Chapter 3: SETS

"Naively", a set is a collection of uniquely determined abstract objects.

The concepts of:

- set
- element
- belongs $\in$ binary predicate

are primitive
we don't define them

<u>A set</u> can be given by:

- enumerating the elements: $A = \{a, b, c\}$
- a property (predicate)
  $A = \{ x \mid P(x) \}$

e.g. $\{x \mid x \neq x\} = \emptyset$ the <u>empty set</u>

it is unique!

- two sets are equal if they have the same elements
  $A = B \iff (\forall x (x \in A \leftrightarrow x \in B))$
- A is a <u>subset</u> of B if the elements of A are also elements of B
  $A \subseteq B \iff (\forall x (x \in A \to x \in B))$
- the <u>powerset</u> of A is the set of subsets of A
  $\mathcal{P}(A) \overset{def}{=} \{ X \mid X \subseteq A \}$
  i.e. $X \in \mathcal{P}(A) \iff X \subseteq A$

Example $\mathcal{P}(\emptyset) = \{\emptyset\}$

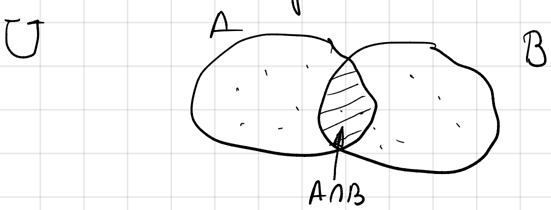
$$\mathcal{P}(\mathcal{P}(\emptyset)) = \{\emptyset, \{\emptyset\}\}$$

$$\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset))) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$$

In what follows, we consider that our sets are subsets of set U

Operations with sets (i.e. ops in $\mathcal{P}(U)$)

Euler–Venn diagrams



1) Union of sets : $A \cup B \overset{def}{=} \{x \mid x \in A \underset{\vee}{\text{ or }} x \in B\}$

2) Intersection : $A \cap B \overset{def}{=} \{x \mid x \in A \underset{\wedge}{\text{ and }} x \in B\}$

3) Difference : $A \setminus B \overset{def}{=} \{x \mid x \in A \text{ and } \underbrace{x \notin B}_{\neg(x \in B)}\}$

In particular, the complement of A is:

$$C_A = U \setminus A = \{x \mid \underbrace{x \notin A}_{\neg(x\in A)}\}$$

We get $\underline{A \setminus B = A \cap C_B}$

4. Symmetric difference

$$A \Delta B \overset{def}{=} (A \setminus B) \cup (B \setminus A)$$
$$= (A \cap C_B) \cup (B \cap C_A)$$
$$= (A \cup B) \setminus (A \cap B)$$
$$= \{x \mid \text{either } x \in A \text{ or } x \in B$$
$$= \{x \mid x \in A \underset{\hookrightarrow \text{ exclusive or}}{\text{ xor }} x \in B\}$$

Ordered pair of elements

We want to define the pair (a,b) such that we have the full property

$$(a,b) = (c,d) \iff a=c \text{ and } b=d$$

Note that if we take $\{a,b\}$, then $\{a,b\} = \{c,d\}$ $\not\Rightarrow$ $a=c$, $b=d$

Def (Kuratowski)

$$(a,b) = \{\{a\}, \{a,b\}\}$$

By recursion, we define ordered n-uples:

$$(a,b,c) \overset{\text{def}}{=} ((a,b),c)$$

$$(a_1,\dots,a_n) \overset{\text{def}}{=} ((a_1,\dots,a_{n-1}), a_n)$$

Cartesian product (Descarte Cartesius) (17^{th} cent.)

$$A \times B \overset{\text{def}}{=} \{(a,b) \mid a \in A \text{ and } b \in B\}$$

$$A_1 \times \dots \times A_n \overset{\text{def}}{=} \{(a_1,\dots,a_n) \mid \forall i=1,\dots n \quad a_i \in A_i\}$$

Russell's paradox

Bertrand Russell

The naive theory of Cantor and Frege permits the formation of "the set of all sets" $=: \mathcal{S}$

Question: $\mathcal{S} \in \mathcal{S}$?

