Homework: Topology on $\mathbb{R}$

EX 1.

$(-1,2]$ ✓ $\quad (-1,1)$ ✗ $\quad [-1,1]$ ✓ $\quad \mathbb{R}\setminus\{1\}$ ✗ $\quad \{1,2,3\}$ ✓ $\quad \mathbb{R}\setminus(0,1)$ ✓ $\quad \mathbb{Z}$ ✓ $\quad \mathbb{Q}$ ✓ $\quad \mathbb{R}\setminus\mathbb{Q}$ ✓ $\quad \mathbb{R}$ ✓

Ex 2.

$$A = \bigcup_{n\in\mathbb{N}\setminus\{1\}} \left(-1+\frac{1}{n}, 1-\frac{1}{n}\right) = \bigcup_{n\in\mathbb{N}\setminus\{1\}} \left(\frac{1-n}{n}, \frac{n-1}{n}\right) =$$

$$= \left(\frac{1-2}{2}, \frac{2-1}{2}\right) \cup \left(\frac{1-3}{3}, \frac{3-1}{3}\right) \cup \dots \cup \left(\frac{1-n}{n}, \frac{n-1}{n}\right) =$$

$$= \left(-\frac{1}{2}, \frac{1}{2}\right) \cup \left(-\frac{2}{3}, \frac{2}{3}\right) \cup \dots \cup \left(\frac{1-n}{n}, \frac{n-1}{n}\right)$$

$$\left(-\frac{1}{2}, \frac{1}{2}\right) \subseteq \left(-\frac{2}{3}, \frac{2}{3}\right) \subseteq \left(-\frac{3}{4}, \frac{3}{4}\right) \subseteq \dots \subseteq \left(\frac{1-n}{n}, \frac{n-1}{n}\right)$$

$$\lim_{n\to\infty} \frac{1-n}{n} = \lim_{n\to\infty} \frac{n\left(\frac{1}{n}-1\right)}{n} = -1$$

$$\cancel{\lim_{n\to\infty} \frac{1+n\;n-1}{n} = \lim_{n\to\infty} \dots} \qquad \lim_{n\to\infty} \frac{n-1}{n} = \lim_{n\to\infty} \frac{n\left(1-\frac{1}{n}\right)}{n} = 1$$

$$\Rightarrow \lim_{n\to\infty} \left(\frac{1-n}{n}, \frac{n-1}{n}\right) = (-1, 1)$$

$$\Rightarrow A = \bigcup_{n\in\mathbb{N}\setminus\{1\}} \left(-1+\frac{1}{n}, 1-\frac{1}{n}\right) = \left(\frac{1-n}{n}, \frac{n-1}{n}\right)$$

| | LB | UB | inf | sup | min | max |
|---|---|---|---|---|---|---|
| $\left(\frac{1-n}{n}, \frac{n-1}{n}\right)$ | $\left(-\infty, \frac{1-n}{n}\right]$ | $\left[\frac{n-1}{n}, \infty\right)$ | $\frac{1-n}{n}$ | $\frac{n-1}{n}$ | $\cancel{\frac{1-n}{n}}$ ∄ | $\cancel{\frac{n-1}{n}}$ ∄ |

$$B = \bigcup_{n\in\mathbb{N}} \left[-1+\frac{1}{n}, 1-\frac{1}{n}\right] = \bigcup_{n\in\mathbb{N}} \left[\frac{1-n}{n}, \frac{n-1}{n}\right] =$$

$$= \left[\frac{1-1}{1}, \frac{1-1}{1}\right] \cup \left[\frac{1-2}{2}, \frac{2-1}{2}\right] \cup \dots \cup \left[\frac{1-n}{n}, \frac{n-1}{n}\right] =$$

$$= \{0\} \cup \left[-\frac{1}{2}, \frac{1}{2}\right] \cup \dots \cup \left[\frac{1-n}{n}, \frac{n-1}{n}\right]$$

$$\{0\} \subseteq \left[-\frac{1}{2}, \frac{1}{2}\right] \subseteq \left[-\frac{2}{3}, \frac{2}{3}\right] \subseteq \dots \subseteq \left[\frac{1-n}{n}, \frac{n-1}{n}\right]$$

$$\Rightarrow B = \left[\frac{1-n}{n}, \frac{n-1}{n}\right]$$

| | LB | UB | inf | sup | min | max |
|---|---|---|---|---|---|---|
| $\left[\frac{1-n}{n}, \frac{n-1}{n}\right]$ | $\left(-\infty, \frac{1-n}{n}\right]$ | $\left[\frac{n-1}{n}, \infty\right)$ | $\frac{1-n}{n}$ | $\frac{n-1}{n}$ | $\frac{1-n}{n}$ | $\frac{n-1}{n}$ |

