Analytic Geometry

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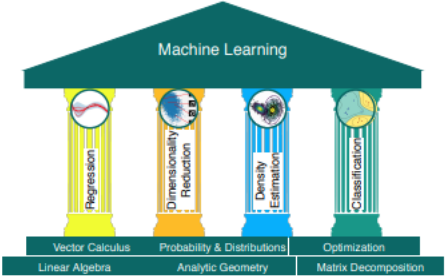
Introduction and motivation



- Given two vectors representing two objects in the real world, we want to make statements about their similarity. The idea is that vectors that are similar should be predicted to have similar outputs by our machine learning algorithm (predictor).

Introduction and motivation

- To formalize the idea of similarity between vectors, we need to introduce operations that take two vectors as input and return a numerical value representing their similarity. The construction of similarity and distances is central to analytic geometry.

Euclidean Geometry is the study of geometric elements (points, lines, planes, etc), relations between elements and configurations. The simplest geometric elements and basic relations between them were introduced for the first time in an axiomatic way by Euclid (IV^{th}-III^{th} centuries B.C.), in his famous book "Elements".

Analytic Geometry is the study of geometric configurations by using the coordinates method, which works not only in $\mathcal{E}_2$ or $\mathcal{E}_3$, but also in a n-dimensional Euclidean space.

The $\mathbb{R}^n$ coordinate space

- Let $\mathbb{R}^n$ be the set of all ordered n-tuples of real numbers, i.e.

$$\mathbb{R}^n = \mathbb{R} \times \ldots \times \mathbb{R}.$$

- Therefore, an element x of $\mathbb{R}^n$ has the form $x = (x_1, \ldots, x_n)$; the real numbers x_1, ..., x_n are called *components* of x. Recall that two n-tuples $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are equal if and only if their components are respectively equal:

$$(x_1, \ldots, x_n) = (y_1, \ldots, y_n) \Longleftrightarrow x_1 = y_1, \ldots, x_n = y_n.$$

The Euclidean 1-space $\mathcal{E}_1$

Let $\mathcal{E}_1$ be the 1-dimensional Euclidean space, i.e. a line d together with one of the above mentioned system of axioms. Let O and A be two different points, fixed on d, such that $OA = 1$.

One obtains an orientation on the line d (from O to A) and can introduce the function

$$f_1 : \mathcal{E}_1 \to \mathbb{R}, \quad f_1(P) = x_P,$$

where $|x_P| = OP$ and $\begin{cases} x_P \geq 0 & \text{if} \quad P \in [OA \\ x_P < 0 & \text{if} \quad P \notin [OA \end{cases}.$

f is bijective and one can associate to any point $P \in \mathcal{E}_1$ a unique real number x_P. One says that Ox is a *Cartesian system of coordinates* on $\mathcal{E}_1$, having the *origin* O and the *axis* Ox, while x_P is said to be the *coordinate* of P. We shall use the notation $P(x_P)$.

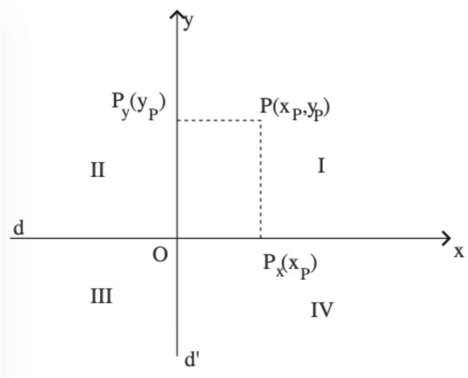
The 2-dimensional Euclidean space $\mathcal{E}_2$

- **Rectangular coordinates**
- Denoted $\mathcal{E}_2$ is the 2-dimensional Euclidean space, i.e. a plane π together with a system of axioms mentioned on the previous slides.
- Let $O \in \mathcal{E}_2$ be a fixed point d and d' be two orthogonal lines, passing through O. One can choose, on each of the lines d and d', a Cartesian system of coordinates, having the same origin O. Suppose they are denoted by Ox respectively Oy.

