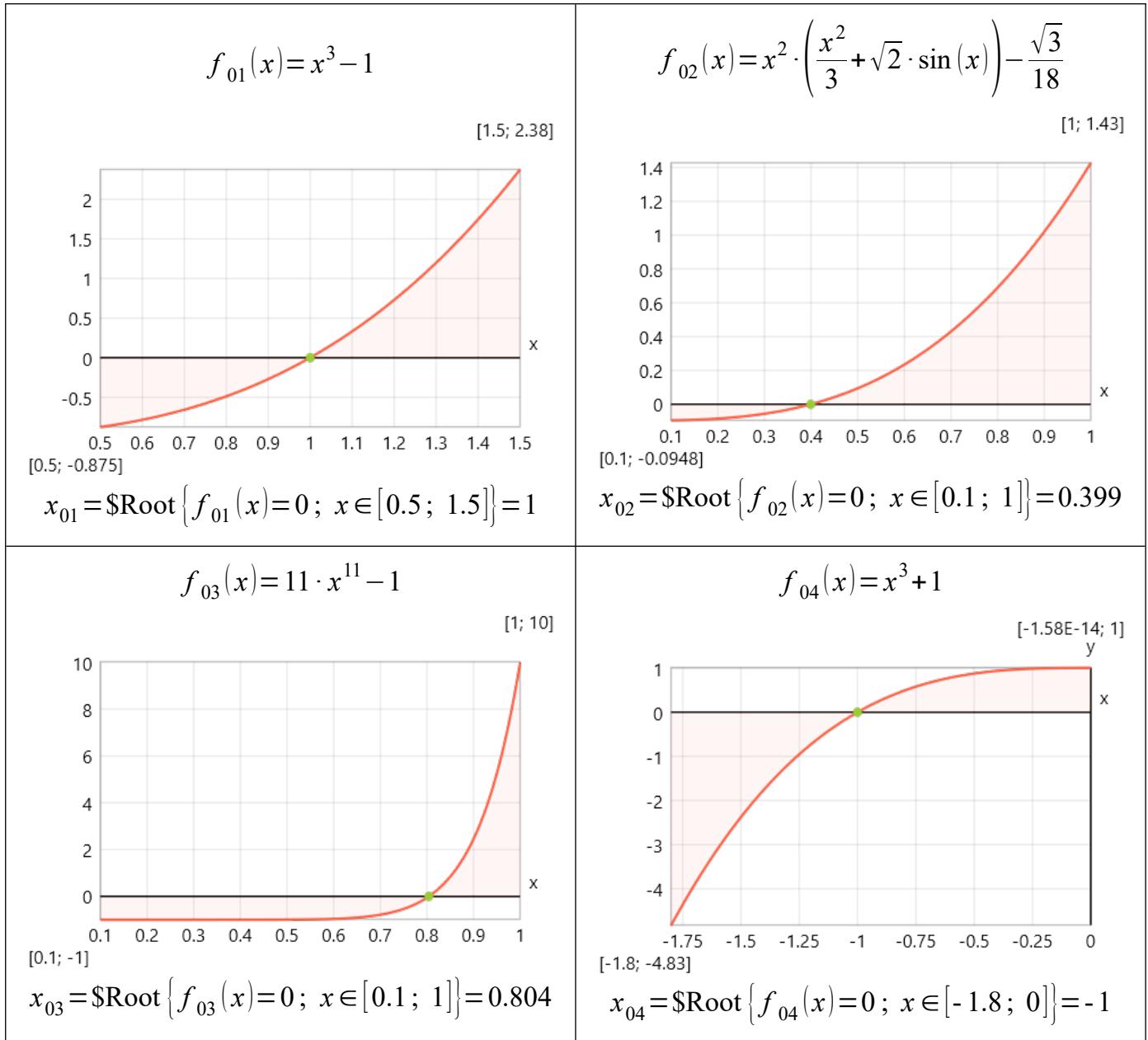


Root-finding Test Functions

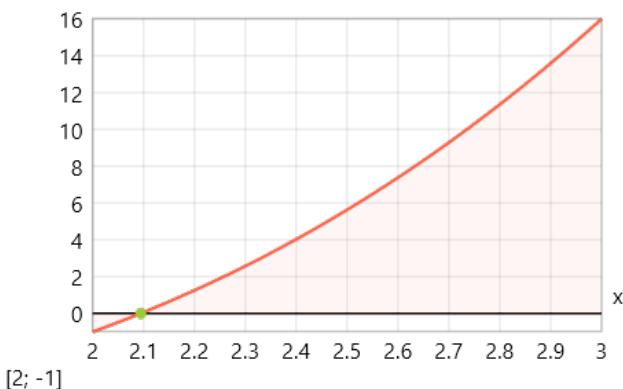
Sérgio Galdino. *A family of regula falsi root-finding methods*. Proceedings of 2011 World Congress on Engineering and Technology. Shanghai, China. IEEE Press. ISBN 978-1-61284-365-0. p. 514-517

<http://sergiogaldino.pbworks.com/w/file/fetch/66011429/0130-1943543>



$$f_{05}(x) = x^3 - 2 \cdot x - 5$$

[3; 16]

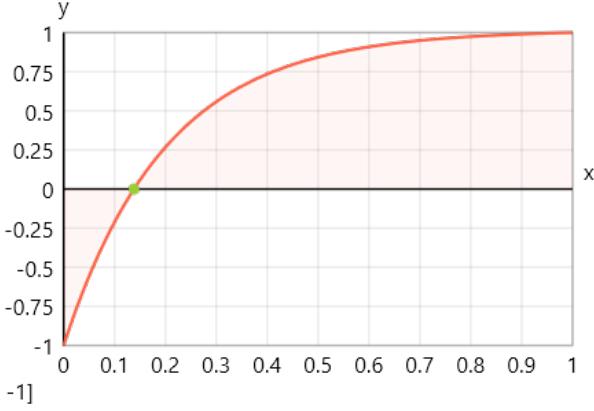


[2; -1]

$$x_{05} = \$\text{Root}\{f_{05}(x)=0; x \in [2; 3]\} = 2.09$$

$$f_{06}(x) = 2 \cdot x \cdot e^{-5} + 1 - 2 \cdot e^{-5 \cdot x}$$

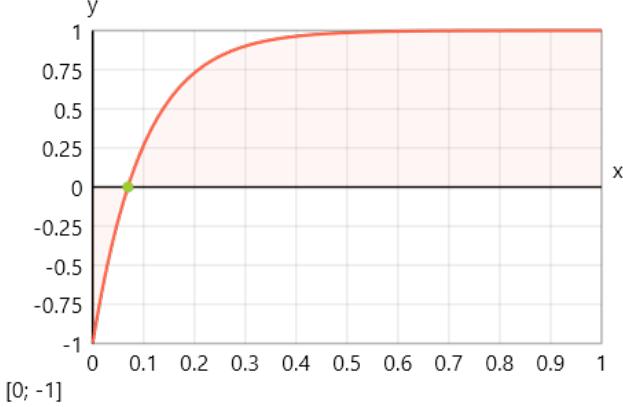
[1; 1]



$$x_{06} = \$\text{Root}\{f_{06}(x)=0; x \in [0; 1]\} = 0.138$$

$$f_{07}(x) = 2 \cdot x \cdot e^{-10} + 1 - 2 \cdot e^{-10 \cdot x}$$

[1; 1]

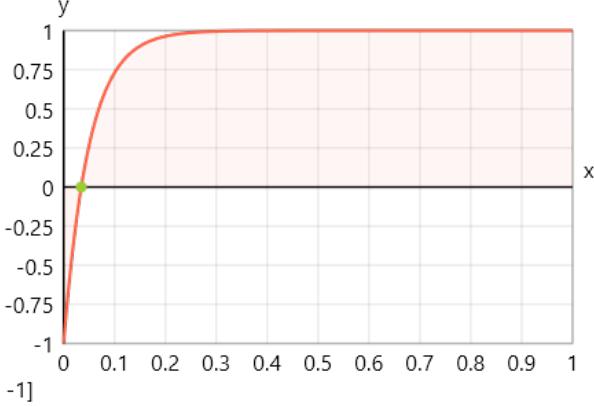


[0; -1]

$$x_{07} = \$\text{Root}\{f_{07}(x)=0; x \in [0; 1]\} = 0.0693$$

$$f_{08}(x) = 2 \cdot x \cdot e^{-20} + 1 - 2 \cdot e^{-20 \cdot x}$$

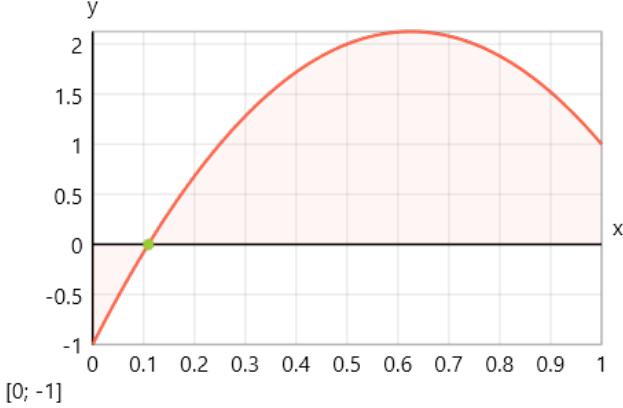
[1; 1]



$$x_{08} = \$\text{Root}\{f_{08}(x)=0; x \in [0; 1]\} = 0.0347$$

$$f_{09}(x) = (1 + (1 - 5)^2) \cdot x^2 - (1 - 5 \cdot x)^2$$

[1; 2.12]

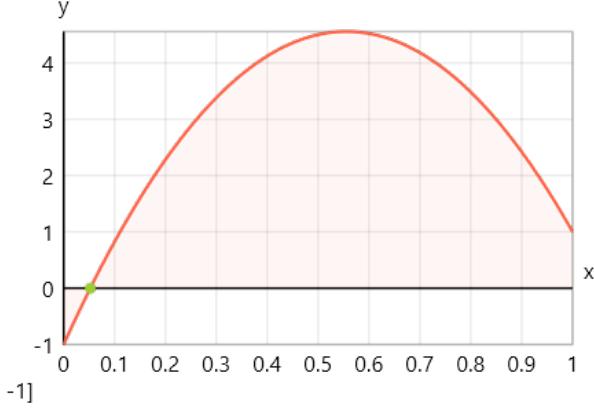


[0; -1]

$$x_{09} = \$\text{Root}\{f_{09}(x)=0; x \in [0; 1]\} = 0.11$$

$$f_{10}(x) = (1 + (1 - 10)^2) \cdot x^2 - (1 - 10 \cdot x)^2$$

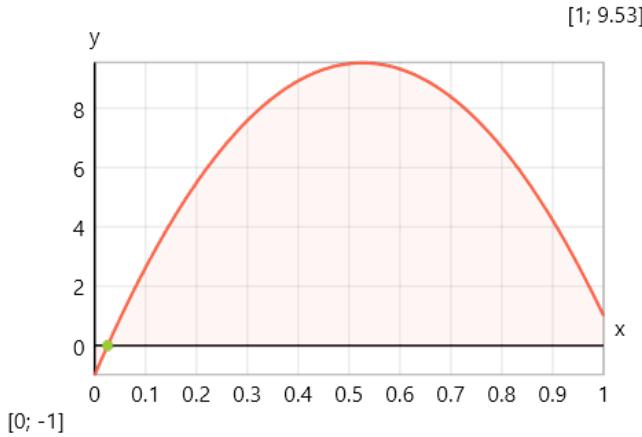
[1; 4.56]



[0; -1]