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$$C=\bigcap_{n\in\mathbb{N}\setminus\{1\}}\left(-1+\frac{1}{n},1-\frac{1}{n}\right)=\bigcap_{n\in\mathbb{N}\setminus\{1\}}\left(\frac{1-n}{n},\frac{n-1}{n}\right)=$$

$$=\left(-\frac{1}{2},\frac{1}{2}\right)\cap\left(\frac{-2}{3},\frac{2}{3}\right)\cap\dots\cap\left(\frac{1-n}{n},\frac{n-1}{n}\right)=\left(-\frac{1}{2},\frac{1}{2}\right)$$

| | LB | UB | inf | sup | min | max |
|---|---|---|---|---|---|---|
| $\left(-\frac{1}{2},\frac{1}{2}\right)$ | $(-\infty,-\frac{1}{2}]$ | $[\frac{1}{2},\infty)$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | $\nexists$ | $\nexists$ |

$$D=\bigcap_{n\in\mathbb{N}}\left[-1+\frac{1}{n},1-\frac{1}{n}\right]=\bigcap_{n\in\mathbb{N}}\left[\frac{1-n}{n},\frac{n-1}{n}\right]=$$

$$=\{0\}\cap\left[-\frac{1}{2},\frac{1}{2}\right]\cap\left[-\frac{2}{3},\frac{2}{3}\right]\cap\dots\cap\left[\frac{1-n}{n},\frac{n-1}{n}\right]=\{0\}$$

| | LB | UB | inf | sup | min | max |
|---|---|---|---|---|---|---|
| $\{0\}$ | $(-\infty,0]$ | $[0,\infty)$ | $0$ | $0$ | $\cancel{\nexists}\ 0$ | $\cancel{\nexists}\ 0$ |

$$E=\bigcup_{n\in\mathbb{N}}\left[-1-\frac{1}{n},1+\frac{1}{n}\right]=\bigcup_{n\in\mathbb{N}}\left[-\frac{n+1}{n},\frac{n+1}{n}\right]=$$

$$=\cancel{[-1,1]\cup[}\ \{0\}\cup[-2,2]\cup\left[-\frac{3}{2},\frac{3}{2}\right]\cup\dots\cup\left[-\frac{n+1}{n},\frac{n+1}{n}\right]=$$

$$\lim_{n\to\infty}-\frac{n+1}{n}=-1\quad \lim_{n\to\infty}\frac{n+1}{n}=1\ \rightarrow\ \lim_{n\to\infty}\left[-\frac{n+1}{n},\frac{n+1}{n}\right]=[-1,1]$$

$$\{0\}\subseteq[-1,1]\subseteq\dots\subseteq\left[-\frac{3}{2},\frac{3}{2}\right]\subseteq[-2,2]$$

$$\Rightarrow E=[-2,2]$$

| | LB | UB | inf | sup | min | max |
|---|---|---|---|---|---|---|
| $[-2,2]$ | $(-\infty,-2]$ | $[2,\infty)$ | $-2$ | $2$ | $-2$ | $2$ |

$$F=\bigcap_{n\in\mathbb{N}}\left(-1-\frac{1}{n},1+\frac{1}{n}\right)=\bigcap_{n\in\mathbb{N}}\left(-\frac{n+1}{n},\frac{n+1}{n}\right)=$$

$$=\overset{\emptyset}{\cancel{\{0\}}}\cap(-2,2)\cap\left(-\frac{3}{2},\frac{3}{2}\right)\cap\dots\cap\left(-\frac{n+1}{n},\frac{n+1}{n}\right)=\cancel{\{0\}}\ \emptyset$$

| | LB | UB | inf | sup | min | max |
|---|---|---|---|---|---|---|
| ~~$\{0\}$~~ | ~~$(-\infty,0]$~~ | ~~$[0,\infty)$~~ | ~~$0$~~ | ~~$0$~~ | ~~$0$~~ | ~~$0$~~ |
| $\emptyset$ | $\emptyset$ | $\emptyset$ | $\cancel{-\infty}\ \nexists$ | $\infty$ | $\nexists$ | $\nexists$ |

