23.11.2022

Ordered sets

$\rho = (A, A, R)$ order relation $\Leftrightarrow$ ρ refl, trans + antisymmetric

(A, ρ) ordered set

(A, ρ) tot. ordered $\Leftrightarrow$ $\forall x, y \in A \ (x \rho y) \vee (y \rho x)$

86) $A \neq \emptyset$, $\rho, \rho' \in O(A)$ (set of order relations on A). Prove that:

a) $\rho \cap \rho' \in O(A)$

$\rho \cap \rho'$ refl: $\forall x \in A \overset{\rho, \rho' \text{refl}}{\Longrightarrow} x \rho x \wedge x \rho' x \Rightarrow x (\rho \cap \rho') x \Rightarrow$

$\rho \cap \rho'$ reflx.

$\rho \cap \rho$ trans: Let $x, y, z \in A$ s.t. $A \ (x(\rho \cap \rho')y) \wedge (y(\rho \cap \rho')z) \Rightarrow$

$\Rightarrow \underline{(x \rho y)} \wedge (x \rho' y) \wedge \underline{(y \rho z)} \wedge (y \rho' z) \underset{\rho' t}{\overset{\rho t}{\Longrightarrow}} (x \rho z) \wedge (x \rho' z) \Rightarrow$

$\Rightarrow x (\rho \cap \rho') z \Rightarrow \rho \cap \rho'$ trans

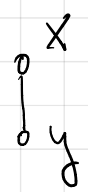
$\rho \cap \rho$ antisy: Let $x, y \in A$ s.t. $x(\rho \cap \rho')y \wedge y(\rho \cap \rho')x \Rightarrow$

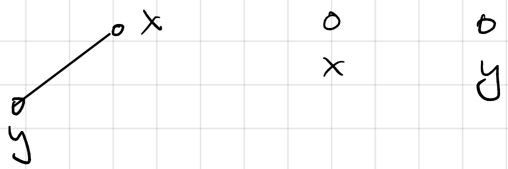
$\Rightarrow \underline{(x \rho y)} \wedge \underline{(x \rho' y)} \wedge \underline{(y \rho x)} \wedge \underline{(y \rho' x)} \underset{\rho' \text{anti}}{\overset{\rho \text{ anti}}{\Longrightarrow}} (x = y) \wedge (x = y)$