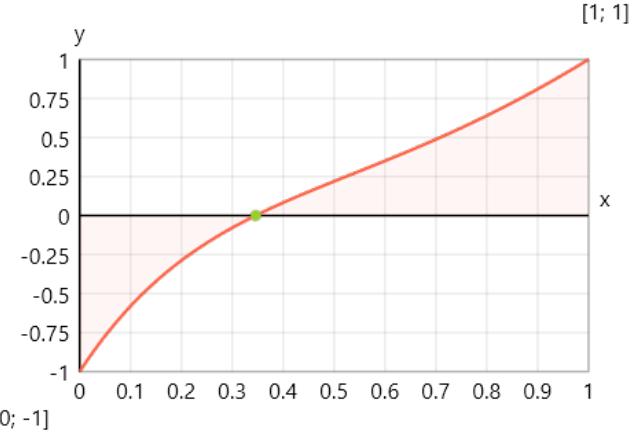
$$x_{10} = \$\text{Root}\{f_{10}(x)=0; x \in [0; 1]\} = 0.0525$$

$$f_{11}(x) = (1 + (1 - 20)^2) \cdot x^2 - (1 - 20 \cdot x)^2$$



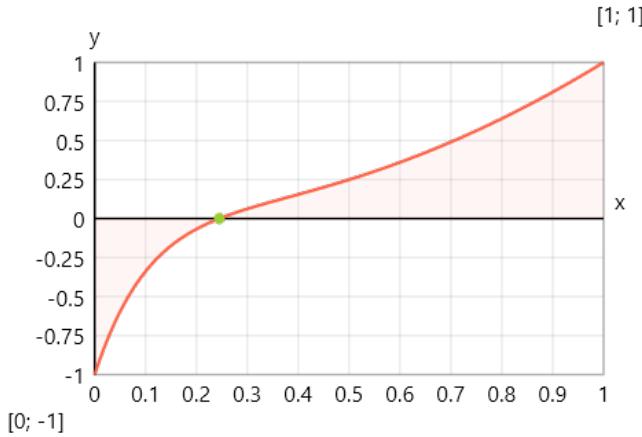
$$x_{11} = \$\text{Root}\{f_{11}(x)=0; x \in [0; 1]\} = 0.0256$$

$$f_{12}(x) = x^2 - (1 - x)^5$$



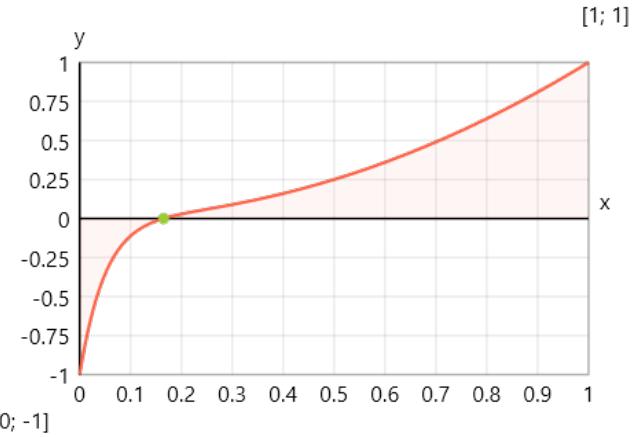
$$x_{12} = \$\text{Root}\{f_{12}(x)=0; x \in [0; 1]\} = 0.346$$

$$f_{13}(x) = x^2 - (1 - x)^{10}$$



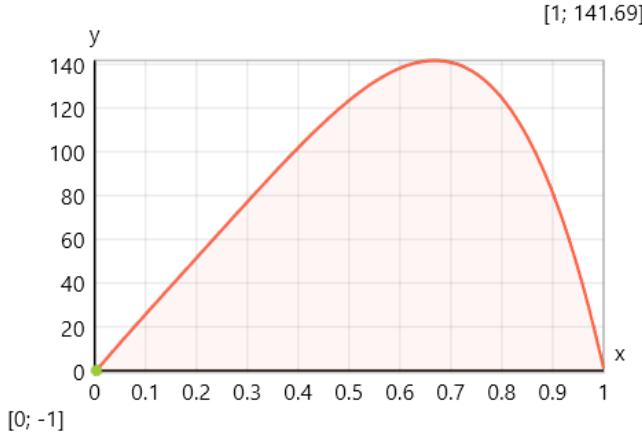
$$x_{13} = \$\text{Root}\{f_{13}(x)=0; x \in [0; 1]\} = 0.245$$

$$f_{14}(x) = x^2 - (1 - x)^{20}$$



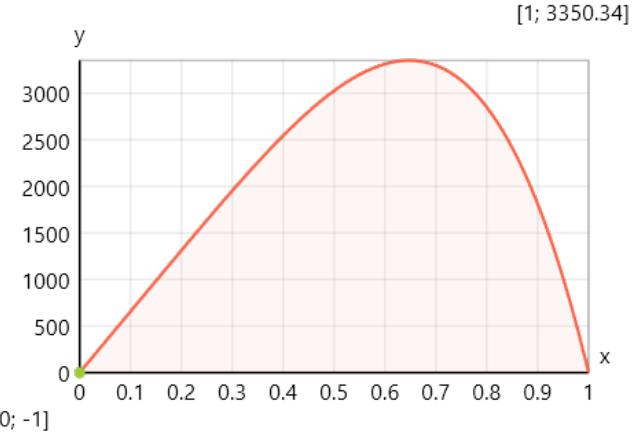
$$x_{14} = \$\text{Root}\{f_{14}(x)=0; x \in [0; 1]\} = 0.165$$

$$f_{15}(x) = (1 + (1 - 5)^4) \cdot x - (1 - 5 \cdot x)^4$$



$$x_{15} = \$\text{Root}\{f_{15}(x)=0; x \in [0; 1]\} = 0.00362$$

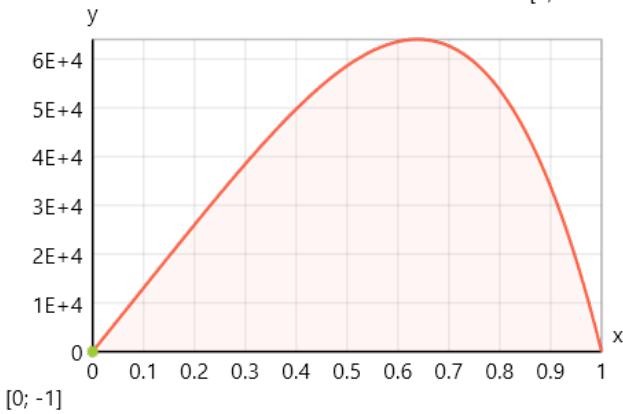
$$f_{16}(x) = (1 + (1 - 10)^4) \cdot x - (1 - 10 \cdot x)^4$$



$$x_{16} = \$\text{Root}\{f_{16}(x)=0; x \in [0; 1]\} = 0.000151$$

$$f_{17}(x) = (1 + (1 - 20)^4) \cdot x - (1 - 20 \cdot x)^4$$

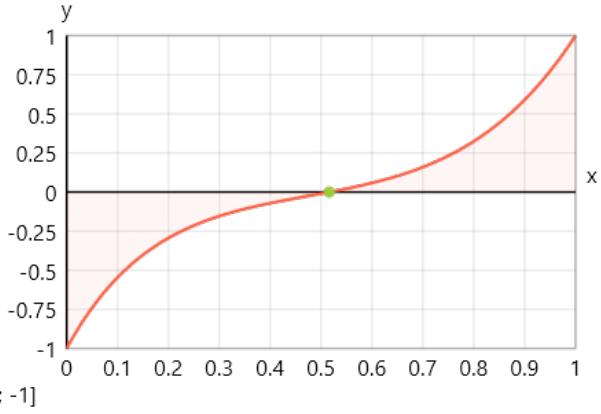
[1; 64019.2]



$$x_{17} = \$\text{Root}\{f_{17}(x)=0; x \in [0; 1]\} = 7.67 \times 10^{-6}$$

$$f_{18}(x) = e^{-5 \cdot x} \cdot (x - 1) + x^5$$

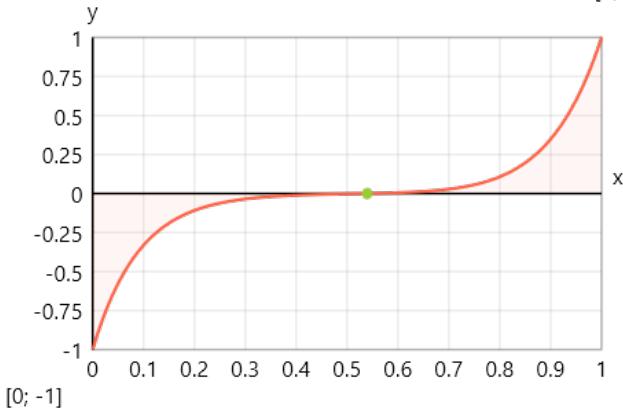
[1; 1]



$$x_{18} = \$\text{Root}\{f_{18}(x)=0; x \in [0; 1]\} = 0.516$$

$$f_{19}(x) = e^{-10 \cdot x} \cdot (x - 1) + x^{10}$$

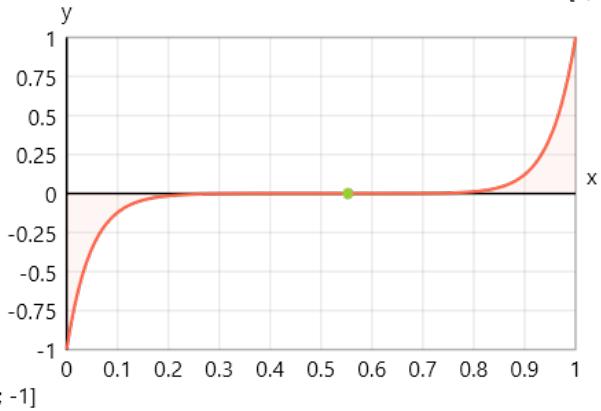
[1; 1]



$$x_{19} = \$\text{Root}\{f_{19}(x)=0; x \in [0; 1]\} = 0.54$$

$$f_{20}(x) = e^{-20 \cdot x} \cdot (x - 1) + x^{20}$$

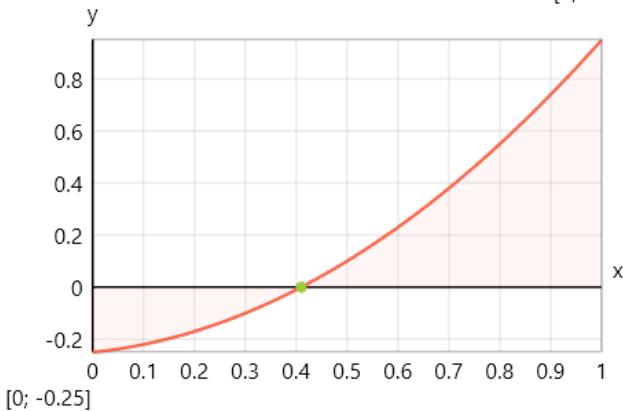
[1; 1]



$$x_{20} = \$\text{Root}\{f_{20}(x)=0; x \in [0; 1]\} = 0.553$$

$$f_{21}(x) = x^2 + \sin\left(\frac{x}{5}\right) - \frac{1}{4}$$

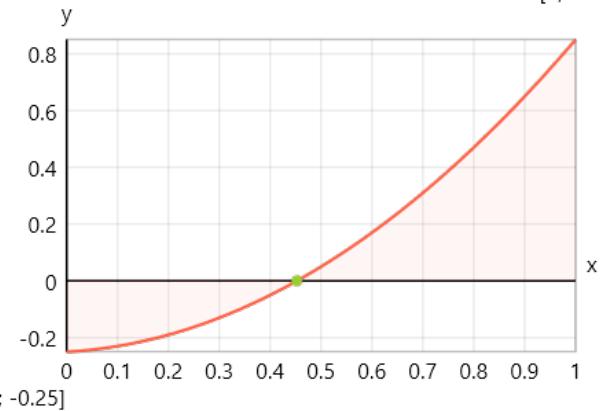
[1; 0.949]



