Relations

A, B - sets

$R \subseteq A \times B = \{(a,b) \mid (a \in A) \wedge (b \in B)\}$

$\rho = (A, B, R)$ - relation

domain codomain graph

a in ρ relation with b

$a \rho b \iff (a,b) \in R$

$\rho = (A, B, R)$

$\sigma = (A, B, S)$

$a(\rho \vee \sigma) b = (a \rho b) \wedge (a \sigma b)$

$a(\rho \cup \sigma) b = (a \rho b) \vee (a \sigma b)$

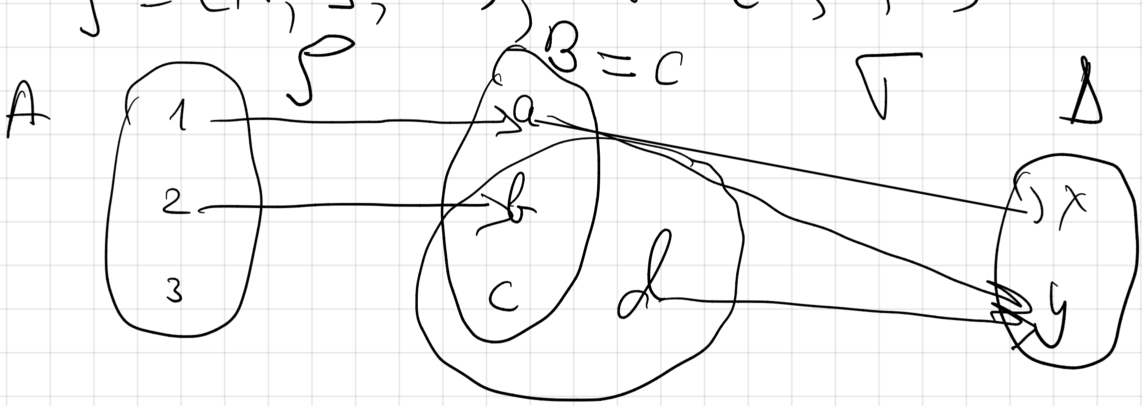
$a(C\rho) b \iff a \not\rho\, b$

$a \rho^{-1} b \iff b \rho a$

$\rho^{-1} = (B, A, R^{-1}), \quad R^{-1} = \{(b,a) \mid (a,b) \in R\}$

Composition of relations:

$\rho = (A, B, R), \quad \sigma = (C, D, S)$



$1 \rho a$

$2 \rho b$

$a \sigma x$

$d \sigma y$

$a \sigma y$

$1 \rho a$

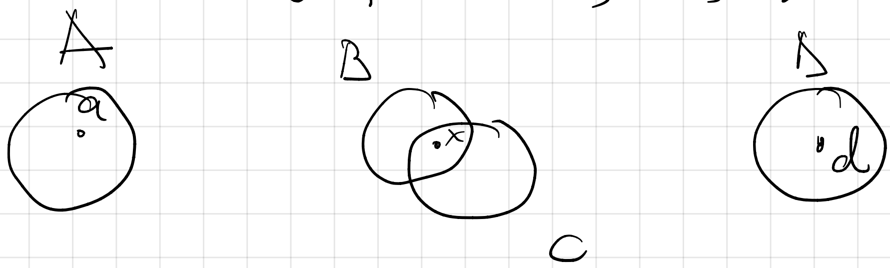
$a \sigma x$

$(1 \rho a) \wedge (a \sigma x) \Rightarrow 1 (\sigma \circ \rho) x$

second

$\sigma \circ \rho = (A, D, S \circ R)$

$S \circ R = \{ (a,d) \mid (a \in A) \wedge (d \in D) \wedge (\exists x \in B \cap C : (d, x) \in R, (x, d) \in S \}$



28) $A = \{1, 2\}$ $\quad C = \{1, 2, 3, 4\}$

$B = \{1, 2, 3\}$

$R_1 = \{(1,2), (1,3), (2,3)\} \subseteq A \times B$

$R_2 = \{(1,4), (3,1), (3,4)\} \subseteq B \times C$

$\rho_1 = (A, B, R_1) \quad \rho_2 = (B, C, R_2)$

<u>Calculate</u>: $\rho_2 \circ \rho_1$, $\rho_1 \circ \rho_2$, ρ_1^{-1}, ρ_2^{-1}, $(\rho_1 \circ \rho_2)^{-1}$, $\rho_2^{-1} \circ \rho_1^{-1}$

