Complements of Geometry

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Optional course for Maths and Computer Science

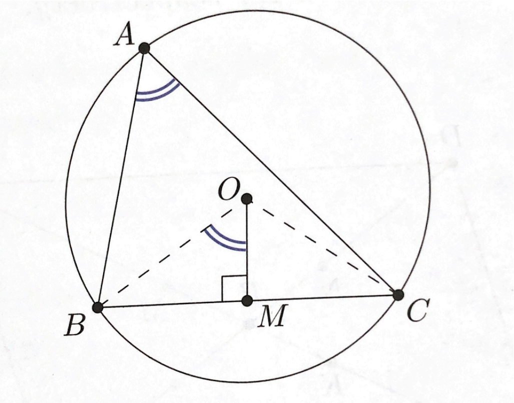
Week 4

Theorem (Extended law of sines)

Let $\triangle ABC$ be a triangle. Then

$$\frac{a}{\sin\angle A} = \frac{b}{\sin\angle B} = \frac{c}{\sin\angle C} = 2R,$$

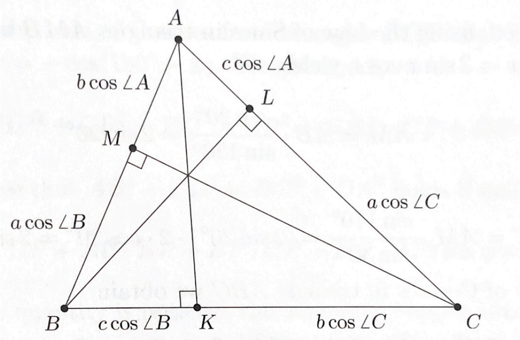
where R is the circumradius of triangle $\triangle ABC$.



Theorem (Cosine theorem)

Let $\triangle ABC$ be a triangle. Then

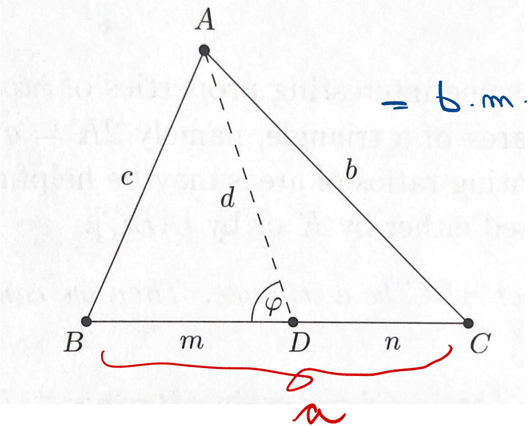
$$a^2 = b^2 + c^2 - 2bc\cos\angle A.$$



Theorem (Stewart)

In the $\triangle ABC$, let D be a point on (BC). Denote the distances BD, DC and AD by m, n and d, respectively. Then

$$a(d^2 + mn) = b^2 m + c^2 n.$$



$= b \cdot m \cdot b + c \cdot n \cdot c.$

Proof

In ΔABD, using cosine theorem we have

$$c^2 = d^2 + m^2 - 2 \cdot d \cdot m \cdot \cos(\varphi) \;\Big|\cdot m \qquad (1)$$

Similarly in ΔACD, using cosine thm we have:

$$b^2 = d^2 + m^2 + 2 \cdot d \cdot m \cdot \cos(\varphi) \;\Big|\cdot m \quad (2)$$

Adding (1) and (2) we get:

$$c^2 \cdot m + b^2 \cdot m = d^2 \cdot m + m^2 \cdot m + d^2 \cdot m + m^2 \cdot m$$

Proof

$$b^2 \cdot m + c^2 \cdot n = d^2 (n+m) + m \cdot n (n+m)$$
$$= d^2 \cdot a + m \cdot n \cdot a$$
$$= a \cdot (d^2 + m \cdot n).$$

Applications

Using Stewart's theorem, we can easily compute

- the lengths of the medians;
- the lengths of the angle bisectors.

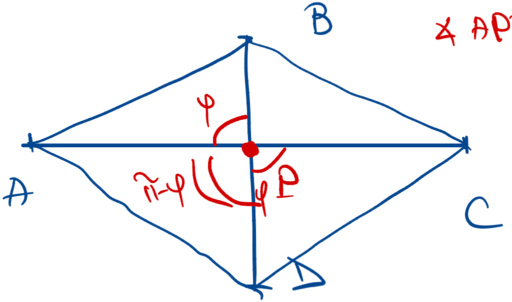
(We will do so in the seminar).

