid's parallel postulate apparently dates to that time, and indeed, father Bolyai published his first work on the subject, The Theory of Parallels, in 1804. He was only one of many who tried to prove that the parallel postulate follows from Euclid's other axioms
- Legendre also persisted in publishing false proofs over many years.

- As for Farkas Bolyai, it was his friend Gauss who pointed out the error in his argument, but he persisted for at least ten more years before giving up in despair.
- No wonder then, when his son, János, who had turned out to be something of a mathematical prodigy, appeared as a teenager to have already been bitten by the parallel-postulate bug, Farkas wrote him the often-quoted feverish admonishment to profit from his own example and guard against this will-o-the-wisp:

  "I have traversed this bottomless night, which extinguished all light and joy in my life. I entreat you, leave the science of parallels alone." (F. Bolyai)

- Bolyai Sr. was right, but not for the reasons he thought.
- János soon concluded that proving the parallel postulate was hopeless, and he gradually became convinced that one could construct a perfectly consistent geometry in which it was not true.

  ““Out of nothing I have created a strange new universe.” (J. Bolyai)
- By 1825, at the age of twenty-three, János was able to show his achievement to his father but not to convince him that there were no hidden flaws. It was not until 1832 that Farkas agreed to publish his son's work in the form of an appendix to his own book.

Gauss' letter

- When Gauss received a copy of the appendix, he wrote back a letter that effectively ended János's brilliant career;
- In the letter Gauss gives an infuriatingly mixed message, saying that he had himself carried out much the same program but had written very little of it down, and how pleased he was that it should turn out to be the son of his old friend who had written it down, thereby sparing him the trouble of doing it himself.

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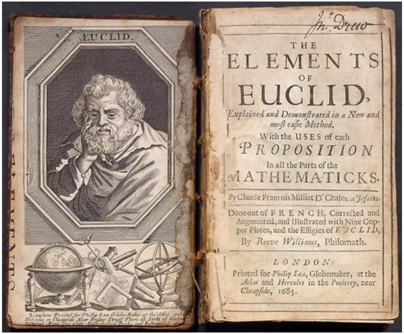
- Had Gauss given his endorsement of the work publicly and brought it to wider attention, it would have changed the course of János's life and career. János not only got no public support but was convinced that his father had betrayed him and revealed what he had been doing to Gauss, who was now claiming it as his own. (This was not the first time that Gauss's silence on the subject of non-Euclidean geometry had a devastating effect – Taurinus 1826)

Lobachevsky made the same discovery

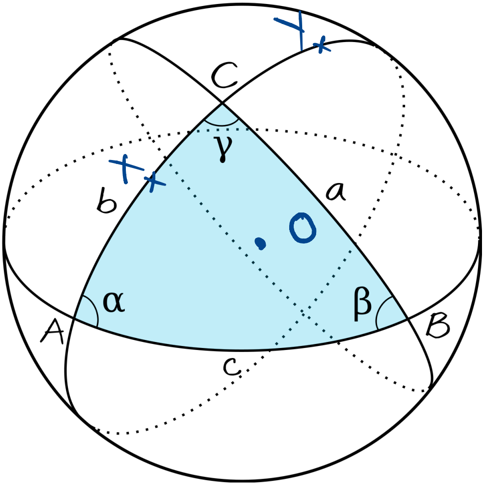
- When Gauss later became aware of Lobachevsky's work, he made no attempt to inform his old friend, Farkas.
- Rumour is that Gauss learned Russian in order to read Lobachevsky's work on the subject.
- Recognition of Bolyai's achievement did not come until too late. Gauss died in 1855, Farkas and Lobachevsky the following year, and János in 1860. As Gauss's correspondence became public, his views on non-Euclidean geometry finally became known. Bolyai's appendix was translated into Italian in 1868, the year and the place that were to be decisive for the future of the new geometry that he had invented.

What is this new geometry?

- There were many opportunities for scientists to discover non-Euclidean geometry way earlier.
- People got side-tracked by Euclid's five postulates
- It is hard to imagine how important Euclid's ideas were to these people.



