

Assignment 3

Deadline: 23 November 2020, 7pm

Total marks: 20

1 Explicit Simpson's method [10 marks]

Consider the Runge-Kutta method whose one-step equation is given by

$$\tilde{y}(t) = y(0) + f(\tilde{y}_1(0)) \frac{t}{6} + f(\tilde{y}_2(\frac{t}{2})) \frac{4t}{6} + f(\tilde{y}_3(t)) \frac{t}{6} \quad (1)$$

where

$$\tilde{y}_1(0) = y(0), \quad \tilde{y}_2(\frac{t}{2}) = y(0) + f(\tilde{y}_1(0)) \frac{t}{2}, \quad \tilde{y}_3(t) = y(0) + f(\tilde{y}_1(0)) t.$$

1. [2 marks] Write down the Butcher tableau for this method.
2. [2 marks] Show that this method is third-order consistent, i.e. show that

$$\tilde{y}(t) - y(t) = O(t^3).$$

Hint. The computations are straightforward but somewhat lengthy. Try to stay as organised as possible.

3. [2 marks] Complete the function `simpson_step()` such that it implements a single time step according to this scheme. Check that your code is correct using the provided function `convergence()`.
4. [2 mark] State one possible reason why this scheme is not listed in https://en.wikipedia.org/wiki/List_of_Runge-Kutta_methods.
5. [2 marks] Determine the stability function $R(z)$ of this method.

2 Implicit Simpson's method [10 marks]

Consider the Runge-Kutta method whose one-step equation is given by

$$\tilde{y}(t) = y(0) + f(\tilde{y}_1(0)) \frac{t}{6} + f(\tilde{y}_2(\frac{t}{2})) \frac{4t}{6} + f(\tilde{y}(t)) \frac{t}{6} \quad (2)$$

where

$$\tilde{y}_1(0) = y(0), \quad \tilde{y}_2(\frac{t}{2}) = y(0) + f(\tilde{y}_1(0)) \frac{t}{2}.$$

Note that equation (2) is obtained from equation (1) by replacing $\tilde{y}_3(t)$ with $\tilde{y}(t)$.

1. [2 marks] Write down the Butcher tableau for this scheme.
2. [4 marks] Complete the function `implicit_simpson_step()`.

You may solve nonlinear equations using the function `find_zero(f,x0)` provided by the `Roots` package. This function returns a number \mathbf{x} such that $\mathbf{f}(\mathbf{x}) = 0$, using $\mathbf{x0}$ as an initial guess for the root-finding algorithm. Use $y(0) + f(\tilde{y}_1(0)) t$ as initial guess for $\tilde{y}(t)$.

Check that your code is correct using the provided function `convergence()`. You may assume without proof that the proposed method is second-order convergent.

3. [2 marks] Determine the stability function $R(z)$ of this scheme.
4. [2 marks] Does this scheme produce a numerical solution which converges to zero when applied to the ODE $\dot{y} = -y$ with a very large time step? Motivate your answer by referring to the stability function determined in [Task 3](#).