$\Rightarrow x = y \Rightarrow$

$\rho \cap \rho'$ anti

$\Rightarrow (A, \rho \cap \rho')$ ord. set

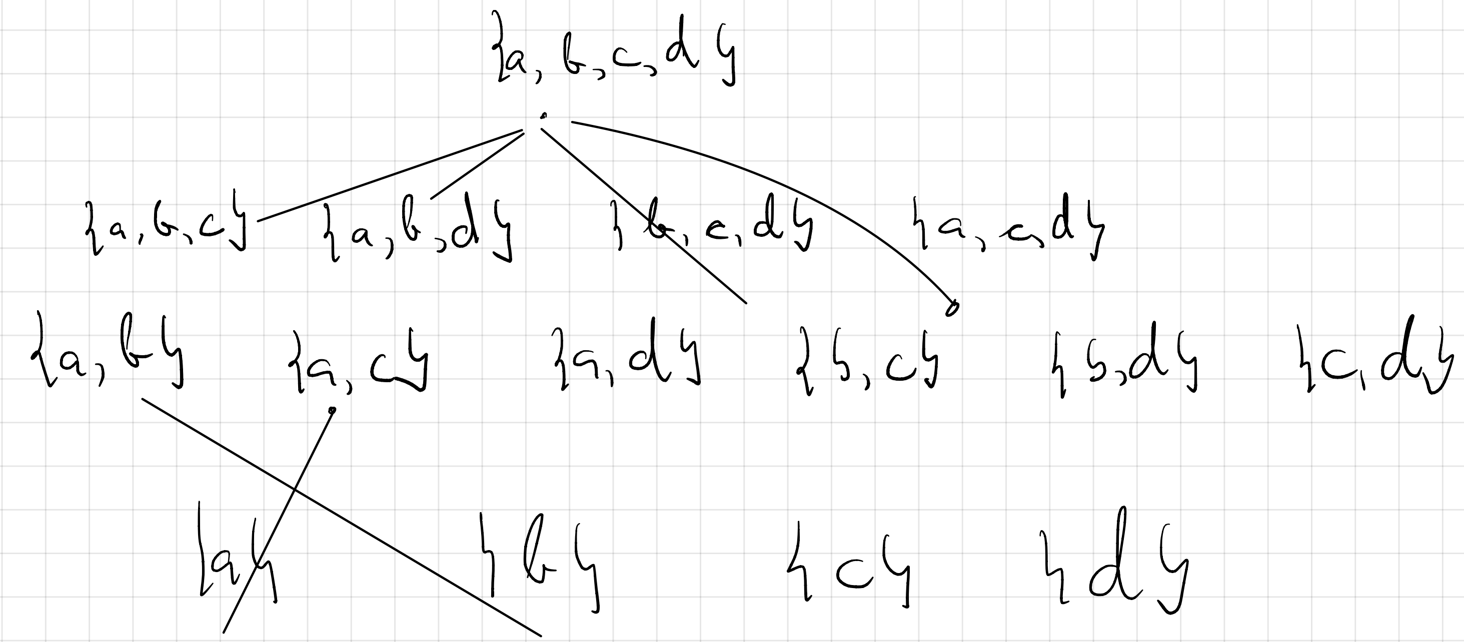
87)  If $y < x$ and $\nexists z : y < z < x$



$(y < x) \Leftrightarrow (y \leq x) \wedge (y \neq x)$

Draw the Hasse diagram for the following ordered sets:

a) $(\mathcal{P}(\{a,b,c,d\}), \subseteq)$

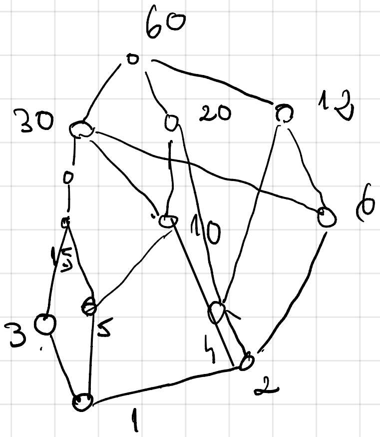


b) $(D_{60}, |)$, $D_{60} = \{n \in \mathbb{N} \mid n \mid 60\}$

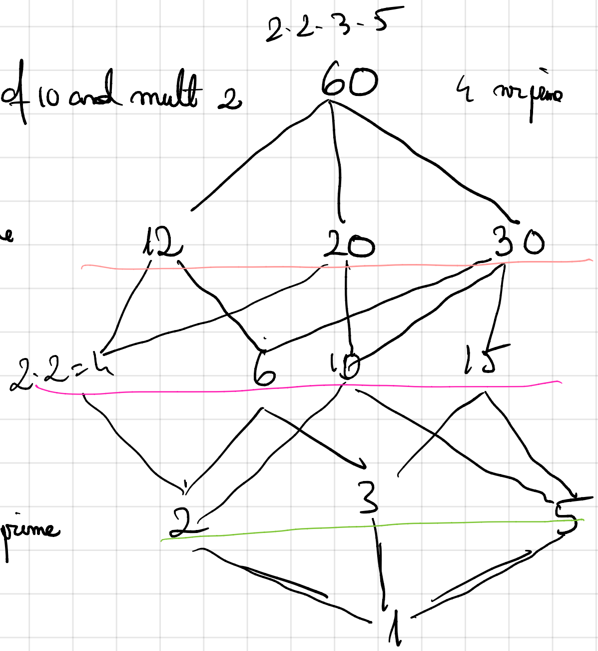
$D_{60} = \{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$

