FORMULE GENERALE

1) ORTOCENTRU (H)
- → intersecția înălțimilor
- → Fie Δ ABC
  - → $OA + OB + OC = OH$
  - → $HA + HB + HC = 2HO$

  O - centrul cercului circumscris

2) CENTRUL CERCULUI CIRCUMSCRIS (O)
- → intersecția mediatoarelor (⊥ & trec prin mij. seg.)
- → $OA = OB = OC$ (raze)

3) CENTRUL CERCULUI ÎNSCRIS (I)
- → intersecția bisectoarelor
- → $r = \frac{S}{p}$ (arie) (semiperimetru)

4) CENTRU DE GREUTATE (G)
- → intersecția medianelor
- → $OG = \frac{OA + OB + OC}{3}$
- → G
  - $\frac{1}{3}$ bază
  - $\frac{2}{3}$ vf

FORMULE GEOMETRIE

DOT PRODUCT

1)
$$\bar{a}\cdot\bar{b} = \begin{cases} |\bar{a}||\bar{b}|\cos\theta, & \text{if } \bar{a}\neq 0 \text{ and } \bar{b}\neq 0 \\ 0, & \text{otherwise} \end{cases},\quad |\bar{a}| = \sqrt{a_1^2 + a_2^2}$$

2)
$$\bar{a}\cdot\bar{b} = a_1 b_1 + a_2 b_2, \quad \bar{a},\bar{b} \in V_2$$

3)
$$\bar{a}\cdot\bar{b} = a_1 b_1 + a_2 b_2 + a_3 b_3, \quad \bar{a},\bar{b} \in V_3$$

4)
$$\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|} \iff \cos\widehat{(\bar{a},\bar{b})} = \frac{a_1 b_1 + a_2 b_2}{\sqrt{a_1^2 + a_2^2}\cdot\sqrt{b_1^2 + b_2^2}}$$

5)
a) *θ is acute if and only if $\bar{u}\cdot\bar{v} > 0$;*
b) *θ is obtuse if and only if $\bar{u}\cdot\bar{v} < 0$;*
c) *$\theta = \frac{\pi}{2}$ if and only if $\bar{u}\cdot\bar{v} = 0$.* $\bar{u} \perp \bar{v}$

6) *The director cosines of a vector $\bar{u}(u_1, u_2, u_3) \in V_3$, $\bar{u} \neq \bar{0}$, are*

$$\cos\alpha = \frac{u_1}{|\bar{u}|},\quad \cos\beta = \frac{u_2}{|\bar{u}|},\quad \cos\gamma = \frac{u_3}{|\bar{u}|}.$$

7) (if $\bar{u} \neq \bar{0}$ $\frac{\bar{u}}{|\bar{u}|}$ is a unit vector ($\iff$ the versor of $\bar{u}$)

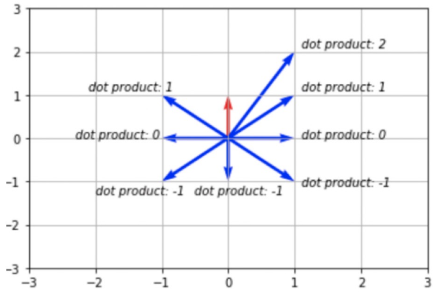
$$\frac{\bar{u}}{|\bar{u}|} = \cos\alpha\cdot\bar{i} + \cos\beta\cdot\bar{j} + \cos\gamma\cdot\bar{k},\ \text{with } (\cos\alpha)^2 + (\cos\beta)^2 + (\cos\gamma)^2 = 1.$$

8)
Given $\bar{a}$, $\bar{b}$, $\bar{c} \in V_3$ (or V_2) and $\lambda \in \mathbb{R}$, one has:

1) $\bar{a}\cdot\bar{b} = \bar{b}\cdot\bar{a}$ (commutativity of the dot product);
2) $\bar{a}\cdot(\bar{b}+\bar{c}) = \bar{a}\cdot\bar{b} + \bar{a}\cdot\bar{c}$ (distributivity of the dot product with respect to the summation of vectors);
3) $\lambda(\bar{a}\cdot\bar{b}) = (\lambda\bar{a})\cdot\bar{b} = \bar{a}\cdot(\lambda\bar{b})$;
4) $\bar{a}\cdot\bar{a} = |\bar{a}|^2$.

$) = a_1 b_1 + b_2 a_2 + a_3 b_3$

9)



- Dot product is larger when the direction of the blue vectors is similar to the direction of the red one.
- Dot product is larger when the magnitude of the blue vector is larger.

10)

If $\overline{u}$ and $\overline{b}$ are vectors in V_2 or V_3 and $\overline{b} \neq 0$, then

1) • *the orthogonal projection of $\overline{u}$ on $\overline{b}$ is* $\underbrace{pr_{\overline{b}}\overline{u}}_{w_1} = \left(\frac{\overline{u} \cdot \overline{b}}{|\overline{b}|^2}\right) \cdot \overline{b}$; constant

2) • *the vector component of $\overline{u}$ orthogonal to $\overline{b}$ is* $\underbrace{\overline{u}\text{-}pr_{\overline{b}}\overline{u}}_{w_2} = \overline{u} - \frac{\overline{u} \cdot \overline{b}}{|\overline{b}|^2} \cdot \overline{b}$.

$$|\mathrm{pr}_{\overline{b}}\overline{u}| = |\overline{u}||\cos\theta|.$$

CROSS PRODUCT

11) $\overline{u}, \overline{v}$ colinear : $\overline{u} \times \overline{v} = \overline{0}$

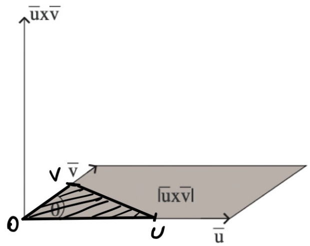
12) the lenght of the new vector

1. $||\overline{u} \times \overline{v}|| = ||\overline{u}|| \cdot ||\overline{v}|| \cdot \sin(\theta)$;
2. $\overline{u} \times \overline{v}$ is perpendicular on $\overline{u}$ and on $\overline{v}$; → $\overline{u} \times \overline{v} \perp \overline{u}$ AND $\overline{u} \times \overline{v} \perp \overline{v}$
3. the orientation of $\overline{u} \times \overline{v}$ is given by the right-hand rule.