One can associate to any point $P \in \mathcal{E}_2$ a unique pair $(x_P, y_P) \in \mathbb{R}^2$, so there's a bijection (a one-to-one correspondence)

$$f_2 : \mathcal{E}_2 \to \mathbb{R}^2, \quad f_2(P) = (x_P, y_P).$$

The *origin* O together with the *axes* Ox and Oy form a *Cartesian system of coordinates* Oxy *on* $\mathcal{E}_2$. We say that the *rectangular coordinates* of P in this system are (x_P, y_P).



- Recall the order of the quadrants

- Recall the order of the quadrants
- The distance formula. Given $P_1(x_1, y_1), P_2(x_2, y_2)$ in rectangular coordinates, we have

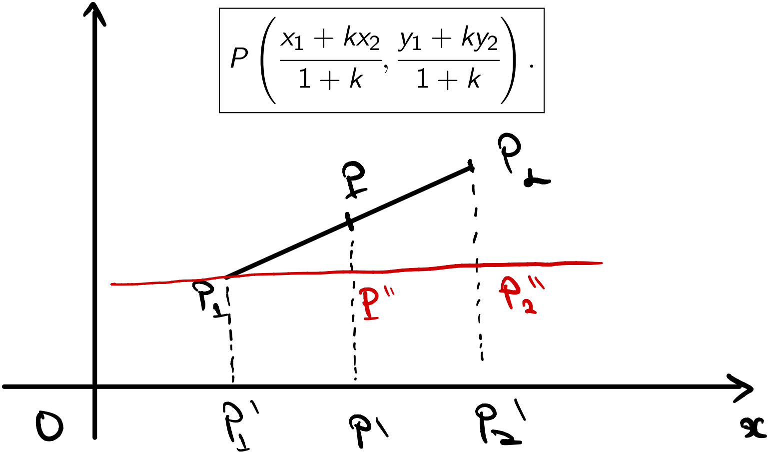
$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Proof: Just apply Pythagoras's thm twice.

The one and only

If the point P divides the segment $[P_1P_2]$ into the ratio k, i.e. $\dfrac{PP_1}{PP_2} = k$, then the coordinates of P are

$$\boxed{P\left(\frac{x_1 + kx_2}{1+k}, \frac{y_1 + ky_2}{1+k}\right).}$$



From Thales thm,

$$\frac{P_1 P''}{P_1 P_2''} = \frac{P_1 P}{P_1 P_2} = \frac{k}{k+1}. \qquad (*)$$

$$\frac{P_1 P_2}{P_1 P} = \frac{P_1 P + P P_2}{P_1 P} = 1 + \frac{1}{k} = \frac{k+1}{k}.$$

From $(*)$ $$\frac{x_P - x_{P_1}}{x_{P_2} - x_{P_1}} = \frac{k}{k+1}$$

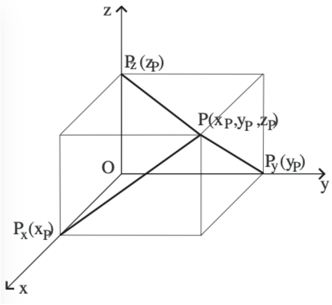
$$x_P = \frac{k}{k+1} \cdot x_{P_2} - \frac{k}{k+1} x_{P_1} + x_{P_1}$$

$$x_P = \frac{1}{k+1} x_{P_1} + \frac{k}{k+1} x_{P_2}.$$

The coordinate y_P can be computed analogously.

The 3-dimensional Euclidean space $\mathcal{E}_3$

- **Rectangular coordinates**
- Let $\mathcal{E}_3$ be the 3-dimensional Euclidean space, O be a fixed point and d, d', d'' be three pairwise orthogonal lines, passing through O. Choose, on the lines d, d' and d'', the Cartesian systems of coordinates Ox, Oy respectively Oz, having the same origin O.



