Benjamin Steinberg
Department of Mathematics
City College of New York
New York, New York
10031, USA
bsteinberg@ccny.cuny.edu

# The Representation Theory of Finite Monoids

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For Ariane and Nicholas

# Preface

This text aims to provide an introduction to the representation theory of finite monoids for advanced graduate students and researchers in algebra, algebraic combinatorics, automata theory and probability theory. As this is the first presentation of the theory in the form of a book, I have made no attempt to be encyclopedic. I also did not hesitate to make assumptions on the ground field that are stronger than strictly necessary in order to keep things simpler. Given that I am targeting a fairly broad audience, a lot of the technical jargon of semigroup theory is deliberately avoided to make the text accessible to as wide a readership as possible.

Much of the impetus behind recent research in the representation theory of finite monoids has come from applications to Markov chains, automata theory and combinatorics. I have devoted three chapters to some of these applications, which I think of as the apex of the text. The first of these chapters proves a theorem of Berstel and Reutenauer [BR90] on the rationality of zeta functions of cyclic regular languages and symbolic dynamical systems. The second is concerned with synchronization problems in automata theory and the Černý conjecture, and also touches on the subject of synchronizing permutation groups. The third of these chapters concerns applications of representation theory to probability theory, namely to Markov chains. It develops Brown's theory of left regular band random walks [Bro00, Bro04], as well as more recent results [Ste06, Ste08, ASST15b]. In particular, the eigenvalues of the transition matrices for the Tsetlin library, the riffle shuffle and the Ehrenfest urn model are computed using monoid representation theory. The author's previously unpublished, simple proof [Ste10a] of Brown's diagonalizability theorem for left regular band walks is provided. Some nice applications that I have not touched upon here is the work of Malandro and Rockmore on fast Fourier transforms for inverse monoids with applications to the spectral analysis of partially ranked data [MR10, Mal10, Mal13].

My viewpoint on the representation theory of finite monoids differs from the classical one found in [CP61, Chapter 5] and [LP69, McA71, RZ91, Okn91]. These authors worked very much with semigroup algebras (not necessarily unital) and explicit matrix representations. In particular, they relied heavily on computations in the algebra of a 0-simple semigroup, expressed as a Rees matrix semigroup (so-called Munn algebras). In my opinion, this both obscured what was really happening and, at the same time, rendered the material essentially inaccessible to non-specialists.

The approach here is highly module theoretic and follows the modern flavor of the theory of quasi-hereditary algebras [CPS88] and stratified algebras [CPS96]. Putcha should be credited for first making the connection between monoid representation theory and quasi-hereditary algebras [Put98]. The viewpoint taken here is most closely reflected in the literature in my joint work with Ganyushkin and Mazorchuk [GMS09]. See also the work of Kuhn [Kuh94b].

The background that is expected throughout the book is comfort with module theory (including categories, functors, tensor products, exact sequences and projective and injective modules) and familiarity with group representation theory in the non-modular setting and with the basics of Wedderburn theory (e.g., semisimple algebras and modules, radicals and composition series). Occasionally, some facts from the representation theory of symmetric groups are needed. Some of the later chapters required more advanced techniques from homological algebra and the theory of finite dimensional algebras. I do not expect the reader to have previously studied semigroup theory. A certain amount of mathematical maturity is required for the book. Some of the more technical algebraic results can be accepted as black boxes for those who are primarily interested in the applications.

The text is divided into seven parts and two appendices. The first part is a streamlined introduction to finite semigroup theory. No prior knowledge beyond elementary group theory is assumed and the approach is a departure from those of standard texts because I am primarily targeting a reader interested in monoids as a tool and not necessarily as an end in themselves. For instance, I have emphasized monoid actions in the proofs of a number of results in which they do not usually appear explicitly to highlight the similarity with ring theory. Chapter 1 develops the basic theory of monoids. This is followed by Chapter 2, which studies $\mathscr{R}$-trivial monoids, which play a key role in applications to Markov chains. Chapter 3 develops the theory of inverse monoids, which form another important class of examples because they capture partial symmetries of mathematical objects.

The second part forms the core material of the book. It gives a modern, module theoretic treatment of the Clifford-Munn-Ponizovskii theory connecting the irreducible representations of monoids with the irreducible representations of their maximal subgroups. The approach used here is based on the theory of idempotents, or what often goes under the name "recollement" [BBD82, CPS88, CPS96], connecting the representation theory of the algebras $A$, $eAe$ and $A/AeA$ for an idempotent $e$. Chapter 4 develops this theory in the abstract in more detail than can be found in, for instance, [Gre80, Chapter 6] or [Kuh94b]. This will be the most technically challenging of the essential chapters for readers who are not pure algebraists and it may be worth accepting the results without being overly concerned with the proofs. Chapter 5 introduces monoid algebras and representations of monoids and then applies the general theory to monoid algebras to deduce the fundamental structure theorem of Clifford-Munn-Ponizovskii for irreducible representations of finite monoids. We then provide Putcha's construction of the irreducible representations of the full transformation monoid as an example of the theory. Afterward, the classical result characterizing semisimplicity of monoid algebras (cf. [CP61, Chapter 5] or [Okn91]) is given a module theoretic proof that avoids entirely the dependence of earlier proofs on Munn algebras and non-unital (*a priori*) algebras. We then furnish the explicit description of the irreducible representations in terms of monomial representations (or Schützenberger representations) that can be found in [LP69, RZ91].

The third part of the book concerns character theory. We first study, in Chapter 6, the Grothendieck ring of a monoid algebra over an arbitrary field. The case of the complex field was studied by McAlister [McA72]. It is shown that the Grothendieck ring of the monoid algebra is isomorphic to the direct product of the Grothendieck rings of its maximal subgroups (one per regular $\mathscr{J}$-class). We also pay careful attention to the subring spanned by the one-dimensional simple modules. This subring encodes the triangularizable representations of a monoid and is important for applications to Markov chains. Chapter 7 briefly introduces Rota's theory of Möbius inversion for partially ordered sets [Rot64]. This is a fundamental technique in algebraic combinatorics. The poset of idempotents of a monoid plays a key role in monoid representation theory and Möbius inversion is an important tool for decoding the information hidden in the fixed subspaces of the idempotents under the representation. Chapter 8 is concerned with the character theory of monoids over the complex numbers as developed by McAlister [McA72] and, independently, Rhodes and Zalcstein [RZ91]. Although much of the theory could be developed over more general fields [MQS15], both the applications and the desire to keep things simple have led me to stick to this restricted setting. The key result here is that the character table is block upper triangular with group character tables as the diagonal blocks. This means that the character table is invertible. The inverse of the character table is a crucial tool for computing the composition factors of a module from its character.

The fourth part of the book is devoted to the representation theory of inverse monoids. Just as groups abstract the notion of permutation groups, inverse monoids abstract the notion of partial permutation monoids and hence form a very natural and important generalization of groups. Like groups, they have semisimple algebras in good characteristic. The approach taken here follows my papers [Ste06, Ste08]. Namely, it is shown that the algebra of an inverse monoid is, in fact, the algebra of an associated groupoid and hence is isomorphic to a direct product of matrix algebras over group algebras. The isomorphism is very explicit, using Möbius inversion, and so one can very efficiently convert group theoretic results into inverse monoid theoretic ones.

In more detail, Chapter 9 develops the basics of the representation theory of finite categories. This theory is essentially equivalent to the representation theory of finite monoids, as can be seen from the results of Webb and his school [Web07]. We explicitly state and prove here a parametrization of the simple modules for a finite category via Clifford-Munn-Ponizovskii theory. The theory is then restricted to groupoid algebras. The actual representation theory of inverse monoids is broached in Chapter 10. A highlight is a formula for decomposing a representation into its irreducible constituents from its character using only group character tables and the Möbius function of the lattice of idempotents.

In part five, I provide an exposition of the theory of the Rhodes radical. This is the congruence on a monoid corresponding to the direct sum of all irreducible representations. It was first studied over the complex numbers by Rhodes [Rho69b], whence the name, and in general by Almeida, Margolis, the author and Volkov [AMSV09]. The Rhodes radical is crucial for determining which monoids have only one-dimensional irreducible representations, or equivalently, have basic algebras. These are precisely the monoids admitting a faithful upper triangular matrix representation. Such monoids are the ones whose representation theory is most useful for analyzing Markov chains. With a heavy heart, I have decided to restrict my attention primarily to the characteristic zero setting. The results in positive characteristic are stated without proof. The reason for this choice is that the applications mostly involve fields of characteristic zero and also the characteristic $p > 0$ case requires a bit more structural semigroup theory. The treatment here differs from [AMSV09] (and in some sense has quite a bit in common with [Rho69b]) in that we take advantage of the bialgebra structure on a monoid algebra and R. Steinberg's lovely theorem [Ste62] on the tensor powers of a faithful representation. Although we do not use the language of bialgebras explicitly, Chapter 11 essentially develops the necessary bialgebraic machinery to prove Steinberg's theorem following the approach of Rieffel [Rie67] (see also [PQ95, Pas14]). Chapter 12 then develops the theory of the Rhodes radical in connection with nilpotent bi-ideals. The characterization of monoids triangularizable over an algebraically closed field of characteristic zero, or equivalently having basic algebras, is given. Here, many of the lines of investigation developed throughout the text are finally interwoven.

Part six consists of the three chapters on applications that I have already discussed previously. They are Chapter 13 on the rationality of zeta functions of cyclic regular languages and symbolic dynamical systems, Chapter 14 on transformation monoids and applications to automata theory and Chapter 15 on Markov chains.

The seventh, and final, part concerns advanced topics. These chapters expect the reader to be familiar with some homological algebra, in particular, with the Ext-functor. Chapter 16 characterizes von Neumann regular monoids with a self-injective algebra, which we believe to be a new result. Chapter 17 proves Nico's upper bound on the global dimension of the algebra of a regular monoid [Nic71, Nic72]. Our approach is inspired by Putcha's observation [Put98] that regular monoids have quasi-hereditary algebras and the machinery of quasi-hereditary algebras [CPS88, DR89]. In Chapter 18, we recall the notion of quivers and their role in modern representation theory. We then provide methods for computing the Gabriel quiver of the algebra of a left regular band and of a $\mathscr{J}$-trivial monoid. These are results of Saliola [Sal07] and Denton, Hivert, Schilling and Thiéry [DHST11], respectively. They are, in fact, special cases of more general results of Margolis and myself, obtained using Hochschild cohomology [MS12a], that we do not present in the current text. The quivers of the algebras of several natural families of monoids are computed as examples. Chapter 19 is a survey, without proofs, of some further developments not covered in the book.

The appendices cover background material concerning finite dimensional algebras and group representation theory. The purpose of the appendices is two-fold: both to review some concepts and results with which we hope that the reader is already familiar or willing to accept and to provide in one location specific statements that we can reference throughout the book. No attempt is made here to be self-contained: proofs are mostly omitted, but references are given to standard texts. Appendix A reviews those elements of the theory of finite dimensional algebras that are used throughout. Most of the book uses nothing more than Wedderburn theory and the Jordan-Hölder theorem. Primitive idempotents are occasionally used. Morita theory and duality appear at a couple of points in the text, most of which can be skipped by readers primarily interested in applications. Projective indecomposable modules and projective covers are only used in the final chapters. Appendix B reviews group representation theory, again without many proofs. Modular representation theory does not appear at all and I have mostly restricted character theory to the field of complex numbers. The last section surveys aspects of the representation theory of the symmetric group in characteristic zero.

As always, I am indebted to John Rhodes who taught me virtually everything I know about finite semigroups and who first interested me in semigroup representation theory. This book is also very much influenced by my collaborators Stuart Margolis, Volodymyr Mazorchuk, Franco Saliola, Anne Schilling and Nicolas Thiéry. Mohan Putcha has been a continuing inspiration to my viewpoint on semigroup representation theory. Discussions with Nicholas Kuhn and Lex Renner have also enlightened me. A wonderful conversation with Persi Diaconis at Banff encouraged my, then, nascent interest in applications of monoid representation theory to Markov chain theory. The following people have provided invaluable feedback concerning the text, itself: Stuart Margolis, Volodymyr Mazorchuk, Don McAlister, Mohan Putcha, Christophe Reutenauer, Anne Schilling and Peter Webb.

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# Contents

**Preface** ........ vii

**List of Figures** ........ xix

**Introduction** ........ 1

**Part I Monoid Theory**

**1 The Structure Theory of Finite Monoids** ........ 5
- 1.1 Basic notions ........ 5
- 1.2 Cyclic semigroups ........ 6
- 1.3 The ideal structure and Green's relations ........ 8
- 1.4 von Neumann regularity ........ 13
- 1.5 Exercises ........ 15

**2 $\mathscr{R}$-trivial Monoids** ........ 19
- 2.1 Lattices and prime ideals ........ 19
- 2.2 $\mathscr{R}$-trivial monoids and left regular bands ........ 22
- 2.3 Exercises ........ 24

**3 Inverse Monoids** ........ 25
- 3.1 Defintions, examples and structure ........ 25
- 3.2 Exercises ........ 31

**Part II Irreducible Representations**

**4 Recollement: The Theory of an Idempotent** ........ 35
- 4.1 A miscellany of functors ........ 35
- 4.2 Idempotents and simple modules ........ 40
- 4.3 Exercises ........ 43

**5 Irreducible Representations** . . . . . 45
5.1 Monoid algebras and representations . . . . . 45
5.2 Clifford-Munn-Ponizovskii theory . . . . . 47
5.3 The irreducible representations of the full transformation monoid . . . . . 51
5.4 Semisimplicity . . . . . 56
5.5 Monomial representations . . . . . 61
5.6 Exercises . . . . . 64

**Part III Character Theory**

**6 The Grothendieck Ring** . . . . . 69
6.1 The Grothendieck ring . . . . . 69
6.2 The restriction isomorphism . . . . . 72
6.3 The triangular Grothendieck ring . . . . . 74
6.4 The Grothendieck group of projective modules . . . . . 74
6.5 Exercises . . . . . 76

**7 Incidence Algebras and Möbius Inversion** . . . . . 77
7.1 The incidence algebra of a poset . . . . . 77
7.2 Exercises . . . . . 79

**8 Characters and Class Functions** . . . . . 81
8.1 Class functions and generalized conjugacy classes . . . . . 81
8.2 Character theory . . . . . 84
8.3 The character table of the full transformation monoid . . . . . 93
8.4 The Burnside-Brauer theorem . . . . . 94
8.4.1 Tensor powers . . . . . 94
8.4.2 Symmetric powers . . . . . 96
8.5 The Cartan matrix . . . . . 98
8.6 Exercises . . . . . 101

**Part IV The Representation Theory of Inverse Monoids**

**9 Categories and Groupoids** . . . . . 105
9.1 Categories . . . . . 105
9.2 Groupoids . . . . . 111
9.3 Exercises . . . . . 113

**10 Representation Theory of Inverse Monoids** . . . . . 115
10.1 The groupoid of an inverse monoid . . . . . 115
10.2 The isomorphism of algebras . . . . . 116
10.3 Decomposing representations of inverse monoids . . . . . 119
10.4 The character table of the symmetric inverse monoid . . . . . 125

10.5 Exercises . . . . . 128

**Part V The Rhodes Radical**

**11 Bi-ideals and Steinberg's Theorem** . . . . . 133
11.1 Annihilators of tensor products . . . . . 133
11.2 Bi-ideals and Steinberg's theorem . . . . . 134
11.3 Exercises . . . . . 139

**12 The Rhodes Radical** . . . . . 141
12.1 The Rhodes radical and nilpotent bi-ideals . . . . . 141
12.2 Triangularizable monoids and basic algebras . . . . . 145
12.3 Exercises . . . . . 148

**Part VI Applications**

**13 Zeta Functions of Languages and Dynamical Systems** . . . . . 153
13.1 Zeta functions . . . . . 153
13.2 Rationality of the zeta function of a cyclic regular language . . . 156
13.3 Computing the zeta function . . . . . 158
13.3.1 Edge shifts . . . . . 159
13.3.2 The even shift . . . . . 160
13.3.3 The Ihara zeta function . . . . . 162
13.4 Exercises . . . . . 163

**14 Transformation Monoids** . . . . . 165
14.1 Transformation monoids . . . . . 165
14.2 Transformation modules . . . . . 166
14.3 The Černý conjecture . . . . . 170
14.4 Exercises . . . . . 177

**15 Markov Chains** . . . . . 179
15.1 Markov Chains . . . . . 179
15.2 Random walks . . . . . 180
15.3 Examples . . . . . 181
15.3.1 The Tsetlin library . . . . . 182
15.3.2 The inverse riffle shuffle . . . . . 183
15.3.3 Ehrenfest urn model . . . . . 186
15.4 Eigenvalues . . . . . 187
15.5 Diagonalizability . . . . . 189
15.6 Examples: revisited . . . . . 192
15.6.1 The Tsetlin library . . . . . 192
15.6.2 The inverse riffle shuffle . . . . . 193
15.6.3 The Ehrenfest urn model . . . . . 194

15.7 Exercises . . . . . 196

**Part VII Advanced Topics**

**16 Self-injective, Frobenius and Symmetric Algebras** . . . . . 199
16.1 Background on self-injective algebras . . . . . 199
16.2 Regular monoids with self-injective algebras . . . . . 200
16.3 Exercises . . . . . 203

**17 Global Dimension** . . . . . 205
17.1 Idempotent ideals and homological algebra . . . . . 205
17.2 Global dimension and homological properties of regular monoids . . . . . 209
17.3 Exercises . . . . . 213

**18 Quivers of Monoid Algebras** . . . . . 215
18.1 Quivers of algebras . . . . . 215
18.2 Projective indecomposable modules for $\mathscr{R}$-trivial monoid algebras . . . . . 216
18.3 The quiver of a left regular band . . . . . 221
18.4 The quiver of a $\mathscr{J}$-trivial monoid . . . . . 223
18.5 Sample quiver computations . . . . . 224
18.5.1 Left regular bands . . . . . 224
18.5.2 $\mathscr{J}$-trivial monoids . . . . . 226
18.6 Exercises . . . . . 230

**19 Further Developments** . . . . . 233
19.1 Monoids of Lie type . . . . . 233
19.2 The representation theory of the full transformation monoid . . . 234
19.3 The representation theory of left regular bands . . . . . 235
19.4 The Burnside problem for linear monoids . . . . . 238

**A Finite Dimensional Algebras** . . . . . 239
A.1 Semisimple modules and algebras . . . . . 239
A.2 Indecomposable modules . . . . . 243
A.3 Idempotents . . . . . 245
A.4 Duality and Morita equivalence . . . . . 247

**B Group Representation Theory** . . . . . 249
B.1 Group algebras . . . . . 249
B.2 Group character theory . . . . . 250
B.3 Permutation modules . . . . . 252
B.4 The representation theory of the symmetric group . . . . . 255
B.5 Exercises . . . . . 257

**References** . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 259

**Index** . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 271

# List of Figures

13.1 The even shift . . . . . . . . . . 161

14.1 The Černý example for $n = 4$ . . . . . . . . . . 171
14.2 An Eulerian automaton . . . . . . . . . . 174

B.1 The Young diagram of $\lambda = (3, 3, 2, 1)$ . . . . . . . . . . 255
B.2 Some standard Young tableaux of shape $(3, 2, 1)$ . . . . . . . . . . 256
B.3 A horizontal strip . . . . . . . . . . 256

# Introduction

To do.

# Part I

# Monoid Theory

# 1

# The Structure Theory of Finite Monoids

This chapter contains those elements of the structure theory of finite monoids that we shall need for the remaining chapters. It also establishes some notation that will be used throughout. More detailed sources for finite semigroup theory include [KRT68, Eil76, Lal79, Alm94, RS09]. Introductory books on the algebraic theory of semigroups, in general, are [CP61, CP67, Hig92, How95]. In this book, all semigroups and monoids will be finite except for endomorphism monoids of vector spaces and free monoids. On a first reading, it may be advisable to skip the proofs in this chapter.

## 1.1 Basic notions

A *semigroup* is a set $S$, possibly empty, with an associative binary operation (usually called *multiplication* and denoted multiplicatively by juxtaposition or by "$\cdot$"). A *monoid* is a semigroup $M$ with an *identity* element, usually written as 1. The identity is necessarily unique. In particular, a monoid is non-empty. We shall primarily be interested here in monoids, but sometimes we shall also need to work with semigroups. A *subsemigroup* of a semigroup is a subset, possibly empty, which is closed under multiplication. A *submonoid* of a monoid is a subsemigroup containing the identity. It is traditional in semigroup theory to call a subsemigroup $G$ of a semigroup $S$, which is a group with respect to the induced binary operation, a *subgroup*. We do *not* require the identity of $G$ to coincide with the identity of $S$ in the case that $S$ is a monoid. This is because most of the subgroups of a monoid that arise in practice will not share its identity. In fact, a recurrent theme in this text is that a finite monoid is a collection of finite groups loosely tied together by a partially ordered set.

An element $m \in M$ is called a *unit* if there exists $m^{-1} \in M$ such that $mm^{-1} = 1 = m^{-1}m$. The units form a group called the *group of units* of $M$. The group of units of $M$ is the unique maximal subgroup of $M$ which is also a submonoid.

An element $z$ of a semigroup $S$ is called a *zero element* (or simply a *zero*) if $sz = z = zs$ for all $s \in S$. It is easy to check that zero elements, if they exist, are unique. Often, a zero element is denoted by 0. However, we shall typically avoid this practice in this text to avoid confusion between the zero of a monoid and its monoid algebra.

As usual, if $S$ is a semigroup and $A, B \subseteq S$, then

$$AB = \{ab \mid a \in A, b \in B\}\,.$$

If $S$ is a semigroup, then $S^{op}$ denotes the *opposite semigroup*. It has the same underlying set as $S$, but the binary operation $*$ is given by $s * t = ts$.

A *homomorphism* $\varphi\colon M \longrightarrow N$ of monoids is a mapping such that $\varphi(1) = 1$ and $\varphi(m_1m_2) = \varphi(m_1)\varphi(m_2)$ for all $m_1, m_2 \in M$. Semigroup homomorphisms are defined analogously. A *congruence* on a monoid $M$ is an equivalence relation $\equiv$ such that $m_1 \equiv m_2$ implies $um_1v \equiv um_2v$ for all $u, v \in M$. It is straightforward to verify that $M/{\equiv}$ then becomes a monoid with respect to the product

$$[m_1]_{\equiv} \cdot [m_2]_{\equiv} = [m_1m_2]_{\equiv}$$

and that the projection $\pi\colon M \longrightarrow M/{\equiv}$ is a surjective homomorphism. If $\varphi\colon M \longrightarrow N$ is a homomorphism, then $\ker\varphi$ is the equivalence relation defined by $m_1 \ker\varphi\, m_2$ if $\varphi(m_1) = \varphi(m_2)$. One easily verifies that $\varphi(M) \cong M/\ker\varphi$.

If $S, T$ are semigroups, then their *direct product* $S \times T$ is a semigroup with coordinate-wise binary operation $(s,t)(s',t') = (ss', tt')$. The direct product of two monoids is again a monoid.

An element $e$ of a semigroup $S$ is an *idempotent* if $e = e^2$. The set of idempotents of a subset $X \subseteq S$ is denoted $E(X)$. Of course, $1 \in E(M)$ for any monoid $M$. If $e \in E(S)$, then $eSe$ is a monoid with respect to the binary operation inherited from $S$ with identity $e$ (because $eese = ese = esee$). The group of units of $eSe$ is denoted by $G_e$ and is called the *maximal subgroup* of $S$ at $e$. It is the unique subgroup with identity $e$ maximal with respect to containment, whence the name. We put $I_e = eSe \setminus G_e$. Note that if $M$ is a monoid, then $G_1$ is the group of units of $M$.

There is a *natural partial order* on the set $E(S)$ of idempotents of a semigroup $S$ defined by putting $e \leq f$ if $ef = e = fe$. Equivalently, $e \leq f$ if and only if $eSe \subseteq fSf$.

The reader should verify that the image of an idempotent under a homomorphism is always an idempotent.

## 1.2 Cyclic semigroups

This section describes the structure of a finite cyclic semigroup. These results will be used later when we study the character theory of a finite monoid.

Fix a finite semigroup $S$ and let $s \in S$. Then there exists a smallest positive integer $c$, called the *index* of $s$, such that $s^c = s^{c+d}$ for some $d > 0$. The smallest possible choice of $d$ is called the *period* of $s$. Notice that $s^c = s^{c+dq}$ for any $q \geq 0$.

**Proposition 1.1.** *Let $s \in S$ have index $c$ and period $d$. Then $s^i = s^j$ if and only if $i = j$ or $i, j \geq c$ and $i \equiv j \bmod d$.*

*Proof.* Without loss of generality, assume that $i < j$. If $i, j \geq c$ and $j = i + qd$ with $q > 0$, then $s^j = s^{c+qd+(i-c)} = s^{c+i-c} = s^i$. Conversely, if $s^i = s^j$ with $i < j$, then by definition of the index, we have $i \geq c$. Also if $j - i = qd + r$ with $0 \leq r < d$ and $q \geq 0$, then since $s^i = s^j$ we have that

$$s^{c+r} = s^{c+qd+r} = s^{c+j-i} = s^{c+id+j-i} = s^j s^{c+id-i} = s^i s^{c+id-i} = s^{c+id} = s^c.$$

By definition of the period, we conclude that $r = 0$ and so $i \equiv j \bmod d$. $\square$

Let $\langle s \rangle = \{s^n \mid n \geq 1\}$ be the *cyclic subsemigroup* generated by $s$. It follows from the proposition that $\langle s \rangle = \{s, s^2, \ldots, s^{c+d-1}\}$, where $c$ is the index of $s$ and $d$ is the period, and that these elements are distinct.

**Corollary 1.2.** *Let $s \in S$ have index $c$ and period $d$. Then the subsemigroup $C = \{s^n \mid n \geq c\}$ of $\langle s \rangle$ is a cyclic group of order $d$. The identity of $C$, denoted $s^\omega$, is the unique idempotent in $\langle s \rangle$ and is given by $s^m$ where $m \geq c$ and $m \equiv 0 \bmod d$. If $s^{\omega+1} = s^\omega s$, then $C = \langle s^{\omega+1} \rangle$.*

*Proof.* Clearly $C$ is a subsemigroup. The mapping $\varphi\colon C \longrightarrow \mathbb{Z}/d\mathbb{Z}$ given by $\varphi(s^n) = n + d\mathbb{Z}$ for $n \geq c$ is well defined and injective by Proposition 1.1. It is surjective because $n + d\mathbb{Z} = \varphi(s^{cd+n})$ for $n \geq 0$. Trivially, $\varphi$ is a homomorphism. Thus $\varphi$ is an isomorphism of semigroups and hence $C$ is a cyclic group of order $d$ with identity $s^m$ where $m \geq c$ and $m \equiv 0 \bmod d$.

If $s^k \in E(\langle s \rangle)$, then $s^k = s^{2k}$ and hence $k \geq c$ by definition of the index. Because $C$ is a group, it has a unique idempotent and so $s^k = s^\omega$. The final statement is clear because $\varphi$ is an isomorphism. $\square$

Notice that $\langle s \rangle$ is a group if and only if $s^{\omega+1} = s$.

*Remark 1.3.* If $S$ is a finite semigroup of order $n$, then $s^{n!} = s^\omega$ for all $s \in S$. Indeed, if $s$ is of index $c$ and period $d$, then obviously $c \leq n$ and $d \leq n$. Thus $n! \geq c$ and $n! \equiv 0 \bmod d$ and so $s^{n!} = s^\omega$ by Corollary 1.2. Consequently, $s^{\omega+1} = s^{n!+1}$ for all $s \in S$.

A fundamental consequence of Corollary 1.2 is that non-empty finite semigroups have idempotents.

**Corollary 1.4.** *A non-empty finite semigroup contains an idempotent.*

*Proof.* If $S$ is a finite semigroup and $s \in S$, then $\langle s \rangle \subseteq S$ contains the idempotent $s^\omega$. $\square$

Corollary 1.4 admits a refinement, sometimes known as the *pumping lemma*.

**Lemma 1.5.** *Let $S$ be a semigroup of order $n$. Then $S^n = SE(S)S$. Equivalently, if $s \in S^n$, then $s = uev$ with $e \in S$ an idempotent and $u, v \in S$.*

*Proof.* The equivalence of the two statements is clear. Trivially, $SE(S)S \subseteq S^n$. Suppose $s \in S^n$. Write $s = s_1 \cdots s_n$ with $s_i \in S$ for $i = 1, \ldots, n$. If the elements $s_1, s_1s_2, \ldots, s_1s_2 \cdots s_n$ of $S$ are all distinct, then they constitute all the elements of $S$. Because $S$ contains an idempotent by Corollary 1.4, we have that $s_1 \cdots s_i \in E(S)$ for some $i$ and hence $s = (s_1 \cdots s_i e)e(es_{i+1} \cdots s_n)$ with $e = s_1 \cdots s_i$ an idempotent. So assume that they are not all distinct and hence there exist $i < j$ such that $s_1 \cdots s_i = s_1 \cdots s_i s_{i+1} \cdots s_j$. Then by induction $s_1 \cdots s_i = s_1 \cdots s_i (s_{i+1} \cdots s_j)^k$ for all $k \geq 0$. In particular, if $e = (s_{i+1} \cdots s_j)^{\omega}$, then $s_1 \cdots s_i e = s_1 \cdots s_i$. Therefore, we have that $s = s_1 \cdots s_i s_{i+1} \cdots s_n = (s_1 \cdots s_i e)e(es_{i+1} \cdots s_n)$, as required. $\square$

We end this section with another useful consequence of the existence of idempotents in finite semigroups.

**Lemma 1.6.** *Let $\varphi \colon S \longrightarrow T$ be a surjective homomorphism of finite semigroups. Then $\varphi(E(S)) = E(T)$.*

*Proof.* The inclusion $\varphi(E(S)) \subseteq E(T)$ is clear. To establish the reverse inclusion, let $e \in E(T)$. Then $\varphi^{-1}(e)$ is a non-empty finite semigroup and hence contains an idempotent $f$. $\square$

## 1.3 The ideal structure and Green's relations

An important role in monoid theory is played by ideals and the closely related notion of Green's relations [Gre51]. Let $M$ be a finite monoid throughout this section.

A *left ideal* (respectively, *right ideal*) of $M$ is a non-empty subset $I$ such that $MI \subseteq I$ (respectively, $IM \subseteq I$). A (two-sided) *ideal* is a non-empty subset $I \subseteq M$ such that $MIM \subseteq I$. Any left, right or two-sided ideal of $M$ is a subsemigroup and hence contains an idempotent by Corollary 1.4. If $I_1, \ldots, I_n$ are all the ideals of $M$, then $I_1 I_2 \cdots I_n$ is an ideal of $M$ and is contained in all other ideals. Consequently, each finite monoid $M$ has a unique *minimal ideal*. Notice that the union of two ideals is again an ideal.

If $m \in M$, then $Mm$, $mM$ and $MmM$ are the *principal*, respectively, left, right and two-sided ideals generated by $m$. It will be convenient to put

$$I(m) = \{s \in M \mid m \notin MsM\}.$$

If $I(m) \neq \emptyset$, then it is an ideal. Also, $I(m) = \emptyset$ if and only if $m$ belongs to the minimal ideal of $M$.

We shall need the following three equivalence relations, called *Green's relations*, associated to the ideal structure of $M$. Put, for $m_1, m_2 \in M$,

(i) $m_1 \mathscr{J} m_2$ if and only if $Mm_1M = Mm_2M$;
(ii) $m_1 \mathscr{L} m_2$ if and only if $Mm_1 = Mm_2$;
(iii) $m_1 \mathscr{R} m_2$ if and only if $m_1M = m_2M$.

The $\mathscr{J}$-class of an element $m$ is denoted by $J_m$, and similarly the $\mathscr{L}$-class and $\mathscr{R}$-class of $m$ are denoted $L_m$ and $R_m$, respectively. Green's relations (of which there are two others that appear implicitly, but not explicitly, throughout this text) were introduced by Green in his celebrated Annals paper [Gre51] and have ever since played a crucial role in all of semigroup theory.

For example, if $\Bbbk$ is a field, then two $n \times n$ matrices $A, B \in M_n(\Bbbk)$ are $\mathscr{J}$-equivalent if and only if they have the same rank. They are $\mathscr{L}$-equivalent if and only if they are row equivalent and they are $\mathscr{R}$-equivalent if and only if they are column equivalent. If $T_n$ denotes the monoid of all self-maps of an $n$-element set, then two mappings $f, g$ are $\mathscr{J}$-equivalent if and only if they have the same *rank* (meaning that their images have the same cardinality). They are $\mathscr{L}$-equivalent if and only if they induce the same partition into fibers over their images and they are $\mathscr{R}$-equivalent if and only if they have the same range.

A monoid $M$ is called *$\mathscr{R}$-trivial* if $mM = nM$ implies $m = n$, that is, the $\mathscr{R}$-relation is equality. The notion of $\mathscr{L}$-triviality is defined dually. A monoid is *$\mathscr{J}$-trivial* if $MmM = MnM$ implies that $m = n$. Note that $\mathscr{J}$-trivial monoids are both $\mathscr{R}$-trivial and $\mathscr{L}$-trivial. The converse is also true by Corollary 1.13, below.

A useful property of Green's relations is that they are compatible with passing from $M$ to $eMe$ with $e \in E(M)$.

**Lemma 1.7.** *Let $e \in E(M)$ and let $m_1, m_2 \in eMe$. If $\mathscr{K}$ is one of Green's relations $\mathscr{J}$, $\mathscr{L}$ or $\mathscr{R}$, then $m_1 \mathscr{K} m_2$ in $M$ if and only if $m_1 \mathscr{K} m_2$ in $eMe$.*

*Proof.* It is clear that if $(eMe)m_1(eMe) = (eMe)m_2(eMe)$, then $Mm_1M = Mm_2M$. Conversely, if $Mm_1M = Mm_2M$, then

$$(eMe)m_1(eMe) = eMm_1Me = eMm_2Me = (eMe)m_2(eMe).$$

This handles the case of $\mathscr{J}$. Similarly, $eMem_1 = eMem_2$ implies $Mm_1 = Mm_2$, and $Mm_1 = Mm_2$ implies $eMem_1 = eMm_1 = eMm_2 = eMem_2$, establishing the result for $\mathscr{L}$. The result for $\mathscr{R}$ is dual. □

To continue our investigation of the ideal structure and Green's relations, it will be helpful to introduce the notion of an $M$-set. A (left) *$M$-set* is a set $X$ together with a mapping $M \times X \longrightarrow X$, written $(m, x) \mapsto mx$, such that $1x = x$ and $m_1(m_2x) = (m_1m_2)x$ for all $x \in X$ and $m_1, m_2 \in M$. Right $M$-sets are defined dually. For example, each left ideal of $M$ is naturally an $M$-set. An $M$-set is said to be *faithful* if $mx = m'x$ for all $x \in X$ implies $m = m'$ for $m, m' \in M$.

A mapping $\varphi\colon X \longrightarrow Y$ of $M$-sets is said to be *$M$-equivariant* if $\varphi(mx) = m\varphi(x)$ for all $m \in M$ and $x \in X$. If $\varphi$ is bijective, we say that $\varphi$ is an *isomorphism of $M$-sets*. The inverse of an isomorphism is again an isomorphism. As usual, $X$ is said to be *isomorphic* to $Y$, denoted $X \cong Y$, if there is an isomorphism between them. We write $\mathrm{Hom}_M(X,Y)$ for the set of all $M$-equivariant mappings from $X$ to $Y$. There is, of course, a category of $M$-sets and $M$-equivariant mappings. We use the obvious notation for endomorphism monoids and automorphism groups of $M$-sets.

**Proposition 1.8.** *Let $e \in E(M)$ and let $X$ be an $M$-set. Then $\mathrm{Hom}_M(Me, X)$ is in bijection with $eX$ via the mapping $\varphi \mapsto \varphi(e)$. Moreover, one has that $\mathrm{End}_M(Me) \cong (eMe)^{op}$ and $\mathrm{Aut}_M(Me) \cong G_e^{op}$.*

*Proof.* Let $\varphi\colon Me \longrightarrow X$ be $M$-equivariant and let $m \in Me$. Then $\varphi(m) = \varphi(me) = m\varphi(e)$. In particular, $\varphi(e) = e\varphi(e) \in eX$ and $\varphi$ is uniquely determined by $\varphi(e)$. It remains to show that if $x \in eX$, then there exists $\varphi\colon Me \longrightarrow X$ with $\varphi(e) = x$. Define $\varphi(m) = mx$ for $m \in Me$. Then $\varphi(m'm) = (m'm)x = m'(mx) = m'\varphi(m)$ for $m \in Me$ and $m' \in M$. This establishes that $\varphi$ is $M$-equivariant. Also we have that $\varphi(e) = ex = x$. This proves the first statement.

The second statement follows from the first and the observation that if $\varphi, \psi \in \mathrm{End}_M(Me)$, then $\varphi(\psi(e)) = \varphi(\psi(e)e) = \psi(e)\varphi(e)$ and hence $\alpha \mapsto \alpha(e)$ provides an isomorphism of $\mathrm{End}_M(Me)$ and $(eMe)^{op}$. $\square$

Since $L_e$ is the set of generators of $Me$ and $\mathrm{Aut}_M(Me) \cong G_e^{op}$, it follows that $G_e$ acts on the right of $L_e$. More precisely, we have the following proposition, where we recall that a group $G$ acts *freely* on a set $X$ if the stabilizer of each element of $X$ is trivial.

**Proposition 1.9.** *If $e \in E(M)$, then $G_e$ acts freely on both the right of $L_e$ and the left of $R_e$ via multiplication.*

*Proof.* Let $m \in L_e$ and $g \in G_e$. Then $mge = mg$ and so $Mmg \subseteq Me$. If $ym = e$, then $g^{-1}ymg = g^{-1}eg = e$ and hence $Me = Mmg$. Therefore, $mg \in L_e$ and so $G$ acts on the right of $L_e$ via right multiplication. To see that the action is free, suppose that $mg = m$ and $e = ym$. Then $g = eg = ymg = ym = e$. Thus the action is free. The result for $R_e$ is dual. $\square$

We now connect the ideal structure near an idempotent with $M$-sets.

**Theorem 1.10.** *Let $e, f \in E(M)$. Then the following are equivalent.*

*(i)* $Me \cong Mf$.
*(ii)* $eM \cong fM$.
*(iii) There exist $a, b \in M$ with $ab = e$ and $ba = f$.*
*(iv) There exist $x, x' \in M$ with $xx'x = x$, $x'xx' = x'$, $x'x = e$ and $xx' = f$.*
*(v)* $MeM = MfM$.

*Proof.* First, we establish the equivalence of (iii) and (iv). Clearly, (iv) implies (iii) by taking $a = x'$ and $b = x$. Suppose that (iii) holds and put $x = fbe$ and $x' = eaf$. Then $x'x = eaffbe = eababe = e^4 = e$ and $xx' = fbeeaf = fbabaf = f^4 = f$, whence $xx'x = ffbe = fbe = x$ and $x'xx' = eeaf = eaf = x'$. This yields the equivalence of (iv) and (iii).

Next we prove the equivalence of (i) and (iii). The equivalence of (ii) and (iii) is then dual. Assume first that $Me \cong Mf$. Let $\varphi\colon Me \longrightarrow Mf$ and $\psi\colon Mf \longrightarrow Me$ be inverse isomorphisms. Put $a = \varphi(e) \in eMf$ and $b = \psi(f) \in fMe$ (by Proposition 1.8). Then $ab = a\psi(f) = \psi(af) = \psi(a) = \psi(\varphi(e)) = e$ and $ba = b\varphi(e) = \varphi(be) = \varphi(b) = \varphi(\psi(f)) = f$. This shows that (i) implies (iii).

Assume next that (iii), and hence (iv), holds. Let $x, x' \in M$ with $xx'x = x$, $x'xx' = x'$, $x'x = e$ and $xx' = f$. Note that $x' \in eMf$ and $x \in fMe$ and hence, by Proposition 1.8, there exist unique $M$-equivariant maps $\varphi\colon Me \longrightarrow Mf$ and $\psi\colon Mf \longrightarrow Me$ such that $\varphi(e) = x' = x'f$ and $\psi(f) = x = xe$. Then $\psi(\varphi(e)) = \psi(x'f) = x'\psi(f) = x'x = e$ and $\varphi(\psi(f)) = \varphi(xe) = x\varphi(e) = xx' = f$. Therefore, $\varphi$ and $\psi$ are inverses by another application of Proposition 1.8. This completes the proof that (i) and (iii) are equivalent.

Trivially, (iii) implies $MeM = MababM \subseteq MbaM = MfM$ and dually $MfM \subseteq MeM$. Therefore, (iii) implies (v). Assume now that $MeM = MfM$. Write $f = xey$ with $x, y \in M$. Then observe that $f = xeyf$ and hence $Mf = Meyf$. By Proposition 1.8, there is a unique $M$-equivariant map $\varphi\colon Me \longrightarrow Mf$ with $\varphi(e) = eyf \in eMf$. As $\varphi(Me) = M\varphi(e) = Meyf = Mf$, we conclude that $\varphi$ is surjective and hence $|Mf| \leq |Me|$. A symmetric argument shows that $|Me| \leq |Mf|$. Therefore, $\varphi$ is in fact a bijection and hence an isomorphism. Thus we have that (v) implies (i), thereby completing the proof of the theorem. □

Isomorphic $M$-sets obviously have isomorphic endomorphism monoids and automorphism groups. Proposition 1.8 and Theorem 1.10 then have the following important consequence.

**Corollary 1.11.** *Let $e, f \in E(M)$ with $MeM = MfM$. Then $eMe \cong fMf$ and $G_e \cong G_f$.*

The next theorem exhibits another crucial property of finite monoids, called *stability*. It provides a certain degree of cancellativity in finite monoids.

**Theorem 1.12.** *Let $m, x \in M$. Then we have that*

$$MmM = MxmM \iff Mm = Mxm; \tag{1.1}$$

$$MmM = MmxM \iff mM = mxM. \tag{1.2}$$

*In other words, for $m \in M$, one has that $J_m \cap Mm = L_m$ and $J_m \cap mM = R_m$.*

*Proof.* We just prove (1.1), as (1.2) is dual. Trivially, $Mm = Mxm$ implies $MmM = MxmM$. For the converse, assume that $m = uxmv$. Clearly, we have $Mxm \subseteq Mm \subseteq Mxmv$. On the other hand, $|Mxmv| \leq |Mxm|$ because $z \mapsto zv$ is a surjective map from $Mxm$ to $Mxmv$. It follows that both containments are equalities and hence $Mxm = Mm$. The final statement is clearly equivalent to (1.1) and (1.2). $\square$

In other words, Theorem 1.12 says that if $xm \in J_m$, then $xm \in L_m$ and if $mx \in J_m$, then $mx \in R_m$.

Our first consequence is a reformulation of $\mathscr{J}$-equivalence.

**Corollary 1.13.** *Let $m_1, m_2 \in M$. Then the following are equivalent.*

*(i) $Mm_1M = Mm_2M$.*
*(ii) There exists $r \in M$ such that $Mm_1 = Mr$ and $rM = m_2M$.*
*(iii) There exists $s \in M$ such that $m_1M = sM$ and $Ms = Mm_2$.*

*Proof.* It is clear that both (ii) and (iii) imply (i). Let us show that (i) implies (ii), as the argument for (i) implies (iii) is similar. Write $m_1 = um_2v$ and $m_2 = xm_1y$. Let $r = xm_1$. Then $uryv = m_1$ and so $Mxm_1M = MrM = Mm_1M$. Stability (Theorem 1.12) then yields $Mr = Mxm_1 = Mm_1$. Also $m_2 = ry$ and $r = xm_1 = xum_2v$ and so $MrM = Mm_2M = MryM$. Theorem 1.12 then implies that $rM = ryM = m_2M$. This completes the proof. $\square$

Another important consequence of stability is that the set of non-units of a finite monoid is an ideal (if non-empty).

**Corollary 1.14.** *Let $M$ be a finite monoid with group of units $G$. Then $G = J_1$ and $M \setminus G$, if non-empty, is an ideal of $M$.*

*Proof.* Trivially, $G \subseteq J_1$. Suppose that $m \in J_1$, that is, $MmM = M$. Then since $1m = m = m1$, it follows from stability (Theorem 1.12) that $mM = M = Mm$. Thus we can write $xm = 1 = my$ for some $x, y \in M$ and so $m$ has both a left and a right inverse. Therefore, $m \in G$ and so $J_1 = G$. Note that since $MmM \subseteq M = M1M$ for all $m \in$M, it follows that

$$I(1) = \{m \in M \mid 1 \notin MmM\} = M \setminus J_1 = M \setminus G,$$

completing the proof. $\square$

Recall that if $e \in E(M)$, then $I_e = eMe \setminus G_e$.

**Corollary 1.15.** *Let $e \in E(M)$. Then $J_e \cap eMe = G_e$. In particular, $I_e = eI(e)e$ and is an ideal of $eMe$ (if non-empty).*

*Proof.* By Lemma 1.7, we have that $J_e \cap eMe$ is the $\mathscr{J}$-class of $e$ in $eMe$. But this is the group of units $G_e$ of $eMe$ by Corollary 1.14. This establishes that $J_e \cap eMe = G_e$. Clearly, $eI(e)e$ is an ideal of $eMe$ (if non-empty) and $eI(e)e \subseteq I_e$. Conversely, if $m \in I_e = eMe \setminus G_e$, then $m \notin J_e$ by the first part of the corollary. Since $m \in MeM$, we conclude that $e \notin MmM$. Therefore, $m \in I(e)$ and so $m = eme \in eI(e)e$. $\square$

As a consequence, we can now describe the orbits of $G_e$ on $L_e$ and $R_e$.

**Corollary 1.16.** *Let $e \in E(M)$. Then two elements $m, n \in L_e$ belong to the same right $G_e$-orbit if and only if $mM = nM$. Similarly, $m, n \in R_e$ belong to the same left $G_e$-orbit if and only if $Mm = Mn$.*

*Proof.* We prove only the first statement. Trivially, if $mG_e = nG_e$, then $mM = nM$. Suppose that $mM = nM$ and write $m = nu$. Then, as $me = m$ and $ne = n$, we conclude that $m = n(eue)$. But $eue \in eMe \cap J_e = G_e$ and so $mG_e = nG_e$. □

Corollary 1.15 also illuminates the structure of the minimal ideal of a finite monoid.

**Corollary 1.17.** *Let $I$ be the minimal ideal of $M$ and let $e \in E(I)$. Then $eIe = eMe = G_e$.*

*Proof.* Obviously, $G_e \subseteq eIe \subseteq eMe$. Since $I$ is an ideal, $eMe = e(eMe)e \subseteq eIe$. Therefore, $eMe = eIe \subseteq I$. Since $I = J_e$, we have $G_e = eMe \cap J_e = eIe \cap I = eIe$. This concludes the proof. □

Another important concept that we shall need from ideal theory is that of a principal series. A *principal series* for $M$ is an unrefinable chain of ideals

$$\emptyset = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_s = M. \tag{1.3}$$

Of course, principal series exist for any finite monoid. It turns out that the differences $I_k \setminus I_{k-1}$ are precisely that $\mathscr{J}$-classes of $M$.

**Proposition 1.18.** *Let* (1.3) *be a principal series for $M$. Then each difference $I_k \setminus I_{k-1}$ with $1 \leq k \leq s$ is a $\mathscr{J}$-class and each $\mathscr{J}$-class arises exactly once in this manner.*

*Proof.* First note that if $m, m' \in I_k \setminus I_{k-1}$, then we have by definition of a principal series that $MmM \cup I_{k-1} = I_k = Mm'M \cup I_{k-1}$ and hence $m \in Mm'M$ and $m' \in MmM$. Therefore, $MmM = Mm'M$. If $MmM = MnM$, then clearly $n \in I_k \setminus I_{k-1}$ and thus $I_k \setminus I_{k-1} = J_m$.

If $m \in M$ and $j$ is minimal with $m \in I_j$, then $m \in I_j \setminus I_{j-1}$ and so $J_m = I_j \setminus I_{j-1}$ by the previous paragraph. Since the sets $I_k \setminus I_{k-1}$ with $1 \leq k \leq s$ are obviously disjoint, we conclude that each $\mathscr{J}$-class occurs exactly once. □

## 1.4 von Neumann regularity

A fundamental role in semigroup theory is played by the notion of von Neumann regularity. We will formulate things for a monoid $M$, although much of the same holds for semigroups. An element $m \in M$ is said to be (von Neumann) *regular* if $m = mam$ for some $a \in M$, i.e., $m \in mMm$. We say that the monoid $M$ is (von Neumann) *regular* if all of its elements are regular. Of course, every group $G$ is regular because $gg^{-1}g = g$ for $g \in G$. Many naturally occurring monoids are regular, such as the monoid of all self-maps on a finite set and the monoid of all endomorphisms of a finite dimensional vector space. We continue to assume that $M$ is a finite monoid.

**Proposition 1.19.** *Let $e \in E(M)$ and $m \in eMe$. Then $m$ is regular in $M$ if and only if it is regular in $eMe$. Consequently, $M$ regular implies $eMe$ regular.*

*Proof.* This follows from the observation that $mMm = meMem$. □

Next we describe some equivalent conditions to an element being regular.

**Proposition 1.20.** *The following are equivalent for $m \in M$.*

*(i) $m$ is regular.*
*(ii) $Mm = Me$ for some idempotent $e \in E(M)$.*
*(iii) $mM = eM$ for some idempotent $e \in E(M)$.*
*(iv) $MmM = MeM$ for some idempotent $e \in E(M)$.*

*Proof.* Assume that $m$ is regular, say that $m = mam$. Then $ma = mama$ and $am = amam$ are idempotents and $Mm = Mam$, $mM = maM$. Thus (i) implies (ii) and (iii). If $Mm = Me$ with $e \in E(M)$, then write $m = ae$ and $e = bm$. Then $mbm = me = aee = ae = m$ and so $m$ is regular. Thus (ii) implies (i) and the proof that (iii) implies (i) is similar. Trivially, (ii) implies (iv). Suppose that (iv) holds. Then we have by Corollary 1.13 that there exists $r \in M$ such that $Mm = Mr$ and $rM = eM$. But then $r$ is regular by (iii) implies (i) and hence $Mm = Mr = Mf$ for some $f \in E(M)$ by (i) implies (ii). Therefore, $m$ is regular by (ii) implies (i). This completes the proof. □

A $\mathscr{J}$-class containing an idempotent is called a *regular $\mathscr{J}$-class*. Proposition 1.20 says that a $\mathscr{J}$-class is regular if and only if all its elements are regular.

**Proposition 1.21.** *Let $I$ be an ideal of $M$. Then $I^2 = I$ if and only if $I$ is generated (as an ideal) by idempotents, that is, $I = ME(I)M = IE(I)I$.*

*Proof.* The equality $IE(I)I = ME(I)M$ is straightforward. If $I = ME(I)M$, then trivially $I = I^2$ because $uev = (ue)(ev) \in I^2$ for $e \in E(I)$. Conversely, suppose that $I^2 = I$ and let $n = |I|$. Then $I = I^n = IE(I)I$ by Lemma 1.5. This completes the proof. □

**Corollary 1.22.** *If $m \in M$, then $(MmM)^2 = MmM$ if and only if $MmM = MeM$ for some $e \in E(M)$, that is, if and only if $m$ is regular.*

*Proof.* Assume that $(MmM)^2 = MmM$. Proposition 1.21 then implies that $m \in MeM$ for some $e \in E(MmM)$ and so $MeM = MmM$. Thus $M$ is regular by Proposition 1.20. The converse is clear. □

A consequence of the previous three results is that $M$ is regular if and only if each of its ideals is idempotent.

**Corollary 1.23.** *A finite monoid $M$ is regular if and only if $I = I^2$ for every ideal $I$ of $M$.*

## 1.5 Exercises

**1.1.** Prove that a non-empty semigroup $S$ is a group if and only if $aS = S = Sa$ for all $a \in S$.

**1.2.** Let $S$ be a non-empty finite semigroup satisfying both the left and right cancellation laws $xy = xz$ implies $y = z$ and $yx = zx$ implies $y = z$. Prove that $S$ is a group.

**1.3.** Prove that a finite monoid is a group if and only if it has a unique idempotent.

**1.4.** Let $\varphi\colon M \longrightarrow N$ be a monoid homomorphism. Prove that $M/\ker\varphi \cong \varphi(M)$.

**1.5.** Let $\varphi\colon M \longrightarrow N$ be a monoid homomorphism and let $e \in E(M)$. Putting $f = \varphi(e) \in E(N)$, prove that:

(a) $\varphi(eMe) \subseteq fNf$;
(b) $\varphi|_{eMe}\colon eMe \longrightarrow fNf$ is a homomorphism of monoids;
(c) $\varphi(G_e) \subseteq G_f$.

**1.6.** Let $I$ be an ideal of a monoid $M$. Define an equivalence relation $\equiv_I$ on $M$ by $m \equiv_I n$ if and only if $m = n$ or $m, n \in I$. Prove that $\equiv_I$ is a congruence and that the class of $I$ is a zero element of the quotient. The quotient of $M$ by $\equiv_I$ is denoted $M/I$ and called the *Rees quotient* of $M$ by $I$.

**1.7.** Suppose that $a, b, c \in M$. Prove that $a \mathscr{L} b$ implies $ac \mathscr{L} bc$ and $a \mathscr{R} b$ implies $ca \mathscr{R} cb$.

**1.8.** Let $N$ be a submonoid of $M$ and let $a, b$ be regular elements of $N$. Prove that $Na = Nb$ if and only if $Ma = Mb$. Given an example showing that the hypothesis that $a, b$ are regular is necessary.

**1.9.** Prove that if $M$ is a finite monoid and $a, b \in M$, then $(ab)^\omega \mathscr{J} (ba)^\omega$.

**1.10.** Let $M$ be a finite monoid. Prove that $\mathscr{J}$ is the join of $\mathscr{R}$ and $\mathscr{L}$ in the lattice of equivalence relations on $M$.

**1.11.** Let $T_n$ be the monoid of all self-maps on an $n$-element set. Prove that if $f, g \in T_n$, then $f \mathscr{J} g$ if and only if they have the same rank, $f \mathscr{R} g$ if and only if they have the same range and $f \mathscr{L} g$ if and only if they induce the same partition into fibers over their images.

**1.12.** Let $A, B \in M_n(\Bbbk)$ for a field $\Bbbk$. Show that $A \mathscr{J} B$ if and only if they have the same rank.

**1.13.** Prove that $T_n$ and $M_n(\Bbbk)$ with $\Bbbk$ a field are regular.

**1.14.** Let $J$ be a $\mathscr{J}$-class of a finite monoid $M$ and suppose that $R \subseteq J$ is an $\mathscr{R}$-class and $L \subseteq J$ is an $\mathscr{L}$-class. Prove that $R \cap L \neq \emptyset$.

**1.15.** Show that if $e \in E(M)$, then $G_e = R_e \cap L_e$.

**1.16.** Let $M$ be a finite monoid and $e \in E(M)$. Prove that $J_e \subseteq L_e R_e$.

**1.17.** Let $M$ be a finite monoid and suppose that $a, b \in M$ with $a \mathscr{L} b$. Suppose that $xa = b$ and $yb = a$ with $x, y \in M$. Define $\varphi_x \colon R_a \longrightarrow R_b$ by $\varphi_x(m) = xm$ and $\varphi_y \colon R_b \longrightarrow R_a$ by $\varphi_y(m) = ym$. This is well defined by Exercise 1.7. Prove that $\varphi_x$ is a bijection with inverse $\varphi_y$ such that $\varphi_x(m) \mathscr{L} m$ for all $m \in R_a$.

**1.18.** Let $M$ be a finite monoid and $a, b \in M$ with $MaM = MbM$. Prove that $|L_a \cap R_a| = |L_b \cap R_b|$ using Exercise 1.17 and its dual.

**1.19.** Let $M$ be a finite monoid and $a \mathscr{J} b$. Prove that $ab \mathscr{J} a \mathscr{J} b$ if and only if $L_a \cap R_b$ contains an idempotent.

**1.20 (Rhodes).** Let $\varphi \colon M \longrightarrow N$ be a surjective homomorphism of finite monoids and let $J$ be a $\mathscr{J}$-class of $N$. Let $X$ be the set of all $\mathscr{J}$-classes $J'$ of $M$ such that $\varphi(J') \subseteq J$ and partially order $X$ by $J' \leq J''$ if $MJ'M \subseteq MJ''M$.

(a) Prove that if $J' \in X$ is minimal, then $\varphi(J') = J$.
(b) Prove that if $J$ is regular, then $X$ has a unique minimal element $J'$ and that $J'$ is regular.

**1.21.** Let $M$ be a finite monoid such that each regular $\mathscr{J}$-class of $M$ is a subsemigroup. Prove that each regular $\mathscr{J}$-class of each quotient of $M$ is also a subsemigroup. (Hint: use Exercise 1.20.)

**1.22.** This exercise proves a special case of Rees's theorem. Let $M$ be a finite monoid with minimal ideal $I$. Fix $e \in E(M)$ and let $G = eIe$, which is a group by Corollary 1.17. Using Exercise 1.14, we can choose $\lambda_1, \ldots, \lambda_s \in L_e$ such that $R_{\lambda_1}, \ldots, R_{\lambda_s}$ are the $\mathscr{R}$-classes of $I$ and $\rho_1, \ldots, \rho_r \in R_e$ such that $L_{\rho_1}, \ldots, L_{\rho_r}$ are the $\mathscr{L}$-classes of $I$. Define $P \in M_{rs}(\mathbb{C}G)$ by $P_{ij} = \rho_i \lambda_j \in G$. Let $S = \{gE_{ij} \in M_{sr}(\mathbb{C}G) \mid g \in G\}$ where $E_{ij}$ is the elementary matrix unit with 1 in the $ij$-entry and 0 elsewhere.

(a) Verify that $S$ is a semigroup with respect to the product $A \odot B = APB$ for $A, B \in S$.
(b) Prove that $\varphi \colon S \longrightarrow I$ defined by

$$\varphi(gE_{ij}) = \lambda_i g \rho_j$$

is an isomorphism. (Hint: Exercise 1.16 may be helpful for surjectivity.)

**1.23.** Let $B$ be the monoid generated by the mappingss $f, g\colon \mathbb{N} \longrightarrow \mathbb{N}$ given by $f(n) = n + 1$ and

$$g(n) = \begin{cases} n - 1, & \text{if } n > 0 \\ 0, & \text{if } n = 0. \end{cases}$$

Prove that Theorem 1.12 fails for $B$.

# 2

# $\mathscr{R}$-trivial Monoids

This chapter studies $\mathscr{R}$-trivial monoids, that is, monoids where Green's relation $\mathscr{R}$ is the equality relation. They form an important class of finite monoids, which have quite recently found applications in the analysis of Markov chains; see Chapter 15. The first section of this chapter discusses lattices and prime ideals of arbitrary finite monoids, as they shall play an important role in the both the general theory and in the structure of $\mathscr{R}$-trivial monoids. The second section concerns $\mathscr{R}$-trivial monoids and in particular, left regular bands, which are precisely the regular $\mathscr{R}$-trivial monoids.

## 2.1 Lattices and prime ideals

The results of this section can be found for arbitrary semigroups in the work of Petrich [Pet63, Pet64]. These results have their roots in Clifford's paper [Cli41]. Our approach roughly follows that of [MS12a].

Recall that a partially ordered set (*poset*) $P$ is a *lattice* if any two elements $p, q \in P$ have both a *meet* (greatest lower bound) $p \wedge q$ and a *join* (least upper bound) $p \vee q$. A finite poset is a lattice if and only if it has a greatest element and each pair of elements has a meet, which occurs if and only if it has a least element and each pair of elements has a join. In this section all lattices will be finite.

A lattice can be viewed as a monoid with respect to either its join or its meet. It is customary in semigroup theory to use the meet operation as the default monoid structure and so often we write it as juxtaposition, unless we want to emphasize its order theoretic aspect. Notice that if $P$ is a lattice, then $P$ is a commutative monoid with $P = E(P)$ and that the lattice order is the natural partial order because $e \leq f$ if and only if $e \wedge f = e = f \wedge e$. Conversely, if $M$ is a finite commutative monoid such that $M = E(M)$, then $M$ is a lattice with respect to its natural partial order and the product is the meet operation. Indeed, we have $(ef)e = e(ef) = ef$ and so $ef \leq e$. Similarly, we have $ef \leq f$. Moreover, if $m \leq e, f$, then $efm = mef = mf = m$ and so $m \leq ef$. Thus $ef = e \wedge f$. As the identity is the maximum element in the natural partial order, we conclude that $M$ is lattice. In summary, we have proved the following classical proposition.

**Proposition 2.1.** *Finite lattices are precisely finite commutative monoids in which each element is idempotent. The meet is the product and the order is the natural partial order. The maximum element is the identity.*

Semigroups in which each element is idempotent are called *bands*. If $M$ is a finite monoid, then there is a universal homomorphism from $M$ to a lattice. Momentarily, it will be shown that this lattice can be identified with the lattice of prime ideals of $M$.

An ideal $P$ of a monoid $M$ is said to be *prime* if its complement $M \setminus P$ is a submonoid or, equivalently, $P$ is a proper ideal and $ab \in P$ implies $a \in P$ or $b \in P$. Note that $I_1 = M \setminus G_1$ is the unique maximal prime ideal. The union $P \cup Q$ of two prime ideals $P, Q$ is again a prime ideal because $M \setminus (P \cup Q) = (M \setminus P) \cap (M \setminus Q)$ is a submonoid. By convention, we shall also declare the empty set to be a prime ideal. Of course, $M \setminus \emptyset = M$. With this convention, the prime ideals form a lattice $\mathrm{Spec}(M)$ with respect to the inclusion ordering with union as the join.

Let us define an idempotent $e \in E(M)$ to be *coprime* if $I(e)$ is a prime ideal, or equivalently, $e \in MmM \cap Mm'M$ implies $e \in Mmm'M$. Note that it is immediate from the definition that if $e, f \in E(M)$ are $\mathscr{J}$-equivalent, that is, $MeM = MfM$, then $e$ is coprime if and only if $f$ is coprime. Thus it is natural to call a $\mathscr{J}$-class $J$ *coprime* if it contains a coprime idempotent, in which case all its idempotents are coprime. We put

$$\Lambda(M) = \{MeM \mid e \text{ is coprime}\}.$$

It is a poset with respect to inclusion. Note that we can identify $\Lambda(M)$ with the set of coprime $\mathscr{J}$-classes ordered by $J \leq J'$ if $MJM \subseteq MJ'M$.

We remark that if $P$ is a prime ideal of $M$, then each $\mathscr{J}$-class of $M \setminus P$ is a $\mathscr{J}$-class of $M$, as is easily checked. In particular, the minimal ideal of $M \setminus P$ is a $\mathscr{J}$-class of $M$.

**Proposition 2.2.** *Let $M$ be a finite monoid. Then the following are equivalent for a $\mathscr{J}$-class $J$.*

*(i) $J$ is a coprime $\mathscr{J}$-class.*
*(ii) $J$ is a subsemigroup.*
*(iii) $J$ is the minimal ideal of $M \setminus P$ where $P$ is a prime ideal.*

*Proof.* If $J$ is a coprime $\mathscr{J}$-class, then $J$ contains an idempotent $e$ with $I(e)$ a prime ideal by definition. Let $M' = M \setminus I(e)$. Then $J = MeM \setminus I(e) \subseteq M'$ and hence $M'JM' \subseteq M' \cap MeM = (M \setminus I(e)) \cap MeM = MeM \setminus I(e) = J$. Therefore, $J$ is an ideal of $M'$. If $m \in M'$, then $e \in MmM$ and so $J \subseteq MmM$. Therefore, if $a \in J$, then $a = umv$ with $u, v \in M$. But then $e \in MaM \subseteq$

$MuM \cap MvM$ and hence $u, v \in M \setminus I(e) = M'$. Thus $a \in M'mM'$ and so $J \subseteq M'mM'$. It follows that $J$ is the minimal ideal of $M \setminus I(e)$ and so (i) implies (iii).

Since the minimal ideal of any monoid is a subsemigroup, (iii) implies (ii). Finally, if $J$ is a subsemigroup, then $J$ contains an idempotent $e$ by Corollary 1.4. If $e \in MmM \cap Mm'M$, then $umv = e = xm'y$ with $u, v, x, y \in M$. Then $e = eumv = xm'ye$. Therefore, $eum, m'ye \in J$ and hence $eumm'ye \in J$ because $J$ is a subsemigroup. Thus $e \in Mmm'M$ and so $I(e)$ is a prime ideal, that is, $J$ is a coprime $\mathscr{J}$-class. This completes the proof. $\square$

As a corollary, it follows that prime ideals are in bijection with coprime $\mathscr{J}$-classes. More precisely, we have the following.

**Theorem 2.3.** *There is an isomorphism of posets $\Lambda(M) \cong \mathrm{Spec}(M)$. Consequently, $\Lambda(M)$ is a lattice.*

*Proof.* For a prime ideal $P$ of $M$, denote by $J(P)$ be the minimal ideal of $M \setminus P$. By Proposition 2.2, and its proof, we have that $J(P)$ is a coprime $\mathscr{J}$-class and that $\psi\colon \mathrm{Spec}(M) \longrightarrow \Lambda(M)$ given by $\psi(P) = MJ(P)M$ is a bijection with inverse $\tau(MeM) = I(e)$ for $e$ a coprime idempotent. We observe that $\psi$ and $\tau$ are order-preserving. Indeed, if $MeM \subseteq MfM$, then $e \notin MmM$ implies $f \notin MmM$ and hence $I(e) \subseteq I(f)$. Thus $\tau$ is order-preserving. Conversely, if $P, Q$ are prime ideals with $P \subseteq Q$, then $M \setminus Q \subseteq M \setminus P$ and hence $J(Q) \subseteq M \setminus P$. Therefore, $J(P) \subseteq (M \setminus P)J(Q)(M \setminus P) \subseteq MJ(Q)M$ and so $\psi(P) \subseteq \psi(Q)$. Thus $\psi$ is order preserving. $\square$

Let us define a mapping $\sigma\colon M \longrightarrow \Lambda(M)$ by

$$\sigma(m) = \bigvee_{\substack{MeM \in \Lambda(M),\\ MeM \subseteq MmM}} MeM, \tag{2.1}$$

that is, $\sigma(m)$ is the largest element of $\Lambda(M)$ smaller than $MmM$. Note that if $e \in E(M)$ belongs to the minimal ideal of $M$, then $MeM \in \Lambda(M)$, $MeM \subseteq MmM$ and hence the join in (2.1) is over a non-empty index set.

**Theorem 2.4.** *The mapping $\sigma\colon M \longrightarrow \Lambda(M)$ is a homomorphism. Moreover, given any homomorphism $\varphi\colon M \longrightarrow P$ with $P$ a lattice, there is a unique homomorphism $\varphi'\colon \Lambda(M) \longrightarrow P$ such that*

$$\begin{tikzcd} M \arrow[rr, "\sigma"] \arrow[dr, "\varphi"'] & & \Lambda(M) \arrow[dl, "\varphi'"] \\ & P & \end{tikzcd}$$

*commutes.*

*Proof.* Since 1 is coprime, $\sigma(1) = M$ is the maximum of $\Lambda(M)$. If $e \in E(M)$ is coprime, then

$$\begin{aligned} MeM \subseteq \sigma(m) \wedge \sigma(m') &\iff MeM \subseteq MmM \cap Mm'M \\ &\iff MeM \subseteq Mmm'M \\ &\iff MeM \subseteq \sigma(mm'). \end{aligned}$$

It follows that $\sigma(mm') = \sigma(m) \wedge \sigma(m')$.