13)

$$\overline{u} \times \overline{v} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

14)



- If the vectors $\overline{u}, \overline{v}$ are not collinear, then if $\overrightarrow{OU} \in \overline{u}$ and $\overrightarrow{OV} \in \overline{v}$, then $||\overline{u} \times \overline{v}||$ is the area of the parallelogram formed by $\overrightarrow{OU}$ and $\overrightarrow{OV}$.
- The area of the triangle $\cancel{\Delta OAB}$ ΔOUV can be computed as

$$\text{Area}_{\cancel{\Delta OAB}\ \Delta OUV} = \frac{||\overline{u} \times \overline{v}||}{2}.$$

15)

- $\overline{u} \cdot (\overline{u} \times \overline{v}) = 0$, so $\overline{u} \times \overline{v}$ is orthogonal on $\overline{u}$; $\overline{u} \times \overline{v} \perp \overline{u}$

16)

$$|\overline{u} \times \overline{v}|^2 = |\overline{u}|^2|\overline{v}|^2 - (\overline{u} \cdot \overline{v})^2$$

(Lagrange's identity).

Consequind

$|\overline{u}|^2|\overline{v}|^2 - (\overline{u} \cdot \overline{v})^2 \geq 0$, or $|\overline{u} \cdot \overline{v}| \leq |\overline{u}||\overline{v}|$

17) Cauchy-Schwartz:

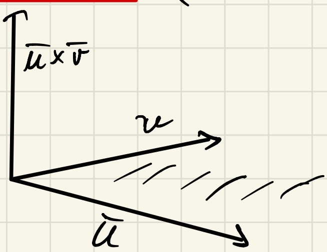
$$|\overline{u}\cdot\overline{v}| = |\overline{u}||\overline{v}| \Leftrightarrow \overline{u}\times\overline{v}=\overline{0} \Leftrightarrow \frac{v_1}{u_1}=\frac{v_2}{u_2}=\frac{v_3}{u_3}=\lambda, \text{ or } \overline{v}=\lambda\overline{u},\ \lambda\in\mathbb{R}^*$$

18) **Theorem**

If $\overline{u}$ and $\overline{v}$ are nonzero vectors in V_3, then $\overline{u}\times\overline{v}=\overline{0}$ if and only if $\overline{u}$ and $\overline{v}$ are parallel.

19) Double cross product: $(\overline{u}\times\overline{v})\times\overline{w} = (\overline{u}\cdot\overline{w})\cdot\overline{v} - (\overline{v}\cdot\overline{w})\cdot\overline{u}$

$\overline{u}\times\overline{v} \Leftrightarrow$



20) Triple scalar product: $(\overline{a},\overline{b},\overline{c}) = \overline{a}\cdot(\overline{b}\times\overline{c}) = (\overline{a}\times\overline{b})\cdot\overline{c}$

$$(\overline{a},\overline{b},\overline{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$(\overline{a},\overline{b},\overline{c}) = (\overline{c},\overline{a},\overline{b}) = (\overline{b},\overline{c},\overline{a})$ $= 0$ – $\overline{a},\overline{b},\overline{c}$ – coplanar

$\neq 0$ volume

Parametric ec. of a line

$$\left.\begin{matrix} P_0(x_0, y_0) \in d \\ \vec{v}(a, b) \parallel d \end{matrix}\right\} \Rightarrow d: \begin{cases} x = x_0 + at \\ y = y_0 + bt \end{cases}, \quad t \in \mathbb{R}$$

Symmetric ec. of a line

- by expressing t:

$$d: \frac{x - x_0}{a} = \frac{y - y_0}{b}$$

The general ec. of a line

$$Ax + By + C = 0 \quad |:A \qquad \Rightarrow \vec{n}(A, B)$$

$$x + \frac{B}{A}y + \frac{C}{A} = 0 \Rightarrow y = \frac{x + \frac{C}{A}}{-\frac{B}{A}} \Rightarrow P_0\left(-\frac{C}{A}, 0\right)$$

$$\vec{v}\left(-\frac{B}{A}, 1\right)$$

$$\Rightarrow Ax + By + C = 0 \Leftrightarrow \begin{cases} x = -\frac{C}{A} - \frac{B}{A}t \\ y = 0 + 1 \cdot t \end{cases} \quad t \in \mathbb{R}$$

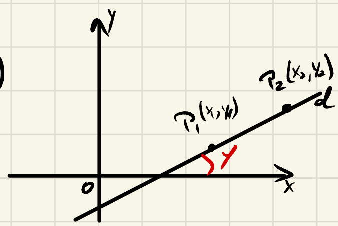
Reduced ep. of a line

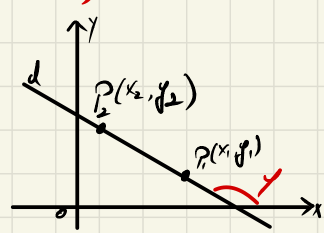
$$Ax + By + C = 0, \qquad m = -\frac{A}{B}$$

$y = mx + n$ (doar în dimensiune 2)