For an arbitrary point in $P \in \mathcal{E}_3$, denote by x_P, y_P and z_P the coordinates of its orthogonal projections P_x, P_y and P_z on d, d' respectively d''. One can define the bijection

$$f_3 : \mathcal{E}_3 \to \mathbb{R}^3, \quad f_3(P) = (x_P, y_P, z_P).$$

We introduced on $\mathcal{E}_3$ the right *rectangular coordinates system* (or the Cartesian system) $Oxyz$, having the following elements:

- the *origin* O;
- the *coordinate lines* (or *axes*) Ox, Oy, Oz;
- the *coordinate planes* Oxy, Oyz, Ozx;

Some examples

- The origin O has the coordinates $(0, 0, 0)$.
- The points situated on Ox, Oy and Oz are of coordinates $(x, 0, 0)$, $(0, y, 0)$, respectively $(0, 0, z)$.
- The points situated on Oxy, Oyz and Ozx have the coordinates $(x, y, 0)$, $(0, y, z)$ respectively $(x, 0, z)$.

The three coordinate planes divide the space $\mathcal{E}_3$ into eight domains, denoted by I, II, ..., VIII. These domains are defined by the signs of the coordinates of the points they contain, as in the following table: (DO NOT MEMORIZE!)

Domain	I	II	III	IV	V	VI	VII	VIII
x_P	$+$	$-$	$-$	$+$	$+$	$-$	$-$	$+$
y_P	$+$	$+$	$-$	$-$	$+$	$+$	$-$	$-$
z_P	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$

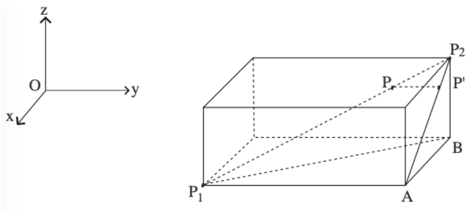
The distance formula

Theorem

The distance between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is given by

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Proof.



Draw a rectangular parallelipiped s.t. its faces are parallel to the coordinate planes and P_1, P_2 are opposite vertices, as above.

$\therefore \ P_1A = |y_{P_1} - y_{P_2}|$ and $AB = |x_{P_1} - x_{P_2}|$

We have,

$$P_1B^2 = P_1A^2 + AB^2.$$

$$P_1P_2^2 = P_1B^2 + BP_2^2 = (y_{P_1} - y_{P_2})^2 + (x_{P_1} - x_{P_2}) + (z_{P_1} - z_{P_2})^2$$

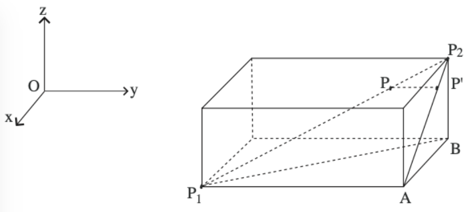
$\boxed{\text{B}}$

Theorem

If the point P divides the segment $[P_1P_2]$, $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, into the ratio k (i.e. $\dfrac{PP_1}{PP_2} = k$), then the coordinates of P are

$$\left(\frac{x_1 + kx_2}{1+k}, \frac{y_1 + ky_2}{1+k}, \frac{z_1 + kz_2}{1+k}\right).$$

$\dfrac{P_1P_2}{PP_2} = k+1.$



Construct P' s.t. $PP' \parallel P_1A$ and $P' \in P_2A$.

