Let $f: A \to B$ a function

f is injective $\iff \forall x_1, x_2 \in A \quad f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

f is surjective $\iff \forall y \in B \quad \exists x \in A: \ f(x) = y \quad (\operatorname{Im} f = f(A) = B)$

f is byective $\iff f$ injective and surjective

50) $f: A \to B$, $X_i \subseteq A$ (exam)

Prove that $f(\bigcap_{i \in I} X_i) = \bigcap_{i \in I} f(X_i)$ if and onlly if f is injective.

"$\subseteq$" Seminar 5 $\qquad x_1 \ f(x_1) = y$, $\quad x_2 \ f(x_2) = y$

"$\supseteq$" Let $y \in \bigcap_{i \in I} f(X_i) \iff \forall i \in I \ y \in f(X_i) \iff \forall i \in I \ \exists x \in X_i$ $f(x) = y \xRightarrow{f \text{ inj}} \exists x \ \forall i \in I \ (x \in X_i) \wedge (f(x) = y) \iff \exists x \in \bigcap_{i \in I} X_i \ f(x) = y$

$\iff y \in f(\bigcap_{i \in I} X_i)$

- Characterisation of injective functions $f: A \to B$. The following are equivalent:

  i) f is injective

  ii) $\forall A'$ set and $\forall \alpha, \beta: A' \to A$ if $f \circ \alpha = f \circ \beta \Rightarrow \alpha = \beta$

  iii) $A \neq \emptyset$ then $\exists r: B \to A: \ r \circ f = 1_A$

  (exam $i \iff ii$, $i \iff iii$ - for ex)

$$\begin{tikzcd} A' \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & A \arrow[r, "f"] & B \arrow[l, bend left, dashed, "r"] \end{tikzcd} \qquad (g \circ f)(x) = g(f(x))$$

- Charact of surj functions

  i) f is surjective

  ii) $\forall B'$ set $\forall \alpha, \beta: B \to B'$ if $\alpha \circ f = \beta \circ f \Rightarrow \alpha = \beta$

  iii) $\exists s: B \to A$, $f \circ s = 1_B$

49) $f: A \to B$, $g: B \to C$ functions

Prove that:

a) if f and g are injective $\Rightarrow$ $g \circ f$ is injective

I: definition

We know
$$\begin{cases} f \text{ injective} \Leftrightarrow \forall x_1, x_2 \in A,\ f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \\ g \text{ injective} \Leftrightarrow \forall y_1, y_2 \in B,\ g(y_1) = g(y_2) \Rightarrow y_1 = y_2 \end{cases}$$

To prove
$$g \circ f \text{ injective} \Leftrightarrow \forall x_1, x_2 \in A\ (g \circ f)(x_1) = (g \circ f)(x_2) \Rightarrow x_1 = x_2$$

Let $x_1, x_2 \in A \Rightarrow (g \circ f)(x_1) = (g \circ f)(x_2) \Rightarrow g(f(x_1)) = g(f(x_2))$
$\overset{g \text{ inj}}{\Longrightarrow} f(x_1) = f(x_2) \overset{f \text{ inj}}{\Longrightarrow} x_1 = x_2$

II: simply feasibility

$$\begin{tikzcd} A' \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \\ & & B' \arrow[u, shift left, "\alpha'"] \arrow[u, shift right, "\beta'"'] \arrow[ur, shift left, "g \circ \alpha'"] \arrow[ur, shift right, "g \circ \beta'"'] & \end{tikzcd}$$

We know
$$f \text{ inj} \Leftrightarrow \forall A' \ \forall \alpha, \beta : A' \to A$$
$$f \circ \alpha = f \circ \beta \Rightarrow \alpha = \beta$$

$$g \text{ inj} \Leftrightarrow \forall B' \ \forall \alpha', \beta' : B' \to B$$
$$g \circ \alpha' = g \circ \beta' \Rightarrow \alpha' = \beta'$$

$$g \circ f \text{ inj} \Leftrightarrow \forall A'' \ \forall \alpha'', \beta'' \to A,\ (g \circ f) \circ \alpha'' = (g \circ f) \circ \beta'' \Rightarrow \alpha'' = \beta'' \leftarrow$$

Let $\alpha'', \beta'' : A'' \to A$ s.t. $(g \circ f) \circ \alpha'' = (g \circ f) \circ \beta'' \Rightarrow g \circ (f \circ \alpha'') = g \circ (f \circ \beta'')$
$\overset{g \text{ inj}}{\Longrightarrow} f \circ \alpha'' = f \circ \beta'' \overset{f \text{ inj}}{\Longrightarrow} \alpha'' = \beta'' \Rightarrow g \circ f$ - inj

III: f inj $\Leftrightarrow \exists r : B \to A$ st $r \circ f = 1_A$
g inj $\Leftrightarrow \exists r' : C \to B$ s.t $r' \circ g = 1_B$