Another application to cosine theorem

Lemma (Carnot)

Let AC and BD be two lines in the plane. Then $AC \perp BD$ if and only if

$$AB^2 + CD^2 = AD^2 + BC^2.$$



$\measuredangle APB = \varphi$

$AB^2 = PA^2 + PB^2 - 2 \cdot PA \cdot PB \cdot \cos\varphi$

Continuation of the proof

$$CD^2 = PC^2 + PD^2 - 2 \cdot PD \cdot PC \cdot \cos(\varphi)$$

$$AD^2 = PA^2 + PD^2 + 2 \cdot PA \cdot PD \cdot \cos(\varphi)$$

$$BC^2 = PB^2 + PC^2 + 2 \cdot PB \cdot PC \cdot \cos(\varphi).$$

$$AB^2 + CD^2 = AD^2 + BC^2 \Longleftrightarrow$$

$$(-2 \cdot PA \cdot PB - 2 \cdot PD \cdot PC) \cdot \cos\varphi =$$

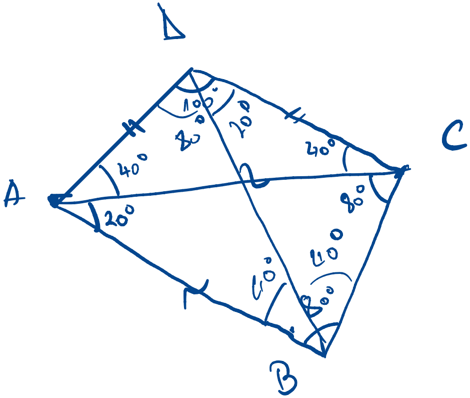
$$(2 \cdot PA \cdot PD + 2 \cdot PB \cdot PC) \cdot \cos\varphi$$

$$\Longleftrightarrow 0 = \cos\varphi \cdot (2 \cdot PA \cdot PB + 2 \cdot PD \cdot PC + 2 \cdot PA \cdot PD + 2 \cdot PB \cdot PC) \Longleftrightarrow \cos\varphi = 0.$$

Some angle chasing

Example

Let $ABCD$ be a quadrilateral such that $AB = AC$, $AD = CD$, $\angle BAC = 20^\circ$ and $\angle ADC = 100^\circ$. Show that $AB = BC + CD$.



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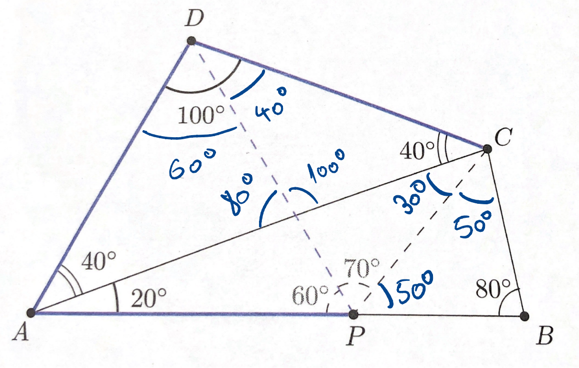


Construct point P on AB s.t. AP = DC.

Proof.

Then $AP = AD \Rightarrow \triangle PAD$ is isosceles,

but $\measuredangle DAP = 60^\circ$, so $\triangle PAD$ is equilateral.

$\triangle DPC$ is isosceles $(DP = DC)$

Then $\measuredangle DPC = \measuredangle DCP = \dfrac{180^\circ - 40^\circ}{2} = 70^\circ$

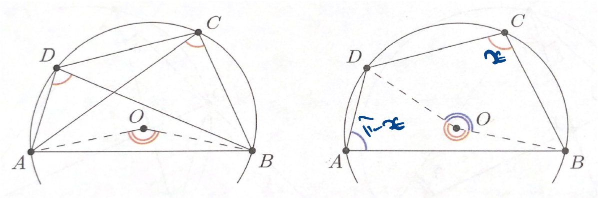
By angle chasing, we show that

$\measuredangle CPB = \measuredangle BCP = 50^\circ \Rightarrow BP = BC.$

So $AB = AP + PB = CD + BC.$

$\blacksquare$

Concyclic points and cyclic quadrilaterals



Def. Points $P_1, \dots, P_n$ are concyclic if $\exists$ a circle $\mathcal{C}$ s.t. $P_1, \dots, P_n \in \mathcal{C}$.

