16.11.2022

Eq. Relation

$\rho = (A, A, R)$ homogenous relation

ρ reflexive $\Leftrightarrow \forall x \in A \quad x \rho x \quad (1_A \subseteq \rho)$

ρ transitive $\Leftrightarrow \forall x, y, z \in A \quad (x \rho y) \wedge (y \rho z) \Rightarrow (x \rho z) \quad (\rho^2 \subseteq \rho)$

ρ symmetric $\Leftrightarrow \forall x, y \in A \quad (x \rho y) \Rightarrow (y \rho x) \quad (\rho = \rho^{-1})$

ρ antisymmetric $\Leftrightarrow \forall x, y \in A \quad (x \rho y) \wedge (y \rho x) \Rightarrow x = y \quad (\rho \cap \rho^{-1} \subseteq 1_A)$

$\rho\langle x \rangle = \{ y \in A \mid x \rho y \}$ eq. class of x

$A/\rho = \{ \rho\langle x \rangle \mid x \in A \}$ factors set (partition of A)

$\mathbb{Z}_n = \{ \hat{0}, \hat{1}, \hat{2}, \dots \}$

$\mathbb{Z}_2 = \{ \hat{0}, \hat{1} \} \quad \hat{0} = \{ \dots, -2, 0, 2, 4, \dots \}$

$\hat{1} = \{ -3, -1, 1, 3, 5, \dots \}$

$x \rho y \Leftrightarrow x \equiv y \mod 2$

$\mathbb{Z}_2 = \mathbb{Z}/\rho \qquad 2 \mid (x-y)$

ρ equivalence relation $\Leftrightarrow \rho$ reflexive, transitive and symmetric

$2 \leq 2 \qquad 2 \mid 4,\ 4 \mid 8 \Rightarrow 2 \mid 8 \qquad 2 = 2$

ρ preorder $\Leftrightarrow \rho$ refl, transitive

ρ order relation $\Leftrightarrow \rho$ refl, trans + antisymmetric

67) $\rho = (A, B, R)$ relation. Prove that:

a) if ρ is reflexive, symmetric and antisymmetric $\Rightarrow \rho = 1_A$

ρ reflexive $\Leftrightarrow \forall x \in A \quad x \rho x \Rightarrow 1_A \subseteq \rho \quad (1)$

ρ symmetric $\Leftrightarrow \forall x, y \in A, \quad x \rho y \Rightarrow y \rho x \Rightarrow \rho = \rho^{-1} \quad (2)$

ρ antisymmetric $\Leftrightarrow \forall x, y \in A \left((x \rho y) \wedge (y \rho x) \right) \Rightarrow x = y \Rightarrow \rho \cap \rho^{-1} \subseteq 1_A$

$$\left.\begin{array}{l} 1_A \subseteq \rho \\ \rho = \rho^{-1} \; (3) \end{array}\right\} \Rightarrow \rho = \rho \cap \rho \overset{(2)}{=} \rho \cap \rho^{-1} \overset{(3)}{\subseteq} 1_A \left.\begin{array}{l} \\ 1_A \subseteq \rho \end{array}\right\} \Rightarrow \rho = 1_A$$

b) ρ reflexive, transitive $\Rightarrow \rho^2 \stackrel{?}{=} \rho$

ρ reflexive $\Leftrightarrow 1_A \subseteq \rho$

ρ transitive $\Leftrightarrow \forall x, y, z \in A, (x \rho y) \wedge (y \rho z) \Rightarrow (x \rho z) \Rightarrow \rho^2 = \rho \circ \rho \subseteq \rho$

$$\left.\begin{array}{l} 1_A \subseteq \rho \,/ \circ \rho \Rightarrow 1_A \circ \rho \subseteq \rho \circ \rho \Rightarrow \rho \subseteq \rho^2 \\ \rho^2 \subseteq \rho \end{array}\right\} \Rightarrow \rho = \rho^2$$

68) $A = \{1, 2, 3, 4\}$

a) $\rho = \{\underbrace{(1,1), (2,2), (3,3), (4,4)}_{\text{reflexive}}, \underbrace{(1,2), (2,1), (3,2), (2,3)}_{\text{symm.}}, (1,3), (3,1)\}$

Determine its corresponding partition $\pi = A/\rho$

$A/\rho = \{\rho\langle x\rangle \mid x \in A\}$

Solution:

$\rho\langle 1\rangle = \{1, 2, 3\} = \rho\langle 2\rangle = \rho\langle 3\rangle$

$\rho\langle 4\rangle = \{4\}$

$A/\rho = \{\{1,2,3\}, \{4\}\} = \{\rho\langle 1\rangle, \rho\langle 4\rangle\}$

b) $\underset{\text{partition}}{\pi} = \{\{1, 2\}, \{3\}, \{4\}\}$ eq. class — one el. from here is a representative of the class