$$x_{21} = \$\text{Root}\{f_{21}(x)=0; x \in [0; 1]\} = 0.41$$

$$f_{22}(x) = x^2 + \sin\left(\frac{x}{10}\right) - \frac{1}{4}$$

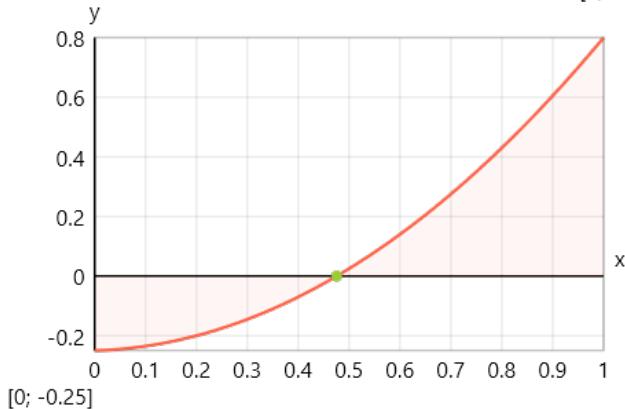
[1; 0.85]



$$x_{22} = \$\text{Root}\{f_{22}(x)=0; x \in [0; 1]\} = 0.453$$

$$f_{23}(x) = x^2 + \sin\left(\frac{x}{20}\right) - \frac{1}{4}$$

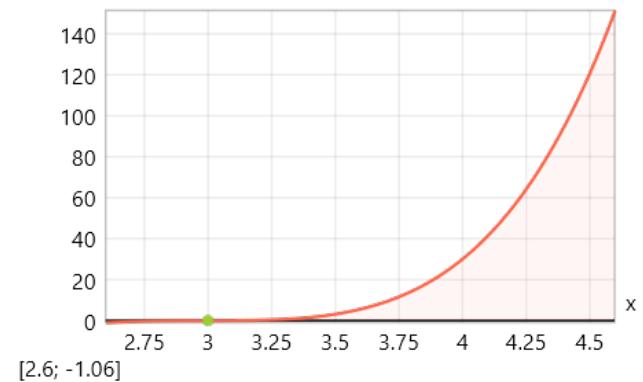
[1; 0.8]



$$x_{23} = \$Root \{f_{23}(x)=0; x \in [0; 1]\} = 0.476$$

$$f_{24}(x) = (x+2) \cdot (x+1) \cdot (x-3)^3$$

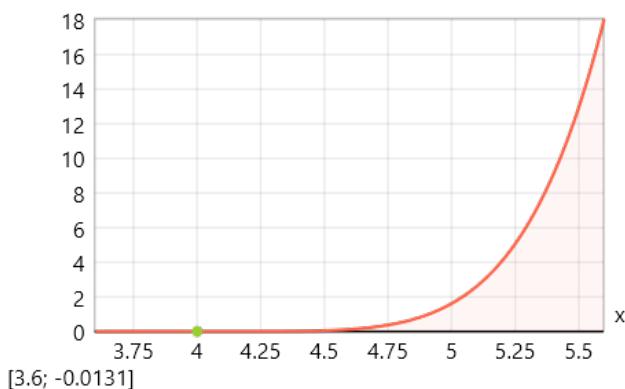
[4.6; 151.39]



$$x_{24} = \$Root \{f_{24}(x)=0; x \in [2.6; 4.6]\} = 3$$

$$f_{25}(x) = (x-4)^5 \cdot \ln(x)$$

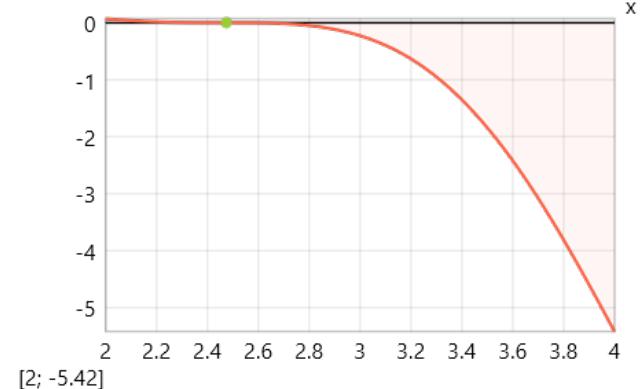
[5.6; 18.06]



$$x_{25} = \$Root \{f_{25}(x)=0; x \in [3.6; 5.6]\} = 4$$

$$f_{26}(x) = \left(\sin(x) - \frac{x}{4} \right)^3$$

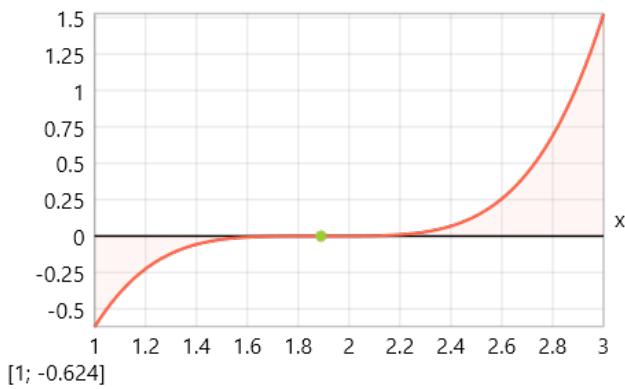
[4; 0.0686]



$$x_{26} = \$Root \{f_{26}(x)=0; x \in [2; 4]\} = 2.47$$

$$f_{27}(x) = \left(81 - p(x) \cdot (108 - p(x) \cdot (54 - p(x) \cdot (12 - p(x))) \right) \cdot \text{sign}(p(x) - 3), \quad p(x) = x + 1.11$$

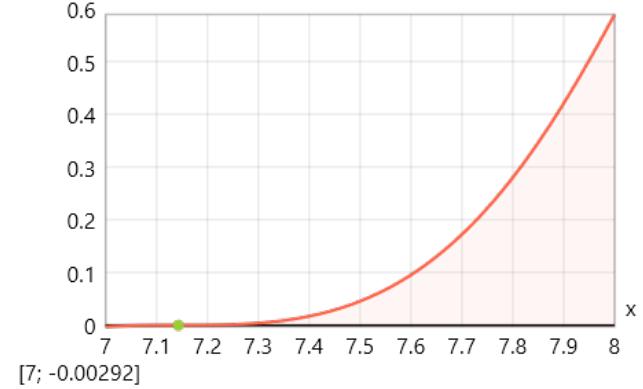
[3; 1.52]



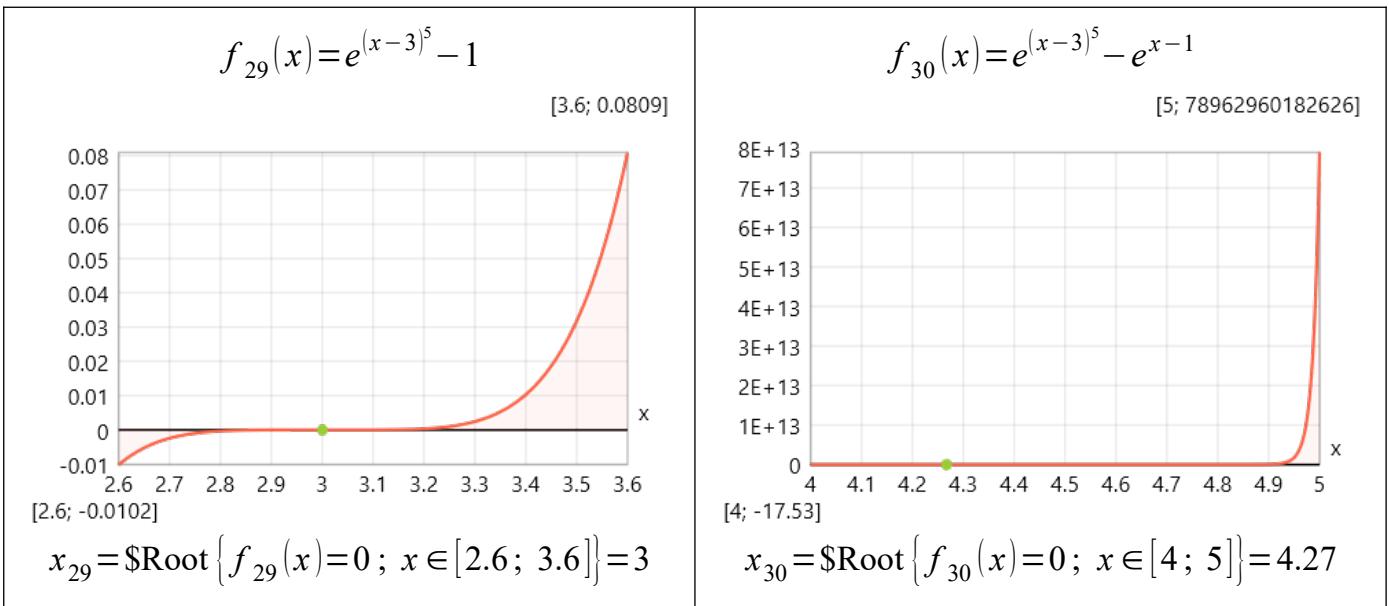
$$x_{27} = \$Root \{f_{27}(x)=0; x \in [1; 3]\} = 1.89$$

$$f_{28}(x) = \sin((x-7.14)^3)$$

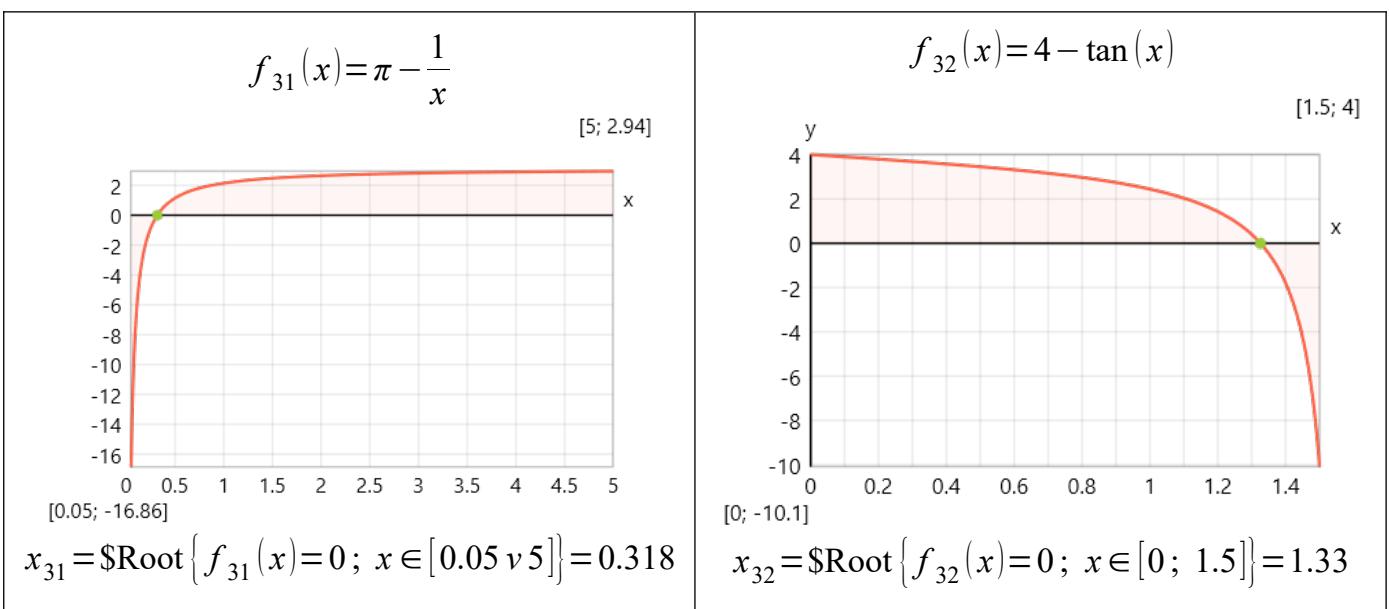
[8; 0.589]



