

b. The x values in the data are between 65 and 75. Ninety is outside of the domain of the observed x values in the data (independent variable), so you cannot reliably predict the final exam score for this student. (Even though it is possible to enter 90 into the equation for x and calculate a corresponding y value, the y value that you get will not be reliable.) To understand really how unreliable the prediction can be outside of the observed x values observed in the data, make the substitution $x = 90$ into the equation.

$$\hat{y} = -173.51 + 4.83(90) = 261.19$$

The final-exam score is predicted to be 261.19. The largest the final-exam score can be is 200.

NOTE

The process of predicting inside of the observed x values observed in the data is called **interpolation**. The process of predicting outside of the observed x values observed in the data is called **extrapolation**.



TRY IT 12.11

Data are collected on the relationship between the number of hours per week practicing a musical instrument and scores on a math test. The line of best fit is as follows:

$$\hat{y} = 72.5 + 2.8x$$

What would you predict the score on a math test would be for a student who practices a musical instrument for five hours a week?

12.6 Outliers

In some data sets, there are values (**observed data points**) called **outliers**. **Outliers are observed data points that are far from the least squares line.** They have large "errors", where the "error" or residual is the vertical distance from the line to the point.

Outliers need to be examined closely. Sometimes, for some reason or another, they should not be included in the analysis of the data. It is possible that an outlier is a result of erroneous data. Other times, an outlier may hold valuable information about the population under study and should remain included in the data. The key is to examine carefully what causes a data point to be an outlier.

Besides outliers, a sample may contain one or a few points that are called **influential points**. Influential points are observed data points that are far from the other observed data points in the horizontal direction. These points may have a big effect on the slope of the regression line. To begin to identify an influential point, you can remove it from the data set and see if the slope of the regression line is changed significantly.

Computers and many calculators can be used to identify outliers from the data. Computer output for regression analysis will often identify both outliers and influential points so that you can examine them.

Identifying Outliers

We could guess at outliers by looking at a graph of the scatterplot and best fit-line. However, we would like some guideline as to how far away a point needs to be in order to be considered an outlier. **As a rough rule of thumb, we can flag any point that is located further than two standard deviations above or below the best-fit line as an outlier.** The standard deviation used is the standard deviation of the residuals or errors.

We can do this visually in the scatter plot by drawing an extra pair of lines that are two standard deviations above and below the best-fit line. Any data points that are outside this extra pair of lines are flagged as potential outliers. Or we can do this numerically by calculating each residual and comparing it to twice the standard deviation. On the TI-83, 83+, or 84+, the graphical approach is easier. The graphical procedure is shown first, followed by the numerical calculations. You would generally need to use only one of these methods.

EXAMPLE 12.12**Problem**

In the [third exam/final exam example](#), you can determine if there is an outlier or not. If there is an outlier, as an exercise, delete it and fit the remaining data to a new line. For this example, the new line ought to fit the remaining data better. This means the **SSE** should be smaller and the correlation coefficient ought to be closer to 1 or -1.

Solution**Graphical Identification of Outliers**

With the TI-83, 83+, 84+ graphing calculators, it is easy to identify the outliers graphically and visually. If we were to measure the vertical distance from any data point to the corresponding point on the line of best fit and that distance were equal to 2s or more, then we would consider the data point to be "too far" from the line of best fit. We need to find and graph the lines that are two standard deviations below and above the regression line. Any points that are outside these two lines are outliers. We will call these lines Y2 and Y3:

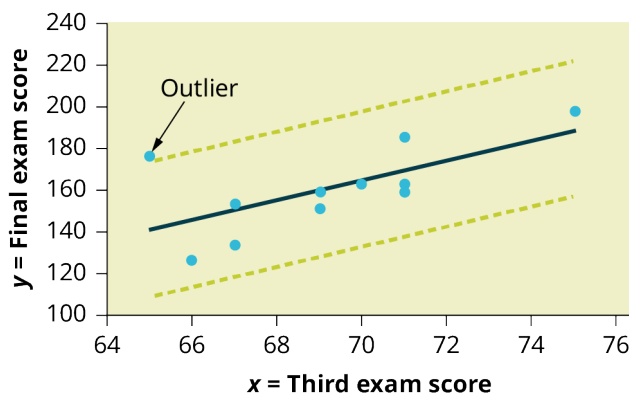
As we did with the equation of the regression line and the correlation coefficient, we will use technology to calculate this standard deviation for us. Using the **LinRegTTest** with this data, scroll down through the output screens to find **s = 16.412**.

Line Y2 = $-173.5 + 4.83x - 2(16.4)$ and line Y3 = $-173.5 + 4.83x + 2(16.4)$

where $\hat{y} = -173.5 + 4.83x$ is the line of best fit. Y2 and Y3 have the same slope as the line of best fit.