Ex 3:

$$A=\bigcup_{n\in\mathbb{N}\setminus\{1\}}\left(-1+\frac{1}{n},1-\frac{1}{n}\right)=\bigcup_{n\in\mathbb{N}\setminus\{1\}}\left(\frac{1-n}{n},\frac{n-1}{n}\right)=\left(\frac{1-n}{n},\frac{n-1}{n}\right)$$

$\left(\frac{1-n}{n},\frac{n-1}{n}\right)$ is open $\Leftrightarrow \forall x\in\left(\frac{1-n}{n},\frac{n-1}{n}\right),\ \left(\frac{1-n}{n},\frac{n-1}{n}\right)\in\upsilon(x)$

$\exists r_x=\cancel{\frac{n-1}{n}>x>0}$ s.t. $\min\left\{\frac{n-1}{n}-x,\ x-\frac{1-n}{n}\right\}>0\quad B(x,r)\subseteq\left(\frac{1-n}{n},\frac{n-1}{n}\right)$

$\Rightarrow A$ is open

$B = \bigcup_{n \in \mathbb{N}} [-1 + \frac{1}{n}, 1 - \frac{1}{n}] = \bigcup_{n \in \mathbb{N}} [\frac{1-n}{n}, \frac{n-1}{n}] =$

$= \{0\} \cup [-\frac{1}{2}, \frac{1}{2}] \cup [-\frac{2}{3}, \frac{2}{3}] \cup \dots \cup [\frac{1-n}{n}, \frac{n-1}{n}] = [\frac{1-n}{n}, \frac{n-1}{n}]$.

$[\frac{1-n}{n}, \frac{n-1}{n}]$ is closed $\Leftrightarrow \mathbb{R} \setminus [\frac{1-n}{n}, \frac{n-1}{n}]$ is open

$\mathbb{R} \setminus [\frac{1-n}{n}, \frac{n-1}{n}] = (-\infty, \frac{1-n}{n}) \cup (\frac{n-1}{n}, \infty)$

$(-\infty, \frac{1-n}{n})$ open $\Leftrightarrow \forall x \in (-\infty, \frac{1-n}{n}), (-\infty, \frac{1-n}{n}) \in \mathcal{V}(x)$

$\exists r_x = \frac{1-n}{n} - x > 0$ s.t. $B(x, r_x) \subseteq (-\infty, \frac{1-n}{n})$ ① $\Rightarrow (-\infty, \frac{1-n}{n})$ open

$(\frac{n-1}{n}, \infty)$ open $\Leftrightarrow \forall x \in (\frac{n-1}{n}, \infty), (\frac{n-1}{n}, \infty) \in \mathcal{V}(x)$

$\exists r_x = \frac{n-1}{n} - x > 0$ s.t. $B(x, r_x) \subseteq (\frac{n-1}{n}, \infty)$ ② $\Rightarrow (\frac{n-1}{n}, \infty)$ open

1, 2 $\Rightarrow (-\infty, \frac{1-n}{n}) \cup (\frac{n-1}{n}, \infty)$ is open $\Rightarrow [\frac{1-n}{n}, \frac{n-1}{n}]$ is closed

$C = \bigcap_{n \in \mathbb{N} \setminus \{1\}} (-1 + \frac{1}{n}, 1 - \frac{1}{n}) = \bigcap_{n \in \mathbb{N} \setminus \{1\}} (\frac{1-n}{n}, \frac{n-1}{n}) =$

$= (-\frac{1}{2}, \frac{1}{2}) \cap (-\frac{2}{3}, \frac{2}{3}) \cap \dots \cap (\frac{1-n}{n}, \frac{n-1}{n}) = (-\frac{1}{2}, \frac{1}{2})$

$(-\frac{1}{2}, \frac{1}{2})$ is open $\Leftrightarrow \forall x \in (-\frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, \frac{1}{2}) \in \mathcal{V}(x)$

$\exists r_x = \min\{\frac{1}{2} - x, x + \frac{1}{2}\} > 0 \quad B(x, r) \subseteq (-\frac{1}{2}, \frac{1}{2})$