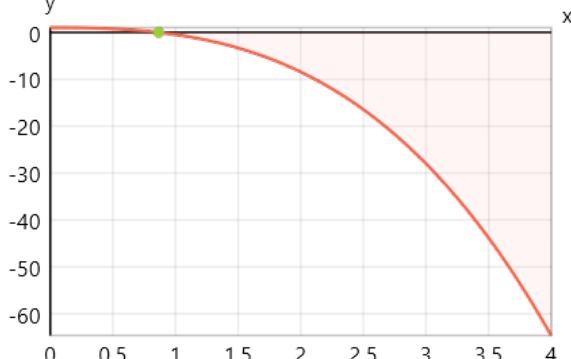
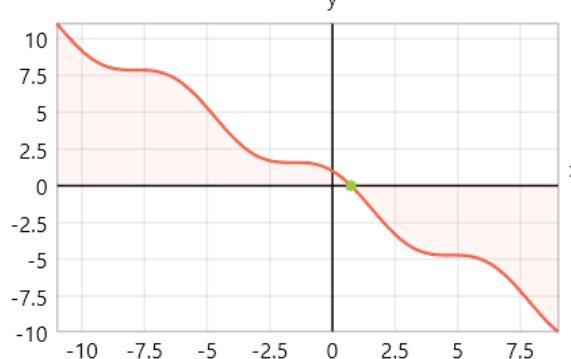
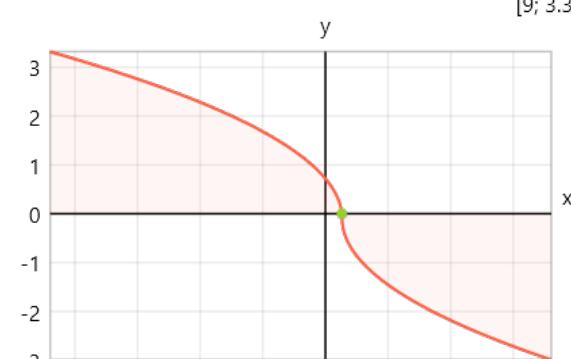
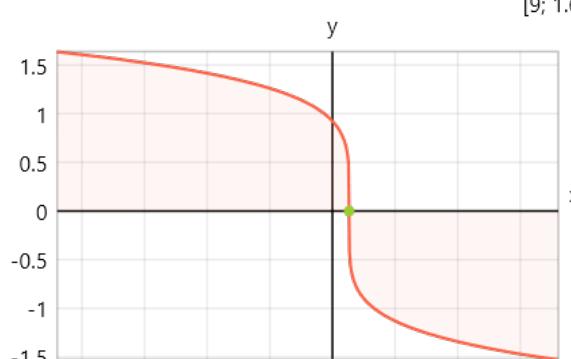
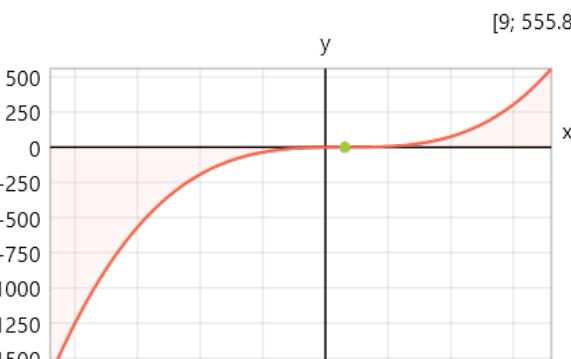
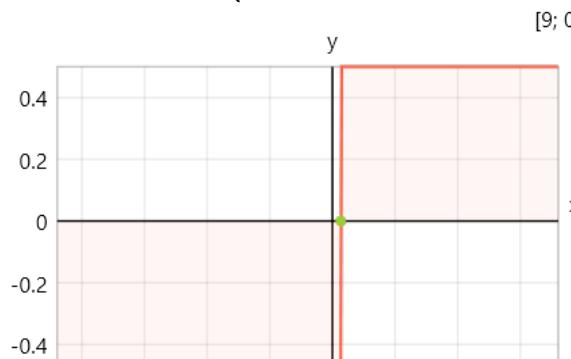
$$x_{28} = \$Root \{f_{28}(x)=0; x \in [7; 8]\} = 7.14$$



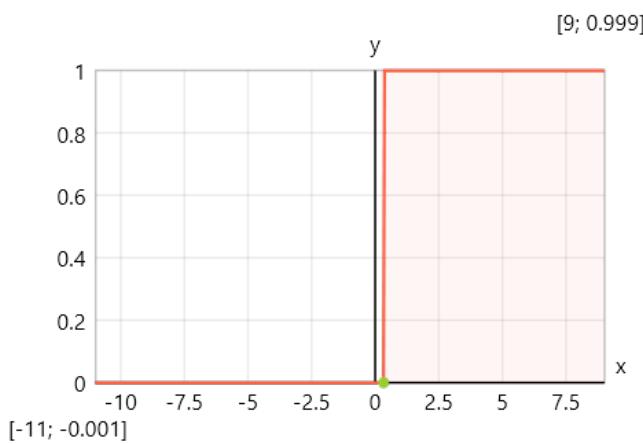
My functions



Steven A. Stage. *Comments on An Improvement to the Brent's Method*. International Journal of Experimental Algorithms (IJEA), Volume (4) : Issue (1) : 2013

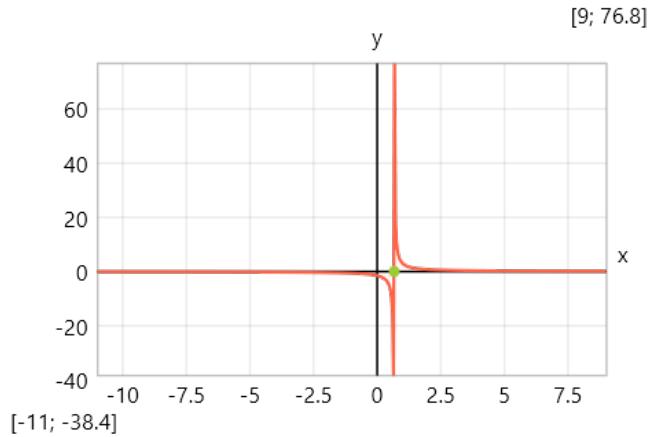
$f_{33}(x) = \cos(x) - x^3$  $x_{33} = \$Root\{f_{33}(x)=0; x \in [0; 4]\} = 0.865$	$f_{34}(x) = \cos(x) - x$  $x_{34} = \$Root\{f_{34}(x)=0; x \in [-11; 9]\} = 0.739$
$f_{35}(x) = \sqrt{\left x - \frac{2}{3}\right } \cdot \begin{cases} \text{if } x \leq \frac{2}{3}: & 1 \\ \text{else:} & -1 \end{cases}$  $x_{35} = \$Root\{f_{35}(x)=0; x \in [-11; 9]\} = 0.657$	$f_{36}(x) = \left(\left x - \frac{2}{3}\right \right)^{0.2} \cdot \begin{cases} \text{if } x \leq \frac{2}{3}: & 1 \\ \text{else:} & -1 \end{cases}$  $x_{36} = \$Find\{f_{36}(x); x \in [-11; 9]\} = 0.667$
$f_{37}(x) = \left(x - \frac{7}{9}\right)^3 + \left(x - \frac{7}{9}\right) \cdot 10^{-3}$  $x_{37} = \$Root\{f_{37}(x)=0; x \in [-11; 9]\} = 0.778$	$f_{38}(x) = \begin{cases} \text{if } x \leq \frac{1}{3}: & -0.5 \\ \text{else:} & 0.5 \end{cases}$  $x_{38} = \$Find\{f_{38}(x); x \in [-11; 9]\} = 0.333$

$$f_{39}(x) = \begin{cases} \text{if } x \leq \frac{1}{3}: & -(10^{-3}) \\ \text{else:} & 1 - 10^{-3} \end{cases}$$



$$x_{39} = \$\text{Find}\{f_{39}(x); x \in [-11; 9]\} = 0.333$$

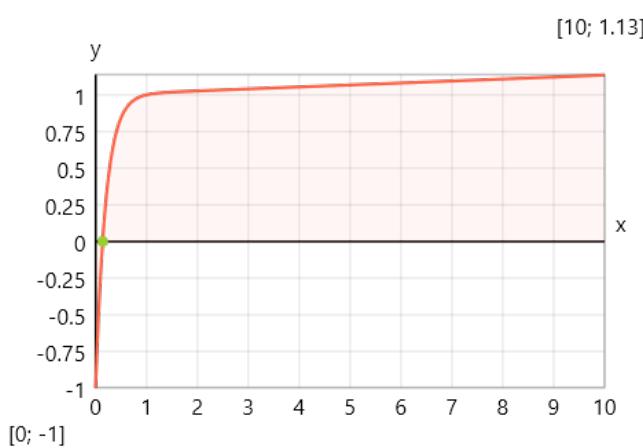
$$f_{40}(x) = \begin{cases} \text{if } x = 0: & 0 \\ \text{else:} & \frac{1}{x - 2/3} \end{cases}$$



$$x_{40} = \$\text{Find}\{f_{40}(x); x \in [-11; 9]\} = 0.667$$

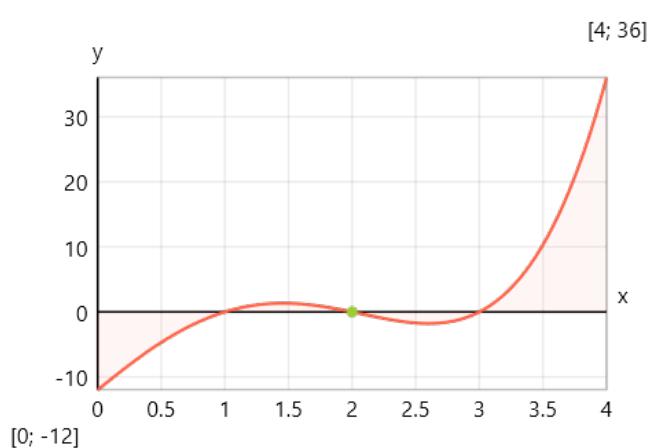
A. Swift, G. R. Lindfield, *Comparison of a continuation method with Brent's method for the numerical solution of a single nonlinear equation*, The Computer Journal, Volume 21, Issue 4, 1978, p. 359–362
<https://doi.org/10.1093/comjnl/21.4.359>

$$f_{41}(x) = 2 \cdot x \cdot e^{-5} - 2 \cdot e^{-5 \cdot x} + 1$$



$$x_{41} = \$\text{Root}\{f_{41}(x) = 0; x \in [0; 10]\} = 0.138$$

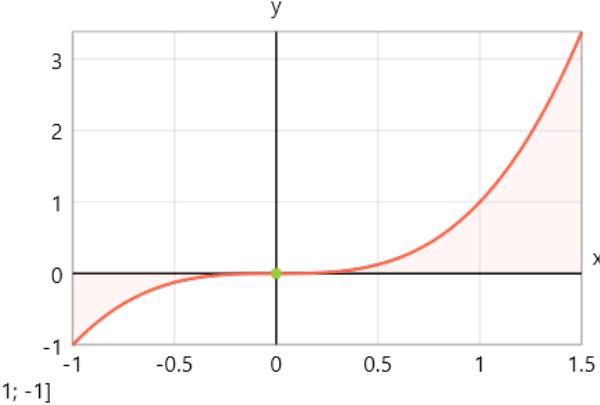
$$f_{42}(x) = (x^2 - x - 6) \cdot (x^2 - 3 \cdot x + 2)$$



$$x_{42} = \$\text{Root}\{f_{42}(x) = 0; x \in [0; 4]\} = 2$$

$$f_{43}(x) = x^3$$

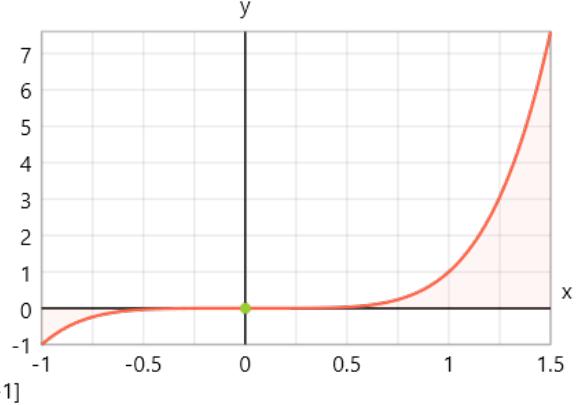
[1.5; 3.38]