Graph the scatterplot with the best fit line in equation Y1, then enter the two extra lines as Y2 and Y3 in the "Y=" equation editor and press ZOOM 9. You will find that the only data point that is not between lines Y2 and Y3 is the point $x = 65$, $y = 175$. On the calculator screen it is just barely outside these lines. The outlier is the student who had a grade of 65 on the third exam and 175 on the final exam; this point is further than two standard deviations away from the best-fit line.

Sometimes a point is so close to the lines used to flag outliers on the graph that it is difficult to tell if the point is between or outside the lines. On a computer, enlarging the graph may help; on a small calculator screen, zooming in may make the graph clearer. Note that when the graph does not give a clear enough picture, you can use the numerical comparisons to identify outliers.

**Figure 12.20****TRY IT 12.12**

Identify the potential outlier in the scatter plot. The standard deviation of the residuals or errors is approximately 8.6.

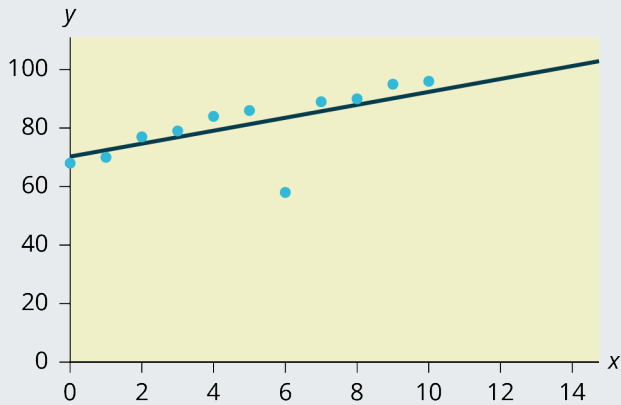


Figure 12.21

Numerical Identification of Outliers

In Table 12.6, the first two columns are the third-exam and final-exam data. The third column shows the predicted \hat{y} values calculated from the line of best fit: $\hat{y} = -173.5 + 4.83x$. The residuals, or errors, have been calculated in the fourth column of the table: observed y value-predicted y value = $y - \hat{y}$.

s is the standard deviation of all the $y - \hat{y} = \epsilon$ values where n = the total number of data points. If each residual is calculated and squared, and the results are added, we get the SSE. The standard deviation of the residuals is calculated from the SSE as:

$$s = \sqrt{\frac{SSE}{n - 2}}$$

NOTE

We divide by $(n - 2)$ because the regression model involves two estimates.

Rather than calculate the value of s ourselves, we can find s using the computer or calculator. For this example, the calculator function LinRegTTest found $s = 16.4$ as the standard deviation of the residuals 35; -17; 16; -6; -19; 9; 3; -1; -10; -9; -1 .

x	y	\hat{y}	$y - \hat{y}$
65	175	140	$175 - 140 = 35$
67	133	150	$133 - 150 = -17$
71	185	169	$185 - 169 = 16$
71	163	169	$163 - 169 = -6$
66	126	145	$126 - 145 = -19$
75	198	189	$198 - 189 = 9$
67	153	150	$153 - 150 = 3$
70	163	164	$163 - 164 = -1$

Table 12.6

x	y	\hat{y}	$y - \hat{y}$
71	159	169	$159 - 169 = -10$
69	151	160	$151 - 160 = -9$
69	159	160	$159 - 160 = -1$

Table 12.6

We are looking for all data points for which the residual is greater than $2s = 2(16.4) = 32.8$ or less than -32.8 . Compare these values to the residuals in column four of the table. The only such data point is the student who had a grade of 65 on the third exam and 175 on the final exam; the residual for this student is 35.

How does the outlier affect the best fit line?

Numerically and graphically, we have identified the point (65, 175) as an outlier. We should re-examine the data for this point to see if there are any problems with the data. If there is an error, we should fix the error if possible, or delete the data. If the data is correct, we would leave it in the data set. For this problem, we will suppose that we examined the data and found that this outlier data was an error. Therefore we will continue on and delete the outlier, so that we can explore how it affects the results, as a learning experience.

Compute a new best-fit line and correlation coefficient using the ten remaining points:

On the TI-83, TI-83+, TI-84+ calculators, delete the outlier from L1 and L2. Using the LinRegTTest, the new line of best fit and the correlation coefficient are:

$$\hat{y} = -355.19 + 7.39x \text{ and } r = 0.9121$$

The new line with $r = 0.9121$ is a stronger correlation than the original ($r = 0.6631$) because $r = 0.9121$ is closer to one. This means that the new line is a better fit to the ten remaining data values. The line can better predict the final exam score given the third exam score.