To prove the second statement it suffices to show that if $\varphi(m) \neq \varphi(m')$, then $\sigma(m) \neq \sigma(m')$. Without loss of generality, assume that $\varphi(m) \ngeq \varphi(m')$. Let $I = \{p \in P \mid p \ngeq \varphi(m')\}$. It is easy to check that $I$ is a prime ideal of $P$ and hence $Q = \varphi^{-1}(I)$ is a prime ideal of $M$ with $m \in Q$ and $m' \notin Q$. Therefore, $J(Q) \nsubseteq MmM$ and $J(Q) \subseteq Mm'M$. It follows that $\sigma(m) \neq \sigma(m')$. This completes the proof. □

It turns out that if a regular $\mathscr{J}$-class consists of either a single $\mathscr{L}$-class or a single $\mathscr{R}$-class, then it is coprime.

**Proposition 2.5.** *Let $J$ be a regular $\mathscr{J}$-class consisting of either a single $\mathscr{L}$-class or a single $\mathscr{R}$-class. Then $J$ is a subsemigroup and hence coprime.*

*Proof.* We just handle the case of a single $\mathscr{L}$-class, as the other is dual. Let $a, b \in J$. Since $b$ is regular, we have $bM = eM$ for some idempotent $e \in J$ by Proposition 1.20. Write $e = by$ with $y \in M$. Because $J$ is an $\mathscr{L}$-class, it follows that $a = xe$ for some $x \in M$. Thus $aby = ae = xee = xe = a$ and so $abM = aM$, whence $ab \in J$. We conclude that $J$ is a subsemigroup. □

## 2.2 $\mathscr{R}$-trivial monoids and left regular bands

Recall that a monoid $M$ is $\mathscr{R}$-trivial if $mM = nM$ implies $m = n$. These are the monoids, which up to now, have played a role in applications to Markov chains. Schocker studied them under the name "weakly ordered semigroups" [Sch08]. Important examples of $\mathscr{R}$-trivial monoids include 0-Hecke monoids [Nor79, DHST11], Catalan monoids [DHST11, MS12b] and hyperplane face monoids [BHR99, Bro04, Sal09]. We shall now investigate $\Lambda(M)$ in this special case.

**Corollary 2.6.** *Let $M$ be an $\mathscr{R}$-trivial monoid.*

*(i) Each regular $\mathscr{J}$-class of $M$ is coprime and hence*

$$\Lambda(M) = \{MeM \mid e \in E(M)\}.$$

*(ii) Each regular element of $M$ is an idempotent.*
*(iii) If $e \in E(M)$, then $em = e$ if and only if $m \notin I(e)$, that is, $e \in MmM$.*
*(iv) $\sigma\colon M \longrightarrow \Lambda(M)$ is given by $\sigma(m) = Mm^{\omega}M$.*

*Proof.* For (i), it suffices, by Proposition 2.5, to observe that each $\mathscr{J}$-class is in fact an $\mathscr{L}$-class. Indeed, if $MmM = MnM$, then by Corollary 1.13 there exists $r \in M$ with $mM = rM$ and $Mr = Mn$. By $\mathscr{R}$-triviality, $r = m$ and so $Mm = Mn$. To prove (ii), if $m \in M$ is regular, then $mM = eM$ for some idempotent by Proposition 1.20. But then $m = e$ by $\mathscr{R}$-triviality. For (iii), trivially $em = e$ implies $m \notin I(e)$. Since $J_e$ is coprime, if $m \notin I(e)$, then $em \in MeM \setminus I(e) = J_e$. By stability (Theorem 1.12), we have $emM = eM$ and hence $em = e$ by $\mathscr{R}$-triviality. To prove (iv), since $\Lambda(M)$ is a lattice, we have that $\sigma(m) = \sigma(m^\omega)$. But $Mm^\omega M$ is obviously the largest element of $\Lambda(M)$ contained in $Mm^\omega M$. This completes the proof. $\square$

An $\mathscr{R}$-trivial monoid $M$ is called a *left regular band* if it is a regular monoid. In this case $M = E(M)$ by Corollary 2.6. Left regular bands are characterized by satisfying the identity $xyx = xy$ for all $x, y \in M$.

**Lemma 2.7.** *A monoid $M$ is a left regular band if and only if it satisfies the identity $xyx = xy$.*

*Proof.* If $M$ is a left regular band, then $xyxy = xy$ and so $xyxM = xyM$. We conclude that $xyx = xy$ by $\mathscr{R}$-triviality. Conversely, suppose that $M$ satisfies the identity $xyx = xy$. Taking $y = 1$, we see that each element of $M$ is idempotent, and so in particular $M$ is regular. If $mM = nM$, then $mu = n$ and $nv = m$ for some $u, v \in M$. As $uvu = uv$, we then have $m = muv = muvu = n$ and so $M$ is $\mathscr{R}$-trivial. This completes the proof. $\square$

Since $M = E(M)$ in a left regular band, we have that

$$\Lambda(M) = \{MmM \mid m \in M\}$$

in this case. Actually, we have the following.

**Proposition 2.8.** *Let $M$ be a left regular band.*

*(i) $MmM = Mm$ for all $m \in M$.*
*(ii) $MmM \cap MnM = MmnM$.*
*(iii) The lattice operation on $\Lambda(M)$ is intersection.*
*(iv) $m \leq n$ in the natural partial order if and only if $mM \subseteq nM$.*

*Proof.* We shall use Lemma 2.7 without comment throughout the proof. For (i), clearly $Mm \subseteq MmM$. For the converse, if $u, v \in M$, then $umv = umvm \in Mm$ and so $MmM \subseteq Mm$. By (i), to prove (ii) we need that $Mm \cap Mn = Mmn$. Clearly $Mmn \subseteq Mn$ and $Mmn = Mmnm \subseteq Mm$. Thus we have $Mmn \subseteq Mm \cap Mn$. Suppose that $x \in Mm \cap Mn$, say $x = um = vn$ with $u, v \in M$. Then we have $xmn = ummn = umn = xn = vnn = vn = x$ and so $x \in Mmn$. Item (iii) is immediate from (ii). To prove (iv), if $m \leq n$, then $nm = m$ and so $mM \subseteq nM$. Conversely, if $mM \subseteq nM$, then $m = nx$ for some $x \in M$ and so $nm = nnx = nx = m$ and $mn = mnm = mm = m$. Thus $m \leq n$. This completes the proof. $\square$

## 2.3 Exercises

**2.1.** Prove the following are equivalent for a finite poset $P$.

(i) $P$ is a lattice.
(ii) $P$ has a maximum element and binary meets.
(iii) $P$ has a minimum element and binary joins.

**2.2.** Let $P$ be a prime ideal of a monoid $M$. Prove that each $\mathscr{J}$-class of $M \setminus P$ is also a $\mathscr{J}$-class of $M$.

**2.3.** Prove that a finite monoid $M$ is $\mathscr{J}$-trivial if and only if it is both $\mathscr{L}$-trivial and $\mathscr{R}$-trivial.

**2.4.** Let $M$ be a finite monoid. Prove that $M$ is $\mathscr{R}$-trivial if and only if $(mn)^{\omega}m = (mn)^{\omega}$ for all $m, n \in M$.

**2.5.** Prove that a finite monoid $M$ is $\mathscr{J}$-trivial if and only if $m^{\omega+1} = m^{\omega}$ and $(mn)^{\omega} = (nm)^{\omega}$ for all $m, n \in M$.

**2.6.** Let $M$ be a finite monoid. Prove that $M$ is $\mathscr{R}$-trivial if and only if each regular $\mathscr{R}$-class is a singleton.

**2.7.** Prove that the class of finite $\mathscr{R}$-trivial monoids is closed under direct product, submonoids and homomorphic images.

**2.8.** Let $M$ be a finite $\mathscr{R}$-trivial monoid, $m \in M$ and $e \in E(M)$. Prove that $MeM \subseteq MmM$ if and only if $Me \subseteq Mm$.

**2.9.** Let $M$ be a finite $\mathscr{R}$-trivial monoid and let $e, f \in E(M)$. Prove that $Me \cap Mf = M(ef)^{\omega} = M(fe)^{\omega}$.

**2.10.** Use Exercises 2.8 and 2.9 to show that if $M$ is a finite $\mathscr{R}$-trivial monoid, then $\Upsilon(M) = \{Me \mid e \in E(M)\}$ is a lattice with intersection as the meet and that $\Upsilon(M) \cong \Lambda(M)$ via $Me \mapsto MeM$.

**2.11.** Let $M$ be a finite $\mathscr{L}$-trivial monoid and $\Omega$ a faithful left $M$-set. Let $e, f \in E(M)$.

(a) Prove that $eM \subseteq fM$ if and only if $e\Omega \subseteq f\Omega$.
(b) Prove that $e\Omega \cap f\Omega = (ef)^{\omega}\Omega = (fe)^{\omega}\Omega$ and hence $\{e\Omega \mid e \in E(M)\}$ is a submonoid of the power set of $\Omega$ equipped with intersection.
(c) Prove that $\Lambda(M) \cong \{e\Omega \mid e \in E(M)\}$ where the latter is ordered by inclusion. (Hint: use the dual of Exercise 2.10.)

**2.12.** Let $M$ be a finite monoid. Prove that $M$ is $\mathscr{L}$-trivial if and only if there is a faithful finite left $M$-set $X$ and a linear ordering on $X$ such that $mx \leq x$ for all $m \in M$ and $x \in X$.

# 3

# Inverse Monoids

An important class of regular monoids is the class of inverse monoids. In fact, many semigroup theorists would assert that inverse monoids form the most important class of monoids outside of groups. They abstract the notion of partial symmetry in much the way that groups abstract the notion of symmetry. For a detailed discussion of this viewpoint, see Lawson [Law98]. From the perspective of this book they provide a natural class of monoids whose representation theory we can understand as well as that of groups. Namely, we shall see in Chapter 10 that the algebra of an inverse monoid can be explicitly decomposed as a direct product of matrix algebras over the group algebras of its maximal subgroups (one per $\mathscr{J}$-class).

A good reference for inverse semigroup theory is the book of Lawson [Law98]. A more encyclopedic reference is Petrich [Pet84]. Paterson's book [Pat99] describes important connections between inverse semigroup theory and the theory of $C^*$-algebras. Here we develop just the basic theory.

## 3.1 Defintions, examples and structure

A monoid $M$ is called an *inverse monoid* if, for all $m \in M$, there exists a unique element $m^*$, called the *inverse* of $m$, such that $mm^*m = m$ and $m^*mm^* = m^*$. Every group is an inverse monoid where $g^* = g^{-1}$. A lattice $E$ is an inverse monoid with respect to its meet operation where $e^* = e$ for all $e \in E$. It is straightforward to verify that if $\varphi\colon M \longrightarrow N$ is a homomorphism of inverse monoids, then $\varphi(m^*) = \varphi(m)^*$ for all $m \in M$.

If $X$ is a set, then a *partial mapping* or *partial transformation* $f\colon X \longrightarrow X$ is a mapping from a subset $\mathrm{dom}(f)$ of $X$, called the *domain* of $f$, to $X$. If $f\colon X \longrightarrow X$ and $g\colon X \longrightarrow X$ are partial mappings, then their composition $f \circ g\colon X \longrightarrow X$ has domain $g^{-1}(\mathrm{dom}(f)) \cap \mathrm{dom}(g)$ and $(f \circ g)(x) = f(g(x))$ for all $x \in g^{-1}(\mathrm{dom}(f)) \cap \mathrm{dom}(g)$. In other words, $f \circ g$ is defined where it makes sense to perform $g$ followed by $f$. The composition of partial mappings is associative and hence the set $PT_X$ of all partial transformations of $X$ is a monoid.

*Example 3.1 (Symmetric inverse monoid).* If $X$ is a set, then the *symmetric inverse monoid* on $X$ is the inverse monoid $I_X$ of all partial injective mappings $f\colon X \longrightarrow X$ with respect to composition of partial mappings. We write $I_n$ for the symmetric inverse monoid on $\{1,\ldots,n\}$. If $f \in I_X$ has domain $A$ and range $B$, then $f\colon A \longrightarrow B$ is a bijection. The inverse bijection $f^{-1}\colon B \longrightarrow A$ can be viewed as a partial injective mapping on $X$ and $f^* = f^{-1}$ in $I_X$. Then *rank* of a partial injective mapping $f$ is the cardinality of its image (or equivalently, of its domain).

The monoid $I_n$ is isomorphic to the *rook monoid* $R_n$ consisting of all $n\times n$ matrices with 0/1-entries and at most one non-zero entry in each row and column [Sol02]. The name rook monoid is used because $|R_n|$ is the number of legal placements of rooks on an $n \times n$-chessboard. The reader should verify that if $B$ denotes the Borel subgroup of all upper triangular matrices of the general linear group $GL_n(\Bbbk)$, then

$$M_n(\Bbbk) = \bigcup_{r\in R_n} BrB$$

with the union being disjoint. This is the analogue of the Bruhat-Chevalley decomposition for $M_n(\Bbbk)$ [Ren05].

A fundamental, but not entirely obvious, property of inverse monoids is that their idempotents commute.

**Theorem 3.2.** *A monoid $M$ is an inverse monoid if and only if it is regular and its idempotents commute.*

*Proof.* Suppose first that $M$ is regular with commuting idempotents. Let $m \in M$. By regularity, $m = mxm$ for some $x \in M$. But if $u = xmx$, then $mum = mxmxm = mxm = m$ and $umu = xmxmxmx = xmxmx = xmx = u$. Suppose that $mvm = m$ and $vmv = v$, as well. Then using that $um, vm, mu, mv$ are commuting idempotents, we have that

$$u = umu = umvmu = vmumu = vmu = vmvmu = vmumv = vmv = v.$$

It follows that $M$ is an inverse monoid.

Conversely, suppose that $M$ is an inverse monoid. Trivially, $M$ is regular. Note that $e^* = e$ for any idempotent $e \in E(M)$ because $eee = e$. Let $e, f \in E(M)$ and put $x = f(ef)^*e$. Then $x^2 = f(ef)^*ef(ef)^*e = f(ef)^*e = x$ is an idempotent and hence $x^* = x$. But also we have

$$\begin{aligned} x(ef)x &= f(ef)^*eeff(ef)^*e = f(ef)^*(ef)(ef)^*e = f(ef)^*e = x \\ (ef)x(ef) &= eff(ef)^*eef = ef(ef)^*(ef) = ef. \end{aligned}$$

Therefore, $ef = x^* = x$ and consequently $ef$ is an idempotent. Exchanging the roles of $e$ and $f$, it follows that $fe$ is also an idempotent. Then we compute

$$(ef)(fe)(ef) = efef = ef$$
$$fe(ef)(fe) = fefe = fe$$

and so $fe = (ef)^* = ef$. This completes the proof. $\square$

Our first corollary is that $E(M)$ is a submonoid of $M$.

**Corollary 3.3.** *Let $M$ be an inverse monoid. Then $E(M)$ is a commutative submonoid. Consequently, if $M$ is finite, then $E(M)$ is lattice with respect to the natural partial order and $ef$ is the meet of $e, f \in E(M)$.*

*Proof.* If $e, f \in E(M)$, then $efef = eeff = ef$ and so $E(M)$ is a commutative submonoid. The rest of the corollary is the content of Proposition 2.1. $\square$

Another consequence of the fact that idempotents commute in an inverse monoid is that $m \mapsto m^*$ is an involution.

**Corollary 3.4.** *Let $M$ be an inverse monoid and $m, n \in M$.*

*(i)* $(m^*)^* = m$.
*(ii)* $mm^*nn^* = nn^*mm^*$.
*(iii)* $(mn)^* = n^*m^*$.
*(iv)* $mem^*$ *is idempotent for* $e \in E(M)$.

*Proof.* The first item is clear. Since $mm^*, nn^*$ are idempotents, (ii) follows from Theorem 3.2. For (iii), we compute $mn(n^*m^*)mn = mm^*mnn^*n = mn$ and $n^*m^*(mn)n^*m^* = n^*nn^*m^*mm^* = n^*m^*$. Thus $(mn)^* = n^*m^*$. The final item follows because $mem^*mem^* = mm^*mem^* = mem^*$. $\square$

Quotients of inverse monoids are again inverse monoids.

**Corollary 3.5.** *Let $M$ be an inverse monoid and let $\varphi\colon M \longrightarrow N$ be a surjective homomorphism of monoids. Then $N$ is an inverse monoid.*

*Proof.* Regularity of $M$ easily implies that $N = \varphi(M)$ is regular and so it remains to show that the idempotents of $N$ commute. Let $e, f \in E(N)$ and suppose that $e = \varphi(x)$ and $f = \varphi(y)$. Then $e = \varphi(xx^*)$ and $f = \varphi(yy^*)$ and so $ef = \varphi(xx^*yy^*) = \varphi(yy^*xx^*) = fe$. This completes the proof. $\square$

A final corollary involves the operation of passing from $M$ to $eMe$ for an idempotent $e \in M$.

**Corollary 3.6.** *If $M$ is an inverse monoid, then so is $eMe$ for each idempotent $e \in E(M)$. Moreover, the equality*

$$G_e = \{m \in M \mid m^*m = e = mm^*\}$$

*holds.*

*Proof.* The monoid $eMe$ is regular by Proposition 1.19 and clearly has commuting idempotents, whence $eMe$ is an inverse monoid. For the second statement, $m^*m = e = mm^*$ implies that $eme = mm^*mm^*m = m$ and hence by the first statement, it suffices to consider the case $e = 1$. Clearly, if $g \in G_1$ is a unit, then $g^{-1} = g^*$ and hence $gg^* = 1 = g^*g$. Conversely, if $m^*m = 1 = mm^*$, then $m$ is a unit. □

Inverse monoids can also be characterized in terms of Green's relations. The hypothesis of finiteness in the following proposition is unnecessary.

**Proposition 3.7.** *Let $M$ be a finite monoid. Then $M$ is an inverse monoid if and only if each $\mathscr{L}$-class and $\mathscr{R}$-class of $M$ contains a unique idempotent.*

*Proof.* Assume first that $M$ is an inverse monoid and $m \in M$. Then $mM = mm^*M$ and $Mm = Mm^*m$ and so each $\mathscr{L}$-class and $\mathscr{R}$-class contains an idempotent. If $e, f$ are $\mathscr{L}$-equivalent idempotents, then $e = xf$ implies $fe = ef = xff = xf = e$ and so $e \leq f$. A dual argument shows that $f \leq e$. Thus $e = f$. A similar argument shows that each $\mathscr{R}$-class contains at most one idempotent.

Next assume that $M$ is a monoid all of whose $\mathscr{L}$-classes and $\mathscr{R}$-classes contain a unique idempotent. Regularity of $M$ is immediate from Proposition 1.20. As in the proof of Theorem 3.2, if $m = mxm$ and $u = xmx$, then $mum = m$ and $umu = u$. Suppose that we have $mvm = m$, $vmv = v$, too. Then $um \mathrel{\mathscr{L}} m \mathrel{\mathscr{L}} vm$ and $mu \mathrel{\mathscr{R}} m \mathrel{\mathscr{R}} mv$, from which it follows by the hypothesis that $um = vm$ and $mu = mv$, as these elements are idempotents. Thus $u = umu = vmu = vmv = v$. It follow that $M$ is an inverse monoid. □

The symmetric inverse monoid $I_X$ on a set $X$ is naturally partial ordered by putting $f \leq g$ if the partial injective mapping $f$ is a restriction of $g$. Observe that $f^*f = 1_{\mathrm{dom}(f)}$ and so $f \leq g$ if and only if $gf^*f = f$. This motivates defining an order on any inverse monoid via $m \leq n$ if $m = nm^*m$. The following lemma will be needed to show that this yields a well-behaved partial order.

**Lemma 3.8.** *Let $M$ be an inverse monoid and $m, n \in M$. Then the following are equivalent.*

*(i)* $m = ne$ *with* $e \in E(M)$.
*(ii)* $m = fn$ *with* $f \in E(M)$.
*(iii)* $m = nm^*m$.
*(iv)* $m = mm^*n$.

*Proof.* Trivially, we have that (iii) implies (i) and (iv) implies (ii). If (i) holds, then $nm^*m = nen^*ne = nn^*ne = ne = m$ and so (iii) holds. Similarly, (ii) implies (iv). It remains to establish the equivalence of (i) and (ii). Suppose that (i) holds. Then $(nen^*)n = nn^*ne = ne = m$ and $nen^* \in E(M)$ by Corollary 3.4, yielding (ii). The proof that (ii) implies (i) is dual. □

The *natural partial order* on an inverse monoid is defined by putting $m \leq n$ if the equivalent conditions of Lemma 3.8 hold. Notice that if $m \leq e$ with $e \in E(M)$, then $m = ef$ with $f \in E(M)$ and hence $m$ is an idempotent because $E(M)$ is a submonoid. Moreover, the restriction of the natural partial order to $E(M)$ is the usual partial order.

**Proposition 3.9.** *Let $M$ be an inverse monoid. Then $\leq$ is a partial order, enjoying the following properties.*

*(i) $m \leq n$ if and only if $m^* \leq n^*$.*
*(ii) $m \leq n$ and $m' \leq n'$ implies $mm' \leq nn'$.*

*Proof.* From $m = mm^*m$, we have $m \leq m$. If $m \leq n$ and $n \leq m$, then $m = ne$ and $n = mf$ with $e, f \in E(M)$. But then $m = ne = mfe = mef = neef = nef = mf = n$. Finally, if $m \leq m' \leq m''$, then there exist $e, f \in E(M)$ such that $m = m'e$ and $m' = m''f$. Then $m = m''fe$ and $fe \in E(M)$. Thus $m \leq m''$. This completes the proof that $\leq$ is a partial order.

If $m = ne$ with $e \in E(M)$, then $m^* = (ne)^* = e^*n^* = en^*$. Thus $m \leq n$ implies $m^* \leq n^*$. The reverse implication is proved dually. Suppose now that $m \leq n$ and $m' \leq n'$. Then $m = fn$ and $m' = n'e$ with $e, f \in E(M)$. Therefore, $mm' = mn'e \leq mn'$ and $mn' = fnn' \leq nn'$. Thus $mm' \leq nn'$. □

Green's relations $\mathscr{L}$ and $\mathscr{R}$ take on a particularly pleasant form for inverse monoids. This will be exploited later in the text to associate a groupoid to each inverse monoid.

**Proposition 3.10.** *Let $M$ be an inverse monoid and $m_1, m_2 \in M$.*

*(i) $m_1 \mathrel{\mathscr{L}} m_2$ if and only if $m_1^*m_1 = m_2^*m_2$.*
*(ii) $m_1 \mathrel{\mathscr{R}} m_2$ if and only if $m_1m_1^* = m_2m_2^*$.*

*Proof.* We just prove (i). Clearly, $m_1^*m_1 = m_2^*m_2$ implies $Mm_1 = Mm_1^*m_1 = Mm_2^*m_2 = Mm_2$. The converse follows because if $m_1 = xm_2$, then $m_1^*m_1 = m_2^*x^*xm_2 \leq m_2^*m_2$ and dually $m_2^*m_2 \leq m_1^*m_1$. Thus $m_1^*m_1 = m_2^*m_2$. □

As a corollary, we can shed some light on the nature of commutative inverse monoids.

**Proposition 3.11.** *Let $M$ be a commutative inverse monoid. Then the $\mathscr{J}$-classes of $M$ are the maximal subgroups.*

*Proof.* If $m \in M$, then by commutativity we have $mm^* = m^*m$ and hence if we call this common idempotent $e$, then $m \in G_e$ by Corollary 3.6. Since $M$ is commutative, we have that $\mathscr{R} = \mathscr{L} = \mathscr{J}$. Hence Proposition 3.10 and Corollary 3.6 immediately imply that $G_e$ is the $\mathscr{J}$-class of $m$. □

An direct consequence is that any submonoid of a finite commutative inverse monoid is again an inverse monoid.

**Corollary 3.12.** *Let $M$ be a finite commutative inverse monoid. Then any submonoid $N$ of $M$ is also a commutative inverse monoid.*

*Proof.* Trivially, $N$ has commuting idempotents and so it remains to show that $N$ is regular. Let $m \in N$. By Proposition 3.11 the $\mathscr{J}$-class of $m$ in $M$ is a maximal subgroup $G$ of $M$. Let $n = |G|$. Then $m^{n-1} \in N$ and $mm^{n-1}m = m$. We deduce that $N$ is regular. This concludes the proof. $\square$

In a finite inverse monoid $\mathscr{J}$-equivalent elements are incomparable.

**Proposition 3.13.** *Let $M$ be a finite inverse monoid and $m, n \in M$ with $MmM = MnM$ and $m \leq n$. Then $m = n$.*

*Proof.* Since $m = nm^*m$, we have that $mM = nM$ by stability (Theorem 1.12). Thus $mm^* = nn^*$ by Proposition 3.10. But then we have $m = mm^*n = nn^*n = n$. This completes the proof. $\square$

The analogue of Cayley's theorem for inverse monoids is the *Preston-Wagner theorem*, which is markedly less trivial because it is not so obvious how to make $M$ act on itself by partial injections. Although we do not use this result in the sequel, we provide a proof to give the reader a flavor of the theory of inverse monoids.

**Theorem 3.14.** *Let $M$ be an inverse monoid. Then $M$ embeds in $I_M$.*

*Proof.* To each $m \in M$, define a mapping $\varphi_m \colon m^*mM \longrightarrow mm^*M$ by $\varphi_m(x) = mx$. Note that $\varphi_{m^*} \colon mm^*M \longrightarrow m^*mM$ satisfies $\varphi_{m^*}(\varphi_m(x)) = m^*mx = x$ for $x \in m^*mM$ and similarly $\varphi_m \circ \varphi_{m^*}$ is the identity on $mm^*M$. Therefore, $\varphi_m, \varphi_{m^*}$ are inverses and so $\varphi_m$ can be viewed as a partial injective map on $M$ with domain $m^*mM$ and range $mm^*M$, that is, $\varphi_m \in I_M$.

Define $\varphi \colon M \longrightarrow I_M$ by $\varphi(m) = \varphi_m$. We must verify that $\varphi_{mn} = \varphi_m \circ \varphi_n$ as partial maps to show that $\varphi$ is a homomorphism. Note that the domain of $\varphi_{mn}$ is $(mn)^*(mn)M = n^*m^*mnM$ and the range of $\varphi_{mn}$ is $(mn)(mn)^*M = mnn^*m^*M$. Moreover, for $x \in n^*m^*mnM$, one has $\varphi_{mn}(x) = mnx$. Clearly, $n^*m^*mnM = n^*n(n^*m^*mnM) \subseteq n^*nM$ and if $x \in n^*m^*mnM$, then we have

$$\varphi_n(x) = nx \in nn^*m^*mnM = m^*mnn^*nM \subseteq m^*mM.$$

Therefore, $\varphi_m(\varphi_n(x))$ is defined and $\varphi_m(\varphi_n(x)) = mnx = \varphi_{mn}(x)$. Conversely, if $\varphi_m(\varphi_n(x))$ is defined, then $x \in n^*nM$ and $nx = \varphi_n(x) \in m^*mM$. Therefore, $m^*mnx = nx$ and so $n^*m^*mnx = n^*nx = x$. Thus $x \in n^*m^*mnM$ and $\varphi_{mn}(x) = mnx = \varphi_m(\varphi_n(x))$. This proves the $\varphi_{mn} = \varphi_m \circ \varphi_n$ as partial mappings, and so $\varphi$ is a homomorphism.

Suppose that $\varphi(m) = \varphi(n)$, that is, $\varphi_m = \varphi_n$. Then the latter two maps have the same domain, whence $m^*mM = n^*nM$. Moreover, $nm^*m = \varphi_n(m^*m) = \varphi_m(m^*m) = mm^*m = m$ and so we have $m \leq n$. A symmetric argument yields $n \leq m$ and hence $m = n$. Thus the mapping $\varphi$ is injective. This establishes the theorem. $\square$

The following property will be reformulated in the language of groupoids in Chapter 9.

**Proposition 3.15.** *Let $M$ be an inverse monoid and suppose that $m^*m = nn^*$. Then $(mn)^*mn = n^*n$ and $mn(mn)^* = mm^*$.*

*Proof.* We compute $(mn)^*mn = n^*m^*mn = n^*nn^*n = n^*n$. The second equality is proved in a similar fashion. □

The next lemma will be crucial in describing the structure of the algebra of an inverse monoid.

**Lemma 3.16.** *Let $M$ be an inverse monoid and let $m, n \in M$. Then there exist unique $m_0 \leq m$ and $n_0 \leq n$ such that $m_0n_0 = mn$ and $m_0^*m_0 = n_0n_0^*$.*

*Proof.* Let $m_0 = mnn^*$ and $n_0 = m^*mn$. Then $m_0n_0 = mnn^*m^*mn = mm^*mnn^*n = mn$ and $m_0 \leq m$, $n_0 \leq n$. Also, we compute that $m_0^*m_0 = nn^*m^*mnn^* = nn^*m^*m$ and $n_0n_0^* = m^*mnn^*m^*m = nn^*m^*m = m_0^*m_0$, establishing the existence of $m_0, n_0$. Suppose that $m_1 \leq m$, $n_1 \leq n$ and $m_1n_1 = mn$ with $m_1^*m_1 = n_1n_1^*$. By Proposition 3.15, we have that

$$m_1m_1^* = (m_1n_1)(m_1n_1)^* = (mn)(mn)^* = mnn^*m^*. \tag{3.1}$$
$$n_1^*n_1 = (m_1n_1)^*(m_1n_1) = (mn)^*(mn) = n^*m^*mn \tag{3.2}$$

By Lemma 3.8 and (3.1), we have that

$$m_1 = m_1m_1^*m = mnn^*m^*m = mm^*mnn^* = mnn^* = m_0$$

and similarly,

$$n_1 = nn_1^*n_1 = nn^*m^*mn = m^*mnn^*n = m^*mn = n_0$$

using (3.2). This establishes the uniqueness. □

## 3.2 Exercises

**3.1.** Prove that an inverse monoid is a group if and only if 1 is its unique idempotent.

**3.2.** Prove that if $\varphi\colon M \longrightarrow N$ is a homomorphism between inverse monoids, then $\varphi(m^*) = \varphi(m)^*$.

**3.3.** Prove that a direct product of inverse monoids is an inverse monoid.

**3.4.** Let $M$ be a monoid with commuting idempotents. Prove that the regular elements of $M$ form a submonoid of $M$, which is an inverse monoid.

**3.5.** Let $M$ be an inverse monoid and $a, b \in M$. Show that $a \mathscr{L} b$ if and only if $a^* \mathscr{R} b^*$.

**3.6.** Let $f, g \in I_n$. Prove that $f \mathscr{J} g$ if and only if they have the same rank, $f \mathscr{L} g$ if and only if they have the same domain and $f \mathscr{R} g$ if and only if they have the same image.

**3.7.** Let $n \geq 0$. Prove that

$$|I_n| = \sum_{r=0}^{n} \binom{n}{r}^2 \cdot r!.$$

**3.8.** Prove that the symmetric inverse monoid $I_n$ is isomorphic to the rook monoid $R_n$ (cf. Example 3.1).

**3.9.** Let $G$ be a group. Define the *wreath product* $G \wr I_n$ to consist of all pairs $(f, \sigma)$ such that $\sigma \in I_n$ and $f \colon \mathrm{dom}(\sigma) \longrightarrow G$ is a mapping. The product is given by $(f, \sigma)(g, \tau) = (h, \sigma\tau)$ where $h(x) = f(\tau(x))g(x)$ for $x \in \mathrm{dom}(\sigma\tau)$. Prove that $G \wr I_n$ is an inverse monoid.

**3.10.** Let $E$ be a finite lattice and let $\{G_e \mid e \in E\}$ be an $E$-indexed family of disjoint finite abelian groups. Suppose that one has homomorphisms $\rho_e^f \colon G_f \longrightarrow G_e$ whenever $e \leq f$ such that:

(i) $\rho_e^e = 1_{G_e}$;
(ii) $\rho_e^f \circ \rho_f^h = \rho_e^h$ whenever $e \leq f \leq h$.

(a) Prove that $M = \bigcup_{e \in G_e}$ is a finite commutative inverse monoid with respect to the product

$$mn = \rho_{e \wedge f}^e(m) \cdot \rho_{e \wedge f}^f(n)$$

for $m \in G_e$ and $n \in G_f$.
(b) Prove that $E(M) \cong E$.

**3.11.** Prove that every finite commutative inverse monoid is isomorphic to one of the form constructed in Exercise 3.10.

**3.12.** Let $G$ be a group. Let $K(G)$ be the set of all cosets $gH$ with $H \leq G$ a subgroup.

(a) Define a binary operation $\odot$ on $K(G)$ by

$$g_1H_1 \odot g_2H_2 = g_1g_2\langle g_2^{-1}H_1g_2, H_2\rangle.$$

Prove that $K(G)$ is an inverse monoid.
(b) Prove that if $\mathbb{K}$ is a finite Galois extension of $\mathbb{F}$ and $G$ is the Galois group of $\mathbb{K}$ over $\mathbb{F}$, then $K(G)$ is isomorphic to the inverse monoid of all partial field automorphisms of $\mathbb{K}$ over $\mathbb{F}$, i.e., the inverse monoid of all isomorphisms between intermediate field extensions between $\mathbb{F}$ and $\mathbb{K}$. (Familiarity with Galois theory is required for this problem.)

# Part II

## Irreducible Representations

# 4

# Recollement: The Theory of an Idempotent

In this chapter we provide an account of the theory connecting the category of modules of a finite dimensional algebra $A$ with the module categories of the algebras $eAe$ and $A/AeA$ for an idempotent $e \in A$, known as *recollement* [BBD82, CPS88, CPS96]. We first learned of this subject from the monograph of Green [Gre80, Chapter 6]. A presentation much closer in spirit to ours is in Kuhn [Kuh94b]. In the next chapter, we shall apply this theory to construct the irreducible representations of a finite monoid and in a later chapter we shall extend the results to finite categories.

In this text all rings are unital. Throughout the book, we shall be considering a finite dimensional algebra $A$ over a field $\Bbbk$. Modules will be assumed to be left modules unless otherwise mentioned. We shall denote by $A$-mod the category of finite dimensional $A$-modules or, equivalently, finitely generated $A$-modules. If $A^{op}$ denotes the *opposite algebra* of $A$ (i.e., it has the same underlying vector space but with product $a * b = ba$), then $A^{op}$-mod can be identified with the category of finite dimensional right $A$-modules. We shall use $M_n(R)$ to denote the ring of $n \times n$ matrices over a ring $R$.

For this chapter we fix a finite dimensional algebra $A$ over a field $\Bbbk$ and an idempotent $e \in A$. As is usual, if $X, Y \subseteq A$, then

$$XY = \left\{ \sum_{i=1}^{n} x_i y_i \mid x_i \in X, y_i \in Y \right\}$$

With this notation, $AeA$ is an ideal of $A$ and $eAe$ is an algebra with identity $e$. In fact, $eAe \cong \mathrm{End}_A(Ae)^{op}$ (cf. Proposition A.20).

## 4.1 A miscellany of functors

In this section we consider several functors associated to $e$. Note that $eA$ is an $eAe$-$A$-bimodule and $Ae$ is an $A$-$eAe$-bimodule via left and right multiplication. This allows us to define functors

$$\begin{aligned}
\mathrm{Ind}_e &\colon eAe\text{-mod} \longrightarrow A\text{-mod} \\
\mathrm{Coind}_e &\colon eAe\text{-mod} \longrightarrow A\text{-mod} \\
\mathrm{Res}_e &\colon A\text{-mod} \longrightarrow eAe\text{-mod} \\
K_e &\colon A\text{-mod} \longrightarrow A\text{-mod} \\
N_e &\colon A\text{-mod} \longrightarrow A\text{-mod}
\end{aligned}$$

by putting

$$\begin{aligned}
\mathrm{Ind}_e(V) &= Ae \otimes_{eAe} V \\
\mathrm{Coind}_e(V) &= \mathrm{Hom}_{eAe}(eA, V) \\
\mathrm{Res}_e(V) &= eV \cong \mathrm{Hom}_A(Ae, V) \cong eA \otimes_A V \\
K_e(V) &= AeV \\
N_e(V) &= \{v \in V \mid eAv = 0\}.
\end{aligned}$$

We call $\mathrm{Ind}_e$ the *induction functor*, $\mathrm{Coind}_e$ the *coinduction functor* and $\mathrm{Res}_e$ the *restriction functor*. Readers familiar with category theory will observe that $\mathrm{Ind}_e$ is left adjoint to $\mathrm{Res}_e$ and $\mathrm{Coind}_e$ is right adjoint to $\mathrm{Res}_e$. More precisely, the usual hom-tensor adjunction yields the following proposition (see Exercise 4.1).

**Proposition 4.1.** *Let $V$ be an $A$-module and $W$ an $eAe$-module. Then there are natural isomorphisms* $\mathrm{Hom}_A(\mathrm{Ind}_e(W), V) \cong \mathrm{Hom}_{eAe}(W, \mathrm{Res}_e(V))$ *and* $\mathrm{Hom}_A(V, \mathrm{Coind}_e(W)) \cong \mathrm{Hom}_{eAe}(\mathrm{Res}_e(V), W)$.

The category $A/AeA$-mod can be identified with the full subcategory of $A$-mod consisting of those modules $V$ with $eV = 0$. Observe that $N_e(V)$ is the largest submodule of $V$ annihilated by $e$, i.e., the largest submodule that is an $A/AeA$-module. Thus $N_e\colon A\text{-mod} \longrightarrow A/AeA\text{-mod}$ can be viewed as the right adjoint of the inclusion functor $A/AeA\text{-mod} \longrightarrow A\text{-mod}$. On the other hand, one can show that the assignment $V \mapsto V/K_e(V)$ gives a functor $A\text{-mod} \longrightarrow A/AeA\text{-mod}$ which is left adjoint to the inclusion functor. We shall not prove these assertions, as they shall not be used in the sequel. The diagram of adjoint functors

$$\begin{tikzcd}
A/AeA\text{-mod} \arrow[r] & A\text{-mod} \arrow[l, bend right, "(-)/K_e(-)"'] \arrow[l, bend left, "N_e"] \arrow[r, "\mathrm{Res}_e"] & eAe\text{-mod} \arrow[l, bend right, "\mathrm{Ind}_e"'] \arrow[l, bend left, "\mathrm{Coind}_e"]
\end{tikzcd}$$

is an example of a what is called a *recollement* of abelian categories (see, for example, [Kuh94b]).

We turn next to exactness properties of these functors.

**Proposition 4.2.** *The functor* $\mathrm{Res}_e$ *is exact. If $Ae$ (respectively, $eA$) is a flat (respectively, projective) $eAe$-module then* $\mathrm{Ind}_e$ *(respectively,* $\mathrm{Coind}_e$*) is exact.*

*Proof.* Exactness of $\operatorname{Res}_e$ follows because $Ae$ is a projective $A$-module and $\operatorname{Res}_e(-) \cong \operatorname{Hom}_A(Ae, -)$ as functors. The second statement follows from the definitions. □

A consequence of the preceding two propositions is that $\operatorname{Ind}_e$ preserves projectivity and $\operatorname{Coind}_e$ preserves injectivity of modules.

**Proposition 4.3.** *The functor* $\operatorname{Ind}_e$ *takes projective modules to projective modules and the functor* $\operatorname{Coind}_e$ *takes injective modules to injective modules.*

*Proof.* Let $P$ be a projective $eAe$-module. Then we have that

$$\operatorname{Hom}_A(\operatorname{Ind}_e(P), -) \cong \operatorname{Hom}_{eAe}(P, \operatorname{Res}_e(-)).$$

Because $P$ is projective and $\operatorname{Res}_e$ is exact, we conclude that $\operatorname{Hom}_A(\operatorname{Ind}_e(P), -)$ is exact. Therefore, $\operatorname{Ind}_e(P)$ is projective. An entirely similar argument shows that if $I$ is an injective $eAe$-module, then $\operatorname{Coind}_e(I)$ is injective. □

Our next proposition shows that $\operatorname{Ind}_e$ and $\operatorname{Coind}_e$ are right inverse to $\operatorname{Res}_e$.

**Proposition 4.4.** *There are natural isomorphisms*

$$\operatorname{Res}_e \circ \operatorname{Ind}_e(V) \cong V \cong \operatorname{Res}_e \circ \operatorname{Coind}_e(V)$$

*for any $eAe$-module $V$.*

*Proof.* This follows from

$$\begin{aligned} \operatorname{Res}_e(\operatorname{Ind}_e(V)) &= e(Ae \otimes_{eAe} V) = eAe \otimes_{eAe} V \cong V \\ \operatorname{Res}_e(\operatorname{Coind}_e(V)) &= e \operatorname{Hom}_{eAe}(eA, V) = \operatorname{Hom}_{eAe}(eAe, V) \cong V. \end{aligned}$$

Explicitly, the first isomorphism is given by $a \otimes v \mapsto av$ for $a \in eAe$ and $v \in V$ and the second takes $\varphi \in \operatorname{Hom}_{eAe}(eAe, V)$ to $\varphi(e) \in V$. □

Let us also determine how restriction interacts with $K_e$ and $N_e$.

**Corollary 4.5.** *Let $V$ be an $A$-module. Then we have* $\operatorname{Res}_e(N_e(V)) = 0$, $\operatorname{Res}_e(K_e(V)) = \operatorname{Res}_e(V)$ *and* $\operatorname{Res}_e(V/N_e(V)) \cong \operatorname{Res}_e(V)$.

*Proof.* First we have $\operatorname{Res}_e(N_e(V)) = eN_e(V) = 0$ by definition. Next we compute that $e(AeV) \subseteq eV \subseteq eAeV = e(AeV)$ and so $\operatorname{Res}_e(K_e(V)) = \operatorname{Res}_e(V)$. Finally, as the restriction functor is exact (Proposition 4.2), we have $\operatorname{Res}_e(V/N_e(V)) \cong \operatorname{Res}_e(V)/\operatorname{Res}_e(N_e(V)) = \operatorname{Res}_e(V)$ because $\operatorname{Res}_e(N_e(V)) = 0$. □

It will be important to link the functors $K_e$ and $N_e$ with $\operatorname{Ind}_e$ and $\operatorname{Coind}_e$.

**Proposition 4.6.** *Let $V$ be an $eAe$-module. Then* $K_e(\operatorname{Ind}_e(V)) = \operatorname{Ind}_e(V)$ *and* $N_e(\operatorname{Coind}_e(V)) = 0$.

*Proof.* The first equality follows because

$$Ae(Ae \otimes_{eAe} V) = Ae(eAe) \otimes_{eAe} V = Ae \otimes_{eAe} V.$$

For the second equality, suppose that $\varphi\colon eA \longrightarrow V$ is an $eAe$-module homomorphism with $eA\varphi = 0$. Then, for $a \in A$, we have $\varphi(ea) = \varphi(eea) = (ea\varphi)(e) = 0$ because $ea\varphi = 0$. Therefore, $\varphi = 0$ and so $N_e(\mathrm{Coind}_e(V)) = 0$. This completes the proof. □

The induction and coinduction functors also preserve indecomposability. We prove this in two steps.

**Proposition 4.7.** *Let $V$ be an $A$-module with $eV$ an indecomposable $eAe$-module. If either $K_e(V) = V$ or $N_e(V) = 0$, then $V$ is indecomposable.*

*Proof.* Suppose that $V = V_1 \oplus V_2$ is a direct sum decomposition of $V$ as an $A$-module. Then $eV = eV_1 \oplus eV_2$. By indecomposability of $eV$, either $eV_1 = 0$ or $eV_2 = 0$. Let us assume without loss of generality that the latter occurs. If $N_e(V) = 0$, then we have $V_2 \subseteq N_e(V) = 0$ and so $V = V_1$. If $K_e(V) = V$, then $V = AeV = AeV_1 \oplus AeV_2 \subseteq V_1$ because $eV_2 = 0$. Therefore, we again have $V = V_1$. We conclude that $V$ is indecomposable. □

Putting Proposition 4.7 together with Proposition 4.4 and Proposition 4.6, we are able to deduce the following corollary.

**Corollary 4.8.** *Let $V$ be an indecomposable $eAe$-module. Then $\mathrm{Ind}_e(V)$ and $\mathrm{Coind}_e(V)$ are indecomposable $A$-modules.*

If $V$ is an $A$-module, then, by Proposition 4.1, the identity mapping $1_{eV} \in \mathrm{Hom}_{eAe}(\mathrm{Res}_e(V), \mathrm{Res}_e(V))$ corresponds to a pair of homomorphisms

$$\alpha\colon \mathrm{Ind}_e(eV) \longrightarrow V \text{ and } \beta\colon V \longrightarrow \mathrm{Coind}(eV).$$

We connect $K_e(V)$ and $N_e(V)$ to these maps.

**Proposition 4.9.** *Let $V$ be an $A$-module. Let $\alpha\colon \mathrm{Ind}_e(eV) \longrightarrow V$ and $\beta\colon V \longrightarrow \mathrm{Coind}_e(eV)$ be given by*

$$\alpha(a \otimes v) = av, \text{ for } a \in Ae,\ v \in eV$$
$$\beta(v)(a) = av, \text{ for } a \in eA,\ v \in V.$$

*Then $K_e(V) = \alpha(\mathrm{Ind}_e(eV))$ and $N_e(V) = \ker \beta$. Furthermore, $\ker \alpha \subseteq N_e(\mathrm{Ind}_e(eV))$ and $K_e(\mathrm{Coind}_e(eV)) \subseteq \beta(V)$.*

*Proof.* By Proposition 4.6, we have that

$$\alpha(\mathrm{Ind}_e(eV)) = \alpha(K_e(\mathrm{Ind}_e(eV))) \subseteq K_e(V).$$

One the other hand, if $a \in A$ and $v \in V$, then $aev = \alpha(ae \otimes v)$. It follows that the image of $\alpha$ is $AeV = K_e(V)$. From $e(a\otimes v) = ea\otimes v = e\otimes eav = e\otimes e\alpha(a\otimes v)$

for $a \in Ae$ and $v \in eV$, we deduce that $ex = e \otimes e\alpha(x)$ for all $x \in \mathrm{Ind}_e(eV)$. It follows that if $x \in \ker\alpha$ and $a \in A$, then $eax = e \otimes e\alpha(ax) = e \otimes ea\alpha(x) = 0$ and so $\ker\alpha \subseteq N_e(\mathrm{Ind}_e(eV))$.

Similarly, by Proposition 4.6, we have that

$$\beta(N_e(V)) \subseteq N_e(\mathrm{Coind}_e(eV)) = 0$$

and so $N_e(V) \subseteq \ker\beta$. But if $v \in \ker\beta$, then $eAv = \beta(v)(eA) = 0$. Thus $v \in N_e(V)$ and so $\ker\beta = N_e(V)$. Let $\varphi \in \mathrm{Coind}_e(eV)$ with $e\varphi = \varphi$ and put $v = \varphi(e)$. Then, for $a \in eA$, we have $\varphi(a) = (e\varphi)(a) = \varphi(ae) = ae\varphi(e) = a\varphi(e) = av = \beta(v)(a)$ and so $\varphi = \beta(v)$. Therefore, $e\,\mathrm{Coind}_e(eV) \subseteq \beta(V)$ and hence $K_e(\mathrm{Coind}_e(eV)) = Ae\,\mathrm{Coind}_e(eV) \subseteq A\beta(V) = \beta(V)$. This completes the proof. □

If $W$ is an $eAe$-module, then $e\,\mathrm{Ind}_e(W) \cong W \cong e\,\mathrm{Coind}_e(W)$ by Proposition 4.4 and so the identity map $1_W$ corresponds to a homomorphism $\varphi\colon \mathrm{Ind}_e(W) \longrightarrow \mathrm{Coind}_e(W)$, which in fact is the same map with respect to either isomorphism in Proposition 4.1.

**Corollary 4.10.** *Let $W$ be an $eAe$-module and let*

$$\varphi\colon \mathrm{Ind}_e(W) \longrightarrow \mathrm{Coind}_e(W)$$

*be given by*

$$\varphi(a \otimes w)(b) = baw$$

*for $a \in Ae$, $b \in eA$ and $w \in W$. Then $\varphi(\mathrm{Ind}_e(W)) = K_e(\mathrm{Coind}_e(W))$ and $\ker\varphi = N_e(\mathrm{Ind}_e(W))$. Thus there is an isomorphism*

$$\mathrm{Ind}_e(W)/N_e(\mathrm{Ind}_e(W)) \cong K_e(\mathrm{Coind}_e(W))$$

*of $A$-modules.*

*Proof.* This follows by observing that $e\,\mathrm{Ind}_e(W) \cong W \cong e\,\mathrm{Coind}_e(W)$ and by applying Proposition 4.9 twice: once with $V = \mathrm{Coind}_e(W)$ and $\varphi = \alpha$ and once with $V = \mathrm{Ind}_e(W)$ and $\varphi = \beta$.

Indeed, the isomorphism $W \longrightarrow e\,\mathrm{Coind}_e(W)$ sends $w \in W$ to the mapping $\varphi_w\colon eA \longrightarrow W$ given by $\varphi_w(b) = bew$ for $b \in eA$. Then $\alpha(a \otimes \varphi_w) = a\varphi_w$ for $a \in Ae$. But $(a\varphi_w)(b) = \varphi_w(ba) = baew = baw$ and so $\alpha(a \otimes \varphi_w) = \varphi(a \otimes w)$.

Similarly, $W$ is isomorphic to $e\,\mathrm{Ind}_e(W)$ via $w \mapsto e \otimes w$. Then, for $a \in Ae$ and $b \in eA$, we have that $\beta(a \otimes w)(b) = ba \otimes w = e \otimes baw = e \otimes \varphi(a \otimes w)(b)$. This completes the proof. □

We end this section with a criterion for when $\mathrm{Ind}_e$ is an equivalence of categories (and hence $A$ is Morita equivalent to $eAe$). The reader is referred to Section A.4 for the definitions of an equivalence of categories and Morita equivalence.

**Theorem 4.11.** *Let $A$ be a finite dimensional algebra and $e \in A$ an idempotent. Then the functor*

$$\mathrm{Ind}_e \colon eAe\text{-mod} \longrightarrow A\text{-mod}$$

*is an equivalence of categories if and only if $A = AeA$.*

*Proof.* Assume first that $\mathrm{Ind}_e$ is an equivalence of categories. Then $A \cong \mathrm{Ind}_e(V)$ for some $eAe$-module $V$. Proposition 4.6 yields $K_e(\mathrm{Ind}_e(V)) = \mathrm{Ind}_e(V)$ and so $AeA = K_e(A) = A$.

Conversely, if $AeA = A$, then $K_e(V) = AeV = AeAV = AV = V$ for all $A$-modules $V$. Also, if $v \in V$ with $eAv = 0$, then $Av = AeAv = 0$ and so $v = 0$. Therefore, $N_e(V) = 0$ for all $A$-modules $V$. Thus the natural map $\alpha\colon \mathrm{Ind}_e(eV) \longrightarrow V$ from Proposition 4.9 is an isomorphism by the self-same proposition and hence $\mathrm{Ind}_e$ is essentially surjective. From the isomorphisms $\mathrm{Hom}_A(\mathrm{Ind}_e(V), \mathrm{Ind}_e(V)) \cong \mathrm{Hom}_{eAe}(V, \mathrm{Res}_e(\mathrm{Ind}_e(V))) \cong \mathrm{Hom}_{eAe}(V, V)$, provided by Proposition 4.1 and Proposition 4.4, we conclude that $\mathrm{Ind}_e$ is fully faithful. Thus $\mathrm{Ind}_e$ is an equivalence of categories. $\square$

## 4.2 Idempotents and simple modules

We continue to assume that $A$ is a finite dimensional $\Bbbk$-algebra and that $e \in A$ is an idempotent. A crucial lemma is that $\mathrm{Res}_e$ takes simple modules to simple modules or to zero.

**Lemma 4.12.** *Let $S$ be a simple $A$-module. Then either $eS = 0$ or $eS$ is a simple $eAe$-module.*

*Proof.* If $0 \neq v \in eS$, then $eAev = eAv = eS$ because $Av = S$ by simplicity of $S$. Thus $eS$ is simple. $\square$

**Corollary 4.13.** *If $V$ is a semisimple $A$-module, then $\mathrm{Res}_e(V)$ is a semisimple $eAe$-module.*

As another corollary, we obtain the following.

**Corollary 4.14.** *Let $V$ be a finite dimensional $A$-module with composition factors $S_1, \ldots, S_m$ (with multiplicities). Then the composition factors of $eV$ are the non-zero elements of $eS_1, \ldots, eS_m$ (with multiplicities).*

*Proof.* Let $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_m = V$ be a composition series with $V_i/V_{i-1} \cong S_i$ for $i = 1, \ldots, m$. Then since restriction is exact, we have that $eV_i/eV_{i-1} \cong e(V_i/V_{i-1}) = eS_i$. By Lemma 4.12, each non-zero $eS_i$ is simple. It follows that after removing the repetitions from the series

$$eV_0 \subseteq eV_1 \subseteq \cdots \subseteq eV_m = eV,$$

we obtain a composition series for $eV$ whose composition factors are the non-zero elements of $eS_1, \ldots, eS_m$. $\square$

The radical of a module $V$ over a finite dimensional algebra is denoted $\operatorname{rad}(V)$. Let us relate the radicals of $A$ and $eAe$.

**Proposition 4.15.** *One has that* $\operatorname{rad}(eAe) = e\operatorname{rad}(A)e$. *In particular, if $A$ is semisimple, then $eAe$ is also semisimple.*

*Proof.* Put $R = \operatorname{rad}(A)$ and let $S$ be a simple $A$-module. Then $eS = 0$ or $eS$ is a simple $eAe$-module by Lemma 4.12. In either case, $\operatorname{rad}(eAe)S = \operatorname{rad}(eAe)eS = 0$. Therefore, we have $\operatorname{rad}(eAe) \subseteq R$ by Theorem A.5 and hence $\operatorname{rad}(eAe) \subseteq eRe$. For the converse, it is evident that $eRe$ is an ideal of $eAe$. We show that it is nilpotent. Indeed, if $R^n = 0$, then $(eRe)^n \subseteq R^n = 0$. It follows that $eRe \subseteq \operatorname{rad}(eAe)$ by Theorem A.5, completing the proof. $\square$

Our next goal is to investigate the radical and socle of induced and coinduced modules.

**Proposition 4.16.** *Let $V$ be a finite dimensional $A$-module.*

*(i) If $V = K_e(V)$, then $N_e(V) \subseteq \operatorname{rad}(V)$. Moreover, equality holds if $eV$ is semisimple.*
*(ii) If $N_e(V) = 0$, then $\operatorname{soc}(V) \subseteq K_e(V)$. Furthermore, equality holds if $eV$ is semisimple.*

*Proof.* Let $R = \operatorname{rad}(A)$. To prove (i), let $M$ be a maximal submodule of $V$. If $N_e(V) \nsubseteq M$, then $N_e(V) + M = V$ and hence $eV = e(N_e(V) + M) = eM$. From this we obtain $M \supseteq AeM = AeV = V$, a contradiction. Thus $N_e(V) \subseteq M$ and hence, since $M$ was arbitrary, we have $N_e(V) \subseteq \operatorname{rad}(V)$. If $eV$ is semisimple, then from $\operatorname{rad}(V) = RV$ (cf. Theorem A.5) and $eRe = \operatorname{rad}(eAe)$, we obtain $e\operatorname{rad}(V) = eRV = eRAeV \subseteq eReV = \operatorname{rad}(eAe)eV = 0$ because $eV$ is semisimple. Thus $\operatorname{rad}(V) \subseteq N_e(V)$, completing the proof of the first item.

Turning to (ii), let $S \subseteq V$ be a simple module. Because $N_e(V) = 0$, we deduce $0 \neq eS \subseteq eV$ and hence $S = AeS \subseteq AeV = K_e(V)$. It follows that $\operatorname{soc}(V) \subseteq K_e(V)$. If $eV$ is semisimple, then $e\operatorname{rad}(AeV) = eRAeV \subseteq eReeV = 0$ because $eRe = \operatorname{rad}(eAe)$. Thus $\operatorname{rad}(AeV) \subseteq N_e(V) = 0$, i.e., $K_e(V)$ is semisimple. Therefore, $K_e(V) \subseteq \operatorname{soc}(V)$. $\square$

Next we analyze, to some extent, $A$-modules $V$ with $eV \neq 0$ simple.

**Proposition 4.17.** *Let $V$ be an $A$-module.*

*(i) If $V = K_e(V)$ and $eV$ is simple, then $V/\operatorname{rad}(V) = V/N_e(V)$ is simple.*
*(ii) If $N_e(V) = 0$ and $eV$ is simple, then $\operatorname{soc}(V) = K_e(V)$ is simple.*

*Proof.* For (i), note that $N_e(V) = \operatorname{rad}(V)$ by Proposition 4.16. If $v \notin N_e(V)$, then $0 \neq eAv \subseteq eV$ is an $eAe$-submodule and so $eAv = eV$ by simplicity of $eV$. Moreover, $Av \supseteq AeAv = AeV = V$. Thus $A(v + N_e(V)) = V/N_e(V)$ and we conclude that $V/N_e(V)$ is simple.

To prove (ii), we have that $\operatorname{soc}(V) = K_e(V)$ by Proposition 4.16. If $0 \neq v \in K_e(V) = AeV$, then $eAv \neq 0$ because $N_e(V) = 0$. Therefore, $eAv = eV$ by simplicity of $eV$. We conclude that $Av \supseteq AeAv = AeV = K_e(V)$. It follows that $K_e(V)$ is simple. $\square$

We now apply the proposition to induced and coinduced modules.

**Corollary 4.18.** *Let $V$ be a semisimple $eAe$-module. Then*

$$\mathrm{Ind}_e(V)/\operatorname{rad}(\mathrm{Ind}_e(V)) \cong \operatorname{soc}(\mathrm{Coind}_e(V)).$$

*Moreover, if $V$ is simple, then $\mathrm{Ind}_e(V)/\operatorname{rad}(\mathrm{Ind}_e(V))$ and $\operatorname{soc}(\mathrm{Coind}_e(V))$ are isomorphic simple modules and there is an isomorphism*

$$\mathrm{Res}_e(\mathrm{Ind}_e(V)/\operatorname{rad}(\mathrm{Ind}_e(V))) \cong V \cong \mathrm{Res}_e(\operatorname{soc}(\mathrm{Coind}_e(V)))$$

*of $eAe$-modules.*

*Proof.* By Proposition 4.6 and Proposition 4.16 we have that $\operatorname{rad}(\mathrm{Ind}_e(V)) = N_e(V)$ and $\operatorname{soc}(\mathrm{Coind}_e(V)) = K_e(\mathrm{Coind}_e(V))$. The first statement then follows from Corollary 4.10. The second statement is immediate from Proposition 4.6, Proposition 4.17, the isomorphisms $e\,\mathrm{Ind}_e(V) \cong V \cong e\,\mathrm{Coind}_e(V)$ (see Proposition 4.4) and Corollary 4.5. □

As a corollary, we can also describe when the natural map in Corollary 4.10 is an isomorphism.

**Corollary 4.19.** *Let $V$ be a semisimple $eAe$-module and let*

$$\varphi\colon\ \mathrm{Ind}_e(V) \longrightarrow \mathrm{Coind}_e(V)$$

*be given by $\varphi(a \otimes v)(b) = bav$ for $a \in Ae$, $b \in eA$ and $v \in V$. Then:*

*(i) $\varphi$ is injective if and only if $\mathrm{Ind}_e(V)$ is semisimple;*
*(ii) $\varphi$ is surjective if and only if $\mathrm{Coind}_e(V)$ is semisimple;*
*(iii) $\varphi$ is an isomorphism if and only if $\mathrm{Ind}_e(V)$ and $\mathrm{Coind}_e(V)$ are both semisimple.*

*Proof.* This is immediate from Corollary 4.10, Proposition 4.6 and Proposition 4.16, which imply that $\ker\varphi = \operatorname{rad}(\mathrm{Ind}_e(V))$ and $\varphi(\mathrm{Ind}_e(V)) = \operatorname{soc}(\mathrm{Coind}_e(V))$. □

Now we set up a bijection between isomorphism classes of simple $eAe$-modules and isomorphism classes of simple $A$-modules not annihilated by $e$.

**Theorem 4.20.** *There is a bijection between isomorphism classes of simple $eAe$-modules and isomorphism classes of simple $A$-modules not annihilated by $e$ given by*

$$\begin{aligned} V \longmapsto\ & \mathrm{Ind}_e(V)/N_e(\mathrm{Ind}_e(V)) = \mathrm{Ind}_e(V)/\operatorname{rad}(\mathrm{Ind}_e(V)) \\ & \cong \operatorname{soc}(\mathrm{Coind}_e(V)) = K_e(\mathrm{Coind}_e(V)) \\ S \longmapsto\ & \mathrm{Res}_e(S) = eS \end{aligned}$$

*for $V$ a simple $eAe$-module and $S$ a simple $A$-module with $eS \neq 0$.*

*Consequently, there is a bijection between the set of isomorphism classes of simple $A$-modules and the disjoint union of the sets of isomorphism classes of simple $eAe$-modules and of simple $A/AeA$-modules.*

*Proof.* Applying Lemma 4.12, Proposition 4.6, Proposition 4.16 and Corollary 4.18 we have that the two maps are well defined and that

$$e\operatorname{soc}(\operatorname{Coind}_e(V)) \cong V.$$

It remains to show that if $S$ is a simple $A$-module with $eS \neq 0$, then we have that $S \cong \operatorname{soc}(\operatorname{Coind}_e(eS))$. Indeed, we have by Proposition 4.1 isomorphisms

$$\operatorname{Hom}_A(S, \operatorname{soc}(\operatorname{Coind}_e(eS))) \cong \operatorname{Hom}_A(S, \operatorname{Coind}_e(eS)) \cong \operatorname{Hom}_{eAe}(eS, eS) \neq 0.$$

Because $eS$ is simple by Lemma 4.12, and hence $\operatorname{soc}(\operatorname{Coind}_e(eS))$ is simple by Corollary 4.18, we conclude that $S \cong \operatorname{soc}(\operatorname{Coind}_e(eS))$ by Schur's lemma.

The final statement follows from what we have just proved and the observation that simple $A$-modules annihilated by $e$ are exactly the same thing as simple $A/AeA$-modules. □

We shall need later the following lemma connecting primitive idempotents in $A$ and $eAe$.

**Lemma 4.21.** *Suppose that $S$ is a simple $A$-module such that $eS \neq 0$, and hence $eS$ is a simple $eAe$-module. If $f \in eAe$ is a primitive idempotent with $eAef/e\operatorname{rad}(A)ef \cong eS$, then $f$ is a primitive idempotent of $A$ and $Af/\operatorname{rad}(A)f \cong S$.*

*Proof.* Let $f \in eAe$ be as above. Then $fAf = feAef$ and so $E(fAf) = E(feAef) = \{0, f\}$ by Proposition A.22. Thus $f$ is primitive in $A$ by another application of Proposition A.22. Because $0 \neq \operatorname{Hom}_{eAe}(eAef, eS) = feS = fS$, it follows that $\operatorname{Hom}_A(Af/\operatorname{rad}(A)f, S) \cong \operatorname{Hom}_A(Af, S) \cong fS \neq 0$. But $Af/\operatorname{rad}(A)f$ is simple, as is $S$, and so we conclude by Schur's lemma that $Af/\operatorname{rad}(A)f \cong S$, as required. □

## 4.3 Exercises

**4.1.** Give a detailed proof of Proposition 4.1.

**4.2.** Prove that if $V$ is an $A$-module and $W$ is an $A/AeA$-module, then there are isomorphisms

$$\begin{aligned}\operatorname{Hom}_A(W, V) &\cong \operatorname{Hom}_{A/AeA}(W, N_e(V))\\ \operatorname{Hom}_A(V, W) &\cong \operatorname{Hom}_{A/AeA}(V/K_e(V), W).\end{aligned}$$

**4.3.** Prove that $\operatorname{Res}_e(V/K_e(V)) = 0$.

**4.4.** Let $V$ be an $eAe$-module. Prove that the natural map

$$\varphi\colon \operatorname{Ind}_e(V) \longrightarrow \operatorname{Coind}_e(V)$$

from Corollary 4.10 is an isomorphism if and only if $\operatorname{Ind}_e(V) \cong \operatorname{Coind}_e(V)$.

**4.5.** Let $A$ be a finite dimensional algebra and $e \in A$ an idempotent. Put $J = AeA$. Prove that $eAe$ is semisimple if and only if $J \operatorname{rad}(A) J = 0$.

**4.6.** Let $A$ be a finite dimensional algebra and $e \in A$ an idempotent. Let $V$ be an $A$-module. Prove that

$$K_e(V) = AeV = \sum_{\varphi \in \operatorname{Hom}_A(Ae,V)} \varphi(Ae).$$

**4.7.** Let $A$ be a finite dimensional $\Bbbk$-algebra and let $e \in E(A)$. Prove that $A = AeA$ if and only if every projective indecomposable $A$-module is isomorphic to a direct summand in $Ae$.

**4.8.** Prove that $M_n(A)$ is Morita equivalent to $A$ for all $n \geq 1$.

**4.9.** Prove that every finite dimensional $\Bbbk$-algebra $A$ is Morita equivalent to an algebra $B$ such that $B/\operatorname{rad}(B)$ is isomorphic to a direct product of division algebras.

# 5

# Irreducible Representations

Clifford-Munn-Ponizovskii theory, developed in Clifford [Cli42], Munn [Mun55, Mun57b, Mun60] and Ponizovsky [Pon56] (and in further detail in [LP69, RZ91]), gives a bijection between equivalence classes of irreducible representations of a finite monoid and equivalence classes of irreducible representations of its maximal subgroups (taken one per regular $\mathscr{J}$-class). We follow here the approach of [GMS09], using the techniques of Chapter 4. Let us commence by introducing formally the notion of a representation of a monoid.

## 5.1 Monoid algebras and representations

Let $M$ be a monoid and $\Bbbk$ a field. A *representation* of $M$ on a $\Bbbk$-vector space $V$ is a homomorphism $\rho\colon M \longrightarrow \mathrm{End}_{\Bbbk}(V)$. We call $\dim V$ the *degree* of $\rho$. Two representations $\rho\colon M \longrightarrow \mathrm{End}_{\Bbbk}(V)$ and $\psi\colon M \longrightarrow \mathrm{End}_{\Bbbk}(W)$ are said to be *equivalent* if there is a vector space isomorphism $T\colon V \longrightarrow W$ such that $T^{-1}\psi(m)T = \rho(m)$ for all $m \in M$. A representation $\rho$ is *faithful* if it is injective.

A *matrix representation* of $M$ over $\Bbbk$ is a homomorphism $\rho\colon M \longrightarrow M_n(\Bbbk)$ for some $n \geq 0$. Two matrix representations $\rho, \psi\colon M \longrightarrow M_n(\Bbbk)$ are *equivalent* if there is an invertible matrix $T \in M_n(\Bbbk)$ such that $T^{-1}\psi(m)T = \rho(m)$ for all $m \in M$. Of course, we can view each matrix representation as a representation by identifying $M_n(\Bbbk)$ with $\mathrm{End}_{\Bbbk}(\Bbbk^n)$ in the usual way. Conversely, if $\rho\colon M \longrightarrow \mathrm{End}_{\Bbbk}(V)$ is a representation of $M$ on a finite dimensional vector space $V$, then we can obtain a matrix representation by choosing a basis. If we choose a different basis, this amounts to choosing an equivalent matrix representation. In fact, it is straightforward to verify that equivalence classes of degree $n$ representations of $M$ are in bijection with equivalence classes of matrix representations $\rho\colon M \longrightarrow M_n(\Bbbk)$. Since we only care about representations up to equivalence, we shall no longer distinguish in terminology between representations on finite dimensional vector spaces and matrix representations.