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \operatorname{tg} \varphi \text{ (the slope)}$$

$$d: y - y_0 = m(x - x_0)$$





Line det. by 2 points

$$\left.\begin{array}{l} P_1(x_1, y_1) \in d \\ P_2(x_2, y_2) \in d \end{array}\right\} \Rightarrow \overline{P_1P_2}\,(x_2 - x_1,\ y_2 - y_1) \rightarrow \text{director vector}$$

$$\Rightarrow d: \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$$

Intersection of 2 lines

$$d_1: a_1x + b_1y + c_1 = 0$$
$$d_2: a_2x + b_2y + c_2 = 0$$

$$\Leftrightarrow \begin{cases} a_1x + b_1y + c_1 = 0 \\ a_2x + b_2y + c_2 = 0 \end{cases} \Leftrightarrow \overbrace{\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}}^{A} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -c_1 \\ -c_2 \end{pmatrix}$$

1) $\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \Rightarrow$ syst. with uniqe sol $(x_0, y_0) \Rightarrow$ uniqe intersection point $P(x_0, y_0) \Rightarrow d_1, d_2$ are secant

$\Updownarrow$

$\det A \neq 0 \Leftrightarrow A$ has inverse

2) $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2} \Rightarrow$ syst. not compatible and the lines have no point in comm. $\Rightarrow d_1 \parallel d_2$

$\Updownarrow$

$\operatorname{rank} A < \operatorname{rank} \bar{A}$

3) $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \Rightarrow$ syst. has infinite sol. and the line coincide $\Rightarrow d_1 = d_2$

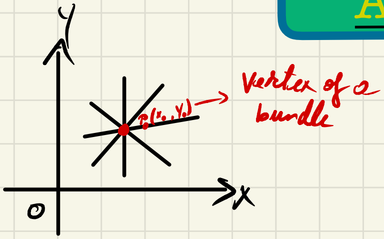
$\Updownarrow$

$$\begin{cases} \det A = 0 \\ \operatorname{rank} A = \operatorname{rank} \bar{A} \end{cases}$$

3 lines are concurrent if:

$$\operatorname{Rank} \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} = \operatorname{Rank} \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = 2$$

A bundle of lines



$$\{P_0\} = d_1 \cap d_2, \quad \begin{cases} d_1: a_1x + b_1y + c_1 = 0 \\ d_2: a_2x + b_2y + c_2 = 0 \end{cases}$$

$$\xRightarrow{\text{the eq. of the bundle of lines through } P_0} r(a_1x + b_1y + c_1) + s(a_2x + b_2y + c_2) = 0$$

$$(r, s) \in \mathbb{R}^2 \setminus \{(0,0)\}$$

Det. a plane by a **point** and a **normal vector**

π - plane in dim. 3

$P_0(x_0, y_0, z_0) \in \pi$

$\overline{n}(a, b, c)$ - normal vector of π ($\Leftrightarrow \overline{n} \perp \pi$)

- A point $P(x, y, z) \in \pi \Leftrightarrow \overline{n} \perp \overline{P_0P}$ or $\overline{n} \cdot \overline{P_0P} = 0$
- $P_0P = (x - x_0, y - y_0, z - z_0)$

$$\Rightarrow \pi: \underbrace{a(x - x_0) + b(y - y_0) + c(z - z_0)}_{\text{The general eq. of the plane}}$$

Det. a plane by **3 points**

$A(x_A, y_A, z_A) \in \pi$
$B(x_B, y_B, z_B) \in \pi$, A, B, C – noncollinear
$C(x_C, y_C, z_C) \in \pi$

$$\begin{vmatrix} x & y & z & 1 \\ x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \end{vmatrix} = 0$$

- Analitic eq. of a plane det. by 3 noncollinear points
- 4 points A, B, C, D are coplanar if:

$$\begin{vmatrix} x_D & y_D & z_D & 1 \\ x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \end{vmatrix} = 0$$

The **parametric ec.** of a plane

$\overline{v_1}(p_1, q_1, r_1) \in \pi$, $\overline{v_1} \nparallel \overline{v_2}$
$\overline{v_2}(p_2, q_2, r_2) \in \pi$
$A(x_A, y_A, z_A) \in \pi$

$$\begin{cases} x = x_A + \alpha p_1 + \beta p_2 \\ y = y_A + \alpha q_1 + \beta q_2 \\ z = z_A + \alpha r_1 + \beta r_2 \end{cases}, \quad \alpha, \beta \in \mathbb{R}$$

The **analytic ec.** of a plane

$$\begin{vmatrix} x-x_A & y-y_A & z-z_A \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} = 0$$

$\to A(x_A, y_A, z_A) \in \pi$

$\to \overline{v_1} \in \pi$

$\to \overline{v_2} \in \pi$

$\overline{v_1} \nparallel \overline{v_2}$

The **parametric ec.** of line in 3D

$P_0(x_0, y_0, z_0) \in d$

$\overline{v}(p, q, r) \to$ the director vector of a line

$$d: \begin{cases} x = x_0 + pt \\ y = y_0 + qt \\ z = z_0 + rt \end{cases}, t \in \mathbb{R}$$

The **symmetric ec.** of line in 3D

$\overline{v}(p, q, r)$, $p, q, r \in \mathbb{R}^*$

$$d: \frac{x-x_0}{p} = \frac{y-y_0}{q} = \frac{z-z_0}{r}$$

The **ec.** of line in 3D det. by **3 points**

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_2}{z_2-z_1}$$

Lines as intersections of planes

$\pi_1 : A_1x + B_1y + C_1z + D_1 = 0$
$\pi_2 : A_2x + B_2y + C_2z + D_2 = 0$, $\pi_1 \nparallel \pi_2$

