14.12.2022

Lattices as algebraic structure

$(A, \wedge, \vee)$ is a lattice $\Leftrightarrow$ i) $\wedge, \vee$ commutative and associative

ii) $\forall a, b \in A \quad a \wedge (a \vee b) = a$
$a \vee (a \wedge b) = a$ absorption

1 is a unit element if $x \wedge 1 = x \quad \forall x \in A$

0 is a unit element if $x \vee 0 = x, \forall x \in A$

$f : (A, \wedge, \vee) \to (B, \wedge, \vee)$ is a lattice morphism $\Leftrightarrow \forall x, y \in A$
$f(x \wedge y) = f(x) \wedge f(y)$ and $f(x \vee y) = f(x) \vee f(y)$

f is an isomorphism if f is a bijective morphism

If $(A, \leq)$ is a lattice an ordered set $\Rightarrow (A, \wedge, \vee)$ is a lattice
where $a \wedge b = \inf \{a, b\}$
$a \vee b = \sup \{a, b\}$ $\quad \forall a, b \in A$

If $(A, \wedge, \vee)$ is a lattice $\Rightarrow (A, \leq)$ is a lattice
where $a \leq b \Leftrightarrow a \vee b = b$

$(\mathbb{N}, |, \wedge, \vee)$ $\quad a \wedge b = \inf \{a, b\} = \gcd(a, b)$
$a \vee b = \sup \{a, b\} = \mathrm{lcm}[a, b]$

102) $A = \{1, 2, 3\}$

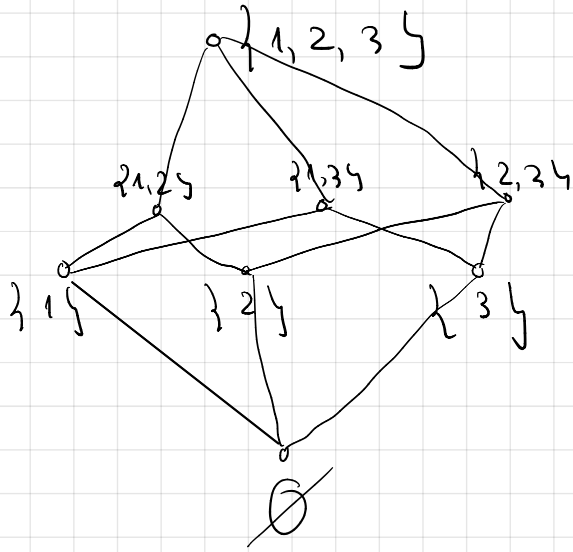
$B = \{ d \in \mathbb{N} \mid d \mid 30 \}$ div of 30

Determine all isomorphisms $f : (P(A), \subseteq) \to (B, \mid)$

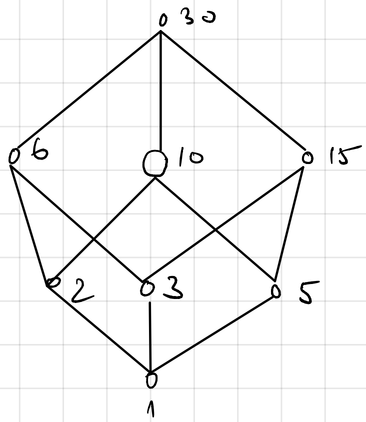
$P(A) = \{ B \mid B \subseteq A \}$

$B = \{1, 2, 3, 5, 6, 10, 15, 30\}$

P(A)



(B, |)



$\boxed{f_i(\emptyset) = 1 \text{ (minimum)}}$ $\boxed{f_i(\{1, 2, 3\}) = 30}$

$f(X \wedge Y) = f(X) \wedge f(Y)$

$f(x \wedge \emptyset) = f(\underset{\underset{f(\emptyset)}{\parallel}}{x \cap \emptyset}) = f(x) \wedge f(\emptyset) = \gcd(f(x), 1) = 1$

? $\underbrace{f_i(\{1\}) \in \{2,3,5\}}_{3 \text{ choiche}}$ $\underbrace{f_i(\{2\}) \in \{2,3,5\}}_{2 \text{ choichs}}$ $\underbrace{f_i(\{3\}) \in \{2,3,5\}}_{1 \text{ choiche}}$?