The *monoid algebra* $\Bbbk M$ of a monoid $M$ over a field $\Bbbk$ is the $\Bbbk$-algebra constructed as follows. As a vector space, $\Bbbk M$ has basis $M$: so the elements of $\Bbbk M$ are formal sums $\sum_{m\in M} c_m m$ with $m \in M$, $c_m \in \Bbbk$ and with only finitely many $c_m \neq 0$. In practice, $M$ will be finite and so this last constraint is unnecessary. The product is given by

$$\left(\sum_{m\in M} c_m m\right) \cdot \left(\sum_{m\in M} d_m m\right) = \sum_{m,n\in M} c_m d_n mn.$$

In other words, the product on $M$ is extended to $\Bbbk M$ via the distributive law. When $G$ is a group, $\Bbbk G$ is called the *group algebra*. Note that $\Bbbk M$ is finite dimensional if and only if $M$ is finite.

The monoid algebra has the following universal property.

**Proposition 5.1.** *Let $A$ be a $\Bbbk$-algebra and $M$ a monoid. Then every monoid homomorphism $\varphi\colon M \longrightarrow A$ extends uniquely to a $\Bbbk$-algebra homomorphism $\Phi\colon \Bbbk M \longrightarrow A$.*

*Proof.* Let us define

$$\Phi\left(\sum_{m\in M} c_m m\right) = \sum_{m\in M} c_m \varphi(m).$$

The reader readily verifies that $\Phi$ is as required and is unique. $\square$

If $A$ is a $\Bbbk$-algebra, then an $A$-module $V$ is the same thing as a $\Bbbk$-vector space $V$ together with a $\Bbbk$-algebra homomorphism $\rho\colon A \longrightarrow \mathrm{End}_{\Bbbk}(V)$. It then follows from Proposition 5.1 that a representation of a monoid $M$ on a $\Bbbk$-vector space $V$ is the same thing as a $\Bbbk M$-module structure on $V$ and that two representations are equivalent if and only if the corresponding $\Bbbk M$-modules are isomorphic. More explicitly, if $\rho\colon M \longrightarrow \mathrm{End}_{\Bbbk}(V)$ is a representation, then the $\Bbbk M$-module structure on $V$ is given by

$$\left(\sum_{m\in M} c_m m\right) \cdot v = \sum_{m\in M} c_m \rho(m) v$$

for $v \in V$. We shall say that the $\Bbbk M$-module $V$ *affords* the corresponding representation $\rho\colon M \longrightarrow \mathrm{End}_{\Bbbk}(V)$ or, if we choose a basis for $V$, the corresponding matrix representation $\rho\colon M \longrightarrow M_n(\Bbbk)$.

Terminology for representations and modules tend to differ. For example, representations corresponding to simple $\Bbbk M$-modules are called *irreducible representations* and representations corresponding to semisimple $\Bbbk M$-modules are called *completely reducible*. We shall stick here principally to module theoretic terminology.

The set of isomorphism classes of simple $\Bbbk M$-modules will be denoted $\mathrm{Irr}_{\Bbbk}(M)$. The isomorphism class of a module $V$ will typically be written $[V]$.

The following is an immediate consequence of the theorem of Frobenius and Schur (Theorem A.14), restricted to monoid algebras.

**Corollary 5.2.** *Let $M$ be a finite monoid and $\Bbbk$ an algebraically closed field. Suppose that $S_1, \ldots, S_r$ are a complete set of representatives of the isomorphism classes of simple $\Bbbk M$-modules and that $S_k$ affords the representation $\varphi^{(k)}\colon M \longrightarrow M_{n_k}(\Bbbk)$ with $\varphi^{(k)}(m) = (\varphi^{(k)}_{ij}(m))$. Then the mappings*

$$\varphi^{(k)}_{ij}\colon M \longrightarrow \Bbbk$$

*with $1 \leq k \leq r$ and $1 \leq i, j \leq n_k$ are linearly independent.*

*Proof.* This follows immediately from Theorem A.14 and the observation that $\mathrm{Hom}_{\Bbbk}(\Bbbk M, \Bbbk) \cong \Bbbk^M$ via $\psi \mapsto \psi|_M$. $\square$

It is usual to identify $M_1(\Bbbk)$ with the multiplicative monoid of the field $\Bbbk$. The representation $\rho\colon M \longrightarrow \Bbbk$ given by $\rho(m) = 1$ for all $m \in M$ is called the *trivial representation*. We call $\Bbbk$, equipped with the corresponding module structure, the *trivial module*.

If $V$ and $W$ are $\Bbbk M$-modules, then their *tensor product* $V \otimes_{\Bbbk} W$ (usually written $V \otimes W$) becomes a $\Bbbk M$-module by defining $m(v \otimes w) = mv \otimes mw$ for $m \in M$, $v \in V$ and $w \in W$.

*Remark 5.3.* If $M$ is a monoid containing a zero element $z$, then $\Bbbk z$ is a one-dimensional central ideal of $\Bbbk M$ and the algebra $\Bbbk M/\Bbbk z$ is called the *contracted monoid algebra* of $M$. One can identify $\Bbbk M/\Bbbk z$-mod with the full subcategory of $\Bbbk M$-mod consisting of those modules $V$ with $zV = 0$. One advantage of working with constracted monoid algebras is that if $I$ is an ideal of $M$, then $\Bbbk M/\Bbbk I$ is isomorphic to the contracted monoid algebra of $M/I$. Notice that since $z$ is a central idempotent, we can identify $\Bbbk M/\Bbbk z$ with $\Bbbk M(1-z)$ and we have $\Bbbk M = \Bbbk z \times \Bbbk M(1-z) \cong \Bbbk \times \Bbbk M/\Bbbk z$. Therefore, the representation theory of these two algebras is not very different. A disadvantage of contracted monoid algebras is that one no longer has the trivial module and so the Grothendieck ring will not have an identity. Essentially for this reason, we avoid working explicitly with contracted monoid algebras in this text.

## 5.2 Clifford-Munn-Ponizovskii theory

We fix a finite monoid $M$ and a field $\Bbbk$. If $V$ is a $\Bbbk M$-module, $X \subseteq M$ and $Y \subseteq V$, then we put

$$XY = \left\{ \sum_{i=1}^{n} k_i x_i y_i \mid k_i \in \Bbbk, x_i \in X, y_i \in Y \right\}.$$

Also, we let $\Bbbk X$ denote the $\Bbbk$-linear span of $X$ in $\Bbbk M$. Note that if $I \subseteq M$ is a left (respectively, right or two-sided) ideal of $M$, then $\Bbbk I$ is a left (respectively, right or two-sided) ideal of $\Bbbk M$.

Let $S$ be a simple $\Bbbk M$-module. We say that an idempotent $e \in E(M)$ is an *apex* for $S$ if $eS \neq 0$ and $I_e S = 0$ where we recall that $I_e = eMe \setminus G_e$. Note that $I_e S = I_e eS$. Let us also recall the notation $I(e) = \{m \in M \mid e \notin MmM\}$ and that $eI(e)e = I_e$ (cf. Corollary 1.15).

We establish some basic properties of apexes.

**Proposition 5.4.** *Let $S$ be a simple $\Bbbk M$-module with apex $e \in E(M)$.*

*(i) $I(e) = \{m \in M \mid mS = 0\}$.*
*(ii) If $f \in E(M)$ is another apex, then $MeM = MfM$.*

*Proof.* Since $S$ is simple and $eS \neq 0$, we have $MeS = S$ and hence $I(e)S = I(e)MeS = I(e)eS$. Suppose that $I(e)S \neq 0$ (and in particular, $I(e) \neq \emptyset$). Then by simplicity, and because $I(e)$ is an ideal, we obtain that $I(e)eS = I(e)S = S$ and so $eS = eI(e)eS = I_e S = 0$, a contradiction. We conclude that $I(e)S = 0$. If $e \in MmM$, then $0 \neq eS \subseteq MmMS = MmS$ shows that $mS \neq 0$, establishing (i). If $f \in E(M)$ with $fS \neq 0$, then $f \notin I(e)$ by (i) and so $MeM \subseteq MfM$. It follows by symmetry that if $f$ is another apex, then $MeM = MfM$. □

Fix an idempotent $e \in E(M)$ and put $A_e = \Bbbk M/\Bbbk I(e)$. Observe that $eA_e e \cong \Bbbk[eMe]/\Bbbk I_e \cong \Bbbk G_e$ by Corollary 1.15. Also note that, by Proposition 5.4, a simple $\Bbbk M$-module $S$ with apex $e$ is the same thing as a simple $A_e$-module $S$ with $eS \neq 0$. We can then apply the theory of Chapter 4 to classify these modules.

Notice that, as a $\Bbbk$-vector space, $A_e e \cong \Bbbk L_e$ and $eA_e \cong \Bbbk R_e$. Indeed, Theorem 1.12 shows that $J_e \cap Me = L_e$ and $J_e \cap eM = R_e$ and therefore $Me \setminus L_e \subseteq I(e)$ and $eM \setminus R_e \subseteq I(e)$. The corresponding left $\Bbbk M$-module structure on $\Bbbk L_e$ is defined by

$$m \odot \ell = \begin{cases} m\ell, & \text{if } m\ell \in L_e \\ 0, & \text{else} \end{cases}$$

for $m \in M$ and $\ell \in L_e$. From now on we will omit the symbol "$\odot$." The right $\Bbbk M$-module structure on $\Bbbk R_e$ is defined dually. Note that $\Bbbk L_e$ is also a free right $\Bbbk G_e$-module and $\Bbbk R_e$ is a free left $\Bbbk G_e$-module by Proposition 1.9 and Exercise 5.3. If $V$ is a $\Bbbk G_e$-module, then

$$\mathrm{Hom}_{eA_e e}(eA_e, V) \cong \mathrm{Hom}_{\Bbbk G_e}(\Bbbk R_e, V) \cong \mathrm{Hom}_{G_e}(R_e, V)$$

where $\mathrm{Hom}_{G_e}(R_e, V)$ denotes the vector space of $G_e$-equivariant mappings $\varphi\colon R_e \longrightarrow V$. In the semigroup theory literature, $\Bbbk L_e$ and $\Bbbk R_e$ are known as the left and right *Schützenberger representations* associated to $J_e$ due to the close connection with the paper of Schützenberger [Sch58].

Using the induction and coinduction functors associated to $A_e$ and $eA_e e$ we define functors

$$\begin{aligned}\mathrm{Ind}_{G_e}&\colon \Bbbk G_e\text{-mod} \longrightarrow \Bbbk M\text{-mod}\\ \mathrm{Coind}_{G_e}&\colon \Bbbk G_e\text{-mod} \longrightarrow \Bbbk M\text{-mod}\\ \mathrm{Res}_{G_e}&\colon \Bbbk M\text{-mod} \longrightarrow \Bbbk G_e\text{-mod}\\ K_e&\colon \Bbbk M\text{-mod} \longrightarrow \Bbbk M\text{-mod}\\ N_e&\colon \Bbbk M\text{-mod} \longrightarrow \Bbbk M\text{-mod}\end{aligned}$$

by putting

$$\begin{aligned}\mathrm{Ind}_{G_e}(V) &= A_e e \otimes_{eA_e e} V = \Bbbk L_e \otimes_{\Bbbk G_e} V\\ \mathrm{Coind}_{G_e}(V) &= \mathrm{Hom}_{eA_e e}(eA_e, V) = \mathrm{Hom}_{G_e}(R_e, V)\\ \mathrm{Res}_{G_e}(V) &= eV\\ K_e(V) &= \Bbbk MeV = MeV\\ N_e(V) &= \{v \in V \mid e\Bbbk Mv = 0\} = \{v \in V \mid eMv = 0\}.\end{aligned}$$

Notice that the functors $\mathrm{Ind}_{G_e}$ and $\mathrm{Coind}_{G_e}$ are exact by Proposition 4.2 because $\Bbbk L_e$ and $\Bbbk R_e$ are free $\Bbbk G_e$-modules. We can now state the main theorem of this section, which completely describes the irreducible representations of a finite monoid in terms of the irreducible representations of its maximal subgroups. It is the fundamental theorem of Clifford-Munn-Ponizovskii theory, as formulated in [GMS09].

**Theorem 5.5.** *Let $M$ be a finite monoid and $\Bbbk$ a field.*

*(i) There is a bijection between isomorphism classes of simple $\Bbbk M$-modules with apex $e \in E(M)$ and isomorphism classes of simple $\Bbbk G_e$-modules given by*

$$\begin{aligned}S &\longmapsto \mathrm{Res}_{G_e}(S) = eS\\ V &\longmapsto V^\sharp = \mathrm{Ind}_{G_e}(V)/N_e(\mathrm{Ind}_{G_e}(V)) = \mathrm{Ind}_{G_e}(V)/\operatorname{rad}(\mathrm{Ind}_{G_e}(V))\\ &\qquad \cong \operatorname{soc}(\mathrm{Coind}_{G_e}(V)) = K_e(\mathrm{Coind}_{G_e}(V))\end{aligned}$$

*for $S$ a simple $\Bbbk M$-module with apex $e$ and $V$ a simple $\Bbbk G_e$-module.*

*(ii) Every simple $\Bbbk M$-module has an apex (unique up to $\mathscr{J}$-equivalence).*

*(iii) If $V$ is a simple $\Bbbk G_e$-module, then every composition factor of $\mathrm{Ind}_{G_e}(V)$ and $\mathrm{Coind}_{G_e}(V)$ has apex $f$ with $MeM \subseteq MfM$. Moreover, $V^\sharp$ is the unique composition factor of these two modules with apex $e$.*

*Proof.* Since a simple $\Bbbk M$-module with apex $e$ is the same thing as a simple $A_e$-module $S$ with $eS \neq 0$, the first item follows from Theorem 4.20 applied to $A_e$ and $eA_e e \cong \Bbbk G_e$.

To prove (ii) let $S$ be a simple $\Bbbk M$-module and let $I$ be minimal amongst ideals of $M$ satisfying $IS \neq 0$, that is, not annihilating $S$. Then $IS = S$ by simplicity of $S$ and so $I^2S = IS = S \neq 0$. Therefore, $I^2 = I$ by minimality. Choose $m \in I$ with $mS \neq 0$. Then $MmMS \neq 0$ and so $I = MmM$ by minimality. By Corollary 1.22 there is an idempotent $e \in E(M)$ with $I = MeM$. We claim that $e$ is an apex for $S$. First note that from $S = IS = MeMS = MeS$ we conclude that $eS \neq 0$. Also, since $MI_eM = MeI(e)eM \subsetneq MeM = I$ by Corollary 1.15, it follows that $MI_eMS = 0$ by minimality of $I$. Therefore, $I_eS = 0$. We conclude that $e$ is an apex for $S$. Uniqueness up to $\mathscr{J}$-equivalence follows from Proposition 5.4.

If $f \in E(M)$ with $f \in I(e)$, then from $f\operatorname{Ind}_{G_e}(V) = 0 = f\operatorname{Coind}_{G_e}(V)$ and Corollary 4.14, we conclude that $f$ annihilates each composition factor of $\operatorname{Ind}_{G_e}(V)$ and $\operatorname{Coind}_{G_e}(V)$ and hence none of their composition factors have apex $f$. Also, from $e\operatorname{Ind}_{G_e}(V) = V = e\operatorname{Coind}_{G_e}(V)$ and Corollary 4.14, we deduce that $V^\sharp$ is the only composition factor with apex $e$. $\square$

As a corollary, we obtain the following parameterization of the irreducible representations of $M$.

**Corollary 5.6.** *Let $e_1, \ldots, e_s$ be a complete set of idempotent representatives of the regular $\mathscr{J}$-classes of $M$. Then there is a bijection between $\operatorname{Irr}_\Bbbk(M)$ and the disjoint union $\bigcup_{i=1}^{s} \operatorname{Irr}_\Bbbk(G_{e_i})$.*

*Proof.* The corollary follows from Theorem 5.5 and the observation that each simple $\Bbbk M$-module $V$ has a unique apex amongst $e_1, \ldots, e_s$. $\square$

As an example, we compute the irreducible representations of an $\mathscr{R}$-trivial monoid.

**Corollary 5.7.** *Let $M$ be an $\mathscr{R}$-trivial monoid. The simple $\Bbbk M$-modules are in bijection with regular $\mathscr{J}$-classes of $M$. More precisely, for each regular $\mathscr{J}$-class $J_e$ with $e \in E(M)$, there is a one-dimensional simple $\Bbbk M$-module $S_{J_e}$ with corresponding representation $\chi_{J_e}\colon M \longrightarrow \Bbbk$ given by*

$$\chi_{J_e}(m) = \begin{cases} 1, & \text{if } J_e \subseteq MmM \\ 0, & m \in I(e). \end{cases}$$

*Proof.* Since $M$ is $\mathscr{R}$-trivial, it is immediate that $G_e = \{e\} = R_e$ for all $e \in E(M)$. Therefore, if $\Bbbk$ is the unique simple $\Bbbk G_e$-module and $S_{J_e} = \operatorname{Coind}_{G_e}(\Bbbk)$, then $S_{J_e} = \operatorname{Hom}_{G_e}(\Bbbk e, \Bbbk) \cong \Bbbk$ as a $\Bbbk$-vector space via the map $\varphi \mapsto \varphi(e)$. Trivially, $S_{J_e}$ is a simple module, being one-dimensional. It is annihilated by $I(e)$ and if $m \notin I(e)$, then $(m\varphi)(e) = \varphi(em) = \varphi(e)$ because $em = e$ for all $m \notin I(e)$ by Corollary 2.6. Therefore, elements of $M \setminus I(e)$ act trivially on $S_{J_e}$. The result follows. $\square$

*Example 5.8.* Let $P$ be a finite lattice. Then $P$ is a $\mathscr{J}$-trivial monoid whose regular $\mathscr{J}$-classes are precisely the singletons of $P$. Hence, according to Corollary 5.7, all of the simple $\Bbbk P$-modules are one-dimensional. More precisely, for each $p \in P$, there is a degree one representation $\chi_p\colon P \longrightarrow \Bbbk$ given by

$$\chi_p(q) = \begin{cases} 1, & \text{if } q \geq p \\ 0, & \text{else} \end{cases}$$

for $q \in P$.

The simple modules for a number of $\mathscr{R}$-trivial monoids are studied in detail in Chapter 15 in the context of Markov chains.

## 5.3 The irreducible representations of the full transformation monoid

As an example of the abstract theory we have just developed, we construct the irreducible representations of the full transformation monoid $T_n$ of degree $n$. The representation theory of $T_n$ has a very long history, beginning with the work of Hewitt and Zuckerman [HZ57], which was a precursor to the work of Munn [Mun57b]. In this section, we give explicit constructions of the simple $\Bbbk T_n$-modules over a field $\Bbbk$ of characteristic 0. In fact, the simple modules are all defined over $\mathbb{Q}$, as is the case with the symmetric group. These results are due to Putcha [Put96, Theorem 2.1], except that he did not make explicit the connection with exterior powers of which he only became aware later on (private communcation). The proofs given here are module theoretic and are a significant simplification of Putcha's approach.

Fix a field $\Bbbk$ of characteristic 0 for the entire section. For $r \geq 0$, we put $[r] = \{1, \ldots, r\}$. The reader should consult Section B.4 for background and terminology concerning partitions and the representation theory of the symmetric group, as we shall use the notation from there. Let us quickly recall that $\lambda = (\lambda_1, \ldots, \lambda_m)$ is a *partition* of $r$ if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 1$ and $\lambda_1 + \cdots + \lambda_m = r$. We put $|\lambda| = r$ to indicate that $\lambda$ is a partition of $r$ as we shall be dealing simultaneously with partitions of several integers. If $\lambda$ is a partition, then $S_\lambda$ will denote the Specht module associated to $\lambda$ and $c_\lambda$ will denote the corresponding Young symmetrizer.

Let $e_r \in T_n$, for $1 \leq r \leq n$, be the idempotent given by

$$e_r(i) = \begin{cases} i, & \text{if } i \leq r \\ 1, & \text{if } i > r \end{cases}$$

and note that $e_1, \ldots, e_n$ form a complete set of idempotent representatives of the $\mathscr{J}$-classes of $T_n$. Moreover, $e_r T_n e_r \cong T_r$ and hence $G_{e_r} \cong S_r$. Indeed, $e_r T_n e_r$ leaves $[r]$ invariant and the map $f \mapsto f|_{[r]}$ is an isomorphism of $e_r T_n e_r$ with $T_r$, as the reader can easily check. We will from now on identify $G_{e_r}$ with $S_r$ when convenient.

We begin by identifying the simple modules $S^\sharp_{(1^r)}$ for $1 \leq r \leq n$. First observe that $\Bbbk^n$ is a $\Bbbk T_n$-module, which we term the *natural module*, in the following way. Let $v_1, \ldots, v_n$ be the standard basis for $\Bbbk^n$ and define $f v_i = v_{f(i)}$ for $f \in T_n$. Notice that the action of $f$ on a vector preserves the sum of its coefficients. Hence if we put

$$\mathrm{Aug}(\Bbbk^n) = \{(x_1, \ldots, x_n) \in \Bbbk^n \mid x_1 + \cdots + x_n = 0\}$$

then $\operatorname{Aug}(\Bbbk^n)$ is a $\Bbbk T_n$-submodule of $\Bbbk^n$. In fact, $\operatorname{Aug}(\Bbbk^n)$ is the kernel of the $\Bbbk T_n$-module homomorphism $\theta\colon \Bbbk^n \longrightarrow \Bbbk$ sending $v_i$ to 1 for all $1 \leq i \leq n$ where $\Bbbk$ is given the trivial $\Bbbk T_n$-module structure. Note that $\dim \operatorname{Aug}(\Bbbk^n) = n-1$.

**Theorem 5.9.** *Let $\Bbbk$ be a field of characteristic 0, $n \geq 1$ and $V = \operatorname{Aug}(\Bbbk^n)$. Then, for $1 \leq r \leq n$, the exterior power $\Lambda^{r-1}(V)$ is a simple $\Bbbk T_n$-module of dimension $\binom{n-1}{r-1}$ with apex $e_r$ and with $e_r\Lambda^{r-1}(V)$ the sign representation $S_{(1^r)}$ of $G_{e_r} \cong S_r$, that is, $S^{\sharp}_{(1^r)} \cong \Lambda^{r-1}(V)$. More generally, if $r \leq m \leq n$, then $e_m\Lambda^{r-1}(V) = \Lambda^{r-1}(e_mV)$ is the simple module $\Bbbk S_m$-module $S_{(m-r+1,1^{r-1})}$.*

*Proof.* If $1 \leq m \leq n$, then clearly $e_m\Bbbk^n \cong \Bbbk^m$ and $e_m \operatorname{Aug}(\Bbbk^n) \cong \operatorname{Aug}(\Bbbk^m)$ under the identification of $e_m\Bbbk T_n e_m = \Bbbk[e_mT_ne_m]$ with $\Bbbk T_m$ induced by restricting an element of $e_mT_ne_m$ to $[m]$. In particular, as $e_m\Lambda^{r-1}(V) = \Lambda^{r-1}(e_mV)$ and $\dim e_mV = \dim \operatorname{Aug}(\Bbbk^m) = m-1$, we conclude that $e_m\Lambda^{r-1}(V) = 0$ if $m < r$.

Assume now that $m \geq r$. By [FH91, Proposition 3.12], the exterior power $\Lambda^{r-1}(e_mV)$ is a simple $\Bbbk S_m$-module of degree $\binom{m-1}{r-1}$ and by [FH91, Exercise 4.6] it is the Specht module $S_{(m-r+1,1^{r-1})}$. In particular, we have that $e_r\Lambda^{r-1}(V) = S_{(1^r)}$. It follows that $W = \Lambda^{r-1}(V)$ is a simple $\Bbbk T_n$-module with apex $e_r$ and that $W \cong S^{\sharp}_{(1^r)}$. □

To construct the remaining simple $\Bbbk T_n$-modules, we introduce some notation. For $1 \leq r \leq n$, let $T_{n,r}$ denote the set of all mappings $f\colon [r] \longrightarrow [n]$. Notice that $T_n$ acts on the left of $T_{n,r}$ by post-composition and $S_r$ acts on the right by precomposition. Moreover, these two actions commute and so $\Bbbk T_{n,r}$ is a $\Bbbk T_n$-$\Bbbk S_r$-bimodule. Let $I_{n,r} \subseteq T_{n,r}$ be the subset of injective mappings and let $L_{n,r} = T_{n,r} \setminus I_{n,r}$. Then $\Bbbk L_{n,r}$ is a sub-bimodule and so $\Bbbk T_{n,r}/\Bbbk L_{n,r}$ is a $\Bbbk T_n$-$\Bbbk S_r$-bimodule which can be identified as a $\Bbbk$-vector space with $\Bbbk I_{n,r}$. Under this identification, the right action of $S_r$ is still by precomposition, but the left action of $T_n$ is now given by

$$f \odot g = \begin{cases} fg, & \text{if } fg \in I_{n,r} \\ 0, & \text{else} \end{cases}$$

for $f \in T_n$ and $g \in I_{n,r}$. We drop from now on the notation "$\odot$." The reader should verify that $\Bbbk L_{e_r} \cong \Bbbk I_{n,r}$ as a $\Bbbk T_n$-$\Bbbk S_r$-bimodule.

The action of $S_r$ on the right of $I_{n,r}$ is free because if $f\colon [r] \longrightarrow [n]$ is injective and $g \in S_r$, then $fg = f = f1_{[r]}$ implies $g = 1_{[r]}$. It is easy to see that two mappings are in the same $S_r$-orbit if and only if they have the same image. If $Y \subseteq [n]$ has cardinality $r$, then let $h_Y\colon [r] \longrightarrow Y$ be the unique order-preserving bijection. Viewing $h_Y$ as an element of $I_{n,r}$, we have that the $h_Y$ form a complete set of representatives of the $S_r$-orbits on $I_{n,r}$. Thus $\Bbbk I_{n,r}$ is a free right $\Bbbk S_r$-module with basis the $h_Y$ with $Y \subseteq [n]$ and $|Y| = r$. Note that $h_{[r]} = 1_{[r]}$.

**Theorem 5.10.** *Let $\Bbbk$ be a field of characteristic 0 and $\lambda$ a partition of $r$ with $1 \leq r \leq n$ and $\lambda \neq (1^r)$. Then there is an isomorphism of $\Bbbk T_n$-modules*

$$S_\lambda^\sharp \cong \Bbbk I_{n,r} \otimes_{\Bbbk S_r} S_\lambda \cong \Bbbk I_{n,r} c_\lambda$$

*where $I_{n,r}$ is the set of injective mappings from $[r]$ to $[n]$. Moreover,*

$$\dim S_\lambda^\sharp = \binom{n}{r} f_\lambda$$

*where $f_\lambda = \dim S_\lambda$ is the number of standard Young tableaux of shape $\lambda$.*

*Proof.* Put $W_\lambda = \Bbbk I_{n,r} \otimes_{\Bbbk S_r} S_\lambda$. Note that $W_\lambda = \Bbbk I_{n,r} \otimes_{\Bbbk S_r} \Bbbk S_r c_\lambda \cong \Bbbk I_{n,r} c_\lambda$. Observe that since $\Bbbk I_{n,r}$ is a free right $\Bbbk S_r$-module with basis the $h_Y$ with $Y \subseteq [n]$ and $|Y| = r$, we have the $\Bbbk$-vector space decomposition

$$W_\lambda = \bigoplus_{Y \subseteq [n], |Y|=r} h_Y \otimes S_\lambda$$

and so $\dim W_\lambda = \binom{n}{r} f_\lambda$. The action of $f \in T_n$ on $h_Y \otimes v_Y$ is given by

$$\begin{aligned} f(h_Y \otimes v_Y) &= \begin{cases} fh_Y \otimes v_Y, & \text{if } |f(Y)| = r \\ 0, & \text{else} \end{cases} \\ &= \begin{cases} h_{f(Y)} \otimes (h_{f(Y)}^{-1} f h_Y) v_Y, & \text{if } |f(Y)| = r \\ 0, & \text{else.} \end{cases} \end{aligned} \tag{5.1}$$

In particular, we have that $W_\lambda$ is annihilated by all mappings of rank less than $r$ and $e_r W_\lambda = 1_{[r]} \otimes S_\lambda \cong S_\lambda$ as a $\Bbbk S_r$-module (recall that $h_{[r]} = 1_{[r]}$). Thus to prove the theorem, it suffices to show that $W_\lambda$ is simple.

We prove the theorem by induction on the quantity $n - r$ (where we do not hold $n$ fixed). If $n - r = 0$, then $I_{n,r} = S_n$ and $W_\lambda = S_\lambda$ made into a $\mathbb{C}T_n$-module by extending the $S_n$-action to $T_n \setminus S_n$ by having the non-permutations annihilate $S_\lambda$. Thus $W_\lambda$ is simple in this case.

Assume that the result is true when $n - r = k$ and suppose that $n - r = k + 1$. If we view $I_{n-1,r}$ as a subset of $I_{n,r}$ in the natural way, then we have $e_{n-1} \Bbbk I_{n,r} = \Bbbk I_{n-1,r}$. Moreover, if we identify $e_{n-1} T_n e_{n-1}$ with $T_{n-1}$ via the restriction to $[n-1]$, then $e_{n-1} \Bbbk I_{n,r} \cong \Bbbk I_{n-1,r}$ as a $\Bbbk T_{n-1}$-$\Bbbk S_r$-submodule. Therefore,

$$e_{n-1} W_\lambda = (e_{n-1} \Bbbk I_{n,r}) \otimes_{\Bbbk S_r} S_\lambda \cong \Bbbk I_{n-1,r} \otimes_{\Bbbk S_r} S_\lambda$$

and hence, because $n - 1 - r = k$, we have that $e_{n-1} W_\lambda$ is a simple $e_{n-1} \Bbbk T_n e_{n-1}$-module by induction. Note that $1_{[r]} \otimes S_\lambda \subseteq e_{n-1} W_\lambda$.

For each $Y \subseteq [n]$ with $|Y| = r$, let $g_Y \in S_n$ be the unique permutation whose restriction to $[r]$ is $h_Y$ and whose restriction to $[n] \setminus [r]$ is the unique order-preserving bijection $[n] \setminus [r] \longrightarrow [n] \setminus Y$. Then $g_Y(1_{[r]} \otimes S_\lambda) = h_Y \otimes S_\lambda$ from which we deduce that $\Bbbk T_n e_{n-1} W_\lambda = W_\lambda$.

Let $v \in W_\lambda$ be non-zero. We must show that $\Bbbk T_n v = W_\lambda$. Write

$$v = \sum_{Y \subseteq [n], |Y|=r} h_Y \otimes v_Y.$$

Suppose that $v_{Y'} \neq 0$. Then replacing $v$ by $g_{Y'}^{-1} v$, which is non-zero because $g_{Y'}$ is invertible in $T_n$, we may assume that $v_{[r]} \neq 0$.

For $1 \leq i < j \leq n$, let $\eta_{i,j} \in T_n$ be the rank $n-1$ idempotent defined by

$$\eta_{i,j}(x) = \begin{cases} x, & \text{if } x \neq j \\ i, & \text{if } x = j \end{cases}.$$

The image of $\eta_{i,j}$ is $[n] \setminus \{j\}$. Note that $\eta_{1,n} = e_{n-1}$ and that if $\sigma_{i,j} = (1\ i)(n\ j)$, then $\eta_{i,j} = \sigma_{i,j} e_{n-1} \sigma_{i,j}$ (where we interpret $(1\ 1)$ and $(n\ n)$ as the identity).

Suppose that $\eta_{i,j} v \neq 0$ for some $1 \leq i < j \leq n$. If we put $w = \sigma_{i,j} \eta_{i,j} v$, then $w = e_{n-1} \sigma_{i,j} v \in e_{n-1} W_\lambda$ is non-zero. Therefore, $e_{n-1} \Bbbk T_n e_{n-1} w = e_{n-1} W_\lambda$ by simplicity of $e_{n-1} W_\lambda$ over $e_{n-1} \Bbbk T_n e_{n-1}$. We conclude that $W_\lambda = \Bbbk T_n e_{n-1} W_\lambda \subseteq \Bbbk T_n w \subseteq \Bbbk T_n v$, as required.

So we are left with the case that $\eta_{i,j} v = 0$ for all $1 \leq i < j \leq n$. We will derive a contradiction to the assumption $\lambda \neq (1^r)$. Let $Y \subseteq [n]$ have cardinality $r$. Then the following computation is straightforward from (5.1):

$$\eta_{i,j}(h_Y \otimes v_Y) = \begin{cases} h_Y \otimes v_Y, & \text{if } j \notin Y \\ h_{(Y \cup \{i\}) \setminus \{j\}} \otimes (h^{-1}_{(Y \cup \{i\}) \setminus \{j\}} \eta_{i,j} h_Y) v_Y, & \text{if } j \in Y,\ i \notin Y \\ 0, & \text{if } \{i,j\} \subseteq Y. \end{cases} \tag{5.2}$$

Let us put $Y_m = [r+1] \setminus \{m\}$ for $1 \leq m \leq r+1$ and put

$$W_m = \bigoplus_{Y \neq Y_m} h_Y \otimes S_\lambda.$$

Note that $Y_{r+1} = [r]$. Let $1 \leq i < j \leq r+1$. Then it follows from (5.2) that

$$\begin{aligned} \eta_{i,j}(h_{Y_j} \otimes v_{Y_j}) &= h_{Y_j} \otimes v_{Y_j} \\ \eta_{i,j}(h_{Y_i} \otimes v_{Y_i}) &= h_{Y_j} \otimes (i\ i+1\ \cdots\ j-1) v_{Y_i} \\ \eta_{i,j}(h_Y \otimes v_Y) &\in W_j,\ \text{if } Y \notin \{Y_i, Y_j\} \end{aligned} \tag{5.3}$$

since $\eta_{i,j} h_{Y_i}$ and $h_{Y_j}(i\ i+1\ \cdots\ j-1)$ are both the map $h \colon [r] \longrightarrow [n]$ given by

$$h(x) = \begin{cases} x, & \text{if } 1 \leq x < i \\ x+1, & \text{if } i \leq x < j-1 \text{ or } j \leq x \leq r \\ i, & \text{if } x = j-1. \end{cases}$$

In particular, if $2 \leq j \leq r+1$, then we have that

$$0 = \eta_{j-1,j}v = h_{Y_j} \otimes (v_{Y_{j-1}} + v_{Y_j}) + w$$

with $w \in W_j$ by (5.3). It follows that $v_{Y_{j-1}} = -v_{Y_j}$ and hence

$$v_{Y_m} = (-1)^{r+1-m} v_{Y_{r+1}} = (-1)^{r+1-m} v_{[r]} \tag{5.4}$$

for $1 \leq m \leq r+1$ because $Y_{r+1} = [r]$.

Next we compute that, for $1 \leq i \leq r-1$,

$$0 = \eta_{i,r+1}v = h_{Y_{r+1}} \otimes ((i\ i+1\ \cdots\ r)v_{Y_i} + v_{Y_{r+1}}) + w$$

with $w \in W_{r+1}$ by (5.3). Therefore, by (5.4), we have that

$$0 = (i\ i+1\ \cdots\ r)(-1)^{r+1-i} v_{[r]} + v_{[r]}$$

and so

$$(i\ i+1\ \cdots\ r)v_{[r]} = (-1)^{r-i} v_{[r]} = \mathrm{sgn}((i\ i+1\ \cdots\ r))v_{[r]} \tag{5.5}$$

for all $1 \leq i \leq r-1$. Since $(r-1\ r)$ and $(1\ 2\ \cdots\ r)$ generate $S_r$ and $v_{[r]} \neq 0$, we conclude from (5.5) that $\Bbbk S_r v_{[r]} \cong S_{(1^r)}$ contradicting that $\lambda \neq (1^r)$. This contradiction completes the proof. $\square$

The reader will be asked in the exercises to prove that $\mathrm{Ind}_{G_{e_r}}(S_\lambda) \cong \Bbbk I_{n,r} \otimes_{\Bbbk S_r} S_\lambda$.

Let $M$ be a monoid. If $\mathbb{F}$ is a subfield of $\Bbbk$, then we say that a $\Bbbk M$-module $V$ is *defined over* $\mathbb{F}$ if there is an $\mathbb{F}M$-module $W$ such that $V \cong \Bbbk \otimes_{\mathbb{F}} W$. Equivalently, $V$ is defined over $\mathbb{F}$ if there is a basis for $V$ such that the matrix representation $\rho\colon M \longrightarrow M_n(\Bbbk)$ afforded by $V$ with respect to this basis takes values in $M_n(\mathbb{F})$.

**Corollary 5.11.** *Let $\Bbbk$ be a field of characteristic $0$. Then each simple $\Bbbk T_n$-module is defined over $\mathbb{Q}$.*

*Proof.* Clearly, $\Lambda^r(\mathrm{Aug}(\Bbbk^n)) = \Bbbk \otimes_{\mathbb{Q}} \Lambda^r(\mathrm{Aug}(\mathbb{Q}^n))$ for $0 \leq r \leq n-1$. Since $c_\lambda \in \mathbb{Q}S_r$ for $\lambda$ a partition of $r$, it follows that $\Bbbk I_{n,r}c_\lambda \cong \Bbbk \otimes_{\mathbb{Q}} \mathbb{Q}I_{n,r}c_\lambda$. We conclude that the simple $\Bbbk T_n$-modules are defined over $\mathbb{Q}$ by Theorem 5.9 and Theorem 5.10. $\square$

The following corollary computes the restriction of $e_m S_\lambda^\sharp$ to $\Bbbk G_{e_m}$ for the case $m \geq r$ and $\lambda \neq (1^r)$. It will also be used when we compute the character table of the symmetric inverse monoid. The reader is referred to Section B.4 for the definition of the outer product $V \boxtimes W$ of symmetric group representations.

**Corollary 5.12.** *Let $\Bbbk$ be a field of characteristic $0$ and $n \geq 1$. Let $\lambda$ be a partition of $r$ with $0 \leq r \leq n$. Then $\Bbbk I_{n,r} \otimes_{\Bbbk S_r} S_\lambda \cong S_\lambda \boxtimes S_{(n-r)}$ as a $\Bbbk S_n$-module. Consequently, if $1 \leq r \leq m \leq n$ and $\lambda \neq (1^r)$ is a partition of $r$, then $e_m S_\lambda^\sharp \cong S_\lambda \boxtimes S_{(m-r)}$ as a $\Bbbk S_m$-module.*

*Proof.* The case $r = 0$ is trivial as $\Bbbk I_{n,0} \cong S_{(n)}$ is the trivial $\Bbbk S_n$-module and $S_\lambda$ is the trivial $\Bbbk S_0$-module. So assume that $r \geq 1$. We retain the notation $h_Y$ and $g_Y$ from the proof of Theorem 5.10 for an $r$-element subset $Y$ of $[n]$.

The group $S_n$ acts transitively on the set $A$ of $r$-element subsets of $[n]$ and $S_r \times S_{n-r}$ is the stabilizer of $[r]$. It follows that the set of $g_Y$ with $Y \in A$ is a set of left coset representatives of $S_r \times S_{n-r}$ in $S_n$. Thus $\Bbbk S_n$ is a free right $\Bbbk[S_r \times S_{n-r}]$-module with with basis the $g_Y$ and hence there is a $\Bbbk$-vector space decomposition

$$S_\lambda \boxtimes S_{(n-r)} = \bigoplus_{Y \in A} g_Y \otimes S_\lambda \otimes S_{(n-r)}.$$

Moreover, since $S_{(n-r)}$ is the trivial module, the $S_n$-action is given by

$$\begin{aligned} f(g_Y \otimes v \otimes 1) &= fg_Y \otimes v \otimes 1 \\ &= g_{f(Y)} \otimes (g_{f(Y)}^{-1} f g_Y)(v \otimes 1) \\ &= g_{f(Y)} \otimes (h_{f(Y)}^{-1} f h_Y) v \otimes 1 \end{aligned}$$

for $f \in S_n$. Comparing with (5.1), we conclude that $g_Y \otimes v \otimes 1 \mapsto h_Y \otimes v$, for $Y \in A$, provides a $\Bbbk S_n$-module isomorphism of $S_\lambda \boxtimes S_{(n-r)}$ with the restriction of $\Bbbk I_{n,r} \otimes_{\Bbbk S_r} S_\lambda$ to $\Bbbk S_n$. This completes the proof.

The final statement follows from Theorem 5.10 and the previous case via the observation $e_m \Bbbk I_{n,r} = \Bbbk I_{m,r}$ where we view $I_{m,r}$ as a subset of $I_{n,r}$. □

## 5.4 Semisimplicity

In this section, we characterize when $\Bbbk M$ is semisimple for a finite monoid $M$. The approach we pursue here differs from the classical approach in [CP61, Chapter 5] in that it is module theoretic and uses the machinery we have been developing, rather than the theory of semigroup algebras and Munn algebras.

If $J_a$ is the $\mathscr{J}$-class of $a \in M$, then $I = MaM \setminus J_a$ is an ideal of $M$. Therefore, $\Bbbk[MaM]/\Bbbk I$ is an ideal of $\Bbbk M/\Bbbk I$. As a vector space it is $\Bbbk J_a$. The left $\Bbbk M$-module structure is given by

$$m \odot x = \begin{cases} mx, & \text{if } mx \in J_a \\ 0, & \text{if } mx \notin J_a \end{cases}$$

for $m \in M$ and $x \in J_a$. We omit the symbol "$\odot$" from now on. We shall repeatedly need the following lemma.

**Lemma 5.13.** *Let $e \in E(M)$ and let $n$ be the number of $\mathscr{L}$-classes in $J_e$. Then $\Bbbk J_e \cong n \cdot \Bbbk L_e$ as a left $\Bbbk M$-module.*

*Proof.* By Theorem 1.12, if $f \in E(J_e)$, then $\Bbbk L_f$ is a submodule of $\Bbbk J_e$. Moreover, since $J_e$ is the disjoint union of its $\mathscr{L}$-classes and each $\mathscr{L}$-class contains an idempotent (by Proposition 1.20), it suffices to show that $MeM = MfM$ implies that $\Bbbk L_e \cong \Bbbk L_f$. But $MeM = MfM$ implies that $I(e) = I(f)$ and that $Me$ and $Mf$ are isomorphic $M$-sets by Theorem 1.10. Moreover, stability (Theorem 1.12) implies that $Me \setminus I(e)e = L_e$ and $Mf \setminus I(f)f = L_f$. Since any isomorphism $Me \longrightarrow Mf$ of $M$-sets takes the set $L_e$ of generators of $Me$ to the set $L_f$ of generators of $Mf$, we conclude that $\Bbbk L_e = \Bbbk Me/\Bbbk I(e)e \cong \Bbbk Mf/\Bbbk I(f)f = \Bbbk L_f$, as required. □

Let $e \in E(M)$ and let $W$ be a $\Bbbk G_e$-module. Then, putting $A_e = \Bbbk M/\Bbbk I(e)$, we have the natural homomorphism of $A_e$-modules

$$\varphi_W\colon \operatorname{Ind}_{G_e}(W) \longrightarrow \operatorname{Coind}_{G_e}(W)$$

from Corollary 4.10. For $\ell \in L_e$, $w \in W$ and $r \in R_e$, it is given by

$$\varphi_W(\ell \otimes w)(r) = (r \diamond \ell)w$$

where

$$r \diamond \ell = \begin{cases} r\ell, & \text{if } r\ell \in G_e \\ 0, & \text{else} \end{cases}$$

(note that $r\ell \in eMe$).

**Theorem 5.14.** *Let $M$ be a finite monoid and $\Bbbk$ a field. Then $\Bbbk M$ is semisimple if and only if:*

*(a) $M$ is regular;*
*(b) the characteristic of $\Bbbk$ does not divide the order of $G_e$ for any $e \in E(M)$;*
*(c) the natural homomorphism $\varphi_{\Bbbk G_e}\colon \operatorname{Ind}_{G_e}(\Bbbk G_e) \longrightarrow \operatorname{Coind}_{G_e}(\Bbbk G_e)$ is an isomorphism for all $e \in E(M)$.*

*Proof.* Let us first assume that $\Bbbk M$ is semisimple. If $M$ is not regular, then there is an ideal $I$ with $I^2 \subsetneq I$ by Corollary 1.23. Then $\Bbbk I/\Bbbk I^2$ is a non-zero nilpotent ideal of $\Bbbk M/\Bbbk I^2$ and hence $\Bbbk M/\Bbbk I^2$ is not semisimple. As quotients of semisimple algebras are semisimple (Proposition A.8), we conclude that $\Bbbk M$ is not semisimple, a contradiction. It follows that $M$ is regular. Let $e \in E(M)$. Then $A_e = \Bbbk M/\Bbbk I(e)$ is semisimple by Proposition A.8 and hence $eA_e e \cong \Bbbk G_e$ is semisimple by Proposition 4.15. Therefore, the characteristic of $\Bbbk$ does not divide $|G_e|$ by Maschke's theorem. Finally, since $\Bbbk G_e$ is a semisimple $eA_e e$-module and $\operatorname{Ind}_{G_e}(\Bbbk G_e)$, $\operatorname{Coind}_{G_e}(\Bbbk G_e)$ are semisimple $A_e$-modules, we have that $\varphi_{\Bbbk G_e}$ is an isomorphism by Corollary 4.19.

For the converse, Maschke's theorem yields that $\Bbbk G_e$ is semisimple for all $e \in E(M)$. Note that, for $e \in E(M)$, we have

$$\operatorname{Ind}_{G_e}(\Bbbk G_e) = \Bbbk L_e \otimes_{\Bbbk G_e} \Bbbk G_e \cong \Bbbk L_e.$$

Consider a principal series

$$\emptyset = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_s = M$$

for $M$. Let $J_k = I_k \setminus I_{k-1}$ for $k = 1, \ldots, s$. Then $J_k$ is a regular $\mathscr{J}$-class (cf. Proposition 1.18) and $\Bbbk J_k \cong \Bbbk I_k/\Bbbk I_{k-1}$. Fix $e_k \in E(J_k)$ and let $n_k$ be the number of $\mathscr{L}$-classes in $J_k$. Then by Lemma 5.13 we have an isomorphism

$$\Bbbk I_k/\Bbbk I_{k-1} \cong \Bbbk J_k \cong n_k \cdot \Bbbk L_{e_k} \tag{5.6}$$

of $\Bbbk M$-modules.

We claim that

$$\Bbbk M/\Bbbk I_{k-1} \cong \Bbbk J_k \oplus \Bbbk M/\Bbbk I_k \tag{5.7}$$

as a $\Bbbk M$-module for $1 \leq k \leq s$. Indeed, let $A = \Bbbk M/\Bbbk I_{k-1}$; it is a finite dimensional $\Bbbk$-algebra. Then we have that $e_k A e_k \cong \Bbbk G_{e_k}$ (by Corollary 1.15), $\mathrm{Ind}_{e_k}(\Bbbk G_{e_k}) = \mathrm{Ind}_{G_{e_k}}(\Bbbk G_{e_k})$ and $\mathrm{Coind}_{e_k}(\Bbbk G_{e_k}) = \mathrm{Coind}_{G_{e_k}}(\Bbbk G_{e_k})$. Because $\Bbbk G_{e_k}$ is semisimple, all of its modules are injective. Therefore, $\mathrm{Coind}_{G_{e_k}}(\Bbbk G_{e_k})$ is an injective $A$-module by Proposition 4.3. Since $\varphi_{\Bbbk G_{e_k}}$ is an isomorphism we conclude that $\mathrm{Ind}_{G_{e_k}}(\Bbbk G_{e_k}) \cong \Bbbk L_{e_k}$ is an injective $A$-module. From (5.6) we then conclude that $\Bbbk J_k$ is an injective $A$-module. Thus the exact sequence of $A$-modules

$$0 \longrightarrow \Bbbk J_k \longrightarrow \Bbbk A \longrightarrow \Bbbk M/\Bbbk I_k \longrightarrow 0$$

splits, establishing (5.7).

Applying (5.6) and (5.7) repeatedly, we obtain

$$\Bbbk M \cong \bigoplus_{k=1}^{s} \Bbbk J_k \cong \bigoplus_{k=1}^{s} n_k \cdot \Bbbk L_{e_k}. \tag{5.8}$$

But $\Bbbk L_e = \mathrm{Ind}_{G_e}(\Bbbk G_e)$ is semisimple, for $e \in E(M)$, by Corollary 4.19 because $\Bbbk G_e$ is semisimple and $\varphi_{\Bbbk G_e}$ is an isomorphism. It follows that $\Bbbk M$ is a semisimple $\Bbbk M$-module and hence $\Bbbk M$ is a semisimple algebra. This completes the proof. $\square$

In the case that $\Bbbk M$ is semisimple, if $V$ is a simple $\Bbbk G_e$-module, then $V^\sharp = \mathrm{Ind}_{G_e}(V)/\operatorname{rad}(\mathrm{Ind}_{G_e}(V)) = \mathrm{Ind}_{G_e}(V)$ and similarly, $V^\sharp = \operatorname{soc}(\mathrm{Coind}_{G_e}(V)) = \mathrm{Coind}_{G_e}(V)$.

Note that Exercise 4.4 shows that $\varphi_{\Bbbk G_e}$ is an isomorphism if and only if $\Bbbk L_e \cong \mathrm{Ind}_{G_e}(\Bbbk G_e) \cong \mathrm{Coind}_{G_e}(\Bbbk G_e) = \mathrm{Hom}_{G_e}(R_e, \Bbbk G_e)$. In Chapter 17 a homological proof of Theorem 5.14 will be given.

Classically, Theorem 5.14 is stated in the language of sandwich matrices. Let $e \in E(M)$. Then, by Proposition 1.9, $L_e$ is a free right $G_e$-set and $R_e$ is a free left $G_e$-set. Let $\lambda_1, \ldots, \lambda_\ell$ be a complete set of representatives for $L_e/G_e$ and $\rho_1, \ldots, \rho_r$ be a complete set of representatives for $G_e \backslash R_e$. Note that $\ell$ is the number of $\mathscr{R}$-classes contained in $J_e$ and $r$ is the number of $\mathscr{L}$-classes contained in $J_e$ by Corollary 1.16 and Exercise 1.14. As before, for $\lambda \in L_e$ and $\rho \in R_e$, put

$$\rho \diamond \lambda = \begin{cases} \rho\lambda, & \text{if } \rho\lambda \in G_e \\ 0, & \rho\lambda \in I_e. \end{cases}$$

Let $P(e)$ be the $r \times \ell$ matrix over $\Bbbk G_e$ with

$$P(e)_{ij} = \rho_i \diamond \lambda_j \in G_e \cup \{0\}.$$

One calls $P(e)$ a *sandwich matrix* for the $\mathscr{J}$-class $J_e$. One can prove that it depends on the choices that we have made only up to left and right multiplication by monomial matrices over $G_e$ (cf. [CP61, KRT68]).

Observe that $\Bbbk L_e = \mathrm{Ind}_{G_e}(\Bbbk G_e)$ is a free right $\Bbbk G_e$-module with basis $\lambda_1, \ldots, \lambda_\ell$ via the right action of $G_e$ on $L_e$. Also $\mathrm{Coind}_{G_e}(\Bbbk G_e) = \mathrm{Hom}_{G_e}(R_e, \Bbbk G_e)$ is a right $\Bbbk G_e$-module via $(\psi g)(x) = \psi(x)g$ for $g \in G_e$. Moreover, since $R_e$ is a free left $G_e$-set with orbit representatives $\rho_1, \ldots, \rho_r$, it follows that $\mathrm{Coind}_{G_e}(\Bbbk G_e)$ is a free $\Bbbk G_e$-module with basis $\rho_1^*, \ldots, \rho_r^*$ where

$$\rho_i^*(g\rho_j) = \begin{cases} g, & \text{if } i = j \\ 0, & \text{else} \end{cases}$$

for $g \in G_e$.

**Lemma 5.15.** *Let $\varphi_{\Bbbk G_e}\colon \mathrm{Ind}_{G_e}(\Bbbk G_e) \longrightarrow \mathrm{Coind}_{G_e}(\Bbbk G_e)$ be the natural homomorphism. Then the sandwich matrix $P(e)$ is the matrix for $\varphi_{\Bbbk G_e}$, as a homomorphism of free right $\Bbbk G_e$-modules, with respect to the bases $\lambda_1, \ldots, \lambda_\ell$ for $\Bbbk L_e = \mathrm{Ind}_{G_e}(\Bbbk G_e)$ and $\rho_1^*, \ldots, \rho_r^*$ for $\mathrm{Hom}_{G_e}(R_e, \Bbbk G_e) = \mathrm{Coind}_{G_e}(\Bbbk G_e)$.*

*Proof.* We compute

$$\varphi_{\Bbbk G_e}(\ell_j)(g\rho_k) = (g\rho_k) \diamond \ell_j = g(\rho_k \diamond \ell_j) = gP(e)_{kj} = \sum_{i=1}^{r} \rho_i^*(g\rho_k)P(e)_{ij}$$

for $g \in G$. Therefore, we conclude that

$$\varphi_{\Bbbk G_e}(\ell_j) = \sum_{i=1}^{r} \rho_i^* \cdot P(e)_{ij},$$

as was required. $\square$

Lemma 5.15 allows us to reformulate Theorem 5.14 in its classical form.

**Theorem 5.16.** *Let $M$ be a finite monoid and $\Bbbk$ a field. Then $\Bbbk M$ is semisimple if and only if:*

*(a) $M$ is regular;*

*(b) the characteristic of* $\Bbbk$ *does not divide the order of* $G_e$ *for any* $e \in E(M)$*;*
*(c) the sandwich matrix* $P(e)$ *is invertible over* $\Bbbk G_e$ *for all* $e \in E(M)$*.*

In principle, this theorem reduces determining semisimplicity of a monoid algebra to checking invertibility of a matrix over a group algebra, but the latter problem is not a simple one.

*Example 5.17.* If $\Bbbk$ is any field and $n \geq 2$, then $\Bbbk T_n$ is not semisimple because the sandwich matrix $P(e_1)$ is $1 \times n$ where $e_1$ is the constant mapping to 1.

*Example 5.18.* Let $\Bbbk$ be a field of characteristic 0 and $M$ a finite commutative monoid. Then because $\mathscr{L} = \mathscr{R} = \mathscr{J}$, each sandwich matrix $P(e)$ is $1 \times 1$ and hence invertible over $\Bbbk G_e$. We conclude from Theorem 5.16 that $\Bbbk M$ is semisimple if and only if $M$ is regular (or equivalently an inverse monoid by Theorem 3.2).

*Example 5.19.* Let $I_2$ be the symmetric inverse monoid of degree 2 and $\Bbbk$ a field of characteristic 0. Then $I_2$ has three $\mathscr{J}$-classes $J_0, J_1, J_2$ where $J_i$ consists of the rank $i$ partial injective mappings. Let $e_i = 1_{[i]}$ for $i = 0, 1, 2$. Then the $e_i$ are idempotent representatives of the $\mathscr{J}$-classes of $I_2$ and $G_{e_i} \cong S_i$. Clearly $J_0 = \{e_0\}$ and hence $P(e_0) = (e_0)$, which is invertible over $\Bbbk G_{e_0}$. Also, $J_2 = S_2 = G_{e_2}$ contains exactly one $\mathscr{R}$-class and one $\mathscr{L}$-class. If we take $\lambda_1 = e_2 = \rho_1$, then $P(e_2) = (e_2)$, which is invertible over $\Bbbk S_2$. Finally, if we let $p_{ij}$ denote the rank one partial mapping taking $j$ to $i$, then, for $J_1$, we may take $\lambda_1 = e_1$, $\lambda_2 = p_{21}$, $\lambda_3 = p_{31}$, $\rho_1 = e_1 = p_{11}$, $\rho_2 = p_{12}$ and $\rho_3 = p_{13}$. Then

$$\rho_i \diamond \lambda_j = \begin{cases} e_1, & \text{if } i = j \\ 0, & \text{else.} \end{cases}$$

and so $P(e_1)$ is an identity matrix and hence invertible over $\Bbbk G_{e_1}$. We conclude that $\Bbbk I_2$ is semisimple by Theorem 5.16.

In Exercise 5.16, the reader will be asked to show that if $M$ is an inverse monoid, then each sandwich matrix may be chosen to be an identity matrix and hence $\Bbbk M$ will be semisimple provided that the characteristic of $\Bbbk$ does not divide the order of any maximal subgroup of $M$. Chapter 10 will provide a different approach to the semisimplicity of inverse monoid algebras in good characteristic.

We end this section with a description of $\Bbbk M$ in the semisimple case.

**Theorem 5.20.** *Let* $M$ *be a finite monoid and* $\Bbbk$ *a field such that* $\Bbbk M$ *is semisimple. Let* $e_1, \ldots, e_s$ *be a complete set of representatives of the* $\mathscr{J}$*-classes of idempotents of* $M$ *and suppose that* $J_{e_i}$ *contains* $n_i$ $\mathscr{L}$*-classes. Then there is an isomorphism*

$$\Bbbk M \cong \prod_{i=1}^{s} M_{n_i}(\Bbbk G_{e_i})$$

*of* $\Bbbk$*-algebras.*

*Proof.* Under the hypotheses of the theorem, we have the isomorphism of $\Bbbk M$-modules $\Bbbk M \cong \bigoplus_{i=1}^{s} n_i \cdot \Bbbk L_{e_i}$ by (5.8). Let $V_1, \ldots, V_r$ be a complete set of representatives of the isomorphism classes of simple $\Bbbk G_{e_i}$-modules. From the decomposition $\Bbbk G_{e_i} = \bigoplus_{j=1}^{r} \dim V_j \cdot V_j$ (coming from the semisimplicity of $\Bbbk G_{e_i}$ and Theorem A.7), we obtain that

$$\Bbbk L_{e_i} = \mathrm{Ind}_{G_{e_i}}(\Bbbk G_{e_i}) = \bigoplus_{j=1}^{r} \dim V_j \cdot \mathrm{Ind}_{G_{e_i}}(V_j)$$

is the decomposition of $\Bbbk L_{e_i}$ into a direct sum of simple $\Bbbk M$-modules, all of which have apex $e_i$. It follows that $\mathrm{Hom}_{\Bbbk M}(\Bbbk L_{e_i}, \Bbbk L_{e_j}) = 0$ if $i \neq j$. We conclude that

$$\Bbbk M^{op} \cong \mathrm{End}_{\Bbbk M}(\Bbbk M) \cong \prod_{i=1}^{s} M_{n_i}(\mathrm{End}_{\Bbbk M}(\Bbbk L_{e_i})).$$

But by Proposition 4.1, we have that

$$\begin{aligned}\mathrm{End}_{\Bbbk M}(\Bbbk L_{e_i}) &= \mathrm{Hom}_{A_{e_i}}(\mathrm{Ind}_{e_i}(\Bbbk G_{e_i}), \mathrm{Ind}_{e_i}(\Bbbk G_{e_i}))\\ &\cong \mathrm{Hom}_{\Bbbk G_{e_i}}(\Bbbk G_{e_i}, \mathrm{Res}_{e_i}(\mathrm{Ind}_{e_i}(\Bbbk G_{e_i})))\\ &\cong \mathrm{End}_{\Bbbk G_{e_i}}(\Bbbk G_{e_i}) \cong \Bbbk G_{e_i}^{op}.\end{aligned}$$

The desired isomorphism now follows. □

The conclusion of Theorem 5.20 holds more generally if $M$ is regular and the sandwich matrix $P(e)$ is invertible over $\Bbbk G_e$ for all $e \in E(M)$, as we shall see in Chapter 16.

## 5.5 Monomial representations

In this section, we show that modules of the form $\mathrm{Ind}_{G_e}(V)$ and $\mathrm{Coind}_{G_e}(V)$ afford representations with a monomial form. This can be viewed as a module theoretic treatment of the results of [LP69, RZ91]. We will work out in detail the case of $\mathrm{Ind}_{G_e}(V)$ and merely state the result for $\mathrm{Coind}_{G_e}(V)$, leaving the details to the reader in the exercises.

Fix an idempotent $e \in E(M)$. Let $\lambda_1, \ldots, \lambda_\ell$ be a complete set of representatives for $L_e/G_e$. Let $m \in M$. Then either $m\lambda_j \notin L_e$ or $m\lambda_j = \lambda_i g$ for a unique $g \in G_e$ and $i \in \{1, \ldots, \ell\}$. Let us define an $\ell \times \ell$ matrix $\Theta_e(m)$ over $G_e \cup \{0\} \subseteq \Bbbk G_e$ by

$$\Theta_e(m)_{ij} = \begin{cases} g, & \text{if } m\lambda_j = \lambda_i g \\ 0, & m\lambda_j \notin L_e. \end{cases}$$

Then $\Theta_e \colon M \longrightarrow M_\ell(\Bbbk G_e)$ is precisely the matrix representation afforded by $\Bbbk L_e$, viewed as a $\Bbbk M$-$\Bbbk G_e$-bimodule which is free over $\Bbbk G_e$. Classically, $\Theta_e$ is called the *left Schützenberger representation* of $M$ associated to the $\mathscr{J}$-class $J_e$. Note that $\Theta_e$ has at most one non-zero entry in each column. Such a matrix is called *column monomial*.

Suppose now that $V$ is a finite dimensional $\Bbbk G_e$-module. Then as a $\Bbbk$-vector space, we have that

$$\mathrm{Ind}_{G_e}(V) = \Bbbk L_e \otimes_{\Bbbk G_e} V = \bigoplus_{j=1}^{\ell} \lambda_j \otimes V$$

because $\Bbbk L_e$ is a free right $\Bbbk G_e$-module with basis $\lambda_1, \ldots, \lambda_\ell$. Moreover, if $v_1, \ldots, v_n$ is a basis for $V$, then the $\lambda_j \otimes v_k$ with $1 \leq j \leq \ell$ and $1 \leq k \leq n$ is a basis for $\mathrm{Ind}_{G_e}(V)$.

**Proposition 5.21.** *If $m \in M$ and $v \in V$, then*

$$m(\lambda_j \otimes v) = \sum_{i=1}^{\ell} \lambda_i \otimes \Theta_e(m)_{ij} v.$$

*Equivalently, the equality*

$$m(\lambda_j \otimes v) = \begin{cases} \lambda_i \otimes gv, & \text{if } m\lambda_j = \lambda_i g \\ 0, & \text{if } m\lambda_j \notin L_e \end{cases}$$

*holds.*

*Proof.* Indeed, if $m\lambda_j \notin L_e$, then $m\lambda_j = 0$ in $\Bbbk L_e$ and so $m(\lambda_j \otimes v) = m\lambda_j \otimes v = 0$. If $m\lambda_j = \lambda_i g$ with $g \in G_e$, then $m(\lambda_j \otimes v) = m\lambda_j \otimes v = \lambda_i g \otimes v = \lambda_i \otimes gv$, as required. □

As a consequence, we obtain the following block column monomial form for induced modules.

**Theorem 5.22.** *Let $V$ be a $\Bbbk G_e$-module affording a matrix representation $\alpha \colon G_e \longrightarrow M_n(\Bbbk)$. Then $\mathrm{Ind}_{G_e}(V)$ affords the matrix representation*

$$\mathrm{Ind}_{G_e}(\alpha) \colon M \longrightarrow M_{\ell n}(\Bbbk)$$

*given by the $\ell \times \ell$-block form*

$$\mathrm{Ind}_{G_e}(\alpha)(m)_{ij} = \alpha(\Theta_e(m)_{ij})$$

*where $\alpha(0)$ is the $n \times n$ zero matrix.*

*Proof.* Let $v_1, \ldots, v_n$ be a basis for $V$ giving rise to $\alpha$. Consider the basis

$$\{\lambda_j \otimes v_k \mid 1 \leq j \leq \ell, 1 \leq k \leq n\}$$

for $\mathrm{Ind}_{G_e}(V)$ with ordering $\lambda_1\otimes v_1,\ldots,\lambda_1\otimes v_n,\ldots,\lambda_\ell\otimes v_1,\ldots,\lambda_\ell\otimes v_n$. Then by Proposition 5.21 we have that

$$m(\lambda_j\otimes v_k)=\sum_{i=1}^{\ell}\lambda_i\otimes\Theta_e(m)_{ij}v_k=\sum_{i=1}^{\ell}\sum_{t=1}^{n}\alpha(\Theta_e(m)_{ij})_{tk}(\lambda_i\otimes v_t).$$

The theorem follows. □

The corresponding result for $\mathrm{Coind}_{G_e}(V)$ is as follows. Let $\rho_1,\ldots,\rho_r$ be a complete set of representatives for $G_e\backslash R_e$. Define $\Upsilon_e\colon M\longrightarrow M_r(\Bbbk G_e)$ by

$$\Upsilon_e(m)_{ij}=\begin{cases} g, & \text{if } \rho_i m=g\rho_j\\ 0, & \rho_i m\notin R_e.\end{cases}$$

Then $\Upsilon$ is the representation afforded by viewing $\Bbbk R_e$ as a $\Bbbk G_e$-$\Bbbk M$-bimodule which is free as a left $\Bbbk G_e$-module. It is called the *right Schützenberger representation* of $M$ associated to the $\mathscr{J}$-class $J_e$. Note that each row of $\Upsilon_e(m)$ contains at most one non-zero entry. Such a matrix is called *row monomial*. The corresponding block row monomial matrix form for coinduced modules is encapsulated in the next theorem.

**Theorem 5.23.** *Let $V$ be a $\Bbbk G_e$-module affording a matrix representation $\alpha\colon G_e\longrightarrow M_n(\Bbbk)$. Then $\mathrm{Coind}_{G_e}(V)$ affords the matrix representation*

$$\mathrm{Coind}_{G_e}(\alpha)\colon M\longrightarrow M_{rn}(\Bbbk)$$

*given by the $r\times r$-block form*

$$\mathrm{Coind}_{G_e}(\alpha)(m)_{ij}=\alpha(\Upsilon_e(m)_{ij})$$

*where $\alpha(0)$ is the $n\times n$ zero matrix.*

In the exercises, the reader will also prove the following theorem.

**Theorem 5.24.** *Let $V$ be a $\Bbbk G_e$-module affording a matrix representation $\alpha\colon G_e\longrightarrow M_n(\Bbbk)$. Let $P(e)$ be an $r\times\ell$ sandwich matrix for $J_e$ and let $P(e)\otimes\alpha$ be the block $rn\times\ell n$ matrix given by*

$$[P(e)\otimes\alpha]_{ij}=\alpha(P(e)_{ij})$$

*where $\alpha(0)$ is the $n\times n$ zero matrix. Then $P(e)\otimes\alpha$ is the matrix for the natural map $\varphi_V\colon\mathrm{Ind}_{G_e}(V)\longrightarrow\mathrm{Coind}_{G_e}(V)$ with respect to an appropriate basis. Therefore, $N_e(\mathrm{Ind}_{G_e}(V))$ can be identified with the null space of $P(e)\otimes\alpha$ and $K_e(\mathrm{Coind}_{G_e}(V))$ can be identified with the column space of $P(e)\otimes\alpha$.*

By considering the particular case where $V$ is simple, Theorem 5.24 provides a matrix theoretic description of the irreducible representations of $M$ in light of Theorem 5.5.