$\Leftrightarrow \overline{n_1}(A_1, B_1, C_1)$, $\overline{n_1} \nparallel \overline{n_2}$
$\overline{n_2}(A_2, B_2, C_2)$

$\pi_1 \parallel \pi_2 \Leftrightarrow \overline{n_1} \parallel \overline{n_2}$

$$d = \pi_1 \cap \pi_2 \Leftrightarrow \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 2$$

Intersections of lines

$$d_1 : \begin{cases} x = x_1 + p_1 t \\ y = y_1 + q_1 t, \ t \in \mathbb{R} \\ z = z_1 + r_1 t \end{cases} \qquad d_2 : \begin{cases} x : x_2 + p_2 s \\ y : y_2 + q_2 s, \ s \in \mathbb{R} \\ z : z_2 + r_2 s \end{cases}$$

The set of intersection points of d_1 & d_2 is given by the set of sol (t, s) of the syst:

$$\begin{cases} x_1 + p_1 t = x_2 + p_2 s \\ y_1 + q_1 t = y_2 + q_2 s \\ z_1 + r_1 t = z_2 + r_2 s \end{cases}$$

$\overline{v_1}$ & $\overline{v_2}$ lin. ind
$\Updownarrow$

- $\text{rank } A = \text{rank } \bar{A} = 2 \Rightarrow$ unique sol. $\Rightarrow d_1 \cap d_2 = \{P_0\}$
- $\text{rank } A = \text{rank } \bar{A} = 1 \Rightarrow$ infinitely sol. $\Rightarrow d_1 = d_2$
  $\Updownarrow$
  $\overline{v_1} = \alpha \overline{v_2}, \alpha \in \mathbb{R}$
- $1 = \text{rank } A < \text{rank } \bar{A} = 2 \Rightarrow \nexists$ sol. $\Rightarrow \overline{v_1}, \overline{v_2}$ lin. dep. $\Rightarrow d_1 \parallel d_2$
- $2 = \text{rank } A < \text{rank } \bar{A} = 3 \Rightarrow \nexists$ sol. $\Rightarrow \overline{v_1}, \overline{v_2}$ lin. indep $\begin{cases} d_1 \nparallel d_2 \\ d_1 \cap d_2 = \emptyset \end{cases}$

Relative position of 2 planes

$$\pi_1 \cap \pi_2 : \begin{cases} \pi_1 = a_1 x + b_1 y + c_1 z + d_1 = 0, & \bar{n}_1(a_1, b_1, c_1) \neq 0 \\ \pi_2 = a_2 x + b_2 y + c_2 z + d_2 = 0, & \bar{n}_2(a_2, b_2, c_2) \neq 0 \end{cases}$$

$$\Downarrow$$

$$\bar{A} = \left(\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{array} \right)$$

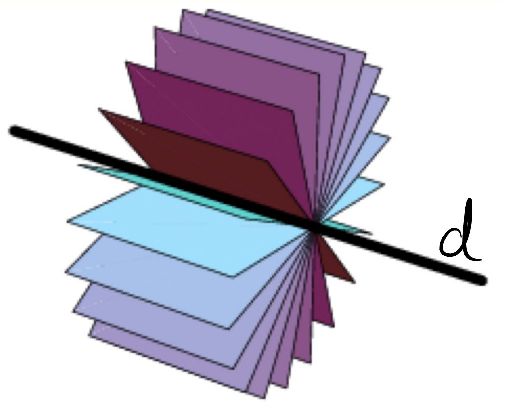
rank $A = 2 \Rightarrow$ compatible sys. $\Rightarrow \pi_1 \cap \pi_2 = d$

rank $A = 1 \Rightarrow$ rows of A are lin. dep. $\Rightarrow \bar{n}_1, \bar{n}_2$ are lin dep

$\Downarrow$

- $1 = \text{rank } A < \text{rank } \bar{A} = 2 \Rightarrow$ sys. not comp. $\Rightarrow \pi_1 \parallel \pi_2$
- $\text{rank } A = \text{rank } \bar{A} = 1 \Rightarrow \pi_1 = \pi_2$

Bundle of planes



$$\begin{cases} \pi_1 = a_1 x + b_1 y + c_1 z + d_1 = 0 \\ \pi_2 = a_2 x + b_2 y + c_2 z + d_2 = 0 \end{cases}$$

with

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} = 2$$

Ec. of a bundle of planes

$$\lambda_1(a_1x+b_1y+c_1z+d_1)+\lambda_2(a_2x+b_2y+c_2z+d_2)=0$$

$$(\lambda_1,\lambda_2)\in\mathbb{R}^2\setminus\{(0,0)\}$$

$$\Updownarrow$$

$$\lambda_1\pi_1+\lambda_2\pi_2=0$$

Reduced ec. of a bundle

$$\lambda_1\pi_1+\lambda_2\pi_2=0 \;|:\lambda_1$$

$$\pi_1+\lambda\pi_2=0,\quad \lambda=\frac{\lambda_2}{\lambda_1}$$ (contains all planes except π_2)

Relative position between a line & a plane

$$d:\begin{cases} x=x_1+pt\\ y=y_1+qt\\ z=z_1+rt\end{cases},\quad p^2+q^2+t^2>0$$

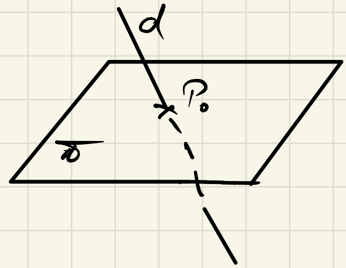
$\bar{v}(p,q,r)$ director vector

$$\pi: ax+by+cz+d=0,\quad a^2+b^2+c^2>0$$

$\bar{n}(a,b,c)$ normal vector

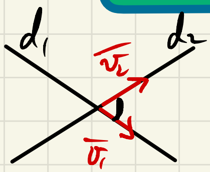
$$d\cap\pi = a(x_1+pt)+b(y_1+qt)+c(z_1+rt)+d=0$$