$\rho\langle 1\rangle = \{1, 2\} \Rightarrow (1,2), (2,1), (1,1), (2,2) \in \rho$

$\rho\langle 2\rangle = \{1, 2\} \Rightarrow (2,1), (1,2), (1,1), (2,2) \in \rho$

$\rho\langle 3\rangle = \{3\} \Rightarrow (3,3) \in \rho$

$\rho\langle 4\rangle = \{4\} \Rightarrow (4,4) \in \rho$

$\rho = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\}$

set $\{-2, 0, 2, 4, \dots\}$

$$\mathbb{Z}_2 = \{\hat{0}, \hat{1}\} = \{\hat{2}, \hat{7}\} = \{\hat{10}, \hat{11}\}$$

$\hat{0} + \hat{1} = \hat{2} + \hat{7} = \hat{9} = \hat{1}$

$\hat{1} = \hat{9}$ (=) $1 \equiv 9 \mod 2$

69) Get all equiv. relations with 1, 2, 3, 4 elements respectively

$A_1 = \{1\}$

$\pi_1 = \{\{1\}\} \Rightarrow \rho_1 = \{(1,1)\}$

$A_2 = \{1,2\}$

$\pi_2 = \{\{1\}, \{2\}\} = \{(1,1), (2,2)\}$

$\pi_3 = \{\{1,2\}\} = \{(1,1), (2,2), (1,2), (2,1)\}$

$A_3 = \{1, 2, 3\}$:

$\pi_4 = \{\{1\}, \{2\}, \{3\}\} = \{(1,1), (2,2), (3,3)\}$

$\pi_5 = \{\{1, 2, 3\}\} = \{(1,1), (2,2), (3,3)$

$\pi_6 = \{\{1\}, \{2,3\}\} =$

$\pi_7 = \{\{1,2\}, \{3\}\} = \{(1,2), (2,1), (1,1), (2,2), (3,3)\}$

$\pi_8 = \{\{1,3\}, \{2\}\} =$

$A_4 = \{1, 2, 3, 4\}$:

$\pi_9 = \{\{1\}, \{2\}, \{3\}, \{4\}\}$

$\pi_{10} = \{\{1, 2, 3, 4\}\}$

$A_4 = \{1,2,3,4\}$

$\pi_9 = \{\{1\}, \{2\}, \{3\}, \{4\}\}$

$\pi_{10} = \{\{1,2,3,4\}\}$

$\pi_{11} = \{\{1,2,3\}, \{4\}\}$

$\pi_{12} = \{\{1,2,4\}, \{3\}\}$

$\pi_{13} = \{\{1,3,4\}, \{2\}\}$

$\pi_{14} = \{\{2,3,4\}, \{1\}\}$

$\pi_{15} = \{\{1,2\}, \{3,4\}\}$

$\pi_{16} = \{\{1,3\}, \{2,4\}\}$

$\pi_{17} = \{\{1,4\}, \{2,3\}\}$

$\pi_{18} = \{\{1,2\}, \{3\}, \{4\}\}$

$\pi_{19} = \{\{1,3\}, \{2\}, \{4\}\}$

$\pi_{20} = \{\{1,4\}, \{2\}, \{3\}\}$

$\pi_{21} = \{\{2,3\}, \{1\}, \{4\}\}$

$\pi_{22} = \{\{2,4\}, \{1\}, \{3\}\}$

$\pi_{23} = \{\{3,4\}, \{1\}, \{2\}\}$

70) Prove that:

a) $(\mathbb{Z}, |)$ is a preordered set, "$|$" is neither symmetric nor antisymmetric

$a|b \Leftrightarrow \exists c \in \mathbb{Z} : b = a \cdot c$

$|$ is reflexive $\Leftrightarrow \forall a \in \mathbb{Z}$ $a|a$ true because $a = 1 \cdot a$

$|$ is transitive $\Leftrightarrow \forall a, b, c \in \mathbb{Z}$ $(a|b) \wedge (b|c) \Rightarrow a|c$

$$\left.\begin{array}{l} a|b : \exists k_1 \in \mathbb{Z} \text{ s.t. } b = a \cdot k_1 \\ b|c : \exists k_2 \in \mathbb{Z} \text{ s.t. } c = b \cdot k_2 \end{array}\right\} \Rightarrow c = b \cdot k_2 = a \cdot \underbrace{k_1 \cdot k_2}_{\in \mathbb{Z}} \Rightarrow a|c$$

$1|3$ but $3 \nmid 1$ $\Rightarrow$ it is not symmetric

$2|-2$ and $-2|2$ but $-2 \neq 2$ $\Rightarrow$ it is not antisymmetric

b) $(\mathbb{N}, |)$ ordered set

reflexivity + transivity similar to a) (1)

$$\left.\begin{array}{l} a|b \Rightarrow \exists k_1 \in \mathbb{N} : b = a \cdot k_1 \\ b|a \Rightarrow \exists k_2 \in \mathbb{N} : a = b \cdot k_2 \end{array}\right\} \Rightarrow b = a \cdot k_1 = b \cdot k_2 \cdot k_1 \Rightarrow$$