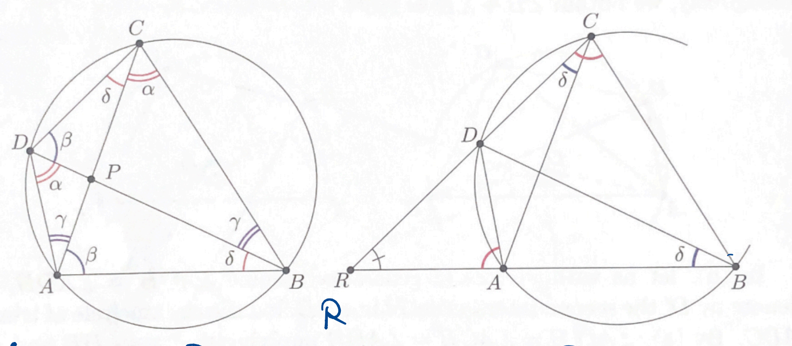
Def: A quadrilateral $ABCD$ is cyclic if A, B, C, D are concyclic.

Corollary

Let ABCD be a cyclic quadrilateral and denote by P the intersection of its diagonals. Denote by R ithe intersection of the lines BA and CD. Then, we have the following pairs of similar triangles:

(a) $\triangle ABP \sim \triangle DCP$ *and* $\triangle BCP \sim \triangle ADP$;

(b) $\triangle RAD \sim \triangle RCB$ *and* $\triangle RAC \sim \triangle RDB$.



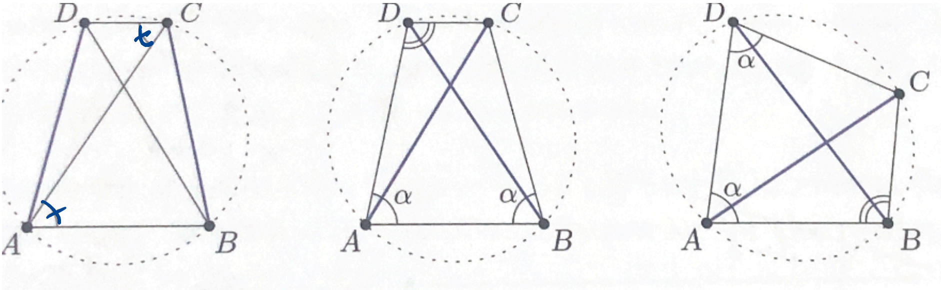
R

(b) $\measuredangle RAD = \pi - \measuredangle DAB = \measuredangle RCB$

Remarks

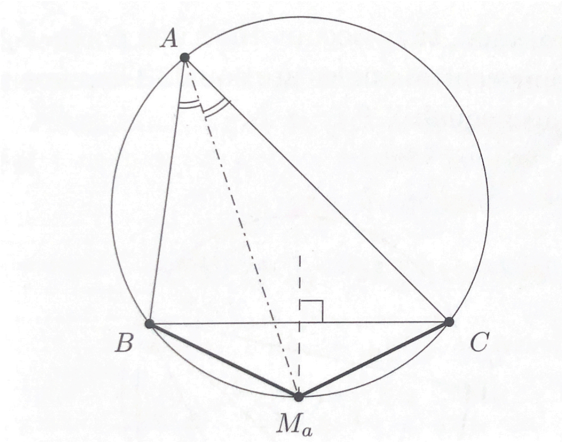
Let $ABCD$ be a cyclic quadrilateral.

- If $AD = BC$, then $ABCD$ is a trapezoid;
- I~~d~~f $AC = BD$, then $ABCD$ is a trapezoid.



Proposition

Let ABC be a triangle and ω its circumcircle. Denote by $M_a \in \omega$ the midpoint of the arc BC which does not contain the point A. Then, the angle bisector of $\angle BAC$ and the perpendicular bisector of $[BC]$ pass through M_a.



