SEMINAR 9

Lattices

$(A, \le)$ ordered set, $x \in A$

x is the minimum of $A \iff \forall a \in A \quad x \le a \quad (x = \min A)$

x is the maximum of $A \iff \forall a \in A \quad a \le x \quad (x = \max A)$

$B \subseteq A$

x is a lower bound of B $(x \in LB_A B) \iff \forall b \in B \quad x \le b$

x is an upper bound of B $(x \in UB_A B) \iff \forall b \in B \quad b \le x$

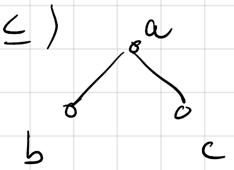
x is the infimum of B $(x = \inf_A B) \iff$ x is the largest lower bound

x is the supremum of B $(x = \sup_A B) \iff$ x is the smallest upper bound

x is a minimal element of $A \iff \forall a \in A \quad a \le x \Rightarrow a = x$

x is a maximal element of $A \iff \forall a \in A \quad x \le a \Rightarrow a = x$

$(A, \le)$



$\max A = a$

$\min A \not\exists$

b, c are minimal

a is maximal

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93) Let $(A, \le)$ be an ordered set

Prove that if $a = \max A$, then A is the unique maximal el of A

$\to a = \max A \iff \forall x \in A \quad x \le a$

a - maximal $\iff \forall x \in A, \quad x \ge a \Rightarrow a = x$

Let $a' =$ maximal of $A \iff \forall x \in A, \; x \ge a' \Rightarrow x = a'$

Choose $x = a \Rightarrow a \geq a' \Rightarrow a' = a$

$\Rightarrow$ a unique maximal element

$(A, \leq)$ is a lattice $\Leftrightarrow \forall x, y \in A \quad \exists \inf \{x, y\}, \exists \sup \{x, y\}$

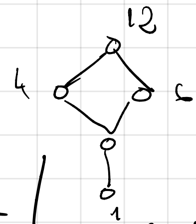
$(\mathbb{N}, |)$

$\sup \{4, 6\} = 12$ – it divides all the other divis

$\inf \{4, 6\} = 2$

$ub(\{4,6\}) = \{12, 24, 36, \dots\}$

$lb(\{4,6\}) = \{1, 2\}$



In general $(\mathbb{N}, |)$ $\sup \{x, y\} = lcm[x, y]$
↳ least common mult

$\inf \{x, y\} = gcd(x, y)$

$\Rightarrow (\mathbb{N}, |)$ is a lattice

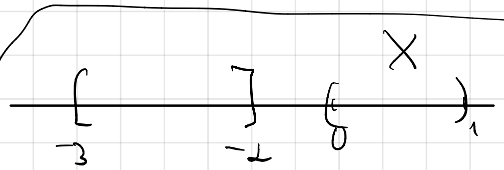
94) Let $(A, \leq)$ be an ordered set

$X \subseteq B \subseteq A$

a) Prove that if $\exists \inf_B X$ and $\exists \inf_A X \Rightarrow \inf_B X \leq \inf_A X$

$\inf_B X \in lb_B X = \{b \in B \mid \forall x \in X \ \ b \leq x\}$

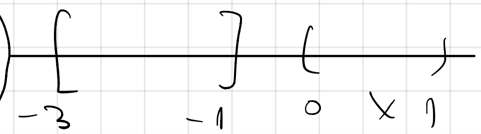
$B \subseteq A$



$\Rightarrow lb_B X \leq lb_A X$

$\inf_B X \leq lb_B X$

$\Rightarrow \inf_B X \in lb_A X \Rightarrow$



$A = [-3, -1] \cup [0, 1)$

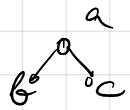
$B = [-3, -2] \cup [0, 1)$

$\Rightarrow \inf_B X \leq \inf_A X$

lower bound largest lower bound

$\inf_A X = -1 \quad \inf_B X = -2$

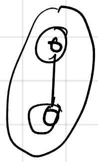
95) Determine all lattices with 1, 2, 3, 4, 5 elements respectively using Hasse diagrams.



not a lattice

$\sup\{b, c\} = a$

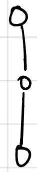
$\nexists \inf\{b, c\}$

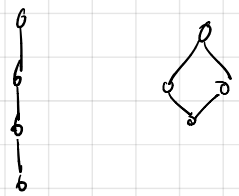


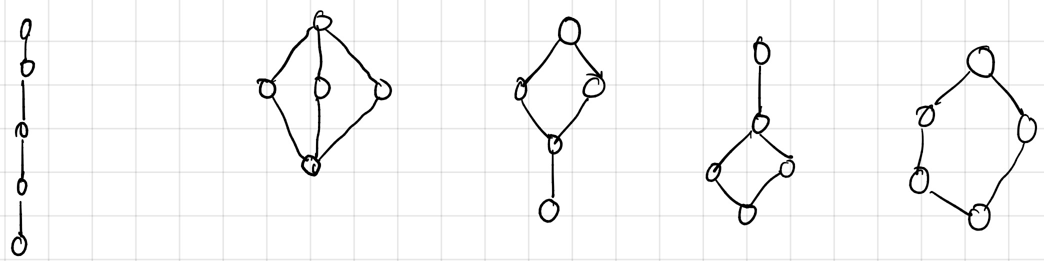
$\inf\{x, y\}$

1 element: 

2 elements: 

3 elements: 

4 elem: 

5 elem: 