**Corollary 5.25.** *Let $V$ be a simple $\Bbbk G_e$-module affording an irreducible matrix representation $\alpha\colon G_e \longrightarrow M_n(\Bbbk)$. Let $P(e)$ be an $r \times \ell$ sandwich matrix for $J_e$ and let $P(e) \otimes \alpha$ be the block $rn \times \ell n$ matrix given by*

$$[P(e) \otimes \alpha]_{ij} = \alpha(P(e)_{ij})$$

*where $\alpha(0)$ is the $n \times n$ zero matrix. Then the simple module $V^\sharp$ can be identified with the quotient of $\Bbbk^{\ell n}$ by the null space of $P(e) \otimes \alpha$ and with the column space of $P(e) \otimes \alpha$.*

## 5.6 Exercises

**5.1.** Let $M$ be a monoid and $\Bbbk$ a field. Let $I$ be an ideal of $M$ and let $\Bbbk I$ be the $\Bbbk$-subspace spanned by $I$. Prove that $\Bbbk I$ is an ideal.

**5.2.** Verify that if $V, W$ are $\Bbbk M$-modules, then so is $V \otimes W$.

**5.3.** Let $G$ be a group acting freely on a set $X$ and let $\Bbbk$ be a field. Let $T \subseteq X$ be a complete set of representatives of the orbits of $G$ on $X$. Prove that $\Bbbk X$ is a free $\Bbbk G$-module with basis $T$.

**5.4.** Let $M$ be a monoid and $\Bbbk$ a field. Prove that $M$ is finitely generated as a monoid if and only if $\Bbbk M$ is finitely generated as a $\Bbbk$-algebra.

**5.5.** Let $M$ be a finite monoid and $\Bbbk$ a field. Suppose that $G_e = \{e\}$ and $J_e$ is a subsemigroup of $M$. Prove that the unique simple $\Bbbk M$-module with apex $e$ affords the one-dimensional representation $\chi_{J_e}\colon M \longrightarrow \Bbbk$ given by

$$\chi_{J_e}(m) = \begin{cases} 1, & \text{if } e \in MmM \\ 0, & \text{if } m \in I(e). \end{cases}$$

Deduce that if each maximal subgroup of $M$ is trivial and each regular $\mathscr{J}$-class of $M$ is a subsemigroup, then each simple $\Bbbk M$-module is one-dimensional.

**5.6.** Compute the dimension of $\mathrm{rad}(\mathbb{C}T_n)$.

**5.7.** Let $\rho\colon M \longrightarrow M_n(\Bbbk)$ be an irreducible representation with apex $e \in E(M)$. Extending $\rho$ to $\Bbbk M$, prove that the identity matrix belongs to $\rho(\Bbbk I)$ where $I = MeM$. (Hint: $A = \rho(\Bbbk M)$ is simple by Proposition A.12.)

**5.8.** Suppose that $M$ is a non-trivial finite monoid with a faithful irreducible representation $\rho\colon M \longrightarrow M_n(\Bbbk)$ with apex $e \in E(M)$.

(a) Prove $I = MeM$ is the unique minimal non-zero ideal of $M$.
(b) Prove that $M$ acts faithfully on both the left and right of $I$. (Hint: use Exercise 5.7.)

**5.9.** Prove that $\mathrm{Coind}_{G_e}(\Bbbk G_e) \cong D(\Bbbk R_e) = \mathrm{Hom}_{\Bbbk}(\Bbbk R_e, \Bbbk) \cong \Bbbk^{R_e}$. (Hint: let $\tau\colon \Bbbk G_e \longrightarrow \Bbbk$ be the functional defined by

$$\tau(g) = \begin{cases} 1, & \text{if } g \neq e \\ 0, & \text{else;} \end{cases}$$

map $\varphi \in \mathrm{Hom}_{G_e}(R_e, \Bbbk G_e)$ to $\tau \circ \varphi\colon R_e \longrightarrow \Bbbk$.)

**5.10.** Let $r, n \geq 1$. Prove that $\Bbbk T_{n,r} \cong (\Bbbk^n)^{\otimes r}$ as $\Bbbk T_n$-$\Bbbk S_r$-bimodules where $S_r$ acts on the right of $(\Bbbk^n)^{\otimes r}$ by permuting the tensor factors.

**5.11.** Let $n \geq 1$ and let $e_r$ be the idempotent from Section 5.3 where $1 \leq r \leq n$. Prove that $\mathrm{Ind}_{G_{e_r}}(V) \cong \Bbbk I_{n,r} \otimes_{\Bbbk S_r} V$ for a $\Bbbk S_r$-module $V$.

**5.12.** Let $1 \leq r \leq n$ and let $\Bbbk$ be a field of characteristic 0. We retain the notation of Section 5.3.

(a) Prove that $\Bbbk I_{n,r} \otimes_{\Bbbk S_r} S_{(1^r)} \cong \Lambda^r(\Bbbk^n)$ as $\Bbbk T_n$-modules.
(b) Prove that $\Lambda^r(\Bbbk^n)/\Lambda^r(\mathrm{Aug}(\Bbbk^n)) \cong \Lambda^{r-1}(\mathrm{Aug}(\Bbbk^n))$ as $\Bbbk T_n$-modules.
(c) Deduce that $\mathrm{rad}(\Lambda^r(\Bbbk^n)) = \Lambda^r(\mathrm{Aug}(\Bbbk^n))$.
(d) Prove that $\Lambda^r(\Bbbk^n) \cong \Bbbk T_n \varepsilon$ where

$$\varepsilon = \frac{1}{r!} \sum_{g \in G_{e_r}} \mathrm{sgn}(g|_{[r]})g$$

is an idempotent. (Hint: Exercise 5.10 may be useful.)
(e) Deduce that $\Lambda^r(\Bbbk^n)$ is a projective indecomposable module with simple top $\Lambda^{r-1}(\mathrm{Aug}(\Bbbk^n))$.

**5.13.** Let $\Bbbk$ be a field of characteristic 0. Adapt the proof of Theorem 5.10 to show that if $e_r$ is a rank $r$ idempotent of $PT_n$ (respectively, $I_n$) and $S_\lambda$ is a simple $\Bbbk S_r$-module, then $\mathrm{Ind}_{G_{e_r}}(S_\lambda)$ is a simple $\Bbbk PT_n$-module (respectively, $\Bbbk I_n$-module).

**5.14.** Let $M$ be a finite monoid and $e \in E(M)$.

(a) Prove that $J_e$ is a subsemigroup if and only if the sandwich matrix $P(e)$ has no zeroes.
(b) Use Corollary 5.25 to give another proof of Exercise 5.5.

**5.15.** Prove Theorem 5.24 and Corollary 5.25.

**5.16.** Let $M$ be finite inverse monoid and $e \in E(M)$. Prove that $P(e)$ can be chosen to be an identity matrix. Deduce that $\Bbbk M$ is semisimple as long as the characteristic of $\Bbbk$ does not divide the order of any maximal subgroup of $M$. (Hint: use Corollary 1.16 and Exercise 3.5.)

**5.17.** Compute sandwich matrices for each $\mathscr{J}$-class of $T_2$, $T_3$ and $T_4$.

**5.18.** Compute matrix representations afforded by the simple $\mathbb{C}T_2$ and $\mathbb{C}T_3$-modules.

**5.19.** Let $M_2(\mathbb{F}_2)$ be the multiplicative monoid of all $2 \times 2$ matrices over the two-element field. Prove that $\Bbbk M_2(\mathbb{F}_2)$ is semisimple if the characteristic of $\Bbbk$ is neither 2 nor 3.

**5.20 (Putcha).** An ideal $I$ of a finite dimensional algebra $A$ is called a *heredity ideal* if the following hold.

(i) $I = AeA$ for some idempotent $e \in A$.
(ii) $eAe$ is semisimple.
(iii) $I$ is a projective (left) $A$-module.

An algebra $A$ is *quasi-hereditary* if there is a filtration

$$\{0\} = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_n = A \tag{5.9}$$

of $A$ by ideals such that $I_k/I_{k-1}$ is a heredity ideal of $A/I_{k-1}$ for $1 \leq k \leq n$. One calls (5.9) a *heredity chain*.

Prove that if $M$ is a regular monoid and $\Bbbk$ is a field of characteristic 0, then $\Bbbk M$ is quasi-hereditary. (Hint: use a principal series for $M$ to construct a heredity chain for $\Bbbk M$; can you find a shorter chain?)

# Part III

# Character Theory

# 6

# The Grothendieck Ring

In this chapter, we introduce the Grothendieck ring of a finite monoid $M$ over a field $\Bbbk$. The main result is that the Grothendieck ring of $M$ is isomorphic to the direct product of the Grothendieck rings of its maximal subgroups (one per regular $\mathscr{J}$-class). This result was first proved by McAlister for $\Bbbk = \mathbb{C}$ in the language of virtual characters [McA72]. Throughout this chapter we hold fixed a finite monoid $M$ and a field $\Bbbk$.

## 6.1 The Grothendieck ring

If $V$ is a $\Bbbk M$-module, then we denoted by $[V]$ the isomorphism class of $V$. As an abelian group, the *Grothendieck ring* $G_0(\Bbbk M)$ of $M$ with respect to $\Bbbk$ is the free abelian group on the set of isomorphism classes of finite dimensional $\Bbbk M$-modules modulo the relations $[V] = [U]+[W]$ if there is an exact sequence of the form

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0.$$

Notice that $[V] + [W] = [V \oplus W]$, from which it follows that every element of $G_0(\Bbbk M)$ can be written in the form $[V] - [W]$ with $V, W$ finite dimensional $\Bbbk M$-modules. A ring structure is defined by putting $[V][W] = [V \otimes W]$ and extending linearly, where all unlabelled tensor products are over the ground field $\Bbbk$. The identity is the trivial module $\Bbbk$.

**Proposition 6.1.** *Let $M$ be a monoid and $\Bbbk$ a field. Then $G_0(\Bbbk M)$ is a commutative ring.*

*Proof.* The tensor product of modules is commutative and associative up to isomorphism. Also, the natural map $\Bbbk \otimes V \longrightarrow V$ given by $c \otimes v \mapsto cv$ is a $\Bbbk M$-module isomorphism if $\Bbbk$ is given the structure of the trivial module. Therefore, the free abelian group on the set of isomorphism classes of finite dimensional $\Bbbk M$-modules can be made a unital commutative ring by putting

$[V][W] = [V \otimes W]$ and extending linearly where the trivial module is the multiplicative identity.

We then just need to show that the subgroup generated by the set of elements of the form $[V] - ([U] + [W])$ for which there is an exact sequence of the form

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$$

is an ideal. But this is clear because tensoring over a field is exact. It follows that $G_0(\Bbbk M)$ is a ring. □

As an abelian group, $G_0(\Bbbk M)$ turns out to be free with basis the simple modules. To prove this, let us first establish some notation and a basic lemma. If $V$ is a finite dimensional $\Bbbk M$-module and $S$ is a simple $\Bbbk M$-module, we denote by $[V : S]$ the multiplicity of $S$ as a composition factor of $V$.

**Lemma 6.2.** *Suppose that*

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$$

*is an exact sequence of finite dimensional* $\Bbbk M$*-modules and that* $S$ *is a simple* $\Bbbk M$*-module. Then* $[V : S] = [U : S] + [W : S]$.

*Proof.* Without loss of generality, we may assume that $U$ is contained in $V$ and $W = V/U$. Let $0 = U_0 \subseteq \cdots \subseteq U_r = U$ and $0 = W_0/U \subseteq \cdots \subseteq W_s/U = W$ be composition series with $U \subseteq W_i$. Then we have that

$$0 = U_0 \subseteq \cdots \subseteq U_r \subseteq W_1 \subseteq \cdots \subseteq W_s = V$$

is a composition series for $V$. From the isomorphism

$$W_{i+1}/W_i \cong (W_{i+1}/U)/(W_i/U)$$

and the Jordan-Hölder theorem, we conclude $[V : S] = [U : S] + [W : S]$. □

Now we prove that the isomorphism classes of simple $\Bbbk M$-modules form a basis for $G_0(\Bbbk M)$.

**Proposition 6.3.** *The additive group of* $G_0(\Bbbk M)$ *is free abelian with basis the set of isomorphism classes of simple* $\Bbbk M$*-modules. Moreover, if* $V$ *is a finite dimensional* $\Bbbk M$*-module, then the decomposition*

$$[V] = \sum_{[S] \in \mathrm{Irr}_{\Bbbk}(M)} [V : S][S] \tag{6.1}$$

*holds.*

*Proof.* We prove that (6.1) holds by induction on the number of composition factors of $V$. If $V = 0$, there is nothing to prove. Otherwise, let $U$ be a maximal proper submodule of $V$ and consider the exact sequence

$$0 \longrightarrow U \longrightarrow V \longrightarrow V/U \longrightarrow 0.$$

Then $V/U$ is simple and by induction $[U] = \sum_{[S]\in \mathrm{Irr}_{\Bbbk}(M)}[U:S][S]$. Applying Lemma 6.2, we conclude that

$$\begin{aligned}[V] &= [U] + [V/U] \\ &= ([U:V/U]+1)[V/U] + \sum_{[S]\in \mathrm{Irr}_{\Bbbk}(M)\setminus\{[V/U]\}} [U:S] \\ &= \sum_{[S]\in \mathrm{Irr}_{\Bbbk}(M)} [V:S][S]\end{aligned}$$

as required. It follows that $G_0(\Bbbk M)$ is generated by $\mathrm{Irr}_{\Bbbk}(M)$.

Fix a simple $\Bbbk M$-module $S$. By Lemma 6.2 the mapping $[V] \mapsto [V:S]$ extends to a homomorphism $f_S\colon G_0(\Bbbk M) \longrightarrow \mathbb{Z}$. Moreover, if $S, S'$ are simple $\Bbbk M$-modules, then

$$f_{S'}([S]) = \begin{cases} 1, & \text{if } [S] = [S'] \\ 0, & \text{else.} \end{cases}$$

Suppose that $0 = \sum_{[S]\in \mathrm{Irr}_{\Bbbk}(M)} n_{[S]} \cdot [S]$ with the $n_{[S]} \in \mathbb{Z}$. If $S'$ is a simple $\Bbbk M$-module, then

$$0 = f_{S'}\left(\sum_{[S]\in \mathrm{Irr}_{\Bbbk}(M)} n_{[S]} \cdot [S]\right) = n_{[S']}.$$

Thus the set $\mathrm{Irr}_{\Bbbk}(M)$ is linearly independent. This completes the proof. $\square$

It is worth mentioning that there is another important ring associated to the representation theory of $M$ over $\Bbbk$, called the *representation ring* of $M$, denoted $\mathcal{R}_{\Bbbk}(M)$. Namely, one can take the free abelian group $\mathcal{R}_{\Bbbk}(M)$ on the isomorphism classes of finite dimensional $\Bbbk M$-modules modulo the relations $[V]+[W] = [V \oplus W]$. One again defines multiplication by $[V][W] = [V \otimes W]$. Notice that there is a natural surjective ring homomorphism

$$\mathcal{R}_{\Bbbk}(M) \longrightarrow G_0(\Bbbk M).$$

As an abelian group, $\mathcal{R}_{\Bbbk}(M)$ is easily checked (using the Krull-Schmidt theorem) to be a free abelian group on the set of isomorphism classes of finite dimensional indecomposable $\Bbbk M$-modules. This may very well be an infinite set. Since many natural finite monoids are of so-called 'wild representation type,' meaning that classifying their indecomposable modules is as hard as classifying the indecomposable modules of any finitely generated $\Bbbk$-algebra, it seems unlikely that we will be able to understand $\mathcal{R}_{\Bbbk}(M)$ in much generality.

## 6.2 The restriction isomorphism

Now let $e \in E(M)$ be an idempotent. The restriction functor

$$\mathrm{Res}_{G_e} \colon \Bbbk M\text{-mod} \longrightarrow \Bbbk G_e\text{-mod}$$

given by $V \mapsto eV$ is an exact functor and hence induces a homomorphism $\mathrm{Res}_{G_e} \colon G_0(\Bbbk M) \longrightarrow G_0(\Bbbk G_e)$ of abelian groups defined by $\mathrm{Res}_{G_e}([V]) = [eV]$.

**Proposition 6.4.** *Let $e \in E(M)$. Then $\mathrm{Res}_{G_e} \colon G_0(\Bbbk M) \longrightarrow G_0(\Bbbk G_e)$ is a ring homomorphism.*

*Proof.* Clearly, if $\Bbbk$ denotes the trivial module, then we have that $\mathrm{Res}_{G_e}([\Bbbk]) = [e\Bbbk] = [\Bbbk]$. Let $V, W$ be $\Bbbk M$-modules and $v_1, \ldots, v_m \in V$ and $w_1, \ldots, w_m \in W$. We compute that

$$e \cdot \left( \sum_{i=1}^{m} v_i \otimes w_i \right) = \sum_{i=1}^{m} ev_i \otimes ew_i$$

from which it follows that $e(V \otimes W) = eV \otimes eW$. Thus $\mathrm{Res}_{G_e}$ is a ring homomorphism. $\square$

Fix now a complete set of representatives $e_1, \ldots, e_s$ of the $\mathscr{J}$-classes of idempotents of $M$. Assume that we have chosen the ordering so that $Me_iM \subseteq Me_jM$ implies $i \leq j$. In particular, $e_s = 1$ and $e_1$ belongs to the minimal ideal of $M$. We fix a total ordering $\preceq$ on $\mathrm{Irr}_{\Bbbk}(M)$ such that if $e_i$ is the apex of $S$ and $e_j$ is the apex of $S'$ with $i < j$, then $[S] \prec [S']$. We put a corresponding total ordering, also denoted $\preceq$, on the disjoint union $\bigcup_{i=1}^{s} \mathrm{Irr}_{\Bbbk}(G_e)$ by putting $[S_1] \preceq [S_2]$ if and only if $[S_1^\sharp] \preceq [S_2^\sharp]$, where we retain the notation from Theorem 5.5. The main theorem of this chapter is the following.

**Theorem 6.5.** *Let $e_1, \ldots, e_s$ be a complete set of representatives of the $\mathscr{J}$-classes of idempotents of $M$. Then the mapping*

$$\mathrm{Res} \colon G_0(\Bbbk M) \longrightarrow \prod_{i=1}^{s} G_0(\Bbbk G_{e_i})$$

*given by*

$$\mathrm{Res}([V]) = ([e_1 V], \ldots, [e_s V])$$

*is a ring isomorphism. Moreover, if we takes as bases $\mathrm{Irr}_{\Bbbk}(M)$ for $G_0(\Bbbk M)$ and $\bigcup_{i=1}^{s} \mathrm{Irr}_{\Bbbk}(G_e)$ for $\prod_{i=1}^{s} G_0(\Bbbk G_{e_i})$, ordered as above, then the matrix $L$ for* Res *is lower triangular with ones along the diagonal (i.e., unipotent lower triangular).*

*Proof.* It follows from Proposition 6.4 that Res is a ring homomorphism. Let $W^\sharp$ be a simple $\Bbbk M$-module with apex $e_j$ corresponding to a simple $\Bbbk G_{e_j}$-module $W$ under Theorem 5.5. Then $e_jW^\sharp \cong W$ and $e_iW^\sharp = 0$ unless $Me_jM \subseteq Me_iM$ by Proposition 5.4. Hence $e_iW^\sharp = 0$ if $i < j$. Therefore, we have

$$\begin{aligned}\mathrm{Res}([W^\sharp]) &= \sum_{i=1}^{s}[e_iW^\sharp] = \sum_{i=1}^{s}\sum_{[V]\in \mathrm{Irr}_{\Bbbk}(G_{e_i})}[e_iW^\sharp : V][V]\\ &= [W] + \sum_{i=j+1}^{s}\sum_{[V]\in \mathrm{Irr}_{\Bbbk}(G_{e_i})}[e_iW^\sharp : V][V].\end{aligned}$$

This establishes that $L$ is lower triangular with ones along the diagonal. In particular, $L$ is invertible. □

We call the matrix $L$ from Theorem 6.5 the *decomposition matrix* of $M$ over $\Bbbk$.

Let us compute the decomposition matrix for the full transformation monoid $T_n$. We use the idempotents $e_1, \ldots, e_n$ from Section 5.3. For $1 \leq r \leq n$, denote by $\mathcal{P}_r$ the set of all partitions of $r$ and put $\mathcal{P}(n) = \bigcup_{r=1}^{n}\mathcal{P}_r$. We can view $L$ as a $\mathcal{P}(n) \times \mathcal{P}(n)$ matrix where $L_{\lambda,\mu} = [e_mS_\mu^\sharp : S_\lambda]$ for $\mu$ a partition of $r$ and $\lambda$ a partition of $m$ with $1 \leq m, r \leq n$. It is convenient to order $\mathcal{P}(n)$ so that $\alpha \prec \beta$ if $|\alpha| < |\beta|$, or $|\alpha| = |\beta|$ and there exists $j$ such that $\alpha_i = \beta_i$ for $i < j$ and $\alpha_j < \beta_j$ where $\alpha = (\alpha_1, \ldots, \alpha_r)$ and $\beta = (\beta_1, \ldots, \beta_s)$.

**Proposition 6.6.** *Let $\Bbbk$ be a field of characteristic 0. The decomposition matrix $L$ for the full transformation monoid $T_n$ over $\Bbbk$ is given by*

$$L_{\lambda,\mu} = \begin{cases} 1, & \text{if } \mu = (1^r) \text{ and } \lambda = (m-r+1, 1^{r-1}) \text{ with } m \geq r\\ 1, & \text{if } \mu \neq (1^r),\ \mu \subseteq \lambda \text{ and } \lambda \setminus \mu \text{ is a horizontal strip}\\ 0, & \text{else.}\end{cases}$$

*Proof.* The result for $\mu = (1^r)$ follows directly from Theorem 5.9; the remaining cases follow from Corollary 5.12 and Pieri's rule (Theorem B.13). □

*Example 6.7.* We compute here the decomposition matrix $L$ for $T_3$ using Proposition 6.6. In this case $\mathcal{P}(3) = (1), (1^2), (2), (1^3), (2,1), (3)$. Then $(3) \setminus (2)$ and $(2,1) \setminus (2)$ are horizontal strips and so we have

$$L = \begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 1&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&1&1&0&1&0\\ 1&0&1&0&0&1\end{bmatrix}.$$

## 6.3 The triangular Grothendieck ring

If $S, T$ are one-dimensional simple $\Bbbk M$-modules, then $S \otimes T$ is also one-dimensional. Hence the one-dimensional simple $\Bbbk M$-modules form a basis for a unital subring $G_0^{\triangledown}(\Bbbk M)$ of $G_0(\Bbbk M)$, which we call the *triangular Grothendieck ring* of $M$ over $\Bbbk$. The reason for this terminology is based on the following lemma.

**Lemma 6.8.** *Let $V$ be a finite dimensional $\Bbbk M$-module. Then $V$ affords a representation $\varphi\colon M \longrightarrow M_n(\Bbbk)$ by upper triangular matrices if and only if each composition factor of $V$ is one-dimensional, i.e., $[V] \in G_0^{\triangledown}(\Bbbk M)$.*

*Proof.* Suppose first that $V$ affords a representation $\varphi\colon M \longrightarrow M_n(\Bbbk)$ by upper triangular matrices. Let $e_1, \ldots, e_n$ denote the standard unit vectors and let $V_i$ be the subspace of $\Bbbk^n$ spanned by $e_1, \ldots, e_i$ for $i = 0, \ldots, n$. Then because $\varphi$ is a representation by upper triangular matrices, we have that each $V_i$ is a $\Bbbk M$-submodule. Moreover, $V_i/V_{i-1}$ is one-dimensional for $1 \leq i \leq n$. Thus

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V \tag{6.2}$$

is a composition series with $V_i/V_{i-1}$ one-dimensional for $i = 1, \ldots, n$.

Conversely, suppose that (6.2) is a composition series for $V$ with one-dimensional composition factors. Let $v_1 \in V_1$ be a non-zero vector. Assume inductively that we have found vectors $v_1, \ldots, v_i$ such that $v_1, \ldots, v_j$ is a basis for $V_j$ for all $0 \leq j \leq i$. Choose $v_{i+1} \in V_{i+1}$ such that $v_{i+1} + V_i \neq V_i$. Then $v_1, \ldots, v_j$ is a basis for $V_j$ for $0 \leq j \leq i+1$. It follows that we have a basis $v_1, \ldots, v_n$ for $V$ such that $v_1, \ldots, v_j$ is a basis for $V_j$ for $0 \leq j \leq n$. Then the matrix representation $\varphi\colon M \longrightarrow M_n(\Bbbk)$ afforded by $V$ with respect to the ordered basis $v_1, \ldots, v_n$ is clearly by upper triangular matrices. □

We shall call a $\Bbbk M$-module that satisfies the equivalent conditions of Lemma 6.8 a *triangularizable module*.

**Corollary 6.9.** *The tensor product of triangularizable modules is again triangularizable and the trivial module is triangularizable.*

*Proof.* This follows from Lemma 6.8 and Proposition 6.3 because $G_0^{\triangledown}(\Bbbk M)$ is a subring of $G_0(\Bbbk M)$. □

## 6.4 The Grothendieck group of projective modules

We end this chapter by considering the *Grothendieck group* $K_0(\Bbbk M)$. By the Krull-Schmidt theorem, the set of isomorphism classes of finite dimensional projective $\Bbbk M$-modules is a free commutative monoid on the isomorphism classes of projective indecomposable $\Bbbk M$-modules with respect to the binary operation of direct sum. The group $K_0(\Bbbk M)$ is then the corresponding group of fractions. It consists of all formal differences $[P]-[Q]$ of finite dimensional projective $\Bbbk M$-modules with the operation

$$[P]-[Q]+[P']-[Q']=[P\oplus P']-[Q\oplus Q'].$$

It is a free abelian group with basis the set of isomorphism classes of projective indecomposable modules. Of course, if $\Bbbk M$ is semisimple, then $K_0(\Bbbk M)=G_0(\Bbbk M)$.

There is a natural abelian group homomorphism

$$C\colon K_0(\Bbbk M)\longrightarrow G_0(\Bbbk M)$$

given by

$$C([P])=[P]=\sum_{[S]\in \mathrm{Irr}_{\Bbbk}(M)}[P:S][S]$$

for a finite dimensional projective module $P$. If $P_1,\ldots,P_r$ is a complete set of representatives of the isomorphism classes of projective indecomposable $\Bbbk M$-modules and $S_i=P_i/\operatorname{rad}(P_i)$ is the corresponding simple module, for $1\leq i\leq r$, then the matrix for $C$ with respect to the bases $[P_1],\ldots,[P_r]$ and $[S_1],\ldots,[S_r]$ of $K_0(\Bbbk M)$ and $G_0(\Bbbk M)$, respectively, is called the *Cartan matrix* of $\Bbbk M$. We shall return to the Cartan matrix, and how to compute it, later in the text.

Notice that if $P$ is a finite dimensional projective module, then because the functor $\mathrm{Hom}_{\Bbbk M}(P,-)$ is exact, there is a well-defined functional $f_P\colon G_0(\Bbbk M)\longrightarrow\mathbb{Z}$ given by $f_P([V])=\dim\mathrm{Hom}_{\Bbbk M}(P,V)$. Moreover, one has $f_{P\oplus Q}([V])=f_P([V])+f_Q([V])$. Thus we have a bilinear pairing

$$\langle -,-\rangle\colon K_0(\Bbbk M)\times G_0(\Bbbk M)\longrightarrow\mathbb{Z}$$

given by

$$\langle [P],[V]\rangle=\dim\mathrm{Hom}_{\Bbbk M}(P,V).$$

If $\Bbbk$ is an algebraically closed field and $P_i=\Bbbk Mf_i$ is a projective indecomposable with corresponding primitive idempotent $f_i$ and with simple quotient $P_i/\operatorname{rad}(P_i)=S_i$, then

$$\langle [P_i],[V]\rangle=\dim\mathrm{Hom}_{\Bbbk M}(P_i,V)=\dim f_iV=[V:S_i] \tag{6.3}$$

(cf. Proposition A.24). It follows that, in the setting of an algebraically closed field, $[P_1],\ldots,[P_r]$ and $[S_1],\ldots,[S_r]$ are dual bases with respect to the pairing.

The reader will verify in Exercise 18.15 that the projective $\Bbbk M$-modules are not in general closed under tensor product and so $K_0(\Bbbk M)$ is not usually a ring.

## 6.5 Exercises

**6.1.** Let $M$ be a finite monoid and $\Bbbk$ a field. Let $e_1, \ldots, e_s$ be a complete set of idempotent representatives of the regular $\mathscr{J}$-classes of $M$. Prove that elements of the form $[\mathrm{Ind}_{G_{e_i}}(V)]$ with $V \in \mathrm{Irr}_{\Bbbk}(G_{e_i})$ form a basis for $G_0(\Bbbk M)$.

**6.2.** Let $M$ be a finite $\mathscr{R}$-trivial monoid. Prove that the submonoid of $G_0(\mathbb{C}M)$ consisting of the isomorphism classes of simple $\mathbb{C}M$-modules is isomorphic to the lattice $\Lambda(M)$ made a monoid via the join operation.

**6.3.** Compute the decomposition matrix for the symmetric inverse monoid $I_3$ over $\mathbb{C}$.

**6.4.** Explicitly compute the decomposition matrix for the full transformation monoid $T_4$ over $\mathbb{C}$.

# 7

# Incidence Algebras and Möbius Inversion

Before continuing our development of the character theory of finite monoids, we take a brief interlude to develop Möbius inversion for posets, as this will play an important role throughout the remainder of the text. The theory of incidence algebras and Möbius inversion for posets was developed by Rota [Rot64] and can be considered as part of the origins of algebraic combinatorics. It provides a highly conceptual generalization of the principle of inclusion-exclusion. A thorough introduction, including techniques for computing the Möbius function of a poset and connections with algebraic topology, can be found in Stanley's classic text [Sta97, Chapter 3]. Throughout this chapter, $P$ will denote a finite poset.

## 7.1 The incidence algebra of a poset

Let $\Bbbk$ be a field. The *incidence algebra* $I(P, \Bbbk)$ of the poset $P$ over $\Bbbk$ is the set of all mappings $f\colon P \times P \longrightarrow \Bbbk$ such that $f(p, q) = 0$ if $p \nleq q$. One can profitably think of $f$ as a $P \times P$-upper triangular matrix. Matrix multiplication then translates into the following associative multiplication on $I(P, \Bbbk)$:

$$f * g(p, q) = \sum_{p \leq r \leq q} f(p, r) g(r, q). \tag{7.1}$$

The proof of the following proposition is left to the reader as an exercise.

**Proposition 7.1.** *The incidence algebra $I(P, \Bbbk)$ of a finite poset $P$ is a finite dimensional $\Bbbk$-algebra with respect to pointwise addition of functions and the product* (7.1). *The multiplicative identity is given by the Kronecker function*

$$\delta(p, q) = \begin{cases} 1, & \textit{if } p = q \\ 0, & \textit{else} \end{cases}$$

*for $p, q \in P$.*

The *zeta function* of $P$ is the mapping $\zeta\colon P \times P \longrightarrow \Bbbk$ given by

$$\zeta(p,q) = \begin{cases} 1, & \text{if } p \leq q \\ 0, & \text{else.} \end{cases}$$

It is easy to see that if $k$ is the length of the longest chain in $P$, then one has that $(\delta - \zeta)^{k+1} = 0$, and so $\zeta = \delta - (\delta - \zeta)$ is a unit with inverse

$$\mu = \sum_{n=0}^{k} (\delta - \zeta)^n.$$

The mapping $\mu$ is called the *Möbius function* of $P$. We shall use the notation $\zeta_P$ and $\mu_P$ if the poset $P$ is not clear from context.

A recursive formula for $\mu$ can also be given, which is essentially the classical adjoint formula for the inverse of a matrix restricted to the special case of an upper triangular matrix. See [Sta97, Chapter 3, Section 7] for details.

**Proposition 7.2.** *The formula*

$$\mu(p,q) = \begin{cases} 1, & \text{if } p = q \\ -\sum_{p \leq r < q} \mu(p,r), & \text{if } p < q \\ 0, & \text{else} \end{cases}$$

*is valid.*

*Example 7.3.* If $P = \{0,1\}$ with $0 < 1$, then $\mu(0,0) = 1 = \mu(1,1)$ and $\mu(0,1) = -1$.

The Möbius function is compatible with the formation of direct products. Recall that if $P, Q$ are posets, then $P \times Q$ is a poset with $(p,q) \leq (p',q')$ if and only if $p \leq p'$ and $q \leq q'$. The following can be found in [Sta97, Proposition 3.8.2].

**Proposition 7.4.** *Let $P$ and $Q$ be finite posets. Then $\mu_{P\times Q}((p,q),(p',q')) = \mu_P(p,p')\mu_Q(q,q')$.*

Since the power set of an $n$-element set is isomorphic to $\{0,1\}^n$, Proposition 7.4 and Example 7.3 combine to yield the following result.

**Corollary 7.5.** *Let $\mathscr{P}(X)$ be the power set of a finite set $X$. Then one has $\mu(Y,Z) = (-1)^{|Z|-|Y|}$ for $Y \subseteq Z$.*

The Möbius function of a poset is a fundamental invariant in poset theory, which also has connections with algebraic topology. In this text, our primary interest is due to Rota's Möbius inversion theorem. We shall state it in the following form.

**Theorem 7.6.** *Let $P$ be a finite poset and $V$ a $\Bbbk$-vector space. Let $f\colon P \longrightarrow V$ be a function and define $g\colon P \longrightarrow V$ by*

$$g(p) = \sum_{q \leq p} f(q).$$

*Then we have that*

$$f(p) = \sum_{q \leq p} g(q)\mu(q,p)$$

*for all $p \in P$.*

*Proof.* Let $W$ be the $\Bbbk$-vector space of all mappings $h\colon P \longrightarrow V$ (with pointwise operations). We make $W$ into a right $I(P,\Bbbk)$-module by putting

$$h * a(p) = \sum_{q \leq p} h(q)a(q,p)$$

for $h \in W$, $a \in I(P,\Bbbk)$ and $p \in P$. Then we compute

$$f * \zeta(p) = \sum_{q \leq p} f(q) = g(p)$$

and so $f = f * \zeta * \mu = g * \mu$. But, for $p \in P$, we have

$$g * \mu(p) = \sum_{q \leq p} g(q)\mu(q,p).$$

The theorem follows. $\square$

## 7.2 Exercises

**7.1.** Compute the Möbius function of the poset $\{1, \ldots, n\}$.

**7.2.** Prove Proposition 7.2.

**7.3.** Prove Proposition 7.4.

**7.4.** Prove Corollary 7.5.

**7.5.** This exercise requires some familiarity with algebraic topology. Let $P$ be a finite poset. If $p \leq q$, let $\Delta(p,q)$ be the simplicial complex with vertex set $(p,q) = \{x \in P \mid p < x < q\}$ and with simplices the chains in $(p,q)$. Prove that $\mu(p,q) = \chi(\Delta(p,q)) - 1$ where $\chi(K)$ denotes the Euler characteristic of a simplicial complex $K$.

**7.6.** Prove that the decomposition matrix of an $\mathscr{R}$-trivial monoid $M$ can be identified with the zeta function of the lattice $\Lambda(M)$.

# 8

# Characters and Class Functions

In this chapter, we work exclusively over $\mathbb{C}$, although most of the results hold in greater generality [MQS15]. We study the ring $\mathrm{Cl}(M)$ of class functions on a finite monoid $M$. It turns out that $\mathrm{Cl}(M) \cong \mathbb{C} \otimes_{\mathbb{Z}} G_0(\mathbb{C}M)$. The character table of a monoid is defined and shown to be invertible. In fact, it is block upper triangular with group character tables on the diagonal blocks. Inverting the character table allows us to determine, in principle, the composition factors of a representation directly from its character. The fundamental results of this chapter are due to McAlister [McA72] and, independently, to Rhodes and Zalcstein [RZ91].

## 8.1 Class functions and generalized conjugacy classes

A *class function* on a finite monoid $M$ is a mapping $f\colon M \longrightarrow \mathbb{C}$ such that the following two properties hold for all $m, n \in M$:

(a) $f(mn) = f(nm)$;
(b) $f(m^{\omega+1}) = f(m)$.

We recall that if $m$ has index $c$ and period $d$, then $m^{\omega+1} = m^k$ where $k \geq c$ and $k \equiv 1 \bmod d$.

It is clear that the set $\mathrm{Cl}(M)$ of class functions is a $\mathbb{C}$-algebra with respect to pointwise operations. We call $\mathrm{Cl}(M)$ the *ring of class functions* on $M$. The next proposition verifies that our notion of class function coincides with the usual notion for finite groups.

**Proposition 8.1.** *If $G$ is a finite group, then $f\colon G \longrightarrow \mathbb{C}$ is a class function if and only if $f$ is constant on conjugacy classes.*

*Proof.* If $f$ is a class function on $G$, then $f(gxg^{-1}) = f(xg^{-1}g) = f(x)$ and so $f$ is constant on conjugacy classes. Conversely, if $f\colon G \longrightarrow \mathbb{C}$ is constant on conjugacy classes, then $f(gh) = f(g^{-1}(gh)g) = f(hg)$. Also, since $g^{\omega+1} = g$, we have $f(g^{\omega+1}) = f(g)$. Thus $f$ is a class function. $\square$

It will be convenient to have an analogue of conjugacy for monoids such that class functions are precisely those functions which are constant on conjugacy classes. We define an equivalence relation $\sim$ on $M$ by $m \sim n$ if and only if there exist $x, x' \in M$ such that $xx'x = x$, $x'xx' = x'$, $x'x = m^\omega$, $xx' = n^\omega$ and $xm^{\omega+1}x' = n^{\omega+1}$. One should think of $x'$ as a generalized inverse of $x$ and therefore think of $m^{\omega+1}$ and $n^{\omega+1}$ as being conjugate.

**Proposition 8.2.** *The relation $\sim$ is an equivalence relation on $M$.*

*Proof.* To see that $m \sim m$, take $x = m^\omega = x'$. Suppose that $m \sim n$ and that $xm^{\omega+1}x' = n^{\omega+1}$ with $xx'x = x$, $x'xx' = x'$, $x'x = m^\omega$, $xx' = n^\omega$. Then $x'n^{\omega+1}x = x'xm^{\omega+1}x'x = m^\omega m^{\omega+1}m^\omega = m^{\omega+1}$ and so $n \sim m$. To verify transitivity, suppose that $a \sim b$ and $b \sim c$. Then we can find $x, x', y, y' \in M$ such that $xx'x = x$, $x'xx' = x'$, $x'x = a^\omega$, $xx' = b^\omega = y'y$, $yy'y = y$, $y'yy' = y'$, $yy' = c^\omega$, $xa^{\omega+1}x' = b^{\omega+1}$ and $yb^{\omega+1}y' = c^{\omega+1}$. Let $z = yx$ and $z' = x'y'$. Then we compute that $zz'z = yxx'y'yx = yy'yy'yx = yx = z$, $z'zz' = x'y'yxx'y' = x'xx'xx'y' = x'y' = z'$, $z'z = x'y'yx = x'xx'x' = a^\omega$ and $zz' = yxx'y' = yy'yy' = c^\omega$. Also, we have $za^{\omega+1}z' = yxa^{\omega+1}x'y' = yb^{\omega+1}y' = c^{\omega+1}$. This establishes that $a \sim c$, completing the proof. $\square$

We shall call $\sim$-classes by the name *generalized conjugacy classes* and denote the $\sim$-class of an element $m$ by $[m]_\sim$. The next proposition shows that the class functions are exactly the functions which are constant on generalized conjugacy classes.

**Proposition 8.3.** *Let $f\colon M \longrightarrow \mathbb{C}$ be a mapping. Then $f$ is a class function if and only if it is constant on generalized conjugacy classes.*

*Proof.* Suppose first that $f$ is a class function and that $m \sim n$. Let $x, x' \in M$ with $xx'x = x$, $x'xx' = x'$, $x'x = m^\omega$, $xx' = n^\omega$ and $xm^{\omega+1}x' = n^{\omega+1}$. Then we have that $f(n) = f(n^{\omega+1}) = f(xm^{\omega+1}x') = f(x'xm^{\omega+1}) = f(m^\omega m^{\omega+1}) = f(m^{\omega+1}) = f(m)$ and so $f$ is constant on generalized conjugacy classes.

Conversely, suppose that $f$ is constant on generalized conjugacy classes. Since $m \sim m^{\omega+1}$ (by taking $x = m^\omega = x'$), it follows that $f(m^{\omega+1}) = f(m)$. Next, let $m, n \in M$. Choose $k > 0$ such that $x^k = x^\omega$ for all $x \in M$ (see Remark 1.3). Note that $x^\omega x^r = x^r$ for any $r \geq k$ as a consequence of Corollary 1.2. Let $x = n(mn)^{2k-1} = (nm)^{2k-1}n$ and $x' = (mn)^k m = m(nm)^k$. Then we compute

$$\begin{aligned} x'x &= (mn)^k mn(mn)^{2k-1} = (mn)^{3k} = (mn)^\omega \\ xx' &= (nm)^{2k-1}nm(nm)^k = (nm)^{3k} = (nm)^\omega \\ xx'x &= (nm)^\omega (nm)^{2k-1}n = (nm)^{2k-1}n = x \\ x'xx' &= (mn)^\omega (mn)^k m = (mn)^k m = x' \end{aligned}$$

and $x(mn)^{\omega+1}x' = n(mn)^{2k-1}(mn)^{k+1}(mn)^k m = n(mn)^{4k}m = nm(nm)^{4k} = (nm)^{\omega+1}$. Therefore, $mn \sim nm$ and so $f(mn) = f(nm)$, completing the proof that $f$ is a class function. $\square$

Proposition 8.3 essentially says that $\sim$ is the smallest equivalence relation on $M$ such that $mn \sim nm$ and $m^{\omega+1} \sim m$ for all $m, n \in M$.

Let $e_1, \ldots, e_s$ be a complete set of idempotent representatives of the regular $\mathscr{J}$-classes. Then it turns out that each generalized conjugacy class of $M$ intersects exactly one maximal subgroup $G_{e_i}$, and in one of its conjugacy classes. If $G$ is a group, then $\mathrm{Conj}(G)$ will denote the set of conjugacy classes of elements of $G$. The class of an element $g \in G$ will be denote by $\mathrm{cl}(g)$.

**Proposition 8.4.** *Let $e_1, \ldots, e_s$ be a complete set of idempotent representatives of the regular $\mathscr{J}$-classes. Then the restriction of $\sim$ to $G_{e_i}$ is conjugacy and the natural map*

$$\psi\colon \bigcup_{i=1}^{s} \mathrm{Conj}(G_{e_i}) \longrightarrow M/{\sim}$$

*given by $\psi(\mathrm{cl}(g)) = [g]_\sim$ is a bijection.*

*Proof.* If $g, h \in G_{e_i}$ and $h = xgx^{-1}$, then putting $x' = x^{-1}$, we have $xx'x = x$, $x'xx' = x'$, $g^\omega = e_i = h^\omega = x'x = xx'$ and $h^{\omega+1} = h = xgx^{-1} = xg^{\omega+1}x'$. Therefore, $g \sim h$. Conversely, suppose that $g \sim h$ and that $x, x'$ satisfy $xx'x = x$, $x'xx' = x'$, $x'x = g^\omega = e_i = h^\omega = xx'$, and $xg^{\omega+1}x' = h^{\omega+1}$. Then $x, x' \in e_iMe_i$ and are inverses, hence $x, x' \in G_{e_i}$ and $x' = x^{-1}$. As $g = g^{\omega+1}$ and $h = h^{\omega+1}$, it follows that $h = xgx^{-1}$ and so $h, g$ are conjugate in $G_{e_i}$. Therefore, $\sim$ restricts to $G_{e_i}$ as conjugacy and hence $\psi$ is a well-defined mapping, injective on each $\mathrm{Conj}(G_{e_i})$. If $g \in G_{e_i}$ and $h \in G_{e_j}$ with $g \sim h$, then there exist $x, x' \in M$ with $x'x = g^\omega = e_i$ and $xx' = h^\omega = e_j$. But then $Me_iM = Me_jM$ by Theorem 1.10 and so $e_i = e_j$. Thus $\psi$ is injective. It remains to show that it is surjective.

Let $m \in M$ and suppose that $Mm^\omega M = Me_iM$. By Theorem 1.10, there exist $x, x' \in M$ such that $xx'x = x$, $x'xx' = x'$, $x'x = m^\omega$ and $xx' = e_i$. Let $g = xm^{\omega+1}x'$. Then $e_ige_i = xx'xm^{\omega+1}x'xx' = xm^{\omega+1}x' = g$ and so $g \in e_iMe_i$. Also, we have that $MgM = Mm^\omega M = Me_iM$ because $x'gx = m^\omega m^{\omega+1}m^\omega = m^{\omega+1}$ and $Mm^{\omega+1}M = Mm^\omega M$. Therefore, $g \in G_{e_i}$ by Corollary 1.15. But $g \sim m$. Therefore, $[m]_\sim = \psi(\mathrm{cl}(g))$, as required. □

As an immediate consequence, we can identify the ring of class functions on $M$ with the product of the rings of class functions on the $G_{e_i}$ with $i = 1, \ldots, s$.

**Corollary 8.5.** *Let $e_1, \ldots, e_s$ be a complete set of idempotent representatives of the regular $\mathscr{J}$-classes of $M$. Then the restriction map*

$$\mathrm{res}\colon \mathrm{Cl}(M) \longrightarrow \prod_{i=1}^{s} \mathrm{Cl}(G_{e_i})$$

*given by*

$$\mathrm{res}(f) = (f|_{G_{e_1}}, \ldots, f|_{G_{e_s}})$$

*is an isomorphism of $\mathbb{C}$-algebras.*

*Proof.* Identifying $\mathrm{Cl}(M)$ with the algebra $A$ of mappings $f\colon M/{\sim} \longrightarrow \mathbb{C}$ and identifying $\prod_{i=1}^{s} \mathrm{Cl}(G_{e_i})$ with the algebra $B$ of mappings

$$f\colon \bigcup_{i=1}^{s} \mathrm{Conj}(G_{e_i}) \longrightarrow \mathbb{C}$$

(using Proposition 8.1), we see that

$$\mathrm{res}\colon \mathrm{Cl}(M) \longrightarrow \prod_{i=1}^{s} \mathrm{Cl}(G_{e_i})$$

corresponds to the algebra homomorphism $\psi^*\colon A \longrightarrow B$ given by $f \mapsto f \circ \psi$, where $\psi$ is the bijection from Proposition 8.4. Thus the mapping res is an isomorphism. □

As a corollary, we may compute the dimension of $\mathrm{Cl}(M)$.

**Corollary 8.6.** *Let $M$ be a finite monoid and let $e_1, \ldots, e_s$ be a complete set of idempotent representatives of the regular $\mathscr{J}$-classes of $M$. Then the equality*

$$\dim \mathrm{Cl}(M) = \kappa_1 + \cdots + \kappa_s$$

*holds where $\kappa_i = |\,\mathrm{Conj}(G_{e_i})|$.*

*Proof.* This is immediate from Proposition 8.1 and Corollary 8.5. □

A natural basis for $\mathrm{Cl}(M)$ consists of the indicator functions $\delta_C\colon M \longrightarrow \mathbb{C}$ of the generalized conjugacy classes, where for a class $C$

$$\delta_C(m) = \begin{cases} 1, & \text{if } m \in C \\ 0, & \text{else.} \end{cases} \tag{8.1}$$

In the next section we shall see that the irreducible characters of $M$ form another basis for $\mathrm{Cl}(M)$. This in turn will allow us to identify $\mathrm{Cl}(M)$ with $\mathbb{C} \otimes_{\mathbb{Z}} G_0(\mathbb{C}M)$.

## 8.2 Character theory

Let $\rho\colon M \longrightarrow M_n(\mathbb{C})$ be a representation of a monoid $M$. Then the *character* of $\rho$ is the mapping $\chi_\rho\colon M \longrightarrow \mathbb{C}$ defined by

$$\chi_\rho(m) = \mathrm{Tr}(\rho(m))$$

where $\mathrm{Tr}(A)$ is the trace of a matrix $A$. If $\rho$ is equivalent to $\varphi$, then $\chi_\rho = \chi_\varphi$. Indeed, if $T$ is invertible with $\rho(m) = T\varphi(m)T^{-1}$ for all $m \in M$, then

$$\chi_\rho(m) = \mathrm{Tr}(\rho(m)) = \mathrm{Tr}(T\varphi(m)T^{-1}) = \mathrm{Tr}(\varphi(m)) = \chi_\varphi(m).$$

Hence if $V$ is a finite dimensional $\mathbb{C}M$-module, then we can put $\chi_V = \chi_\rho$ where $\rho$ is the representation afforded by $V$ with respect to some basis. If $V \cong W$, then clearly $\chi_V = \chi_W$ because they afford equivalent representations. We sometimes say that $V$ *affords* the character $\chi_V$. A character of an irreducible representation will be called an *irreducible character*. The zero mapping will be considered the character of the zero module. We shall call a mapping $f\colon M \longrightarrow \mathbb{C}$ a *character* if $f = \chi_V$ for some finite dimensional $\mathbb{C}M$-module $V$. One says that $f$ is a *virtual character* if $f = \chi_V - \chi_W$ for some $\mathbb{C}M$-modules $V$ and $W$, that is, if $f$ is a difference of two characters.

The following is an immediate consequence of the theorem of Frobenius and Schur (Corollary 5.2).

**Theorem 8.7.** *Let $V_1, \ldots, V_r$ be a set of pairwise non-isomorphic simple $\mathbb{C}M$-modules. Then the characters $\chi_{V_1}, \ldots, \chi_{V_r}$ are linearly independent as complex-valued functions on $M$.*

*Proof.* Suppose that $\varphi^{(k)}\colon M \longrightarrow M_{d_k}(\mathbb{C})$ is a representation afforded by $V_k$ for $k = 1, \ldots, r$. If $c_1\chi_{V_1} + \cdots + c_r\chi_{V_r} = 0$ with $c_1, \ldots, c_r \in \mathbb{C}$, then

$$0 = \sum_{k=1}^{r} \sum_{i=1}^{d_k} c_k \varphi_{ii}^{(k)}$$

and so $c_1 = \cdots = c_r = 0$ by Corollary 5.2. □

We shall need the following algebraic properties of character values in the sequel.

**Lemma 8.8.** *Let $\rho\colon M \longrightarrow M_n(\mathbb{C})$ be a representation and let $m \in M$ have index $c$ and period $d$.*

*(i) The minimal polynomial of $\rho(m)$ divides $x^c(x^d - 1)$.*
*(ii) Every non-zero eigenvalue of $\rho(m)$ is a $d^{th}$-root of unity.*
*(iii) $\chi_\rho(m)$ is an algebraic integer.*
*(iv) $\chi_\rho(m) = n$ if and only if $\rho(m) = I$.*

*Proof.* From $m^c = m^{c+d}$, we obtain that $\rho(m)^c = \rho(m)^{c+d}$ and hence (i) follows. Item (ii) is an immediate consequence of (i). Since $\mathrm{Tr}(\rho(m))$ is the sum of the eigenvalues with multiplicities, (iii) is a consequence of (ii). For the final statement, let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $\rho(m)$ with multiplicities. By (ii) and the Cauchy-Schwarz inequality we have that

$$\left|\sum_{i=1}^{n} \lambda_i\right| \leq \sum_{i=1}^{n} |\lambda_i| \leq n$$

with equality if and only if $\lambda_1 = \cdots = \lambda_n \neq 0$. Therefore, if $n = \chi_\rho(m) = \mathrm{Tr}(\rho(m)) = \sum_{i=1}^{n} \lambda_i$, then $\lambda_1 = \cdots = \lambda_n = 1$ and so the minimal polynomial of $\rho(m)$ divides both $(x-1)^n$ and $x^c(x^d-1)$. But the greatest common divisor of these two polynomials is $x-1$, whence $\rho(m) = I$, completing the proof. □

As a consequence, we deduce that characters are class functions.

**Proposition 8.9.** *Characters of representations are class functions.*

*Proof.* Let $\rho\colon M \longrightarrow M_n(\mathbb{C})$ be a representation. Then we compute that $\chi_\rho(m_1m_2) = \mathrm{Tr}(\rho(m_1)\rho(m_2)) = \mathrm{Tr}(\rho(m_2)\rho(m_1)) = \chi_\rho(m_2m_1)$. Next let $m \in M$ have index $c$ and period $d$. Then $m^{\omega+1} = m^k$ where $k \geq c$ and $k \equiv 1 \bmod d$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $\rho(m)$ with multiplicity. Then $\lambda_1^k, \ldots, \lambda_n^k$ are the eigenvalues of $\rho(m)^k = \rho(m^k) = \rho(m^{\omega+1})$. By Lemma 8.8 each nonzero eigenvalue $\lambda$ of $\rho(m)$ is a $d^{th}$-root of unity and hence satisfies $\lambda^k = \lambda$. Since this latter equality is obviously true for $\lambda = 0$, we conclude that $\lambda_i = \lambda_i^k$ for all $i = 1, \ldots, n$. Thus we have

$$\chi_\rho(m) = \mathrm{Tr}(\rho(m)) = \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \lambda_i^k = \mathrm{Tr}(\rho(m)^k) = \chi_\rho(m^{\omega+1}).$$

This completes the proof that $\chi_\rho$ is a class function. □

We may now deduce that the irreducible characters form a basis for $\mathrm{Cl}(M)$.

**Theorem 8.10.** *Let $V_1, \ldots, V_r$ be a complete set of representatives of the isomorphism classes of simple $\mathbb{C}M$-modules. Then the characters $\chi_{V_1}, \ldots, \chi_{V_r}$ form a basis for $\mathrm{Cl}(M)$. In particular, the number of isomorphism classes of simple $\mathbb{C}M$-modules coincides with the number of generalized conjugacy classes of $M$.*

*Proof.* The characters $\chi_{V_1}, \ldots, \chi_{V_r}$ are linearly independent by Theorem 8.7. Also, we know that $\dim \mathrm{Cl}(M)$ is the number of generalized conjugacy classes. Fix a complete set of idempotent representatives $e_1, \ldots, e_s$ of the regular $\mathscr{J}$-classes of $M$. Put $\kappa_i = |\operatorname{Conj} G_{e_i}|$. Then $\dim \mathrm{Cl}(M) = \kappa_1 + \cdots + \kappa_s$ by Corollary 8.6. But $\kappa_i = |\operatorname{Irr}_{\mathbb{C}}(G_{e_i})|$ by Corollary B.5 and hence $\kappa_1 + \cdots + \kappa_s = r$ by Corollary 5.6. Therefore, $\chi_{V_1}, \ldots, \chi_{V_r}$ is a basis for $\mathrm{Cl}(M)$. □

We define the *character table* $X(M)$ to be the transpose of the change of basis matrix between the basis of irreducible characters and the basis of indicator functions $\{\delta_C \mid C \in M/{\sim}\}$, cf. (8.1), for $\mathrm{Cl}(M)$. In other words,

$$X(M)_{\chi,C} = \chi(C)$$

where $\chi(C)$ is the value taken by the character $\chi$ on the generalized conjugacy class $C$. Notice that $X(M)$ is invertible by Theorem 8.10. It seems unfortunate to use the transpose of the change of basis matrix, but this tradition is too deeply entrenched to change. From the definition of $X(M)$ as the transpose of the change of basis matrix, the following proposition is immediate.

**Proposition 8.11.** *If $f\colon M \longrightarrow \mathbb{C}$ is a class function and $\chi_1, \ldots, \chi_r$ are the irreducible characters of $M$, then*

$$f = \sum_{i=1}^{r} \sum_{C \in M/\sim} f(C) X(M)^{-1}_{C,\chi_i} \cdot \chi_i$$

*where $f(C)$ denotes the value of $f$ on the generalized conjugacy class $C$.*

To further enhance our understanding of the ring of class functions and the character table, it will be invaluable to connect $\mathrm{Cl}(M)$ with $G_0(\mathbb{C}M)$.

**Proposition 8.12.** *The assignment $V \mapsto \chi_V$ enjoys the following properties.*

*(i) The character of the trivial representation is identically* $1$.

*(ii) If* $0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$ *is an exact sequence of* $\mathbb{C}M$*-modules, then* $\chi_V = \chi_U + \chi_W$.

*(iii)* $\chi_{V\otimes W} = \chi_V \cdot \chi_W$.

*Proof.* The first item is clear. To prove (ii), without loss of generality assume that $U \leq V$ and $W = V/U$. Let $u_1, \ldots, u_k$ be a basis for $U$ and extend it to a basis $u_1, \ldots, u_k, v_1, \ldots, v_n$ for $V$ so that $v_1 + U, \ldots, v_n + U$ is a basis for $V/U$. Let $\rho\colon M \longrightarrow M_k(\mathbb{C})$ be the representation afforded by $U$ with respect to the basis $u_1, \ldots, u_k$ and let $\psi\colon M \longrightarrow M_n(\mathbb{C})$ be the representation afforded by $V/U$ with respect to the basis $v_1 + U, \ldots, v_n + U$. Then the representation $\varphi\colon M \longrightarrow M_{k+n}(\mathbb{C})$ afforded by $V$ has the block form

$$\varphi(m) = \begin{bmatrix} \rho(m) & * \\ 0 & \psi(m) \end{bmatrix}$$

and so

$$\chi_V(m) = \chi_\varphi(m) = \mathrm{Tr}(\varphi(m)) = \mathrm{Tr}(\rho(m)) + \mathrm{Tr}(\psi(m)) = \chi_U(m) + \chi_W(m),$$

as required.

Let us turn to (iii). Let $v_1, \ldots, v_r$ be a basis for $V$ and let $w_1, \ldots, w_s$ be a basis for $W$. Then $\{v_i \otimes w_j \mid 1 \leq i \leq r, 1 \leq j \leq s\}$ is a basis for $V \otimes W$. Let $\rho\colon M \longrightarrow M_r(\mathbb{C})$ and $\psi\colon M \longrightarrow M_s(\mathbb{C})$ be the representations afforded by $V$ and $W$, respectively, using these bases. Clearly, we have

$$\begin{aligned} m(v_i \otimes w_j) = mv_i \otimes mw_j &= \sum_{k=1}^{r} \rho(m)_{ki}(v_k \otimes mw_j) \\ &= \sum_{k=1}^{r} \sum_{\ell=1}^{s} \rho(m)_{ki} \psi(m)_{\ell j}(v_k \otimes w_\ell). \end{aligned}$$

In particular, the coefficient of $v_i \otimes w_j$ is $\rho(m)_{ii}\psi(m)_{jj}$ and so

$$\begin{aligned} \chi_{V\otimes W}(m) &= \sum_{i=1}^{r} \sum_{j=1}^{s} \rho(m)_{ii}\psi(m)_{jj} = \left(\sum_{i=1}^{r} \rho(m)_{ii}\right)\left(\sum_{j=1}^{s} \psi(m)_{jj}\right) \\ &= \chi_V(m)\chi_W(m), \end{aligned}$$

as required. $\square$

It follows from the proposition that mapping sending $[V]$ to $\chi_V$ induces a ring homomorphism $G_0(\mathbb{C}M) \longrightarrow \mathrm{Cl}(M)$.

**Corollary 8.13.** *The assignment $[V] \mapsto \chi_V$ induces a ring homomorphism $\Delta_M\colon G_0(\mathbb{C}M) \longrightarrow \mathrm{Cl}(M)$. The image of $\Delta_M$ is the subring of virtual characters. Moreover, $\Delta$ induces an isomorphism $\mathbb{C} \otimes_{\mathbb{Z}} G_0(\mathbb{C}M) \cong \mathrm{Cl}(M)$.*

*Proof.* The first two statements are clear. The final statement follows because if $\mathrm{Irr}_{\mathbb{C}}(M) = \{[V_1], \ldots, [V_r]\}$, then $[V_1], \ldots, [V_r]$ is a basis for $G_0(\mathbb{C}M)$ as a free abelian group and $\chi_{V_1}, \ldots, \chi_{V_r}$ is a basis for $\mathrm{Cl}(M)$ as a $\mathbb{C}$-vector space by Theorem 8.10. □

To complete the picture, we will show that the maps Res and res from Theorem 6.5 and Corollary 8.5 are intertwined by the maps sending a representation to its character.

**Proposition 8.14.** *Suppose that the $\mathbb{C}M$-module $V$ affords the character $\chi_V$ and $e \in E(M)$. Then $\chi_{eV} = (\chi_V)|_{eMe}$ as a character of $eMe$ and $\chi_{eV} = (\chi_V)|_{G_e}$ as a character of $G_e$.*

*Proof.* The direct sum decomposition $V = eV \oplus (1-e)V$ is into $eMe$-invariant subspaces. Moreover, $eMe$ annihilates $(1-e)V$. Choose a basis $B$ for $eV$ and $B'$ for $(1-e)V$. Let $\psi\colon eMe \longrightarrow M_r(\mathbb{C})$ be the representation afforded by $eV$ with respect to the basis $B$. Choosing the basis $B \cup B'$ for $V$ affords a matrix representation $\rho\colon M \longrightarrow M_n(\mathbb{C})$ with

$$\rho(m) = \begin{bmatrix} \psi(m) & 0 \\ 0 & 0 \end{bmatrix}$$

for $m \in eMe$. Therefore, $\chi_V(m) = \mathrm{Tr}(\rho(m)) = \mathrm{Tr}(\psi(m)) = \chi_{eV}(m)$ for $m \in eMe$, as was required. □

We now present the fundamental theorem of this section.

**Theorem 8.15.** *Let $M$ be a finite monoid. Let $e_1, \ldots, e_s$ be a complete set of idempotent representatives of the regular $\mathscr{J}$-classes of $M$. Then there is a commutative diagram*

$$\begin{array}{ccc} G_0(\mathbb{C}M) & \xrightarrow{\mathrm{Res}} & \prod_{i=1}^{s} G_0(\mathbb{C}G_{e_i}) \\ {\scriptstyle \Delta_M}\big\downarrow & & \big\downarrow{\scriptstyle \prod_{i=1}^{s} \Delta_{G_{e_i}}} \\ \mathrm{Cl}(M) & \xrightarrow[\mathrm{res}]{} & \prod_{i=1}^{s} \mathrm{Cl}(G_{e_i}) \end{array} \tag{8.2}$$

*where*

$$\begin{aligned} \mathrm{Res}([V]) &= ([e_1V], \ldots, [e_sV]), \\ \mathrm{res}(f) &= (f|_{G_{e_1}}, \ldots, f|_{G_{e_s}}) \end{aligned}$$

*and where $\Delta_M$, $\Delta_{G_{e_i}}$ are given by $[V] \mapsto \chi_V$. The horizontal maps are isomorphisms and the vertical maps are monomorphisms inducing a commutative diagram*

$$\begin{array}{ccc} \mathbb{C}\otimes_{\mathbb{Z}} G_0(\mathbb{C}M) & \xrightarrow{\mathrm{Res}} & \prod_{i=1}^{s} \mathbb{C}\otimes_{\mathbb{Z}} G_0(\mathbb{C}G_{e_i}) \\ \downarrow & & \downarrow \\ \mathrm{Cl}(M) & \xrightarrow[\mathrm{res}]{} & \prod_{i=1}^{s} \mathrm{Cl}(G_{e_i}). \end{array} \tag{8.3}$$

*of isomorphisms of $\mathbb{C}$-algebras.*

*Proof.* Proposition 8.14 implies that the diagram commutes. The remainder of the theorem follows directly from Theorem 6.5, Corollary 8.5 and Corollary 8.13. □

An important corollary of Theorem 8.15 is a simple characterization of the virtual characters of a finite monoid.

**Corollary 8.16.** *Let $M$ be a finite monoid and let $e_1, \ldots, e_s$ be a complete set of idempotent representatives of the regular $\mathscr{J}$-classes of $M$. Then the following are equivalent for a class function $f\colon M \longrightarrow \mathbb{C}$.*

*(i) $f$ is a virtual character.*
*(ii) $f|_{G_e}$ is a virtual character of $G_e$ for each $e \in E(M)$*
*(iii) $f|_{G_{e_i}}$ is a virtual character of $G_{e_i}$ for $i = 1, \ldots, s$.*

*Proof.* The restriction of a virtual character to a maximal subgroup is again a virtual character by Proposition 8.14 and so (i) implies (ii). Obviously, (ii) implies (iii). Assume that (iii) holds. Suppose that $f|_{G_{e_i}} = \chi_{V_i} - \chi_{W_i}$ with $V_i, W_i$ finite dimensional $\Bbbk G_{e_i}$-modules for $i = 1, \ldots, s$. Then by Theorem 6.5, there exist finite dimensional $\Bbbk M$-modules $V, W$ with

$$\mathrm{Res}([V] - [W]) = ([V_1] - [W_1], \ldots, [V_s] - [W_s]).$$

By commutativity of the diagram (8.2) from Theorem 8.15, it follows that

$$\mathrm{res}(\chi_V - \chi_W) = (\chi_{V_1} - \chi_{W_1}, \ldots, \chi_{V_s} - \chi_{W_s}) = \mathrm{res}(f)$$

and so $f = \chi_V - \chi_W$ because res is an isomorphism. This establishes that (iii) implies (i) and completes the proof of the theorem. □

As in Section 6.2, we fix a complete set of idempotent representatives $e_1, \ldots, e_s$ of the regular $\mathscr{J}$-classes of $M$ and assume that we have chosen the ordering so that $Me_iM \subseteq Me_jM$ implies $i \leq j$. Again, we fix a total ordering $\preceq$ on $\mathrm{Irr}_{\Bbbk}(M)$ such that if $e_i$ is the apex of $S$ and $e_j$ is the apex of $S'$ with $i < j$, then $[S] \prec [S']$. We put a corresponding total ordering, also denoted $\preceq$, on the disjoint union $\bigcup_{i=1}^{s} \mathrm{Irr}_{\Bbbk}(G_e)$ by putting $[S_1] \preceq [S_2]$ if and only if $[S_1^\sharp] \preceq [S_2^\sharp]$, where we retain the notation from Theorem 5.5. Let us choose a total ordering $\leq$ on the generalized conjugacy classes of $M$ such that if $C \cap G_{e_i} \neq \emptyset$ and $C' \cap G_{e_j} \neq \emptyset$ with $i < j$, then $C < C'$. We totally order $\bigcup_{i=1}^{s} \mathrm{Conj}(G_{e_i})$ by transporting $\leq$ along $\psi^{-1}$ where $\psi$ is the bijection in Proposition 8.4.

Retaining all this notation, we take $\mathrm{Irr}_{\mathbb{C}}(M)$ as a basis for $\mathbb{C} \otimes_{\mathbb{Z}} G_0(\mathbb{C}M)$, $\bigcup_{i=1}^{s} \mathrm{Irr}_{\mathbb{C}}(G_{e_i})$ as a basis for $\prod_{i=1}^{s} \mathbb{C} \otimes_{\mathbb{Z}} G_0(\mathbb{C}G_{e_i})$, the set of indicator functions of generalized conjugacy classes of $M$ as a basis for $\mathrm{Cl}(M)$ and the union of the sets of indicator functions of the conjugacy classes of the $G_{e_i}$ as a basis for $\prod_{i=1}^{s} \mathrm{Cl}(G_{e_i})$. These bases are ordered using the orderings $\preceq$ in the first two cases and using the orderings coming from $\leq$ in the latter two. Then $X(M)^T$ is the matrix of left hand vertical arrow in (8.3), the matrix of res is the identity matrix, the matrix of Res is the decomposition matrix (cf. Theorem 6.5) and the matrix of the right hand vertical map is

$$\begin{bmatrix} X(G_{e_1})^T & 0 & \cdots & 0 \\ 0 & X(G_{e_2})^T & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & X(G_{e_s})^T \end{bmatrix}.$$

Taking transposes, we have thus proved the following result.