$$x_{43} = \text{Root}\{f_{43}(x)=0; x \in [-1; 1.5]\} = -1.46 \times 10^{-11}$$

$$f_{44}(x) = x^5$$

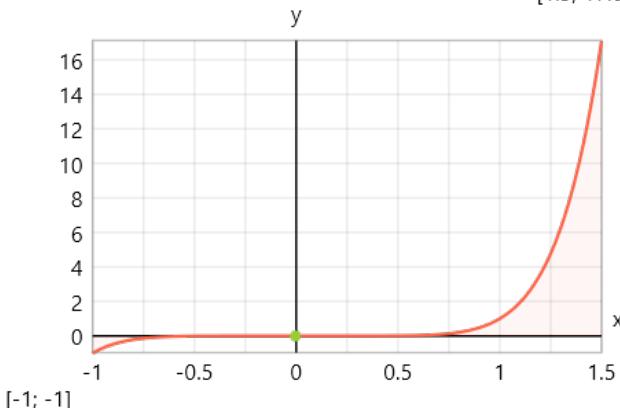
[1.5; 7.59]



$$x_{44} = \text{Root}\{f_{44}(x)=0; x \in [-1; 1.5]\} = 2.38 \times 10^{-7}$$

$$f_{45}(x) = x^7$$

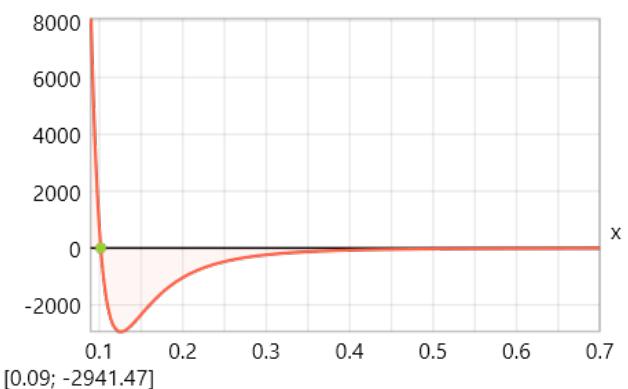
[1.5; 17.09]



$$x_{45} = \text{Root}\{f_{45}(x)=0; x \in [-1; 1.5]\} = -1.53 \times 10^{-5}$$

$$f_{46}(x) = \frac{e^{-5 \cdot x} - x - 0.5}{x^5}$$

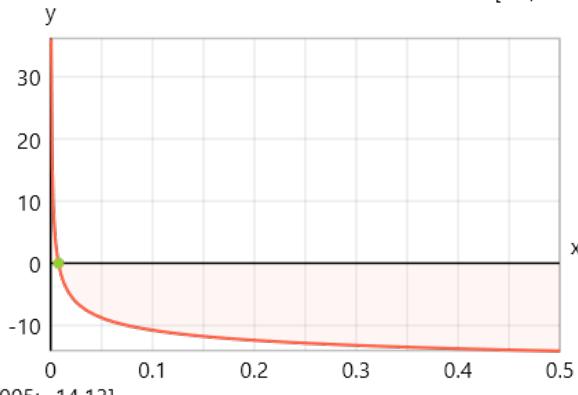
[0.7; 8065.87]



$$x_{46} = \text{Root}\{f_{46}(x)=0; x \in [0.09; 0.7]\} = 0.102$$

$$f_{47}(x) = \frac{1}{\sqrt{x}} - 2 \cdot \ln(5 \cdot 10^3 \cdot \sqrt{x}) + 0.8$$

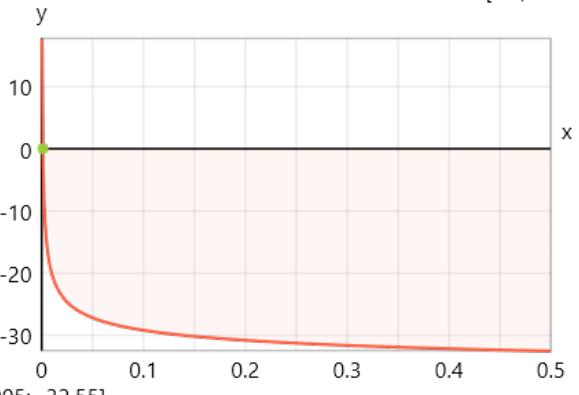
[0.5; 36.09]



$$x_{47} = \text{Root}\{f_{47}(x)=0; x \in [0.0005; 0.5]\} = 0.00773$$

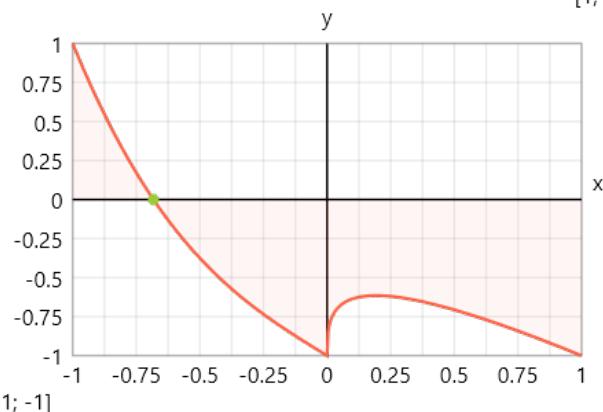
$$f_{48}(x) = \frac{1}{\sqrt{x}} - 2 \cdot \ln(5 \cdot 10^7 \cdot \sqrt{x}) + 0.8$$

[0.5; 17.67]



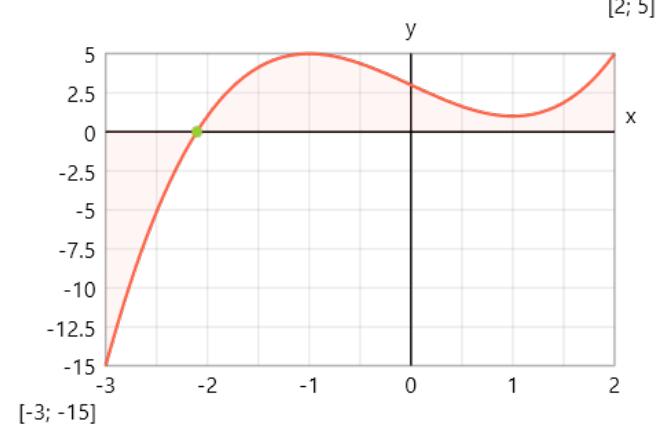
$$x_{48} = \text{Root}\{f_{48}(x)=0; x \in [0.0005; 0.5]\} = 0.00128$$

$$f_{49}(x) = \begin{cases} \text{if } x \leq 0 : & -(x^3) - x - 1 \\ \text{else:} & x^{\frac{1}{3}} - x - 1 \end{cases}$$



$$x_{49} = \$\text{Root}\{f_{49}(x)=0; x \in [-1; 1]\} = -0.682$$

$$f_{50}(x) = x^3 - 2 \cdot x - x + 3$$



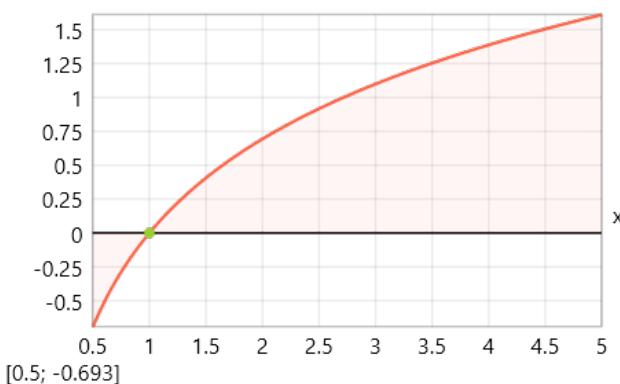
$$x_{50} = \$\text{Root}\{f_{50}(x)=0; x \in [-3; 2]\} = -2.1$$

Alojz Suhadolnik, Combined bracketing methods for solving nonlinear equations, Applied Mathematics Letters, Volume 25, Issue 11, 2012, Pages 1755-1760, ISSN 0893-9659

<https://doi.org/10.1016/j.aml.2012.02.006>

$$f_{51}(x) = \ln(x)$$

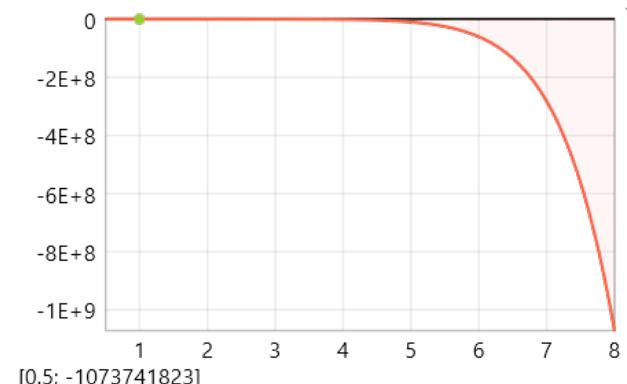
[5; 1.61]



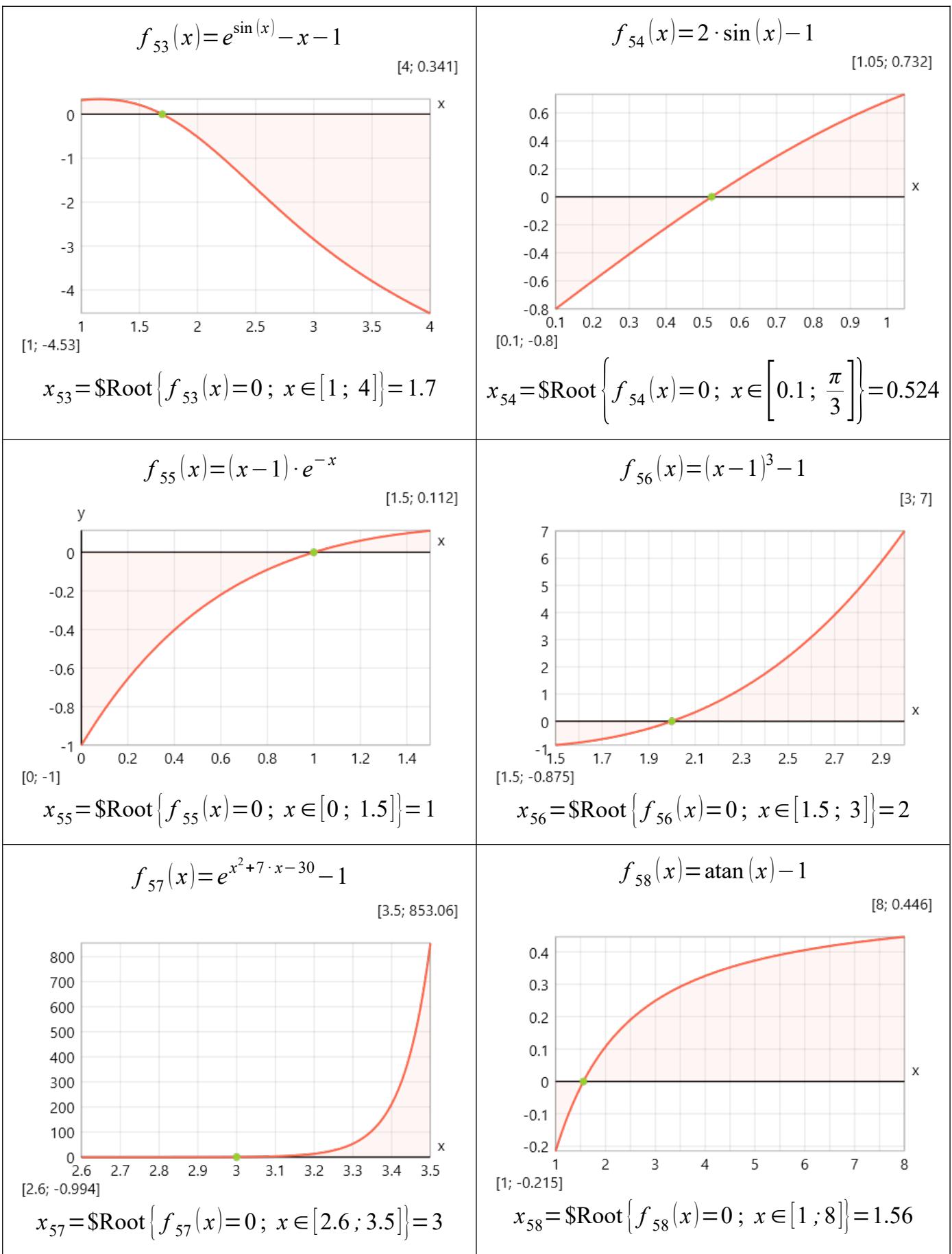
$$x_{51} = \$\text{Root}\{f_{51}(x)=0; x \in [0.5; 5]\} = 1$$