Numerical Identification of Outliers: Calculating s and Finding Outliers Manually

If you do not have the function LinRegTTest, then you can calculate the outlier in the first example by doing the following.

First, **square each $|y - \hat{y}|$**

The squares are 35^2 ; 17^2 ; 16^2 ; 6^2 ; 19^2 ; 9^2 ; 3^2 ; 1^2 ; 10^2 ; 9^2 ; 1^2

Then, add (sum) all the $|y - \hat{y}|$ squared terms using the formula

$$\sum_{i=1}^{11} (|y_i - \hat{y}_i|)^2 = \sum_{i=1}^{11} \epsilon_i^2 \text{ (Recall that } y_i - \hat{y}_i = \epsilon_i.)$$

$$= 35^2 + 17^2 + 16^2 + 6^2 + 19^2 + 9^2 + 3^2 + 1^2 + 10^2 + 9^2 + 1^2$$

$$= 2440 = \text{SSE. The result, SSE is the Sum of Squared Errors.}$$

Next, calculate s , the standard deviation of all the $y - \hat{y} = \epsilon$ values where n = the total number of data points.

$$\text{The calculation is } s = \sqrt{\frac{\text{SSE}}{n-2}}.$$

$$\text{For the third exam/final exam problem, } s = \sqrt{\frac{2440}{11-2}} = 16.47.$$

Next, multiply s by 2:

$$(2)(16.47) = 32.94$$

32.94 is 2 standard deviations away from the mean of the $y - \hat{y}$ values.

If we were to measure the vertical distance from any data point to the corresponding point on the line of best fit and that distance is at least $2s$, then we would consider the data point to be "too far" from the line of best fit. We call that point a **potential outlier**.

For the example, if any of the $|y - \hat{y}|$ values are **at least** 32.94, the corresponding (x, y) data point is a potential outlier.

For the third exam/final exam problem, all the $|y - \hat{y}|$'s are less than 31.29 except for the first one which is 35.

$35 > 31.29$ That is, $|y - \hat{y}| \geq 2(s)$

The point which corresponds to $|y - \hat{y}| = 35$ is (65, 175). **Therefore, the data point (65,175) is a potential outlier.** For this example, we will delete it. (Remember, we do not always delete an outlier.)

NOTE

When outliers are deleted, the researcher should either record that data was deleted, and why, or the researcher should provide results both with and without the deleted data. If data is erroneous and the correct values are known (e.g., student one actually scored a 70 instead of a 65), then this correction can be made to the data.

The next step is to compute a new best-fit line using the ten remaining points. The new line of best fit and the correlation coefficient are:

$$\hat{y} = -355.19 + 7.39x \text{ and } r = 0.9121$$

EXAMPLE 12.13

? Problem

Using this new line of best fit (based on the remaining ten data points in the [third exam/final exam example](#)), what would a student who receives a 73 on the third exam expect to receive on the final exam? Is this the same as the prediction made using the original line?

✓ Solution

Using the new line of best fit, $\hat{y} = -355.19 + 7.39(73) = 184.28$. A student who scored 73 points on the third exam would expect to earn 184 points on the final exam.

The original line predicted $\hat{y} = -173.51 + 4.83(73) = 179.08$ so the prediction using the new line with the outlier eliminated differs from the original prediction.



TRY IT 12.13

The data points for the graph from the [third exam/final exam example](#) are as follows: (1, 5), (2, 7), (2, 6), (3, 9), (4, 12), (4, 13), (5, 18), (6, 19), (7, 12), and (7, 21). Remove the outlier and recalculate the line of best fit. Find the value of \hat{y} when $x = 10$.

EXAMPLE 12.14

The Consumer Price Index (CPI) measures the average change over time in the prices paid by urban consumers for consumer goods and services. The CPI affects nearly all Americans because of the many ways it is used. One of its biggest uses is as a measure of inflation. By providing information about price changes in the Nation's economy to government, business, and labor, the CPI helps them to make economic decisions. The President, Congress, and the Federal Reserve Board use the CPI's trends to formulate monetary and fiscal policies. In the following table, x is the year and y is the CPI.

x	y	x	y
1915	10.1	1979	72.6
1926	17.7	1980	82.4
1935	13.7	1986	109.6
1940	14.7	1991	130.7
1947	24.1	1999	166.6
1952	26.5	2010	219.2
1964	31.0	2020	258
1969	36.7	2023	299.2
1975	49.3		

Table 12.7 Data

❓ **Problem**

- Draw a scatterplot of the data.
- Calculate the least squares line. Write the equation in the form $\hat{y} = a + bx$.
- Draw the line on the scatterplot.
- Find the correlation coefficient. Is it significant?
- What is the average CPI for the year 1990?