**Corollary 8.17.** *With respect to the above ordering of simple modules and generalized conjugacy classes, we have that $X(M) = UX$ where*

$$X = \begin{bmatrix} X(G_{e_1}) & 0 & \cdots & 0 \\ 0 & X(G_{e_2}) & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & X(G_{e_s}) \end{bmatrix}$$

*and $U$ is a unipotent upper triangular matrix (the transpose of the decomposition matrix). Hence $X(M)$ is block upper triangular with the diagonal blocks the character tables of the maximal subgroups of $M$, one per $\mathscr{J}$-class of idempotents.*

We shall work out the character table of the full transformation monoid in the next section.

Recall that if $f, h\colon G \longrightarrow \mathbb{C}$ are class functions on a group $G$, then their inner product is given by

$$\langle f, h \rangle_G = \frac{1}{|G|} \sum_{g \in G} f(g)\overline{h(g)}.$$

Also, recall that the irreducible characters form an orthonormal basis for $\mathrm{Cl}(G)$ with respect to this inner product (cf. Theorem B.4).

**Corollary 8.18.** *Let $V$ be a finite dimensional $\Bbbk M$-module. Then the equality*

$$\chi_V = \sum_{i=1}^{s} \sum_{S \in \mathrm{Irr}_{\mathbb{C}}(G_{e_i})} \langle \chi_V, \chi_S \rangle_{G_{e_i}} \left( \sum_{S' \in \mathrm{Irr}_{\mathbb{C}}(M)} L^{-1}_{[S'],[S]} \chi_{S'} \right)$$

*holds, where $L$ is the decomposition matrix.*

*Proof.* This follows because we have

$$\begin{aligned}
[V] &= \mathrm{Res}^{-1}\left(\sum_{i=1}^{s} [e_i V]\right) \\
&= \mathrm{Res}^{-1}\left(\sum_{i=1}^{s} \sum_{S \in \mathrm{Irr}_{\mathbb{C}}(G_{e_i})} \langle \chi_V, \chi_S \rangle_{G_{e_i}} [S]\right) \\
&= \sum_{i=1}^{s} \sum_{S \in \mathrm{Irr}_{\mathbb{C}}(G_{e_i})} \langle \chi_V, \chi_S \rangle_{G_{e_i}} \cdot \mathrm{Res}^{-1}([S])
\end{aligned}$$

as the irreducible characters of a finite group form an orthonormal basis for the ring of class functions. □

Corollary 8.18 provides one method to compute the composition factors of a finite dimensional $\mathbb{C}M$-module $V$ from its character $\chi_V$. Another possible way to do this is via primitive idempotents, although computing primitive idempotents for monoid algebras does not seem to be an easy task.

**Proposition 8.19.** *Let $V$ be a finite dimensional $\mathbb{C}M$-module with character $\chi_V$ and $S$ a simple $\mathbb{C}M$-module. Suppose that $S \cong \mathbb{C}Mf/\operatorname{rad}(\mathbb{C}M)f$ where*

$$f = \sum_{m \in M} c_m \cdot m$$

*is a primitive idempotent. Then the equality*

$$[V : S] = \sum_{m \in M} c_m \chi_V(m)$$

*holds.*

*Proof.* By Proposition A.24, we have $[V : S] = \dim fV$. If $\rho\colon M \longrightarrow M_n(\mathbb{C})$ is the representation afforded by $V$, and we extend $\rho$ linearly to $\mathbb{C}M$, then we have

$$\dim fV = \mathrm{Tr}(\rho(f)) = \sum_{m \in M} c_m \,\mathrm{Tr}(\rho(m)) = \sum_{m \in M} c_m \chi_V(m)$$

because $\rho(f)$ is idempotent. This completes the proof. □

Let us compute the character table of an $\mathscr{R}$-trivial monoid. Recall from Corollary 5.7 that if $M$ is $\mathscr{R}$-trivial, then there is a one-dimensional simple module $S_J$ for each regular $\mathscr{J}$-class $J$ of $M$ and the corresponding character is given by

$$\chi_J(m) = \begin{cases} 1, & \text{if } J \subseteq MmM \\ 0, & \text{else.} \end{cases} \tag{8.4}$$

Corollary 2.6 shows that there is a homomorphism $\sigma\colon M \longrightarrow \Lambda(M)$, where

$$\Lambda(M) = \{MeM \mid e \in E(M)\},$$

given by $\sigma(m) = Mm^{\omega}M$. Moreover, the fibers of $\sigma$ are exactly the generalized conjugacy classes of $M$ because $m^{\omega} = m^{\omega+1}$ and each maximal subgroup is trivial. Thus we can identify both the set of simple modules and the set of generalized conjugacy classes with the lattice $\Lambda(M)$. Moreover, we immediately deduce from (8.4) that $X(M)$ is the zeta function of $\Lambda(M)$. That is, if $J$ is a regular $\mathscr{J}$-class and $C$ is the generalized conjugacy class corresponding to $MeM \in \Lambda(M)$, then

$$X(M)_{\chi_J,C} = \chi_J(e) = \begin{cases} 1, & \text{if } MJM \subseteq MeM \\ 0, & \text{else.} \end{cases}$$

We have thus proved the following result.

**Proposition 8.20.** *The character table $X(M)$ of an $\mathscr{R}$-trivial monoid is the zeta function of the lattice $\Lambda(M) = \{MeM \mid e \in E(M)\}$. Hence $X(M)^{-1}$ is the Möbius function $\mu$ of $\Lambda(M)$.*

From Proposition 8.20, we obtain the following specialization of Proposition 8.11.

**Corollary 8.21.** *Let $M$ be an $\mathscr{R}$-trivial monoid and $V$ a finite dimensional $\mathbb{C}M$-module. Then, for a regular $\mathscr{J}$-class $J$, one has that*

$$[V : S_J] = \sum_{\substack{MeM \in \Lambda(M),\\ MeM \subseteq MJM}} \chi_V(e) \cdot \mu(MeM, MJM)$$

*holds.*

*Proof.* Since $\chi_V = \sum_{MJM \in \Lambda(M)} [V : S_J] \cdot \chi_{S_J}$, we have

$$\begin{aligned} [V : S_J] &= \sum_{MeM \in \Lambda(M)} \chi_V(e) X(M)^{-1}_{MeM,MJM} \\ &= \sum_{\substack{MeM \in \Lambda(M),\\ MeM \subseteq MJM}} \chi_V(e) \cdot \mu(MeM, MJM) \end{aligned}$$

by Proposition 8.11. □

## 8.3 The character table of the full transformation monoid

In this section, we use the results of Section 5.3 to compute the character table of the full transformation, a result first achieved by Putcha [Put96]. Fix $n \geq 1$. For $1 \leq r \leq n$, we let $\mathcal{P}_r$ denote the set of all partitions of $r$ and we put $\mathcal{P}(n) = \bigcup_{r=1}^{n} \mathcal{P}_r$. Both the simple $\mathbb{C}T_n$-modules and the generalized conjugacy classes of $T_n$ are indexed by $\mathcal{P}(n)$. We order $\mathcal{P}(n)$ as follows. Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ and $\mu = (\mu_1, \ldots, \mu_m)$ be elements of $\mathcal{P}(n)$. Then we declare $\mu \prec \lambda$ if $|\mu| < |\lambda|$ or $|\mu| = |\lambda|$ and $\mu$ precedes $\lambda$ in lexicographic order, that is, there exists $j$ such that $\mu_i = \lambda_i$ for $i < j$ and $\mu_j < \lambda_j$.

We use the idempotents $e_1, \ldots, e_n$ defined in Section 5.3 as the representatives of the $\mathscr{J}$-classes of $T_n$. Recall that $e_i$ fixes $[i]$ and sends $[n] \setminus [i]$ to 1. We identify $G_{e_r}$ with $S_r$ when convenient (via the restriction). If $\lambda$ is a partition of $r$ with $1 \leq r \leq n$, then $C_\lambda$ will denote the conjugacy class of $G_{e_r}$ consisting of elements whose restriction to $[r]$ has cycle type $\lambda$ and $\overline{C_\lambda}$ will denote the corresponding generalized conjugacy class of $T_n$. We order the simple $\mathbb{C}T_n$-modules by putting $S^\sharp_\mu < S^\sharp_\lambda$ if $\mu \prec \lambda$ and similarly, we order the generalized conjugacy classes if $T_n$ by setting $\overline{C_\mu} < \overline{C_\lambda}$ if $\mu \prec \lambda$.

**Theorem 8.22.** *With respect to the above ordering of simple modules and generalized conjugacy classes, $X(T_n) = UX$ where*

$$X = \begin{bmatrix} X(S_1) & 0 & \cdots & 0 \\ 0 & X(S_2) & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & X(S_n) \end{bmatrix}$$

*and $U$ is the unipotent upper triangular matrix $\mathcal{P}(n) \times \mathcal{P}(n)$ matrix given by*

$$U_{\mu,\lambda} = \begin{cases} 1, & \text{if } \mu = (1^r) \text{ and } \lambda = (m-r+1, 1^{r-1}) \text{ with } m \geq r \\ 1, & \text{if } \mu \neq (1^r),\ \mu \subseteq \lambda \text{ and } \lambda \setminus \mu \text{ is a horizontal strip} \\ 0, & \text{else} \end{cases}$$

*for $\mu, \lambda \in \mathcal{P}(n)$.*

*Proof.* This follows from Corollary 8.17 and Proposition 6.6. □

*Example 8.23.* We compute the character table for $T_3$. We have $\mathcal{P}(3) = \{(1), (1^2), (2), (1^3), (2,1), (3)\}$. The matrix $U$ is the transpose of the matrix $L$ computed in Example 6.7 and so we have

$$X(T_3) = \begin{bmatrix} 1&0&1&0&0&1 \\ 0&1&0&0&1&0 \\ 0&0&1&0&1&1 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{bmatrix} \cdot \begin{bmatrix} 1&0&0&0&0&0 \\ 0&1&-1&0&0&0 \\ 0&1&1&0&0&0 \\ 0&0&0&1&-1&1 \\ 0&0&0&2&0&-1 \\ 0&0&0&1&1&1 \end{bmatrix} = \begin{bmatrix} 1&1&1&1&1&1 \\ 0&1&-1&2&0&-1 \\ 0&1&1&3&1&0 \\ 0&0&0&1&-1&1 \\ 0&0&0&2&0&-1 \\ 0&0&0&1&1&1 \end{bmatrix}.$$

## 8.4 The Burnside-Brauer theorem

A classical result of Burnside [Bur55] says that if a finite dimensional $\mathbb{C}G$-module affords a faithful representation of a finite group $G$, then every simple $\mathbb{C}G$-module is a composition factor of some tensor power of $V$. Brauer [Bra64] refined this result by bounding the number of tensor powers needed in terms of the number of distinct values taken on by $\chi_V$. Here we prove the analogues for finite monoids; this is original to the text. In Chapter 11, we will prove a result of R. Steinberg [Ste62], which says that the direct sum of the tensor powers of a faithful representation of a monoid provides a faithful module for the monoid algebra. This also implies the result of Burnside, but without a good bound on the number of tensor powers needed. We also prove here an analogue of the Burnside-Brauer theorem for symmetric powers. These kinds of results for representations of finite monoids over finite fields can be found in [Kuh94a, KK94].

### 8.4.1 Tensor powers

We begin with a well-known lemma about idempotents of group algebras.

**Lemma 8.24.** *Let $G$ be a finite group. Suppose that $e = \sum_{g\in G} c_g g$ in $\mathbb{C}G$ is a nonzero idempotent. Then $c_1 \neq 0$.*

*Proof.* Because $e \neq 0$, we have $\dim e\mathbb{C}G > 0$. Let $\chi$ be the character of the regular representation of $G$ over $\mathbb{C}$, which we extend linearly to $\mathbb{C}G$. Then

$$\dim e\mathbb{C}G = \chi(e) = \sum_{g\in G} c_g\chi(g) = c_1 \cdot |G|$$

since by (B.2)

$$\chi(g) = \begin{cases} |G|, & \text{if } g = 1 \\ 0, & \text{else.} \end{cases}$$

Therefore, $c_1 = (\dim e\mathbb{C}G)/|G| \neq 0$. □

We now prove the monoid analogue of the Burnside-Brauer theorem. We shall, in fact, weaken the faithfulness hypothesis. Let us say that a homomorphism $\varphi\colon M \longrightarrow N$ of monoids is an **LI**-*morphism* if $\varphi$ separates $e$ from $eMe \setminus \{e\}$ for all idempotents $e \in E(M)$, that is, $\varphi^{-1}(\varphi(e)) \cap eMe = \{e\}$. Obviously an injective homomorphism is an **LI**-morphism and the converse holds for a group homomorphism. The notion of an **LI**-morphism will become fundamental in Chapter 11.

*Example 8.25.* Let $M = \{1, x, x^2\}$ where $x^2 = x^3$. Then $\varphi\colon M \longrightarrow \mathbb{C}$ given by $\varphi(1) = 1$ and $\varphi(x) = 0 = \varphi(x^2)$ is an **LI**-morphism but is not injective.

**Theorem 8.26.** *Let $M$ be a finite monoid and $V$ a finite dimensional $\mathbb{C}M$-module affording a representation $\rho\colon M \longrightarrow \mathrm{End}_{\mathbb{C}}(V)$ which is an* **LI**-*morphism, e.g., if $\rho$ is faithful. Suppose that the character $\chi$ of $V$ takes on $r$ distinct values. Then every simple $\mathbb{C}M$-module is a composition factor of $V^{\otimes i}$ for some $0 \leq i \leq r-1$.*

*Proof.* Let $S$ be a simple $\mathbb{C}M$-module with apex $e \in E(M)$. Put $A = \mathbb{C}M$ and let $R = \mathrm{rad}(A)$. Observe that $eAe = \mathbb{C}[eMe]$. As $eS \neq 0$, there is a primitive idempotent $f$ of $eAe$ such that $f$ is primitive in $A$ and $S \cong Af/Rf$ by Lemma 4.21. Write

$$f = \sum_{m \in eMe} c_m m.$$

By definition of an apex $I_e S = 0$. On the other hand, $fS \neq 0$. Thus $f \notin \mathbb{C}I_e$. Define a homomorphism $\varphi\colon eAe \longrightarrow \mathbb{C}G_e$ by

$$\varphi(m) = \begin{cases} m, & \text{if } m \in G_e \\ 0, & \text{if } m \in I_e \end{cases}$$

for $m \in eMe$ and note that $\ker \varphi = \mathbb{C}I_e$. Therefore,

$$\varphi(f) = \sum_{g \in G_e} c_g g$$

is a nonzero idempotent of $\mathbb{C}G_e$ and hence $c_e \neq 0$ by Lemma 8.24.

Let $\chi_1, \ldots, \chi_r$ be the distinct values taken on by $\chi$ and let

$$M_j = \{m \in eMe \mid \chi(m) = \chi_j\}.$$

Without loss of generality assume that $\chi_1 = \chi(e) = \dim eV$. Put

$$b_j = \sum_{m \in M_j} c_m.$$

Suppose now that $[V^{\otimes i} : S] = 0$ for all $0 \leq i \leq r-1$. We follow here the convention that $\chi_j^0 = 1$ even if $\chi_j = 0$. As $\chi_{V^{\otimes i}}(m) = \chi_V(m)^i$ for $m \in M$, an application of Proposition 8.19 yields

$$0 = [V^{\otimes i} : S] = \sum_{m \in eMe} c_m \chi^i(m) = \sum_{j=1}^{r} \chi_j^i \sum_{m \in M_j} c_m = \sum_{j=1}^{r} \chi_j^i b_j$$

for all $0 \leq i \leq r-1$. By nonsingularity of the Vandermonde matrix, we conclude that $b_j = 0$ for all $1 \leq j \leq r$.

Applying Proposition 8.14, we have that

$$M_1 = \{m \in eMe \mid \chi_{eV}(m) = \dim eV\}.$$

Because $V$ affords a representation of $M$ which is an **LI**-morphism, it follows that $eV$ affords a representation of $\psi\colon eMe \longrightarrow \mathrm{End}_{\mathbb{C}}(eV)$ such that $\psi^{-1}(I) = \{e\}$. Lemma 8.8(iv) then implies that $M_1 = \{e\}$. Thus $0 = b_1 = c_e \neq 0$. This contradiction concludes the proof. $\square$

We remark that it is necessary to include the trivial representation $V^{\otimes 0}$ because if $M$ is a monoid with a zero element $z$ and if $zV = 0$, then $zV^{\otimes i} = 0$ for all $i > 0$ and so the trivial representation is not a composition factor of any positive tensor power of $V$. The proof of Theorem 8.26 can be modified to show that if $S$ is not the trivial module, or if $M$ has no zero element, then $S$ appears as a composition factor of $V^{\otimes i}$ with $1 \leq i \leq r$. The key point is that only the trivial representation can have the zero element of $M$ as an apex and so in either of these two cases, $\chi(e) \neq 0$.

If $G$ is a finite group and $V$ is a finite dimensional $\mathbb{C}G$-module affording a faithful representation of $G$ whose character takes on $r$ distinct values, then $\bigoplus_{i=0}^{r-1} V^{\otimes i}$ contains every simple $\mathbb{C}G$-module as a composition factor by the Burnside-Brauer theorem and hence is a faithful $\mathbb{C}G$-module because $\mathbb{C}G$ is semisimple. We observe that the analogous result fails in a very strong sense for monoids.

Let $N_t = \{0, 1, \ldots, t\}$ where 1 is the identity and $xy = 0$ for $x, y \in N_t \setminus \{1\}$. Define a faithful two-dimensional representation $\rho\colon N_t \longrightarrow M_2(\mathbb{C})$ by

$$\rho(0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \ \rho(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \rho(j) = \begin{bmatrix} 0 & j \\ 0 & 0 \end{bmatrix}, \ \text{for } 2 \leq j \leq t.$$

Let $V$ be the corresponding $\mathbb{C}N_t$-module. The character of $\rho$ takes on 2 values, 0 and 1. However, $V^{\otimes 0} \oplus V^{\otimes 1}$ is 3-dimensional and so cannot be a faithful $\mathbb{C}N_t$-module for $t \geq 9$ by dimension considerations. In fact, given any integer $k \geq 0$, we can choose $t$ sufficiently large so that $\bigoplus_{i=0}^{k} V^{\otimes i}$ is not a faithful $\mathbb{C}N_t$-module (again by dimension considerations). Thus, the minimum $k$ such that $\bigoplus_{i=0}^{k} V^{\otimes i}$ is a faithful $\mathbb{C}N_t$-module cannot be bounded as a function of only the number of values assumed by the character $\chi_V$ (independently of the monoid in question).

### 8.4.2 Symmetric powers

This subsection assumes familiarity with the symmetric powers $\mathrm{Sym}^d(V)$ of a vector space $V$ over $\mathbb{C}$. Readers not familiar with symmetric powers should feel free to skip it.

Fix a finite monoid $M$. If $V$ is a finite dimensional $\mathbb{C}M$-module, then $\mathrm{Sym}^d(V)$ is naturally a $\mathbb{C}M$-submodule of $V^{\otimes d}$. It is well known that if $\rho\colon M \longrightarrow \mathrm{End}_{\mathbb{C}}(V)$ is the representation afforded by $V$, then the character $\chi_{\mathrm{Sym}^d(V)}$ afforded by $\mathrm{Sym}^d(V)$ is given by

$$\chi_{\mathrm{Sym}^d(V)}(m) = h_d(\lambda_1, \ldots, \lambda_n)$$

where $h_d(x_1, \ldots, x_n)$ is the complete symmetric polynomial of degree $d$, $\dim V = n$ and $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $\rho(m)$ with multiplicities; see [FH91, Page 77]. We shall also need the well-known identity [FH91, Appendix A]:

$$\sum_{i=0}^{\infty} h_i(x_1,\ldots,x_n)t^i = \prod_{j=1}^{n} \frac{1}{1-tx_i}. \tag{8.5}$$

**Theorem 8.27.** *Let $M$ be a finite monoid and let $V$ be a finite dimensional $\mathbb{C}M$-module affording a representation $\rho\colon M \longrightarrow \mathrm{End}_{\mathbb{C}}(V)$ which is an* **LI**-*morphism. Then every simple $\mathbb{C}M$-module is a composition factor of one of $\mathrm{Sym}^0(M),\ldots,\mathrm{Sym}^{r-1}(M)$ with $r = \dim V \cdot s$ where $s$ is the number of distinct characteristic polynomials of the elements $\rho(m)$ with $m \in M$.*

*Proof.* We proceed as in the proof of Theorem 8.26. Let $S$ be a simple $\mathbb{C}M$-module and let $e \in M$ be an apex of $S$. Since $\mathrm{Sym}^0(V)$ is the trivial module, we may assume without loss of generality that $S$ is not the trivial module. Then $eMe \neq \{e\}$ and so $eV \neq 0$ because $\rho$ is an **LI**-morphism. Put $A = \mathbb{C}M$ and let $R = \mathrm{rad}(A)$. As $eS \neq 0$, there is a primitive idempotent $f$ of $eAe$ such that $f$ is primitive in $A$ and $S \cong Af/Rf$ by Lemma 4.21. Write

$$f = \sum_{m\in eMe} c_m m.$$

The proof of Theorem 8.26 shows that $c_e \neq 0$.

Let $a_i = [\mathrm{Sym}^i(V) : S]$ and let

$$g(t) = \sum_{i=0}^{\infty} a_i t^i$$

be the corresponding generating function. We prove that $g(t)$ is a non-zero rational function with denominator of degree at most $r$ by establishing a formula in the spirit of Molien.

Let $n = \dim V$ and let $p_m(t)$ be the characteristic polynomial of $\rho(m)$ for $m \in M$. Let $q_1(t),\ldots,q_s(t)$ be the $s$ characteristic polynomials of the endomorphisms $\rho(m)$ with $m \in M$.

Notice that $e\,\mathrm{Sym}^i(V) = \mathrm{Sym}^i(eV)$ as an $eAe$-module because $e(V^{\otimes i}) = (eV)^{\otimes i}$. Let $\rho'\colon eMe \longrightarrow \mathrm{End}_{\mathbb{C}}(eV)$ be the representation afforded by $eV$. Note that if $m \in eMe$, then

$$t^n p_m(1/t) = \det(I - t\rho(m)) = \det(I - t\rho'(m)) \tag{8.6}$$

because if we write $V = eV \oplus (1-e)V$ and choose a basis for $V$ accordingly, we then have the block form

$$I - t\rho(m) = \begin{bmatrix} I - t\rho'(m) & 0 \\ 0 & I \end{bmatrix}.$$

Let $M_j = \{m \in eMe \mid p_m(t) = q_j(t)\}$ and assume that $q_1(t) = p_e(t)$. Let

$$b_j = \sum_{m\in M_j} c_m.$$

Note that if $M_j = \emptyset$, then $b_j = 0$. Observe that

$$t^n q_1(1/t) = \det(I - t\rho'(e)) = \det(I - tI) = (1-t)^k$$

where $k = \dim eV$. On the other hand, because $\rho'$ is an **LI**-morphism if $m \in eMe \setminus \{e\}$, it follows from Lemma 8.8(iv) that not all eigenvalues of $\rho'(m)$ are 1. Therefore, $t^n p_m(1/t) = \det(I - t\rho'(m))$ is a degree $k$ polynomial whose roots are not all equal to 1. In particular, $M_1 = \{e\}$ and so $b_1 = c_e \neq 0$.

Let $m \in eMe$ and let $\lambda_1, \ldots, \lambda_k$ be the eigenvalues of $\rho'(m)$ with multiplicities. Then, using (8.5), we have that

$$\sum_{i=0}^{\infty} \chi_{\mathrm{Sym}^i(eV)}(m)t^i = \sum_{i=0}^{\infty} h_i(\lambda_1, \ldots, \lambda_k)t^i = \prod_{j=1}^{k} \frac{1}{1 - t\lambda_i} = \frac{1}{\det(I - t\rho'(m))}.$$

Therefore, applying (8.6) and Proposition 8.19 yields

$$\begin{aligned} g(t) &= \sum_{i=0}^{\infty} \sum_{m \in eMe} c_m \chi_{\mathrm{Sym}^i(eV)}(m) t^i \\ &= \sum_{m \in eMe} \frac{c_m}{\det(I - t\rho'(m))} \\ &= \sum_{j=1}^{s} \frac{b_j}{t^n q_j(1/t)} \\ &= \frac{b_1}{(1-t)^k} + \sum_{j=2}^{s} \frac{b_j}{t^n q_j(1/t)}. \end{aligned}$$

Since, for all $j = 2, \ldots, s$ with $b_j \neq 0$, the polynomial $t^n(q_j(1/t))$ has degree $k$ and not all roots equal to 1 and since $b_1 = c_e \neq 0$, we conclude 1 is a pole of $g(t)$ of order $k$. Therefore, $g(t) \neq 0$ and, moreover, $g(t) = h(t)/q(t)$ where $\deg q(t) \leq ks \leq \dim V \cdot s = r$. Thus the sequence $a_i$ is not identically zero and satisfies a linear recurrence of degree $r$ (cf. [Sta97, Theorem 4.1.1]). Consequently, there exists $0 \leq i \leq r-1$ such that $a_i \neq 0$. We conclude that $S$ is a composition factor of one of $\mathrm{Sym}^0(V), \ldots, \mathrm{Sym}^{r-1}(V)$. $\square$

*Remark 8.28.* Using Newton's identities, the characteristic polynomial of $\rho(m)$ is determined by $\chi_V(m), \ldots, \chi_V(m^{n-1})$ where $n = \dim V$, and hence $s$ can be bounded in terms of the number of values assumed by $\chi_V$.

## 8.5 The Cartan matrix

In this section we provide, following an idea of Thiéry [Thi12], a method to compute the Cartan matrix of $\mathbb{C}M$ for a finite monoid $M$.

If $A, B$ are $\Bbbk$-algebras, then $A \otimes B$ is a $\Bbbk$-algebra with the product given on basic tensors by $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$. If $V$ is an $A$-module and $W$ is a $B$-module, then $V \otimes W$ is an $A \otimes B$-module via $(a \otimes b)(v \otimes w) = av \otimes bw$ on basis tensors.

If $A$ is a finite dimensional $\Bbbk$-algebra and $M$ is a finite dimensional left $A$-module, then $D(M) = \mathrm{Hom}_{\Bbbk}(M, \Bbbk)$ is a right $A$-module, i.e., an $A^{op}$-module. Moreover, it is known that $D$ takes simples to simples, projectives to injectives, injectives to projectives and preserves indecomposability. In fact, $D$ provides a contravariant equivalence $A\text{-mod} \longrightarrow A^{op}\text{-mod}$ (cf. Theorem A.25). The *enveloping algebra* of $A$, defined by $A^e = A \otimes A^{op}$, has the property that its left modules correspond to $A$-$A$-bimodules [CE99]. We shall need the following consequence of [SY11, Proposition 11.3] and its proof, which is beyond the scope of this text.

**Theorem 8.29.** *Let $\Bbbk$ be an algebraically closed field and let $A$ be a finite dimensional algebra over $\Bbbk$. Suppose that $e_1, \ldots, e_n$ is a complete set of orthogonal primitive idempotents for $A$. Then the $e_i \otimes e_j$ with $1 \leq i, j \leq n$ form a complete set of orthogonal primitive idempotents of $A^e = A \otimes A^{op}$. Moreover, if $S_i = Ae_i / \mathrm{rad}(A)e_i$ is the simple module corresponding to $e_i$, then $A^e(e_i \otimes e_j)/\mathrm{rad}(A^e)(e_i \otimes e_j) \cong S_i \otimes D(S_j)$.*

As a corollary, we can reinterpret the Cartan matrix.

**Corollary 8.30.** *Let $\Bbbk$ be an algebraically closed field and let $A$ be a finite dimensional algebra over $\Bbbk$. Retaining the notation of Theorem 8.29, if $C$ is the Cartan matrix of $A$, then $C_{ij} = [A : S_i \otimes D(S_j)]$.*

*Proof.* By definition $C_{ij} = [Ae_j : S_i] = \dim \mathrm{Hom}_A(Ae_i, Ae_j) = \dim e_i A e_j$. But $e_i A e_j = (e_i \otimes e_j)A$ if we view $A$ as an $A^e$-module. Therefore,

$$C_{ij} = \dim \mathrm{Hom}_{A^e}(A^e(e_i \otimes e_j), A) = [A : S_i \otimes D(S_j)]$$

by Theorem 8.29 and Proposition A.24. This completes the proof. □

Next we observe that the enveloping algebra of $\mathbb{C}M$ is nothing more than $\mathbb{C}[M \times M^{op}]$.

**Proposition 8.31.** *Let $\Bbbk$ be a field and let $M, N$ be finite monoids. Then $\Bbbk[M \times N] \cong \Bbbk M \otimes \Bbbk N$. In particular, $\Bbbk M^e \cong \Bbbk[M \times M^{op}]$.*

*Proof.* The isomorphism takes $(m, n)$ to $m \otimes n$. The details are left to the reader. □

Let $e_1, \ldots, e_s \in E(M)$ be a complete set of idempotent representatives of the regular $\mathscr{J}$-classes of $M$. Then the $(e_i, e_j)$ with $1 \leq i, j \leq s$ form a complete set of idempotent representatives of the regular $\mathscr{J}$-classes of $M \times M^{op}$ and $G_{(e_i, e_j)} = G_{e_i} \times G_{e_j}^{op}$. The simple $\mathbb{C}[M \times M^{op}]$-modules are then the modules

$$V^\sharp \otimes D(W^\sharp) = (V \otimes D(W))^\sharp$$

with $V \in \mathrm{Irr}_{\mathbb{C}}(G_{e_i})$, $W \in \mathrm{Irr}_{\mathbb{C}}(G_{e_j})$ because $(e_i, e_j)$ is an apex for $V^\sharp \otimes D(W^\sharp)$ and $e_i V^\sharp \otimes D(W^\sharp) e_j = e_i V^\sharp \otimes D(e_j W^\sharp) = V \otimes D(W)$.

**Theorem 8.32.** *Let $M$ be a finite monoid and $e_1, \ldots, e_s$ a complete set of idempotent representatives of the regular $\mathscr{J}$-classes of $M$. Denote by $L$ the decomposition matrix of $M$ over $\mathbb{C}$. Let $B$ be a $\mathbb{C}M$-$\mathbb{C}M$-bimodule. Define matrices $Y(B)$ and $Z(B)$ by*

$$Y(B)_{V,W} = [B : V \otimes D(W)]$$

*for $V, W \in \mathrm{Irr}_{\mathbb{C}}(M)$ and*

$$Z(B)_{S,S'} = [e_i B e_j : S \otimes D(S')]$$

*for $S \in \mathrm{Irr}_{\mathbb{C}}(G_{e_i})$ and $S' \in \mathrm{Irr}_{\mathbb{C}}(G_{e_j})$. Then $Z(B) = LY(B)L^T$ holds.*

*Proof.* Since $Y$ and $Z$ are additive as functions of $[B]$ on $G_0(\mathbb{C}[M \times M^{op}])$, it is enough to handle the case that $B$ is simple, i.e., $B = V \otimes D(W)$ with $V, W$ simple. Note that $Y(V \otimes D(W))$ has 1 in position $V, W$ and 0 in all other positions. Then we have

$$\begin{aligned}
[e_i(V \otimes D(W))e_j] &= [e_i V \otimes D(e_j W)] \\
&= \sum_{S \in \mathrm{Irr}_{\mathbb{C}}(G_{e_i})} \sum_{S' \in \mathrm{Irr}_{\mathbb{C}}(G_{e_j})} [e_i V : S][e_j W : S'] \cdot [S \otimes D(S')] \\
&= \sum_{S \in \mathrm{Irr}_{\mathbb{C}}(G_{e_i})} \sum_{S' \in \mathrm{Irr}_{\mathbb{C}}(G_{e_j})} L_{S,V} L_{S',W} \cdot [S \otimes D(S')] \\
&= \sum_{\substack{S \in \mathrm{Irr}_{\mathbb{C}}(G_{e_i}), \\ S' \in \mathrm{Irr}_{\mathbb{C}}(G_{e_j})}} (LY(V \otimes D(W))L^T)_{S,S'} \cdot [S \otimes D(S')].
\end{aligned}$$

We conclude that $Z(V \otimes D(W)) = LY(V \otimes D(W))L^T$. □

It follows from Corollary 8.30 that $Y(\mathbb{C}M)$ is the Cartan matrix of $\mathbb{C}M$. There is an isomorphism $G_{e_i} \times G_{e_j} \longrightarrow G_{e_i} \times G_{e_j}^{op}$ given by $(g, h) \mapsto (g, h^{-1})$. The character $\chi$ of $e_i \mathbb{C}M e_j$ as a $\mathbb{C}[G_{e_i} \times G_{e_j}]$-module is given by

$$\chi(g,h) = |\{m \in e_i M e_j \mid gmh^{-1} = m\}| = |\{m \in e_i M e_j \mid gm = mh\}|.$$

The character $\theta$ of $S \otimes D(S')$ as a $\mathbb{C}[G_{e_i} \times G_{e_j}]$-module is given by $\theta(g,h) = \chi_S(g)\overline{\chi_{S'}(h)}$ using Proposition B.3. Therefore, we have that

$$Z(\mathbb{C}M)_{S,S'} = \frac{1}{|G_{e_i}||G_{e_j}|} \sum_{g \in G_{e_i}} \sum_{h \in G_{e_j}} |\{m \in e_i M e_j \mid gm = mh\}| \cdot \overline{\chi_S(g)} \chi_{S'}(h). \tag{8.7}$$

As a consequence of Theorem 8.32 and the above discussion, we obtain the following result.

**Theorem 8.33.** *Let $M$ be a finite monoid and $e_1, \ldots, e_s$ a complete set of idempotent representatives of the regular $\mathscr{J}$-classes of $M$. Then the Cartan matrix $C$ of $\mathbb{C}M$ is given by $L^{-1}Z(\mathbb{C}M)(L^{-1})^T$ where $L$ is the decomposition matrix of $M$ over $\mathbb{C}$ and $Z(\mathbb{C}M)$ is as in (8.7).*

For example, suppose that each maximal subgroup of $M$ is trivial. Let $e_1, \ldots, e_s$ be a complete set of idempotent representatives of the regular $\mathscr{J}$-classes of $M$ and let $S_i$ be the unique simple module for $\Bbbk G_{e_i} \cong \Bbbk$. Then $Z(\mathbb{C}M)_{S_i,S_j} = |e_i M e_j|$ and $L_{S_i,S_j^\sharp} = \dim e_i S_j^\sharp$.

## 8.6 Exercises

**8.1.** Give an alternative proof of Theorem 8.7 using Proposition 8.19 (which does not depend on it).

**8.2.** Let $M$ be a monoid. Show that the one-dimensional irreducible characters of $M$ form a commutative inverse monoid under pointwise product. (Hint: show that $\chi^*(m) = \overline{\chi(m)}$.)

**8.3.** Prove that generalized conjugacy is the smallest equivalence relation $\sim$ on a finite monoid $M$ such that $mn \sim nm$ and $m \sim m^{\omega+1}$ for all $m, n \in M$.

**8.4.** Let $M$ be a finite monoid. Prove that $m, n \in M$ are generalized conjugates if and only if $\chi(m) = \chi(n)$ for every irreducible character $\chi$ of $M$.

**8.5.** Let $M$ be a finite monoid and $m, n \in M$. Prove that $m, n$ are generalized conjugates if and only if there exist $x, x' \in M$ such that $xx'x = x$, $x'xx' = x'$, $xm^{\omega+1}x' = n^{\omega+1}$ and $x'n^{\omega+1}x = m^{\omega+1}$.

**8.6.** Prove that $f, g \in I_n$ belong to the same generalized conjugacy class if and only if $f^{\omega+1}$ and $g^{\omega+1}$ have the same cycle type, viewed as permutations of their domains.

**8.7.** Compute the character table of $I_3$.

**8.8.** Let $f \in T_n$. Define $i \in \{1, \ldots, n\}$ to be *recurrent* if $f^m(i) = i$ for some $m > 0$. Denote by $\mathrm{rec}(f)$ the set of recurrent points of $f$.

(a) Prove that $\mathrm{rec}(f)$ is the image of $f^\omega$.
(b) Prove that $f$ leaves $\mathrm{rec}(f)$ invariant and that $f|_{\mathrm{rec}(f)}$ is a permutation.
(c) Prove that two mappings $f, g \in T_n$ are in the same generalized conjugacy class if and only if $|\mathrm{rec}(f)| = |\mathrm{rec}(g)|$ and the permutations $f|_{\mathrm{rec}(f)}$ and $g|_{\mathrm{rec}(g)}$ have the same cycle type.

**8.9.** Compute the character table of $T_4$.

**8.10.** Let $M, N$ be monoids and $\Bbbk$ a field. Prove that $\Bbbk[M \times N] \cong \Bbbk M \otimes \Bbbk N$.

**8.11.** Let $M$ be a finite monoid and let $e, f \in E(M)$. Prove that $MeM = MfM$ if and only if, for any representation $\rho\colon M \longrightarrow M_n(\mathbb{C})$ with $n \geq 1$, there exists an invertible matrix $A \in M_n(\mathbb{C})$ with $A\rho(e)A^{-1} = \rho(f)$.

**8.12.** Compute the Cartan matrix of $\mathbb{C}T_3$.

**8.13 (Kudryavtseva-Mazorchuk [KM09]).** Let $M$ be a finite regular monoid. Prove that $f\colon M \longrightarrow \mathbb{C}$ is a class function if and only if $f(mn) = f(nm)$ for all $m, n \in M$.

**8.14.** Let $V$ be a finite dimensional $\mathbb{C}M$-module for a finite monoid $M$ and let $S$ be a simple $\mathbb{C}M$-module. Prove that the generating function

$$f(t) = \sum_{n=0}^{\infty} [V^{\otimes n} : S] t^n$$

is a rational function.

**8.15.** Let $M, N$ be finite monoids. Describe the character table $X(M \times N)$ in terms of $X(M)$ and $X(N)$.

**8.16.** Let $A$ be a finite dimensional $\Bbbk$-algebra and let $V, W$ be finite dimensional left $A$-modules. Prove that $V \otimes D(W) \cong \mathrm{Hom}_{\Bbbk}(W, V)$ as an $A^e$-module.

# Part IV

# The Representation Theory of Inverse Monoids

# 9

# Categories and Groupoids

In this chapter, we consider a generalization of monoid algebras that will be used in the next chapter to study inverse monoid algebras, namely the algebra of a small category. Further examples include incidence algebras and path algebras of quivers. We show that the Clifford-Munn-Ponizovskii theory applies equally well to categories. The parameterization of the simple modules for the algebra of a finite category given here could also be obtained from a result of Webb [Web07], reducing to the monoid case, and the Clifford-Munn-Ponizovskii theory, but we give a direct proof. A basic reference on category theory is Mac Lane [Mac98]. Category algebras were considered at least as far back as Mitchell [Mit72].

## 9.1 Categories

A *small category* $C$ is a category whose objects and arrows each form a set. We use $C_0$ to denote the object set of $C$ and $C_1$ to denote the set of arrows. The domain and range mappings will be denoted by $\boldsymbol{d}$ and $\boldsymbol{r}$, respectively. The identity at $c$ will be denoted $1_c$ and the hom set of arrows from $c$ to $d$ will be denoted $C(c,d)$. Sometimes we write $C_c$ for the endomorphism monoid $C(c,c)$.

Let us consider some key examples.

*Example 9.1 (Monoids).* A monoid $M$ can be viewed as a category with a single object $*$ and with arrow set $M$. Then $\boldsymbol{d}(m) = * = \boldsymbol{r}(m)$ for all $m \in M$, $1_* = 1$ and the composition is the product in $M$. Functors between monoids, viewed as categories, are exactly homomorphisms.

*Example 9.2 (Posets).* A poset $P$ can be considered a category by taking the object set to be $P$ and the arrow set to be $\{(p,q) \mid p \leq q\}$. Here $\boldsymbol{d}(p,q) = p$ and $\boldsymbol{r}(p,q) = q$. Composition is given by $(q,r)(p,q) = (p,r)$ and $1_p = (p,p)$. The category is finite if and only if the poset is finite. Functors between posets, viewed as categories, are precisely order-preserving maps.

*Example 9.3 (Path categories).* Let $Q$ be a finite directed graph with vertex set $Q_0$ and edge set $Q_1$, called a *quiver* in this context. Then the *path category* of $Q$, or *free category* on $Q$, is the category $Q^*$ with object set $Q_0$ and with hom set $Q^*(v,w)$ consisting of all paths from $v$ to $w$, including an empty path $1_v$ if $v = w$. Composition is given by concatenation of paths, where we follow the convention that paths are concatenated from right to left, that is, if $e_1, \ldots, e_n$ are edges with $\boldsymbol{d}(e_{i+1}) = \boldsymbol{r}(e_i)$, then their concatenation is

$$e_n e_{n-1} \cdots e_1 \colon \boldsymbol{d}(e_1) \longrightarrow \boldsymbol{r}(e_n).$$

For example, if $Q$ has a single vertex, then $Q^*$ is the free monoid on $Q_1$, viewed as a one-object category. The path category of $Q$ is finite if and only if $Q$ is *acyclic*, that is, has no directed cycles.

The category $Q^*$ has the following universal property. Given a category $C$ and a pair of maps $f_0 \colon Q_0 \longrightarrow C_0$ and $f_1 \colon Q_1 \longrightarrow C_1$ such that, for each edge $e \colon v \longrightarrow w$ of $Q$, one has $f_1(e) \colon f_0(v) \longrightarrow f_0(w)$, there is a unique functor $F \colon Q^* \longrightarrow C$ such that $F|_{Q_0} = f_0$ and $F|_{Q_1} = f_1$, that is, the diagram

$$\begin{tikzcd} Q \arrow[rr, "\iota"] \arrow[dr, "f"'] & & Q^* \arrow[dl, "F"] \\ & C & \end{tikzcd}$$

commutes, where $\iota$ is the inclusion and edges are viewed as paths of length 1.

*Example 9.4 (Injective maps).* Let $FI_n$ be the category whose objects are subsets of $\{1, \ldots, n\}$ and whose morphisms are injective mappings. The study of the representation theory of the direct limit of the categories $FI_n$ has become quite popular in recent years, cf. [CEFN14].

A number of the above examples are *EI-categories*, that is, categories in which each endomorphism is an isomorphism. A monoid is an EI-category if and only if it is a group. The category associated to a poset and the path category of an acyclic quiver are EI-categories, as is $FI_n$. There is a significant amount of literature on EI-categories [Lüc89, Web07, Web08, Li11, MS12a, Li14].

Fix now a category $C$ and a field $\Bbbk$. The *category algebra* of $C$, denoted $\Bbbk C$, has underlying $\Bbbk$-vector space $\Bbbk C_1$. The product on basis elements is given by

$$fg = \begin{cases} f \circ g, & \text{if } \boldsymbol{d}(f) = \boldsymbol{r}(g) \\ 0, & \text{else} \end{cases}$$

and is then extended linearly.

*Example 9.5.* If $M$ is a monoid, viewed as a one-object category, then the category algebra of $M$ is the usual monoid algebra $\Bbbk M$. If $P$ is a finite poset, viewed as a category, then the reader should check that $\Bbbk P \cong I(P,\Bbbk)^{op}$, the opposite of the incidence algebra of $P$ over $\Bbbk$. If $Q$ is a quiver, then $\Bbbk Q^*$ is usually denoted by $\Bbbk Q$ and is called the *path algebra* of $Q$.

**Proposition 9.6.** *Let $C$ be a category with finitely many objects and $\Bbbk$ a field. Then $\Bbbk C$ is a unital $\Bbbk$-algebra with identity element $\sum_{c\in C_0} 1_c$. Moreover, the identity elements $\{1_c \mid c \in C_0\}$ form a complete set of orthogonal idempotents.*

*Proof.* We just verify that $\Bbbk C$ is unital. The remaining assertions are straightforward. Indeed, if $f$ is an arrow of $C$, then

$$\sum_{c\in C_0} 1_c \cdot f \cdot \sum_{c\in C_0} 1_c = 1_{\boldsymbol{r}(f)} f 1_{\boldsymbol{d}(f)} = f$$

and the result follows. □

Next we provide a concrete interpretation of what are $\Bbbk C$-modules. Let $\mathrm{rep}_{\Bbbk}(C)$ be the category of all functors $F\colon C \longrightarrow \Bbbk\text{-mod}$. The morphisms are natural transformations. We remind the reader that if $F,G\colon C \longrightarrow D$ are two functors, then a *natural transformation* $\eta\colon F \longrightarrow G$ is a collection of morphisms $\eta_c\colon F(c) \longrightarrow G(c)$, for each object $c$ of $C$, such that the diagram

$$\begin{tikzcd} F(c) \arrow[r, "F(f)"] \arrow[d, "\eta_c"'] & F(c') \arrow[d, "\eta_{c'}"] \\ G(c) \arrow[r, "G(f)"'] & G(c') \end{tikzcd}$$

commutes for each arrow $f\colon c \longrightarrow c'$ of $C$.

For example, if $M$ is a monoid viewed as a one-object category, then a functor $F\colon M \longrightarrow \Bbbk\text{-mod}$ is the same thing as a homomorphism from $M$ into the endomorphism monoid of a finite dimensional $\Bbbk$-vector space. If $Q$ is a quiver, then a functor $F\colon Q^* \longrightarrow \Bbbk\text{-mod}$ is the same thing as an assignment of a finite dimensional $\Bbbk$-vector space $F(v)$ to each vertex $v$ and of a linear transformation $F(e)\colon F_v \longrightarrow F_w$ for each edge $e\colon v \longrightarrow w$ of $Q$ by the universal property in Example 9.3.

**Theorem 9.7.** *Let $C$ be a category with finitely many objects and $\Bbbk$ a field. Then the categories $\mathrm{rep}_{\Bbbk}(C)$ and $\Bbbk C\text{-mod}$ are equivalent.*

*Proof.* First let $V$ be a $\Bbbk C$-module. Define a functor $F\colon C \longrightarrow \Bbbk\text{-mod}$ by putting $F(c) = 1_c V$ for an object $c \in C_0$ and if $g\colon c \longrightarrow d$, then

$$F(g)\colon 1_c V \longrightarrow 1_d V$$

is given by $F(g)v = gv$. Note that $1_d g = g$ implies that $gv \in 1_d V$. It is straightforward to verify that $F$ is a functor.

Conversely, suppose that $F\colon C \longrightarrow \Bbbk\text{-mod}$ is a functor and put

$$V = \bigoplus_{c \in C_0} F(c).$$

If $g\colon c \longrightarrow d$ is an arrow of $C$ and $v \in F(c)$, define $gv = F(g)(v)$. If $v \in F(c')$ with $c' \neq c$, put $gv = 0$. This yields a mapping $\varphi\colon C_1 \longrightarrow \mathrm{End}_{\Bbbk}(V)$ whose natural extension $\Phi\colon \Bbbk C \longrightarrow \mathrm{End}_{\Bbbk}(V)$ is a $\Bbbk$-algebra homomorphism, thereby making $V$ into a $\Bbbk C$-module. Moreover, the reader can check that the above two constructions are inverses up to natural isomorphism using that $\{1_c \mid c \in C_0\}$ is a complete set of orthogonal idempotents. $\square$

As a consequence, it follows that equivalent categories have Morita equivalent algebras.

**Corollary 9.8.** *Let $C$ and $D$ be equivalent categories with finitely many objects and let $\Bbbk$ be a field. Then $\Bbbk C$-mod is equivalent to $\Bbbk D$-mod.*

*Proof.* Clearly, if $C$ and $D$ are equivalent, then $\mathrm{rep}_{\Bbbk}(C)$ and $\mathrm{rep}_{\Bbbk}(D)$ are equivalent. The corollary then follows from Theorem 9.7. $\square$

The set of idempotents in a category $C$ will be denoted $E(C)$. We can define Green's relations on a category $C$ exactly as we do for a monoid. Put, for $f, g \in C_1$,

(i) $f \mathrel{\mathscr{J}} g$ if and only if $C_1 f C_1 = C_1 g C_1$
(ii) $f \mathrel{\mathscr{L}} g$ if and only if $C_1 f = C_1 g$
(iii) $f \mathrel{\mathscr{R}} g$ if and only if $f C_1 = g C_1$.

If $C$ is a category, then we can define a monoid $M(C)$ by

$$M(C) = C_1 \cup \{1\} \cup \{z\}$$

where 1 is an identity element, $z$ is a zero element and

$$fg = \begin{cases} f \circ g, & \text{if } \boldsymbol{d}(f) = \boldsymbol{r}(g) \\ z, & \text{else} \end{cases}$$

defines the product of elements of $f, g \in C_1$. Note that if $M$ is a monoid, viewed as a 1-element category, then $M \neq M(C)$.

**Proposition 9.9.** *Let $C$ be a category and $f, g \in C_1$.*

*(i) $C_1 f C_1 \subseteq C_1 g C_1$ if and only if $M(C) f M(C) \subseteq M(C) g M(C)$*
*(ii) $C_1 f \subseteq C_1 g$ if and only if $M(C) f \subseteq M(C) g$*
*(iii) $f C_1 \subseteq g C_1$ if and only if $f M(C) \subseteq g M(C)$.*

*In particular, Green's relations on $C$ coincide with the restriction of Green's relations on $M(C)$ to $C_1$.*

*Proof.* We prove only (i), as the other cases are similar. Trivially, if $C_1fC_1 \subseteq C_1gC_1$, then $M(C)fM(C) \subseteq M(C)gM(C)$. Conversely, if $f = ugv$ with $u, v \in M(C)$, then $u, v \neq z$. If $u, v \in C_1$, there is nothing to prove. If $u = 1$ or $v = 1$, then we can replace $u$ by $1_{\boldsymbol{r}(g)}$, respectively, $v$ by $1_{\boldsymbol{d}(g)}$ and maintain the equality. Thus $f \in C_1gC_1$. □

It follows that properties of Green's relations in finite monoids, like Theorem 1.10 and Theorem 1.12, have analogues in finite categories. If $e \in E(C)$ is an idempotent, then $eC_1e$ is a monoid with identity $e$ and we denote by $G_e$ the group of units of this monoid. Again we call $G_e$ the *maximal subgroup* of $C$ at $e$. Note that if $e\colon c \longrightarrow c$, then $eC_1e = eC_ce$ (where $C_c = C(c, c)$) and hence $G_e$ is the maximal subgroup at $e$ of the monoid $C_c$. We put $I_e = eC_1e \setminus G_e = eC_ce \setminus G_e$ for an idempotent $e\colon c \longrightarrow c$.

The following lemma shows that $\Bbbk C$ and $\Bbbk M(C)$ are very closely related.

**Lemma 9.10.** *Let $C$ be a finite category. Then the elements $z$,*

$$\varepsilon = \sum_{c \in C_0} (1_c - z)$$

*and $1-\varepsilon-z$ form a complete set of orthogonal central idempotents of $\Bbbk M(C)$. Moreover, one has the following isomorphisms.*

*(i)* $\Bbbk M(C)\varepsilon = \varepsilon \Bbbk M(C)\varepsilon \cong \Bbbk C$.
*(ii)* $\Bbbk M(c)z = z\Bbbk M(C)z \cong \Bbbk \cong (1-\varepsilon-z)\Bbbk M(C)(1-\varepsilon-z) = \Bbbk M(C)(1-\varepsilon-z)$
*(iii)* $\Bbbk M(C) \cong \Bbbk^2 \times \Bbbk C$.

*Proof.* It is immediate that $\{z\} \cup \{1_c - z \mid c \in C_0\}$ is a set of orthogonal idempotents and hence $\varepsilon, z$ are orthogonal idempotents. It then follows that $\varepsilon$, $z$, $1 - \varepsilon - z$ form a complete set of orthogonal idempotents. Trivially, we have that $z$ is a central idempotent and that $\Bbbk M(C)z = \Bbbk z$. If $g \in C_1$, then

$$(1_c - z)g = \begin{cases} g - z, & \text{if } c = \boldsymbol{r}(g) \\ 0, & \text{else} \end{cases}$$

and

$$g(1_c - z) = \begin{cases} g - z, & \text{if } c = \boldsymbol{d}(g) \\ 0, & \text{else.} \end{cases}$$

It follows that $\varepsilon g = g - z = g\varepsilon$ and hence $g(1 - \varepsilon - z) = g - (g - z) - z = 0$. Since $1, z$ are central and $\varepsilon$ commutes with elements of $C_1$, we conclude that $\varepsilon$ is central. Therefore, $\varepsilon$, $z$ and $1 - \varepsilon - z$ form a complete set of orthogonal central idempotents and $\Bbbk M(C)(1 - \varepsilon - z) = \Bbbk(1 - \varepsilon - z)$.

Finally, it is immediate from the definitions that there is a surjective homomorphism of $\Bbbk$-algebras $\varphi\colon M(C) \longrightarrow \Bbbk C$ given $\varphi(1) = \sum_{c \in C_0} 1_c$, $\varphi(z) = 0$ and $\varphi(g) = g$ for $g \in C_1$. It is clear that $z, 1 - \varepsilon - z \in \ker\varphi$ and hence form a basis for $\ker\varphi$ by dimension considerations (or direct computation). We conclude that $\varphi|_{\Bbbk M(C)\varepsilon}\colon \Bbbk M(C)\varepsilon \longrightarrow \Bbbk C$ is an isomorphism.

Item (iii) follows from (i) and (ii). □

Let $C$ be a finite category. If $S$ is a simple $\Bbbk C$-module, we say that an idempotent $e\colon c \longrightarrow c$ is an *apex* for $S$ if $eS \neq 0$ and $I_eS = 0$. Note that in this case, $1_cS \neq 0$ is a simple $\Bbbk C_c$-module with apex $e$.

If $e \in E(C)$, then we can define the functors $\mathrm{Ind}_{G_e}(V) = \Bbbk L_e \otimes_{\Bbbk G_e} V$ and $\mathrm{Coind}_{G_e}(V) = \mathrm{Hom}_{\Bbbk G_e}(\Bbbk R_e, V)$ exactly as for monoids, using Green's relations in $C$. Also we define

$$\mathrm{Res}_{G_e}(V) = eV, \quad K_e(V) = \Bbbk CeV, \quad N_e(V) = \{v \mid e\Bbbk CV = 0\},$$

as was the case for monoids. The following is a generalization of Theorem 5.5 to finite categories. It can also be deduced from a result of [Web07] and Clifford-Munn-Ponizovskii theory.

**Theorem 9.11.** *Let $C$ be a finite category and $\Bbbk$ a field.*

*(i) There is a bijection between isomorphism classes of simple $\Bbbk C$-modules with apex $e \in E(C)$ and isomorphism classes of simple $\Bbbk G_e$-modules given by*

$$\begin{aligned} S &\longmapsto \mathrm{Res}_{G_e}(S) = eS \\ V &\longmapsto V^\sharp = \mathrm{Ind}_{G_e}(V)/N_e(\mathrm{Ind}_{G_e}(V)) = \mathrm{Ind}_{G_e}(V)/\operatorname{rad}(\mathrm{Ind}_{G_e}(V)) \\ &\qquad \cong \operatorname{soc}(\mathrm{Coind}_{G_e}(V)) = K_e(\mathrm{Coind}_{G_e}(V)) \end{aligned}$$

*for $S$ a simple $\Bbbk C$-module with apex $e$ and $V$ a simple $\Bbbk G_e$-module.*

*(ii) Every simple $\Bbbk C$-module has an apex (unique up to $\mathscr{J}$-equivalence).*

*(iii) If $V$ is a simple $\Bbbk G_e$-module, then every composition factor of $\mathrm{Ind}_{G_e}(V)$ and $\mathrm{Coind}_{G_e}(V)$ has apex $f$ with $C_1eC_1 \subseteq C_1fC_1$. Moreover, $V^\sharp$ is the unique composition factor of these two modules with apex $e$.*

*Proof.* By Lemma 9.10, we can identify simple $\Bbbk C$-modules with simple $\Bbbk M(C)$-modules that are not annihilated by $\varepsilon$. Applying Theorem 5.5, these are in bijection with simple $\Bbbk M(C)$-modules whose apex is different from 1 and $z$. In light of Proposition 9.9, which lets us identify Green's relations on $M(C) \cap C_1$ with Green's relations on $C$, the result follows. $\square$

This theorem specializes to the following result for EI-categories [Lüc89, Web07].

**Corollary 9.12.** *Let $C$ be a finite EI-category and $\Bbbk$ a field. Let $c_1, \ldots, c_n$ represent the distinct isomorphism classes of objects of $C$ and let $G_{c_i}$ be the automorphism group at the object $c_i$. Then there is a bijection between simple $\Bbbk C$-modules and simple $\Bbbk G_{c_i}$-modules for $i = 1, \ldots, n$. The bijection is as follows. If $V$ is a simple $\Bbbk C$-module, then there exists a unique $i$ such that $1_{c_i}V \neq 0$ and this latter module is a simple $\Bbbk G_{c_i}$-module. Conversely, if $V$ is a simple $\Bbbk G_{c_i}$-module, then*

$$\left(\Bbbk C1_{c_i} \otimes_{\Bbbk G_{c_i}} V\right) / \operatorname{rad}(\Bbbk C1_{c_i} \otimes_{\Bbbk G_{c_i}} V)$$

*is the corresponding simple $\Bbbk C$-module.*

*Proof.* Since $C$ is an EI-category, the identities are its only idempotents. The result then follows from Theorem 9.11 once one observes that two identities $1_c, 1_d$ of $C$ are $\mathscr{J}$-equivalent if and only if $c, d$ are isomorphic. Trivially, if $c, d$ are isomorphic, then $1_c, 1_d$ are $\mathscr{J}$-equivalent. Conversely, if $1_c, 1_d$ are $\mathscr{J}$-equivalent, then by the category version of Theorem 1.10, there exist $a, b \in C_1$ with $ab = 1_c$ and $ba = 1_d$. Thus $c, d$ are isomorphic. $\square$

## 9.2 Groupoids

A *groupoid* $G$ is a category in which each arrow is an isomorphism. That means, for each arrow $f\colon c \longrightarrow d$, there is an arrow $f^{-1}\colon d \longrightarrow c$ such that $f^{-1}f = 1_c$ and $ff^{-1} = 1_d$. The inverse arrow is unique. For each object $c \in G_0$, the endomorphism monoid $G_c$ is, in fact, a group. A group is, of course, the same thing as a one-object groupoid.

Assume from now on that $G$ is a finite groupoid. We shall write $\overline{c}$ for the isomorphism class of an object $c \in G_0$. Fix representatives $c_1, \ldots, c_s$ of the isomorphism classes of objects on $G$. For each object $c \in G_0$ with $\overline{c} = \overline{c}_i$, fix an arrow $p_c\colon c_i \longrightarrow c$. We may take $p_{c_i} = 1_{c_i}$.

**Proposition 9.13.** *Let $G$ be a groupoid and let $c_1, \ldots, c_s$ be representatives of the isomorphism classes of objects on $G$. Then $G$ is equivalent to the disjoint union groupoid $\coprod_{i=1}^{s} G_{c_i}$. Hence if $\Bbbk$ is a field, $\Bbbk G$-mod is equivalent to $\prod_{i=1}^{s} \Bbbk G_{c_i}$-mod.*

*Proof.* The second statement is a consequence of the first, Theorem 9.7 and Corollary 9.8. To prove the first statement, define $F\colon G \longrightarrow \coprod_{i=1}^{s} G_{c_i}$ by $F(c) = c_i$ if $\overline{c} = \overline{c}_i$ on objects and on arrows, for $g\colon c \longrightarrow d$, define $F(g) = p_d^{-1} g p_c$. It is straightforward to verify that $F$ and the inclusion functor are quasi-inverses. $\square$

Let us make the second equivalence in Proposition 9.13 more explicit. Let $A = \Bbbk G$ and $B = \Bbbk\left[\coprod_{i=1}^{s} G_{c_i}\right]$. Set $e = \sum_{i=1}^{s} 1_{c_i}$. Then $e$ is an idempotent with $eAe = B$ and $AeA = A$. Thus $\mathrm{Res}_e$ gives an equivalence between $A$-mod and $B$-mod with quasi-inverse $\mathrm{Ind}_e$ by Theorem 4.11. But in $B$, the idempotents $1_{c_1}, \ldots, 1_{c_s}$ form a complete set of orthogonal central idempotents and also $1_{c_i} \cdot B \cdot 1_{c_i} = \Bbbk G_{c_i}$. Thus $B \cong \prod_{i=1}^{s} \Bbbk G_{c_i}$ and, therefore, we have the following corollary.

**Corollary 9.14.** *Let $G$ be a groupoid and let $c_1, \ldots, c_s$ be representatives of the isomorphism classes of objects of $G$. Then there is an equivalence of categories*

$$\Bbbk G\text{-mod} \longrightarrow \prod_{i=1}^{s} \Bbbk G_{c_i}\text{-mod}$$

*sending a $\Bbbk G$-module $V$ to $(1_{c_1}V, \ldots, 1_{c_s}V)$. The quasi-inverse equivalence is given on objects by*

$$(V_1,\ldots,V_s)\longmapsto\bigoplus_{i=1}^{s}\Bbbk L_i\otimes_{\Bbbk G_{c_i}}V_i$$

*where $L_i=\boldsymbol{d}^{-1}(c_i)$ (which is a free right $G_{c_i}$-set). In particular, the simple $\Bbbk G$-modules are the modules of the form $\Bbbk L_i\otimes_{\Bbbk G_{c_i}} S$ with $S$ a simple $\Bbbk G_{c_i}$-module.*

*Proof.* Note that $\Bbbk G1_{c_i}=\Bbbk L_i$ and so $\mathrm{Ind}_{1_{c_i}}(V)=\Bbbk L_i\otimes_{\Bbbk G_{c_i}}V$ for $V$ a $\Bbbk G_{c_i}$-module. Also $L_i$ is a free right $G_{c_i}$-set because of the cancellation law in the groupoid $G$. The result now follows from the discussion preceding the corollary. □

We now prove the analogue of Proposition 9.13 at the level of algebras.

**Theorem 9.15.** *Let $G$ be a finite groupoid and $\Bbbk$ a field. Let $c_1,\ldots,c_s$ be representatives of the isomorphism classes of objects of $G$. Let $n_i$ be the cardinality of the isomorphism class of $c_i$. Then we have*

$$\Bbbk G\cong\prod_{i=1}^{s}M_{n_i}(\Bbbk G_{c_i})$$

*and hence $\Bbbk G$ is semisimple if and only if $|G_{c_i}|$ is not divisible by the characteristic of $\Bbbk$ for $i=1,\ldots,s$.*

*Proof.* Let $e_i=\sum_{c\in\overline{c}_i}1_c$. Then the $e_1,\ldots,e_s$ are orthogonal central idempotents of $\Bbbk G$ and $1=\sum_{i=1}^{s}e_i$. Moreover, if $H_i$ denotes the full subgroupoid of $G$ on the objects of $\overline{c}_i$ (with all morphisms of $G$ between these objects), then $e_i\Bbbk Ge_i=\Bbbk H_i$. It follows that $\Bbbk G\cong\prod_{i=1}^{s}\Bbbk H_i$ and hence it suffices to show that $\Bbbk H_i\cong M_{n_i}(\Bbbk G_{c_i})$. Thus, without loss of generality, we may assume that all objects of $G$ are isomorphic.

So suppose from now on that $G_0=\{x_1,\ldots,x_n\}$ and that $p_i\colon x_1\longrightarrow x_i$ is an arrow for each $i=1,\ldots,n$. We may choose $p_1=1_{x_1}$. Let $E_{ij}$ be the standard matrix unit of $M_n(\Bbbk G_{x_1})$. Define $\alpha\colon\Bbbk G\longrightarrow M_n(\Bbbk G_{x_1})$ on a basis element $g\colon x_i\longrightarrow x_j$ by

$$\alpha(g)=p_j^{-1}gp_iE_{ji}.$$

Observe that if $g\colon x_i\longrightarrow x_j$ and $h\colon x_k\longrightarrow x_\ell$, then

$$\alpha(g)\alpha(h)=p_j^{-1}gp_ip_\ell^{-1}hp_kE_{ji}E_{\ell k}=\begin{cases}p_j^{-1}ghp_kE_{jk}, & \text{if } i=\ell\\ 0, & \text{else.}\end{cases}$$

Therefore, $\alpha(g)\alpha(h)=\alpha(gh)$ and so $\alpha$ is a homomorphism, as required.

The inverse of $\alpha$ is given by

$$\beta((a_{ij}))=\sum_{i,j}p_ia_{ij}p_j^{-1}$$

where we identify $\Bbbk G_{x_1}$ with $1_{x_1}\Bbbk G1_{x_1}$. Indeed, if $g\colon x_i\longrightarrow x_j$, then $\beta(\alpha(g))=\beta(p_j^{-1}gp_iE_{ji})=p_jp_j^{-1}gp_ip_i^{-1}=g$. Conversely, $\alpha(\beta(gE_{ij}))=\alpha(p_igp_j^{-1})=p_i^{-1}p_igp_j^{-1}p_jE_{ij}=gE_{ij}$. This completes the proof. □

In summary the representation theory of finite groupoids is essentially the same as the representation theory of finite groups.

## 9.3 Exercises

**9.1.** Show that if $P, Q$ are posets, then order-preserving maps from $P$ to $Q$ are in bijection with functors from the category associated to $P$ to the category associated to $Q$.

**9.2.** Let $P$ be a poset and $\Bbbk$ a field. Prove that $\Bbbk P \cong I(P, \Bbbk)^{op}$.

**9.3.** Prove the universal property of $Q^*$ in Example 9.3.

**9.4.** Prove that if $C$ and $D$ are equivalent categories and $A$ is any category, then the categories $A^C$ and $A^D$ of functors from $C$ to $A$ and $D$ to $A$, respectively, are equivalent.

**9.5.** Let $\Bbbk$ be a field and $Q = (Q_0, Q_1)$ an acyclic quiver.

(a) Prove that $\{1_v \mid v \in Q_0\}$ is a complete set of orthogonal primitive idempotents of $\Bbbk Q$.
(b) Prove that $\operatorname{rad}(\Bbbk Q)$ is the ideal spanned by paths of length at least 1 and that $\Bbbk Q/\operatorname{rad}(\Bbbk Q) \cong \Bbbk^{|Q_0|}$.
(c) Describe explicitly the simple $\Bbbk Q$-modules.
(d) Describe explicitly the projective indecomposable $\Bbbk Q$-modules.

**9.6.** Let $P$ be a finite poset and $\Bbbk$ a field.

(a) Prove that $\{(p, p) \mid p \in P\}$ is a complete set of orthogonal primitive idempotents of $\Bbbk P$.
(b) Prove that $\operatorname{rad}(\Bbbk P)$ is spanned by all arrows $(p, q)$ with $p < q$ and $\Bbbk P/\operatorname{rad}(\Bbbk P) \cong \Bbbk^{|P|}$.
(c) Describe explicitly the simple $\Bbbk P$-modules.
(d) Describe explicitly the projective indecomposable $\Bbbk P$-modules.

**9.7.** Let $P$ be a finite poset and $\Bbbk$ a field. Let $Q$ be the Hasse diagram of $P$. Express $\Bbbk P$ as a quotient of $\Bbbk Q$.

**9.8.** Let $\Bbbk$ be a field of characteristic 0. Prove that

$$\Bbbk FI_n/\operatorname{rad}(\Bbbk FI_n) \cong \prod_{r=0}^{n} M_{\binom{n}{r}}(\Bbbk S_r).$$

(Hint: if $G$ is the groupoid of bijections between subsets of $[n]$, prove that $\Bbbk FI_n/\operatorname{rad}(\Bbbk FI_n) \cong \Bbbk G$.)

