Proof of Inner Approximation in

"Coordinated Scheduling of Multiple Frequency Services in Electricity-Gas-Hydrogen Systems Based on Federated Warm Starts"

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This document intends to prove the inner approximation nature of the linear inequality constraint used to reformulate the Naidr frequency requirement.

1. Directly solve the differential swing equation

$$\frac{2\mathrm{H}^t}{f_0}\frac{\mathrm{d}\Delta f(\tau)}{\mathrm{d}\tau} + D\Delta f(\tau) = \Delta P^t_{\mathrm{adj}}(\tau) - \Delta P^t_{\mathrm{dis}}. \tag{S1}$$

   in the range of $[T_{\mathrm{DB}} + T^{\mathrm{Ele}}_{\mathrm{DL}}, T_{\mathrm{DB}} + T^{\mathrm{Gen}}_{\mathrm{DL}}]$ where the frequency nadir resides, which leads to:

$$\begin{aligned}\Delta f(\tau) =& \Delta f_{\mathrm{DB}} + \frac{1}{D}(\mathrm{e}^{-\frac{f_0 D}{2\mathrm{H}^t}(\tau - T_{\mathrm{DB}})} - 1)(\Delta P^t_{\mathrm{dis}} + \frac{2\mathrm{H}^t R^t_{\mathrm{Gen}}}{f_0 D T^{\mathrm{Gen}}_{\mathrm{DL}}}) + \frac{R^t_{\mathrm{Gen}}}{D T^{\mathrm{Gen}}_{\mathrm{DL}}}(\tau - T_{\mathrm{DB}}) \\ &+ \frac{R^t_{\mathrm{Ele}}}{D}[1 + \frac{2\mathrm{H}^t}{D f_0 T_{\mathrm{Ele}}}(\mathrm{e}^{-\frac{D f_0}{2\mathrm{H}^t}(\tau - T_{\mathrm{DB}})} - \mathrm{e}^{-\frac{D f_0}{2\mathrm{H}^t}(\tau - T_{\mathrm{DB}} - T_{\mathrm{Ele}})})]\end{aligned} \tag{S2}$$

2. The time τ where the frequency nadir is reached can be solved by setting $\frac{\mathrm{d}\Delta f(\tau)}{\mathrm{d}\tau}$=0:

$$T_{\mathrm{Nadir}} = T_{\mathrm{DB}} + \frac{T_{\mathrm{Gen}}}{R^t_{\mathrm{Gen}}}[\Delta P^t_{\mathrm{dis}} - R^t_{\mathrm{Ele}} - D(|\Delta f_{\mathrm{Nadir}} - \Delta f_{\mathrm{DB}}|)] \tag{S3}$$

3. One can easily see from Eq. (S2) that the value of $\Delta f(\tau)$ monotonically increases with R^t_{Gen}. This means that the minimum R^t_{Gen} required for meeting the frequency nadir requirements can be resolved by setting the left-hand side of Eq. (S2) to $\Delta f^{\max}_{\mathrm{Nadir}}$. This gives us an implicit function $R^t_{\mathrm{Gen}}(D)$:

$$(\mathrm{e}^{-\gamma} - 1)(\Delta P^t_{\mathrm{dis}} + \frac{2\mathrm{H}^t R^t_{\mathrm{Gen}}}{D f_0 T_{\mathrm{Gen}}}) + \Delta P^t_{\mathrm{dis}} + \frac{2\mathrm{H}^t R^t_{\mathrm{Ele}}}{D f_0 T_{\mathrm{Ele}}}\mathrm{e}^{-\gamma}(1 + \mathrm{e}^{\frac{D f_0}{2\mathrm{H}^t}T_{\mathrm{Ele}}}) = 0 \tag{S4}$$

   where

$$\gamma = (\Delta P^t_{\mathrm{dis}} - R^t_{\mathrm{Ele}} - D|\Delta f^{\max}_{\mathrm{Nadir}} - \Delta f_{\mathrm{DB}}|)\frac{D f_0 T_{\mathrm{Gen}}}{2\mathrm{H}^t R^t_{\mathrm{Gen}}} \tag{S5}$$