✓ **Solution**

- See [Figure 12.22](#).
- $\hat{y} = -3204 + 1.662x$ is the equation of the line of best fit.
- $r = 0.8694$
- The number of data points is $n = 14$. Use the 95% Critical Values of the Sample Correlation Coefficient table at the end of Chapter 12. $n - 2 = 12$. The corresponding critical value is 0.532. Since $0.8694 > 0.532$, r is significant.
 $\hat{y} = -3204 + 1.662(1990) = 103.4$ CPI
- Using the calculator LinRegTTest, we find that $s = 25.4$; graphing the lines $Y_2 = -3204 + 1.662X - 2(25.4)$ and $Y_3 = -3204 + 1.662X + 2(25.4)$ shows that no data values are outside those lines, identifying no outliers. (Note that the year 1999 was very close to the upper line, but still inside it.)

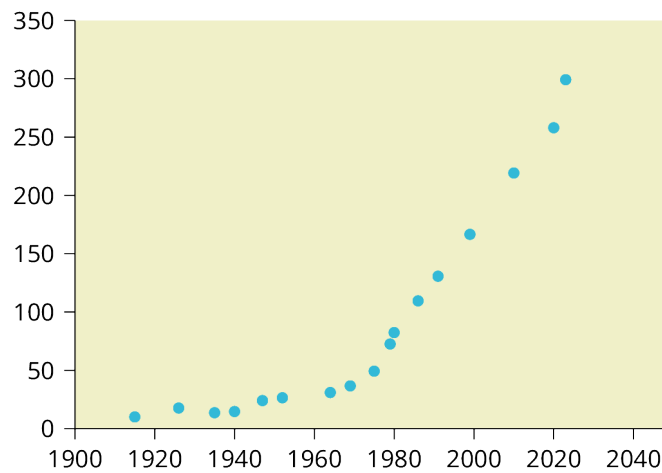


Figure 12.22

NOTE

In the example, notice the pattern of the points compared to the line. Although the correlation coefficient is significant, the pattern in the scatterplot indicates that a curve would be a more appropriate model to use than a line. In this example, a statistician should prefer to use other methods to fit a curve to this data, rather than model the data with the line we found. In addition to doing the calculations, it is always important to look at the scatterplot when deciding whether a linear model is appropriate.

If you are interested in seeing more years of data, visit the Bureau of Labor Statistics CPI website <ftp://ftp.bls.gov/pub/special.requests/cpi/cpiai.txt>; our data is taken from the column entitled "Annual Avg." (third column from the right). For example you could add more current years of data. Try adding the more recent years: 2010: CPI = 219.2; 2020: CPI = 258.0; 2023: CPI = 299.2. See how it affects the model. (Check: $\hat{y} = -5030 + 2.598x$; $r = 0.9067$. Is r significant? Is the fit better with the addition of the new points?)

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TRY IT 12.14

The following table shows economic development measured in per capita income PCINC.

Year	PCINC	Year	PCINC
1870	340	1920	1050
1880	499	1930	1170
1890	592	1940	1364
1900	757	1950	1836
1910	927	1960	2132

Table 12.8

- What are the independent and dependent variables?
- Draw a scatter plot.

- c. Use regression to find the line of best fit and the correlation coefficient.
- d. Interpret the significance of the correlation coefficient.
- e. Is there a linear relationship between the variables?
- f. Find the coefficient of determination and interpret it.
- g. What is the slope of the regression equation? What does it mean?
- h. Use the line of best fit to estimate PCINC for 1900, for 2000.
- i. Determine if there are any outliers.

95% Critical Values of the Sample Correlation Coefficient Table

Degrees of Freedom: $n - 2$	Critical Values: (+ and -)
1	0.997
2	0.950
3	0.878
4	0.811
5	0.754
6	0.707
7	0.666
8	0.632
9	0.602
10	0.576
11	0.555
12	0.532
13	0.514
14	0.497
15	0.482
16	0.468
17	0.456
18	0.444
19	0.433
20	0.423

Table 12.9

Degrees of Freedom: $n - 2$	Critical Values: (+ and -)
21	0.413
22	0.404
23	0.396
24	0.388
25	0.381
26	0.374
27	0.367
28	0.361
29	0.355
30	0.349
40	0.304
50	0.273
60	0.250
70	0.232
80	0.217
90	0.205
100	0.195

Table 12.9

12.7 Regression (Distance from School)



Stats Lab

Regression (Distance from School)

Class Time:

Names:

Student Learning Outcomes

- The student will calculate and construct the line of best fit between two variables.
- The student will evaluate the relationship between two variables to determine if that relationship is significant.

Collect the Data

Use eight members of your class for the sample. Collect bivariate data (distance an individual lives from school, the cost of supplies for the current term).