**9.9.** Fill in the missing details of the proof of Theorem 9.7.

**9.10.** Let $G$ be a finite group acting on a finite set $X$. Let $H$ be the groupoid with $H_0 = X$, $H_1 = G \times X$, $\boldsymbol{d}(g,x) = x$, $\boldsymbol{r}(g,x) = gx$, product

$$(g', gx)(g, x) = (gg', x),$$

identities $(1, x)$ and inverses $(g, x)^{-1} = (g^{-1}, gx)$ where $x \in X$ and $g, g' \in G$. Let $\Bbbk$ be a field.

(a) Express $\Bbbk H$ as a direct product of matrix algebras over group algebras of subgroups of $G$.
(b) Prove that $\varphi\colon \Bbbk G \longrightarrow \Bbbk H$ given by

$$\varphi(g) = \sum_{x \in X} (g, x)$$

for $g \in G$ is a ring homomorphism.

**9.11.** Let $M$ be a finite monoid. Define a category $K(M)$ (called the *Karoubi envelope* of $M$) by $K(M)_0 = E(M)$ and by

$$K(M)_1 = \{(e, m, f) \in E(M) \times M \times E(M) \mid fme = m\}.$$

Here $\boldsymbol{d}(e, m, f) = e$ and $\boldsymbol{r}(e, m, f) = f$. The product is given by

$$(f, n, g)(e, m, f) = (g, nm, e).$$

(i) Verify that $K(M)$ is a category.
(ii) Prove that two idempotents $e, f \in E(M)$ are isomorphic in $K(M)$ if and only if $MeM = MfM$.
(iii) Prove that $K(M)(e, e) \cong eMe$.
(iv) Prove that $\Bbbk K(M)$ is Morita equivalent to $\Bbbk M$ for any field $\Bbbk$.

# 10

# Representation Theory of Inverse Monoids

In this chapter we develop the representation theory of inverse monoids following the approach of the author [Ste06, Ste08] using groupoid algebras and Möbius inversion. In fact, this work has a precursor in the work of Rukolaĭne [Ruk78, Ruk80], who used alternating sums of idempotents to achieve the same effect as Möbius inversion and used Brandt inverse semigroups instead of groupoids. The author only became aware of the work of Rukolaĭne after [Ste06, Ste08] was published. However, our more explicit approach lets one take advantage of the detailed knowledge of the Möbius function for a number of naturally occurring lattices. We will, for instance, exploit this for our simple character theoretic proof of Solomon's computation [Sol02] of the tensor powers of the natural module for the symmetric inverse monoid.

In this chapter, we work with inverse monoids because we provided an exposition in Chapter 3 of the apparatus of inverse semigroup theory in the context of monoids. But, in fact, finite inverse semigroups have unital semigroup algebras and everything in this chapter works *mutatis mutandis* for inverse semigroups.

## 10.1 The groupoid of an inverse monoid

Canonically associated to each inverse monoid is a groupoid. The author learned of this construction from Lawson's book [Law98]. The *groupoid* $G(M)$ of an inverse monoid $M$ has object set $E(M)$ and arrow set $\{[m] \mid m \in M\}$, a set in bijection with $M$ via $m \mapsto [m]$. One defines $\boldsymbol{d}([m]) = m^*m$ and $\boldsymbol{r}([m]) = mm^*$. The product is given by $[m][n] = [mn]$ if $\boldsymbol{d}([m]) = \boldsymbol{r}([n])$. The identity at an object $e \in E(S)$ is $1_e = [e]$ and the inverse operation is given by $[m]^{-1} = [m^*]$.

**Proposition 10.1.** *If $M$ is a finite inverse monoid, then $G(M)$ is a finite groupoid. Moreover, the automorphism group $G(M)_e$ at an object $e \in E(M)$ can be identified with the maximal subgroup $G_e$ and $\boldsymbol{d}^{-1}(e)$ can be identified with the $\mathscr{L}$-class $L_e$ as a right $G_e$-set.*

*Proof.* First observe that if $\boldsymbol{d}([m]) = \boldsymbol{r}([n])$, then $\boldsymbol{d}([mn]) = \boldsymbol{d}([n])$ and $\boldsymbol{r}([mn]) = \boldsymbol{r}([m])$ by Proposition 3.15. Associativity follows from the associativity of multiplication in $M$. If $[m]\colon e \longrightarrow f$, then $e = m^*m$ and $f = mm^*$. Thus $[m][e] = [me] = [m]$ and $[f][m] = [fm] = [m]$, whence $1_e = [e]$ and $1_f = [f]$. We deduce that $G(M)$ is a category. Finally, if $m \in M$, then $[m]\colon m^*m \longrightarrow mm^*$, $[m^*]\colon mm^* \longrightarrow m^*m$ and

$$[m^*][m] = [m^*m] = 1_{m^*m} \quad \text{and} \quad [m][m^*] = [mm^*] = 1_{mm^*}.$$

It follows that $[m^*] = [m]^{-1}$ and so $G(M)$ is a groupoid, as required.

Note that $[m] \in G(M)_e$ if and only if $\boldsymbol{d}([m]) = e = \boldsymbol{r}([m])$, if and only if $m^*m = e = mm^*$, if and only if $m \in G_e$ by Corollary 3.6. It is then immediate that $g \mapsto [g]$ gives an isomorphism between $G_e$ and $G(M)_e$. Also it follows that $m \mapsto [m]$ is a bijection between $L_e$ and $\boldsymbol{d}^{-1}(e)$ by Proposition 3.10 and clearly this bijection identifies these two sets as right $G_e$-sets. $\square$

For example, the groupoid $G(I_n)$ of the symmetric inverse monoid $I_n$ is isomorphic to the groupoid whose objects are subsets of $\{1, \ldots, n\}$ and whose arrows are bijections. If $M = E(M)$ is a lattice, then each element of $G(M)$ is an identity.

**Proposition 10.2.** *Let $M$ be a finite inverse monoid. Then idempotents $e, f \in E(M)$ are isomorphic in $G(M)$ if and only if $MeM = MfM$.*

*Proof.* If $e$ is isomorphic to $f$ in $G(M)$, then there exists $m \in M$ with $m^*m = e$ and $mm^* = f$. But then $MeM = MmM = MfM$. Conversely, if $MeM = MfM$, then by Corollary 1.13, there exists $m \in M$ with $Me = Mm$ and $mM = fM$. By Proposition 3.10, we then have $m^*m = e$ and $mm^* = f$. Thus $[m]\colon e \longrightarrow f$ is an isomorphism. $\square$

## 10.2 The isomorphism of algebras

The main result of this section is that the algebras of an inverse monoid $M$ and of its groupoid $G(M)$ are isomorphic. In the next section, we shall exploit this isomorphism to obtain a better understanding of the representation theory of the inverse monoid $M$. An analogue of this theorem for infinite inverse semigroups appears in [Ste10c].

**Theorem 10.3.** *Let $M$ be a finite inverse monoid and $\Bbbk$ a field. Then there is an isomorphism $\alpha\colon \Bbbk M \longrightarrow \Bbbk G(M)$ given by*

$$\alpha(m) = \sum_{n \leq m} [n]$$

*for $m \in M$. The inverse isomorphism is given by*

$$\beta([m]) = \sum_{n\leq m} \mu(n,m)n$$

*for $m \in M$, where $\mu$ is the Möbius function of the poset $(M, \leq)$.*

*Proof.* We first verify that $\alpha$ is a homomorphism. That is, we must verify that if $m_1, m_2 \in M$, then $\alpha(m_1)\alpha(m_2) = \alpha(m_1m_2)$. We begin by computing

$$\alpha(m_1)\alpha(m_2) = \sum_{n_1\leq m_1} [n_1] \sum_{n_2\leq m_2} [n_2] = \sum_{\substack{n_1\leq m_1, n_2\leq m_2,\\ n_1^*n_1 = n_2n_2^*}} [n_1n_2]. \tag{10.1}$$

But Lemma 3.16 implies that each $n \leq m_1m_2$ can be uniquely factored as $n = n_1n_2$ with $n_1 \leq m_1$, $n_2 \leq m_2$ and $n_1^*n_1 = n_2n_2^*$. Therefore, the right hand side of (10.1) is precisely

$$\sum_{n\leq m_1m_2} [n] = \alpha(m_1m_2).$$

This proves that $\alpha$ is a homomorphism.

If $m \in M$, we compute

$$\begin{aligned}
\beta(\alpha(m)) &= \sum_{n\leq m}\sum_{n'\leq n} \mu(n',n)n' = \sum_{n'\leq m}\left(\sum_{n'\leq n\leq m} \mu(n',n)\zeta(n,m)\right) n' \\
&= \sum_{n'\leq m} \delta(n',m)n' = m \\
\alpha(\beta([m])) &= \sum_{n\leq m}\sum_{n'\leq n} \mu(n,m)[n'] = \sum_{n'\leq m}\left(\sum_{n'\leq n\leq m} \zeta(n',n)\mu(n,m)\right)[n'] \\
&= \sum_{n'\leq m} \delta(n',m)[n'] = [m].
\end{aligned}$$

This completes the proof that $\alpha, \beta$ are inverse isomorphisms. □

As a corollary, we obtain that $\Bbbk M$ is isomorphic to a product of matrix algebras over the group algebras of its maximal subgroups.

**Corollary 10.4.** *Let $M$ be a finite inverse monoid and $\Bbbk$ a field. Let $e_1, \ldots, e_s$ be idempotent representatives of the $\mathscr{J}$-classes of $M$ and let $n_i = |E(J_{e_i})|$. Then there is an isomorphism*

$$\Bbbk M \cong \prod_{i=1}^{s} M_{n_i}(\Bbbk G_{e_i}).$$

*Consequently, $\Bbbk M$ is semisimple if and only if the characteristic of $\Bbbk$ does not divide the order of any of the maximal subgroups $G_{e_i}$.*

*Proof.* By Proposition 10.1, Proposition 10.2 and Theorem 9.15, we have an isomorphism

$$\Bbbk G(M) \cong \prod_{i=1}^{s} M_{n_i}(\Bbbk G_{e_i}).$$

The required isomorphism now follows from Theorem 10.3. The final statement follows from Maschke's theorem and the fact that a finite dimensional algebra $A$ is semisimple if and only if $M_n(A)$ is semisimple (for any $n \geq 1$) as $A$ and $M_n(A)$ are Morita equivalent. □

Let us specialize to the case of a lattice, which was considered early on by Solomon [Sol67].

**Corollary 10.5.** *Let $L$ be a finite lattice and $\Bbbk$ a field. Then $\Bbbk L \cong \Bbbk^L$ is a commutative semisimple $\Bbbk$-algebra. If, for $x \in L$, we put*

$$e_x = \sum_{y \leq x} \mu(y, x) y$$

*with $\mu$ the Möbius function of $L$, then we have that $\{e_x \mid x \in L\}$ is a complete set of orthogonal primitive idempotents of $\Bbbk L$.*

*Proof.* The groupoid $G(L)$ consists of just the identities $[x]$ and these identities form a complete set of orthogonal central primitive idempotents. Moreover, $[x]\Bbbk G(L)[x] \cong \Bbbk G_x \cong \Bbbk$ and so $\Bbbk G(L) \cong \Bbbk^L$. As $e_x = \beta([x])$ (using the notation of Theorem 10.3), the result follows from Theorem 10.3. □

Next we consider the case of a more general commutative inverse monoid.

**Corollary 10.6.** *Let $M$ be a finite commutative inverse monoid and let $\Bbbk$ be a field. Then the isomorphism*

$$\Bbbk M \cong \prod_{e \in E(M)} \Bbbk G_e$$

*holds.*

*Proof.* Proposition 3.11 says that the $\mathscr{J}$-classes of a commutative inverse monoid are its maximal subgroups, and hence no two distinct idempotents of $M$ are $\mathscr{J}$-equivalent. The corollary now follows from Corollary 10.4. □

Define the *rank* of a partial injective map to be the cardinality of its image. Then it is straightforward to check that two elements of the symmetric inverse monoid $I_n$ are $\mathscr{J}$-equivalent if and only if they have the same rank. As usual, for $k \geq 0$, put $[k] = \{1, \ldots, k\}$ (and so $[0] = \emptyset$). Then $1_{[0]}, \ldots, 1_{[n]}$ are idempotent representatives of the $\mathscr{J}$-classes of $I_n$. There are $\binom{n}{k}$ idempotents of rank $k$ and the maximal subgroup at $1_{[k]}$ can be identified with the symmetric group $S_k$ of degree $k$; indeed, it consists of all bijections from $[k]$ to $[k]$. We then deduce the following corollary of Theorem 10.3, which was written in an explicit fashion by Solomon [Sol02].

**Corollary 10.7.** *Let $\Bbbk$ be a field. Then there is an isomorphism*

$$\Bbbk I_n \cong \prod_{k=0}^{n} M_{\binom{n}{k}}(\Bbbk S_k)$$

*where $I_n$ is the symmetric inverse monoid of degree $n$ and $S_k$ is the symmetric group of degree $k$.*

## 10.3 Decomposing representations of inverse monoids

Let $M$ be a finite inverse monoid. Then $\mathbb{C}M$ is semisimple by Corollary 10.4. Proposition 8.19 provides a method to compute the composition factors of a finite dimensional $\mathbb{C}M$-module from its character by inverting the character table. However, in practice, computing the character table of a monoid is not an easy task. We present here, for inverse monoids, an alternative means for computing composition factors from the character that exploits the isomorphism $\mathbb{C}M \cong \mathbb{C}G(M)$. This method requires only knowing the character tables of the maximal subgroups and the Möbius function of the lattice $E(M)$ of idempotents. As an application, we provide simpler character theoretic proofs of results of Solomon [Sol02] decomposing the exterior and tensor powers of the standard module for the symmetric inverse monoid.

We first need a lemma relating the Möbius function $\mu_M$ of $M$ and the Möbius function $\mu_{E(M)}$ of $E(M)$.

**Lemma 10.8.** *Let $m \in M$. Then the equality*

$$\sum_{n \leq m} \mu_M(n,m)n = \sum_{f \leq m^*m} \mu_{E(M)}(f, m^*m)mf$$

*holds.*

*Proof.* If $P$ is a partially ordered set and $p \leq q$, then it is well known and easy to show that $\mu_P(p,q)$ only depends on the isomorphism type of the poset $[p,q] = \{r \in P \mid p \leq r \leq q\}$. Let $P = \{n \in M \mid n \leq m\}$ and let $Q = \{f \in E(M) \mid f \leq m^*m\}$. Define $\psi \colon P \longrightarrow Q$ and $\gamma \colon Q \longrightarrow P$ by $\psi(n) = n^*n$ and $\gamma(f) = mf$. Then $n_1 \leq n_2 \leq m$ implies $n_1^*n_1 \leq n_2^*n_2 \leq m^*m$ by Proposition 3.9, and so $\psi$ is a well-defined order-preserving map. Similarly, $f \leq e \leq m^*m$ implies that $mf \leq me \leq m$, again by Proposition 3.9, and so $\gamma$ is order-preserving. We claim that $\psi$ and $\gamma$ are inverses. Indeed, if $n \leq m$ and $f \leq m^*m$, then $\gamma(\psi(n)) = mn^*n = n$ and $\psi(\gamma(f)) = (mf)^*mf = fm^*mf = f$.

It then follows that, for $f \leq m^*m$, we have

$$\mu_M(\gamma(f), m) = \mu_M(\gamma(f), \gamma(m^*m)) = \mu_{E(M)}(f, m^*m).$$

We may now compute

$$\begin{aligned}\sum_{n\leq m}\mu_M(n,m)n &= \sum_{n\leq m}\mu_M(\gamma(\psi(n)),m)\gamma(\psi(n))\\ &= \sum_{f\leq m^*m}\mu_M(\gamma(f),m)\gamma(f)\\ &= \sum_{f\leq m^*m}\mu_{E(M)}(f,m^*m)mf\end{aligned}$$

as required. □

We are now ready to give a formula for decomposing a $\mathbb{C}M$-module from its character. We retain the notation from Theorem 5.5. Note that if $S$ is a simple $\mathbb{C}G_e$-module, for $e \in E(M)$, then $S^\sharp = \mathrm{Ind}_{G_e}(S)$ because $\mathbb{C}M$ is semisimple (see the discussion following Theorem 5.14).

**Theorem 10.9.** *Let $M$ be a finite inverse monoid and let $V$ be a finite dimensional $\mathbb{C}M$-module. Let $e_1,\ldots,e_s$ be a complete set of idempotent representatives of the $\mathscr{J}$-classes of $M$. Then one has that*

$$V = \bigoplus_{i=1}^{s}\bigoplus_{[S]\in\mathrm{Irr}_{\mathbb{C}}(G_{e_i})} m_S\cdot\mathrm{Ind}_{G_{e_i}}(S) = \bigoplus_{i=1}^{s}\bigoplus_{[S]\in\mathrm{Irr}_{\mathbb{C}}(G_{e_i})} m_S\cdot S^\sharp$$

*is the decomposition of $V$ into a direct sum of simple modules where*

$$m_S = \frac{1}{|G_{e_i}|}\sum_{g\in G_{e_i}}\overline{\chi_S(g)}\sum_{f\leq e_i}\chi_V(gf)\cdot\mu(f,e_i)$$

*with $\mu$ the Möbius function of $E(M)$, for $S\in\mathrm{Irr}_{\mathbb{C}}(G_{e_i})$. In particular, $V$ is determined up to isomorphism by $\chi_V$.*

*Proof.* Let $\beta\colon \mathbb{C}G(M)\longrightarrow\mathbb{C}M$ be the isomorphism from Theorem 10.3. Then, applying Lemma 10.8, we have that

$$\beta([m]) = \sum_{n\leq m} n\cdot\mu_M(n,m) = \sum_{f\leq m^*m} mf\cdot\mu(f,m^*m).$$

We can view every $\mathbb{C}M$-module as a $\mathbb{C}G(M)$-module via the isomorphism $\beta$. Observe that if $S$ is a simple $\mathbb{C}G_e$-module with $e\in E(M)$, then since $e$ is an apex of $S^\sharp$, $h<g$ implies $MhM \subsetneq MgM$ (by Proposition 3.13) and $\mu(g,g)=1$ for $g\in G_e$, we have that

$$[g]v = \sum_{h\leq g}\mu(h,g)hv = \mu(g,g)gv = gv$$

for all $g\in G_e$. Hence $[e]S^\sharp \cong eS^\sharp \cong S$ as a $\Bbbk G_e$-module. Therefore, $S^\sharp$ is the simple $\mathbb{C}G(M)$-module corresponding to $S$ in Corollary 9.14 and $[V : S^\sharp] =$

$[[e]V : S]$ (again by Corollary 9.14). It follows that if $[S] \in \mathrm{Irr}_{\mathbb{C}}(G_{e_i})$, then (extending $\chi_V$ linearly to $\mathbb{C}M$) we have that

$$\begin{aligned}[V : S^\sharp] &= [[e]V : S] \\ &= \langle \chi_V \circ \beta|_{G_{e_i}}, \chi_S \rangle_{G_{e_i}} \\ &= \frac{1}{|G_{e_i}|} \sum_{g \in G_{e_i}} \chi_V(\beta([g]))\overline{\chi_S(g)} \\ &= \frac{1}{|G_{e_i}|} \sum_{g \in G_{e_i}} \sum_{f \leq e_i} \chi_V(gf)\mu(f, e_i)\overline{\chi_S(g)} \\ &= \frac{1}{|G_{e_i}|} \sum_{g \in G_{e_i}} \overline{\chi_S(g)} \sum_{f \leq e_i} \chi_V(gf) \cdot \mu(f, e_i)\end{aligned}$$

as required. □

We now prove two results of Solomon concerning the tensor and exterior powers of the natural module for the symmetric inverse monoid [Sol02]. The original proofs of Solomon involved inverting the character table and applying some symmetric function theory. The proof here is elementary and based on Theorem 10.9. A more general version of these results appears in [Ste08].

Let $I_n$ be the symmetric inverse monoid of degree $n$. The *natural module* for $\mathbb{C}I_n$ is $V = \mathbb{C}^n$ where if $e_1, \ldots, e_n$ is the standard basis and $m \in I_n$, then

$$me_i = \begin{cases} e_{m(i)}, & \text{if } m(i) \text{ is defined} \\ 0, & \text{else.} \end{cases}$$

We continue to identify the maximal subgroup at $1_{[r]}$ with the symmetric group $S_r$ in the natural way for $0 \leq r \leq n$.

Let $S(p, r)$ be the *Stirling number of the second kind*. By definition, $S(p, r)$ is the number of partitions of a $p$-element set into $r$ non-empty subsets. A well-known formula for it is

$$S(p, r) = \frac{1}{r!} \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} k^p.$$

See [Sta97, Page 34].

**Lemma 10.10.** *If $X$ is a set, then the equality*

$$\sum_{Y \subseteq X} (-1)^{|X|-|Y|} = \begin{cases} 1, & \textit{if } X = \emptyset \\ 0, & \textit{else} \end{cases}$$

*holds.*

*Proof.* If $X = \emptyset$, the result is clear. If $|X| = n \geq 1$, then

$$\sum_{Y \subseteq X} (-1)^{|X|-|Y|} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} = (1-1)^n = 0$$

as required. □

The following theorem, decomposing the tensor powers of the natural module, is due to Solomon [Sol02].

**Theorem 10.11.** *Let $V$ be the natural module for $\mathbb{C}I_n$ and let $[S] \in \mathrm{Irr}_{\mathbb{C}}(S_r)$ for $0 \leq r \leq n$. Then one has that $[V^{\otimes p} : \mathrm{Ind}_{S_r}(S)] = \dim S \cdot S(p,r)$.*

*Proof.* Throughout this proof, we shall use that $E(I_n)$ is isomorphic to the power set $\mathscr{P}([n])$ under the inclusion ordering via $X \mapsto 1_X$ and that

$$\mu(X,Y) = (-1)^{|Y|-|X|}$$

if $X \subseteq Y$ by Corollary 7.5. Let $\theta$ be the character afforded by $V$. If $m \in I_n$, we put $\mathrm{Fix}(m) = \{i \in [n] \mid m(i) = i\}$. It is easy to see that $\theta(m) = |\mathrm{Fix}(m)|$ and hence $\chi_{V^{\otimes p}}(m) = \theta^p(m) = |\mathrm{Fix}(m)|^p$.

Using Theorem 10.9 we conclude that

$$[V^{\otimes p} : \mathrm{Ind}_{S_r}(S)] = \frac{1}{r!} \sum_{g \in S_r} \overline{\chi_S(g)} \sum_{Y \subseteq [r]} (-1)^{r-|Y|} \theta^p(g|_Y). \tag{10.2}$$

Let $g \in S_r$. Observe that $\theta^p(g|_Y) = |\mathrm{Fix}(g|_Y)|^p = |\mathrm{Fix}(g) \cap Y|^p$. Let $X = \mathrm{Fix}(g) \subseteq [r]$ and $Z = [r] \setminus X$. Performing the change of variables $Y \mapsto (Y \cap X, Y \cap Z)$ (with inverse $(U,V) \mapsto U \cup V$), we obtain

$$\begin{aligned} \sum_{Y \subseteq [r]} (-1)^{r-|Y|} \theta^p(g|_Y) &= \sum_{U \subseteq X, V \subseteq Z} (-1)^{r-|U|-|V|} |U|^p \\ &= \sum_{U \subseteq X} (-1)^{|X|-|U|} |U|^p \sum_{V \subseteq Z} (-1)^{|Z|-|V|} \\ &= \begin{cases} \displaystyle\sum_{U \subseteq [r]} (-1)^{r-|U|} |U|^p, & \text{if } Z = \emptyset \\ 0, & \text{else} \end{cases} \end{aligned} \tag{10.3}$$

where the last equality follows from Lemma 10.10. But $Z = \emptyset$ if and only if $[r] = X = \mathrm{Fix}(g)$, if and only if $g = 1_{[r]}$. Substituting the rightmost term of (10.3) into (10.2) and using that $\chi_S(1_{[r]}) = \dim S$ yields

$$\begin{aligned} [V^{\otimes p} : \mathrm{Ind}_{S_r}(S)] &= \frac{\dim S}{r!} \sum_{U \subseteq [r]} (-1)^{r-|U|} |U|^p \\ &= \dim S \cdot \frac{1}{r!} \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} k^p \\ &= \dim S \cdot S(p,r) \end{aligned}$$

as required. □

If $\Bbbk$ is any field, we can define the *natural module* $V = \Bbbk^n$ for $\Bbbk I_n$ exactly as we did for the field of complex numbers. We now present a direct sum decomposition of the tensor powers of $V$ at this level of generality which immediately implies Theorem 10.11 for $\mathbb{C}$. Essentially, we are finding the decomposition of $V^{\otimes p}$ into $M_{\binom{n}{r}}(\Bbbk S_r)$-modules coming from the direct product decomposition in Corollary 10.7. This result seems to be new.

**Theorem 10.12.** *Let $\Bbbk$ be a field and let $V$ be the natural module for $\Bbbk I_n$. Then we have the isomorphism*

$$V^{\otimes p} \cong \bigoplus_{r=0}^{p} S(p,r) \cdot \Bbbk L_{1_{[r]}}.$$

*Consequently, if the characteristic of $\Bbbk$ is $0$ or greater than $n$, then*

$$[V^{\otimes p} : \mathrm{Ind}_{S_r}(S)] = \dim S \cdot S(p,r)$$

*for $S \in \mathrm{Irr}_{\Bbbk}(S_r)$.*

*Proof.* First we observe that the basis for $V^{\otimes p}$ can be indexed by mappings $f\colon [p] \longrightarrow [n]$ by putting $e_f = e_{f(1)} \otimes \cdots \otimes e_{f(p)}$. If $g \in I_n$, then one checks that

$$ge_f = \begin{cases} e_{gf}, & \text{if } f([p]) \subseteq \mathrm{dom}(g) \\ 0, & \text{else} \end{cases} \tag{10.4}$$

where $\mathrm{dom}(g)$ denotes the domain of $g$. Notice that because $g$ is injective, $gf$ and $f$ have the same associated partition into fibers over elements of its range. Let $\Pi_{p,r}$ be the set of all set partitions of $[p]$ into $r$ non-empty subsets; so $\Pi_{p,r}$ has $S(p,r)$ elements. Then, for $\pi \in \Pi_{p,r}$, set $V_\pi$ to be the $\Bbbk$-span of all $e_f$ such that $\pi$ is the partition associated to $f$. We have $V_\pi$ is a $\Bbbk I_n$-submodule of $V^{\otimes p}$ by (10.4) and the preceding discussion. Moreover,

$$V^{\otimes p} = \bigoplus_{r=0}^{p} \bigoplus_{\pi \in \Pi_{p,r}} V_\pi \tag{10.5}$$

where we note that if $p > 0$, then $\Pi_{p,0} = \emptyset$. If $p = 0$, one should make the obvious conventions about functions from the empty set and partitions (equal equivalence relations) on the empty set.

From (10.5) it is clear that to prove the first statement of the theorem it suffices to show that if $\pi \in \Pi_{p,r}$, then $V_\pi \cong \Bbbk L_{1_{[r]}}$. Let $X_\pi$ be the set of all mappings $f\colon [p] \longrightarrow [n]$ with associated partition $\pi$. Then elements of $X_\pi$ are in bijection with injective mappings $f'\colon [p]/\pi \longrightarrow [n]$. On the other hand, the $\mathscr{L}$-class $L_{1_{[r]}}$ consists of all injective mappings $f\colon [r] \longrightarrow [n]$ (viewed as partial injective mappings on $[n]$). So if we fix a bijection of $h\colon [p]/\pi \longrightarrow [r]$ and let $k\colon [p] \longrightarrow [p]/\pi$ be the canonical projection, then we have a bijection $\alpha\colon L_{1_{[r]}} \longrightarrow X_\pi$ given by $\alpha(f) = fhk$. Moreover, if $g \in I_n$, then $gf \in L_{1_{[r]}}$

if and only if the range of $f$ is contained in the domain of $g$. If the latter occurs, then $\alpha(gf) = gfhk = g\alpha(f)$. It now follows easily from (10.4) that the mapping $\varphi\colon \Bbbk L_{1_{[r]}} \longrightarrow V_\pi$ defined on the basis $L_{1_{[r]}}$ by $\varphi(f) = e_{\alpha(f)}$ is a $\Bbbk I_n$-module isomorphism.

To deduce the second statement, we have that $\Bbbk S_r$ is semisimple, for all $0 \leq r \leq n$, under the hypothesis on $\Bbbk$. Therefore, $\Bbbk I_n$ is also semisimple. Moveover, one has that

$$\begin{aligned}\Bbbk L_{1_{[r]}} &\cong \Bbbk L_{1_{[r]}} \otimes_{\Bbbk S_r} \Bbbk S_r \cong \Bbbk L_{1_{[r]}} \otimes_{\Bbbk S_r} \left(\bigoplus_{S\in \mathrm{Irr}_{\Bbbk}(S_r)} \dim S \cdot S\right) \\ &\cong \bigoplus_{S\in \mathrm{Irr}_{\Bbbk}(S_r)} \dim S \cdot \left(\Bbbk L_{1_{[r]}} \otimes_{\Bbbk S_r} S\right) = \bigoplus_{S\in \mathrm{Irr}_{\Bbbk}(S_r)} \dim S \cdot \mathrm{Ind}_{S_r}(S)\end{aligned}$$

from which the second statement follows. □

Next we consider the exterior powers of $V$. Again, the result is due to Solomon [Sol02].

**Theorem 10.13.** *Let $V$ be the standard module for $\mathbb{C}I_n$. Then the exterior power $V^{\wedge p}$ is simple, for any $0 \leq p \leq n$. Moreover, $V^{\wedge p} = \mathrm{Ind}_{S_p}(S_{(1^p)})$ where $S_{(1^p)}$ affords the sign representation of $S_p$.*

*Proof.* Let $e_1, \ldots, e_n$ be the standard basis for $V$. For each $p$-element subset $X \subseteq [n]$ with $X = \{i_1 < \cdots < i_p\}$, put $e_X = e_{i_1} \wedge \cdots \wedge e_{i_p}$. Then the $e_X$ with $|X| = p$ form a basis for $V^{\wedge p}$. If $m \in I_n$, one verifies directly that

$$me_X = \begin{cases} \mathrm{sgn}(m|_X)e_X, & \text{if } m|_X \in S_X \\ \pm e_{m(X)}, & \text{if } |m(X)| = p,\ X \neq m(X) \\ 0, & \text{else.} \end{cases} \tag{10.6}$$

If $\theta$ is the character afforded by $V^{\wedge p}$, then we deduce that

$$\theta(m) = \sum_{\substack{1_Y \leq m^*m, \\ |Y|=p, m|_Y \in S_Y}} \mathrm{sgn}(m|_Y).$$

Let $S$ be a simple $\mathbb{C}S_r$-module with $0 \leq r \leq n$. Then by Theorem 10.9, we have that

$$\begin{aligned}[V^{\wedge p} : \mathrm{Ind}_{S_r}(S)] &= \frac{1}{r!} \sum_{g\in S_r} \overline{\chi_S(g)} \sum_{X\subseteq [r]} (-1)^{r-|X|} \sum_{\substack{Y\subseteq X, \\ |Y|=p, g|_Y \in S_Y}} \mathrm{sgn}(g|_Y) \\ &= \frac{1}{r!} \sum_{g\in S_r} \overline{\chi_S(g)} \sum_{\substack{Y\subseteq [r], \\ |Y|=p, g|_Y \in S_Y}} \mathrm{sgn}(g|_Y) \sum_{Y\subseteq X\subseteq [r]} (-1)^{r-|X|} \\ &= \frac{1}{r!} \sum_{g\in S_r} \overline{\chi_S(g)} \sum_{\substack{Y\subseteq [r], \\ |Y|=p, g|_Y \in S_Y}} \mathrm{sgn}(g|_Y) \sum_{U\subseteq [r]\setminus Y} (-1)^{r-|Y|-|U|}.\end{aligned}$$

Using Lemma 10.10, we see that the rightmost term of the above equation vanishes unless $p = r$ and in this case it is

$$\frac{1}{p!}\sum_{g\in S_p} \mathrm{sgn}(g)\overline{\chi_S(g)} = \langle \chi_{S_{(1^p)}}, \chi_S\rangle_{S_p}.$$

We conclude that $V^{\wedge p} \cong \mathrm{Ind}_{S_p}(S_{(1^p)})$, completing the proof. □

One could also prove Theorem 10.13 by giving an explicit isomorphism between $V^{\wedge p}$ and $\mathrm{Ind}_{S_p}(S_{(1^p)})$. Analogues of the previous results for wreath products $G \wr I_n$ with $G$ a finite group are given in [Ste08].

## 10.4 The character table of the symmetric inverse monoid

The character table of the symmetric inverse monoid was first computed by Munn [Mun57a]. We follow here the approach of Solomon [Sol02]. A generalization of this technique to arbitrary inverse monoids was given by the author [Ste08]. Let us fix $n \geq 0$. We shall present the character table $X(I_n)$ of the symmetric inverse monoid $I_n$ (over $\mathbb{C}$) as a product of a block diagonal matrix consisting of the character tables of symmetric groups of degree at most $n$ and an explicit unipotent upper triangular matrix on both the left and the right.

The reader is referred to Section B.4 for the representation theory of the symmetric group and basics about partitions. We simply recall that $\lambda = (\lambda_1, \ldots, \lambda_s)$ is a *partition* of $k \geq 1$ if $\lambda_1 \geq \cdots \geq \lambda_s \geq 1$ and $\lambda_1 + \cdots + \lambda_s = k$. We also consider there to be an empty partition of 0. We write $|\lambda| = k$ to indicate that $\lambda$ is a partition of $k$. We denote by $\mathcal{P}_k$ be the set of partitions of $k$ and we put $\mathcal{Q} = \bigcup_{k=0}^{n} \mathcal{P}_k$. We totally order $\mathcal{Q}$ by $\alpha \prec \beta$ if $|\alpha| < |\beta|$ or $|\alpha| = |\beta|$ and $\alpha$ precedes $\beta$ in lexicographic order meaning that if $\alpha = (\alpha_1, \ldots, \alpha_m)$ and $\lambda = (\lambda_1, \ldots, \lambda_s)$, then $\alpha$ lexicographically precedes $\lambda$ if there exists $i$ such that $\alpha_j = \lambda_j$ for all $j < i$ and $\alpha_i < \lambda_i$.

It is well known from elementary group theory that the conjugacy classes of the symmetric group $S_k$ are indexed by the elements of $\mathcal{P}_k$. The conjugacy class $C_\lambda$ corresponding to $\lambda = (\lambda_1, \ldots, \lambda_s)$ is the set of permutations in $S_k$ of cycle type $\lambda$. Let us take $1_{[k]}$ with $0 \leq k \leq n$ as idempotent representatives of the $\mathscr{J}$-classes of $I_n$ and identify $G_{1_{[k]}}$ with $S_k$ as we have done before. Then it follows that the generalized conjugacy classes of $I_n$ can be indexed by $\mathcal{Q}$ where $\overline{C}_\lambda$, for $|\lambda| = k$, is the generalized conjugacy class of $I_n$ that intersects $S_k$ in $C_\lambda$.

The simple $\mathbb{C}S_k$-modules can be naturally parameterized by $\mathcal{P}_k$, as well; see Section B.4. We shall not explicitly use the nature of the parametrization here except in Theorem 10.16. Let $S_\lambda$ be the simple module corresponding to $\lambda \in \mathcal{P}_k$ and let $\chi_\lambda$ be its character. We set $\zeta_\lambda$ to be the character of $\mathrm{Ind}_{S_k}(S_\lambda)$, the corresponding simple module for $\mathbb{C}I_n$. Let $X(S_k)$ be the character table of $S_k$, which we index by $\mathcal{P}_k$, so that

$$X(S_k)_{\alpha,\beta} = \chi_\alpha(C_\beta).$$

We index the character table $X(I_n)$ by $\mathcal{Q}$, so that

$$X(I_n)_{\alpha,\beta} = \zeta_\alpha(\overline{C}_\beta).$$

If $\alpha, \beta \in \mathcal{Q}$, we put

$$\binom{\alpha}{\beta} = \prod_{i\geq 0} \binom{a_i}{b_i}$$

where $a_i$ is the number of parts of $\alpha$ equal to $i$ and $b_i$ is the number of parts of $\beta$ equal to $i$ with the convention $\binom{0}{0} = 1$ and $\binom{n}{m} = 0$ if $n < m$.

**Proposition 10.14.** *If $\binom{\alpha}{\beta} > 0$, then $\beta \preceq \alpha$. Moreover, $\binom{\alpha}{\alpha} = 1$.*

*Proof.* If $\binom{\alpha}{\beta} > 0$, then $a_i \geq b_i$ for all $i \geq 1$ (retaining the above notation). As $|\alpha| = \sum_{i\geq 1} ia_i$ and $|\beta| = \sum_{i\geq 1} ib_i$, we conclude that $|\beta| \leq |\alpha|$ and that if $|\alpha| = |\beta|$, then $\alpha = \beta$. Thus $\beta \preceq \alpha$. The second statement is immediate from the definition. □

All the preparations are finished to compute the character table $X(I_n)$ of the symmetric inverse monoid of degree $n$.

**Theorem 10.15.** *Let $n \geq 0$. Then $X(I_n) = XT$ where*

$$X = \begin{bmatrix} X(S_0) & 0 & \cdots & 0 \\ 0 & X(S_1) & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & X(S_n) \end{bmatrix}$$

*and $T$, given by*

$$T_{\beta,\alpha} = \binom{\alpha}{\beta},$$

*is unipotent upper triangular.*

*Proof.* The fact that $T$ is unipotent upper triangular is immediate from Proposition 10.14.

For $K \subseteq [n]$ with $|K| = k$, let $p_K\colon [k] \longrightarrow K$ be the unique order-preserving bijection. Note that $p_{[k]} = 1_{[k]}$. Then since $L_{1_{[k]}}$ consists of the partial injective mappings with domain $[k]$ and two mappings are in the same right $S_k$-orbit if and only if they have the same range, it follows that the mappings $p_K$ with $|K| = k$ form a transversal for $L_{1_{[k]}}/S_k$. Therefore, if $S_\lambda$ with $\lambda \in \mathcal{P}_k$ is a simple $\mathbb{C}S_k$-module, then as a $\mathbb{C}$-vector space

$$\mathrm{Ind}_{S_k}(S_\lambda) = \mathbb{C}L_{1_{[k]}} \otimes_{\mathbb{C}S_k} S_\lambda = \bigoplus_{K\subseteq[n],|K|=k} p_K \otimes S_\lambda.$$

Hence, if $v_1,\ldots,v_{f_\lambda}$ is a basis for $S_\lambda$, then the $p_K \otimes v_j$ with $K \subseteq [n]$, $|K| = k$ and $1 \leq j \leq f_\lambda$ form a basis for $\mathrm{Ind}_{S_k}(S_\lambda)$.

Let $\alpha \in \mathcal{Q}$ with $|\alpha| = r$ and let $g \in C_\alpha \subseteq S_r$. We know that $g$ annihilates $\mathrm{Ind}_{S_k}(S_\lambda)$ unless $r \geq k$, so let us assume this. Then $g$ preserves the summand $p_K \otimes S_\lambda$ if and only if $K \subseteq [r]$ and $g(K) = K$, in which case

$$g(p_K \otimes v_j) = p_K(p_K^* g|_K p_K) \otimes v_j = p_K \otimes (p_K^* g|_K p_K)v_j.$$

We conclude from this that

$$\zeta_\lambda(\overline{C}_\alpha) = \sum_{\substack{K\subseteq[r],\\ |K|=k, g(K)=K}} \chi_\lambda(p_K^* g|_K p_K). \tag{10.7}$$

Note that the partition associated to $p_K^* g|_K p_K$ in $S_k$ is the same as the partition associated to $g|_K$ in $S_K$. Now $K \subseteq [r]$ is invariant under $g$ if and only if $K$ is a union of orbits of $g$ on $[r]$. If $g|_K$ is to have associated partition $\beta \in \mathcal{P}_k$, then (retaining the above notation) $g|_K$ must have $b_i$ orbits on $K$ of size $i$. As $g$ has $a_i$ orbits of size $i$ on $[r]$, there are $\binom{a_i}{b_i}$ ways to choose these $b_i$ orbits of size $i$ and hence $\binom{\alpha}{\beta}$ ways to choose $K \subseteq [r]$ such that $g(K) = K$ and $g|_K$ has associated partition $\beta$. As $\chi_\lambda(p_K^* g|_K p_K) = \chi_\lambda(C_\beta)$ for all such $K$, we conclude from (10.7) that

$$\zeta_\lambda(\overline{C}_\alpha) = \sum_{\beta\in\mathcal{P}_k} \binom{\alpha}{\beta}\chi_\lambda(C_\beta) = \sum_{\beta\in\mathcal{P}_k} X(S_k)_{\lambda,\beta}T_{\beta,\alpha} = \sum_{\beta\in\mathcal{Q}} X_{\lambda,\beta}T_{\beta,\alpha}$$

from which it follows that $X(I_n) = XT$. □

Solomon also computed the unipotent upper triangular matrix $U$ from Corollary 8.17 for $I_n$ in [Sol02, Proposition 3.11]. Equivalently, he computes the decomposition matrix, i.e., the matrix of the isomorphism

$$\mathrm{Res}\colon G_0(\mathbb{C}I_n) \longrightarrow \prod_{r=0}^{n} G_0(\mathbb{C}S_r).$$

An equivalent result can be found in Putcha [Put96, Theorem 2.5].

**Theorem 10.16.** *If $\alpha, \beta \in Q$, then*

$$U_{\alpha,\beta} = \begin{cases} 1, & \text{if } \alpha \subseteq \beta \text{ and } \beta \setminus \alpha \text{ is a horizontal strip} \\ 0, & \text{else.} \end{cases}$$

*Proof.* Let $r = |\alpha|$ and $m = |\beta|$. We retain the notation of Section 5.3. Note that $1_{[m]}L_{1_{[r]}}$ can be identified with the set $I_{m,r}$ of injective mappings from $[r]$ to $[m]$. Thus $1_{[m]}\mathbb{C}L_{1_{[r]}} \otimes_{\mathbb{C}S_r} S_\alpha \cong \mathbb{C}I_{m,r} \otimes_{\mathbb{C}S_r} S_\alpha$ as a $\mathbb{C}S_m$-module. Corollary 5.12 shows that this module is isomorphic to $S_\alpha \boxtimes S_{m-r}$ as a $\mathbb{C}S_m$-module. The result now follows from Pieri's rule (Theorem B.13). □

As an example, we compute the character table of $I_3$.

*Example 10.17.* For $n = 3$, we have that

$$\mathcal{Q} = \{(), (1), (1^2), (2), (1^3), (2,1), (3)\}.$$

It is a routine exercise in group representation theory to compute the character tables of $S_0, S_1, S_2, S_3$. One obtains

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Direct computation yields

$$T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 3 & 1 & 0 \\ 0 & 0 & 1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad X(I_3) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 3 & 1 & 0 \\ 0 & 0 & 1 & -1 & 3 & -1 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

In the exercises, we shall ask the reader to compute the character table of $I_4$.

## 10.5 Exercises

**10.1.** Let $P$ and $Q$ be finite posets and suppose that $[p, p'] \cong [q, q']$. Prove that $\mu_P(p, p') = \mu_Q(q, q')$.

**10.2.** Let $M$ be a finite commutative inverse monoid and $\Bbbk$ an algebraically closed field of characteristic 0. Prove that $\Bbbk M \cong \Bbbk^M$. For $\Bbbk = \mathbb{C}$ describe a complete set of orthogonal primitive idempotents of $\Bbbk M$ in terms of the characters of maximal subgroups and the Möbius function of the lattice of idempotents of $M$.

**10.3.** Verify directly that if $V$ is the natural module for $\mathbb{C}I_n$, then $V^{\wedge p} \cong \mathrm{Ind}_{S_p}(S_{(1^p)})$.

**10.4.** Compute the character table of $I_4$.

**10.5.** Let $M$ be a finite inverse monoid and let $d_1, \ldots, d_s$ be the degrees of a complete set of inequivalent irreducible representations of $M$ over $\mathbb{C}$. Prove that $|M| = d_1^2 + \cdots + d_s^2$.

**10.6.** Let $PT_n$ be the monoid of all partial mappings of $[n]$ and $\Bbbk$ a field of characteristic 0. Prove that $\Bbbk PT_n / \operatorname{rad}(\Bbbk PT_n) \cong \Bbbk I_n$.

**10.7 (I. Stein).** Prove that $\mathbb{C}PT_n \cong \mathbb{C}FE_n$ where $FE_n$ is the EI-category with objects the subsets of $[n]$ and with arrows surjective mappings. (Hint: imitate the proof of Theorem 10.3.)

# Part V

# The Rhodes Radical

# 11

# Bi-ideals and Steinberg's Theorem

In this chapter, we prove Steinberg's Theorem [Ste62] that the direct sum of the tensor powers of a faithful representation of a monoid yields a faithful representation of the monoid algebra. We also commence the study of a special family of ideals, called bi-ideals, which will be at the heart of the next chapter.

The results of this chapter should more properly be viewed as about bialgebras, but we have chosen not to work at that level of generality in order to keep things more concrete. The approach we follow here is influenced by Passman [Pas14]. The bialgebraic approach was pioneered by Rieffel [Rie67].

## 11.1 Annihilators of tensor products

Fix a field $\Bbbk$ for the chapter. If $A, B$ are $\Bbbk$-algebras, recall that their tensor product $A \otimes B$ (over $\Bbbk$) is a $\Bbbk$-algebra with respect to the product

$$(a \otimes b)(a' \otimes b') = aa' \otimes bb'$$

on basic tensors. Moreover, if $V$ is an $A$-module and $W$ is a $B$-module, then $V \otimes W$ is an $A \otimes B$-module via the action $(a \otimes b)(v \otimes w) = av \otimes bw$.

**Lemma 11.1.** *Let $A$ and $B$ be finite dimensional $\Bbbk$-algebras and let $I$ and $J$ be ideals of $A$ and $B$, respectively. Then the kernel of the natural projection $\pi\colon A \otimes B \longrightarrow A/I \otimes B/J$, given on basic tensors by $\pi(a \otimes b) = (a+I) \otimes (b+J)$, is $A \otimes J + I \otimes B$.*

*Proof.* It is clear that $\pi$ is a homomorphism and $A \otimes J + I \otimes B \subseteq \ker \pi$. Choose a basis $a_1, \ldots, a_n$ for $A$ such that $a_1, \ldots, a_i$ is a basis for $I$ and $a_{i+1}+I, \ldots, a_n+I$ is a basis for $A/I$ and, similarly, choose a basis $b_1, \ldots, b_m$ for $B$ such that $b_1, \ldots, b_j$ is a basis for $J$ and $b_{j+1}+J, \ldots, b_m+J$ is a basis for $B/J$. Suppose that

$$u = \sum_{r=1}^{n} \sum_{s=1}^{m} c_{rs}(a_r \otimes b_s)$$

belongs to $\ker \pi$, where $c_{rs} \in \Bbbk$ for all $r, s$. Then we have

$$0 = \sum_{r=i+1}^{n} \sum_{s=j+1}^{m} c_{rs}(a_r + I) \otimes (b_s + J)$$

and so $c_{rs} = 0$ if $(r, s) \in \{i+1, \ldots, n\} \times \{j+1, \ldots, m\}$ because the elements $(a_r + I) \otimes (b_s + J)$ with $i + 1 \leq r \leq n$ and $j + 1 \leq s \leq m$ form a basis for $A/I \otimes B/J$. Therefore, we have $u \in A \otimes J + I \otimes B$, as required. □

The next lemma connects the annihilator ideal of a tensor product of modules with the annihilators of the factors.

**Lemma 11.2.** *Let $A, B$ be finite dimensional $\Bbbk$-algebras, $V$ an $A$-module with annihilator ideal $I$ and $W$ a $B$-module with annihilator ideal $J$. Then the annihilator of $V \otimes W$ in $A \otimes B$ is $A \otimes J + I \otimes B$.*

*Proof.* By Lemma 11.1, the ideal $A \otimes J + I \otimes B$ of $A \otimes B$ is the kernel of the natural surjective homomorphism $A \otimes B \longrightarrow A/I \otimes B/J$. Note that $V$ is a faithful $A/I$-module, $W$ is a faithful $B/J$-module and what we are trying to prove is that $V \otimes W$ is a faithful $A/I \otimes B/J$-module. Thus without loss of generality, we may assume that $I = 0 = J$ and we aim to prove $V \otimes W$ is a faithful $A \otimes B$-module.

Suppose that $c \in A \otimes B$ annihilates $V \otimes W$ and write $c = \sum_{i=1}^{k} a_i \otimes b_i$ where we may assume without loss of generality that $b_1, \ldots, b_k$ are linearly independent over $\Bbbk$. Suppose that $\gamma \colon V \longrightarrow \Bbbk$ is a functional. Then we can define a linear mapping $\Gamma \colon V \otimes W \longrightarrow W$ by $\Gamma(v \otimes w) = \gamma(v)w$.

We have, for all $v \in V$ and $w \in W$, that

$$0 = c(v \otimes w) = \sum_{i=1}^{k} a_i v \otimes b_i w.$$

Applying $\Gamma$ then yields that $\sum_{i=1}^{k} \gamma(a_i v) b_i w = 0$. Fixing $v$, we see that $\sum_{i=1}^{k} \gamma(a_i v) b_i$ annihilates $W$. Since $W$ is a faithful $B$-module and the $b_i$ are assumed linearly independent, we must have that $\gamma(a_i v) = 0$ for all $i$. As this holds for all functionals $\gamma$, we conclude that $a_i v = 0$ for all $v \in V$ and thus $a_i$ annihilates $V$. As $V$ is faithful, we have $a_i = 0$ for all $i$. This shows that $c = 0$. □

Note that we did not assume that the modules $V$ and $W$ were finite dimensional in Lemma 11.2.

## 11.2 Bi-ideals and Steinberg's theorem

We continue to hold fixed the field $\Bbbk$ and now fix a finite monoid $M$ for the remainder of the chapter. Put $A = \Bbbk M$ and note that we have a homomorphism

$$\Delta\colon A \longrightarrow A \otimes A,$$

called the *comultiplication*, given on elements $m \in M$ by $\Delta(m) = m \otimes m$. Notice that if $V$ and $W$ are $A$-modules, then the $A$-module structure on $V \otimes W$ is induced by its $A \otimes A$-module structure and the homomorphism $\Delta$.

The trivial representation of $M$ induces a homomorphism $\varepsilon\colon \Bbbk M \longrightarrow \Bbbk$ of $\Bbbk$-algebras given by $\varepsilon(m) = 1$ for all $m \in M$, which is sometimes called the *augmentation map*. The *augmentation ideal* of $\Bbbk M$ is then

$$\mathrm{Aug}(\Bbbk M) = \ker \varepsilon = \Bbbk\{m - m' \mid m, m' \in M\}$$

where the second equality is easily checked (cf. the proof that (iii) implies (ii) of Proposition 11.3 below).

Motivated by Lemma 11.2, we make the following definition. An ideal $I$ of $A = \Bbbk M$ is a *bi-ideal*[1] if $\Delta(I) \subseteq A \otimes I + I \otimes A$ and $I \subseteq \mathrm{Aug}(\Bbbk M)$. The next proposition characterizes bi-ideals in monoid theoretic terms.

**Proposition 11.3.** *Let $I$ be an ideal of $\Bbbk M$. Then the following are equivalent.*

*(i) $I$ is a bi-ideal.*
*(ii) $I = \Bbbk\{m - m' \mid m, m' \in M, m - m' \in I\}$.*
*(iii) There is a monoid homomorphism $\varphi\colon M \longrightarrow N$ such that the kernel of the induced homomorphism $\Phi\colon \Bbbk M \longrightarrow \Bbbk N$ is $I$.*

*Proof.* We continue to let $A = \Bbbk M$. To see that (iii) implies (ii), we may assume without loss of generality that $\varphi$ is onto. For each $n \in N$, fix $\widetilde{n} \in M$ with $\varphi(\widetilde{n}) = n$. Suppose that $a \in I = \ker \Phi$. Then

$$a = \sum_{m \in M} c_m m = \sum_{n \in N} \sum_{m \in \varphi^{-1}(n)} c_m m.$$

Therefore, if we put

$$b_n = \sum_{m \in \varphi^{-1}(n)} c_m,$$

then $0 = \varphi(a) = \sum_{n \in N} b_n n = 0$. Consequently, each $b_n = 0$ and hence we have that

$$\begin{aligned} a &= \sum_{n \in N} \sum_{m \in \varphi^{-1}(n)} c_m m = \sum_{n \in N} \sum_{m \in \varphi^{-1}(n)} c_m m - \sum_{n \in N} b_n \widetilde{n} \\ &= \sum_{n \in N} \sum_{m \in \varphi^{-1}(n)} c_m (m - \widetilde{n}). \end{aligned}$$

Thus $I$ is spanned by differences of elements of $M$.

To see that (ii) implies (i), suppose that $m - m' \in I$ with $m, m' \in M$. Then we calculate

[1] These are precisely the ideals for which $A/I$ is a quotient bialgebra.

$$\begin{aligned}\varDelta(m-m') &= \varDelta(m)-\varDelta(m') = m\otimes m - m'\otimes m' \\ &= m\otimes m - m\otimes m' + m\otimes m' - m'\otimes m' \\ &= m\otimes(m-m') + (m-m')\otimes m'\end{aligned}$$

which is in $A\otimes I + I\otimes A$. Trivially, $I\subseteq \mathrm{Aug}(\Bbbk M)$. This shows that $I$ is a bi-ideal.

Suppose now that $I$ is a bi-ideal. Define a congruence $\equiv$ on $M$ by $m\equiv m'$ if and only if $m-m'\in I$. Let $N = M/{\equiv}$ and let $\varphi\colon M\longrightarrow N$ be the canonical homomorphism. Let $\varPhi\colon \Bbbk M\longrightarrow \Bbbk N$ be the extension. Then we have a commutative diagram

$$\begin{tikzcd} \Bbbk M \arrow[rr, "\pi"] \arrow[dr, "\varPhi"'] & & \Bbbk M/I \\ & \Bbbk N \arrow[ur, "\alpha"'] & \end{tikzcd}$$

of surjective homomorphisms. Our goal is to show that $I=\ker\varPhi$ or, equivalently, that $\alpha$ is an isomorphism. Because $I\subseteq \mathrm{Aug}(\Bbbk M)$, we have that $N\cap\ker\alpha=\emptyset$, i.e., $0\notin\alpha(N)$. Since $\alpha|_N\colon N\longrightarrow \Bbbk M/I$ is an injective homomorphism of monoids by construction, it suffices to show that $\alpha(N)$ is linearly independent. Assume that this is not the case. Then by considering a minimum size dependence relation in $\alpha(N)$, we can find $n,n_1,\dots,n_k\in N$ such that $\alpha(n_1),\dots,\alpha(n_k)$ are linearly independent, $n\neq n_i$ for all $i$ and

$$\alpha(n) = \sum_{i=1}^{k} c_i\alpha(n_i) \tag{11.1}$$

with the $c_i\in\Bbbk\setminus\{0\}$. Choose $m,m_1,\dots,m_k\in M$ with $\varphi(m)=n$ and $\varphi(m_i)=n_i$ for $i=1,\dots,k$ and observe that $m-\sum_{i=1}^{k}c_im_i\in I$ by (11.1).

Consider the $A$-module $V=A/I\otimes A/I$. By Lemma 11.2 and the assumption that $I$ is a bi-ideal, we have that $I$ annihilates $V$. Thus $m-\sum_{i=1}^{k}c_im_i$ annihilates $V$. Let $e=1+I\in A/I$. Then we compute

$$m(e\otimes e) = me\otimes me = \alpha(n)\otimes\sum_{i=1}^{k}c_i\alpha(n_i) = \sum_{i=1}^{k}c_i(\alpha(n)\otimes\alpha(n_i))$$

$$m(e\otimes e) = \sum_{i=1}^{k}c_im_i(e\otimes e) = \sum_{i=1}^{k}c_i(\alpha(n_i)\otimes\alpha(n_i)).$$

Subtracting these equations yields

$$0 = \sum_{i=1}^{k}c_i(\alpha(n)-\alpha(n_i))\otimes\alpha(n_i).$$

As $\alpha(n_1), \ldots, \alpha(n_k)$ are linearly independent, it follows that

$$\sum_{i=1}^{k} A/I \otimes \alpha(n_i) = \bigoplus_{i=1}^{k} A/I \otimes \alpha(n_i)$$

and so $c_i(\alpha(n) - \alpha(n_i)) = 0$ for all $i$. But then since $c_i \neq 0$, we conclude $\alpha(n) = \alpha(n_i)$, contradicting the injectivity of $\alpha|_N$. Thus $\alpha$ is an isomorphism and $I = \ker \Phi$. This completes the proof. $\square$

Proposition 11.3 sets up a bijection between bi-ideals of $\Bbbk M$ and congruences on $M$. More explicitly, if $\equiv$ is a congruence on $M$, then

$$I_{\equiv} = \Bbbk\{m - m' \mid m \equiv m'\}$$

is a bi-ideal. Conversely, if $I$ is a bi-ideal, then $\equiv_I$ defined by $m \equiv_I m'$ if and only if $m - m' \in I$ is a congruence on $M$.

Next we consider a module theoretic way of obtaining bi-ideals.

**Proposition 11.4.** *Let $\mathcal{F}$ be a family of $\Bbbk M$-modules closed under tensor product and containing the trivial module. Let $I$ be the intersection of the annihilators of the modules in $\mathcal{F}$. Then $I$ is a bi-ideal.*

*Proof.* Let $U = \bigoplus_{V \in \mathcal{F}} V$. Then $I$ is the annihilator of $U$. Also, by hypothesis $I \subseteq \mathrm{Aug}(\Bbbk M)$. Closure of $\mathcal{F}$ under tensor products guarantees that $I$ annihilates $U \otimes U = \bigoplus_{V,W \in \mathcal{F}} V \otimes W$ and hence $\Delta(I) \subseteq A \otimes I + I \otimes A$ by Lemma 11.2. Therefore, $I$ is a bi-ideal. $\square$

All the preparation has now been completed in order to prove Steinberg's theorem [Ste62].

**Theorem 11.5.** *Let $\Bbbk$ be a field and $M$ a finite monoid. Suppose that $V$ is a $\Bbbk M$-module affording a faithful representation $\varphi\colon M \longrightarrow \mathrm{End}_{\Bbbk}(V)$. Then*

$$\mathcal{T}(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}$$

*is a faithful $\Bbbk M$-module.*

*Proof.* The family $\mathcal{F}$ of modules $V^{\otimes n}$ with $n \geq 0$ is closed under tensor product and contains the trivial module. Thus the intersection $I$ of the annihilator ideals of the tensor powers of $V$ is a bi-ideal by Proposition 11.4. But $I$ is precisely the annihilator ideal of $\mathcal{T}(V)$. Trivially, $I$ is contained in the annihilator of $V$. By Proposition 11.3 and faithfulness of $\varphi$, we conclude that

$$I = \Bbbk\{m - n \in I \mid m, n \in M\} \subseteq \Bbbk\{m - n \mid \varphi(m) = \varphi(n)\} = 0,$$

as required. $\square$

We can, in fact, replace the infinite direct sum with a finite one.

**Corollary 11.6.** *Let $M$ be a finite monoid and $\Bbbk$ a field. Let $V$ be a $\Bbbk M$-module affording a faithful representation $\varphi\colon M \longrightarrow \mathrm{End}_{\Bbbk}(V)$. Then there exists $r \geq 0$ such that $\bigoplus_{n=0}^{r} V^{\otimes n}$ is a faithful $\Bbbk M$-module.*

*Proof.* We have that $\mathcal{T}(V)$ is faithful $\Bbbk M$-module by Theorem 11.5. Let $I_k$ be the annihilator of $\bigoplus_{n=0}^{k} V^{\otimes n}$. Then $I_0 \supseteq I_1 \supseteq \cdots$ and $\bigcap_{k=0}^{\infty} I_k = 0$. As $\Bbbk M$ is finite dimensional, there exists $r$ such that $I_r = 0$. This completes the proof. $\square$

In fact, it is not difficult to refine the proof of Corollary 11.6 to show that one can take $r = |M|$. As a further corollary, we obtain the following result, which should be contrasted with Theorem 8.26.

**Corollary 11.7.** *Let $M$ be a finite monoid and $\Bbbk$ a field. Let $V$ be a $\Bbbk M$-module affording a faithful representation $\varphi\colon M \longrightarrow \mathrm{End}_{\Bbbk}(V)$. Then every simple $\Bbbk M$-module is a composition factor of a tensor power $V^{\otimes n}$ for some $n \geq 0$.*

*Proof.* Let $S$ be a simple $A$-module and write $S = Ae/\operatorname{rad}(A)e$ where $e$ is a primitive idempotent of $A$. By Corollary 11.6, there exists $r \geq 0$ such that $\bigoplus_{n=0}^{r} V^{\otimes n}$ is faithful. Therefore, $0 \neq \bigoplus_{n=0}^{r} eV^{\otimes n}$ and hence $eV^{\otimes n} \neq 0$ for some $n \geq 0$. But this means that $S$ is a composition factor of $V^{\otimes n}$ by Proposition A.24. $\square$

We end this chapter with an application of Steinberg's theorem that will be explored further in the next chapter. Namely, we characterize those monoids admitting a faithful representation over $\Bbbk$ by upper triangular matrices as precisely those monoids whose irreducible representations over $\Bbbk$ are all one-dimensional.

**Proposition 11.8.** *Let $M$ be a finite monoid and $\Bbbk$ a field. The $M$ admits a faithful representation by upper triangular matrices over $\Bbbk$ if and only if each simple $\Bbbk M$-module is one-dimensional.*

*Proof.* If each simple $\Bbbk M$-module is one-dimensional, then the regular module $\Bbbk M$ has only one-dimensional composition factors and hence affords a representation by upper triangular matrices according to Lemma 6.8. Since the regular module affords a faithful representation, this completes one direction of the proof.

For the converse, suppose that $M$ admits a faithful representation by upper triangular matrices over $\Bbbk$, afforded by a module $V$. Then by Corollary 11.7 each simple $\Bbbk M$-module is a composition factor of a tensor power $V^{\otimes n}$ with $n \geq 0$. But these tensor powers are triangularizable modules by Corollary 6.9, and hence have only one-dimensional composition factors by Lemma 6.8. $\square$

It follows from Proposition 11.8 and Lemma 6.8 that if $M$ has a faithful representation over $\Bbbk$ by upper triangular matrices, then each representation of $M$ over $\Bbbk$ is equivalent to one by upper triangular matrices. We say that a monoid $M$ is *triangularizable* over $\Bbbk$ if the equivalent conditions of Proposition 11.8 hold.

**Corollary 11.9.** *If $M$ is a finite $\mathscr{R}$-trivial monoid and $\Bbbk$ is a field, then $M$ admits a faithful representation over $\Bbbk$ by upper triangular matrices. Moreover, every representation of $M$ is equivalent to one by upper triangular matrices.*

*Proof.* This follows from Proposition 11.8 because every irreducible representation of $M$ over $\Bbbk$ is one-dimensional by Corollary 5.7. □

## 11.3 Exercises

**11.1.** Let $P, Q$ be finite posets. Prove that $I(P \times Q, \Bbbk) \cong I(P, \Bbbk) \otimes I(Q, \Bbbk)$.

**11.2.** Let $M$ be a finite monoid and $\varphi\colon M \longrightarrow N$ a surjective monoid homomorphism. Let $\Phi\colon \Bbbk M \longrightarrow \Bbbk N$ be the induced homomorphism where $\Bbbk$ is a field. Fix, for each $n \in N$, an element $\overline{n} \in M$ with $\varphi(\overline{n}) = n$. Prove that the set of non-zero elements of the form $m - \overline{\varphi(m)}$ is a basis for $\ker \Phi$.

**11.3.** Let $M$ be a finite monoid and $\Bbbk$ a field. Let $I$ be an ideal of $\Bbbk M$ and put $J = \Bbbk\{m - m' \in I \mid m, m' \in M\}$.

(a) Prove that $J$ is a bi-ideal.
(b) Prove that $J$ is the largest bi-ideal contained in $I$.
(c) Prove that $J$ is the intersection of the annihilators of the tensor powers $V^{\otimes n}$ with $n \geq 0$.

**11.4.** Prove Theorem 11.5 is valid for infinite monoids.

**11.5.** Suppose that $\rho\colon M \longrightarrow \mathrm{End}_{\Bbbk}(V)$ is a faithful representation of a finite monoid $M$ on a finite dimensional vector space $V$. Prove that $\bigoplus_{i=0}^{n} V^{\otimes i}$ is faithful where $|M| = n$.

**11.6.** Let $M$ be a finite monoid and $\Bbbk$ a field of characteristic 0. Prove that there is a faithful representation $\rho\colon M \longrightarrow M_n(\Bbbk)$ where $n = |M| - 1$.

**11.7 (G. Bergman).** Let $M$ be a monoid and $\Bbbk$ a field. Suppose that $L$ is a left ideal of $\Bbbk M$ with simple socle $S$ containing a non-zero element of the form $m - m'$ with $m, m' \in M$. Prove that if $V$ is a $\Bbbk M$-module affording a faithful representation of $M$, then $V$ contains a submodule isomorphic to $L$.

**11.8.** Let $V$ be a $\mathbb{C}T_n$-module affording a faithful representation of $T_n$. Prove that $V$ contains a submodule isomorphic to the natural module $\mathbb{C}^n$. (Hint: Apply Exercise 11.7 to the left ideal spanned by the constant mappings.)

# 12

# The Rhodes Radical

In this chapter we provide a correspondence between nilpotent bi-ideals and a certain class of congruences on a finite monoid. We characterize the largest nilpotent bi-ideal, which is called the *Rhodes radical* because it was first characterized by Rhodes [Rho69b] in the case of an algebraically closed field of characteristic 0. For simplicity, we only give complete details in characteristic 0. As an application, we characterize those monoids with a faithful representation by upper triangular matrices over an algebraically closed field $\Bbbk$ (the general case will be left to the reader in the exercises). These are precisely the monoids with a basic algebra over $\Bbbk$ and so it also characterizes these monoids. The chapter mostly follows the work of Almeida, Margolis, the author and Volkov [AMSV09].

## 12.1 The Rhodes radical and nilpotent bi-ideals

Let $\Bbbk$ be a field and $M$ a finite monoid. We saw in the Chapter 11 that bi-ideals in $\Bbbk M$ are precisely kernels of homomorphisms $\Phi\colon \Bbbk M \longrightarrow \Bbbk N$ induced by monoid homomorphisms $\varphi\colon M \longrightarrow N$. It follows from Proposition 11.3 that each ideal $I$ of $\Bbbk M$ contains a largest bi-ideal, namely the $\Bbbk$-span of all differences $m - n$ with $m, n \in M$ and $m - n \in I$; see Exercise 11.3. In particular, there is a largest nilpotent bi-ideal, namely

$$\mathrm{rad}_{\Bbbk}(M) = \Bbbk\{m - n \mid m, n \in M,\ m - n \in \mathrm{rad}(\Bbbk M)\}.$$

We term $\mathrm{rad}_{\Bbbk}(M)$ the *Rhodes radical* of $M$ with respect to $\Bbbk$. The corresponding congruence $\equiv_{\Bbbk}$ on $M$ is defined by

$$m \equiv_{\Bbbk} n \iff m - n \in \mathrm{rad}(\Bbbk M)$$

and is called the *Rhodes radical congruence* on $M$. Equivalently, $m \equiv_{\Bbbk} n$ if and only if $\rho(m) = \rho(n)$ for every irreducible representation $\rho$ of $M$ over $\Bbbk$.

In particular, every semisimple $\Bbbk M$-module is, in fact, a semisimple $\Bbbk[M/{\equiv_\Bbbk}]$-module.