$\{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$

no div of 60 and multp of 20



2 – 10 no div of 10 and mult 2



Ex: $(A, \leq), (B, \leq)$ ordered sets

$f: A \to B$ increasing $\Leftrightarrow \forall x, y \in A$

$x \leq y \Rightarrow f(x) \leq f(y)$

$f: A \to B$ decreasing $\Leftrightarrow \forall x, y \in A$

$x \leq y \Rightarrow f(x) \geq f(x) \geq f(y)$

a) If $f: A \to B$, $g: B \to C$ are increasing $\Rightarrow g \circ f$ increasing

Let $x, y \in A$: $x \leq y \overset{f \text{ incr}}{\Rightarrow} f(x) \leq f(y) \overset{g \text{ incr}}{\Rightarrow} g(f(x)) \leq g(f(y)) \Rightarrow$

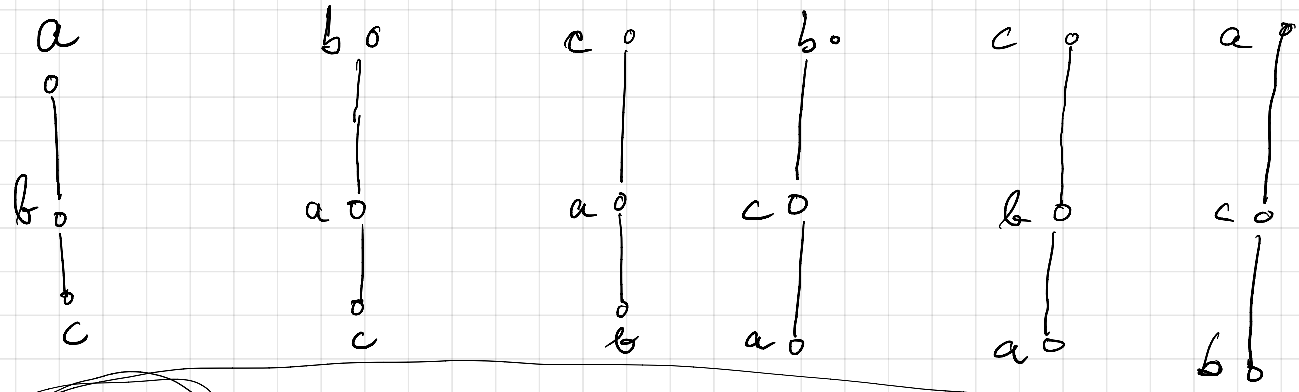
$\Rightarrow (g \circ f)(x) \leq (g \circ f)(y) \Rightarrow g \circ f$ is increasing

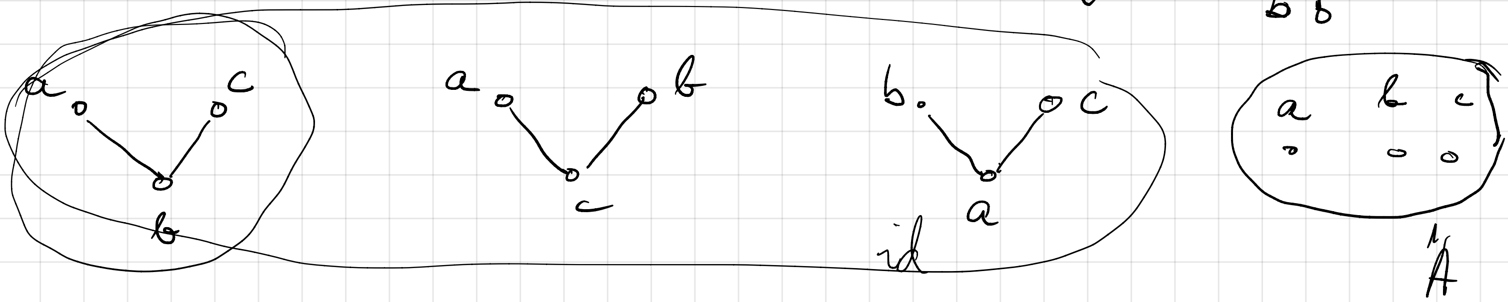
b) f incr, g decr $\Rightarrow g \circ f$ decreasing

