Complements of Geometry

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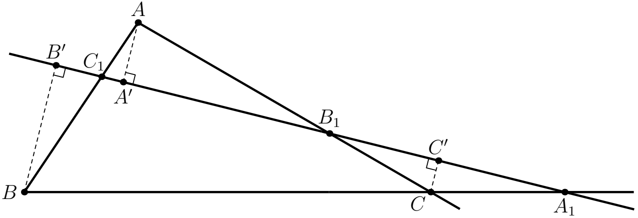
Optional course for Maths and Computer Science

Week 3

Theorem (Menelaus's theorem)

Let l be a line which intersects the sides (or their extensions) of the triangle ABC in the points A_1, B_1, C_1. Then the following relation holds:

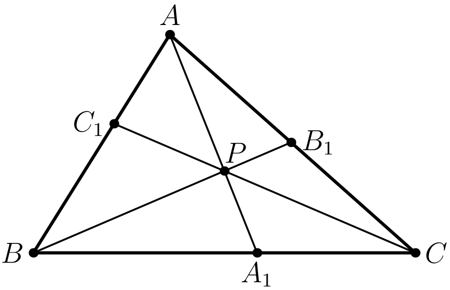
$$\frac{A_1B}{A_1C} \cdot \frac{B_1C}{B_1A} \cdot \frac{C_1A}{C_1B} = 1. \qquad (*)$$



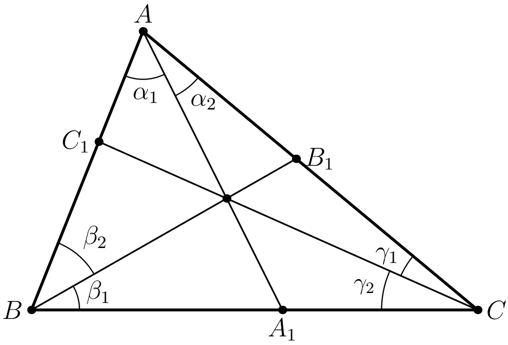
Theorem (Ceva)

Let AA_1, BB_1, CC_1, be concurrent cevians in $\triangle ABC$. Then we have the relation:

$$\frac{A_1B}{A_1C} \cdot \frac{B_1C}{B_1A} \cdot \frac{C_1A}{C_1B} = 1.$$



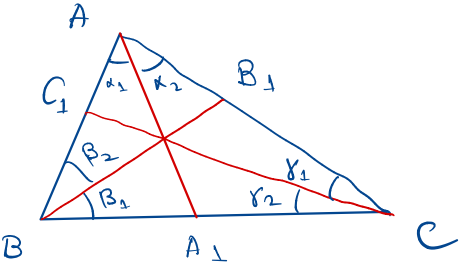
Trigonometric form of Ceva's theorem



Theorem (Trig. Ceva)

Using the above notations, the cevians AA_1, BB_1, CC_1 are concurrent if and only if

$$\frac{\sin\alpha_1}{\sin\alpha_2}\cdot\frac{\sin\gamma_1}{\sin\gamma_2}\cdot\frac{\sin\beta_1}{\sin\beta_2}=1.$$



Proof

By Ceva's Theorem (and its reciprocal), it follows that AA_1, BB_1 and CC_1 are concurrent if and only if:

$$\frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} \cdot \frac{AC_1}{BC_1} = 1. \qquad (*)$$

$$\frac{BA_1}{A_1C} = \frac{BA \cdot AA_1 \cdot \sin\alpha_1}{CA \cdot AA_1 \cdot \sin\alpha_2} = \frac{BA}{CA} \cdot \frac{\sin\alpha_1}{\sin\alpha_2} \qquad (1)$$

$$\frac{CB_1}{B_1A} = \frac{CB}{BA} \cdot \frac{\sin\beta_1}{\sin\beta_2} \ (2) \quad \text{and} \quad \frac{AC_1}{BC_1} = \frac{CA \cdot \sin\gamma_1}{CB \cdot \sin\gamma_2} \ (3)$$

Replacing (1), (2) and (3) in $(*)$ we get that AA_1, BB_1, CC_1 are concurrent $\Leftarrow$)

Proof

$$\frac{\cancel{BA}}{\cancel{CA}} \cdot \frac{\sin\alpha_1}{\sin\alpha_2} \cdot \frac{\cancel{CB}}{\cancel{BA}} \cdot \frac{\sin\beta_1}{\sin\beta_2} \cdot \frac{\cancel{CA}}{\cancel{CB}} \cdot \frac{\sin\gamma_1}{\sin\gamma_2} = 1.$$

$$\Longleftrightarrow \quad \frac{\sin\alpha_1}{\sin\alpha_2} \cdot \frac{\sin\gamma_1}{\sin\gamma_2} \cdot \frac{\sin\beta_1}{\sin\beta_2} = 1.$$

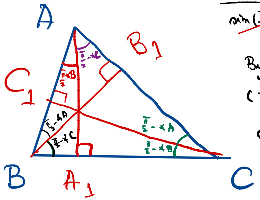
$\boxtimes$

Consequences of Ceva's theorem

The reciprocal version of Ceva's theorem (in both metric or trigonometric forms) is a valuable important instrument for proving that certain cevians are concurrent in a triangle.