From Thales' thm:

$$\frac{PP'}{P_1A} = \frac{AP_2}{P_2P_1} = \frac{1}{k+1}$$

But $$\frac{PP'}{P_1A} = \frac{|y_2 - y_P|}{|y_2 - y_1|} = \frac{1}{k+1}.$$

Suppose $y_2 > y_1$ and $y_P > y_1$

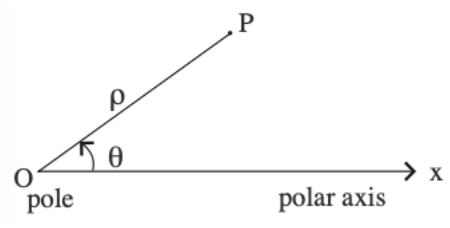
$$\Rightarrow \quad y_P = \frac{y_1 + k y_2}{1 + k}. \quad \blacksquare$$

Other coordinate systems in $\mathcal{E}_2$ and $\mathcal{E}_3$

So far we used a rectangular coordinate system to pin-point a point P in the plane $\mathcal{E}_2$ or in the space $\mathcal{E}_3$. There are many situations when this makes computation difficult.

The Polar Coordinate System (PS)

As an alternative to a rectangular coordinate system (RS) one considers in the plane $\mathcal{E}_2$ a fixed point O, called *pole* and a half-line directed to the right of O, called *polar axis*.



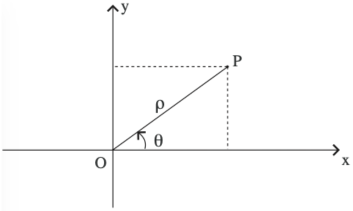
- By specifying a directed distance ρ from O to a point P and an angle θ (measured in radians), whose "initial" side is the polar axis and whose "terminal" side is the ray OP, the *polar coordinates* of the point P are (ρ, θ).
- One obtains a bijection

$$\mathcal{E}_2 \setminus \{O\} \to \mathbb{R}_+ \times [0, 2\pi), \quad P \to (\rho, \theta)$$

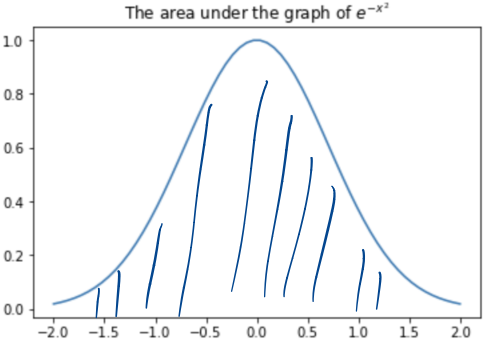
which associates to any point P in $\mathcal{E}_2 \setminus \{O\}$ the pair (ρ, θ) (suppose that $O(0,0)$). The positive real number ρ is called the *polar ray* of P and θ is called the *polar angle* of P.

- Consider RS to be the rectangular coordinate system in $\mathcal{E}_2$, whose origin O is the pole and whose positive half-axis Ox is the polar axis. The following transformation formulas give the connection between the coordinates of an arbitrary point in the two systems of coordinates.
- **PS**$\rightarrow$ **RS** Let $P(\rho, \theta)$ be a point in the system PS. It is immediate that

$$\begin{cases} x_P = \rho \cos \theta \\ y_P = \rho \sin \theta \end{cases} .$$

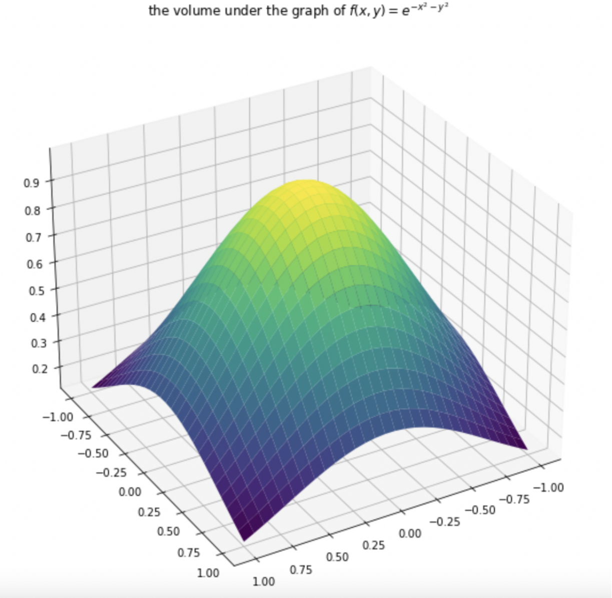


Motivation. An impossible integral



$$A = \int_{-\infty}^{+\infty} e^{-x^2}\,dx.$$

The problem becomes easier if we "complicate" it



Let us pretend we know A is finite and compute $A^2 = \left(\int_{-\infty}^{+\infty} e^{-x^2} dx\right)^2 = \int_{-\infty}^{+\infty} e^{-x^2} dx \cdot \int_{-\infty}^{+\infty} e^{-y^2} dy$