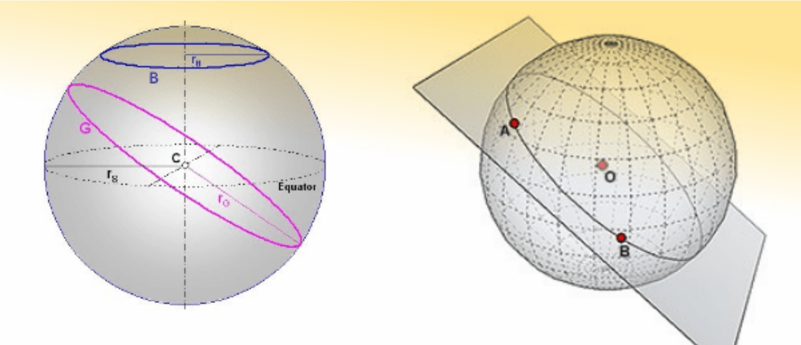
Here's a non-Euclidean geometry



- One can talk about lines, triangles, distances, angles, areas, proportions.

This geometry has been studied for millennia

- Line = a *great* circle



Great Circle Path (Magenta)
Small Circle Path (Blue)

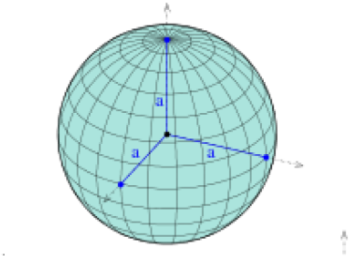
Great Circle Path formed by intersection of a plane passing through center of sphere

- There are many “lines” passing through two antipodal points (contradicting one of Euclid’s axioms);

- There is another problem regarding being able to draw a circle of any given radius on the sphere. (contradicting another one of the Euclid's postulates)
- For these reasons people said "No, that can't be a non-Euclidean geometry". Which was wrong thinking!
- Once one understands that spherical geometry is a proper geometry, it doesn't take much to understand that there is another geometry around, *hyperbolic geometry*.

A model for hyperbolic geometry

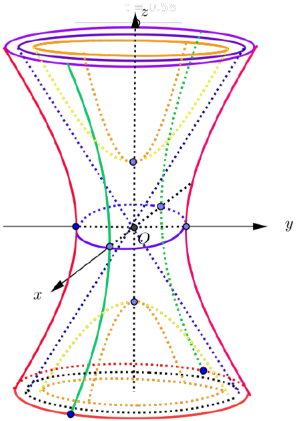
- We will present a model due to Eugenio Beltrami (1835–1900), an italian mathematician notable for his work concerning differential geometry and mathematical physics.
- Key is to generalize the sphere:



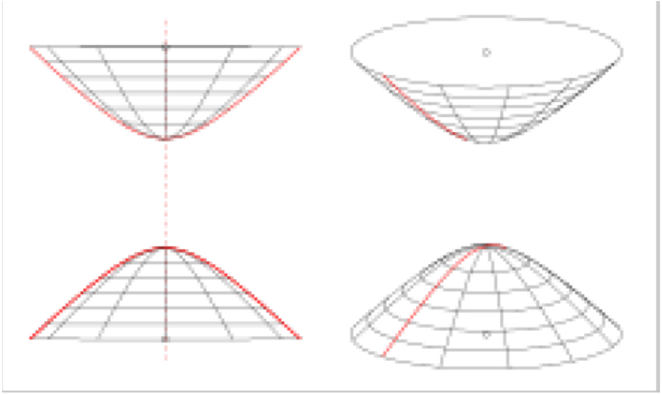
$$x^2 + y^2 + z^2 = K$$

- By changing a sign in the defining equation

$$x^2 + y^2 - z^2 = K$$

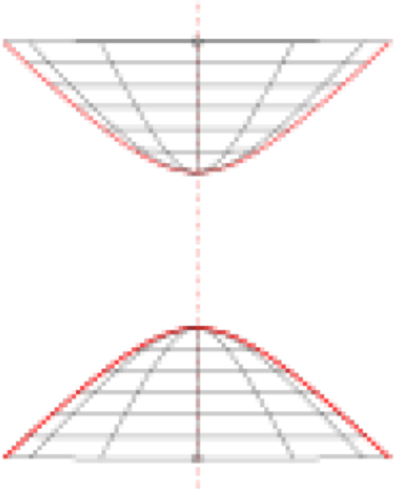


Beltrami focused on K = -1



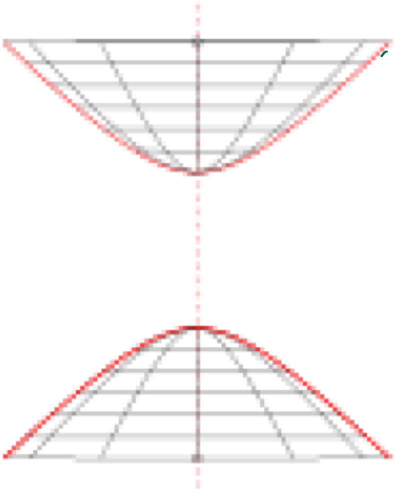
- What are the lines?

It becomes a little bit hard to draw on this thing



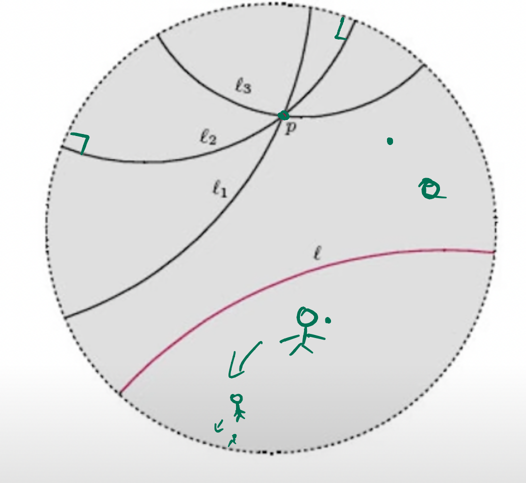
- This is nowadays called the Beltrami-Klein model.

Alternatively

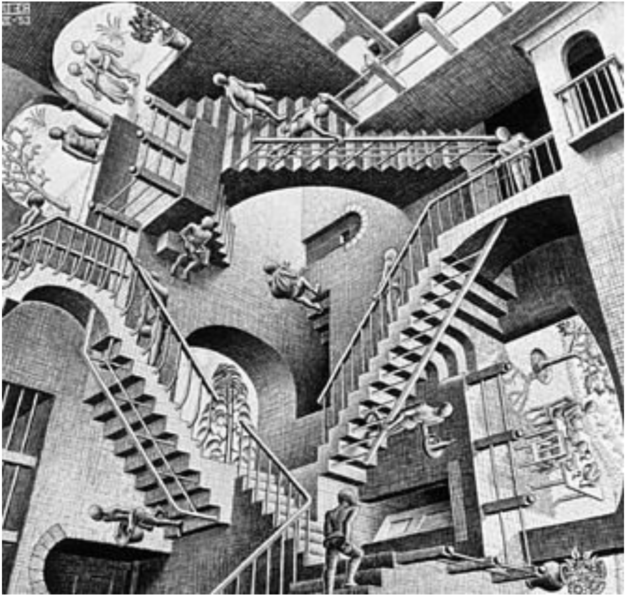


- This is nowadays called the Beltrami-Poincaré model.

Poincaré model



Maurits Cornelis Escher's art



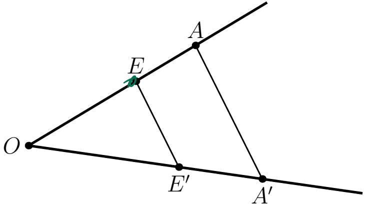
Similar triangles

Let $A, B, C \in \mathcal{P}$ be non-coliniar points. Then, the geometric figure consisting of the union of the closed segments $[AB]$, $[BC]$, $[CA]$ is called a triangle. We denote this by $\triangle ABC$.