We can immediately show that the following lines are concurrent

- internal angle bisectors; (metric or trigonometric)
- altitudes;



$$\frac{\cancel{\sin\left(\frac{\pi}{2}-\measuredangle B\right)}}{\cancel{\sin\left(\frac{\pi}{2}-\measuredangle C\right)}} \cdot \frac{\cancel{\sin\left(\frac{\pi}{2}-\measuredangle A\right)}}{\cancel{\sin\left(\frac{\pi}{2}-\measuredangle B\right)}} \cdot \frac{\cancel{\sin\left(\frac{\pi}{2}-\measuredangle C\right)}}{\cancel{\sin\left(\frac{\pi}{2}-\measuredangle A\right)}} = 1.$$

By (the reciprocal) of Ceva's theorem (trig. form), the altitudes are concurrent. □

Consequences of Ceva's theorem

The reciprocal version of Ceva's theorem (in both metric or trigonometric forms) is a valuable important instrument for proving that certain cevians are concurrent in a triangle.

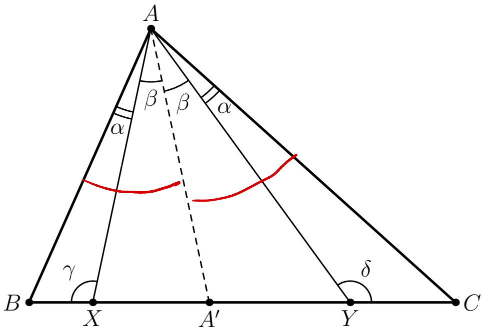
We can immediately show that the following lines are concurrent

- internal angle bisectors;
- altitudes;
- medians. (use the metric form of Ceva's theorem)

Izogonal Cevians

Definition

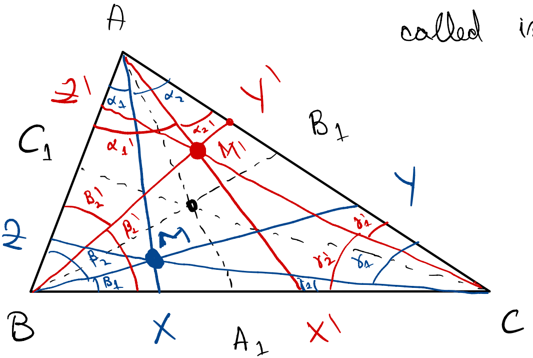
In the triangle ABC, the cevians AX, AY are called *izogonal* if they are symmetric with respect to the angle bisector of A, namely if $\angle XAA' \equiv \angle YAA'$.



Theorem

Let (AX, AX'), (BY, BY'), (CZ, CZ') be pairs of izogonal cevians. Then AX, BY, CZ are concurrent if and only if AX', BY', CZ' are concurrent.

Proof.



Obs: M and M' are called izogonal conjugates.

Theorem

Let (AX, AX'), (BY, BY'), (CZ, CZ') be pairs of izogonal cevians. Then AX, BY, CZ are concurrent if and only if AX', BY', CZ' are concurrent.

Proof. Suppose AX, BY and CZ are concurrent.
By trig Ceva's theorem, we get:

$$\frac{\sin \alpha_1}{\sin \alpha_2} \cdot \frac{\sin \gamma_1}{\sin \gamma_2} \cdot \frac{\sin \beta_1}{\sin \beta_2} = 1,$$

Following the angles on the diagram, this yields:

$$\frac{\sin \alpha_2'}{\sin \alpha_1'} \cdot \frac{\sin \gamma_2'}{\sin \gamma_1'} \cdot \frac{\sin \beta_2'}{\sin \beta_1'} = 1$$

$\Rightarrow$ By the reciprocal of Ceva's theorem, AX', BY', CZ' are conc.