- unique sol. $t_0 \Rightarrow d \cap \pi = \{P_0\}$



- infinitely sol $\Rightarrow d \cap \pi = d \Rightarrow d \subset \pi \Leftrightarrow \bar{n} \perp \bar{v}$
- no sol $\Rightarrow d \parallel \pi$

Angle between **lines** in **3D**



$$m(\widehat{d_1, d_2}) = \begin{cases} m(\widehat{\bar{v}_1, \bar{v}_2}), & \bar{v}_1 \cdot \bar{v}_2 \geq 0 \\ \pi - m(\widehat{\bar{v}_1, \bar{v}_2}), & \bar{v}_1 \cdot \bar{v}_2 < 0 \end{cases}$$

$$\Updownarrow$$

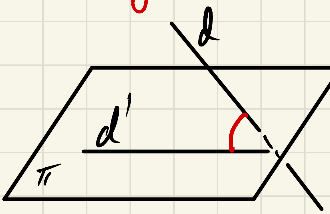
$$m(\widehat{d_1, d_2}) = \begin{cases} \arccos \dfrac{\bar{v}_1 \cdot \bar{v}_2}{\|\bar{v}_1\| \cdot \|\bar{v}_2\|}, & \bar{v}_1 \cdot \bar{v}_2 \geq 0 \\ \pi - \arccos \dfrac{\bar{v}_1 \cdot \bar{v}_2}{\|\bar{v}_1\| \cdot \|\bar{v}_2\|}, & \bar{v}_1 \cdot \bar{v}_2 < 0 \end{cases}$$

If $\bar{v}_1 \cdot \bar{v}_2 = 0 \Rightarrow d_1 \perp d_2$

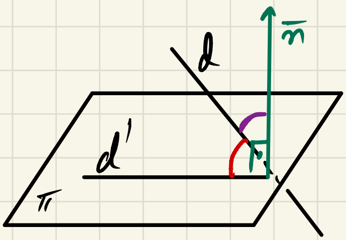
Angle between a line and a plane

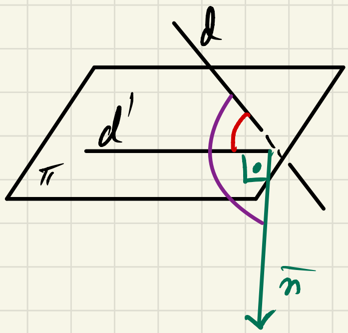
$\Longleftrightarrow$

Angle between a line and its orthogonal projection



(*) $m(\widehat{d,\pi}) = \begin{cases} \frac{\pi}{2} - m(\bar{v}, \bar{n}), & \bar{v}\cdot\bar{n} \geq 0 \\ m(\bar{v}, \bar{n}) - \frac{\pi}{2}, & \bar{v}\cdot\bar{n} < 0 \end{cases}$





(*) $\Leftrightarrow \; m(\widehat{d,\pi}) = \begin{cases} \frac{\pi}{2} - \arccos \frac{\bar{v}\cdot\bar{n}}{\|v\|\cdot\|n\|}, & \bar{v}\cdot\bar{n} \geq 0 \\ \arccos \frac{\bar{v}\cdot\bar{n}}{\|v\|\cdot\|n\|} - \frac{\pi}{2}, & \bar{v}\cdot\bar{n} < 0 \end{cases}$

- $d \parallel \pi \Leftrightarrow \bar{v} \perp \bar{n} \Leftrightarrow \bar{v}\cdot\bar{n} = 0$
- $d \perp \pi \Leftrightarrow \bar{v} \parallel \bar{n} \Leftrightarrow \alpha\bar{v} = \bar{n}, \; \alpha \in \mathbb{R}$

Angle between planes in 3D

Same as for 2 lines but we use the normal vectors of the plan insted of director vectors

Distance between a plane and a point

$P_0(x_0, y_0, z_0)$

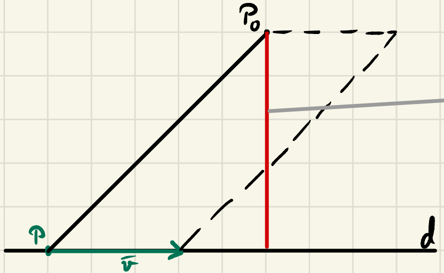
$\pi : xa + yb + zc + d = 0$

$$d(\pi, P_0) = \frac{|a x_0 + b y_0 + c z_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Distance between a line and a point

$P_0(x_0, y_0, z_0)$ $\quad d : \begin{cases} x + pt \\ y + qt \\ z + rt \end{cases}, t \in \mathbb{R}$

take a point $P(p, q, r) \in d$ && director vector $v(p, q, r)$



$d(P_0, d) = h_{paral} \Rightarrow \dfrac{|\vec{v} \times \vec{PP_0}|}{\|\vec{v}\|}$

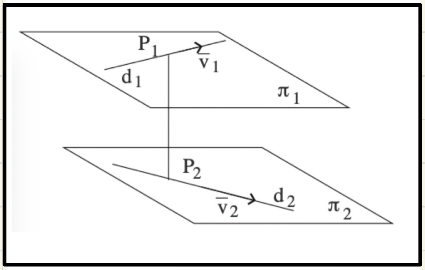
- If d_1 and d_2 are skew, there exists a unique line which is orthogonal on both d_1 and d_2 and intersects both d_1 and d_2. The length of the segment determined by these intersection points is the distance between the skew lines.