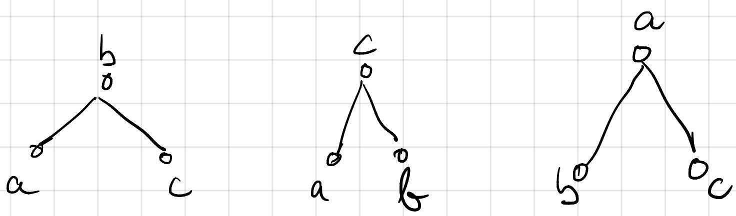
Let $x, y \in A$: $x \leq y \overset{f \text{ incr}}{\Rightarrow} f(x) \leq f(y) \overset{g \text{ decr}}{\Rightarrow} g(f(x)) \geq g(f(y)) \Rightarrow$

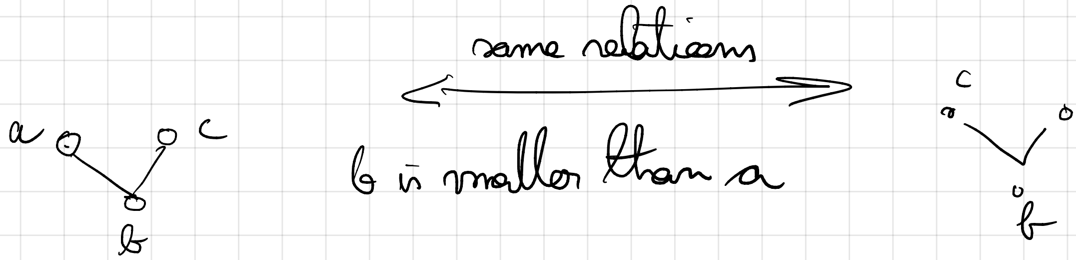
$\Rightarrow (g \circ f)(x) \geq (g \circ f)(y) \Rightarrow g \circ f$ decr.

88) list all the order relations on $A = \{a, b, c\}$ using Haase diagrams









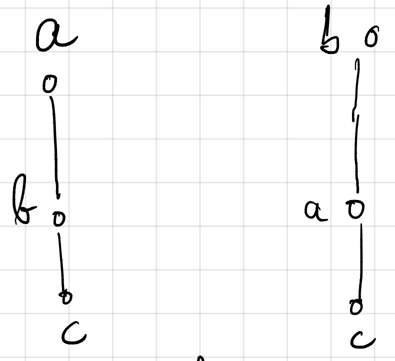
$\rho = \{(b,a),(b,c),(a,a),(b,b),(c,c)\}$ reflx ✓

trans ✓

antisy ✓

(we have (b,a), but not (a,b))

(If I have (a,b) and (b,a) → simmetrique)



$(A, \leq) \xrightarrow{f} (B, \leq)$

$x \leq y \Rightarrow f(x) \leq f(y)$

$f : (A, \leq) \to (B, \leq)$ is an isomorphism $\Leftrightarrow$ f is bijective, increasing and f^{-1} is also increasing

$f(a) = b$
$f(b) = a$
$f(c) = c$

finite set $\to$ bij

$b \leq a \to f(b) \leq f(a)$
$ \;\; a \;\; \leq' \;\; b$

increasing ✓

$\Rightarrow$ isomorphism

90) Let $(A, \leq), (B, \leq)$ ordered sets and $f : A \to B$ a bij and incr. function

a) Prove that if A is tot. ord, then f^{-1} is increasing and B is tot. ord

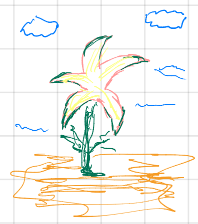
$(\mathbb{N}, \leq)$ tot. ordered

$(\mathbb{N}, |)$ not tot ord
$2 \nmid 3$
$3 \nmid 2$

i) Let $b_1, b_2 \in B$ s.t. $b_1 \leq b_2 \overset{f \text{ bij}}{\Longrightarrow} \exists! \, a_1, a_2 \in A$:

$f(a_1) = b_1$
$f(a_2) = b_2$

$\overset{(A, \leq) \text{ tot ord}}{\Longrightarrow} (a_1 \leq a_2)$ or $(a_2 \leq a_1)$



I case: $a_1 \leq a_2 \overset{f \text{ incr.}}{\Longrightarrow} f(a_1) \leq f(a_2)$

$a_1 \leq a_2 \Rightarrow a_1 = f^{-1}(b_1) \leq f^{-1}(b_2) = a_2 \Rightarrow f^{-1}$ increasing

II case $a_2 \leq a_1 \overset{f \text{ incr.}}{\Longrightarrow} f(a_2) \leq f(a_1)$

$ \underset{b_2}{\parallel} \leq \underset{b_1}{\parallel}$

$b_2 \leq b_1$
(hyp) $b_1 \leq b_2$

$\Rightarrow b_1 = b_2 \Rightarrow f^{-1}(b_1) = f^{-1}(b_2)$

ii) Prove B tot. ord

$(B, \le)$ tot. ord $\Leftrightarrow \forall b_1, b_2 \in B \quad (b_1 \le b_2) \vee (b_2 \le b_1)$

$\forall b_1, b_2 \in B, \ \exists! a_1, a_2 \in A$ s.t. $f(a_1) = b_1$, $f(a_2) = b_2$ $\overset{A \text{ tot. ord}}{\Longrightarrow}$ $a_1 \le a_2$ or $a_2 \le a_1$

I case : $a_1 \le a_2 \overset{f \text{ incr}}{\Longrightarrow} \underset{b_1}{\underset{\|}{f(a_1)}} \le \underset{b_2}{\underset{\|}{f(a_2)}}$

II case $a_2 \le a_1 \overset{f \text{ incr}}{\Longrightarrow} \underset{b_2}{\underset{\|}{f(a_2)}} \le \underset{b_1}{\underset{\|}{f(a_1)}}$

$\Rightarrow b_1 \le b_2$ or $b_2 \le b_1 \Rightarrow B$ tot ord

b) Prove that $1_{\mathbb{N}^*} : (\mathbb{N}^*, |) \to (\mathbb{N}^*, \le)$ is not an ordered isomorphism

$\iota_{\mathbb{N}^*}$

$2 \le 3 \not\Rightarrow 2 \nmid 3$

$\underset{a}{\underset{\|}{1_{\mathbb{N}^*}(a)}} \le \underset{b}{\underset{\|}{1_{\mathbb{N}^*}(b)}} \overset{?}{\not\Rightarrow} a \mid b$

Ex 92: $\rho = (A, A, R)$ preorder $< \begin{matrix} \text{refl} \\ \text{tran} \end{matrix}$, then $\Gamma = \rho \cap \rho^{-1}$ is an eq rel on A