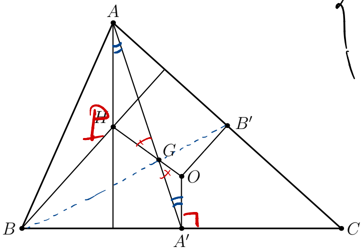
The Euler line

Theorem (Euler line)

In every triangle ABC the points O, G, H are collinear. Moreover, $\dfrac{OG}{GH} = \dfrac{1}{2}$.

Proof. Let A' be the midpoint of the side $[BC]$.

G - int. of medians
H - int. of altitudes.
O - int. of perp. bisect.



$OA' \perp BC$ (since OA' is perp. bisector)

On the half-line $[OG$ construct the point P such that

$$\frac{OG}{GP} = \frac{1}{2}.$$

! We will show that $P = H$. From

seminar 1 (or lecture 2)

$$\frac{OG}{GP} = \frac{GA'}{GA} = \frac{1}{2} \quad \text{and} \quad \angle OGA' \equiv \angle PGA$$, by S.A.S.

it follows that $\triangle OGA' \sim \triangle PGA$. Therefore, $\angle OA'G \equiv \angle PAG$.
From $OA' \| AP$ and $OA' \perp BC$, it follows that $\boxed{AP \perp BC}$, hence P belongs to the height from A.
Analogously, we can show that P belongs to the altitude from B, therefore it is necessary that $P = H$. □

We start our presentation with the following useful result.

Proposition (Equal tangents)

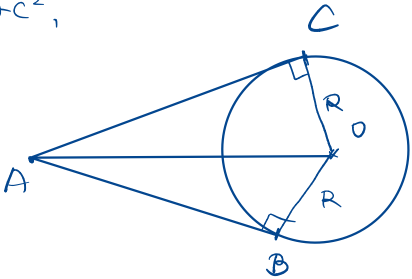
Two tangent lines to the given circle ω intersect at A. Denote by B, C the points of tangency with the circle. Then $AB = AC$.

$AB^2 = AO^2 - OB^2 = AO^2 - R^2 =$

$= AO^2 - OC^2 = AC^2,$

using Pytagora's in

$\triangle AOC$ and $\triangle AOB$.

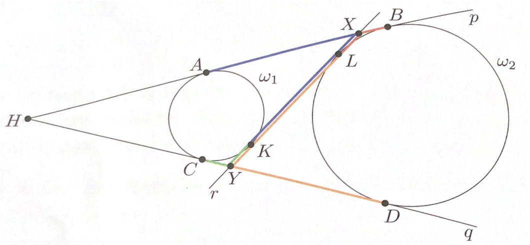


Proposition

Let p, q be common external tangents of circles ω_1 and ω_2. Denote by A, B the points of tangency of p with ω_1 and ω_2, respectively. Similarly, we denote by C and D the tangency points of the line q with ω_1 and ω_2, respectively. Then:

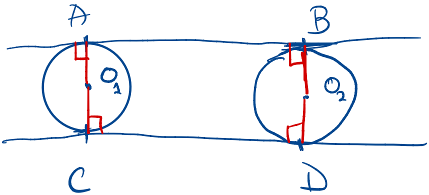
(a) $AB = CD$.

(b) *If the two circles are nonintersecting and their common internal tangent r intersects p and q at points X and Y, respectively, we have $AB = CD = XY$.*



Proof (a)

I) Suppose $AB \parallel CD$.



Note that $ABDC$ is a rectangle, so $\boxed{AB = CD}$.

II) Let $\{H\} = AB \cap CD$.

$HB = HD$ (since they are common tangents to ω_2)

$HA = HC$ (- - - - - - - - - - - - to ω_1).

$AB = HB - HA = HD - HC = CD$. $\square$

Proof (b)

$$2 \cdot XY = XY + XY = (YK + KX) + (XL + LY)$$

$$= YC + AX + XB + YD$$

$$= CD + AB = 2 \cdot AB$$

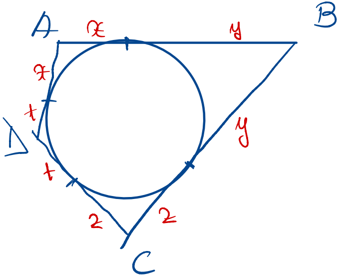
where we used repeatedly equal tangents.

Theorem (Pitot)

Let $ABCD$ be a circumscriptible quadrilateral. Then

$$AB + CD = BC + DA.$$

$AB + CD = x + y + z + t = BC + AD$.