$\Rightarrow C$ is open

$D = \bigcap_{n \in \mathbb{N}} [-1 + \frac{1}{n}, 1 - \frac{1}{n}] = \{0\}$

$\{0\}$ is closed $\Leftrightarrow \mathbb{R} \setminus \{0\}$ is open

$\mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$

$(-\infty, 0)$ open $\Leftrightarrow \forall x \in (-\infty, 0), (-\infty, 0) \in \mathcal{V}(x)$

$\exists r_x = 0 - x > 0$ s.t. $B(x, r_x) \subseteq (-\infty, 0)$ ① $\Rightarrow (-\infty, 0)$ open

$(0, \infty)$ open $\Leftrightarrow \forall x \in (0, \infty), (0, \infty) \in \mathcal{V}(x)$

$\exists r_x = x - 0 > 0$ s.t. $B(x, r_x) \subseteq (0, \infty)$ ② $\Rightarrow (0, \infty)$ open

1, 2 $\Rightarrow (-\infty, 0) \cup (0, \infty)$ is open $\Rightarrow \{0\}$ is closed

$E = \bigcup_{n\in\mathbb{N}} [-1-\frac{1}{n}, 1+\frac{1}{n}] = \bigcup_{n\in\mathbb{N}} [-\frac{n+1}{n}, \frac{n+1}{n}] = [-2,2]$

$[-2,2]$ is closed $\Leftrightarrow \mathbb{R}\setminus[-2,2]$ is open

$\mathbb{R}\setminus[-2,2] = (-\infty,-2)\cup(2,\infty)$

$(-\infty,-2)$ is open $\Leftrightarrow \forall x\in(-\infty,-2),\ (-\infty,-2)\in V(x)$

$\exists r_x = -2-x>0$ s.t. $B(x,r_x) \subseteq (-\infty,-2)$ ① $\Rightarrow (-\infty,-2)$ open

$\Leftrightarrow (2,\infty)$ is open $\Leftrightarrow \forall x\in(2,\infty),\ (2,\infty)\in V(x)$

$\exists r_x = x-2>0$ s.t. $B(x,r_x) \subseteq (2,\infty)$ ② $\Rightarrow (2,\infty)$ open

$1,2 \Rightarrow (-\infty,-2)\cup(2,\infty)$ is open $\Rightarrow [-2,2]$ is closed

$F = \bigcap_{n\in\mathbb{N}} (-1-\frac{1}{n}, 1+\frac{1}{n}) = \bigcap_{n\in\mathbb{N}} (-\frac{n+1}{n}, \frac{n+1}{n}) = \emptyset$

$\emptyset$ is closed $\Leftrightarrow \mathbb{R}\setminus\emptyset$ is open

$\mathbb{R}\setminus\emptyset = \mathbb{R}$ open $\Leftrightarrow \forall x\in\mathbb{R},\ \mathbb{R}\in V(x)$

$\exists r = 12>0 \Rightarrow B(x,12) \subseteq \mathbb{R}$

Ex. 4:

| $A$ | $LB(A)$ | $\inf A$ | $\min A$ | $UB(A)$ | $\sup A$ | $\max A$ |
|---|---|---|---|---|---|---|
| $(-\infty,-1]\cup(2,\infty)$ | $\emptyset$ | $-\infty$ | $\nexists$ | $\emptyset$ | $\infty$ | $\nexists$ |
| $(-1,9]\cup[10,20)$ | $(-\infty,-1]$ | $-1$ | $\nexists$ | $[20,\infty)$ | $20$ | $\nexists$ |
| $((-1,9]\cup[10,20))\cap\mathbb{N}$ | $(-\infty,0]$ | $0$ | $0$ | $[20,\infty)$ | $20$ | $\nexists$ |
| $\{1,2,3\}$ | $(-\infty,1]$ | $1$ | $1$ | $[3,\infty)$ | $3$ | $3$ |
| $\mathbb{N}$ | $(-\infty,0]$ | $0$ | $0$ | $\emptyset$ | $\infty$ | $\nexists$ |
| $\mathbb{R}\setminus\{1,2,3\}$ | $\emptyset$ | $-\infty$ | $\nexists$ | $\emptyset$ | $\infty$ | $\nexists$ |
| $\mathbb{R}\setminus\mathbb{N}$ | $\emptyset$ | $-\infty$ | $\nexists$ | $\emptyset$ | $\infty$ | $\nexists$ |
| $\mathbb{Z}$ | $\emptyset$ | $-\infty$ | $\nexists$ | $\emptyset$ | $\infty$ | $\nexists$ |
| $\mathbb{R}\setminus\mathbb{Z}$ | $\emptyset$ | $-\infty$ | $\nexists$ | $\emptyset$ | $\infty$ | $\nexists$ |
| $\mathbb{Q}$ | $\emptyset$ | $-\infty$ | $\nexists$ | $\emptyset$ | $\infty$ | $\nexists$ |
| $\mathbb{R}\setminus\mathbb{Q}$ | $\emptyset$ | $-\infty$ | $\nexists$ | $\emptyset$ | $\infty$ | $\nexists$ |
| $\mathbb{R}$ | $\emptyset$ | $-\infty$ | $\nexists$ | $\emptyset$ | $\infty$ | $\nexists$ |