  Suppose that

$$d_1 : \begin{cases} x = x_1 + p_1 t \\ y = y_1 + q_1 t \\ z = z_1 + r_1 t \end{cases}, t \in \mathbb{R} \text{ and } d_2 : \begin{cases} x = x_2 + p_2 s \\ y = y_2 + q_2 s \\ z = z_2 + r_2 s \end{cases}, s \in \mathbb{R}$$

  are, respectively, the parametric equations of the lines of director vectors $\overline{v}_1(p_1, q_1, r_1) \neq \overline{0}$, respectively $\overline{v}_2(p_2, q_2, r_2) \neq \overline{0}$.

  One can determine the equations of two parallel planes $\pi_1 \parallel \pi_2$, such that $d_1 \subset \pi_1$ and $d_2 \subset \pi_2$. The normal vector $\overline{n}$ of these planes has to be orthogonal on both $\overline{v}_1$ and $\overline{v}_2$, hence $\overline{n} = \overline{v}_1 \times \overline{v}_2$.



Then $\overline{n}(A, B, C)$, with $A = \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix}$, $B = \begin{vmatrix} r_1 & p_1 \\ r_2 & p_2 \end{vmatrix}$ and $C = \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix}$.

The equations of the planes π_1 and π_2 are:

$$\pi_1 : A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$$

$$\pi_2 : A(x - x_2) + B(y - y_2) + C(z - z_2) = 0.$$

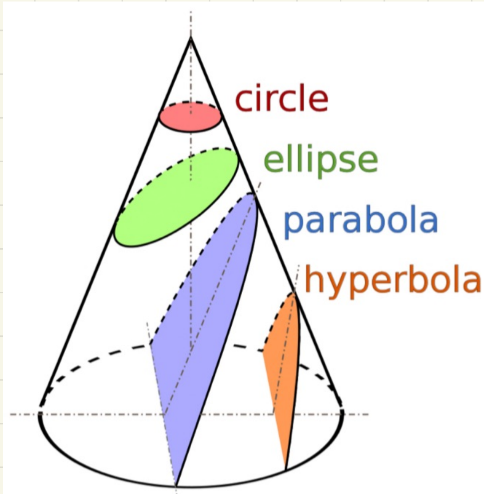
Now, the distance between d_1 and d_2 is the distance between the parallel planes π_1 and π_2; $d(d_1, d_2) = d(\pi_1, \pi_2)$, and one has the following theorem.

Theorem

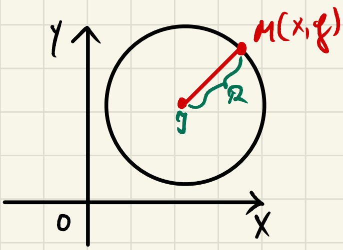
The distance between two skew lines d_1 and d_2 is given by

$$d(d_1, d_2) = \frac{|A(x_1 - x_2) + B(y_1 - y_2) + C(z_1 - z_2)|}{\sqrt{A^2 + B^2 + C^2}}. \tag{16}$$

Conic sections



The circle



Fie $\mathcal{C}(I,R)$, $I(a,b)$

$M(x,y) \in \mathcal{C} \iff |MI| = R$

$\iff \sqrt{(x-a)^2+(y-b)^2} = R \quad |()^2$

$\iff (x-a)^2+(y-b)^2 = R^2$

THE EC. OF A CIRCLE

$\iff x^2+y^2-2ax-2by+\underbrace{a^2+b^2-R^2}_{=c}=0$

R: In xoy the ec. of a circle represents
- circle
- point
- empty set

The circle det. by 3 points

• There is a unique circle det. by: $M_1(x_1, y_1, z_1) \in \mathscr{C}$, $M_2(x_2, y_2, z_2) \in \mathscr{C}$, $M_3(x_3, y_3, z_3) \in \mathscr{C}$

$$\mathscr{C}: x^2 + y^2 - 2ax - 2by + c = 0$$

$$\begin{cases} x^2 + y^2 - 2ax - 2by + c = 0 \\ x_1^2 + y_1^2 - 2ax_1 - 2by_1 + c = 0 \\ x_2^2 + y_2^2 - 2ax_2 - 2by_2 + c = 0 \\ x_3^2 + y_3^2 - 2ax_3 - 2by_3 + c = 0 \end{cases} \Longleftrightarrow \begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

R: for 4 points → replace first col. with other coord.

Intersection of a circle & line

$$\mathscr{C}: \begin{cases} x^2 + y^2 - R^2 = 0 \\ y = mx + n \end{cases} \Longleftrightarrow x^2 + m^2x^2 + 2mxn + n^2 - R^2 = 0$$

$$\Longleftrightarrow x^2(1 + m^2) + 2mnx + n^2 - R^2 = 0$$

$$\Rightarrow \Delta = 4m^2n^2 - 4 - 4m^2 - 4n^2 + 4R^2$$

$$\Delta = 4(R^2 + m^2R^2 - n^2)$$

I $\Delta < 0$ — no intersection point

II $\Delta = 0$ — tangent $\Rightarrow$ the coord of tg point $\left(\frac{-mn}{1+m^2}, \frac{n}{1+m^2}\right)$

III $\Delta > 0$ — the line intersects $\mathscr{C}$ in 2 points

The tangent of slope m

$\mathscr{C}: x^2 + y^2 - R^2 = 0$, $m \in \mathbb{R}$ | There are 2 tg. with this slope

Since: a line is tg. $\Longleftrightarrow R^2 + m^2R^2 - n^2 = 0 \Rightarrow mx + R^2\sqrt{1 + m^2}$