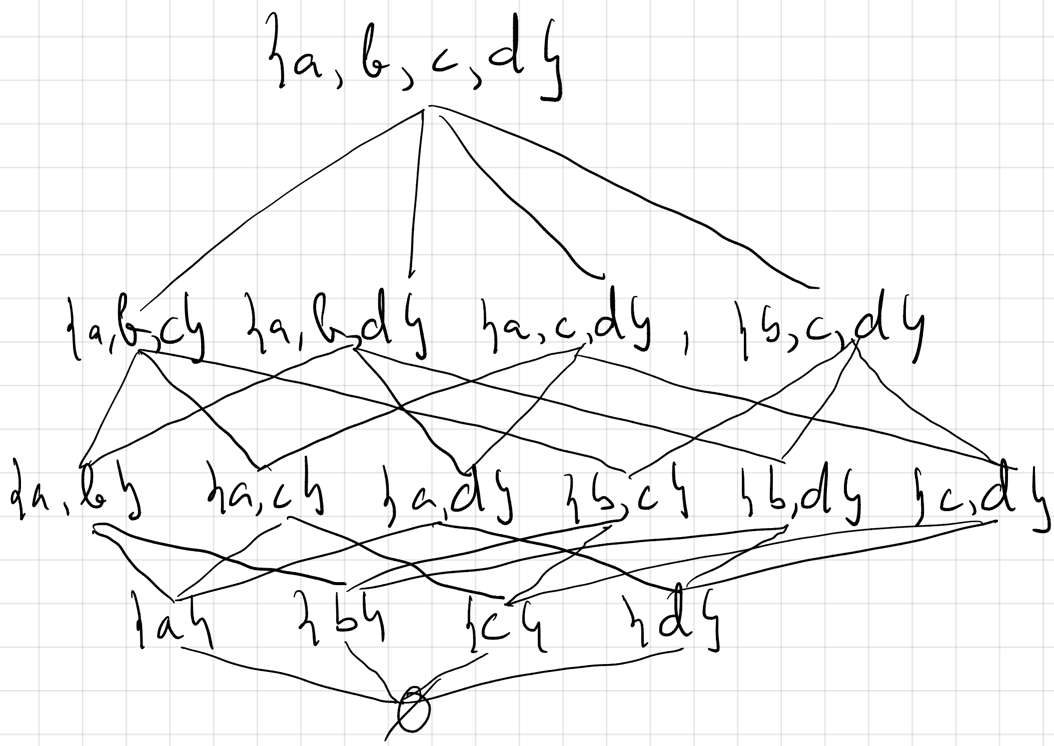
and $(A/_\Gamma, \le)$ ordered set, where $\rho\langle x\rangle \le \rho\langle y\rangle \Leftrightarrow x \rho y$

Apply the theorem for $(\mathbb{Z}, |)$

$A/_\Gamma \simeq \mathbb{N}$

$\rho\langle x\rangle = \{x, -x\}$

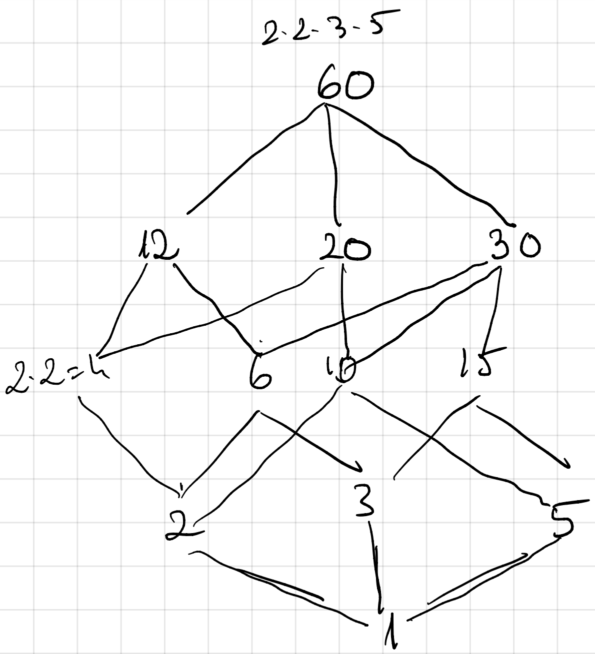
Example $\rho\langle 2\rangle = \{2, -2\} = \rho\langle -2\rangle$
I don't care about - minus


{a, b, c, d}
{a,b,c}
{a,b,d}
{a,c,d},
{b,c,d}
{a,b}
{a,c}
{a,d}
{b,c}
{b,d}
{c,d}
{a}
{b}
{c}
{d}
∅

3)

2)

1)


2·2·3·5
60
12
20
30
2·2=4
6
10
15
2
3
5
1