Ex 5:

| $A$ | $\text{int}A$ | $\text{bd}A$ | $\text{cl}A$ | $\text{ext}A$ | $\text{iso}A$ |
|---|---|---|---|---|---|
| $(-\infty,-1]\cup(2,\infty)$ | $(-\infty,-1)\cup(2,\infty)$ | $\{-1,2\}$ | $(-\infty,-1]\cup[2,\infty)$ | $(-1,2]$ | $\emptyset$ |
| $(-1,9]\cup[10,20)$ | $(-1,9)\cup(10,20)$ | $\{-1,9,10,20\}$ $\cancel{\{9,10\}}$ | $[-1,9]\cup[10,20]$ | $(-\infty,-1)\cup(9,10)\cup(20,\infty)$ | $\emptyset$ |
| $((-1,9]\cup[10,20))\cap\mathbb{N}$ | $\emptyset$ | $\{0,1,2,\dots,20\}$ | $\{0,1,\dots,20\}$ $\cancel{\{0,1,2,3,5,6,7,8,9,\dots,20\}}$ | $\mathbb{R}\setminus\{0,1,\dots,20\}$ | $\{0,1,2,\dots,20\}$ |
| $\{1,2,3\}$ | $\emptyset$ | $\{1,2,3\}$ | $\{1,2,3\}$ | $\mathbb{R}\setminus\{1,2,3\}$ | $\{1,2,3\}$ |
| $\mathbb{N}$ | $\emptyset$ | $\mathbb{N}$ | $\mathbb{N}\cancel{\cup\{\}}$ | $\mathbb{R}\setminus\mathbb{N}$ | $\mathbb{N}$ |
| $\mathbb{R}\setminus\{1,2,3\}$ | $(-\infty,1)\cup(1,2)\cup(2,3)$ | $\{1,2,3\}$ | $\cancel{(-\infty,1]\cup}\,\mathbb{R}$ | $\{1,2,3\}$ | $\emptyset$ |
| $\mathbb{R}\setminus\mathbb{N}$ | $\mathbb{R}\setminus\mathbb{N}$ | $\mathbb{N}$ | $\cancel{\mathbb{R}\ \mathbb{N}}\,\mathbb{R}$ | $\mathbb{N}$ | $\emptyset$ |
| $\mathbb{Z}$ | $\emptyset$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{R}\setminus\mathbb{Z}$ | $\mathbb{Z}$ |
| $\mathbb{R}\setminus\mathbb{Z}$ | $\mathbb{R}\setminus\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{R}$ | $\mathbb{Z}$ | $\emptyset$ |
| $\mathbb{Q}$ | $\emptyset$ | $\mathbb{Q}$ | $\mathbb{Q}$ | $\mathbb{R}\setminus\mathbb{Q}$ | $\cancel{\emptyset}\ \mathbb{Q}$ |
| $\mathbb{R}\setminus\mathbb{Q}$ | $\mathbb{R}\setminus\mathbb{Q}$ | $\mathbb{Q}$ | $\mathbb{R}$ | $\mathbb{Q}$ | $\emptyset$ |
| $\mathbb{R}$ | $\mathbb{R}$ | $\emptyset$ | $\mathbb{R}$ | $\emptyset$ | $\emptyset$ |