Let us first investigate when the Rhodes radical coincides with the Jacobson radical.

**Proposition 12.1.** *Let $M$ be a finite monoid and $\Bbbk$ a field. Then the following are equivalent.*

*(i)* $\mathrm{rad}_\Bbbk(M) = \mathrm{rad}(\Bbbk M)$.
*(ii)* $\Bbbk M/\mathrm{rad}(\Bbbk M)$ *is the monoid algebra of* $M/{\equiv_\Bbbk}$.
*(iii)* $\Bbbk[M/{\equiv_\Bbbk}]$ *is semisimple.*
*(iv) The semisimple* $\Bbbk M$*-modules are closed under tensor product.*

*Proof.* Let $N = M/{\equiv_\Bbbk}$. The implication (i) implies (ii) is clear from Proposition 11.3. Trivially, (ii) implies (iii). Assume that (iii) holds. As every semisimple $\Bbbk M$-module is, in fact, a $\Bbbk N$-module and the $\Bbbk N$-modules are closed under tensor product, (iv) follows because each $\Bbbk N$-module is semisimple by assumption. Finally, to see that (iv) implies (i), suppose that the family $\mathcal{F}$ of semisimple $\Bbbk M$-modules is closed under tensor product. Then since the trivial module is simple and $\mathrm{rad}(\Bbbk M)$ is the intersection of the annihilators of the elements of $\mathcal{F}$, it follows from Proposition 11.4 that $\mathrm{rad}(\Bbbk M)$ is a bi-ideal and hence coincides with $\mathrm{rad}_\Bbbk(M)$. $\square$

Since every simple module for a triangularizable monoid is one-dimensional by Proposition 11.8 and one-dimensional modules are closed under tensor products, it follows that Proposition 12.1 applies to triangularizable monoids.

**Corollary 12.2.** *If $M$ is a triangularizable monoid over $\Bbbk$, then* $\mathrm{rad}(\Bbbk M) = \mathrm{rad}_\Bbbk(M)$ *and hence $M/{\equiv_\Bbbk}$ has a semisimple algebra over $\Bbbk$.*

Recall from Section 8.4 that a monoid homomorphism $\varphi\colon M \longrightarrow N$ is an **LI**-morphism if $\varphi$ separates $e$ from $eMe \setminus \{e\}$ for all $e \in E(M)$, that is, $\varphi^{-1}(\varphi(e)) \cap eMe = \{e\}$ for all $e \in E(M)$. Let us say that a congruence $\equiv$ on $M$ is an **LI**-*congruence* if the canonical projection $\pi\colon M \longrightarrow M/{\equiv}$ is an **LI**-morphism. Our aim is to show that if $\Bbbk$ is a field of characteristic 0, then nilpotent bi-ideals of $\Bbbk M$ are in bijection with **LI**-congruences. In particular, $\equiv_\Bbbk$ will be the largest **LI**-congruence on $M$. We first need some algebraic properties of **LI**-morphisms.

**Lemma 12.3.** *Let $\varphi\colon M \longrightarrow N$ be an* **LI***-morphism and suppose $e, m, f \in M$ with $\varphi(e) = \varphi(m) = \varphi(f)$ and $e, f \in E(M)$. Then $emf = ef$.*

*Proof.* First note that since $fef \in fMf$ and $\varphi(fef) = \varphi(f)$, we have $fef = f$. Similarly, since $emfe \in eMe$ and $\varphi(emfe) = \varphi(e)$, we have $emfe = e$. Thus $emf = emfef = ef$. $\square$

Now we can prove the main result of this section.

**Theorem 12.4.** *Let $\Bbbk$ be a field of characteristic* 0 *and $\varphi\colon M \longrightarrow N$ a homomorphism of finite monoids. Let $\Phi\colon \Bbbk M \longrightarrow \Bbbk N$ be the extension of $\varphi$. Then the following are equivalent.*

*(i) $\varphi$ is an* **LI***-morphism.*
*(ii) $I_\varphi = \ker \Phi$ is nilpotent.*

*Moreover, the implication (i) implies (ii) remains valid over any field $\Bbbk$.*

*Proof.* First suppose that $I_\varphi$ is nilpotent (and that the characteristic of $\Bbbk$ is 0). Let $e \in E(M)$ and $m \in M$ such that $\varphi(e) = \varphi(eme)$. Then $eme - e \in I_\varphi$ and hence $(eme - e)^k = 0$ for some $k > 0$. Thus the minimal polynomial $f(x)$ of $eme$, as an element of the unital $\Bbbk$-algebra $\Bbbk[eMe]$ with identity $e$, divides the polynomial $p(x) = (x-1)^k$. But if $eme$ has index $c$ and period $d$, then $(eme)^c = (eme)^{c+d}$ and so if $q(x) = x^c(x^d - 1)$, then $q(eme) = 0$. Therefore, $f(x)$ divides $(x-1)^k$ and $x^c(x^d-1)$, and consequently $f(x) = x - 1$, as the polynomial $x^d - 1$ has no repeated roots over a field $\Bbbk$ of characteristic 0. Thus $eme = e$ (as $f(eme) = 0$), completing the proof that $\varphi$ is an **LI**-morphism.

Suppose now that $\varphi$ is an **LI**-morphism. We drop the assumption that $\Bbbk$ is of characteristic 0. By a theorem of Wedderburn (Theorem A.11), to prove that $I_\varphi$ is nilpotent, it suffices to show that each element of its spanning set $\{m_1 - m_2 \mid \varphi(m_1) = \varphi(m_2)\}$ is nilpotent. We shall need the following subresult. Let $e \in E(N)$ and let $S = \varphi^{-1}(e)$. Note that $S$ is a subsemigroup of $M$. Suppose that $e_1, e_2 \in E(S)$ and $a \in \mathrm{Aug}(\Bbbk M) \cap \Bbbk S$. We claim that $e_1 a e_2 = 0$. Indeed, we can write $a = \sum_{s\in S} c_s s$ with $\sum_{s \in S} c_s = 0$. Therefore, by Lemma 12.3, we compute

$$e_1 a e_2 = \sum_{s\in S} c_s e_1 s e_2 = \sum_{s \in S} c_s e_1 e_2 = 0.$$

Now let $\varphi(m_1) = n = \varphi(m_2)$ and put $e = n^\omega$. Fix $k > 0$ such that $n^k = e$ and put $S = \varphi^{-1}(e)$. Let $r = |S|$. We claim that $(m_1 - m_2)^{(2r+1)k} = 0$. First note that $(m_1 - m_2)^k \in \Bbbk S$ and hence $(m_1 - m_2)^{kr} \in \Bbbk S^r$. Applying Lemma 1.5, it follows that

$$(m_1 - m_2)^{kr} = \sum_{i=1}^{d} c_i a_i e_i b_i$$

with $a_i, b_i \in S$ and $e_i \in E(S)$. As $(m_1 - m_2)^k \in \mathrm{Aug}(\Bbbk M) \cap \Bbbk S$, it follows from the claim in the previous paragraph that $e_i b_i (m_1 - m_2)^k a_j e_j = 0$ for all $i, j$ and hence

$$\begin{aligned}(m_1 - m_2)^{(2r+1)k} &= (m_1 - m_2)^{kr}(m_1 - m_2)^k (m_1 - m_2)^{kr} \\ &= \sum_{i,j=1}^{d} c_i c_j a_i e_i b_i (m_1 - m_2)^k a_j e_j b_j \\ &= 0.\end{aligned}$$

This completes the proof. $\square$

**Corollary 12.5.** *Let $M$ be a finite monoid and $\Bbbk$ a field. Then there is an inclusion-preserving bijection between* **LI**-*congruences and nilpotent bi-ideals. In particular, $\equiv_{\Bbbk}$ is the largest* **LI**-*congruence on $M$.*

The reader is referred to [RS09, Chapter 6] for an explicit description of the largest **LI**-congruence on a finite monoid. Let us give an application to the case of $\mathscr{R}$-trivial monoids. Recall from Corollary 2.6 that if $M$ is an $\mathscr{R}$-trivial monoid, then the set

$$\Lambda(M) = \{MeM \mid e \in E(M)\}$$

is a lattice ordered by inclusion, which we view as a monoid via its meet.

**Corollary 12.6.** *Let $M$ be a finite $\mathscr{R}$-trivial monoid and $\Bbbk$ a field. Then the surjective homomorphism $\overline{\sigma}\colon \Bbbk M \longrightarrow \Bbbk\Lambda(M)$ induced by the natural homomorphism $\sigma\colon M \longrightarrow \Lambda(M)$ given by $\sigma(m) = Mm^{\omega}M$ is the semisimple quotient, i.e.,* $\ker\overline{\sigma} = \mathrm{rad}(\Bbbk M)$.

*Proof.* By Corollary 2.6 the map $\sigma$ is a surjective homomorphism. If $e \in E(M)$, then $M(eme)^{\omega}M = MeM$ implies $eme \in G_e = \{e\}$ by Corollary 1.15. It follows that $\sigma$ is an **LI**-morphism and hence $\ker\overline{\sigma}$ is nilpotent by Theorem 12.4. Thus it suffices to observe that $\Bbbk\Lambda(M)$ is semisimple by Corollary 10.5 and apply Corollary A.10. $\square$

We end this section by stating without proof the analogue of Theorem 12.4 in positive characteristic. Details can be found in [AMSV09] and the exercises. Let $p$ be a prime. Then a homomorphism $\varphi\colon M \longrightarrow N$ of finite monoids is called an $\mathbf{LG}_p$-morphism if, for each $e \in E(M)$, one has that $\varphi^{-1}(\varphi(e)) \cap eMe$ is a $p$-group. Notice that each **LI**-morphism is an $\mathbf{LG}_p$-morphism. By an $\mathbf{LG}_p$-*congruence*, we mean a congruence $\equiv$ on $M$ such that the natural projection $\pi\colon M \longrightarrow M/{\equiv}$ is an $\mathbf{LG}_p$-morphism.

**Theorem 12.7.** *Let $\Bbbk$ be a field of characteristic $p > 0$ and $\varphi\colon M \longrightarrow N$ a homomorphism of finite monoids. Let $\Phi\colon \Bbbk M \longrightarrow \Bbbk N$ be the extension of $\varphi$. Then the following are equivalent.*

*(i) $\varphi$ is an $\mathbf{LG}_p$-morphism.*
*(ii) $I_{\varphi} = \ker\Phi$ is nilpotent.*

**Corollary 12.8.** *Let $M$ be a finite monoid and let $\Bbbk$ be a field of characteristic $p > 0$. Then there is an inclusion-preserving bijection between $\mathbf{LG}_p$-congruences and nilpotent bi-ideals. In particular, $\equiv_{\Bbbk}$ is the largest $\mathbf{LG}_p$-congruence on $M$.*

The largest $\mathbf{LG}_p$-congruence on a finite monoid was determined by Rhodes and Tilson [Rho69a, Til69] (see also [AMSV09]).

## 12.2 Triangularizable monoids and basic algebras

Let $\Bbbk$ be an algebraically closed field. A finite dimensional $\Bbbk$-algebra $A$ is said to be *basic* if $A/\operatorname{rad}(A) \cong \Bbbk^n$ for some $n > 0$. Equivalently, by Wedderburn's theorem, $A$ is basic if every simple $A$-module is one-dimensional. Basic algebras play an important role in representation theory because each finite dimensional $\Bbbk$-algebra is Morita equivalent to a unique (up to isomorphism) basic algebra, cf. [DK94, ARS97, Ben98, ASS06]. Our aim is to characterize monoids with basic algebras. We provide all the details in characteristic 0; for the proofs in characteristic $p > 0$ the reader is referred to [AMSV09] or the exercises.

First we reformulate Proposition 11.8 in this language.

**Proposition 12.9.** *Let $\Bbbk$ be an algebraically closed field and $M$ a finite monoid. Then the following are equivalent.*

*(i)* $\Bbbk M$ *is basic.*
*(ii)* $M$ *admits a faithful representation by upper triangular matrices over* $\Bbbk$.
*(iii) Each representation of $M$ over $\Bbbk$ is equivalent to one by upper triangular matrices.*

If $\Bbbk M$ is basic, then the simple (and hence semisimple) $\Bbbk M$-modules are closed under tensor product and so $\operatorname{rad}(\Bbbk M) = \operatorname{rad}_{\Bbbk}(M)$ by Proposition 12.1. It follows, again by Proposition 12.1, that $M/{\equiv_{\Bbbk}}$ has a semisimple algeba over $\Bbbk$. This leads to our first characterization of triangularizable monoids over $\Bbbk$.

**Proposition 12.10.** *Let $\Bbbk$ be an algebraically closed field of characteristic* 0 *and $M$ a finite monoid. Then the following are equivalent.*

*(i)* $\Bbbk M$ *is basic.*
*(ii)* $M$ *admits a faithful representation by upper triangular matrices over* $\Bbbk$.
*(iii) There is an* **LI**-*morphism* $\varphi\colon M \longrightarrow N$ *with $N$ a commutative inverse monoid.*
*(iv)* $M/{\equiv_{\Bbbk}}$ *is a commutative inverse monoid.*

*Proof.* If (iv) holds, then since $\equiv_{\Bbbk}$ is an **LI**-congruence on $M$, it follows that (iii) holds. For the converse, assume that (iii) holds. Then $\varphi(M)$ is a commutative inverse monoid by Corollary 3.12 and hence, without loss of generality, we may assume that $\varphi$ is surjective. But then since $\equiv_{\Bbbk}$ is the largest **LI**-congruence on $M$ by Corollary 12.5, whence $\ker\varphi \subseteq {\equiv_{\Bbbk}}$, and quotients of commutative inverse monoids are again commutative inverse monoids by Corollary 3.5, we may conclude that (iv) holds. This yields the equivalence of (iii) and (iv).

We already have the equivalence of (i) and (ii). Suppose that (iv) holds. We shall prove (i). Let us put $N = M/{\equiv_{\Bbbk}}$. Then $\Bbbk N \cong \prod_{e\in E(N)} \Bbbk G_e$ by Corollary 10.6 and hence $\Bbbk N$ is semisimple. Therefore, we have $\Bbbk M/\operatorname{rad}(\Bbbk M) \cong \Bbbk N$

by Proposition 12.1. But if $G$ is a finite abelian group, then $\Bbbk G \cong \Bbbk^{|G|}$ by Proposition B.2 and hence $\Bbbk N \cong \Bbbk^{|N|}$. We conclude that $\Bbbk M$ is basic.

Now suppose that (i) holds. Then, as discussed above, the semisimple $\Bbbk M$-modules are closed under tensor product and hence $\Bbbk M/\operatorname{rad}(\Bbbk M) \cong \Bbbk N$, where $N = M/{\equiv_{\Bbbk}}$, by Proposition 12.1. In particular, $\Bbbk N$ is semisimple and commutative because $\Bbbk M$ is basic. We conclude that $N$ is commutative and regular, where the latter conclusion uses Theorem 5.14. But a commutative regular monoid must be an inverse monoid by Theorem 3.2. Thus (iv) holds, completing the proof. □

We also state here the corresponding result in characteristic $p$. A proof can be found in [AMSV09].

**Proposition 12.11.** *Let $\Bbbk$ be an algebraically closed field of characteristic $p > 0$ and $M$ a finite monoid. Then the following are equivalent.*

*(i)* *$\Bbbk M$ is basic.*
*(ii)* *$M$ admits a faithful representation by upper triangular matrices over $\Bbbk$.*
*(iii)* *There is a surjective $\mathbf{LG}_p$-morphism $\varphi\colon M \longrightarrow N$ with $N$ a commutative inverse monoid whose maximal subgroups have order prime to $p$.*
*(iv)* *$M/{\equiv_{\Bbbk}}$ is a commutative inverse monoid whose maximal subgroups have order prime to $p$.*

We now wish to describe in purely algebraic terms when a monoid admits a surjective **LI**-morphism to a commutative inverse monoid. We say that a finite monoid $M$ is *rectangular*[1] if, for all $e, f \in E(M)$ with $MeM = MfM$, one has $efe = e$.

*Example 12.12.* Let us show that bands are rectangular monoids. Let $M$ be a band (i.e., $M = E(M)$) If $MeM = MfM$, then there exists $a \in M$ such that $Ma = Me$ and $aM = fM$ by Proposition 1.20. Then $ea = e$ and $fa = a$, whence $efa = ea = e$. Therefore, we have $MeM = MefM$. But $efef = ef$ and so $MeM = MefM = MefeM$. We conclude, using Corollary 1.15 , that $efe \in J_e \cap MeM = G_e$ and so $efe = e$ as $G_e = \{e\}$.

Let us explore the structure of rectangular monoids.

**Proposition 12.13.** *Let $M$ be a rectangular monoid.*

*(i)* *If $e, f \in E(M)$ with $MeM = MfM$, then $ef, fe$ are idempotents belonging to $J_e = J_f$.*
*(ii)* *Each regular $\mathscr{J}$-class of $M$ is a subsemigroup.*
*(iii)* *If $e \in E(M)$ and $e \in MmM \cap MnM$, then $eme, ene, emne \in G_e$ and $emene = emne$.*

[1] The class of all rectangular monoids is often denoted by **DO** in the literature.

*Proof.* By definition of a rectangular monoid, $efe = e$ and so $efef = ef$ and $fefe = fe$. Clearly $ef, fe \in J_e$. This proves (i). To prove (ii), let $J$ be a regular $\mathscr{J}$-class and $m, n \in J$. Then there exist $u, v \in M$ such that $um = e$ and $nv = f$ are idempotents by Proposition 1.20. Note that $umnv = ef \in J$ by (i) and hence $mn \in J$. This establishes (ii).

Turning to (iii), we have that $eme, ene, emne \in J_e$ by (ii) and Proposition 2.2. Hence these elements belong to $G_e$ by Corollary 1.15. Since $em, ne \in J_e$, we can find by Proposition 1.20 idempotents $f, f' \in E(J_e)$ such that $Mf = Mem$ and $f'M = neM$. Notice that this implies $emf = em$ and $f'ne = ne$. Then $fef = f$ and $f'ef' = f'$ by definition of a rectangular monoid. Also, as $ff' \in E(J_e)$ by (i), we have that $eff'e = e$. Consequently,

$$emne = emff'ne = emfeff'ef'ne = emeff'ene = emene,$$

as required. □

If $G$ is a group, we put $G^0 = G \cup \{0\} \subseteq \mathbb{C}G$. Then $G^0$ is an inverse monoid that is commutative if $G$ is commutative.

**Proposition 12.14.** *Let $M$ be a rectangular monoid. Let $e_1, \ldots, e_s$ be a complete set of idempotent representatives of the regular $\mathscr{J}$-classes of $M$. Then the mapping*

$$\varphi\colon M \longrightarrow \prod_{i=1}^{s} G_{e_i}^0$$

*defined by*

$$\varphi(m)_i = \begin{cases} e_i m e_i, & \text{if } e_i \in MmM \\ 0, & \text{if } m \in I(e_i) \end{cases}$$

*is an* **LI**-*morphism.*

*Proof.* By Proposition 12.13, if $e_i \in MmM$, then $e_i m e_i \in G_{e_i}$ and so $\varphi$ makes sense as a mapping. Since $I(e_i)$ is an ideal, the only thing required to check that $\varphi$ is a homomorphism is that if $e_i \in MmM \cap MnM$, then $e_i m e_i n e_i = e_i m n e_i \in G_{e_i}$. But this is the content of Proposition 12.13(iii). Let us proceed to verify that $\varphi$ is an **LI**-morphism.

Suppose that $e \in E(M)$. Then $MeM = Me_iM$ for some $i$. Let $m \in eMe$ and assume that $\varphi(m) = \varphi(e)$. Then $e_i m e_i = \varphi(m)_i = \varphi(e)_i = e_i e e_i = e_i$ (using that $M$ is a rectangular monoid). As $M$ is a rectangular monoid and $eme = m$, we compute that

$$e = ee_ie = ee_ime_ie = ee_iemee_ie = eme = m.$$

This concludes the proof that $\varphi$ is an **LI**-morphism. □

We are now prepared to state and prove the main theorem of this chapter.

**Theorem 12.15.** *Let $\Bbbk$ be an algebraically closed field of characteristic* 0 *and $M$ a finite monoid. Then the following are equivalent.*

*(i)* $\Bbbk M$ *is basic.*
*(ii)* $M$ *admits a faithful representation by upper triangular matrices over* $\Bbbk$.
*(iii) There is an* **LI**-*morphism* $\varphi\colon M \longrightarrow N$ *with* $N$ *a commutative inverse monoid.*
*(iv)* $M/{\equiv_{\Bbbk}}$ *is a commutative inverse monoid.*
*(v)* $M$ *is a rectangular monoid with abelian maximal subgroups.*

*Proof.* The equivalence of (i)-(iv) is the content of Proposition 12.10. We check the equivalence of (iii) and (v). If $M$ is rectangular with abelian maximal subgroups, then Proposition 12.14 furnishes an **LI**-morphism to a commutative inverse monoid. Suppose that $\varphi\colon M \longrightarrow N$ is an **LI**-morphism with $N$ a commutative inverse monoid. Let $e \in E(M)$. Then $\ker \varphi|_{G_e} = \{e\}$ by definition of an **LI**-morphism. Therefore, $\varphi|_{G_e}$ is injective and we may deduce that $G_e$ is abelian. Let $e, f \in E(M)$ with $MeM = MfM$. Then $N\varphi(e)N = N\varphi(f)N$. But the $\mathscr{J}$-classes of a commutative inverse monoid are maximal subgroups, by Proposition 3.11, and hence contain a unique idempotent. Therefore, we must have $\varphi(e) = \varphi(f)$. But then $\varphi(efe) = \varphi(e)$ and so, by definition of a **LI**-morphism, we conclude that $efe = e$. This shows that $M$ is a rectangular monoid and completes the proof of the theorem. □

It is also straightforward to describe the irreducible representations of a rectangular monoid, and hence of a triangularizable monoid.

**Proposition 12.16.** *Let $M$ be a rectangular monoid and $\Bbbk$ a field. Suppose that $e \in E(M)$ and let $V \in \mathrm{Irr}_{\Bbbk}(G_e)$. Suppose that $V$ affords the representation $\rho\colon G_e \longrightarrow M_n(\Bbbk)$. Then the representation $\rho^\sharp\colon M \longrightarrow M_n(\Bbbk)$ given by*

$$\rho^\sharp(m) = \begin{cases} \rho(eme), & \text{if } e \in MmM \\ 0, & \text{if } m \in I(e) \end{cases}$$

*is afforded by the simple module $V^\sharp \in \mathrm{Irr}_{\Bbbk}(M)$ corresponding to $V$.*

*Proof.* One can verify that $\rho^\sharp$ is a representation of $M$ as a straightforward consequence of Proposition 12.13. It is obvious that $\rho^\sharp(e) = I$ and $\rho^\sharp|_{G_e} = \rho$. Hence $\rho^\sharp$ is irreducible by irreducibility of $\rho$. Also, by construction, $\rho^\sharp(I_e) = 0$. We conclude that $e$ is an apex for $\rho^\sharp$. It follows that the module corresponding to $\rho^\sharp$ is $V^\sharp$. □

## 12.3 Exercises

**12.1.** A finite semigroup $S$ is said to be *locally trivial* if $eSe = \{e\}$ for all $e \in E(S)$. Prove that a homomorphism $\varphi\colon M \longrightarrow N$ of finite monoids is an **LI**-morphism if and only if $\varphi^{-1}(f)$ is a locally trivial semigroup for each $f \in E(N)$.

**12.2.** Let $\Bbbk$ be a field. Prove that any finite submonoid of the multiplicative monoid of $\Bbbk^n$ is a commutative inverse monoid.

**12.3.** Prove that a finite monoid $M$ is triangularizable over an algebraically closed field of characteristic 0 if and only if:

(i) the regular $\mathscr{J}$-classes of $M$ are subsemigroups;
(ii) if $J$ is a regular $\mathscr{J}$-class, then $\langle E(J)\rangle = E(J)$;
(iii) each maximal subgroup of $M$ is abelian.

**12.4.** Let $M$ be a finite monoid. Define $m \equiv m'$ if, for all regular $\mathscr{J}$-classes $J$ of $M$ and all $x, y \in J$, one has

$$xmy \in J \iff xm'y \in J$$

and moreover, if this occurs, then $xmy = xm'y$. Prove that $\equiv$ is the largest **LI**-congruence on $M$.

**12.5.** Prove that a finite monoid $M$ has a faithful completely reducible representation over $\mathbb{C}$ if and only if the largest **LI**-congruence on $M$ is the equality relation.

**12.6 (J. Rhodes).** Prove that a finite monoid $M$ has a faithful irreducible representation over $\mathbb{C}$ if and only if it has an ideal $I$ such that $M$ acts faithfully on both the left and right of $I$ and there is an idempotent $e \in I$ with $I = IeI$, $|I(e)| \leq 1$ and such that the maximal subgroup $G_e$ has a faithful irreducible representation over $\mathbb{C}$. (Hint: you may find it helpful to use Exercise 12.4 and Exercise 5.8.)

**12.7.** Prove Theorem 12.7.

**12.8.** Let $M$ be a finite monoid and $\Bbbk$ a field of characteristic 0. Prove that $M$ admits a faithful representation over $\Bbbk$ by upper triangular matrices if and only if $M$ is a rectangular monoid with abelian maximal subgroups and $x^n - 1$ splits over $\Bbbk$ into linear factors where $n$ is the least common multiple of the orders of the elements of the maximal subgroups of $M$.

**12.9.** Let $M$ be a finite monoid and $\Bbbk$ a field of characteristic $p > 0$. Prove that $M$ is triangularizable over $\Bbbk$ if and only if:

(i) every regular $\mathscr{J}$-class is a subsemigroup;
(ii) the maximal subgroups of $\langle E(M)\rangle$ are $p$-groups;
(iii) each maximal subgroup $G$ of $M$ has normal $p$-Sylow subgroup $N$ such that $G/N$ is abelian and if $n$ is the least common multiple of the orders of elements of $G/N$, then $x^n - 1$ splits over $\Bbbk$ into distinct linear factors.

**12.10.** Prove that a finite band is triangularizable over any field.

**12.11.** Let $\Bbbk$ be an algebraically closed field of characteristic 0 and $A$ a finite dimensional $\Bbbk$-algebra. Let $P_1, \ldots, P_s$ be a complete set of representatives of the isomorphism classes of projective indecomposable $A$-modules and let $V = P_1 \oplus \cdots \oplus P_s$.

(a) Prove that $B = \mathrm{End}_A(V)^{op}$ is a basic finite dimensional $\Bbbk$-algebra.
(b) Prove that $B$ is Morita equivalent to $A$.
(c) Prove that every basic algebra Morita equivalent to $A$ is isomorphic to $B$.

# Part VI

# Applications

# 13

# Zeta Functions of Languages and Dynamical Systems

In this section, we apply the character theory of finite monoids to provide a proof of a theorem of Berstel and Reutenauer on the rationality of zeta functions of cyclic regular languages [BR90]. This generalizes the rationality of zeta functions of sofic shifts [LM95], an important result in symbolic dynamics. Background on free monoids, formal languages and automata can be found in [Eil74, Eil76, Lot97, Lot02, BPR10, BR11].

## 13.1 Zeta functions

If $A$ is a finite set, then $A^*$ will denote the *free monoid* on the set $A$. Elements of $A^*$ are (possibly empty) words in the alphabet $A$ and the product is concatenation. We write $|w|$ for the length of a word $w$ and we write 1 for the empty word. A subset $L \subseteq A^*$ is called a *(formal) language* (since it consists of a collection of words).

The *zeta function* $\zeta_L$ of a language $L \subseteq A^*$ is the power series defined by

$$\zeta_L(t) = \exp\left(\sum_{n=1}^{\infty} a_n \frac{t^n}{n}\right)$$

where $a_n$ is the number of words of length $n$ in $L$. Notice that $\zeta_L$ does not keep track of whether $1 \in L$, that is, $\zeta_L = \zeta_{L\cup\{1\}}$. Our goal will be to show that $\zeta_L$ is rational if $L$ is a cyclic regular language.

The set of *regular languages* over an alphabet $A$ is the smallest collection of subsets of $A^*$ that contains the finite subsets and is closed under union, product and generation of submonoids. Regular languages play a fundamental role in theoretical computer science and have also entered into group theory, cf. [ECH$^+$92]. A classical theorem of Kleene states that $L$ is regular if and only if it is accepted by a finite state automaton, if and only if there is a homomorphism $\eta\colon A^* \longrightarrow M$ with $M$ a finite monoid such that $\eta^{-1}(\eta(L)) = L$. Replacing $M$ by $\eta(M)$, we may assume without loss of generality that $\eta$ is surjective. There is, in fact, a unique (up to isomorphism) minimal cardinality choice for $M$, called the *syntactic monoid* of $M$. The syntactic monoid can be effectively computed from any of the standard ways of presenting a regular language (via an automaton or a regular expression). We shall use in this book only the formulation of regularity of a language in terms of finite monoids. The algebraic theory of regular languages tries to use finite monoids to classify regular languages [Eil74, Eil76, Pin86, Str94].

A language $L \subseteq A^*$ is called *cyclic* if it satisfies the following two conditions for all $u, v \in A^*$:

(a) $uv \in L \iff vu \in L$;
(b) for all $n \geq 1$, $u \in L \iff u^n \in L$.

One example of a cyclic regular language is the following. Let $M$ be a finite monoid with a zero element $z$ and let $A$ be a generating set for $M$. Let $\varphi\colon A^* \longrightarrow M$ be the canonical surjection and let $L = \{u \in A^* \mid \varphi(u)^\omega \neq z\}$. Then $L$ is a cyclic regular language.

Another example comes from symbolic dynamics. The set $A^{\mathbb{Z}}$ is a topological space, homeomorphic to the Cantor set, if we endow it with the topology of a product of discrete spaces. We view elements of $A^{\mathbb{Z}}$ as bi-infinite words $\cdots a_{-2}a_{-1}.a_0a_1\cdots$ where the decimal point is placed immediately to the left of the image of the origin under the corresponding map $f\colon \mathbb{Z} \longrightarrow A$ given by $f(n) = a_n$. The *shift map* $T\colon A^{\mathbb{Z}} \longrightarrow A^{\mathbb{Z}}$ is defined by

$$T(\cdots a_{-2}a_{-1}.a_0a_1\cdots) = \cdots a_{-2}a_{-1}a_0.a_1\cdots .$$

A *symbolic dynamical system*, or *shift*, is a non-empty, closed, shift-invariant subspace $\mathscr{X} \subseteq A^{\mathbb{Z}}$ for some finite set $A$. The *zeta function* of $\mathscr{X}$ is

$$\zeta_{\mathscr{X}}(t) = \exp\left(\sum_{n=1}^{\infty} a_n \frac{t^n}{n}\right)$$

where $a_n$ is the number of points $x \in \mathscr{X}$ with $T^n(x) = x$, i.e., the number of fixed points of $T^n$. Such a fixed point is of the form $\cdots ww.www\cdots$ where $|w| = n$, and hence $a_n$ is finite. In fact, if

$$P(\mathscr{X}) = \{w \in A^* \mid \cdots ww.www\cdots \in \mathscr{X}\} \cup \{1\},$$

then $P(\mathscr{X})$ is a cyclic language and $\zeta_{P(\mathscr{X})} = \zeta_{\mathscr{X}}$. If $\mathscr{X}$ is a sofic shift, then $P(\mathscr{X})$ will be regular. We shall use here the original definition of Weiss [Wei73] of a sofic shift.

A shift $\mathscr{X} \subseteq A^{\mathbb{Z}}$ is a *sofic shift* if there is a finite monoid $M$ with a zero element $z$ and a homomorphism $\varphi\colon A^* \longrightarrow M$ such that $x \in \mathscr{X}$ if and only if $\varphi(w) \neq z$ for every finite factor (i.e., subword) $w \in A^*$ of $x \in A^{\mathbb{Z}}$. It is then straightforward to see that $P(\mathscr{X}) = \varphi^{-1}(T)$ where

$$T = \{m \in M \mid m^n \neq z,\ \forall n \geq 0\}.$$

Therefore, $P(\mathscr{X})$ is a cyclic regular language. The reader is referred to [BR90, LM95] for more details.

*Example 13.1.* As an example, let $\Gamma = (V, E)$ be a simple undirected finite graph with vertex set $V$ and edge set $E$, which is not a tree. We choose an orientation for $E$ thereby allowing us to identify paths in $\Gamma$ with certain words over the alphabet $E \cup E^{-1}$. Let $\mathscr{X}_\Gamma \subseteq (E \cup E^{-1})^{\mathbb{Z}}$ be the subspace of bi-infinite reduced paths in the graph $\Gamma$ (that is, paths with no subpaths of the form $ee^{-1}$ or $e^{-1}e$ with $e \in E$). Then one can verify that $\mathscr{X}_\Gamma$ is a sofic shift (in fact, it is what is called a shift of finite type).

Indeed, let $M_\Gamma = \{1, z\} \cup (E \cup E^{-1})^2$ where 1 is the identity, $z$ is a zero element and

$$(x, y)(u, v) = \begin{cases} (x, v), & \text{if } yu \text{ is a reduced path} \\ z, & \text{else} \end{cases}$$

for $x, y, u, v \in E \cup E^{-1}$ Define $\eta\colon (E \cup E^{-1})^* \longrightarrow M_\Gamma$ by $\eta(x) = (x, x)$ for $x \in E \cup E^{-1}$. Then it is straightforward to verify that $M_\Gamma$ is a monoid and $x \in (E \cup E^{-1})^{\mathbb{Z}}$ is reduced if and only if $\eta(w) \neq z$ for each finite factor $w$ of $x$.

The zeta function $\zeta_{\mathscr{X}_\Gamma}$ is known as the *Ihara zeta function* of $\Gamma$ [Ter99]. So the rationality of zeta functions of cyclic regular languages recovers the well-known fact that the Ihara zeta function is rational.

The zeta function of a cyclic language admits an Euler product expansion, due to Berstel and Reutenauer [BR90]. Two words in $u, v \in A^*$ are said to be *conjugate* if there exist $r, s \in A^*$ such that $u = rs$ and $v = sr$. This is an equivalence relation on $A^*$. A word $w \in A^*$ is *primitive* if it is not a proper power. Clearly, each conjugate of a primitive word is primitive. A primitive word $w$ of length $n$ has exactly $n$ distinct conjugates; in fact, a word $w$ of length $n$ has exactly $n$ conjugates if and only if it is primitive. This statement relies on the well-known fact that two words commute if and only if they are powers of a common element, cf. [Lot97, Proposition 1.3.2]. Every word is a power of a unique primitive word, called its *primitive root*. By definition, a cyclic language is closed under taking conjugates, primitive roots and positive powers.

**Theorem 13.2.** *Let $L \subseteq A^*$ be a cyclic language. Then we have the Euler product expansion*

$$\zeta_L(t) = \exp\left(\sum_{n=1}^{\infty} a_n \frac{t^n}{n}\right) = \prod_{k=1}^{\infty} \frac{1}{(1 - t^k)^{b_k}} \tag{13.1}$$

*where $a_n$ is the number of words of length $n$ in $L$ and $b_k$ is the number of conjugacy classes of primitive words of length $k$ intersecting $L$. Consequently, $\zeta_L(t)$ has integer coefficients when expanded as a power series.*

*Proof.* First we note that if we count words in $L$ by their primitive roots, then

$$a_n = \sum_{k|n} b_k k. \tag{13.2}$$

We now compute logarithmic derivatives of both sides of (13.1) and show that they are equal. Since both sides agree at $t = 0$, it will then follow that (13.1) is valid. Let $f(t) = \log \zeta_L(t)$. Then we have, using (13.2), that

$$\begin{aligned} tf'(t) &= \sum_{n=1}^{\infty} a_n t^n = \sum_{k=1}^{\infty}\sum_{d=1}^{\infty} b_k k t^{kd} = \sum_{k=1}^{\infty} b_k \frac{kt^k}{1-t^k} \\ &= t\frac{d}{dt}\left[\sum_{k=1}^{\infty} - \log(1-t^k)^{b_k}\right] = t\frac{d}{dt}\log\left(\prod_{k=1}^{\infty}\frac{1}{(1-t^k)^{b_k}}\right). \end{aligned}$$

This completes the proof. $\square$

In the case of a shift, conjugacy classes of primitive words of length $n$ correspond to periodic orbits of size $n$ and the Euler product expansion of Theorem 13.2 is well known [LM95]. The corresponding Euler product expansion for Ihara zeta functions is also well known. Here conjugacy classes of primitive words correspond to primitive cyclically reduced closed paths in the graph, taken up to cyclic conjugacy.

## 13.2 Rationality of the zeta function of a cyclic regular language

Let us slightly generalize the definition of a zeta function of a language. Let $f\colon A^* \longrightarrow \mathbb{C}$ be a mapping. Define the *zeta function* of $f$ by

$$\zeta_f(t) = \exp\left(\sum_{n=1}^{\infty}\sum_{|w|=n} f(w)\frac{t^n}{n}\right).$$

The zeta function of a language $L \subseteq A^*$ is then the zeta function of the indicator function $\delta_L$ of $L$. As usual, a virtual character of $A^*$ is a difference $\chi_\rho - \chi_\psi$ of characters of finite dimensional representations $\rho$ and $\psi$ of $A^*$. Note that one has that $\zeta_{f-g} = \zeta_f/\zeta_g$ for mappings $f, g\colon A^* \longrightarrow \mathbb{C}$.

**Lemma 13.3.** *Let $f\colon A^* \longrightarrow \mathbb{C}$ be a virtual character. Assume that $f = \chi_\rho - \chi_\psi$ where $\rho\colon A^* \longrightarrow M_r(\mathbb{C})$ and $\psi\colon A^* \longrightarrow M_s(\mathbb{C})$ are representations. Let $S = \sum_{a\in A}\rho(a)$ and $T = \sum_{a\in A}\psi(a)$. Then the equality*

$$\zeta_f(t) = \frac{\det(I - tT)}{\det(I - tS)}$$

*holds and hence $\zeta_f$ is a rational function.*

*Proof.* First observe that if $B$ is a $k \times k$ matrix over $\mathbb{C}$, then

$$\exp\left(\sum_{n=1}^{\infty} \mathrm{Tr}(B^n)\frac{t^n}{n}\right) = \frac{1}{\det(I - tB)}. \tag{13.3}$$

Indeed, if $\lambda_1, \ldots, \lambda_k$ are the eigenvalues of $B$ with multiplicities, then

$$\begin{aligned}
\exp\left(\sum_{n=1}^{\infty} \mathrm{Tr}(B^n)\frac{t^n}{n}\right) &= \exp\left(\sum_{n=1}^{\infty} \frac{(\lambda_1^n + \cdots + \lambda_k^n)t^n}{n}\right) \\
&= \prod_{i=1}^{k} \exp\left(\sum_{n=1}^{\infty} \frac{(\lambda_i t)^n}{n}\right) \\
&= \prod_{i=1}^{k} \exp\left(-\log(1 - \lambda_i t)\right) \\
&= \prod_{i=1}^{k} \frac{1}{1 - \lambda_i t} \\
&= \frac{1}{\det(I - tB)}
\end{aligned}$$

as required.

From the equality $\zeta_f = \zeta_{\chi_\rho}/\zeta_{\chi_\psi}$, it suffices to prove $\zeta_{\chi_\rho}(t) = 1/\det(I - tS)$ and $\zeta_{\chi_\psi}(t) = 1/\det(I - tT)$. Since the proofs are identical, we handle only the first case. Observe that

$$\sum_{|w|=n} \rho(w) = \left[\sum_{a \in A} \rho(a)\right]^n = S^n$$

and, therefore,

$$\sum_{|w|=n} \chi_\rho(w) = \sum_{|w|=n} \mathrm{Tr}(\rho(w)) = \mathrm{Tr}\left(\sum_{|w|=n} \rho(w)\right) = \mathrm{Tr}(S^n).$$

The result now follows by applying (13.3) with $B = S$. $\square$

To establish that the zeta function of a cyclic regular language $L \subseteq A^*$ is rational, we shall prove now that the indicator function $\delta_L$ is a virtual character and apply Lemma 13.3.

**Proposition 13.4.** *Let $L \subseteq A^*$ be a cyclic regular language and $\eta\colon A^* \longrightarrow M$ a surjective homomorphism with $M$ finite and $L = \eta^{-1}(\eta(L))$. Set $X = \eta(L)$.*

*(i) $\delta_L = \delta_X \circ \eta$.*
*(ii) $X$ enjoys the following two properties for all $m, m' \in M$.*

*(a)* $mm' \in X \iff m'm \in X$.
*(b) for all* $n \geq 1$, $m \in X \iff m^n \in X$.
*(iii) If* $e \in E(M)$, *then* $X \cap G_e = \emptyset$ *or* $G_e \subseteq X$.
*(iv) The indicator function* $\delta_X \colon M \longrightarrow \mathbb{C}$ *is a virtual character.*

*Proof.* The first item is clear because $w \in L$ if and only if $\eta(w) \in X$. To prove (ii), let $m = \eta(w)$ and $m' = \eta(w')$. Then we have that

$$\begin{aligned} mm' \in X &\iff \eta(ww') \in X \iff ww' \in L \iff w'w \in L \\ &\iff \eta(w'w) \in X \iff m'm \in X, \end{aligned}$$

establishing (a). Similarly, we have that, for all $n \geq 1$,

$$\begin{aligned} m \in X &\iff \eta(w) \in X \iff w \in L \iff w^n \in L \\ &\iff \eta(w^n) \in X \iff m^n \in X, \end{aligned}$$

yielding (b).

We deduce (iii) from (ii) because if $g \in X \cap G_e$ and $n = |G_e|$, then we have $e = g^n \in X$. But if $h \in G_e$, we also have $h^n = e \in X$ and so $h \in X$. Therefore, $G_e \subseteq X$.

To prove (iv), we first note that $\delta_X$ is a class function because $\delta_X(mm') = \delta_X(m'm)$ by (ii)(a) and $\delta_X(m) = \delta_X(m^{\omega+1})$ by (ii)(b). Moreover, if $G_e$ is a maximal subgroup, then by (iii), $(\delta_X)|_{G_e}$ is either identically 0 or 1. In the former case, $(\delta_X)|_{G_e}$ is the character of the zero $\mathbb{C}G_e$-module and in the latter it is the character of the trivial $\mathbb{C}G_e$-module. We conclude that $\delta_X$ is a virtual character by Corollary 8.16. □

If $\eta \colon A^* \longrightarrow M$ is a homomorphism and $\rho \colon M \longrightarrow M_r(\mathbb{C})$ is a representation, then $\rho \circ \eta$ is a representation of $A^*$ and $\chi_{\rho\circ\eta} = \chi_\rho \circ \eta$. Consequently, if $f$ is a virtual character of $M$, then $f \circ \eta$ is a virtual character of $A^*$. Proposition 13.4(i) and (iv), in conjunction with Lemma 13.3, then yield the following theorem of Berstel and Reutenauer.

**Theorem 13.5.** *Let* $L \subseteq A^*$ *be a cyclic regular language. Then the indicator function* $\delta_L \colon A^* \longrightarrow \mathbb{C}$ *is a virtual character and hence* $\zeta_L$ *is a rational function.*

‘ We remark that the proof of Theorem 13.5 is constructive, but not very explicit, because the proof of Corollary 8.16 involves inverting the decomposition matrix (cf. Theorem 6.5).

## 13.3 Computing the zeta function

In this section, we compute some zeta functions of sofic shifts to show that the methods are applicable.

### 13.3.1 Edge shifts

Let $Q$ be a (finite) quiver with vertex set $Q_0$ and edge set $Q_1$. The *edge shift* $\mathscr{X}_Q \subseteq Q_1^{\mathbb{Z}}$ associated to $Q$ is the shift consisting of all bi-infinite paths in $Q$. As in Chapter 9, we shall concatenate the edges in a path from right to left. We assume that $Q$ is not acyclic so that $\mathscr{X}_Q$ is non-empty. Let $A$ be the *adjacency matrix* of $Q$: so the rows and columns of $A$ are indexed by $Q_0$ and $A_{rs}$ is the number of edges from $s$ to $r$.

Let us consider the inverse monoid

$$B_Q = \{1, z\} \cup (Q_0 \times Q_0)$$

where 1 is an identity, $z$ is a zero element and the remaining products are defined by

$$(u,v)(r,s) = \begin{cases} (u,s), & \text{if } v = r \\ z, & \text{else.} \end{cases} \tag{13.4}$$

Note that $E(B_Q) = \{1, z\} \cup \{(v,v) \mid v \in Q_0\}$. Also note that the inverse monoid involution is given by $1^* = 1$, $z^* = z$ and $(u,v)^* = (v,u)$.

Define $\varphi\colon Q_1^* \longrightarrow B_Q$ by $\varphi(a) = (r,s)$ where $a\colon s \longrightarrow r$ is an edge. Then it is straightforward to verify that a bi-infinite word $x$ over $Q_1$ is a path if and only if $\varphi(w) \neq z$ for all finite subwords $w$ of $x$. Therefore, $\mathscr{X}_Q$ is a sofic shift. Also we have that $P(\mathscr{X}_Q) = \varphi^{-1}(F)$ where $F = E(B_Q) \setminus \{z\}$. Note that $\varphi$ is onto if and only if $Q$ is strongly connected. Nonetheless, we will be able to express $\delta_F$ as a virtual character $\chi_V - \chi_W$ of $B_Q$, which is enough to compute the zeta function of the edge shift. Although it is easy to find this representation in an *ad hoc* manner, we follow the systematic method. We fix a vertex $q_0$ and put $e = (q_0, q_0)$. It is straightforward to check that $L_e = \{(r, q_0) \mid r \in Q_0\}$.

We take as our $\mathscr{J}$-classes representatives $\{z, e, 1\}$. The corresponding maximal subgroups, $G_z$, $G_e$ and $G_1$, are trivial. Let $S_z$, $S_e$ and $S_1$ be the trivial modules for these three maximal subgroups over $\mathbb{C}$. Then $S_z^\sharp$ is the trivial $\mathbb{C}B_Q$-module, $S_e^\sharp = \mathbb{C}L_e$ and $S_1^\sharp$ affords the representation sending 1 to 1 and $B_Q \setminus \{1\}$ to 0. We compute the matrix $L$ of

$$\mathrm{Res}\colon G_0(\mathbb{C}B_Q) \longrightarrow G_0(\mathbb{C}G_z) \times G_0(\mathbb{C}G_e) \times G_0(\mathbb{C}G_1)$$

with respect to the ordered bases $\{S_z^\sharp, S_e^\sharp, S_1^\sharp\}$ and $\{S_z, S_e, S_1\}$. As $L$ is lower triangular and unipotent by Theorem 6.5 the only computations to make are that $[fS_z^\sharp] = [S_f]$ for $f = \{e, 1\}$ and $[1S_e^\sharp] = |Q_0| \cdot [S_1]$ as $\dim S_e^\sharp = |L_e| = |Q_0|$. Therefore, we have

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & |Q_0| & 1 \end{bmatrix}.$$

Now $\delta_F$ is a class function and restricts to the zero mapping on $G_z$ and to the trivial character on $G_e$ and $G_1$. Hence it is a virtual character and, using the notation of Theorem 8.15, we have $\delta_F = \Delta_{B_Q}(\mathrm{Res}^{-1}(0, [S_e], [S_1]))$. That is, we must solve $Lx = (0, 1, 1)^T$. The solution is $(0, 1, 1 - |Q_0|)^T$ and so

$$\delta_F = \chi_{S_e^\sharp} - (|Q_0| - 1) \cdot \chi_{S_1^\sharp}.$$

Let $\rho\colon B_Q \longrightarrow M_{|Q_0|}(\mathbb{C})$ be the representation afforded by $S_e^\sharp$ and let $\psi\colon B_q \longrightarrow \mathbb{C}$ be the representation afforded by $(|Q_0| - 1) \cdot S_1^\sharp$. Let $S = \sum_{a \in Q_1} \rho(\varphi(a))$ and $T = \sum_{a \in Q_1} \psi(\varphi(a))$. Then by Lemma 13.3, we have that

$$\zeta_{\mathscr{X}_Q} = \frac{\det(I - tT)}{\det(I - tS)}.$$

Since $\psi$ vanishes on $B_Q \setminus \{1\}$, it follows that $T = 0$. On the other hand, $S_e^\sharp$ has basis $(s, q_0)$ with $s \in Q_0$. Using (13.4) we compute that

$$\sum_{a \in Q_1} \varphi(a)(s, q_0) = \sum_{r \in Q_0} A_{rs}(r, q_0)$$

where $A$ is the adjacency matrix of $Q$. Therefore, $S = A$ and we have arrived at the following classical result of Bowen and Lanford [LM95].

**Theorem 13.6.** *Let $Q$ be a finite quiver that is not acyclic and let $\mathscr{X}_Q$ be the corresponding edge shift. Then the zeta function of $\mathscr{X}_Q$ is given by*

$$\zeta_{\mathscr{X}_Q}(t) = \frac{1}{\det(I - tA)}$$

*where $A$ is the adjacency matrix of $Q$.*

Up to conjugacy, edge shifts associated to quivers are precisely the shifts of finite type [LM95]. We remark that the zeta function is a conjugacy invariant [LM95].

### 13.3.2 The even shift

Next we consider a sofic shift which is not of finite type. Let $A = \{a, b\}$ and let $\mathscr{X}$ consist of all bi-infinite words $x \in A^{\mathbb{Z}}$ such that between any two consecutive occurrences of $b$, there are an even number of occurrences of $a$. In other words, $\mathscr{X}$ consists of all labels of bi-infinite paths in the automaton in Figure 13.1. One easily checks that $\mathscr{X}$ is a shift, called the *even shift* (cf. [LM95]).

Let $I_2$ be the symmetric inverse monoid on $\{1, 2\}$ and put $z = 1_\emptyset$, the zero element of $I_2$. Define $\varphi\colon A^* \longrightarrow I_2$ by $\varphi(a) = (1\ 2)$ and $\varphi(b) = 1_{\{2\}}$. Note that $\varphi$ is surjective. A straightforward computation shows that $\varphi(w) = z$ if and only if $w$ contains an odd number of occurrences of $a$ between some consecutive pair of occurrences of $b$ and hence $\mathscr{X}$ is a sofic shift. Also $P(\mathscr{X}) = \varphi^{-1}(F)$ where $F = S_2 \cup E(I_2) \setminus \{1_\emptyset\}$. Again, we must express $\delta_F$ as a virtual character.

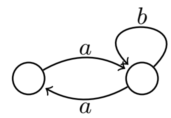



**Fig. 13.1.** The even shift

Let us choose $1, e = 1_{\{2\}}, z$ as our set of $\mathscr{J}$-class representatives. Note that $G_e$ and $G_z$ are trivial, whereas $G_1 = S_2$ is the symmetric group on two symbols. Denote by $S_z$, $S_e$ and $S_1$ the trivial modules for these three maximal subgroups over $\mathbb{C}$ and let $V$ be the sign representation of $G_1 = S_2$. Then $S_z^\sharp$ is the trivial $\mathbb{C}I_2$-module, $S_e^\sharp = \mathbb{C}L_e$, $S_1^\sharp$ affords the representation sending $S_2$ to 1 and $I_2 \setminus S_2$ to 0 and $V^\sharp$ affords the representation which restricts to $S_2$ as the sign representation and annihilates $I_2 \setminus S_2$.

We compute the matrix $L$ of

$$\mathrm{Res}\colon\, G_0(\mathbb{C}I_2) \longrightarrow G_0(\mathbb{C}G_z) \times G_0(\mathbb{C}G_e) \times G_0(\mathbb{C}G_1)$$

with respect to the ordered bases $\{S_z^\sharp, S_e^\sharp, S_1^\sharp, V^\sharp\}$ and $\{S_z, S_e, S_1, V\}$. Given that $L$ is lower triangular and unipotent by Theorem 6.5, it remains to observe that $[fS_z^\sharp] = [S_f]$ for $f = \{e, 1\}$ and that the restriction of $S_e^\sharp$ to $G_1$ is the regular representation. Indeed, $L_e$ consists of the two rank one partial injective mappings of $\{1, 2\}$ with domain $\{2\}$ and the left action by the transposition $(1\ 2)$ interchanges these two maps. Therefore, $[1S_e^\sharp] = [S_1] + [V]$ and hence

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

The mapping $\delta_F$ is a class functions and restricts to the zero mapping on $G_z$ and to the trivial character on both $G_1$ and $G_e$. Therefore, retaining the notation of Theorem 8.15, we have $\delta_F = \Delta_{I_2}(\mathrm{Res}^{-1}(0, [S_e], [S_1], 0))$. That is, we must solve $Lx = (0, 1, 1, 0)^T$. The solution is $(0, 1, 0 - 1)^T$ and so

$$\delta_F = \chi_{S_e^\sharp} - \chi_{V^\sharp}.$$

Let $\rho\colon I_2 \longrightarrow M_2(\mathbb{C})$ be the representation afforded by $S_e^\sharp$ and let $\psi\colon I_2 \longrightarrow \mathbb{C}$ be the representation afforded by $V^\sharp$. Put $S = \rho(\varphi(a)) + \rho(\varphi(b))$ and $T = \psi(\varphi(a)) + \psi(\varphi(b))$. Lemma 13.3, yields that

$$\zeta_{\mathscr{X}} = \frac{\det(I - tT)}{\det(I - tS)}.$$

Since $\psi(\varphi(a)) = -1$ and $\psi(\varphi(b)) = 0$, we have $\det(I - tT) = 1 + t$. On the other hand, $S_e^\sharp$ has basis the two rank one maps with domain $\{2\}$. The transposition $\rho(a)$ swaps them and $\rho(b)$ fixes one and annihilates the other. Therefore, we have in an appropriate basis that

$$S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

and so $\det(I - tS) = 1 - t - t^2$. We have thus proved the following result.

**Theorem 13.7.** *The zeta function of the even shift is* $\dfrac{1+t}{1-t-t^2}$.

### 13.3.3 The Ihara zeta function

Let $\Gamma = (V, E)$ be a simple undirected graph which is not a tree and fix an orientation on $\Gamma$. Put $E^{\pm} = E \cup E^{-1}$. Recall that $\mathscr{X}_\Gamma \subseteq (E^{\pm})^{\mathbb{Z}}$ is the space of bi-infinite reduced paths in $\Gamma$ and that $\zeta_{\mathscr{X}_\Gamma}$ is known as the Ihara zeta function of $\Gamma$.

Define a matrix $T\colon E^{\pm} \times E^{\pm} \longrightarrow \mathbb{C}$, called *Hashimoto's edge adjacency operator*, by

$$T(x, y) = \begin{cases} 1, & \text{if } xy \text{ is a reduced path} \\ 0, & \text{else.} \end{cases}$$

**Theorem 13.8.** *The Ihara zeta function of* $\Gamma = (V, E)$ *is given by*

$$\zeta_{\mathscr{X}_\Gamma}(t) = \frac{1}{\det(I - tT)}$$

*where* $T$ *is Hashimoto's edge adjacency operator.*

*Proof.* Let $\eta\colon (E^{\pm})^* \longrightarrow M_\Gamma$ be as in Example 13.1. Note that

$$E(M_\Gamma) = \{1, z\} \cup \{(x, y) \in (E^{\pm})^2 \mid yx \text{ is a reduced path}\}$$

and that $P(\mathscr{X}_\Gamma) = \eta^{-1}(X)$ where $X = E(M_\Gamma) \setminus \{z\}$. We express $\delta_X$ as a virtual character of $M_\Gamma$. Make $\mathbb{C}E^{\pm}$ a $\mathbb{C}M_\Gamma$-module by putting $1u = u$, $zu = 0$ and

$$(x, y)u = \begin{cases} x, & \text{if } yu \text{ is reduced} \\ 0, & \text{else} \end{cases}$$

for $x, y, u \in E^{\pm}$. In particular, $(x, y)u = u$ if and only if $u = x$ and $yx$ is reduced, that is, if and only if $x = u$ and $(x, y) \in E(M_\Gamma)$. Thus the character $\chi$ of $\mathbb{C}E^{\pm}$ is given by $\chi(1) = |E^{\pm}|$, $\chi(z) = 0$ and

$$\chi(x, y) = \begin{cases} 1, & \text{if } (x, y) \in E(M_\Gamma) \\ 0, & \text{else.} \end{cases}$$

We conclude that $\delta_X = \chi - (|E^{\pm}| - 1)\theta$ where $\theta$ is the character of the degree one representation sending 1 to 1 and $M_\Gamma \setminus \{1\}$ to 0.

Now $\sum_{x \in E^{\pm}} \theta(\eta(x)) = 0$, whereas

$$\sum_{x \in E^{\pm}} \eta(x)y = \sum_{x \in E^{\pm}} T(x,y)x$$

and so if $\rho$ is the representation afforded by $\mathbb{C}E^{\pm}$ with respect to the basis $E^{\pm}$, then $\sum_{x \in E^{\pm}} \rho(\eta(x)) = T$. We conclude from Lemma 13.3 that

$$\zeta_{\mathscr{X}_\Gamma}(t) = \frac{1}{\det(I - tT)},$$

as was required. □

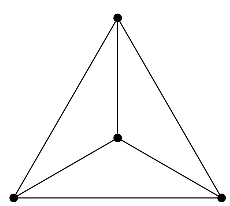
## 13.4 Exercises

**13.1.** Prove that $A^*$ is indeed a free monoid on $A$. That is, given a monoid $M$ and a mapping $\varphi\colon A \longrightarrow M$, prove that there is a unique homomorphism $\Phi\colon A^* \longrightarrow M$ such that $\Phi|_A = \varphi$.

**13.2.** Verify the assertions in Example 13.1.

**13.3.** Let $n \geq 2$. Compute the zeta function of the shift $\mathscr{X}_n$ where $\mathscr{X}_n$ consists of all $x \in \{a,b\}^{\mathbb{Z}}$ such that the number of occurrences of $a$ between two consecutive occurrences of $b$ is divisible by $n$. (Hint: consider the inverse submonoid of $I_n$ generated by $1_{\{n\}}$ and the cyclic permutation $(1\ 2 \cdots n)$.)

**13.4.** Compute the Ihara zeta function of the graph below.



**13.5.** Let $L \subseteq A^*$ be a language. Define an equivalence relation $\equiv_L$ on $A^*$, called the *syntactic congruence* of $L$, by $x \equiv_L y$ if and only if one has

$$uxv \in L \iff uyv \in L$$

for all $u, v \in A^*$.

(a) Prove that $\equiv_L$ is a congruence on $A^*$. The quotient $M_L = A^*/{\equiv_L}$ is called the *syntactic monoid* of $L$.
(b) Prove that if $\varphi\colon A^* \longrightarrow M_L$ is the canonical projection, then $L = \varphi^{-1}(\varphi(L))$.

(c) Prove that if $\eta\colon A^* \longrightarrow M$ is a homomorphism with $\eta^{-1}(\eta(L)) = L$, then $\ker\eta \subseteq \equiv_L$.
(d) Deduce that $L$ is regular if and only if $M_L$ is finite.
(e) Let $G$ be a group and let $\eta\colon A^* \longrightarrow G$ be a surjective homomorphism. Prove that if $L = \eta^{-1}(1)$, then $G \cong M_L$. Deduce that $L$ is regular if and only if $G$ is finite.

**13.6.** A finite *automaton* is a 5-tuple $\mathscr{A} = (Q, A, \delta, i, T)$ where $Q$ is a finite set, $A$ is a finite alphabet, $\delta\colon A \times Q \longrightarrow Q$, $i \in Q$ and $T \subseteq Q$. Define $\delta^*\colon A^* \times Q \longrightarrow Q$ recursively by

$$\begin{aligned} \delta^*(1, q) &= q \\ \delta^*(wa, q) &= \delta^*(w, \delta(a, q)) \end{aligned}$$

for $q \in Q$, $a \in A$ and $w \in A^*$. The language accepted by $\mathscr{A}$ is

$$L(\mathscr{A}) = \{w \in A^* \mid \delta^*(w, i) \in T\}.$$

Prove that $L \subseteq A^*$ is regular if and only if $L = L(\mathscr{A})$ for some finite automaton $\mathscr{A}$.

**13.7 (M.-P. Schützenberger).** Let $L \subseteq A^*$ be a regular language and

$$f_L(t) = \sum_{n=0}^{\infty} a_n t^n$$

where $a_n$ is the number of words of length $n$ belonging to $L$. Prove that $f_L(t)$ is rational. (Hint: let $\eta\colon A^* \longrightarrow M$ be a surjective homomorphism with $M$ a finite monoid and $\eta^{-1}(\eta(L)) = L$ and let $\rho\colon M \longrightarrow M_n(\mathbb{C})$ be the regular regular representation of $M$; put $B = \sum_{a \in A} \rho(\eta(a))$ and show that

$$f_L(t) = \sum_{m \in \eta(L)} \sum_{n=0}^{\infty} (Bt)^n_{1,m} = \sum_{m \in M} (I - tB)^{-1}_{1,m};$$

then use the classical adjoint formula for the inverse of a matrix over a commutative ring.)

# 14

# Transformation Monoids

In this chapter we shall use the representation theory of finite monoids to study finite monoids acting on finite sets. Such actions play an important role in automata theory and we provide here some applications in this direction. In particular, we study connections with the popular Černý conjecture [Č64]; see [Vol08] for a survey. This chapter is primarily based upon the paper [Ste10b].

## 14.1 Transformation monoids

In this section, we discuss transformation monoids. Much of this is folklore. Fix a finite monoid $M$ for the section. The minimal ideal of $M$ shall be denoted by $I(M)$. Let $\Omega$ be a finite set equipped with an action of $M$ on the left. We say that $M$ is *transitive* on $\Omega$ if $M\omega = \Omega$ for all $\omega \in \Omega$, that is, $\Omega$ has no proper, non-empty $M$-invariant subset. The *rank* of $m \in M$ is defined by

$$\mathrm{rk}(m) = |m\Omega|.$$

The rank is constant on $\mathscr{J}$-classes. More precisely, we have the following.

**Proposition 14.1.** *If* $u, v, m \in M$, *then* $\mathrm{rk}(umv) \leq \mathrm{rk}(m)$.

*Proof.* Trivially $|mv\Omega| \leq |m\Omega| = \mathrm{rk}(m)$. But the map $mv\Omega \longrightarrow umv\Omega$ given by $x \mapsto ux$ is surjective. Thus $|umv\Omega| \leq |mv\Omega| \leq \mathrm{rk}(m)$ and hence $\mathrm{rk}(umv) \leq \mathrm{rk}(m)$. □

We now consider elements of minimum rank. Put

$$\mathrm{rk}(\Omega) = \min\{\mathrm{rk}(m) \mid m \in M\}$$

for a finite $M$-set $\Omega$. We show that the minimum rank is achieved on the minimal ideal.

**Lemma 14.2.** *Let $M$ act on a finite set $\Omega$. Let $r = \mathrm{rk}(\Omega)$. Then* $\mathrm{rk}(m) = r$ *for all $m \in I(M)$. If $M$ acts faithfully on $\Omega$, then conversely* $\mathrm{rk}(m) = r$ *implies that $m \in I(M)$.*

*Proof.* Let $\mathrm{rk}(u) = r$ and let $m \in I(M)$. Trivially, $\mathrm{rk}(m) \geq r$. But $m \in MuM$ and so $\mathrm{rk}(m) \leq r$ by Proposition 14.1. The first statement follows.

Suppose now that the action is faithful and let $\mathrm{rk}(m) = r$. We prove that $m \in I(M)$. First note that $r \leq |m^2\Omega| \leq |m\Omega| = r$ and so $m^2\Omega = m\Omega$. Thus $m$ permutes $m\Omega$ and hence $m^\omega$ fixes $m\Omega$. Therefore, $m^\omega m = m$ by faithfulness of the action, whence $MmM = Mm^\omega M$. So without loss of generality, we may assume that $m$ is an idempotent. Let $z \in I(M)$ and consider $e = (mzm)^\omega$. Note that $em = e$ by construction because $m \in E(M)$. Also, we have $e\Omega \subseteq m\Omega$ and both sets have the same size $r$, whence $e\Omega = m\Omega$. Since $e, m$ are idempotent, they both fix their image set. Thus $m\alpha = e(m\alpha) = e\alpha$ for all $\alpha \in \Omega$. Therefore, $m = e \in I(M)$ by faithfulness. $\square$

The pair $(M, \Omega)$ is called a *transformation monoid* if $M$ acts faithfully on the left of $\Omega$. We write $T_\Omega$ for the *full transformation monoid* on $\Omega$, that is, the monoid of all self-maps of $\Omega$. Transformation monoids on $\Omega$ amount to submonoids of $T_\Omega$. The previous lemma shows that the minimal ideal of a transformation monoid consists of the elements of minimal rank.

A useful fact about transitive actions is the following observation.

**Proposition 14.3.** *Let $M$ act transitively on $\Omega$. Then $I(M)\omega = \Omega$ for all $\omega \in \Omega$.*

*Proof.* Trivially, $I(M)\omega$ is a non-empty $M$-invariant subset. $\square$

A *congruence* $\equiv$ on an $M$-set $\Omega$ is an equivalence relation on $\Omega$ such that $\alpha \equiv \beta$ implies $m\alpha \equiv m\beta$ for all $m \in M$. In this case, $\Omega/{\equiv}$ is an $M$-set with action $m[\alpha]_\equiv = [m\alpha]_\equiv$. A transformation monoid $(M, \Omega)$ is said to be *primitive* if $M$ is transitive on $\Omega$ and the only congruences on $\Omega$ are the universal equivalence relation and the equality relation. Note that in [Ste10b], transitivity is not assumed in the definition of primitivity.

## 14.2 Transformation modules

Fix a field $\Bbbk$ for this section and a finite monoid $M$ acting on the left of a finite set $\Omega$. Then $\Bbbk\Omega$ is a left $\Bbbk M$-module by extending the action of $M$ linearly. We call it the *transformation module* associated to the action. Also, we have that $\Bbbk^\Omega = \{f \colon \Omega \longrightarrow \Bbbk\}$ is a right $\Bbbk M$-module by putting $(fm)(\omega) = f(m\omega)$. We have a dual pairing

$$\langle \cdot, \cdot \rangle \colon \Bbbk^\Omega \times \Bbbk\Omega \longrightarrow \Bbbk$$

defined by $\langle f, \omega \rangle = f(\omega)$ for $\omega \in \Omega$. Note that $\langle fm, \omega \rangle = \langle f, m\omega \rangle$ for all $m \in M$.

If $X \subseteq \Omega$, put $[X] = \sum_{\alpha \in X} \alpha \in \Bbbk\Omega$.

**Theorem 14.4.** *Let $(M, \Omega)$ be a transitive transformation monoid. Let*

$$V = \Bbbk\{[m\Omega] - [n\Omega] \mid m, n \in I(M)\}.$$

*Then $V$ is a $\Bbbk M$-submodule. Moreover, we have the following.*

*(i)* $\dim V \leq |\Omega| - \mathrm{rk}(\Omega)$.
*(ii) $mV = 0$ if and only if $m \in I(M)$.*

*Proof.* Let $k = |\Omega|$ and $r = \mathrm{rk}(\Omega)$. If $m \in M$ and $x \in I(M)$, then $|mx\Omega| = r = |x\Omega|$ and so $y \mapsto my$ is a bijection $x\Omega \longrightarrow mx\Omega$. Thus $m[x\Omega] = [mx\Omega]$. It now follows since $I(M)$ is an ideal that $V$ is a $\Bbbk M$-submodule.