Consider the set:

$$\mathcal{R} := \{X \mid X \notin X\}$$

$$\mathcal{R} \in \mathcal{R} ?$$

- If $\mathcal{R} \in \mathcal{R}$, then $\mathcal{R}$ does not satisfy the condition from the def of $\mathcal{R}$, hence $\mathcal{R} \notin \mathcal{R}$, contradiction

. If $\mathcal{R} \notin \mathcal{R}$, then $\mathcal{R}$ satisfies the condition from the def of $\mathcal{R}$, hence $\mathcal{R} \in \mathcal{R}$, contradiction

This leads to the need to develop axiomatic set theory

HW: 19 → 26

Chapter 4: RELATIONS
(correspondences)

Def: Let A and B be two sets. A relation between the element of A and B is a triple $\rho = (A, B, R)$, where $R \subseteq A \times B$

$$\rho = (\underset{\text{dom } \rho}{A}, \underset{\text{codom } \rho}{B}, \underset{\text{graph of } \rho}{R})$$

We usually $\boxed{a \rho b \iff (a, b) \in R}$

e.g. $a = b$

$a < b$

$d_1 | d_2$, $d_1 \parallel d_2$

R: Equality of relations:

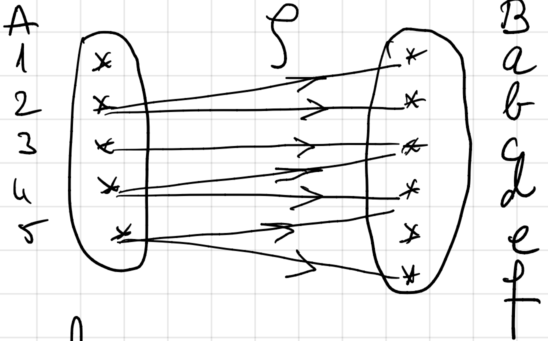
if $\rho = (A, B, R)$ and $\sigma = (C, D, S)$ are relations

then $\rho = \sigma \iff \begin{cases} A = C & \text{: have the same domain} \\ B = D & \text{- — " — codomain} \\ R = S & \text{— " — graph} \end{cases}$

i.e for all $(a, b) \in A \times B$ we have

$$\boxed{a \rho b \iff a \sigma b}$$

- We may represent relations using oriented graphs.



$R = \{(2,a), (2,b), (3,e), (4,c), (4,d), (5,d), (5,f)\}$

Examples: 1) $(A, B, A\times B)$ the <u>universal</u> relation

$(A, B, \emptyset)$ the empty relation

2) $1_A = (A, A, \Delta_A)$ $\qquad$ Δ Delta, δ delta

where $\Delta_A = \{(a,a) \mid a \in A\}$ is the <u>diagonal subset of $A\times A$</u>

We have $a\, 1_A\, b \Leftrightarrow (a,b) \in \Delta_A \Leftrightarrow a = b \in \Delta_A \Leftrightarrow a = b$

So 1_A is just the <u>equality relation</u> on A

ρ - ro

Operations with relations

1) <u>Union</u>: Let $\rho = (A, B, R)$, $\sigma = (A, B, S)$

$\rho \cup \sigma \overset{def}{=} (A, B, R\cup S)$

hence $\forall (a,b) \in A\times B$, we have

$a\, \rho\cup\sigma\, b \Leftrightarrow a\,\rho\, b$ or $a\,\sigma\, b$

2) <u>Intersection</u>: $\rho\cap\sigma \overset{def}{=} (A, B, R\cap S)$

hence, $\forall (a,b) \in A\times B$ we have

$a\, \rho\cap\sigma\, b \overset{def}{\Leftrightarrow} a\,\rho\, b$ and $a\,\sigma\, b$

3) The inverse of a relation.

Let $\rho = (A, B, R)$ a set. Then $\rho^{-1} = (B, A, R^{-1})$

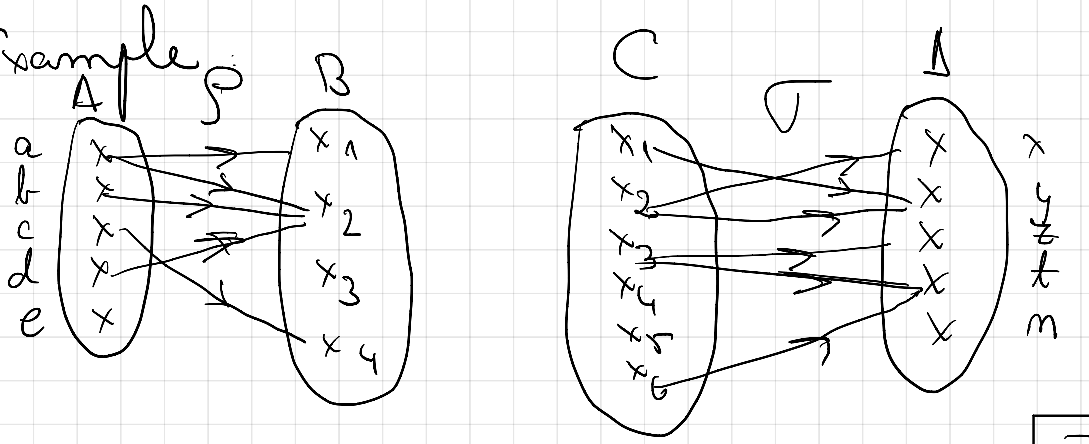
where $R^{-1} \overset{def}{=} \{(b,a) \mid (a,b) \in R\}$