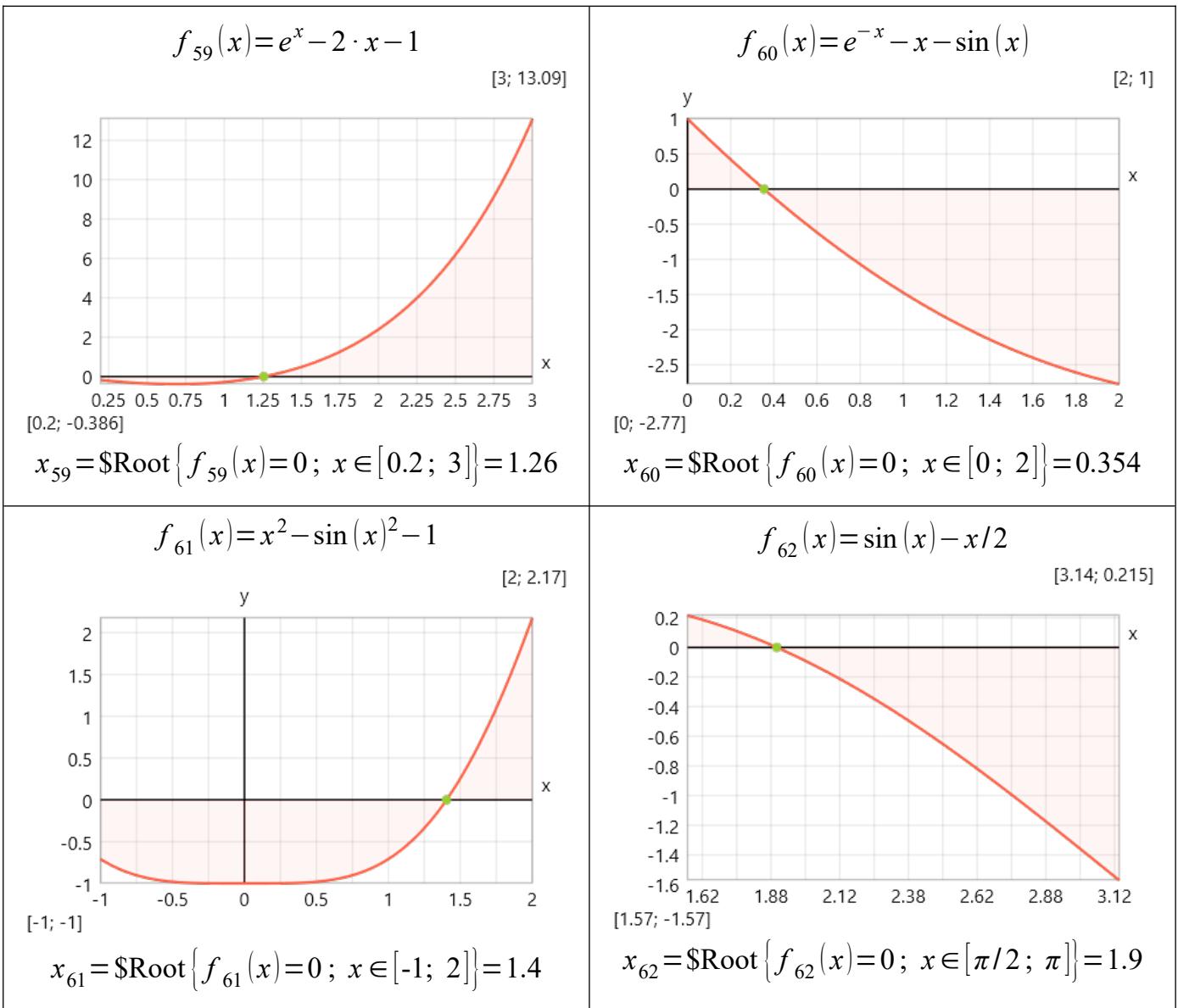
$$f_{52}(x) = (10 - x) \cdot e^{-10 \cdot x} - x^{10} + 1$$

[8; 1.06]



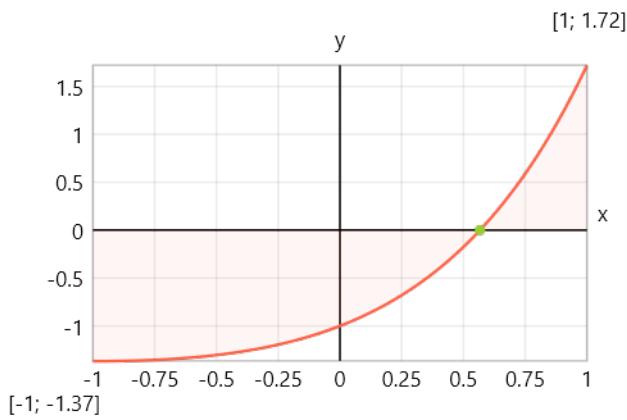
$$x_{52} = \$\text{Root}\{f_{52}(x)=0; x \in [0.5; 8]\} = 1$$





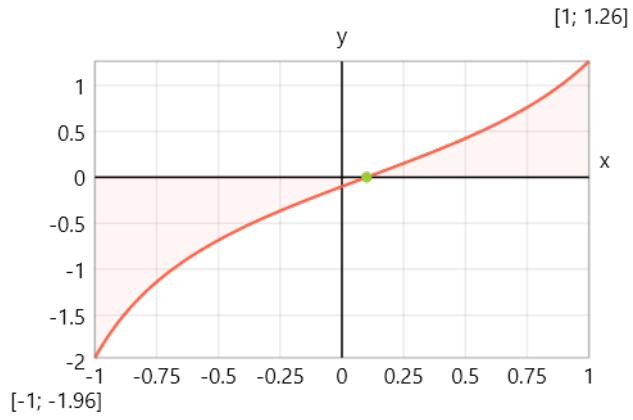
Oliveira I. F. D., Takahashi R. H. C. An Enhancement of the Bisection Method Average Performance Preserving Minmax Optimality

Lambert - $f_{63}(x) = x \cdot \exp(x) - 1$



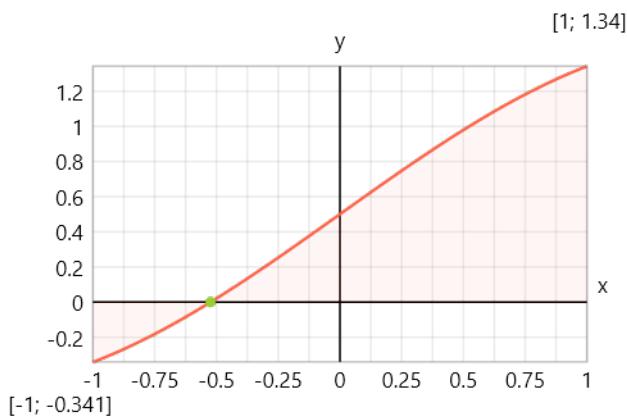
$x_{63} = \$\text{Root}\{f_{63}(x)=0; x \in [-1; 1]\} = 0.567$

Trigonometric 1 - $f_{64}(x) = \tan\left(x - \frac{1}{10}\right)$



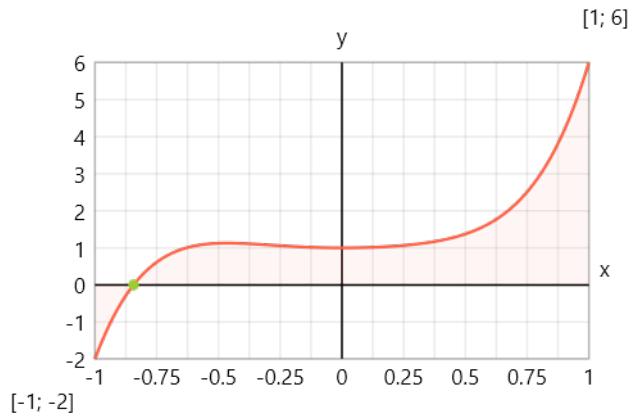
$x_{64} = \$\text{Root}\{f_{64}(x)=0; x \in [-1; 1]\} = 0.1$

Trigonometric 2 - $f_{65}(x) = \sin(x) + 0.5$



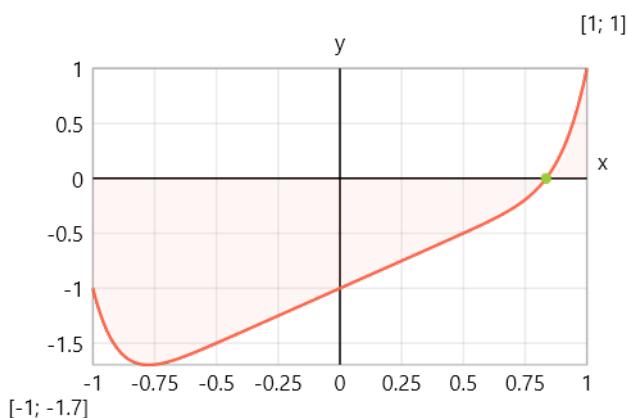
$x_{65} = \$\text{Root}\{f_{65}(x)=0; x \in [-1; 1]\} = -0.524$

Polynomial 1 - $f_{66}(x) = 4 \cdot x^5 + x \cdot x + 1$



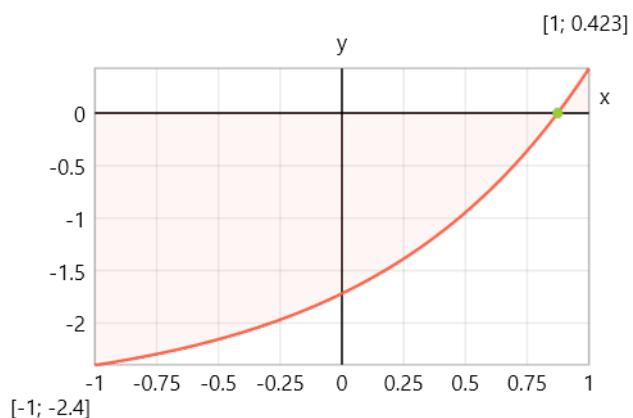
$x_{66} = \$\text{Root}\{f_{66}(x)=0; x \in [-1; 1]\} = -0.844$

Polynomial 2 - $f_{67}(x) = x + x^{10} - 1$



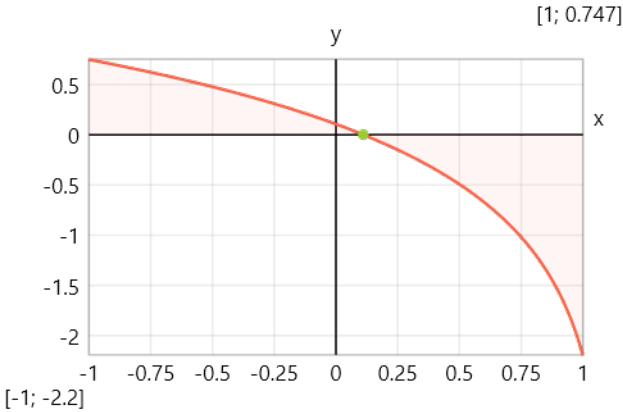
$x_{67} = \$\text{Root}\{f_{67}(x)=0; x \in [-1; 1]\} = 0.835$