$$= \underbrace{\ldots}_{\text{not rigurous}} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-x^2-y^2} dx\, dy := V$$

V represents the volume under the 3D plot above.

We will change coordinates for the range $(x,y) \in [-\infty, +\infty] \times [-\infty, +\infty]$.

In polar coordinates we have the range

$$(r, \theta) \in [0, +\infty) \times [0, 2\pi),$$

where $x = r\cos\theta$, $y = r \cdot \sin\theta$

$$dx\,dy = \left|\begin{vmatrix} \frac{d}{dr}x & \frac{d}{d\theta}x \\ \frac{d}{dr}y & \frac{d}{d\theta}y \end{vmatrix}\right| d\theta\, dr$$

$$= \left|\begin{vmatrix} \cos\theta & -r\cdot\sin\theta \\ \sin\theta & r\cdot\cos\theta \end{vmatrix}\right| d\theta\, dr = r \cdot d\theta\, dr.$$

$$\therefore A^2 = V = \int_0^{+\infty}\int_0^{2\pi} r\cdot e^{-r^2}\, d\theta\, dr = \int_0^{+\infty}\left(-\frac{1}{2}e^{-r^2}\right)' \int_0^{2\pi} 1\, d\theta\, dr$$

$$= 2\pi \cdot \left[-\frac{1}{2} e^{-r^2} \right]_{r=0}^{+\infty} = \pi.$$

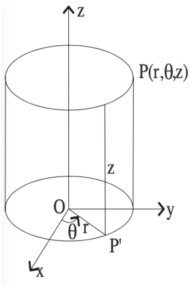
So, the one we wanted is

$$A = \int_{-\infty}^{+\infty} e^{-x^2} \, dx = \sqrt{\pi}.$$

The cylindrical coordinate system

In order to have a valid coordinate system in the 3-dimensional case, each point of the space must be associated to a unique triple of real numbers (the coordinates of the point) and each triple of real numbers must determine a unique point.

Let $P(x, y, z)$ be a point in a rectangular system of coordinates $Oxyz$ and P' be the orthogonal projection of P on xOy. One can associate to the point P the triple (r, θ, z), where (r, θ) are the polar coordinates of P'.



The triple (r, θ, z) gives the *cylindrical coordinates* of the point P. There is the bijection

$$h_1 : \mathcal{E}_3 \setminus \{O\} \to \mathbb{R}_+ \times [0, 2\pi) \times \mathbb{R},\ \ P \to (r, \theta, z)$$

and one obtains a new coordinate system, named the *cylindrical coordinate system* (CS) in $\mathcal{E}_3$.

In the following table, the conversion formulas relative to the cylindrical coordinate system (CS) and the rectangular coordinate system (RS) are presented.

Conversion	Formulas
CS$\rightarrow$RS $(r,\theta,z)\rightarrow(x,y,z)$	$x = r\cos\theta,\ y = r\sin\theta,\ z = z$
RS$\rightarrow$CS $(x,y,z)\rightarrow(r,\theta,z)$	$r=\sqrt{x^2+y^2}$, $z=z$ and θ is given as follows: Case 1. If $x\neq 0$, then $\theta = \arctan\frac{y}{x} + k\pi$, where $k = \begin{cases} 0, \text{if } P\in I\cup(Ox \\ 1, \text{ if } P\in II\cup III\cup(Ox' \\ 2, \text{ if } P\in IV \end{cases}$ Case 2. If $x=0$ and $y\neq 0$, then $\theta = \begin{cases} \frac{\pi}{2} \text{ when } P\in(Oy \\ \frac{3\pi}{2} \text{ when } P\in(Oy' \end{cases}$ Case 3. If $x=0$ and $y=0$, then $\theta=0$.