$a \rho b \overset{def}{\Longleftrightarrow} b \rho^{-1} a$

4) Composition of relations

$a \to 1 \quad 1 \to y$

Example:



$$\underbrace{\sigma}_{2^{nd}} \circ \underbrace{\rho}_{1^{st}} = (\underset{\text{dom } \rho}{A}, \underset{\text{codom } \rho}{D}, S \circ R)$$

$\boxed{\tau \text{ tau}}$

In one example, $S \circ R = \{(a,y), (a,x), (b,x), (b,y), (d,e), (d,y)\}$

In general:

$$S \circ R \overset{def}{=} \{(a,d) \in A \times D \mid \exists x \ x \in B \cap C \text{ and } (a,x) \in R \text{ and } (x,d) \in S\}$$

if $(a,d) \in A \times D$, we have

$$a\, \sigma \circ \rho\, d \overset{def}{\Longleftrightarrow} \exists x (a \in \underset{\text{codom } \rho}{B} \cap \underset{\text{dom } \sigma}{C} \text{ and } a \rho x \text{ and } x \sigma d)$$

Theorem (properties of the composition)

1) The equality relation is a neutral element w.r.t. composition

i.e. if $\rho = (A, B, R)$, then $1_B \circ \rho = \rho \circ 1_A = \rho$

2) Let $\rho = (A, B, R)$, $\sigma = (C, D, S)$, $\tau = (E, F, T)$ be relations. Then:

$\tau \circ (\sigma \circ \rho) = (\tau \circ \sigma) \circ \rho$

("$\circ$" is associative)

<u>Proof</u>: 1) see the notes

2) We have $\sigma \circ \rho = (A, \overset{\text{dom of } \rho}{} D, \overset{\text{cod of } \sigma}{} S \circ R)$

$\tau \circ (\sigma \circ \rho) = (A, F, T \circ (S \circ R))$

$\tau \circ \sigma = (C, F, T \circ S)$

$(\tau \circ \sigma) \circ \rho = (A, F, (T \circ S) \circ R)$

So, the two relation have the same dom and codom.

To show the equality of graph, let a pair $(a, f) \in A \times F$. We have

$a\, \tau \circ (\sigma \circ \rho)\, f \iff \exists y \,(y \in D \cap E \text{ and } a\, \sigma \circ \rho\, y \text{ and } y\, \tau\, f) \iff$

$\iff \exists y \,(y \in D \cap E \text{ and } \exists x \,(x \in B \cap C \text{ and } a \rho x \text{ and } x \sigma y) \text{ and } y \tau f)$

$\iff \exists y \exists x \,(y \in D \cap E \text{ and } x \in B \cap C \text{ and } a \rho x \text{ and } x \sigma y \text{ and } y \tau f)$

$\iff \exists x \exists y \,(x \in B \cap C \text{ and } a \rho x \text{ and } \exists y \; y \in D \cap E \text{ and } x \sigma y \text{ and } y \tau f)$

?

$\iff \exists x \,(x \in B \cap C \text{ and } a \rho x \text{ and } \exists y \,(y \in D \cap E \;\; a \rho x \sigma y \text{ and } y \tau f))$

$\iff \exists x \,(x \in B \cap C \text{ and } a \rho x \text{ and } x \;\; \tau \circ \sigma\, f) \iff$

$\iff a\, (\tau \circ \sigma) \circ \rho\, f$

We used the following tautologies.

- $A \wedge B \iff B \wedge A$
- $(A \wedge B) \wedge C \iff A \wedge (B \wedge C)$
- $\exists x \exists y\, A(x, y) \iff \exists y \exists x\, A(x, y)$
- $\exists x (C \wedge A(x)) \iff C \wedge \exists x\, A(x)$

rew 27-30