$i \in \{1, \dots, 6\}$

Let $f_1(\{1\}) = 5$ $f_1(\{2\}) = 3$, $f_1(\{3\}) = 2$

$f_1(\{1,2\}) = f_1(\{1\} \cup \{2\}) = f_1(\{1\} \vee \{2\}) =$
$= f_1(\{1\}) \vee f_1(\{2\}) = \sup\{f_1(\{1\}), f_1(\{2\})\} =$
$= \text{lcm}[5,3] = 15$ fixed

In total we have 6 diff isomorphisms

103) $(A, \leq, \wedge, \vee)$, $(B, \leq, \wedge, \vee)$ lattices, $f: A \to B$ function

Prove that:

a) if f is a lattice morphism $\Rightarrow f$ is increasing

$f(a \vee b) = f(a) \vee f(b)$ and $f(a \wedge b) = f(a) \wedge f(b)$

Let $x, y \in A$ s.t. $x \leq y \Rightarrow x \vee y = y \Rightarrow f(x \vee y) = f(y) = f(x) \vee f(y)$
$x \wedge y = x \Rightarrow f(x \wedge y) = f(x) = f(x) \wedge f(y)$

$\Rightarrow f(x) \leq f(y) \Rightarrow f$ is increasing

b) f increasing $\not\Rightarrow$ f lattice morphism

$f: (\mathbb{N}, |) \to (\mathbb{N}, \leq)$
$f(x) = x$ $a | b \Rightarrow a \leq b$ incr.

$f(2 \wedge 3) = 1$
$f(2) \wedge f(3) = 2 \wedge 3 = 2$
$= \inf\{2,3\}$
$= \min\{2,3\}$

In $(\mathbb{N}, \leq)$ $\inf\{x,y\} = \min\{x,y\}$
$\sup\{x,y\} = \max\{x,y\}$

c) If A is totally ordered and f is increasing $\Rightarrow f$ is a lattice morphism

Let $x,y \in A \overset{A \text{ tot}}{\underset{\text{ordered}}{\Rightarrow}} x \leq y$ or $y \leq x$

Assume that $x \leq y \Rightarrow x \vee y = y \Rightarrow f(x \vee y) = f(y)$
$x \wedge y = x \Rightarrow f(x \wedge y) = f(x)$

I $x \leq y \overset{f \text{ incr.}}{\Longrightarrow} f(x) \leq f(y) \Rightarrow f(x) \wedge f(y) = f(x)$
$f(x) \vee f(y) = f(y)$ $\Rightarrow$

$\Rightarrow f$ is a lattice morphism

d) If f is a bijective morph $\Rightarrow f^{-1}: B \to A$ is also a morph

f-morphism $\Leftrightarrow \forall x,y \in A$, $f(x \vee y) = f(x) \vee f(y)$
$f(x \wedge y) = f(x) \wedge f(y)$

$f^{-1}: B \to A$ morphism $\Leftrightarrow \forall x', y' \in B$, $f^{-1}(x' \vee y') = f^{-1}(x') \vee f^{-1}(y')$ If
$f^{-1}(x' \wedge y') = f^{-1}(x') \wedge f^{-1}(y')$ (3)

$f(f^{-1}(x' \vee y')) = x' \vee y'$ (1)
$f(f^{-1}(x') \vee f^{-1}(y')) = f(f^{-1}(x')) \vee f(f^{-1}(y')) = x' \vee y'$ (2)

$(1), (2) \Rightarrow f^{-1}(x' \vee y') = f^{-1}(x') \vee f^{-1}(y)$ $\Big\} \Rightarrow$
Similarly (3)

$f(a) = f(b)$
$\Downarrow f$ bij
$a = b$

$\Rightarrow f^{-1}$ morphism

$(A, \wedge, \vee)$ distributive lattice if $\forall a, b, c \in A$

$$(a \vee b) \wedge c = \underbrace{(a \wedge c) \vee (b \wedge c)}_{\text{prove } 2 \Rightarrow 1} \quad (1)$$

104) a) In $(A, \wedge, \vee)$ distributive lattice if $a \leq a'$, $b \leq b'$ $\Big\} \Rightarrow a \vee b \leq a' \vee b'$ (1)

$a \wedge b \leq a' \wedge b'$ (2)