Some examples

1. In the cylindrical coordinate system, the equation $r = r_0$ represents a right circular cylinder of radius r_0, centered on the z-axis.

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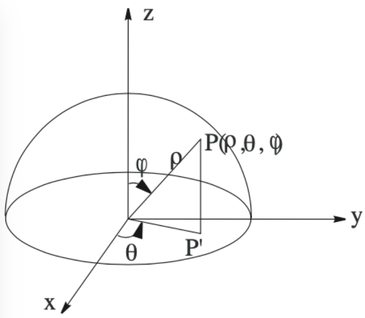
2. The equation $\theta = \theta_0$ describes a half-plane attached along the z-axis and making an angle θ_0 with the positive x-axis.
3. The equation $z = z_0$ defines a plane which is parallel to the coordinate plane xOy.

The Spherical Coordinate system

Another way to associate to each point P in $\mathcal{E}_3$ a triple of real numbers is illustrated below. If $P(x, y, z)$ is a point in a rectangular system of coordinates $Oxyz$ and P' its orthogonal projection on Oxy, let ρ be the length of the segment $[OP]$, θ be the oriented angle determined by $[Ox$ and $[OP'$ and φ be the oriented angle between $[Oz$ and $[OP$.



The triple (ρ, θ, φ) gives the *spherical coordinates* of the point P. This way, one obtains the bijection

$$h_2 : \mathcal{E}_3 \setminus \{O\} \to \mathbb{R}_+ \times [0, 2\pi) \times [0, \pi], P \to (\rho, \theta, \varphi),$$

which defines a new coordinate system in $\mathcal{E}_3$, called the *spherical coordinate system* (SS).

The conversion formulas involving the spherical coordinate system (SS) and the rectangular coordinate system (RS) are presented in the following table.

Conversion	Formulas
SS→RS $(\rho,\theta,\varphi)\to(x,y,z)$	$x=\rho\cos\theta\sin\varphi,\ y=\rho\sin\theta\sin\varphi,\ z=\rho\cos\varphi$
RS→SS $(x,y,z)\to(\rho,\theta,\varphi)$	$\rho=\sqrt{x^2+y^2+z^2},\ \varphi=\arccos\dfrac{z}{\sqrt{x^2+y^2+z^2}}$ θ is given as follows: Case 1. If $x\neq 0$, then $\theta=\arctan\dfrac{y}{x}+k\pi,$ where $k=\begin{cases}0, P'\in I\cup(Ox\\ 1, P'\in II\cup III\cup(Ox'\\ 2, P'\in IV\end{cases}$ Case 2. If $x=0$ and $y\neq 0$, then $\theta=\begin{cases}\dfrac{\pi}{2}, P'\in(Oy\\ \dfrac{3\pi}{2}, P'\in(Oy'\end{cases}$ Case 3. If $x=0$ and $y=0$, then $\theta=0$

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2. As in the cylindrical coordinates, the equation $\theta = \theta_0$ defines a half-plane attached along the z-axis, making an angle θ_0 with the positive x-axis.
3. The equation $\varphi = \varphi_0$ describes the points P for which the angle determined by $[OP$ and $[Oz$ is φ_0. If $\varphi_0 \neq \frac{\pi}{2}$ and $\varphi_0 \neq \pi$, this is a right circular cone, having the vertex at the origin and centered on the z-axis. The equation $\varphi = \frac{\pi}{2}$ defines the coordinate plane xOy. The equation $\varphi = \pi$ describes the negative axis $(Oz'$.

The problem set for this week will be posted soon. Ideally you would think about it before the seminar on Friday.

Thank you very much for your attention!