$g \circ f$ inj $\Leftrightarrow \exists r'' : C \to A$ s.t. $r'' \circ (g \circ f) = 1_A$

let $r'': C \to A$ and $r'' = (r \circ r')$

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[l, bend left, dashed, "r"] & C \arrow[l, bend left, dashed, "r'"] \end{tikzcd}$$

id wit $f \to f$, $1 \cdot a = a$

$$(r \circ r') \circ (g \circ f) = r \circ (r' \circ g) \circ f = r \circ 1_B \circ f =$$

$$\overset{\text{iden}}{=} r \circ f = 1_A \Rightarrow r'' \text{ is a left inverse of } (g \circ f) \Rightarrow$$

$(g \circ f)$ inj

a) f, g surj $\Rightarrow g \circ f$ surj

I definition

f surj $\Leftrightarrow \forall b \in B, \exists a \in A$ st $f(a) = b$

g surj $\Leftrightarrow \forall c \in C, \exists b' \in B$ st $g(b') = c$

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "g"] & C \\ a' \arrow[rr, bend right, "b'", "g \circ f"'] & & c' \end{tikzcd}$$

$g(b') = c'$

$g \circ f$ surj $\Leftrightarrow \forall c' \in C\ \exists a' \in A$ st $(g \circ f)(a') = c'$

Let $c' \in C \overset{g \text{ surj}}{\Rightarrow} \exists b' \in B$ st $g(b') = c' \overset{f \text{ surj}}{\Rightarrow} \exists a' \in A$ st $f(a') = b'$

$\Rightarrow (g \circ f)(a') = g(f(a')) \overset{g \text{ inj}}{=} g(b') = c' \Rightarrow (g \circ f)$ is surj

II $\forall \alpha, \beta : B \to B'$, $\alpha \circ f = \beta \circ f \Rightarrow \alpha = \beta \Leftrightarrow f$ surj

g surj $\Leftrightarrow \forall \alpha', \beta' : C \to C'$, $\alpha' \circ g = \beta' \circ g \Rightarrow \alpha' = \beta' \Leftrightarrow g$ surj

$g \circ f$ surj $\Leftrightarrow \forall \alpha'', \beta'' : C \to C''$, $\alpha'' \circ (g \circ f) = \beta'' \circ (g \circ f)$

$\Rightarrow \alpha'' = \beta'' \Leftrightarrow g \circ f$ surj

Let $\alpha'', \beta'' : C \to C''$ s.t. $\alpha'' \circ (g \circ f) = \beta'' \circ (g \circ f) \Leftrightarrow (\alpha'' \circ g) \circ f = (\beta'' \circ g) \circ f$

$\overset{f \text{ surj}}{\Rightarrow} \alpha'' \circ g = \beta'' \circ g \overset{g \text{ surj}}{\Rightarrow} \alpha'' = \beta'' \Rightarrow g \circ f$ surj

III) if f surjective $\Leftrightarrow \exists\, s: B \to A$ st $f \circ s = 1_B$ $\quad A \xrightarrow{f} B \xrightarrow{g} C$, $\quad A \xleftarrow{s} B \xleftarrow{s'} C$

if g surjective $\Leftrightarrow \exists\, s': C \to B$ s.t. $g \circ s' = 1_C$

$g \circ f$ surj $\Leftrightarrow s'': C \to A$ s.t $(g \circ f) \circ s'' = 1_C$ ← domain of C

Let $s'': C \to A$ and $s'' = (s \circ s') \Rightarrow$

$\Rightarrow (g \circ f) \circ (s \circ s') \Rightarrow g \circ (f \circ s) \circ s' \overset{f\ surj}{\Rightarrow} g \circ 1_B \circ s' = g \circ s' = 1_C$

$\Rightarrow s''$ is a right inverse of $g \circ f \Rightarrow g \circ f$ surjective

! At exam find all the sections (for inj) and retractions (for surj) !

b) $g \circ f$ injective $\Rightarrow f$ injective $\quad A \xrightarrow{f} B \xrightarrow{g} C$

Let $x_1, x_2 \in A$ s.t $| f(x_1) = f(x_2)$

$\Rightarrow g(f(x_1)) = g(f(x_2))^g \Rightarrow (g \circ f)(x_1) = (g \circ f)(x_2) \overset{g \circ f\ inj}{\Rightarrow} x_1 = x_2$

c) $\left.\begin{array}{l} g \circ f \text{ injective} \\ f \text{ surj} \end{array}\right\} \Rightarrow g$ injective

$$A \underset{s}{\overset{f}{\rightleftarrows}} B \xrightarrow{g} C, \quad r: C \to A$$

$s: B \to A: \ f \circ s = 1_B$

$r: C \to A: \ r \circ (g \circ f) = 1_A \ | \circ s \overset{f \circ}{\Rightarrow} | r \circ g = s \Rightarrow \underbrace{f \circ r}_{r'} \circ g = f \circ s = 1_B$

$\Rightarrow r \circ g \circ \underbrace{f \circ s}_{1_B} = s \Rightarrow$

$r' = f \circ r$