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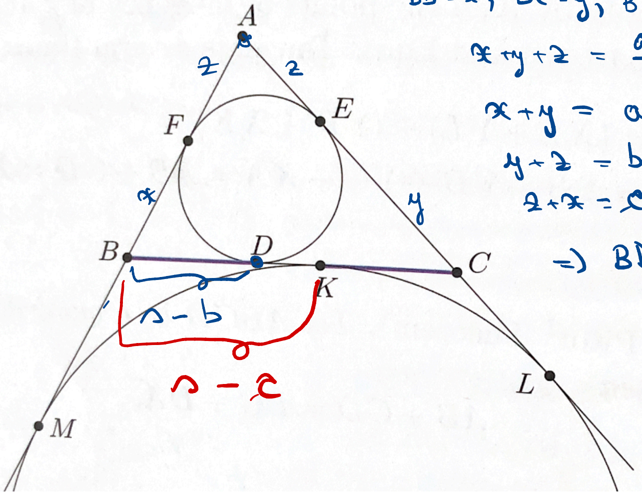
The tangency points with the inscribed and ex-circle



$BD = x$, $DC = y$, $BF = x$

$$x + y + z = \frac{a+b+c}{2} := s.$$

$$x + y = a$$

$$y + z = b$$

$$z + x = c$$

$$\Rightarrow BD = s - b.$$

Computing the distances BD and BK

$$BK = BM$$

$$BK + KC = a$$

$$KC = CL$$

$$AM = AL \Leftrightarrow c + BK = b + KC.$$

$$\Rightarrow \begin{cases} BK + KC = a \\ BK - KC = b - c \end{cases} \Rightarrow BK = a - c.$$

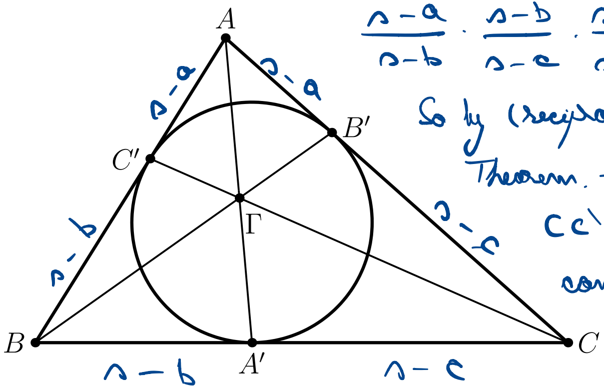
Remark: $BD = BK \Leftrightarrow a - b = a - c \Leftrightarrow b = c \Leftrightarrow \boxed{AC = AB}$

Gergonne's point

Theorem (Gergonne)

The cevians determined by the points of tangency with the incircle are concurrent in a point Γ.

Proof. Let A', B', C' be the three points of tangency.

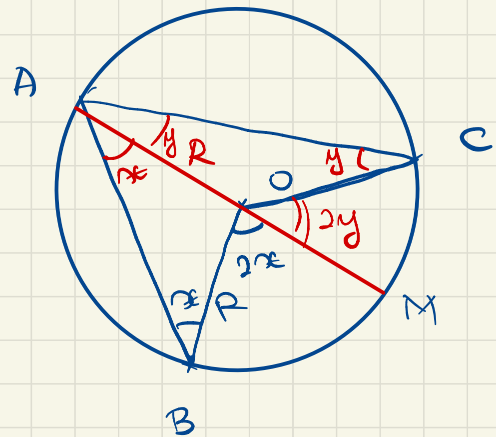


$$\frac{s-a}{s-b}\cdot\frac{s-b}{s-c}\cdot\frac{s-c}{s-a}=1.$$

So by (reciprocal) Ceva's Theorem. AA', BB', CC' are concurrent.

Continuation of the proof

The inscribed angle theorem



$$2 \cdot m(\sphericalangle BAC) = m(\sphericalangle BOC)$$

<u>Proof</u>. $\sphericalangle BOM = \sphericalangle OAB + \sphericalangle OBA = 2x$

$\sphericalangle COM = \sphericalangle CAO + \sphericalangle COA = 2y$

$\Rightarrow 2 \cdot \sphericalangle BAC = 2(x+y) = \sphericalangle BOM + \sphericalangle COM = \sphericalangle BOC.$

Theorem (Extended law of sines)

$a = BC$, $b = CA$, $c = AB$.