Theorem

(Thales) *Consider the angle $\angle EOE'$ and the points $A \in (OE$, $A' \in (OE'$. Then we have $AA' \| EE'$ if and only if*

$$\frac{OA}{OE} = \frac{OA'}{OE'}.$$



The definition of similarity

Definition

The triangles ABC and $A'B'C'$ are *similar* if **both** of the following conditions are satisfied:

- the corresponding angles are congruent:

$$\angle A \equiv \angle A',\ \angle B \equiv \angle B',\ \angle C \equiv \angle C'$$

- the following ratios are equal to some constant $k > 0$:

$$\frac{BC}{B'C'} = \frac{CA}{C'A'} = \frac{AB}{A'B'} = k,$$

  where k is called the similarity ratio.

In this case, we use the notation $\triangle ABC \sim \triangle A'B'C'$. $(\neq \Delta B'A'C')$

We just defined an equivalence relation

1. reflexivity : $\triangle ABC \sim \triangle ABC$

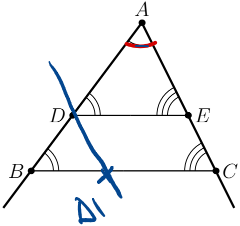
2. transitivity : $\triangle ABC \sim \triangle A'B'C'$, $\triangle A'B'C' \sim \triangle A''B''C''$ $\Rightarrow \triangle ABC \sim \triangle A''B''C''$

3. symmetry : $\triangle ABC \sim \triangle A'B'C' \Leftrightarrow \triangle A'B'C' \sim \triangle ABC$.

Theorem (fundamental theorem of similar triangles)

Let ABC be a triangle and $D \in AB$ an arbitrary point. The parallel through D at BC meets AC in E. Then $\triangle ABC \sim \triangle ADE$.

Proof.



I.) $A - D - B$ (A, D, B - collinear and in this order).

$DE \parallel BC \overset{\text{Thales}}{\Longrightarrow} \dfrac{AD}{AB} = \dfrac{AE}{AC}$. (1)

$\sphericalangle A = \sphericalangle A$, $\sphericalangle ADE = \sphericalangle ABC$, $\sphericalangle AED = \sphericalangle ACB$ ($DE \parallel BC$).

Let $D' \in (BC)$ s.t. $DD' \parallel AC$.

Then $DED'C$ is a par. $\Rightarrow DE = D'C$.

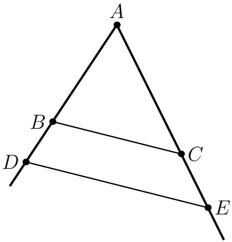
$$\frac{DE}{BC} = \frac{D'C}{BC} = \frac{AD}{AB} \qquad (2)$$

From (1) and (2) it follows that

$$\frac{AD}{AB} = \frac{AE}{AC} = \frac{DE}{BC}.$$

$\Rightarrow \triangle ABC \sim \triangle ADE.$

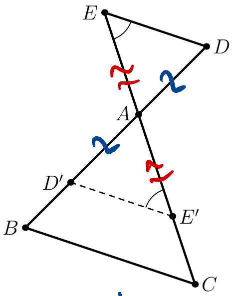
2) $A - B - D$



By Case I, we get $\Delta ABC \sim \Delta ADE$.

Then we use reflexivity of similarity.

III) $D - A - B$.



Let D' be the reflection of D in A and E' be the reflection of E in A.

$$\Delta ADE \equiv \Delta AD'E' \left(\begin{cases} AD = AD' \\ \sphericalangle DAE = \sphericalangle D'AE' \\ AE = AE' \end{cases} (S.A.S) \right)$$

By Case I (or II) $\Delta AD'E' \sim \Delta ABC$.

By transitivity, $\Delta ADE \sim \Delta ABC$.