$\Rightarrow k_1 \cdot k_2 = 1 \Rightarrow k_1 = k_2 = 1 \Rightarrow a = b$

$\Rightarrow$ "$|$" is antisymmetric (2)

(1), (2) $\Rightarrow (\mathbb{N}, |)$ ordered set

71) on $\mathbb{C}$

$\forall z, w \in \mathbb{C}$ $z \rho w \Leftrightarrow |z| = |w|$

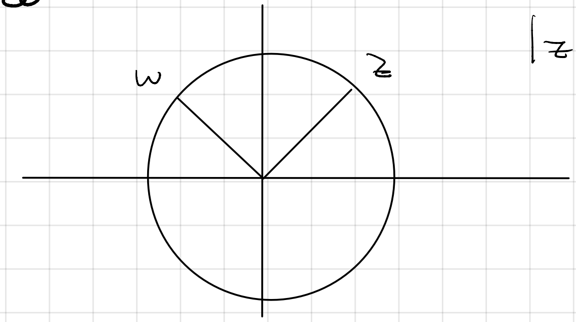
Prove that ρ is an eq relation and det the eq. classes

$\forall z \in \mathbb{C}$ $|z| = |z| \Rightarrow z \rho z \Rightarrow \rho$ is reflexive

$$\forall z, w, t \in \mathbb{C} \text{ if } \left.\begin{array}{l} |z| = |w| \\ |w| = |t| \end{array}\right\} \Rightarrow |z| = |w| = |t| \Rightarrow \rho \text{ transitive}$$

$\forall z, w \in \mathbb{C}$ $|z| = |w| \Rightarrow |w| = |z| \Rightarrow \rho$ is symmetric

$\Rightarrow \rho$ is an eq. rel.

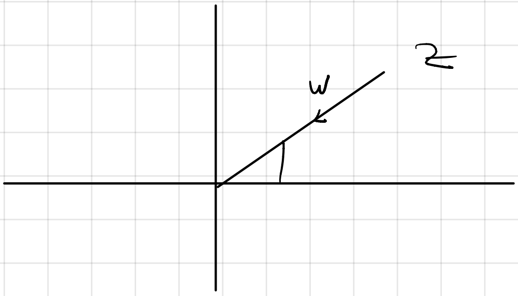


$|z| = |w|$

set of circles

$$C/_{\rho} = \{ C(0, r) \mid r \geq 0 \}$$

circle centered at (0,0) with radius r



$\arg z = \arg w$

72) ρ_1, ρ_2 are eq relation. Prove that

$A \Leftrightarrow B$

$2 = 3$

a) $\rho_1 \cap \rho_2$ is an eq. relation (exam)

reflexivity:

$$\left.\begin{array}{l} \rho_1 \text{ reflexive} \Rightarrow \forall x \in A,\ x \rho_1 x \\ \rho_2 \text{ reflexive} \Rightarrow \forall x \in A,\ x \rho_2 x \end{array}\right\} \Rightarrow \forall x \in A\ (x \rho_1 x) \wedge (x \rho_2 x)$$

$\Rightarrow x(\rho_1 \cap \rho_2) x \Rightarrow \rho_1 \cap \rho_2$ is reflexive (1)

$$\left.\begin{array}{l} \forall x, y, z \in A,\ (x \rho_1 y) \wedge (y \rho_1 z) \Rightarrow x \rho_1 z \\ \qquad (x \rho_2 y) \wedge (y \rho_2 z) \Rightarrow x \rho_2 z \end{array}\right\} \Rightarrow \big((x \rho_1 y) \wedge (y \rho_1 z) \wedge (x \rho_2 y) \wedge (y \rho_2 z) \big) \Rightarrow$$

$$\Rightarrow (x \rho_1 z) \wedge (x \rho_2 z) \Rightarrow \big(x (\rho_1 \cap \rho_2) y \big) \wedge \big(y (\rho_1 \cap \rho_2) z \big) \Rightarrow x (\rho_1 \wedge \rho_2) z$$

$\Rightarrow \rho_1 \cap \rho_2$ is transitive (2)

$\forall x, y \in A$, $x(\rho_1 \cap \rho_2) y \Rightarrow (x \rho_1 y) \wedge (x \rho_2 y) \overset{\rho_1, \rho_2 \text{ symmetric}}{\Rightarrow} (y \rho_1 x) \wedge (y \rho_2 x)$

$\Rightarrow y(\rho_1 \cap \rho_2) x \Rightarrow \rho_1 \cap \rho_2$ is symmetric (3)

(1), (2), (3) $\Rightarrow \rho_1 \cap \rho_2$ is an eq relation