96) Let $(B, \leq)$ be an ordered set, and A a set

On $Hom(A, B)$ we define the relation $f \leq g \Leftrightarrow \forall a \in A \ f(a) \leq g(a)$

$Hom(A, B) = \{f \text{ function} \mid f : A \to B\}$

a) Prove that "$\leq$" is an order relation in $Hom(A, B)$

Reflexivity: $\forall f \in Hom(A, B) \ \forall a \in A \ f(a) \in B$, B ordered $\Big\} \overset{\leq \text{ refl in } B}{\Longrightarrow}$

$\Rightarrow f(a) \leq f(a) \Rightarrow f \leq f$ in $Hom(A, B) \Rightarrow$ "$\leq$" refl in $Hom(A, B)$

<u>Transitivity</u>: Let $f, g, h \in Hom(A, B)$: $(f \le g) \wedge (g \le h) \Rightarrow$

$\Rightarrow \forall a \in A$ $\left.\begin{array}{l} f(a) \le g(a) \\ g(a) \le h(a) \\ B \text{ ordered} \end{array}\right\} \overset{\le \text{ tran in } B}{\Rightarrow} f(a) \le h(a) \Rightarrow f \le h$ in $Hom(A, B)$

$\Rightarrow$ $\le$ trans in $Hom(A, B)$

(2)

<u>Antisymmetry</u>: Let $f, g \in Hom(A, B)$: $(f \le g) \wedge (g \le f) \Rightarrow$

$\Rightarrow \forall a \in A$ $\left.\begin{array}{l} f(a) \le g(a) \\ g(a) \le f(a) \\ B \text{ ordered} \end{array}\right\} \overset{\le \text{ antis. in } B}{\Rightarrow} f(a) = g(a) \Rightarrow$

$\Rightarrow f(a) = g(a) \Rightarrow f = g$ in $Hom(A, B) \Rightarrow$

$\Rightarrow$ "$\le$" antisymetric in $Hom(A, B)$ (3)

b) If B is a lattice $\Rightarrow Hom(A, B)$ is a lattice

Let $f, g \in Hom(A, B)$ and $h \in Hom(A, B)$ s.t. $\forall a \in A$

$h(a) = \inf\{f(a), g(a)\}$ (this exists because $f(a), g(a) \in B$, B is a lattice)

We claim that h is the infimum of f and g

Let's assume that it's not $\Rightarrow \exists h' \in Hom(A, B)$

$h' \le f$
$h' \le g$ $\quad h < h' \Rightarrow \forall a \in A$ $h(a) = \inf\{f(a), g(a)\} < h(a)$

$\left.\begin{array}{l} h'(a) \le f(a) \\ h'(a) \le g(a) \end{array}\right\} \Rightarrow h'(a) \in lb\{f(a), g(a)\}$
$h(a)$ largest lower bound

$\Rightarrow h = \inf\{f, g\} \Rightarrow \text{Hom}(A, B)$ is a lattice

sup similarly

97) Let $(A, \le)$ be a complete lattice and $f: A \to A$ an increasing function.
Prove that $\exists\, a \in A: \ f(a) = a$ (a is a fixed point of f)

$(A, \le)$ complete lattice $\Leftrightarrow \forall\, B \subseteq A \ \exists \sup_A B, \ \exists \inf_A B$

Let $B = \{a \in A \mid a \le f(a)\}$. $B \ne \emptyset$ because $a = \inf_A A \in B$
$\Rightarrow a \le x, \ \forall x \in A$
Choose $x = f(a)$

$$\left.\begin{array}{l} B \ne \emptyset \\ A \text{ complete} \end{array}\right\} \Rightarrow \sup_A B =: a_0 \in A$$

$$\Rightarrow \forall\, b \in B \quad \left.\begin{array}{l} b \le a_0 \overset{f \text{ incr.}}{\Rightarrow} f(b) \le f(a_0) \\ b \le f(b) \end{array}\right\} \Rightarrow b \le f(a_0)$$

$$\left.\begin{array}{l} \Rightarrow f(a_0) \in ub\,B \\ a_0 = \sup_A B \end{array}\right\} \Rightarrow \underbrace{a_0}_{\text{smallest upper bound}} \le \underbrace{f(a_0)}_{\text{upper bound}} \ (1) \Rightarrow \left.\begin{array}{l} a_0 \in B \\ a_0 = \sup_A B \end{array}\right\} \Rightarrow$$

$\Rightarrow a_0 = \max B$

$$a_0 \le f(a_0) \overset{f \text{ incr.}}{\Rightarrow} \underbrace{f(a_0)}_{b_0} \le \underbrace{f(f(a_0))}_{\le f(b_0)} \left.\right\} \Rightarrow b_0 = f(a_0) \in B \left.\begin{array}{l} \\ a_0 = \max_A B \end{array}\right\} \Rightarrow$$

$\Rightarrow f(a_0) \le a_0$ (2) $\qquad (1), (2) \Rightarrow f(a_0) = a_0$

9D) $(A, \leq)$ well-ordered, $f: A \to A$ increasing

a) Prove that $\forall x \in A \quad a \leq f(a)$

$(A, \leq)$ well ordered $\Leftrightarrow \forall B \subseteq A \ \exists \min B$

Assume that $\exists a \in A: \ a > f(a) \Rightarrow f(a) > f(f(a)) \Rightarrow$

$\Rightarrow a > f(a) > f(f(a)) > f(f(f(a))) \ldots$

Let $B = \{a, f(a), f(f(a)) \ldots\} \subseteq A$ then $\nexists \min B$ ↯

$\Rightarrow a \leq f(a) \ \forall a \in A$