Similar triangles

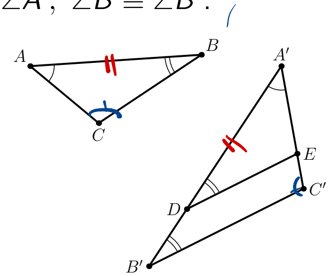
Theorem (AA)

Let ABC and $A'B'C'$ be two triangles If two out of the three relations below

$$\angle A \equiv \angle A', \ \angle B \equiv \angle B', \ \angle C \equiv \angle C'$$

hold, then $\triangle ABC \sim \triangle A'B'C'$.

Proof. Suppose $\angle A \equiv \angle A', \ \angle B \equiv \angle B'$.



Let $D \in (A'B'$ such that $A'D = AB$. Consider $DE \| B'C'$, where $E \in A'C'$.

$\Delta ABC \equiv \Delta A'DE \quad \begin{cases} \sphericalangle A = \sphericalangle A' \\ AB = A'D \\ \sphericalangle A'DE = \sphericalangle A'B'C' = \sphericalangle ABC \end{cases}$

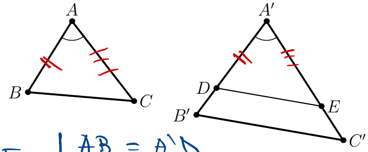
By F. T. of similarity, $\Delta A'DE \sim \Delta A'B'C'$.

Therefore, by transitivity, $\Delta ABC \sim \Delta A'B'C'$.

Theorem (SAS)

Given triangles ABC and $A'B'C'$, if $\angle A \equiv \angle A'$, $\dfrac{AB}{A'B'} = \dfrac{AC}{A'C'}$, then $\triangle ABC \sim \triangle A'B'C'$.

Proof. Construct the points $D \in (A'B'$ such that $A'D = AB$ and $E \in (A'C'$ with $A'E = AC$.



$$\triangle ABC \equiv \triangle A'DE \quad \begin{cases} AB = A'D \\ \measuredangle A = \measuredangle A' \\ AC = A'E. \end{cases}$$

$$\frac{AB}{A'B'} = \frac{A'D}{A'B'} = \frac{A'E}{AC'} = \frac{AC}{A'C'}$$

$$\Rightarrow \frac{A'D}{A'B'} = \frac{A'E}{A'C'} \xRightarrow{R.\ Thales} DE \parallel B'C'.$$

By the fundamental theorem of similar triangles,

$\Delta A'DE \sim \Delta A'B'C'$.

By transitivity, $\Delta ABC \sim \Delta A'B'C'$.

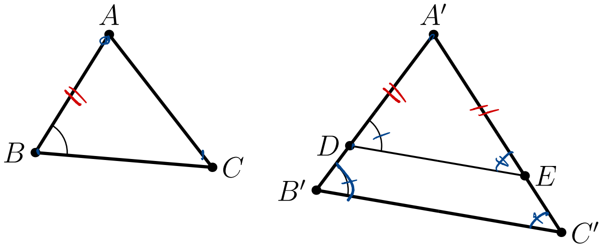
Theorem (SSS)

Let ABC and $A'B'C'$ be triangles. If

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'},$$

then $\triangle ABC \sim \triangle A'B'C'$.

Proof. Construct the points $D \in (A'B'$ such that $A'D = AB$ și $E \in (A'C'$ cu $A'E = AC$.



From Thales thm. it follows that $DE \parallel B'C'$ and hence, by the Fundamental Thm. of similarity we have $\Delta A'DE \sim \Delta A'B'C'$.

Hence

$$\left.\begin{cases} \dfrac{A'D}{A'B'} = \dfrac{A'E}{A'C'} = \dfrac{DE}{B'C'} \\[2ex] \dfrac{A'D}{A'B'} = \dfrac{AB}{A'B'} \underset{\substack{\uparrow \\ \text{hypothesis}}}{=} \dfrac{BC}{B'C'} \end{cases}\right\} \Rightarrow DE = BC.$$

$\therefore \Delta ADE \equiv \Delta ABC$ (S.S.S) hence $\Delta ABC \sim \Delta A'B'C'$.