EC. OF TG. POINTS

The tangent of at a point of the circle

$\mathcal{C}: x^2+y^2-r^2=0$

$A(x_0, y_0) \in \mathcal{C}$

$x_0 x + y_0 y - r^2 = 0$ (lecture 10 for more details)

the tg. at P_0 is $\perp$ on R corresponding to P_0

Intersection of 2 circles

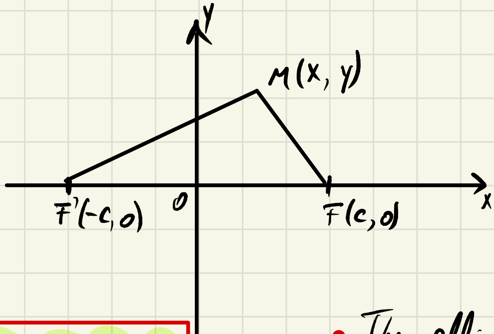
$$\begin{cases} \mathcal{C}_1: x^2+y^2-2a_1x-2b_1y+c_1=0 \\ \mathcal{C}_2: x^2+y^2-2a_2x-2b_2y+c_2=0 \end{cases}$$

$\Leftrightarrow$

$$\begin{cases} x^2+y^2-2a_1x-2b_1y+c_1=0 \\ 2(a_2-a_1)x+2(b_2-b_1)y-(c_2-c_1)=0 \end{cases}$$

- $\Delta>0 \longrightarrow \mathcal{C}_1 \cap \mathcal{C}_2 = \{A(x_A, y_A), B(x_B, y_B)\}$
- $\Delta=0 \longrightarrow \mathcal{C}_1 \cap \mathcal{C}_2 = \{A(x_A, y_A)\}$
- $\Delta<0 \longrightarrow \mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$

The ellipse



$d(F', F) =$ focal distance

EC. OF AN ELLIPSE

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$$

$$\boxed{c^2=a^2-b^2}$$

- The ellipse is symmetric both to Ox & Oy
- The mid point of [F'F] is the center of the ellipse

Sketch the ellipse

Enough to sketch the graph:

$$f : [-a, a] \to \mathbb{R}, \quad f(x) = \frac{b}{a}\sqrt{a^2 - x^2}$$

and to complete the ellipse by symmetry w.r.t Ox

The eccentricity

If $b = a \Rightarrow$ the ec. of a circle

The number $e = \dfrac{c}{a}$ is called the *eccentricity* of the ellipse. Since $a > c$, then $0 \leq e < 1$, hence any ellipse has the eccentricity smaller that 1.

On the other hand, $e^2 = \dfrac{c^2}{a^2} = 1 - \left(\dfrac{b}{a}\right)^2$, so that e gives informations about the shape of the ellipse. When e is closer and closer to 0, then the ellipse is "closer and closer" to a circle; and when e is closer to 1, then the ellipse is flattened to Ox.

Intersection with a line

$$\begin{cases} \mathcal{E}: \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1 = 0 \\ d: \ y = mx + n \end{cases}$$

- $\Delta > 0$, $\mathcal{E} \cap d = \{A(x_A, y_A), B(x_B, y_B)\}$
- $\Delta = 0$, $\mathcal{E} \cap d = \{A(x_A, y_A)\}$
- $\Delta < 0$, $\mathcal{E} \cap d = \emptyset$

Tangent with given slope

$\mathcal{E}: \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$

$m \in \mathbb{R}$ — there exists 2 lines tg to $\mathcal{E}$ with slope m

$tg: y = mx \pm \sqrt{a^2m^2 - b^2}$

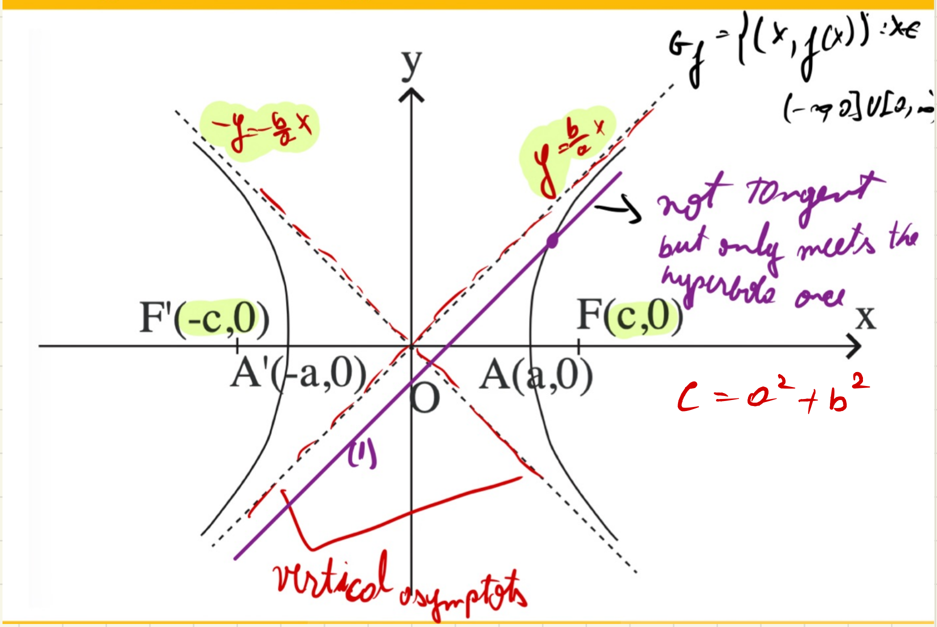
Tangent at a given point

$\mathcal{E}: \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ | $\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} - 1 = 0$ only if $P_0 \in \mathcal{E}$

$P_0(x_0, y_0) \in \mathcal{E}$

The hyperbola

The graph of the hyperbola



The ec. of hyperbola

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0}$$

Sketch the hyperbola

Enough to sketch the function:

$$f : (-\infty, -a] \cup [a, \infty) \to \mathbb{R}, \quad f(x) = \frac{b}{a}\sqrt{x^2 - a^2}$$