Exponential - $f_{68}(x) = \pi^x - e$



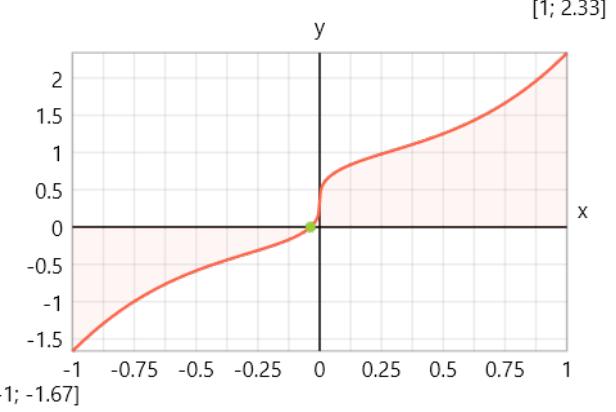
$x_{68} = \$\text{Root}\{f_{68}(x)=0; x \in [-1; 1]\} = 0.874$

Logarithmic - $f_{69}(x) = \ln\left(\left|x - \frac{10}{9}\right|\right)$



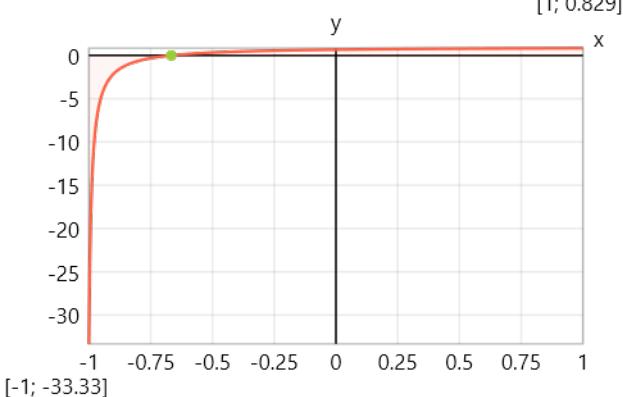
$$x_{69} = \$Root\{f_{69}(x)=0; x \in [-1; 1]\} = 0.111$$

Polynomial - $f_{70}(x) = \frac{1}{3} + \text{sign}(x) \cdot \sqrt[3]{|x|} + x^3$



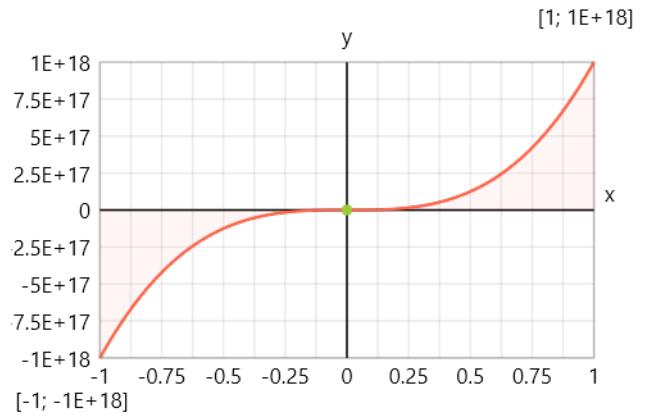
$$x_{70} = \$Root\{f_{70}(x)=0; x \in [-1; 1]\} = -0.037$$

Poly. Frac. - $f_{71}(x) = \frac{x+2/3}{x+101/100}$



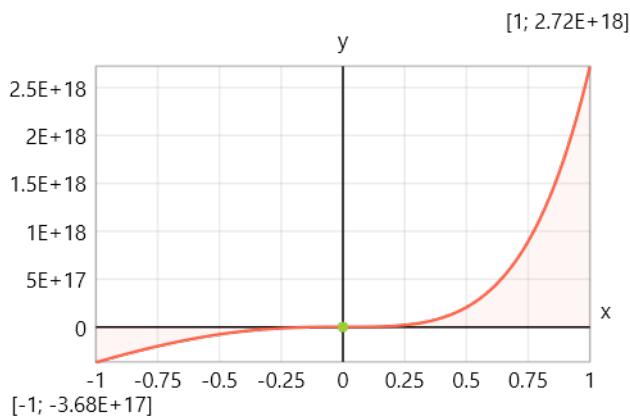
$$x_{71} = \$Root\{f_{71}(x)=0; x \in [-1; 1]\} = -0.667$$

Polynomial 3 - $f_{72}(x) = (x \cdot 10^6 - 1)^3$



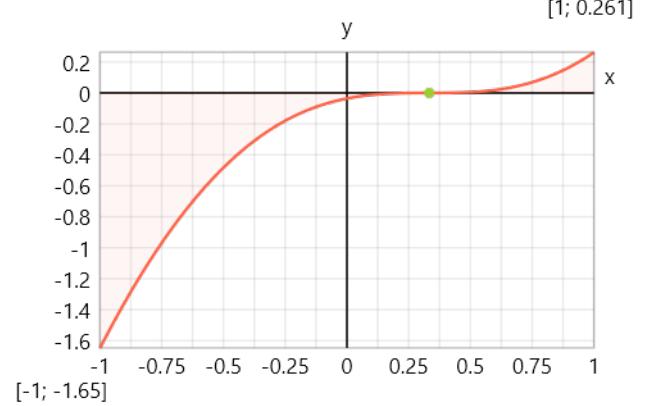
$$x_{72} = \$Root\{f_{72}(x)=0; x \in [-1; 1]\} = 10^{-6}$$

Exp. Poly. - $f_{73}(x) = \exp(x) \cdot (x \cdot 10^6 - 1)^3$



$$x_{73} = \$Root\{f_{73}(x)=0; x \in [-1; 1]\} = 10^{-6}$$

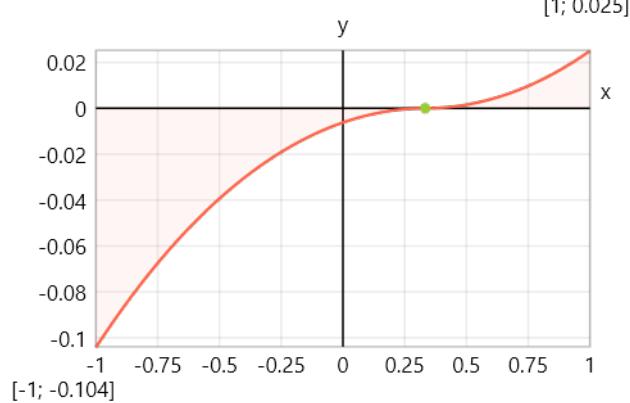
Tan. Poly. - $f_{74}(x) = \left(x - \frac{1}{3}\right)^2 \cdot \left(\text{atan}\left(x - \frac{1}{3}\right)\right)$



$$x_{74} = \$Root\{f_{74}(x)=0; x \in [-1; 1]\} = 0.333$$

Circles

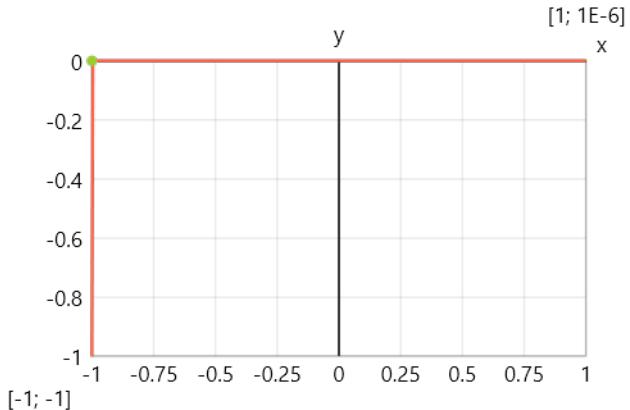
$$f_{75}(x) = \text{sign}(3 \cdot x - 1) \cdot \left(1 - \sqrt{1 - \frac{(3 \cdot x - 1)^2}{81}} \right)$$



$$x_{75} = \$\text{Root}\{f_{75}(x) = 0; x \in [-1; 1]\} = 0.333$$

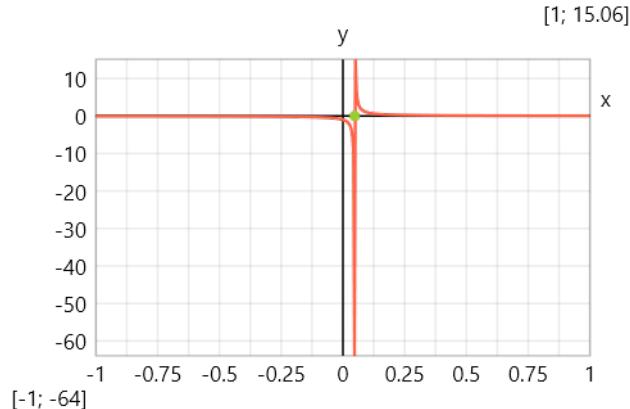
Step Function

$$f_{76}(x) = \begin{cases} \text{if } x > \frac{1-10^6}{10^6}: & \frac{1+10^6}{10^6} - 1 \\ \text{else:} & 0 \end{cases}$$



$$x_{76} = \$\text{Find}\{f_{76}(x); x \in [-1; 1]\} = -1$$

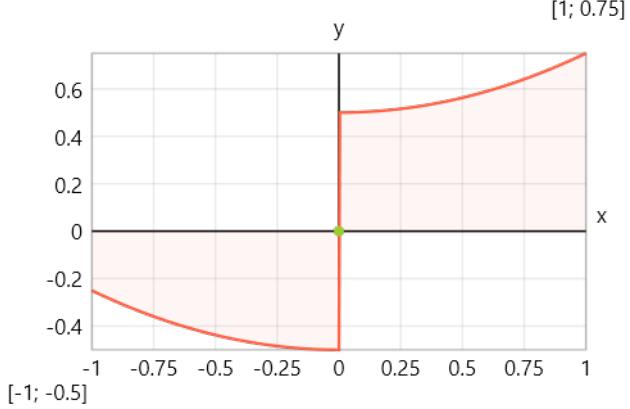
$$\text{Geometric - } f_{77}(x) = \begin{cases} \text{if } x \neq \frac{1}{21}: & \frac{1}{21 \cdot x - 1} \\ \text{else:} & 0 \end{cases}$$



$$x_{77} = \$\text{Find}\{f_{77}(x); x \in [-1; 1]\} = 0.0476$$

Trunc. Poly.

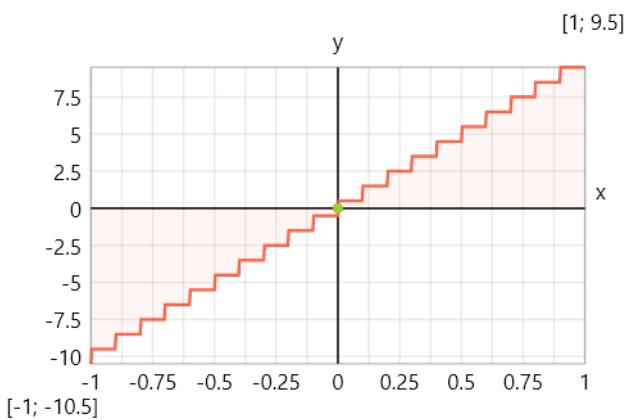
$$f_{78}(x) = \frac{x \cdot x}{4} + \text{ceiling}\left(\frac{x}{2}\right) - 0.5$$



$$x_{78} = \$\text{Find}\{f_{78}(x); x \in [-1; 1]\} = 7.11 \times 10^{-15}$$

Staircase

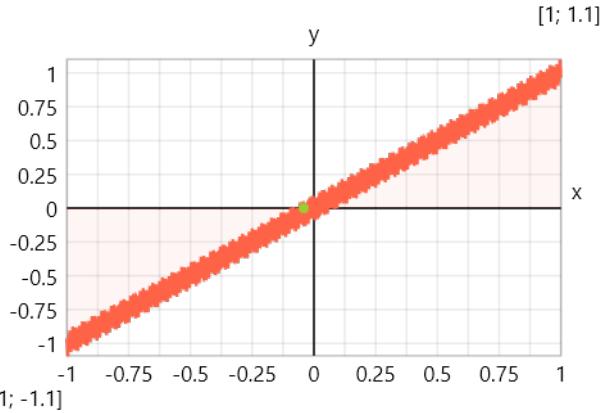
$$f_{79}(x) = \text{ceiling}(10 \cdot x - 1) + 0.5$$