4. When we plot the implicit function $R^t_{\mathrm{Gen}}(D)$ in Fig. S1, we find that the feasible region (gray shadow) is convex and in the form of an epigraph. This means that a linear approximation (dashed line) would create an inner-approximation of the feasible region, which is conservative and beneficial to the secure operation of power system.
5. The remaining task is to determine the mathematical formula for the linear approximation, which can be uniquely determined through Points A and B. Point A falls back to the point

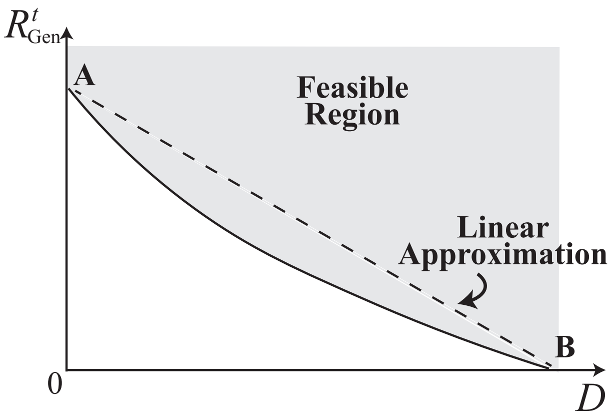

Figure S1: Feasible region of the Nadir constraint (gray shadow), as well as its linear approximation (dashed line).

where there is no damping, which can be derived as:

$$
\begin{aligned}
(\frac{\mathrm{H}^t}{f_0} - \frac{R^t_{\mathrm{Ele}} T^{\mathrm{Ele}}_{\mathrm{DL}}}{4|\Delta f^{\max}_{\mathrm{Nadir}}|} + \frac{T_{\mathrm{DB}} \Delta P^t_{\mathrm{dis}}}{2|\Delta f^{\max}_{\mathrm{Nadir}}|}) R^t_{\mathrm{Gen,A}} &= \frac{T^{\mathrm{Gen}}_{\mathrm{DL}} (\Delta P^t_{\mathrm{dis}} - R^t_{\mathrm{Ele}})^2}{4|\Delta f^{\max}_{\mathrm{Nadir}}|}. \\
R^t_{\mathrm{Gen,A}} &= \frac{T^{\mathrm{Gen}}_{\mathrm{DL}} f_0 (\Delta P^t_{\mathrm{dis}} - R^t_{\mathrm{Ele}})^2}{4\mathrm{H}^t |\Delta f^{\max}_{\mathrm{Nadir}}|}
\end{aligned}
\tag{S6}
$$

Point B can be determined from Eq. (S5), where γ should still be bounded when the denominator R^t_{Gen} approaches zero:

$$
\begin{aligned}
\Delta P^t_{\mathrm{dis}} - R^t_{\mathrm{Ele}} - D_{\mathrm{B}} |\Delta f^{\max}_{\mathrm{Nadir}} - \Delta f_{\mathrm{DB}}| &= 0 \\
D_{\mathrm{B}} &= \frac{P^t_{\mathrm{dis}} - R^t_{\mathrm{Ele}}}{|\Delta f^{\max}_{\mathrm{Nadir}} - \Delta f_{\mathrm{DB}}|}
\end{aligned}
\tag{S7}
$$

6. The rest of the deduction for the linear approximation of is self-explanatory, which can be derived as:

$$
(\frac{\mathrm{H}^t}{f_0} - \frac{R^t_{\mathrm{Ele}} T^{\mathrm{Ele}}_{\mathrm{DL}}}{4|\Delta f^{\max}_{\mathrm{Nadir}}|} + \frac{T_{\mathrm{DB}} \Delta P^t_{\mathrm{dis}}}{2|\Delta f^{\max}_{\mathrm{Nadir}}|}) R^t_{\mathrm{Gen}} \geq \frac{T^{\mathrm{Gen}}_{\mathrm{DL}} (\Delta P^t_{\mathrm{dis}} - R^t_{\mathrm{Ele}})^2}{4|\Delta f^{\max}_{\mathrm{Nadir}}|} - \frac{(\Delta P^t_{\mathrm{dis}} - R^t_{\mathrm{Ele}}) D T^{\mathrm{Gen}}_{\mathrm{DL}}}{4}. \tag{S8}
$$

which is the same as the linear approximation inequality Eq. (47) in the original manuscript.