To know how to represent the H we intersect it with ox and oy

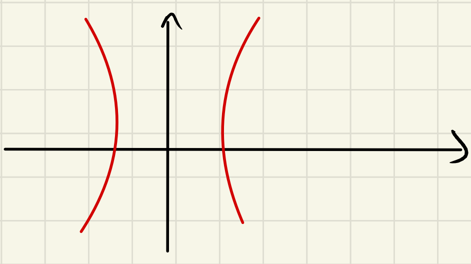
Ex: $\frac{x^2}{25} - \frac{y^2}{36} = 1$

- intersection with ox : $\Rightarrow x = 0$

$$-\frac{y^2}{36} = 1 \Leftrightarrow y^2 = -36 \Rightarrow$$ nu se intersectează

- intersection with oy $\Rightarrow y = 0$

$$\frac{x^2}{25} = 1 \Rightarrow x^2 = 25$$



- The number $e = \frac{c}{a}$ is called the *eccentricity* of the hyperbola. Since $c > a$, then the eccentricity is always greater than 1.
- Moreover,

$$e^2 = \frac{c^2}{a^2} = 1 + \left(\frac{b}{a}\right)^2,$$

hence e gives informations about the shape of the hyperbola. For e closer to 1, the hyperbola has the branches closer to Ox.

Intersection with line

$$\begin{cases} \mathcal{H}: \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0 \\ d: y = mx + n \end{cases}$$

- $\Delta < 0$, $\mathcal{H} \cap d = \{A(x_A, y_A), B(x_B, y_B)\}$
- $\Delta = 0$, $\mathcal{H} \cap d = \{A(x_A, y_A)\}$
- $\Delta > 0$, $\mathcal{H} \cap d = \emptyset$

The tangent

$d: y = mx + n$ tangent $\Leftrightarrow$ $\Delta = 0$ && $m \in \left(-\infty, \frac{b}{a}\right) \cup \left(\frac{b}{a}, \infty\right)$

— having an angular coefficient m:

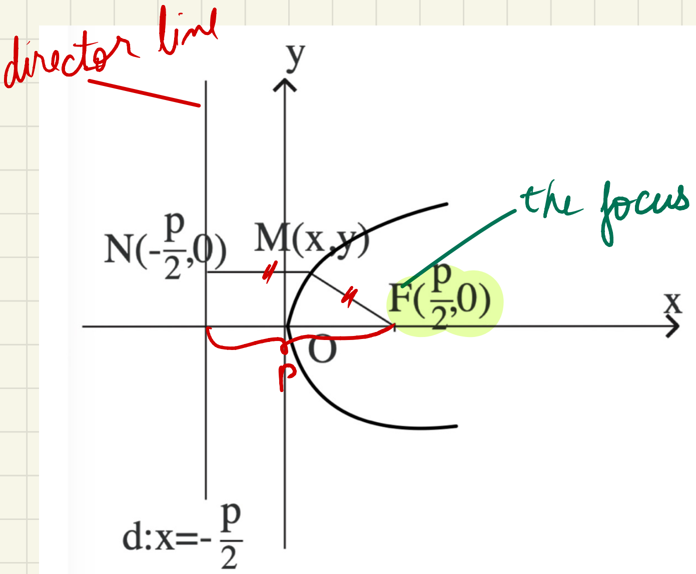
$$y = mx \pm \sqrt{a^2 m^2 - b^2}$$

If $a^2 m^2 - b^2 < 0$ then is no tg.

— tg. of a point:

$$\frac{x x_0}{a^2} - \frac{y y_0}{b^2} - 1 = 0 \qquad \text{If } P_0(x_0, y_0) \in \mathcal{H}$$

The parabola



Ec. of the parabola

$$\boxed{y^2 = 2px}$$

Sketch the parabola

Representing the graph of the function $f : [0, \infty) \to [0, \infty)$ and using the symmetry of the curve with respect to Ox, one obtains the graph of the parabola. One has

$$f'(x) = \frac{p}{\sqrt{2px}}; \qquad f''(x) = -\frac{p}{2x\sqrt{2x}}.$$

x	0				∞
$f'(x)$	$\vert$	$+$	$+$	$+$	$+$
$f(x)$	0		$\nearrow$		∞
$f''(x)$	$-$	$-$	$-$	$-$	$-$

Intersection with a line

$$\begin{cases} \mathcal{P}: y^2 = 2px \\ d: y = mx + n \end{cases} \Leftrightarrow (mx+n)^2 = 2px$$

$$m^2x^2 + 2mnx + n^2 = 2px$$

$$m^2x^2 + 2(mn-p)x + n^2 = 0$$

$$\Delta = 4(mn-p)^2 - 4m^2n^2$$

$$\Delta = \cancel{4m^2n^2} - 8mnp + 4p^2 - \cancel{4m^2n^2}$$

$$\Delta = 4p(2mn-p)$$

- $\Delta < 0$, $\mathcal{P} \cap d = \emptyset$
- $\Delta = 0$, $\mathcal{P} \cap d = \{A(x_A, y_A)\}$ $\longrightarrow$ tangent (1)
- $\Delta > 0$, $\mathcal{P} \cap d = \{A(x_A, y_A), B(x_B, y_B)\}$

Ec. of a tangent with angular coefficient

$$(1) \Rightarrow 4p(2mn-p) = 0 \Rightarrow 2mn = p$$

$$\Rightarrow y = mx + \frac{p}{2m}$$

Ec. of a tangent in a given point

$P_0(x_0, y_0) \in \mathcal{P}$

tg: $yy_0 = p(x - x_0) \Leftrightarrow P_0 \in \mathcal{P}$