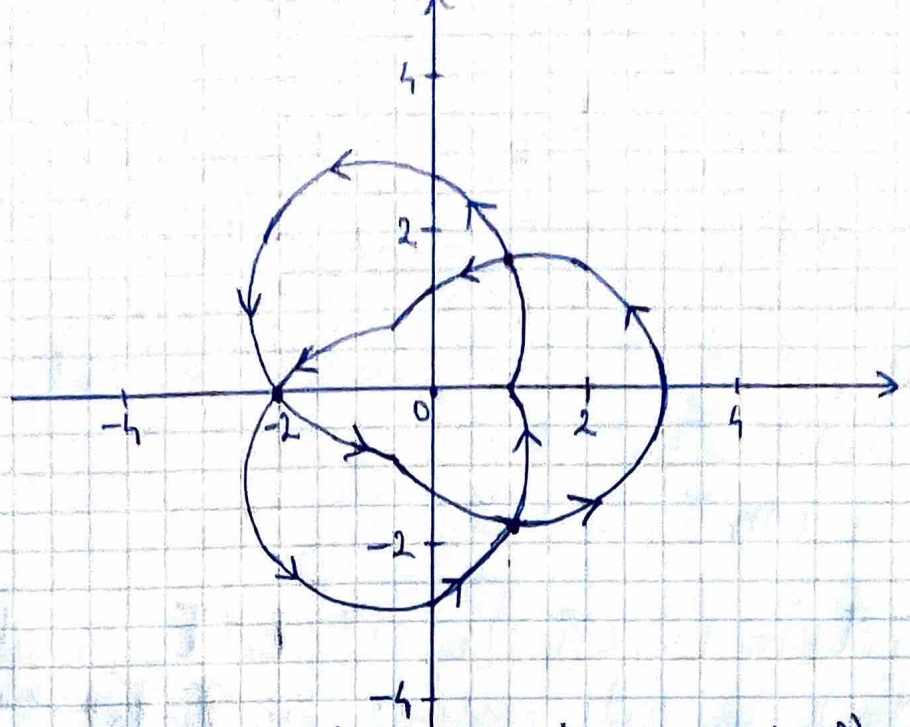
2.



$r(\tau) = 2+\cos(3\tau))\cos(2\tau)i + (2+\cos(3\tau))\sin(2\tau)j, \tau\in[0,2\pi]$

a) Plotting, we can see 3 points of self-intersection

Let us find the pairs (τ_1, τ_2) s.t.

$$\begin{cases}(2+\cos(3\tau_1))\cos(2\tau_1) = (2+\cos(3\tau_2)\cos(2\tau_2)\\ (2+\cos(3\tau_1))\sin(2\tau_1) = (2+\cos(3\tau_2))\sin(2\tau_2)\end{cases}$$

$\tau_1 \neq \tau_2$ and $\tau_1, \tau_2 \in [0, 2\pi]$

Squaring and adding up the equations we get:

$$(2+\cos(3\tau_1))^2(\cos^2(2\tau_1)+\sin^2(2\tau_1)) = (2+\cos(3\tau_2))^2(\cos^2(2\tau_2)+\sin^2(2\tau_2))$$

$$\Leftrightarrow (2+\cos(3\tau_1))^2 = (2+\cos(3\tau_2))^2$$

$$\Leftrightarrow \cancel{4}+4\cos(3\tau_1)+\cos^2(3\tau_1) = \cancel{4}+4\cos(3\tau_2)+\cos^2(3\tau_2)$$

$$4(\cos(3\tau_1)-\cos(3\tau_2)) + (\cos(3\tau_1)-\cos(3\tau_2))(\cos(3\tau_1)+\cos(3\tau_2))$$

$$\Rightarrow (\cos(3\tau_1)-\cos(3\tau_2))\underbrace{(4+(\cos(3\tau_1)+\cos(3\tau_2)))}_{>0} = 0$$

$$\Rightarrow \cos(3\tau_1) = \cos(3\tau_2)$$

Using this in the hypotesis, together with $2+\cos x > 0$ $\forall x \in \mathbb{R}$ we find

$$\begin{cases}\cos(3\tau_1) = \cos(3\tau_2)\\ \cos(2\tau_1) = \cos(2\tau_2)\\ \sin(2\tau_1) = \sin(2\tau_2)\end{cases}$$

From the first equation we have that

$3\tau_1 = 3\tau_2 + 2k\pi$ or $3\tau_1 = -3\tau_2 + 2k\pi$, $k \in \mathbb{Z}$

$\Leftrightarrow \tau_1 = \tau_2 + \frac{2k\pi}{3}$ or $\tau_1 = -\tau_2 + \frac{2k\pi}{3}$, $k \in \mathbb{Z}$

From the second and third we get:

$2\tau_1 = 2\tau_2 + 2m\pi$, $m \in \mathbb{Z}$

$\tau_1 = \tau_2 + m\pi$, $m \in \mathbb{Z}$

As $(\tau_1, \tau_2) \in [0, 2\pi] \times [0, 2\pi]$, $\tau_1 \neq \tau_2$ and $(\tau_1, \tau_2) \neq (0, 2\pi)$ we see that is not possible for $\tau_1 = \tau_2 + \frac{2k\pi}{3} = \tau_2 + m\pi$, $k, m \in \mathbb{Z}$

So, $\tau_1 = -\tau_2 + \frac{2k\pi}{3} = \tau_2 + m\pi$, $m \in \mathbb{Z}$ (*)

Considering $\tau_1, \tau_2 \in [0, 2\pi]$, we can restrict $m \in \{-1, 1\}$

$\overset{(*)}{\Rightarrow} \tau_2 = \frac{(2k - 3m)\pi}{6}$, $k \in \mathbb{Z}$, $m \in \{-1, 1\}$

$\tau_1 = \tau_2 + \pi$

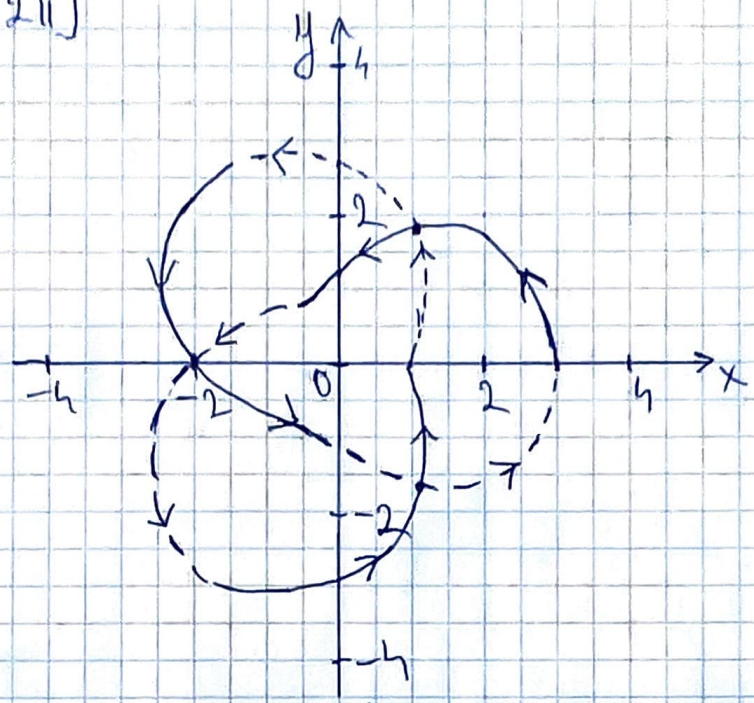
So we find the 3 pairs:

$\left(\pi + \frac{\pi}{6}, \frac{\pi}{6}\right), \left(\pi + \frac{3\pi}{6}, \frac{3\pi}{6}\right), \left(\pi + \frac{5\pi}{6}, \frac{5\pi}{6}\right) =$

$= \left(\frac{7\pi}{6}, \frac{\pi}{6}\right), \left(\frac{3\pi}{2}, \frac{\pi}{2}\right), \left(\frac{11\pi}{6}, \frac{5\pi}{6}\right)$

These give the 3 self-intersecting points that we see on the plot.

b) $r(\tau) = (2+\cos(3\tau))\cos(2\tau)i + (2+\cos(3\tau))\sin(2\tau)j + \sin(3\tau)k$,
$\tau \in [0, 2\pi]$



If the curve is not embedded $\Rightarrow \exists\, \tau_1 \neq \tau_2 \in [0, 2\pi]$ s.t. $r(\tau_1) = r(\tau_2)$

So all three components should coincide so we verify for the three points that we found at point a) if the third component coincide.

$\Leftrightarrow \sin(3\tau_1) = \sin(3\tau_2)$

$\sin\frac{2\pi}{6} = \sin\frac{\pi}{2}$

$\sin(\frac{\pi}{6}+\pi) = \sin\frac{\pi}{6}$

$-\sin\frac{\pi}{6} = \frac{1}{2}$

$-\frac{1}{2} \neq \frac{1}{2}$

$\sin\frac{3\pi}{6} = \sin\frac{5\pi}{2}$

$-\sin\frac{\pi}{6} = \sin(\pi - \frac{\pi}{6})$

$-\frac{1}{2} = \sin\frac{\pi}{6}$

$-\frac{1}{2} \neq \frac{1}{2}$

$\sin\frac{3\pi}{2} = \sin\frac{9\pi}{2}$

$1 \neq -1$

$\Rightarrow$ this curve does not self intersect.