$$x_{79} = \$\text{Find}\{f_{79}(x); x \in [-1; 1]\} = 2.91 \times 10^{-19}$$

Noisy Line

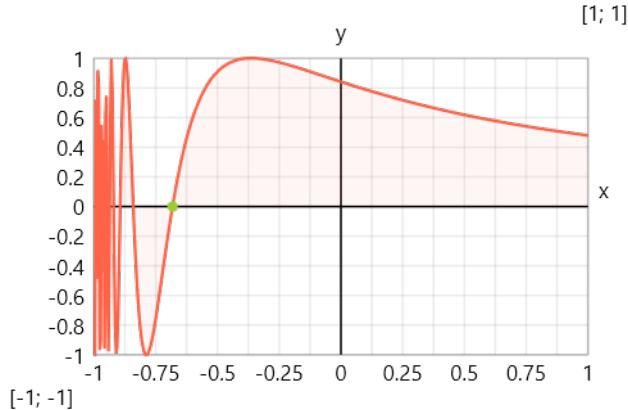
$$f_{80}(x) = x + \frac{\sin(x \cdot 10^6)}{10} + 10^{-3}$$



$$x_{80} = \$\text{Root}\{f_{80}(x)=0; x \in [-1; 1]\} = -0.0415$$

Warsaw

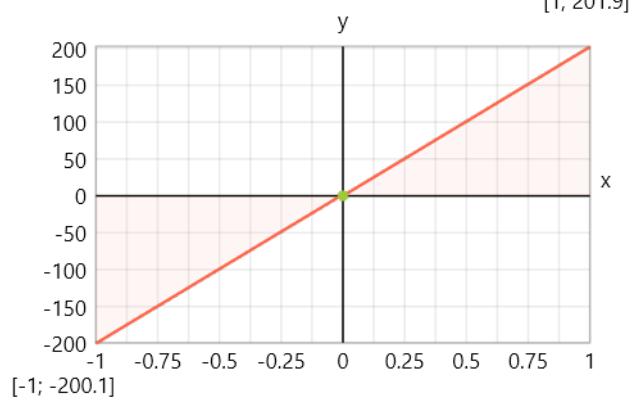
$$f_{81}(x) = \begin{cases} \text{if } x > (-1): & 1 + \sin\left(\frac{1}{x+1}\right) - 1 \\ \text{else:} & 0 \end{cases}$$



$$x_{81} = \$\text{Root}\{f_{81}(x)=0; x \in [-1; 1]\} = -0.682$$

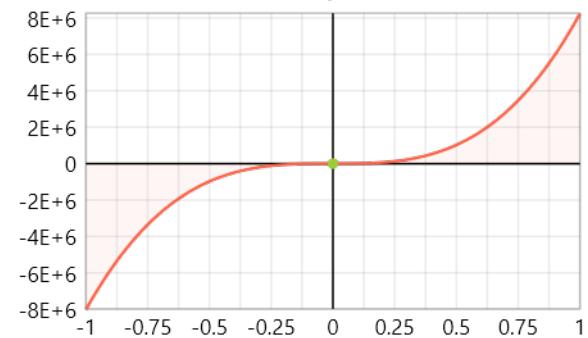
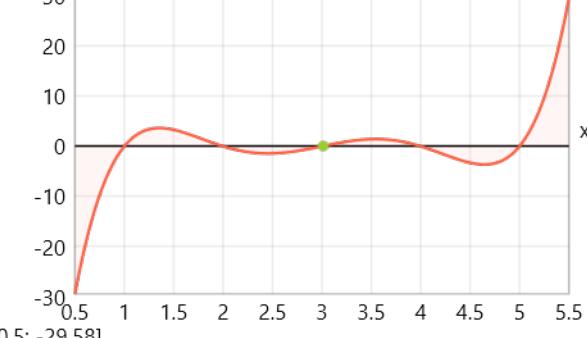
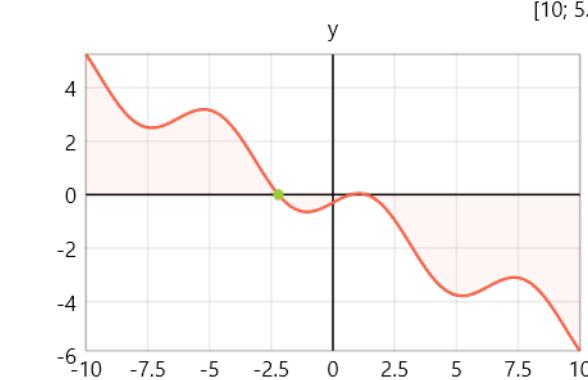
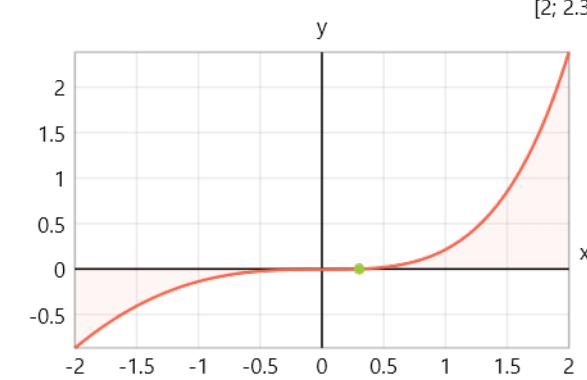
Sawtooth

$$f_{82}(x) = 202 \cdot x - 2 \cdot \text{floor}\left(\frac{2 \cdot x + 10^{-2}}{2} \cdot 10^{-2}\right) - 0.1$$



$$x_{82} = \$\text{Root}\{f_{82}(x)=0; x \in [-1; 1]\} = 0.000495$$

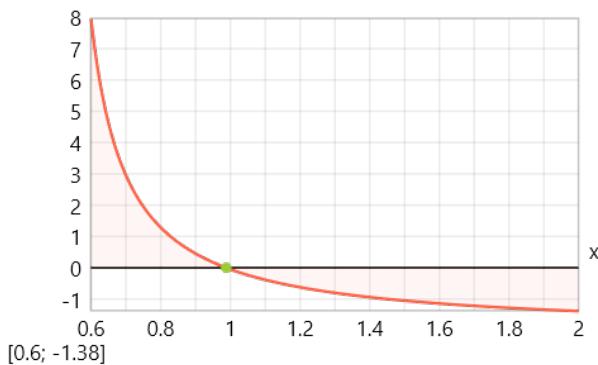
SciML Benchmarks test suite

<p style="text-align: center;">Sawtooth Cube</p> $f_{83}(x) = \left(202 \cdot x - 2 \cdot \text{floor}\left(\frac{2 \cdot x + 10^{-2}}{2} \cdot 10^{-2}\right) - 0.1 \right)^3$  <p style="text-align: right;">[1; 8230173] [-1; -8012006]</p> $x_{83} = \$\text{Root}\{f_{83}(x)=0; x \in [-1; 1]\} = 0.000495$	<p style="text-align: center;">Polynomial with multiple roots</p> $f_{84}(x) = (x-1) \cdot (x-2) \cdot (x-3) \cdot (x-4) \cdot (x-5) - 0.05$  <p style="text-align: right;">[5.5; 29.48] [0.5; -29.58]</p> $x_{84} = \$\text{Root}\{f_{84}(x)=0; x \in [0.5; 5.5]\} = 3.01$
<p style="text-align: center;">Trigonometric with multiple roots</p> $f_{85}(x) = \sin(x) - 0.5 \cdot x - 0.3$  <p style="text-align: right;">[10; 5.24] [-10; -5.84]</p> $x_{85} = \$\text{Root}\{f_{85}(x)=0; x \in [-10; 10]\} = -2.21$	<p style="text-align: center;">Exponential function (sensitive near zero)</p> $f_{86}(x) = \exp(x) - 1 - x - \frac{x \cdot x}{2} - 0.005$  <p style="text-align: right;">[2; 2.38] [-2; -0.87]</p> $x_{86} = \$\text{Root}\{f_{86}(x)=0; x \in [-2; 2]\} = 0.303$

Rational function with pole

$$f_{87}(x) = \frac{1}{x-0.5} - 2 - 0.05$$

[2; 7.95]

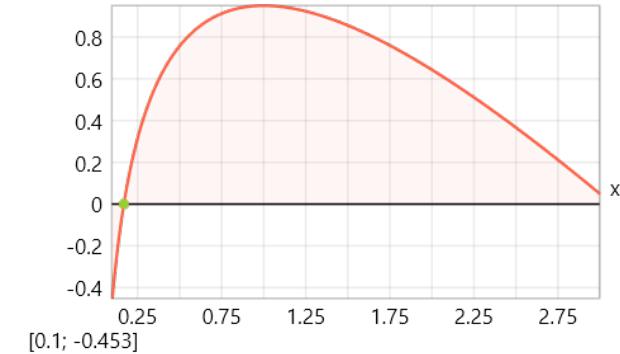


$$x_{87} = \$\text{Root}\{f_{87}(x)=0; x \in [0.6; 2]\} = 0.988$$

Logarithmic function

$$f_{88}(x) = \ln(x) - x + 2 - 0.05$$

[3; 0.95]

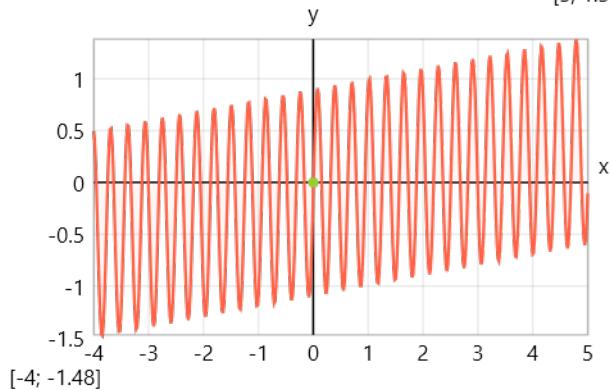


$$x_{88} = \$\text{Root}\{f_{88}(x)=0; x \in [0.1; 3]\} = 0.168$$

High oscillation function

$$f_{89}(x) = \sin(20 \cdot x) + 0.1 \cdot x - 0.1$$

[5; 1.38]

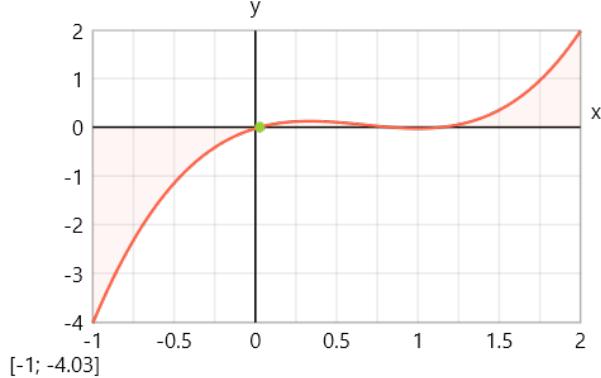


$$x_{89} = \$\text{Root}\{f_{89}(x)=0; x \in [-4; 5]\} = -0.794$$

Function with very flat region

$$f_{90}(x) = x \cdot x \cdot x - 2 \cdot x \cdot x + x - 0.025$$

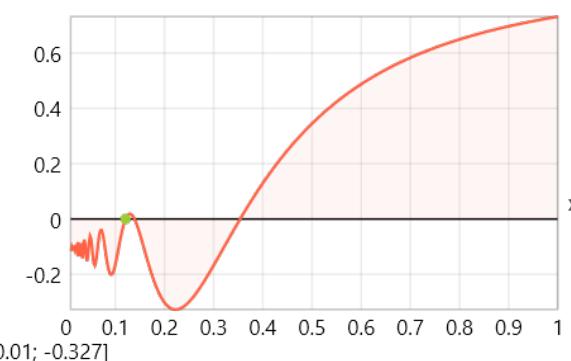
[2; 1.98]



$$x_{90} = \$\text{Root}\{f_{90}(x)=0; x \in [-1; 2]\} = 0.0264$$

Bessel-like function - $f_{91}(x) = x \cdot \sin\left(\frac{1}{x}\right) - 0.1 - 0.01$

[1; 0.731]



$$x_{91} = \$\text{Root}\{f_{91}(x)=0; x \in [0.01; 1]\} = 0.121$$