Let $\triangle ABC$ be a triangle. Then

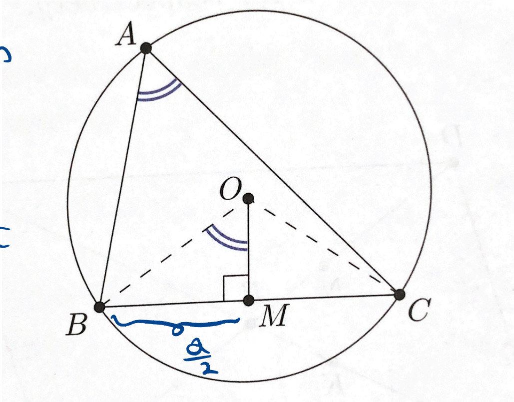
$$\frac{a}{\sin\angle A} = \frac{b}{\sin\angle B} = \frac{c}{\sin\angle C} = 2R,$$

where R is the circumradius of triangle $\triangle ABC$.

I. $\triangle ABC$ is acute.

Let M be the midpoint of $[BC]$.

$BM = \frac{a}{2}$



Proof

$OM \perp BC$, because $OB = OC = R$.

So ΔBOM is a right triangle.

$$\sin \measuredangle BOM = \frac{MB}{OB} = \frac{\frac{a}{2}}{R} = \frac{a}{2R} \quad (*)$$

On the other hand, by the inscribed angle theorem $\measuredangle BOC = 2 \cdot \measuredangle A \Rightarrow$

$\measuredangle BOM = \measuredangle A$.

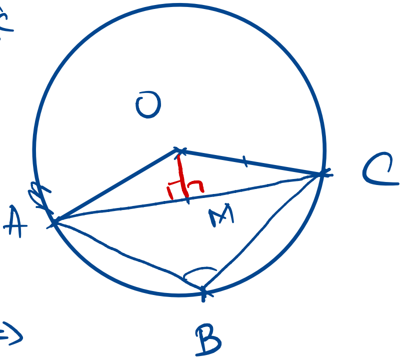
Replacing in $(*)$ we get

$$\sin \measuredangle A = \frac{a}{2R} \Leftrightarrow \frac{a}{\sin \measuredangle A} = 2R.$$

Proof

(II) $\triangle ABC$ is obtuse, then.

$$\measuredangle B = \frac{\text{large arc } \overset{\frown}{AC}}{2}$$
$$= \frac{360^\circ - \text{small arc } \overset{\frown}{AC}}{2}$$
$$= 180^\circ - \frac{\text{small arc } \overset{\frown}{AC}}{2}$$



$$\measuredangle AOM = \frac{1}{2} \cdot \measuredangle AOC$$
$$= \frac{\text{small arc } \overset{\frown}{AC}}{2}$$

$$\measuredangle B = 180^\circ - \measuredangle AOM. \Rightarrow$$

$$\sin(\measuredangle B) = \sin(\measuredangle AOM). = \frac{AM}{OA} = \frac{AC}{2 \cdot R} = \frac{b}{2R}.$$

Proof

$$\Rightarrow \frac{b}{\sin(\measuredangle B)} = 2R.$$

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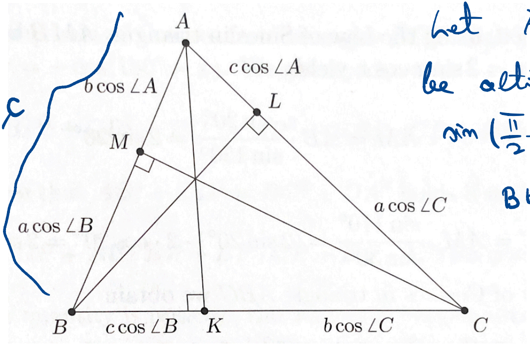
Theorem (Cosine theorem)

Let $\triangle ABC$ be a triangle. Then

$$a^2 = b^2 + c^2 - 2bc\cos\angle A.$$



Let AK, BL, CM be altitudes in $\triangle ABC$.

$\sin\left(\frac{\pi}{2} - x\right) = \cos x$.

$BK = AB \cdot \sin(\measuredangle BAK)$

$= c \cdot \cos(\measuredangle B)$.

$$a^2 = a \cdot (BK + KC) = a \cdot c \cdot \cos\measuredangle B + a \cdot b \cdot \cos\measuredangle C$$

Proof

$$a^2 = c\cdot(a\cdot\cos\sphericalangle B) + b\cdot(a\cdot\cos\sphericalangle C)$$

$$= c\cdot BM + b\cdot CL$$

$$= c\cdot(c - AM) + b\cdot(b - AL)$$

$$= b^2 + c^2 - c\cdot AM - b\cdot AL$$

$$= b^2 + c^2 - c\cdot b\cdot\cos\sphericalangle A - b\cdot c\cdot\cos\sphericalangle A$$

$$= b^2 + c^2 - 2bc\cos\sphericalangle A.$$

Thank you very much for your attention!