$\rho_2 \circ \rho_1 = (A, C, R_2 \circ R_1)$

$R_2 \circ R_1 = \{(1,1), (1,4), (2,1), (2,4)\}$

$\rho_1 \circ \rho_2 = (B, B, R_1 \circ R_2)$

$R_1 \circ R_2 = \{(3,2), (3,3)\}$

$$\rho_1^{-1} = (B, A, R_1^{-1}) \qquad R_1^{-1} = \{(2,1), (3,1), (3,2)\}$$

$$\rho_2^{-1} = (C, B, R_2^{-1}) \qquad R_2^{-1} = \{(4,1), (1,3), (4,3)\}$$

$$(\rho_1 \circ \rho_2)^{-1} = (B, B, (R_1 \circ R_2)^{-1})$$

$$(R_1 \circ R_2)^{-1} = \{(2,3), (3,3)\}$$

$$\rho_2^{-1} \circ \rho_1^{-1} = (B, B, R_2^{-1} \circ R_1^{-1})$$

$$R_2^{-1} \circ R_1^{-1} = \{(2,3), (3,3)\}$$

$$(a \cdot b)^{-1} = b^{-1} \cdot a^{-1}$$

$$\underbrace{(a \cdot b) \cdot b^{-1}}_{1} \cdot a^{-1} = 1$$

30) $A = \{1, 2, 3, 4\}$

$R, S, S' \subseteq A \times A$

$R = \{(1,2), (1,4), (2,3), (4,1), (4,3)\}$

$S = \{(1,1), (2,4), (3,4)\}$

$S' = \{(1,4), (4,4)\}$

$(S \cap S') \circ R$; $(S \circ R) \cap (S' \circ R)$; $R \circ (S \cap S')$;

$(R \circ S) \cap (R \circ S')$

$(S \cap S') \circ R = \emptyset \circ R = \emptyset$

$(S \circ R) \cap (S' \circ R) = \{(1,4), (2,4), (4,1), (4,4)\} \cap$
$\{(1,4), (4,4)\}$

$= \{(1,4), (4,4)\}$

$R \circ (S \cap S') = R \circ \emptyset = \emptyset$

$(R \circ S) \cap (R \circ S') = \{(1,2), (2,1), (2,3), (1,3), (3,1), (3,3)\} \cap$
$\{(1,1), (4,1), (4,3)\} = \emptyset$

31 $\rho = (A, B, R)$
$\rho' = (A, B, R')$

$\sigma = (C, D, S)$

$\sigma' = (C, D, S')$

b) $(\sigma \circ \rho)^{-1} = \rho^{-1} \circ \sigma^{-1}$

$(\sigma \circ \rho)^{-1} = (D, A, (S \circ R)^{-1})$

$\rho^{-1} \circ \sigma^{-1} = (D, A, (R^{-1} \circ S^{-1}))$

Let $(d, a) \in D \times A$

$d (\sigma \circ \rho)^{-1} a \iff a (\sigma \circ \rho) d \overset{\text{def}}{\iff} \exists x \in B \cap C \text{ s.t. } (a \rho x) \wedge (x \sigma d) \overset{\text{def of}}{\underset{?}{\iff}}$

$\iff \exists x \in B \cap C \text{ s.t. } (x \rho^{-1} a) \wedge (d \sigma^{-1} x) \overset{\text{comm}}{\iff} \exists x \in B \cap C \text{ s.t. } (d \sigma^{-1} x)$
$\wedge (x \rho^{-1} a)$

$\Leftrightarrow d(\rho^{-1} \circ \Gamma^{-1}) a \Rightarrow (S \circ R)^{-1} = (R^{-1} \circ S^{-1})$

c) $(\rho \cap \rho')^{-1} = \rho^{-1} \cap (\rho')^{-1}$

Let $(b,a) \in B \times A$

$b(\rho \cap \rho^{-1}) a \Leftrightarrow a(\rho \cap \rho^{-1}) b \Leftrightarrow (a \rho b) \wedge (a \rho^{-1} b)$

$\overset{\text{def } \cap}{\Leftrightarrow} (b \rho^{-1} a) \wedge (b (\rho')^{-1} a) \Leftrightarrow b(\rho^{-1} \cap (\rho')^{-1}) a$

e) $\Gamma \circ (\rho \cap \rho') \subseteq (\Gamma \circ \rho) \cap (\Gamma \circ \rho')$

Let $(a,d) \in A \times D$

$a(\Gamma \circ (\rho \cap \rho')) d \Leftrightarrow \exists x \in B \cap C$ s.t. $(a(\rho \cap \rho') x \wedge (x \Gamma d))$

$\Leftrightarrow \exists x \in B \cap C$ s.t. $(a \rho x) \wedge (a \rho' x) \wedge (x \Gamma d) \underbrace{\wedge (x \Gamma d)}_{\Leftrightarrow}$

$\Leftrightarrow \exists x \in B \cap C$ s.t. $\underbrace{(a \rho x) \wedge (x \Gamma d)}_{A} \wedge \underbrace{(a \rho' x) \wedge (x \Gamma d)}_{B}$

$\Rightarrow \exists x_1 \in B \cap C$ s.t. $(a \rho x_1) \wedge (x_1 \Gamma d) \wedge \exists x_2 \in B \cap C$ s.t. $(a \rho' x_2) \wedge (x_2 \Gamma d)$

$\exists x \, A \wedge B$
$\Rightarrow \exists x_1 A \wedge \exists x_2 B$

$\Leftrightarrow a(\Gamma \circ \rho) d \wedge a(\Gamma \circ \rho') d$

$\Leftrightarrow a((\Gamma \circ \rho) \cap (\Gamma \circ \rho')) d$