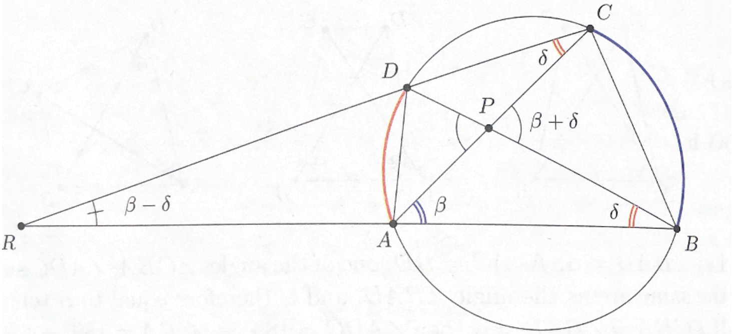
Proof

Proposition

Let ABCD by a cyclic quadrilateral and denote by P the intersection of its diagonals. Suppose that the half-lines (BA and (CD meet in a point R. We write β and δ for the measure of the angles corresponding to the arcs BC and DA (which do not contain A and B), respectively. Then

(a) $\angle BPC = \beta + \delta$;

(b) $\angle BRC = \beta - \delta$.

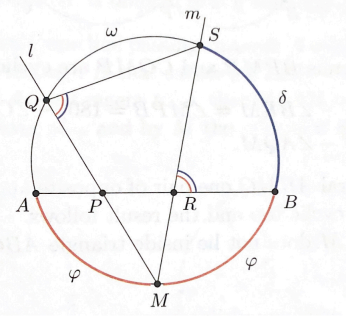


Proof

Corollary

Corollary

Let AB be a chord on a circle ω and denote by M the midpoint of the arc AB. Let l be a line which passes through M and intersects the chord AB in P, and the circle ω for the second time in the point Q. Similarly, let $m \neq l$ be a different line that passes through M and intersects the chord AB in R, and the circle ω, for the second time in S. Show that the points P, Q, S and R are concyclic.



Proof

It is enough to show that $\angle PQS + \angle SRP = 180^\circ$, or equivalently that $\angle MQS = \angle BRS$.

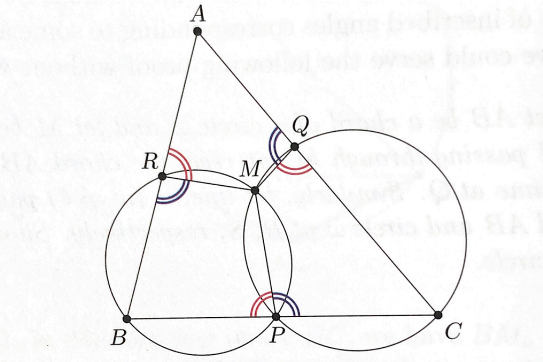
Let us observe that on the circle ω, the shorter arcs MA and MB are equal, and denote by φ the measure of the inscribed angle φ that corresponds to them. Write δ for the measure of the inscribed angle corresponding to the arc BS (which does not contain A). From the previous result we have that

$$\angle BRS = \varphi + \delta = \angle MQS.$$

Miquel's Pivot Theorem

Theorem

Let P, Q, R be arbitrary points on the sides BC, CA and AB of the $\triangle ABC$ respectively. Then the circumcircles of the triangles $\triangle ARQ$, $\triangle BRP$ and $\triangle CQP$ pass through a common point.

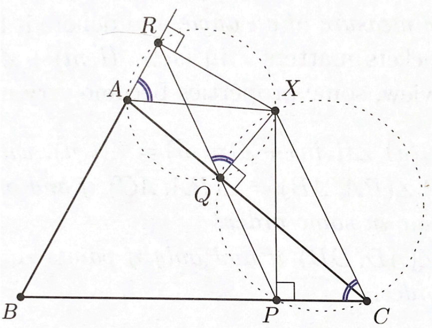


Proof

The Simson line

Theorem

Let ABC be a triangle and X a point in the plane. If we denote by P, Q, R the perpendicular projections of X on the lines BC, CA and AB respectively, then the points P, Q, R are collinear if and only if X belongs to the circumcircle of the triangle $\triangle ABC$.



Proof

Thank you very much for your attention!