$a \leq a' \Rightarrow a \wedge a' = a$, $a \vee a' = a'$

$b \leq b' \Rightarrow \wedge b' = b$, $b \vee b' = b'$

(1) $(a \vee b) \wedge (a' \vee b') = \underbrace{(a \wedge a')}_{a} \vee (a \wedge b') \vee (b \wedge a') \vee \underbrace{(b \wedge a')}_{b}$

$= \underbrace{a \vee (a \wedge b')}_{a} \vee \underbrace{b \vee (b \wedge a')}_{b} = a \vee b$

(2) Same

b) $(A, \wedge, \vee)$ is distributive $\Leftrightarrow \forall a, b, c \in A$

$\underline{(a \wedge b) \vee c} = \underline{(a \vee c) \wedge (b \vee c)}$ (2) (use this for $2 \Rightarrow 1$)

(1) $\Leftarrow$ (2)

(2) $\Rightarrow$ (1) $\quad (a \wedge c) \vee (b \wedge c) = (a \vee (b \wedge c)) \wedge \underbrace{(c \vee (b \wedge c))}_{c} \overset{\text{abs comm}}{\Rightarrow}$

$= ((b \wedge c) \vee a) \wedge c \overset{(2)}{=} (b \vee c) \wedge (c \vee a) \wedge c \underset{\text{abs com}}{=}$

$= (a \vee b) \wedge c$ Sim (1) $\Rightarrow$ (2)

105) a) Prove that if $(A, \leq)$ is tot ordered $\Rightarrow A$ is a dist lattice

$\forall a, b, c \in A$, $\xrightarrow{A \text{ is tot ord}}$ $a \leq b \leq c$ or $a \leq c \leq b$ or

$b \leq a \leq c$ or $b \leq c \leq a$ or

$c \leq a \leq b$ or $c \leq b \leq a$

Let $a \leq b \leq c \Rightarrow (a \wedge b) \vee c = a \vee c = c$

$(a \vee c) \wedge (b \vee c) = c \wedge c = c$

$\Rightarrow A$ is a distr. lattice

Similarly the other 5

at exam, just one

b) $(\mathbb{N}, |)$ is a distr. lattice

$\forall x, y \in \mathbb{N}$, $x \wedge y = \gcd(x, y)$

$x \vee y = \operatorname{lcm}[x, y]$

$12 = 2^2 \cdot 3^1 \cdot 5^0$

$15 = 2^0 \cdot 3^1 \cdot 5^1$

smallest exponent $\gcd(12, 15) = 2^0 \cdot 3 \cdot 5 = 3$

Let $x, y \in \mathbb{N} \Rightarrow x = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$ p_i's are distinct primes

$y = p_1^{\beta_1} \cdots p_n^{\beta_n}$ $\alpha_i, \beta_i, \gamma_i \in \mathbb{N}$, $n \in \mathbb{N}$

$z = p_1^{\gamma_1} \cdots p_n^{\gamma_n}$

We have to prove that $(x \wedge y) \vee z = (x \vee z) \wedge (y \wedge z)$ (1)

$x = y \Leftrightarrow \alpha_i = \beta_i \ \forall i \in \{1, \ldots, n\}$

(1) $\Leftrightarrow [\underbrace{(x, y)}_{\gcd}, z] = (\underbrace{[x, z]}_{\operatorname{lcm}}, [y, z]) \Leftrightarrow$

$\Leftrightarrow \max\{\min\{\alpha_i, \beta_i\}, \gamma_i\} = \min\{\max\{\alpha_i, \gamma_i\}, \max\{\beta_i, \gamma_i\}\}$

$\Leftrightarrow (\alpha_i \wedge \beta_i) \vee \gamma_i = (\alpha_i \vee \gamma_i) \wedge (\beta_i \vee \gamma_i)$ in $(\mathbb{N}, \leq)$

true because $(\mathbb{N}, \leq)$ tot. ordered hence distributive

$(A, \wedge, \vee)$ Boole lattice $\Leftrightarrow$ A is distr.

$\exists\, 0 = \min A$

$\exists\, 1 = \max A$

$\forall a \in A \;\; \exists\, a' \in A: \; a \wedge a' = 0$

$a \vee a' = 1$

$(A, +, \cdot)$ Boole ring $\Leftrightarrow$ $(A, +, \cdot)$ assoc ring

$\forall x \in A \;\; x^2 = x$