We turn to the proof of (i). Fix $e \in E(I(M))$ and, for $\alpha \in e\Omega$, put $P_\alpha = \{\beta \in \Omega \mid e\beta = \alpha\}$. Then $\{P_\alpha \mid \alpha \in e\Omega\}$ is a partition of $\Omega$ into $r$ parts. We claim that $|m\Omega \cap P_\alpha| = 1$ for all $m \in I(M)$ and $\alpha \in e\Omega$. Indeed, if $|m\Omega \cap P_\alpha| > 1$ for some $\alpha \in e\Omega$, then $r = |em\Omega| < |m\Omega| = r$, a contradiction. Thus $|m\Omega \cap P_\alpha| \leq 1$ for each $\alpha \in e\Omega$. But then we have

$$r = |m\Omega| = \left|\bigcup_{\alpha \in e\Omega} (m\Omega \cap P_\alpha)\right| = \sum_{\alpha \in e\Omega} |m\Omega \cap P_\alpha| \leq |e\Omega| = r.$$

We conclude that $|m\Omega \cap P_\alpha| = 1$ for all $\alpha \in e\Omega$.

Let $\delta_{P_\alpha} \colon \Omega \longrightarrow \Bbbk$ be the indicator function of $P_\alpha$ for $\alpha \in e\Omega$. Then we compute, for $m, n \in I(M)$,

$$\begin{aligned}
\langle \delta_{P_\alpha}, [m\Omega] - [n\Omega]\rangle &= \langle \delta_{P_\alpha}, [m\Omega]\rangle - \langle \delta_{P_\alpha}, [n\Omega]\rangle \\
&= \sum_{\beta \in m\Omega} \delta_{P_\alpha}(\beta) - \sum_{\gamma \in n\Omega} \delta_{P_\alpha}(\gamma) \\
&= |m\Omega \cap P_\alpha| - |n\Omega \cap P_\alpha| \\
&= 0
\end{aligned}$$

by the claim. Thus $V \subseteq W^\perp$ where $W$ is the span of the mappings $\delta_{P_\alpha}$ with $\alpha \in e\Omega$. Since the mappings $\delta_{P_\alpha}$ with $\alpha \in e\Omega$ have disjoint supports, they form a linearly independent set. It follows that $\dim W = r$ and hence $\dim V \leq \dim W^\perp = n - r$. This proves the first item.

Suppose now that $m \in I(M)$. Then, for all $x \in I(M)$, we have $mx\Omega \subseteq m\Omega$ and both sets have size $r$. Thus $mx\Omega = m\Omega$ and so $m[x\Omega] = [mx\Omega] = [m\Omega]$. It follows that $mV = 0$. Suppose that $m \notin I(M)$. Then $\mathrm{rk}(m) > r$ by Lemma 14.2. Let $e \in E(I(M))$. Then $me\Omega \subsetneq m\Omega$ because $|me\Omega| = r < |m\Omega|$. Let $\alpha \in m\Omega \setminus me\Omega$ and choose $\beta \in \Omega$ with $m\beta = \alpha$. Since $I(M)\beta = \Omega$ by Proposition 14.3, there exists $n \in I(M)$ such that $n\beta = \beta$. Then $m([n\Omega] - [e\Omega]) = [mn\Omega] - [me\Omega] \neq 0$ because $\alpha \in mn\Omega \setminus me\Omega$. This completes the proof. $\square$

Next we consider the relationship between congruences and transformation modules.

**Proposition 14.5.** *Let $\equiv$ be a congruence on $\Omega$. Then there is a surjective homomorphism $\eta\colon \Bbbk\Omega \longrightarrow \Bbbk[\Omega/{\equiv}]$ given by $\omega \mapsto [\omega]_{\equiv}$ for $\omega \in \Omega$. Moreover,* $\ker \eta$ *is spanned by the differences $\alpha - \beta$ with $\alpha \equiv \beta$.*

*Proof.* It is clear that $\eta$ is a surjective $\Bbbk M$-module homomorphism. Obviously each difference $\alpha - \beta$ with $\alpha \equiv \beta$ is in $\ker \eta$. Fix a set $T$ of representatives for each $\equiv$-class and write $\overline{\alpha}$ for the representative of the class of $\alpha$. We claim that the non-zero elements of the form $\alpha - \overline{\alpha}$ form a basis for $\ker \eta$. There are $|\Omega| - |\Omega/{\equiv}| = \dim \ker \eta$ such elements so it is enough to show that they span $\ker \eta$. Indeed, if $v = \sum_{\alpha \in \Omega} c_\alpha \cdot \alpha$ is in $\ker \eta$, then

$$0 = \eta(v) = \sum_{\beta \in T} \sum_{\overline{\alpha} = \beta} c_\alpha [\beta]_{\equiv}.$$

Thus for each $\beta \in T$, we have that $\sum_{\overline{\alpha}=\beta} c_\alpha = 0$ and so

$$v = \sum_{\beta \in T} \sum_{\overline{\alpha} = \beta} c_\alpha (\alpha - \beta).$$

This completes the proof. $\square$

As a special case, we may consider the universal equivalence relation $\equiv$ on $\Omega$. The corresponding map is $\eta\colon \Bbbk\Omega \longrightarrow \Bbbk$ sending each element of $\Omega$ to 1. This is usually called the *augmentation map* and hence we write $\ker \eta = \mathrm{Aug}(\Bbbk\Omega)$. It is spanned by the differences $\alpha - \beta$ with $\alpha, \beta \in \Omega$. Of course, $\dim \mathrm{Aug}(\Bbbk\Omega) = |\Omega| - 1$.

**Theorem 14.6.** *Let $(M, \Omega)$ be a transitive transformation monoid. Suppose that* $\mathrm{Aug}(\Bbbk\Omega)$ *is a simple* $\Bbbk M$*-module.*

*(i)* $(M, \Omega)$ *is primitive.*
*(ii) Either* $(M, \Omega)$ *is a permutation group or* $I(M)$ *is the set of all constant mappings on* $\Omega$.

*Proof.* To prove (i), observe that if $\equiv$ is a congruence, then the span of the $\alpha - \beta$ with $\alpha \equiv \beta$ is a $\Bbbk M$-submodule of $\mathrm{Aug}(\Bbbk\Omega)$ of codimension $|\Omega/{\equiv}| - 1$ by Proposition 14.5. Hence $\equiv$ is either universal or the equality relation by simplicity of $\mathrm{Aug}(\Bbbk\Omega)$. We conclude that $(M, \Omega)$ is primitive.

Let $r = \mathrm{rk}(\Omega)$. Let $V = \Bbbk\{[m\Omega] - [n\Omega] \mid m, n \in I(M)\}$. Note that $V \subseteq \mathrm{Aug}(\Bbbk\Omega)$ since the value of the augmentation mapping on $[m\Omega]$ with $m \in I(M)$ is $r$. By Theorem 14.4, we have that $V$ is a submodule of $\mathrm{Aug}(\Bbbk\Omega)$ of dimension at most $|\Omega| - r$. By simplicity, we conclude $V = \mathrm{Aug}(\Bbbk\Omega)$ or $V = 0$. In the first case, we must have $r = 1$ and so $I(M)$ consists of constant maps. As $I(M)\beta = \Omega$ for all $\beta \in \Omega$ by Proposition 14.3, in fact, $I(M)$ contains all constant maps. If $V = 0$, then we have $M = I(M)$ by Theorem 14.4 as $M$ annihilates $V$. Thus $M = J_1 = G_1$ and so $M$ is a group. This completes the proof. $\square$

A permutation group $G \leq S_\Omega$ is called a *synchronizing group* if, for each mapping $f \in T_\Omega \setminus S_\Omega$, the monoid $\langle G, f\rangle$ contains a constant map. This notion was introduced by Arnold and the author in [AS06].

**Corollary 14.7.** *If $G \leq S_\Omega$ is 2-transitive, then $G$ is synchronizing.*

*Proof.* Let $f \in T_\Omega \setminus S_\Omega$ and $M = \langle G, f\rangle$. Proposition B.12 says that if $G$ is 2-transitive, then $\mathrm{Aug}(\mathbb{C}\Omega)$ is a simple $\mathbb{C}G$-module and hence a simple $\mathbb{C}M$-module. Since $M$ is not a permutation group, it follows that it contains a constant map by Theorem 14.6. $\square$

*Remark 14.8.* If $G \leq S_\Omega$ is 2-homogeneous, that is, acts transitively on unordered pairs of elements of $\Omega$, then it is known that $\mathrm{Aug}(\mathbb{R}\Omega)$ is a simple $\mathbb{R}G$-module. Thus one can deduce as above that $G$ is synchronizing.

The following theorem was first proved by Pin [Pin78]. The proof below is due to the author and Arnold [AS06].

**Theorem 14.9.** *Let $G \leq S_\Omega$ be transitive with $|\Omega| = p$ prime. Then $G$ is synchronizing.*

*Proof.* First note that $G$ contains a $p$-cycle $g$ where $p = |G|$. Indeed, since $G$ is transitive one has that $p \mid |G|$. But then $G$ has an element $g$ of order $p$, which is necessarily a $p$-cycle. Let $C = \langle g\rangle$. It suffices to prove that $\mathrm{Aug}(\mathbb{Q}\Omega)$ is a simple $\mathbb{Q}C$-module. The result then follows by the argument in the proof of Corollary 14.7.

Note that $\mathbb{Q}C \cong \mathbb{Q}[x]/(x^p-1) \cong \mathbb{Q}(\zeta_p)\times\mathbb{Q}$ where $\zeta_p$ is a primitive $p^{th}$-root of unity. Hence $\mathbb{Q}C$ has two simple modules: the trivial module and one of dimension $p-1$. Since $G$ is transitive, $\mathbb{Q}\Omega$ contains only one copy of the trivial module, spanned by $[\Omega]$ (cf. Proposition B.10). As $\mathrm{Aug}(\mathbb{Q}\Omega)$ has dimension $p-1$, we deduce that $\mathrm{Aug}(\mathbb{Q}\Omega)$ is the $(p-1)$-dimensional simple $\mathbb{Q}C$-module. This completes the proof. $\square$

The above result can also be deduced from the following lemma of Neumann [Neu09].

**Lemma 14.10.** *Let $(M, \Omega)$ be a transformation monoid with group of units $G$. Assume that $G$ acts transitively on $\Omega$. Let $m \in I(M)$. Then $|m^{-1}(m\alpha)| = |\Omega|/\operatorname{rk}(\Omega)$ for all $\alpha \in \Omega$, that is, the partition associated with $m$ is uniform.*

*Proof.* Let $n = |\Omega|$ and $r = \operatorname{rk}(\Omega)$. First note that $MmM = Mm^\omega M$ implies that $m \in G_{m^\omega}$. Indeed, $mM = m^\omega M$ and $Mm = Mm^\omega$ by stability (Theorem 1.12). Thus $m \in m^\omega M m^\omega \cap J_{m^\omega} = G_{m^\omega}$ (using Corollary 1.15). It follows that $m$ and $m^\omega$ induce the same partition of $\Omega$ and so we may assume without loss of generality that $m \in E(M)$. Put $P_\alpha = m^{-1}(m\alpha)$. The proof of Theorem 14.4 shows that $|P_\alpha \cap x\Omega| = 1$ for all $x \in I(M)$ and hence $|P_\alpha \cap gm\Omega| = 1$ for all $g \in G$.

Let $\eta\colon \mathbb{C}\Omega \longrightarrow \mathbb{C}$ be the augmentation mapping and put

$$U = \frac{1}{|G|}\sum_{g\in G} g.$$

Then $U\beta = \frac{1}{n}[\Omega]$ for $\beta \in \Omega$ by Proposition B.10 and so $U[m\Omega] = \frac{r}{n}[\Omega]$ for all $v \in \mathbb{C}\Omega$. Therefore, we have

$$r\frac{|P_\alpha|}{n} = \langle \delta_{P_\alpha}, \frac{r}{n}[\Omega]\rangle = \langle \delta_{P_\alpha}, U[m\Omega]\rangle = \frac{1}{|G|}\sum_{g\in G}\langle \delta_{P_\alpha}, [gm\Omega]\rangle = 1$$

because $|P_\alpha \cap gm\Omega| = 1$ for all $g \in G$. This completes the proof. $\square$

It follows that if $|\Omega|$ is prime and $G$ is transitive, then either $I(M) = G$ or $I(M)$ consists of constant maps.

We end this section with a discussion of partial transformation modules. An *action* of a monoid $M$ on a set $\Omega$ by partial transformations is a partial mapping $A\colon M \times \Omega \longrightarrow M$, written $A(m,\omega) = m\omega$, satisfying $1\omega = \omega$ for all $\omega \in \Omega$ and $m(n\omega) = (mn)\omega$ for all $m, n \in M$ and $\omega \in \Omega$, where equality either means both sides are defined and equal or both sides are undefined.

A typical example is that if $\Lambda$ is an $M$-set and $\Delta$ is an $M$-invariant subset, then $M$ acts on $\Lambda \setminus \Delta$ by partial transformations via restriction of the action. Conversely, if $M$ acts on $\Omega$ by partial transformations and $\theta \notin \Omega$, then $\Omega \cup \{\theta\}$ is a left $M$-set via $m\theta = \theta$ and by putting $m\alpha = \theta$ whenever $m\alpha$ is undefined. Then $\{\theta\}$ is an $M$-invariant subspace and restricting the action to $\Omega$ recovers the original action by partial transformations.

For example, if $e \in E(M)$, then $Me \setminus L_e$ is an $M$-invariant subset of $Me$ and so $M$ acts on $L_e$ by partial transformations via restriction of the action.

If $M$ is a finite monoid acting on a finite set $\Omega$ by partial transformations and $\Bbbk$ is a field, then $\Bbbk\Omega$ becomes a $\Bbbk M$-module by putting

$$m \odot \alpha = \begin{cases} m\alpha, & \text{if } m\alpha \text{ is defined} \\ 0, & \text{else} \end{cases}$$

for $m \in M$ and $\alpha \in \Omega$. We call $\Bbbk\Omega$ a *partial transformation module*. For example, $\Bbbk L_e$ is a partial transformation module for $e \in E(M)$. We shall see later that if $M$ is $\mathscr{R}$-trivial, then each projective indecomposable $\Bbbk M$-module is a partial transformation module. If $\Omega = \Lambda \setminus \Delta$ where $\Lambda$ is an $M$-set and $\Delta$ is an $M$-invariant subset, then $\Bbbk\Omega \cong \Bbbk\Lambda/\Bbbk\Delta$.

## 14.3 The Černý conjecture

Let us say that $A \subseteq T_\Omega$ is *synchronizing* if $M_A = \langle A\rangle$ contains a constant mapping. Synchronizing subsets of $T_\Omega$ correspond to synchronizing automata with state set $\Omega$ in Computer Science. One can think of $\Omega$ as the set of states of the automaton and of $A$ as the input alphabet. See Exercise 13.6 for the

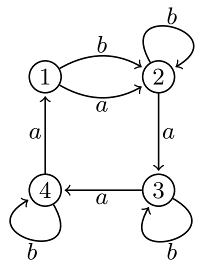



**Fig. 14.1.** The Černý example for $n = 4$

formal definition of an automaton. An automaton is synchronizing if some input sequence "resets" it to a fixed state, regardless of the original state of the automaton. A nice survey on synchronizing automata is [Vol08].

Let $A^*$ denote the free monoid on $A$. A word $w \in A^*$ is called a *reset word* for $A \subseteq T_\Omega$ if its image under the canonical surjection $A^* \longrightarrow M_A$ is a constant map. The following conjecture of Černý [Č64] has been a popular open problem in automata theory for a half-century.

*Conjecture 14.11 (Černý).* Let $A \subseteq T_\Omega$ be synchronizing and let $n = |\Omega|$. Then there is a reset word for $A$ of length at most $(n-1)^2$, that is, $A^{(n-1)^2}$ contains a constant mapping.

The current best upper bound for the length of a reset word is $(n^3 - n)/6$, due to Pin [Pin81]. The exercises sketch a simpler cubic upper bound. Černý himself provided the lower bound of $(n-1)^2$ with the following example.

*Example 14.12.* Let $a = (1\ 2\ \cdots n)$ and let $b$ be the idempotent sending 1 to 2 and fixing all other elements. Then Černý [Č64] proved that $b(a^{n-1}b)^{n-2}$, which has length $(n-1)^2$, is the unique minimum length reset word for $A = \{a, b\}$. See Figure 14.1 for the case $n = 4$.

In this section we give several examples of how representation theory has been used to successfully attack special cases of the Černý conjecture.

**Lemma 14.13.** *Let $(M, \Omega)$ be a transformation monoid and $\Bbbk$ a field. Let $\varepsilon\colon \Bbbk^\Omega \longrightarrow \Bbbk$ be given by*

$$\varepsilon(f) = \langle f, [\Omega]\rangle = \sum_{\omega\in\Omega} f(\omega).$$

*Then* $\ker\varepsilon$ *is a $\Bbbk M$-submodule if and only if $M$ is a group.*

*Proof.* If $m \in M$ is a permutation, then $\varepsilon(fm) = \langle fm, [\Omega]\rangle = \langle f, m[\Omega]\rangle = \langle f, [\Omega]\rangle = \varepsilon(f)$. It follows that if $M$ is a group, then $\varepsilon$ is a homomorphism of right $\Bbbk M$-modules, where $\Bbbk$ is viewed as the trivial right $\Bbbk M$-module, and hence $\ker \varepsilon$ is a submodule. If $m \in M$ is not a permutation, choose $\alpha \in \Omega$ with $|m^{-1}\alpha| \geq 2$. Then

$$\varepsilon\left(\delta_\alpha - \frac{1}{|\Omega|}\delta_\Omega\right) = 0$$

but

$$\varepsilon\left(\delta_\alpha m - \frac{1}{|\Omega|}\delta_\Omega m\right) = \varepsilon\left(\delta_{m^{-1}\alpha} - \frac{1}{|\Omega|}\delta_\Omega\right) = |m^{-1}\alpha| - 1 > 0$$

□

The following two lemmas are used frequently in conjunction to obtain bounds on the length of a reset word.

**Lemma 14.14.** *Let $A \subseteq T_\Omega$ be synchronizing. Suppose that there exists $k > 0$ such that, for each subset $S \subsetneq \Omega$ with $|S| \geq 2$, there exists $w \in A^*$ of length at most $k$ such that $|w^{-1}S| > |S|$. Then there is a reset word for $A$ of length at most $1 + k(n-2)$.*

*Proof.* Since $A$ is synchronizing, there exists $a \in A$ which is not a permutation. Then there exists $\alpha \in \Omega$ with $|a^{-1}\alpha| \geq 2$. Repeated application of the hypothesis results in a sequence $w_1, \ldots, w_s$, with $0 \leq s \leq n-2$, of words of length at most $k$ such that $|w_s^{-1}w_{s-1}^{-1} \cdots w_1^{-1}a^{-1}\alpha| = |\Omega|$. Then $aw_1 \cdots w_s$ is a reset word (with image $\alpha$) of length at most $1 + k(n-2)$. □

The next lemma is obvious, but important. It is the observation that a sum of non-positive numbers, not all of which are 0, cannot be 0.

**Lemma 14.15.** *Let $A \subseteq T_\Omega$ and $S \subsetneq \Omega$. Let $X \subseteq A^*$ be finite. If*

$$\sum_{w \in X} (|w^{-1}S| - |S|) = 0$$

*and $|u^{-1}S| - |S| \neq 0$ for some $u \in X$, then $|v^{-1}S| > |S|$ for some $v \in X$.*

Motivated by the Černý examples, it is natural to consider the conjecture in the special case that one of the transformations is a cyclic permutation. The following theorem is due to Pin [Pin78], which was the first result along this line. The proof here follows [AS06].

**Theorem 14.16.** *Let $|\Omega|$ be a prime $p$. Suppose that $a \in S_\Omega$ is a $p$-cycle and $B \subseteq T_\Omega$ with $B \nsubseteq S_\Omega$. Then $A = \{a\} \cup B$ is synchronizing and there exists a reset word of length at most $(p-1)^2$.*

*Proof.* Let $G = \langle a \rangle$ and $M = \langle A \rangle$. We know that $A$ is synchronizing by Theorem 14.9. As $(p-1)^2 = 1 + p(p-2)$, it suffices to show by Lemma 14.14 that if $S \subsetneq \Omega$ with $|S| \geq 2$, there exists $w \in A^*$ of length at most $p$ such that $|w^{-1}S| > |S|$.

Consider the right $\mathbb{Q}M$-module $\mathbb{Q}^{\Omega}$ and the mapping $\varepsilon\colon \mathbb{Q}^{\Omega} \longrightarrow \mathbb{Q}$ given by $\varepsilon(f) = \sum_{\alpha\in\Omega} f(\alpha)$. Then $V = \ker\varepsilon$ is a $\mathbb{Q}G$-submodule, by Lemma 14.13, of dimension $p-1$. Recall from the proof of Theorem 14.9 that $\mathbb{Q}G$ has two simple modules: the trivial module and one of dimension $p-1$. Clearly $(\delta_\alpha - \frac{1}{p}\delta_\Omega)a = \delta_{a^{-1}\alpha} - \frac{1}{p}\delta_\Omega$ and hence $\ker\varepsilon$ is not a direct sum of copies of the trivial module. Thus $V$ is isomorphic to the non-trivial simple $\mathbb{Q}G$-module. Note that identifying $\mathbb{Q}G$ with

$$\mathbb{Q}[x]/(x^p-1) \cong \mathbb{Q}[x]/(x-1) \times \mathbb{Q}[x]/(1+x+\cdots+x^{p-1})$$

we see that the non-trivial simple $\mathbb{Q}G$-module is $\mathbb{Q}[x]/(1+x+\cdots+x^{p-1})$. It follows that $1+a+\cdots+a^{p-1}$ annihilates $V$. Since $V$ is not a $\mathbb{Q}M$-submodule by Lemma 14.13, there exists $b\in B$ with $Vb \nsubseteq V$.

We claim that if $S \subsetneq \Omega$ with $|S| \geq 2$, then $|b^{-1}a^{-k}S| > |S|$ for some $0 \leq k \leq p-1$. Indeed, put

$$\gamma_S = \delta_S - \frac{|S|}{|\Omega|}\delta_\Omega \in V.$$

As $V$ is a simple $\mathbb{Q}G$-module, we must have $V = \mathbb{Q}\{\gamma_S a^i \mid 0 \leq i \leq p-1\}$. Thus there exists $0 \leq i \leq p-1$ with $\gamma_S a^i b \notin V$. Now, observe that

$$\gamma_S a^i b = \delta_{b^{-1}a^{-i}S} - \frac{|S|}{|\Omega|}\delta_\Omega \notin V$$

and hence $0 \neq \varepsilon(\gamma_S a^i b) = |b^{-1}a^{-i}S| - |S|$.

Using that $1+a+\cdots+a^{p-1}$ annihilates $V$ we have that

$$0 = \varepsilon\left(\gamma_S \sum_{j=0}^{p-1} a^j b\right) = \sum_{j=0}^{p-1}\left(|b^{-1}a^{-j}S| - |S|\right).$$

Thus there exists $0 \leq k \leq p-1$ with $|b^{-1}a^{-k}S| > |S|$ by Lemma 14.15. This completes the proof in light of Lemma 14.14. $\square$

Černý's conjecture was proved in general for the case where some element of $A$ is a cyclic permutation by Dubuc [Dub98].

The next result we present is due to Kari [Kar03]. We shall require a well-known combinatorial lemma about representations of free monoids.

**Lemma 14.17.** *Let $\Bbbk$ be a field and $A$ a set. Suppose that $V$ is a finite dimensional right $\Bbbk A^*$-module and that $W \subseteq U$ are subspaces with $WA^* \nsubseteq U$. Let $\{w_1,\ldots,w_k\}$ be a spanning set for $W$. Then there exist $x \in A^*$ of length at most* $\dim U - \dim W + 1$ *and $i \in \{1,\ldots,k\}$ such that $w_i x \notin W$.*

*Proof.* Let $A^{\leq i} = \{x \in A^* \mid |x| \leq i|\}$ for $i \geq 0$ be the set of words in $A^*$ of length at most $i$. Put $W_i = WA^{\leq i}$. Then we have a chain of subspaces

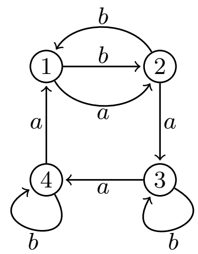



**Fig. 14.2.** An Eulerian automaton

$$W = W_0 \subseteq W_1 \subseteq \cdots$$

whose union is $WA^* \nsubseteq U$. Moreover, if $W_i = W_{i+1}$, then $W_i$ is invariant under $A$, and hence $A^*$, and so $W_i = WA^* \nsubseteq U$. Thus if $j$ is maximal with $W_j \subseteq U$, then

$$W = W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_j \subseteq U$$

and so $j \leq \dim U - \dim W$. But then there exists $x \in A^{\leq j+1}$ and $w_i$ such that $w_i x \notin U$ because $W_{j+1} \nsubseteq U$. This completes the proof. $\square$

A subset $A \subseteq T_\Omega$ is called *Eulerian* if, for each $\alpha \in \Omega$, one has that

$$\sum_{a \in A} |a^{-1}\alpha| = |A|. \tag{14.1}$$

The reason for the terminology is that if we form a digraph by connecting $\alpha$ to $a\alpha$ for each $a \in A$, then condition (14.1) says that we obtain an Eulerian digraph (i.e., a digraph with a directed Euler circuit). See Figure 14.2

**Theorem 14.18.** *Let $A \subseteq T_\Omega$ be Eulerian and synchronizing with $|\Omega| = n$. Then there is a reset word of length at most $n^2 - 3n + 3 \leq (n-1)^2$.*

*Proof.* Let $M = \langle A \rangle$ and $P = \sum_{a \in A} a \in \mathbb{Q}M$. Put $k = |A|$. Recall that $\varepsilon\colon \mathbb{Q}^\Omega \longrightarrow \mathbb{Q}$ is given by $\varepsilon(f) = \sum_{\omega \in \Omega} f(\omega) = \langle f, [\Omega] \rangle$. We compute that

$$P[\Omega] = \sum_{a \in A} \sum_{\omega \in \Omega} a\omega = \sum_{\alpha \in \Omega} \sum_{a \in A} |a^{-1}\alpha|\alpha = k[\Omega]$$

by (14.1). Thus, for any mapping $f \in \mathbb{Q}^\Omega$ and $m \geq 0$, we have

$$\varepsilon(fP^m) = \langle fP^m, [\Omega] \rangle = \langle f, P^m[\Omega] \rangle = k^m \langle f, [\Omega] \rangle = k^m \varepsilon(f). \tag{14.2}$$

We now claim that if $S \subsetneq \Omega$ with $|S| \geq 2$, then there exists a word $v$ of length at most $n-1$ such that $|v^{-1}S| > |S|$. It will then follow by Lemma 14.14 that there is a reset word of length $1 + (n-1)(n-2) = n^2 - 3n + 3$, as required.

Let $U = \ker \varepsilon$ and put $\gamma_S = \delta_S - \frac{|S|}{n}\delta_\Omega \in U$. Let $W = \mathbb{Q}\gamma_S$. If $w \in A^*$ is a reset word, then $w^{-1}S = \Omega$ and so

$$\gamma_S w = \delta_{w^{-1}S} - \frac{|S|}{n}\delta_\Omega = \left(1 - \frac{|S|}{n}\right)\delta_\Omega \notin \ker \varepsilon = U.$$

Applying Lemma 14.17, there is $u \in A^*$ with $|u| \leq \dim U - \dim W + 1 = n - 1$ such that $\gamma_S u \notin U$. But then $0 \neq \varepsilon(\gamma_S u) = |u^{-1}S| - |S|$.

Put $j = |u|$. Then we have by, (14.2), that

$$0 = k^j \varepsilon(\gamma_S) = \varepsilon(\gamma_S P^j) = \sum_{|w|=j} \varepsilon(\gamma_S w) = \sum_{|w|=j} (|w^{-1}S| - |S|).$$

As $|u| = j$ and $0 \neq |u^{-1}S| - |S|$, we conclude that there exists $v$ of length $j \leq n - 1$ with $|v^{-1}S| > |S|$ by Lemma 14.15. This completes the proof. □

Our final application of representation theory to the Černý problem is a result from [AS09], generalizing an earlier result from [AMSV09].

**Lemma 14.19.** *Let $M$ be an $A$-generated finite monoid all of whose regular $\mathscr{J}$-classes are subsemigroups and $\Bbbk$ a field. Let $V$ be a simple $\Bbbk M$-module and suppose that $mV = 0$ for some $m \in M$. Then $aV = 0$ for some $a \in A$.*

*Proof.* Let $e \in E(M)$ be an apex for $V$. Then $I(e)$ is a prime ideal by Proposition 2.2 and $m \in I(e)$. If $m = a_1 \cdots a_r$ with $a_1, \ldots, a_r \in A$, then $a_i \in I(e)$ for some $i$ because $I(e)$ is prime. But then $a_i V = 0$. □

As a consequence, one obtains the following generalization to arbitrary finite dimensional modules.

**Theorem 14.20.** *Let $M$ be a finite $A$-generated monoid in which each regular $\mathscr{J}$-class is a subsemigroup and let $\Bbbk$ be a field. Let $V$ be a $\Bbbk M$-module of length $r$ and suppose $mV = 0$ for some $m \in M$. Then there is a word $w$ over $A$ of length at most $r$ such that $wV = 0$.*

*Proof.* We induct on the length of $V$, the case $r = 1$ being handled by Lemma 14.19. Suppose that $V$ has length $r > 1$. Then there is a maximal submodule $W$ of $V$ such that $W$ has length $r - 1$ and $V/W$ is simple. Since $m(V/W) = 0$, there exists $a \in A$ such that $a(V/W) = 0$, i.e., $aV \subseteq W$ by Lemma 14.19. Since $mW = 0$, there exists by induction a word $u$ of length at most $r - 1$ such that $uW = 0$. Then $uaV \subseteq uW = 0$. As $|ua| \leq r$, this completes the proof. □

As a corollary, we obtain the following Černý-type result.

**Corollary 14.21.** *Let $A \subseteq T_\Omega$ be such that each regular $\mathscr{J}$-class of $M = \langle A \rangle$ is a subsemigroup. Let $r = \operatorname{rk}(\Omega)$ and $n = |\Omega|$. Then there is a word $w \in A^*$ of length at most $n - r$ such that $w$ represents an element of $M$ of rank $r$. In particular, if $A$ is synchronizing, then there is a reset word of length at most $n - 1$.*

*Proof.* Assume first that $M$ is transitive on $\Omega$. Consider the $\mathbb{C}M$-module $\mathbb{C}\Omega$. Let $V = \mathbb{C}\{[m\Omega] - [n\Omega] \mid m, n \in I(M)\}$. Then $V$ is a submodule of dimension at most $n-r$ and $mV = 0$ if and only if $\mathrm{rk}(m) = r$ by Theorem 14.4 and Lemma 14.2. As the length of a module is bounded by its dimension, Theorem 14.20 applied to $V$ provides the required conclusion.

Next suppose that $M$ is not transitive, but that $\Omega$ decomposes as a disjoint union $\Omega = \Omega_1 \cup \cdots \cup \Omega_s$ of transitive $M$-sets $\Omega_i$ for $i = 1, \ldots, s$. Put $r_i = r(\Omega_i)$ and $n_i = |\Omega_i|$. Then one easily checks that

$$n = n_1 + \cdots + n_s \quad \text{and} \quad r = r_1 + \cdots + r_s$$

because elements of $I(M)$ have minimal rank on each $\Omega_i$ by Lemma 14.2. Let $M_i$ be the quotient of $M$ that acts faithfully on $\Omega_i$. By Exercise 1.21, each regular $\mathscr{J}$-class of $M_i$ is a subsemigroup. By the previous case, we can find $w_i \in A^*$ of length at most $n_i - r_i$ with $|w_i\Omega_i| = r_i$ for $i = 1, \ldots, s$. Then putting $w = w_1 \cdots w_s$, we have that

$$|w| \leq (n_1 + \cdots + n_s) - (r_1 + \cdots + r_s) = n - r$$

and

$$|w\Omega| = \sum_{i=1}^{s} |w\Omega_i| = \sum_{i=1}^{s} r_i = r$$

where the penultimate equality uses Proposition 14.1.

Finally, we handle the general case. Let $\Omega' = I(M)\Omega$. Note that $\Omega'$ is $M$-invariant and that $r(\Omega) = r(\Omega')$ as a consequence of Lemma 14.2. Let $n' = |\Omega'|$. We claim that $\Omega'$ decomposes as a disjoint union of transitive actions of $M$. Indeed, if $\alpha \in \Omega'$ and $\beta = m\alpha$ with $m \in M$, then we can find $\gamma \in \Omega$ and $x \in I(M)$ with $x\gamma = \alpha$. Then $mx \in I(M)$ and so $mx \in L_x$ by stability (Theorem 1.12) because $I(M)$ is a $\mathscr{J}$-class. Thus there exists $m' \in M$ with $m'mx = x$ and so $m'\beta = m'm\alpha = m'mx\gamma = x\gamma = \alpha$. Thus $M\alpha$ is a transitive $M$-set for $\alpha \in \Omega'$. It follows that the distinct sets of the form $M\alpha$ with $\alpha \in \Omega'$ partition $\Omega'$ into transitive $M$-invariant subsets. Let $M'$ be the quotient of $M$ acting faithfully on $\Omega'$. Then each regular $\mathscr{J}$-class of $M'$ is a subsemigroup by Exercise 1.21. Thus by the previous case, we can find $w \in A^*$ with $|w| \leq n' - r$ and $|w\Omega'| = r$.

Consider now the module $V = \mathbb{C}\Omega/\mathbb{C}\Omega'$. Then $xV = 0$ for $x \in I(M)$ by construction. Thus, by Theorem 14.20, there exists $u \in A^*$ with $|u| \leq \dim V = n - n'$ such that $uV = 0$, that is, such that $u\Omega \subseteq \Omega'$. Therefore, we have that $r \leq |wu\Omega| \leq |w\Omega'| = r$ and $|wu| \leq n' - r + n - n' = n - r$. This completes the proof. ◻

Let $1 \leq r \leq n$ and define a mapping $a\colon [n] \longrightarrow [n]$ by

$$a(x) = \begin{cases} x + 1, & \text{if } 1 \leq x < n \\ n - r + 1, & \text{if } x = n. \end{cases}$$

Consideration of $A = \{a\} \subseteq T_n$ shows that Corollary 14.21 is sharp.

## 14.4 Exercises

**14.1.** Suppose that $M$ acts transitively on a set $\Omega$ and $e \in E(M)$. Prove that $eMe$ acts transitively on $e\Omega$. Deduce that if $e \in E(I(M))$, then $G_e$ acts transitively on $e\Omega$.

**14.2.** Suppose that $(M, \Omega)$ is a primitive transformation monoid and let $e \in E(I(M))$. Prove that $(G_e, e\Omega)$ is a primitive permutation group.

**14.3.** Let $A \subseteq T_\Omega$ with $|\Omega| = n$.

(a) Show that $A$ is synchronizing if and only if, for any pair of distinct elements $\alpha, \beta \in \Omega$, there exists $w \in A^*$ with $w\alpha = w\beta$.
(b) Prove that if $A$ is synchronizing, then one can always choose $w$ in (a) so that $|w| \leq \binom{n}{2}$. (Hint: consider the action of $A^*$ by partial transformations on the set of pairs of elements of $\Omega$ and use the pigeonhole principle.)
(c) Deduce that there is a reset word of length at most $1 + (n-2)\binom{n}{2}$ for $A$.

**14.4.** Prove that $b(a^{n-1}b)^{n-2}$ is the unique minimal length reset word for Černý's example (Example 14.12).

**14.5.** Prove that the Černý conjecture is true if and only if it is true for all synchronizing $A \subseteq T_\Omega$ with $M_A = \langle A \rangle$ transitive on $\Omega$.

**14.6.** Let us say that $A \subseteq T_\Omega$ is pseudo-Eulerian if there is a strictly positive mapping $f\colon A \longrightarrow \mathbb{R}$ such that

$$\sum_{a \in A} f(a)[\Omega] = \lambda[\Omega]$$

for some $\lambda \in \mathbb{R}$.

(a) Prove that if $A$ is Eulerian, then $A$ is pseudo-Eulerian.
(b) Prove that if $A \subseteq T_\Omega$ is pseudo-Eulerian and synchronizing with $|\Omega| = n$, then there is a reset word of length at most $n^2 - 3n + 3$.
(c) Give an example of a synchronizing subset $A \subseteq T_\Omega$ that is pseudo-Eulerian but not Eulerian for some $\Omega$.

**14.7.** Let $G$ be a finite group of order $n$ and let $G = \langle B \rangle$. Let $A \subseteq T_G$ be such that $B \subseteq A$ and $A$ is synchronizing, where we view $G \subseteq S_G$ via the regular representation. Prove that there is a reset word of length at most $2n^2 - 6n + 5$. (Hint: use that $\sum_{g \in G} g$ annihilates $\ker \varepsilon$ (defined as per Lemma 14.13) and that each element of $G$ can be represented by a word in $B$ of length at most $n - 1$.)

**14.8 (Rystsov).** Let $G$ be a subgroup of $S_n$ and let $e \in E(T_n)$ have rank $n-1$ where $e(i) = e(j)$ for $i < j$. Prove that $G \cup \{e\}$ is synchronizing if and only if the graph $\Gamma$ with vertex set $[n]$ and edge set consisting of all pairs $\{g(i), g(j)\}$ with $g \in G$ is connected.

# 15

# Markov Chains

A Markov chain is a stochastic process on a finite state set such the system evolves from one state to another according to a prescribed probabilistic law. For example, card shuffling can be modelled via a Markov chain. The state set is all 52! orderings of a deck of cards. Each step of the Markov chain corresponds to performing a riffle shuffle to the deck.

The analysis of Markov chains using group representation theory was pioneered to a large extent by Diaconis and the reader is encouraged to look at his beautiful lecture notes [Dia88]. The reader may also consult [CSST08] or [Ste12a, Chapter 11]. The usage of techniques from monoid theory to study Markov chains is a relatively new subject. Work in this area began in a seminal paper of Bidigare, Hanlon and Rockmore [BHR99] on hyperplane walks, and was continued by Brown, Diaconis, Björner, Athanasiadis, Chung and Graham, amongst others [BD98, BBD99, Bro00, Bro04, Bjö08, Bjö09, AD10, CG12, Sal12]. Most of these works concern left regular bands, that is, von Neumann regular $\mathscr{R}$-trivial monoids. In his 1998 ICM lecture [Dia98], Diaconis highlighted these developments and in Section 4.1, entitled *What is the ultimate generalization?*, essentially asked how far can the monoid techniques be taken.

Recently, the author [Ste06, Ste08], Ayyer, Klee, and Schilling [AKS14a, AKS14b], and the author with Ayyer, Schilling and Thiéry in [ASST15a, ASST15b] have extended the techniques beyond left regular bands.

In this chapter we only touch on the basics of this topic. In particular, we focus on the computation of the eigenvalues for the transition matrix of the Markov chain and diagonalizability since it is in this aspect that representation theory most directly intervenes. A good reference on finite state Markov chain theory is [LPW09].

## 15.1 Markov Chains

A *probability* on a finite set $X$ is a mapping $P\colon X \longrightarrow \mathbb{R}$ such that:

(i) $P(x) \geq 0$ for all $x \in X$;
(ii) $\sum_{x \in X} P(x) = 1$.

The *support* of $P$ is $\operatorname{supp}(P) = \{x \in X \mid P(x) \neq 0\}$. We shall systematically identify $P$ with the element $\sum_{x \in X} P(x)x \in \mathbb{C}X$.

A *Markov chain* $\mathscr{M} = (T, \Omega)$ consists of a finite set $\Omega$, called the *state set*, and a *transition matrix* $T \colon \Omega \times \Omega \longrightarrow \mathbb{R}$ satisfying the following properties:

(i) $T(\alpha, \beta) \geq 0$ for all $\alpha, \beta \in \Omega$;
(ii) $\sum_{\alpha \in \Omega} T(\alpha, \beta) = 1$ for all $\beta \in \Omega$.

One interprets $T(\alpha, \beta)$ to be the probability of going from state $\beta$ to state $\alpha$ in one step of the Markov chain[1]. We view $T$ as an operator on $\mathbb{C}\Omega$ via the usual rule

$$T\beta = \sum_{\alpha \in \Omega} T(\alpha, \beta)\alpha$$

for $\beta \in \mathbb{C}\Omega$. Then one can interpret $T^n(\alpha, \beta)$ as the probability of going from state $\beta$ to state $\alpha$ in exactly $n$ steps of the chain. Since we are interested in the dynamics of the operator $T$, it is natural to try and compute the eigenvalues of $T$. We shall use representation theory as a tool.

If $\mathscr{M} = (T, \Omega)$ is a Markov chain and $\nu$ is a probability on $\Omega$, which we think of as an initial distribution, then $T\nu$ is another probability on $\Omega$, as the reader easily checks. One says that $\nu$ is a *stationary distribution* if $T\nu = \nu$. Under mild hypotheses, there is a unique stationary $\pi$ and $T^n\mu$ converges to $\pi$ for any initial distribution $\mu$. One is typically interested in the rate of convergence. For example, the Markov chain might be a model of shuffling a deck of cards and the rate of convergence is how quickly the deck mixes. We shall not, however, focus on this topic here. There are some monoid theoretic techniques for bounding mixing times for monoid random walks in [ASST15b, Section 3], but they do not rely so much on representation theory. In the case of groups, there are Fourier analytic techniques for bounding mixing times, cf. [Dia88].

## 15.2 Random walks

Let $M$ be a finite monoid, $P$ a probability on $M$ and $\Omega$ a finite left $M$-set. The *random walk* of $M$ on $\Omega$ driven by $P$ is the Markov chain $\mathscr{M} = (T, \Omega)$ with transition matrix $T \colon \Omega \times \Omega \longrightarrow \mathbb{R}$ given by

$$T(\alpha, \beta) = \sum_{\{m \in M \mid m\beta = \alpha\}} P(m).$$

[1] In probability theory, it is traditional to use the transpose of what we are calling the transition matrix. However, because we are using left actions and left modules, it is more natural for us to use this formulation.

In other words, $T(\alpha,\beta)$ is the probability that acting upon $\beta$ with a randomly chosen element of $M$, distributed according to $P$, results in $\alpha$.

**Proposition 15.1.** *The pair $(T,\Omega)$ constructed above is a Markov chain.*

*Proof.* Trivially, $T(\alpha,\beta) \geq 0$ for all $\alpha,\beta \in \Omega$. Fix $\beta \in \Omega$. Then we compute

$$\sum_{\alpha\in\Omega} T(\alpha,\beta) = \sum_{\alpha\in\Omega}\sum_{\{m\in M\mid m\beta=\alpha\}} P(m) = \sum_{m\in M} P(m) = 1$$

because, for each $m \in M$, there is exactly one $\alpha \in \Omega$ with $\alpha = m\beta$. We conclude that $(T,\Omega)$ is a Markov chain. $\square$

It is worth noting that every Markov chain $(T,\Omega)$ can be obtained as a random walk of the full transformation monoid $T_\Omega$ on $\Omega$ with respect to an appropriate probability on $T_\Omega$. This is a reformulation of the fact that the extreme points of the polytope of stochastic matrices are the column monomial matrices. We do not give a proof here but refer the reader to [ASST15b, Theorem 2.3].

If $\Omega$ is a left $M$-set, then $\mathbb{C}\Omega$ is a $\mathbb{C}M$-module as per Chapter 14. If we identify a probability $P$ on $M$ with the element

$$\sum_{m\in M} P(m)m$$

of $\mathbb{C}M$, then $P$ is an operator on $\mathbb{C}\Omega$. It turns out that the transition matrix $T$ of the random walk of $M$ on $\Omega$ driven by $P$ is the matrix of the operator $P$ with respect to the basis $\Omega$. This is what underlies the applications of representation theory to Markov chains: $\mathbb{C}M$-submodules of $\mathbb{C}\Omega$ are $T$-invariant subspaces.

**Proposition 15.2.** *Let $P$ be a probability on $M$ and $\Omega$ a left $M$-set. Then the transition matrix $T$ of the random walk of $M$ on $\Omega$ driven by $P$ is the matrix of $P = \sum_{m\in M} P(m)m$ acting on $\mathbb{C}\Omega$ with respect to the basis $\Omega$.*

*Proof.* If $\beta \in \Omega$, then we compute

$$P\beta = \sum_{m\in M} P(m)m\beta = \sum_{\alpha\in\Omega}\sum_{\{m\in M\mid m\beta=\alpha\}} P(m)\alpha = \sum_{\alpha\in\Omega} T(\alpha,\beta)\alpha$$

as required. $\square$

## 15.3 Examples

In this section we describe several famous Markov chains and discuss how to model them as monoid random walks. In Section 15.6, we shall reprise these examples and perform a detailed analysis using the theory that we shall develop. The examples in this section will all be left regular bands walks. The theory of left regular band walks was developed by Brown [Bro00, Bro04], following the ground breaking work of Bidigare, Hanlon and Rockmore [BHR99].

We recall that a left regular band is an $\mathscr{R}$-trivial monoid $M$ in which each element is regular. In this case each element of $M$ is idempotent and

$$\Lambda(M) = \{MmM \mid m \in M\} = \{Mm \mid m \in M\}$$

is a lattice with intersection as the meet. See Proposition 2.8 for details.

### 15.3.1 The Tsetlin library

The *Tsetlin library* is a well-studied Markov chain [Hen72, Pha91, DF95, FH96, Fil96, BHR99]. This was the first Markov chain analyzed via left regular band techniques. Imagine that you have a shelf books that you wish to keep organized, but no time to do it. A simple self-organizing system is to always replace a book at the front of the shelf after using it. In this way, over time, your favorite books will be at the front of the shelf and your least favorite books at the back. Therefore, if you know the relative frequency with which you use a book, then you will know roughly where to find it.

Formally speaking, you have a set $B$ (of books) and a probability $P$ on $B$. The states of the Markov chain are the $|B|!$ linear orderings of the set of books $B$, i.e., the set of possible orderings of the books on the shelf. At each step of the chain, you choose a book $b$ with probability $P(b)$ and place book $b$ at the front of the shelf.

Let us remark that this Markov chain has a card shuffling interpretation in the special case that $P$ is the uniform distribution, that is, $P(b) = 1/|B|$ for all $b \in B$. One can then think of $B$ as the set of cards in a deck and the state set as the set of all possible orderings of the cards. One shuffles the deck by randomly choosing a card and placing it at the top. This is sometimes called the *random-to-top shuffle*. It is the time reversal of the *top-to-random shuffle* where you shuffle a deck by putting the top card in a random position. The transition matrix of the top-to-random shuffle is the transpose of that of the random-to-top shuffle and these chains have the same eigenvalues and mixing time. See [LPW09, Section 4.6] for details.

Let us now model the Tsetlin library as a left regular band random walk. To do this we introduce the *free left regular band* $F(B)$ on the set $B$. The left regular band identity $xyx = xy$ says that one can always remove repetitions from any product. Since the free monoid $B^*$ consists of all words over the alphabet $B$, including the empty one, it follows that $F(B)$ should consist of all words $w \in B^*$ with no repeated letters. The product is $u \odot v = \overline{uv}$ where if $w \in B^*$, then $\overline{w}$ is the result of removing all repetitions of letters as you scan $w$ from left to right. For example, if $B = \{1, 2, 3, 4\}$, then $3 \odot 1342 = 3142$. Notice that left multiplying a word containing all the letters by a generator moves that generator to the front.

Let $\mathscr{P}(B)$ be the power set of $B$, viewed as a monoid via the operation of union. In other words, we view $\mathscr{P}(B)$ as a lattice with respect to the ordering

$\supseteq$. Then there is a monoid homomorphism $c\colon F(B) \longrightarrow \mathscr{P}(B)$ sending a word $w$ to its set $c(w)$ of letters, called the *content* of $w$. For example, $c(312) = \{1,2,3\}$. Let $\Omega_B = \{w \in F(B) \mid c(w) = B\}$ be the set of words with full content.

**Proposition 15.3.** *Let $B$ be a finite set.*

*(i) $uF(B) \subseteq vF(B)$ if and only if $v$ is a prefix of $u$.*
*(ii) $F(B)u \subseteq F(B)v$ if and only if $c(u) \supseteq c(v)$.*

*Proof.* If $v$ is a prefix of $u$, then $u = vx = \overline{vx}$ in $B^*$ and hence $u = v \odot x$. Conversely, if $v \odot x = u$, then since $v$ has no repeated letters, $u = \overline{vx} = vx'$ where $x'$ is obtained from $x$ by deleting some letters. This establishes (i).

To prove (ii), if $u = x \odot v$, then $c(u) = c(x) \cup c(v)$ and hence $c(u) \supseteq c(v)$. Conversely, if $c(u) \supseteq c(v)$, then $u \odot v = \overline{uv} = u$ because each letter of $v$ is a repetition of a letter occurring in $u$. Thus $F(B)u \subseteq F(B)v$, as required. $\square$

As a consequence we deduce that $F(B)$ is indeed a left regular band, and more.

**Corollary 15.4.** *Let $B$ be a finite set. Then $F(B)$ is a left regular band and $(\Lambda(F(B)), \subseteq) \cong (\mathscr{P}(B), \supseteq)$ via $F(B)uF(B) \mapsto c(u)$. Consequently, $\Omega_B$ is the minimal ideal of $F(B)$.*

*Proof.* The first item of Proposition 15.3 implies that $F(B)$ is $\mathscr{R}$-trivial. Trivially, $u \odot u = \overline{uu} = u$ and so $F(B)$ is a left regular band. The identification of $\Lambda(F(B))$ with $\mathscr{P}(B)$ is immediate from the second item of Proposition 15.3 because $MmM = Mm$ in a left regular band $M$. The final statement follows from this identification. $\square$

We now verify that the Tsetlin library is the random walk of $F(B)$ on $\Omega_B$ driven by $P$.

**Proposition 15.5.** *The Tsetlin library Markov chain with set of books $B$ and probability $P$ on $B$ is the random walk of $F(B)$ on $\Omega_B$ driven by $P$.*

*Proof.* Clearly, we can identify $\Omega_B$ with all linear orderings of $B$. The action of a generator $b \in B$ on a word $w \in \Omega_B$ is given by $b \odot w = \overline{bw}$. But $\overline{bw}$ is obtained from $w$ by placing $b$ at the front and removing the occurrence of $b$ in $w$. The proposition then follows. $\square$

### 15.3.2 The inverse riffle shuffle

The standard way of shuffling a deck of cards, by dividing it roughly in half and then interleaving the two halves, is called *riffle shuffling*. The Gilbert-Shannon-Reeds model is a classical Markov chain model of this process (cf. [LPW09, Section 8.3]). In this model, the number of cards in the top half of the deck (after dividing in two) is binomially distributed and then all interleavings are considered equally likely. The time reversal of the riffle shuffle is the *inverse riffle shuffle*. Its transition matrix is the transpose of the transition matrix for the riffle shuffle and they have the same eigenvalues and mixing time. In fact, Bayer and Diaconis used the inverse riffle shuffle to analyze the mixing time of the riffle shuffle in their famous paper [BD92], which established the "7-shuffle" rule of thumb. Let us describe the inverse riffle shuffle in detail since it is a left regular band random walk [BHR99, Bro04].

One has a deck of $n$ cards. The state set is the set of all $n!$ possible orderings of the cards. One shuffles by randomly choosing a subset $A$ of cards (with all subsets equally likely) and moving the cards from $A$ to the front while maintaining the relative ordering of the cards in $A$ (and its complement). Note that if either $A = [n]$ or $A = \emptyset$, the result is to stay at the current state. You should think of $A$ as the set of cards which formed the top half of the deck before interleaving to see this process as performing the riffle shuffle backwards in time. For example, if there are 4 cards in the order 1432 and we choose the subset $A = \{2, 4\}$, then the new ordering of the deck is 4213.

To analyze this Markov chain, we use a combinatorial model of the hyperplane monoid used in Bidigare, Hanlon and Rockmore [BHR99]. Let $\Sigma_n$ be the set of all ordered set partitions of $[n] = \{1, \ldots, n\}$. In other words, an element of $\Sigma_n$ is an $r$-tuple $(P_1, \ldots, P_r)$ of subsets of $[n]$ such that $\{P_1, \ldots, P_r\}$ is a set partition. We define a product by putting

$$(P_1, \ldots, P_r)(Q_1, \ldots, Q_s) = (P_1 \cap Q_1, \ldots, P_1 \cap Q_s, \cdots, P_r \cap Q_1, \ldots, P_r \cap Q_s)^\wedge$$

where if $R_1, \ldots, R_t$ are subsets of $[n]$, then $(R_1, \ldots, R_t)^\wedge$ is the result of removing all empty $R_i$ from the list. For example, if $n = 4$, then

$$(\{2, 4\}, \{1, 3\})(\{1\}, \{4\}, \{3\}, \{2\}) = (\{4\}, \{2\}, \{1\}, \{3\}). \tag{15.1}$$

The reader should check that $\Sigma_n$ is a monoid with identity $([n])$. Let $\Omega_n$ be the set of all ordered set partitions of $[n]$ into singletons. Notice that $\Omega_n$ can be identified with the set of linear orderings of $[n]$. The computation (15.1) indicates how the inverse riffle shuffle can be modelled as a random walk on $\Sigma_n$.

Let $\Pi_n$ denote the lattice of set partitions of $[n]$. As usual, a partition $\{P_1, \ldots, P_r\}$ is smaller than a partition $\{Q_1, \ldots, Q_s\}$ if, for each $i$, there exists $j$ (necessarily unique) with $P_i \subseteq Q_j$. The meet $\{P_1, \ldots, P_r\} \wedge \{Q_1, \ldots, Q_s\}$ is the partition whose blocks consist of all non-empty intersections of the form $P_i \cap Q_j$.

We can define $c\colon \Sigma_n \longrightarrow \Pi_n$ by $c((P_1, \ldots, P_r)) = \{P_1, \ldots, P_r\}$. From the above description of the meet in $\Pi_n$, it is immediate that $c$ is a homomorphism.

**Lemma 15.6.** *Let* $(P_1, \ldots, P_r)$ *and* $(Q_1, \ldots, Q_s)$ *be elements of* $\Sigma_n$. *Then*

$$(P_1, \ldots, P_r)(Q_1, \ldots, Q_s) = (P_1, \ldots, P_r) \tag{15.2}$$

*if and only if* $c((P_1, \ldots, P_r)) \leq c((Q_1, \ldots, Q_s))$.

*Proof.* If (15.2) holds, then $c((P_1,\dots,P_r))\wedge c((Q_1,\dots,Q_s)) = c((P_1,\dots,P_r))$, giving the desired inequality. Conversely, if $c((P_1,\dots,P_r)) \leq c((Q_1,\dots,Q_s))$, then, for each $i$, either $P_i \cap Q_j = \emptyset$ or $P_i \cap Q_j = P_i$ and the equality holds for exactly one index $j$. With this in mind, we easily deduce (15.2) from the definition of the product in $\Sigma_n$. □

We obtain as a consequence that $\Sigma_n$ is a left regular band.

**Corollary 15.7.** *Let $n \geq 1$.*

*(i) $\Sigma_n$ is a left regular band.*
*(ii) One has*
$$\Sigma_n(P_1,\dots,P_r) \leq \Sigma_n(Q_1,\dots,Q_s)$$
*if and only if $c((P_1,\dots,P_r)) \leq c((Q_1,\dots,Q_s))$.*
*(iii) The map $c$ induces an isomorphism $MmM \mapsto c(m)$ of $\Lambda(\Sigma_n)$ and $\Pi_n$.*
*(iv) $\Omega_n$ is the minimal ideal of $\Sigma_n$.*

*Proof.* For (i) we check the identity $xyx = xy$ from Lemma 2.7. Since $c$ is a homomorphism, it is obvious that if $x, y \in \Sigma_n$, then $c(xy) \leq c(x)$. Therefore, $xyx = xy$ by Lemma 15.6. This proves (i). Lemma 15.6 is equivalent to (ii) because $\Sigma_n$ is a left regular band. Since $MmM = Mm$ in a left regular band, (iii) follows from (ii). Finally, (iv) may be deduced from (iii) because the partition into singletons is the smallest element of $\Pi_n$. □

We are now prepared to prove that the inverse riffle shuffle is a random walk of $\Sigma_n$ on $\Omega_n$.

**Proposition 15.8.** *The inverse riffle shuffle Markov chain for a deck of $n$ cards is the random walk of $\Sigma_n$ on $\Omega_n$ driven by the probability $P$ given by*
$$P((Q_1,\dots,Q_s)) = \begin{cases} \frac{1}{2^{n-1}}, & \text{if } s=1,\\ \frac{1}{2^n}, & \text{if } s=2\\ 0, & \text{else}\end{cases}$$
*for $(Q_1,\dots,Q_s) \in \Sigma_n$.*

*Proof.* As was mentioned earlier, we can identify linear orderings of $[n]$ with $\Omega_n$ via
$$i_1 \cdots i_n \longmapsto (\{i_1\},\dots,\{i_n\}).$$
The random walk of $\Sigma_n$ on $\Omega_n$ driven by $P$ can be described as follows. If we are in state $(\{i_1\},\dots,\{i_n\})$, we choose $A \subseteq [n]$ with probability $1/2^n$ and we move to state
$$(A,B)^\wedge(\{i_1\},\dots,\{i_n\}) = (A\cap\{i_1\},\dots,A\cap\{i_n\},B\cap\{i_1\},\dots,B\cap\{i_n\})^\wedge$$
where $B = [n] \setminus A$ (here we have used $(\emptyset,[n])^\wedge = ([n]) = ([n],\emptyset)^\wedge$). But the right hand side of the above equation is the ordered partition into singletons

in which the elements of $A$ appear before the elements of $B$ and within $A$ and $B$ are ordered as per $(\{i_1\}, \ldots, \{i_n\})$. This is exactly the result of moving the cards from $A$ to the front and maintaining the relative ordering in $A$ and $B$. The proposition follows. $\square$

### 15.3.3 Ehrenfest urn model

We consider here a variant of the *Ehrenfest urn model*. In this model there are $n$ balls and two urns: $A$ and $B$. In one of the standard versions of the model you randomly pick a ball and an urn and move that ball to the chosen urn. The state space is then $\{A, B\}^n$ where an $X$ in position $i$ means ball $i$ is in urn $X$.

Alternatively, one can view this as the lazy hypercube random walk. Namely, the hypercube graph is the graph with vertex set $\{A, B\}^n$ in which two vertices are connected if they differ in a single entry. In the lazy random walk the state set is the vertices. With probability $1/2$ the state does not change; otherwise, the state changes to a random neighbor.

In fact, it is not any more difficult to work with the following more general model. We have a probability $Q$ on $\{A, B\} \times [n]$. The state set is $\{A, B\}^n$. With probability $Q(X, i)$ we place ball $i$ in urn $X$. The original Ehrenfest urn model is the case where $Q(X, i) = 1/2n$ for all $(X, i) \in \{A, B\} \times [n]$.

Let $L = \{1, A, B\}$ with the product defined so that 1 is the identity and $XY = X$ for $X, Y \in \{A, B\}$. Trivially, $L$ is a left regular band and hence so is the direct product $L^n$. In fact, $\{A, B\}^n$ is the minimal ideal of $L^n$. To see this, define $c\colon L^n \longrightarrow \mathscr{P}([n])$ by

$$c(x_1, \ldots, x_n) = \{i \mid x_i \neq 1\}.$$

One easily checks that $c$ is a homomorphism, where again $\mathscr{P}([n])$ is made a monoid via union, and hence ordered by reverse inclusion.

**Proposition 15.9.** *Let $n \geq 1$.*

*(i) One has*

$$L^n(x_1, \ldots, x_n) \subseteq L^n(y_1, \ldots, y_n)$$

*if and only if $c(x_1, \ldots, x_n) \supseteq c(y_1, \ldots, y_n)$.*

*(ii) $\Lambda(L^n) \cong \mathscr{P}([n])$ via $L^n m L^n \mapsto c(m)$.*

*(iii) $\{A, B\}^n$ is the minimal ideal of $L^n$.*

*Proof.* Recall that for a left regular band $M$, one has that $Mm \subseteq Mn$ if and only if $mn = m$. We may then deduce (i) as follows. If $x_i = 1$, then $x_i y_i = x_i$ if and only if $y_i = 1$. If $x_i \neq 1$, then $x_i y_i = x_i$ holds independently of the value of $y_i$. Thus $xy = x$ if and only if $c(x) \supseteq c(y)$. The remaining items are immediate from the first. $\square$

We are now prepared to realize the Ehrenfest urn model as a random walk of $L^n$ on $\{A, B\}^n$.

**Proposition 15.10.** *The Ehrenfest urn Markov chain with $n$ balls and probability $Q$ on $\{A,B\} \times [n]$ is the random walk of $L^n$ on $\{A,B\}^n$ driven by the probability $P$ given by*

$$P(x_1,\dots,x_n) = \begin{cases} Q(A,i), & \text{if } c(x_1,\dots,x_n) = \{i\},\ x_i = A \\ Q(B,i), & \text{if } c(x_1,\dots,x_n) = \{i\},\ x_i = B \\ 0, & \text{else} \end{cases}$$

*for $(x_1,\dots,x_n) \in L^n$.*

*Proof.* For $(X,i) \in \{A,B\} \times [n]$, let $e_{X,i}$ be the element of $L^n$ with $X$ in position $i$ and 1 in all other positions. If we are in state $(y_1,\dots,y_n)$, then with probability $Q(X,i)$ we move to state $e_{X,i} \cdot (y_1,\dots,y_n)$. But $e_{X,i} \cdot (y_1,\dots,y_n)$ is obtained from $(y_1,\dots,y_n)$ by replacing $y_i$ by $X$. Thus we have recovered the Ehrenfest urn Markov chain. □

## 15.4 Eigenvalues

We return now to the development of the general theory. In this section, we compute the eigenvalues of the transition matrix of a random walk of a triangularizable monoid on a set. Recall that $M$ is triangularizable over $\mathbb{C}$ if each simple $\mathbb{C}M$-module is one-dimensional. The following result was first proved in [Ste08, Section 8.3].

**Theorem 15.11.** *Let $M$ be a triangularizable monoid over $\mathbb{C}$, $\Omega$ a left $M$-set and $P$ a probability on $M$. Let $\chi_1,\dots,\chi_s$ be the irreducible characters of $M$ and let $m_i$ be the multiplicity of the simple module affording $\chi_i$ as a composition factor of $\mathbb{C}\Omega$. Then the transition matrix $T$ for the random walk of $M$ on $\Omega$ driven by $P$ has eigenvalues $\lambda_1,\dots,\lambda_s$ where*

$$\lambda_i = \sum_{m\in M} P(m)\chi_i(m)$$

*and $\lambda_i$ has multiplicity $m_i$.*

*Proof.* Since $M$ is triangularizable each representation of $M$ is similar to one by upper triangular matrices, as observed after Proposition 11.8. That is, there is a basis $v_1,\dots,v_n$ for $\mathbb{C}\Omega$ such that $V_i = \mathbb{C}\{v_1,\dots,v_i\}$ is a $\mathbb{C}M$-submodule and

$$\mathbb{C}\Omega = V_n \supseteq V_{n-1} \supseteq \cdots \supseteq V_0 = 0$$

is a composition series.

Let $\theta_i$ be the character of the one-dimensional simple module $V_i/V_{i-1}$ for $i = 1,\dots,n$. Then with respect to the basis $v_1,\dots,v_n$, we have that $\mathbb{C}\Omega$ affords a representation of the form $\rho\colon M \longrightarrow M_n(\mathbb{C})$ with

$$\rho(m) = \begin{bmatrix} \theta_1(m) & * & \cdots & * \\ 0 & \theta_2(m) & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \theta_n(m) \end{bmatrix}.$$

Since $T$ is the matrix of the operator $P$ on $\mathbb{C}\Omega$ with respect to the basis $\Omega$ by Proposition 15.2, it follows that $T$ is similar to the matrix

$$\begin{aligned} T' &= \sum_{m\in M} P(m)\rho(m) \\ &= \begin{bmatrix} \sum_{m\in M} P(m)\theta_1(m) & * & \cdots & * \\ 0 & \sum_{m\in M} P(m)\theta_2(m) & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \sum_{m\in M} P(m)\theta_n(m) \end{bmatrix}. \end{aligned}$$

Since $\chi_i$ appears among the $\theta_j$ exactly $m_i$ times, the result follows. $\square$

We note that $m_i = 0$ and $\lambda_i = \lambda_j$ for $i \neq j$ can occur. Since triangularizable monoids are rectangular with abelian maximal subgroups by Theorem 12.15, the characters of a triangularizable monoid $M$ can be explicitly computed using Proposition 12.16 and the character theory of abelian groups. Moreover, the multiplicities can be obtained by inverting the character table. Alternatively, one can use that the semisimple quotient of $M$ is the algebra of a commutative inverse monoid and apply Theorem 10.9 to compute multiplicities. Details can be found in [Ste08, Section 8.3]. Let us specialize to the case of an $\mathscr{R}$-trivial monoid, where we can be more explicit. This result is a special case of results from [Ste06].

**Theorem 15.12.** *Let $M$ be an $\mathscr{R}$-trivial monoid, $\Omega$ a left $M$-set and $P$ a probability on $M$. Let $\Lambda(M) = \{MeM \mid e \in E(M)\}$. Then the transition matrix $T$ for the random walk of $M$ on $\Omega$ driven by $P$ has an eigenvalue*

$$\lambda_{MeM} = \sum_{\{m\in M \mid MmM \supseteq MeM\}} P(m) \tag{15.3}$$

*for each $MeM \in \Lambda(M)$. The multiplicity of $\lambda_{MeM}$ is given by*

$$m_{MeM} = \sum_{\{MfM\in\Lambda(M) \mid MfM\subseteq MeM\}} |f\Omega| \cdot \mu(MfM, MeM) \tag{15.4}$$

*where $\mu$ is the Möbius function of the lattice $\Lambda(M)$. In particular, each eigenvalue of $T$ is non-negative.*

*Proof.* First of all, $\mathscr{R}$-trivial monoids are triangularizable by Corollary 11.9. The characters of $M$ are indexed by $\Lambda(M)$ and given by

$$\chi_{MeM}(m) = \begin{cases} 1, & \text{if } MmM \supseteq MeM \\ 0, & \text{else} \end{cases}$$

by Corollary 5.7. Thus the eigenvalue $\lambda_{MeM}$ associated to $\chi_{MeM}$ according to Theorem 15.11 is as per (15.3).

The character $\theta$ of $\mathbb{C}\Omega$ is given by

$$\theta(m) = |\{\alpha \in \Omega \mid m\alpha = \alpha\}|\,.$$

In particular, if $f \in E(M)$, then $\theta(f) = |f\Omega|$. We can then deduce (15.4) from Corollary 8.21. □

Theorem 15.12 has so far been the principal tool in computing eigenvalues of transition matrices of Markov chains using monoids that are not groups.

## 15.5 Diagonalizability

In this section, we give a proof of Brown's theorem on the diagonalizability of a left regular band random walk [Bro00]. The treatment here is from the author's unpublished note [Ste10a]; a more sophisticated version of this argument, which works for some more general classes of $\mathscr{R}$-trivial monoids, appears in [ASST15b, Theorem 4.3].

Let $M$ be a left regular band and let $\Lambda(M) = \{MmM \mid m \in M\}$ be the lattice of principal ideals. Recall that there is a natural surjective homomorphism $\sigma\colon M \longrightarrow \Lambda(M)$ given by $\sigma(m) = MmM$ where $\Lambda(M)$ is equipped with the meet operation (which turns out to be intersection for a left regular band, although we do not use this fact here). A key property that we shall exploit is that $\sigma(m) \leq \sigma(n)$ if and only if $mn = m$ by Corollary 2.6 since each $m \in M$ is idempotent.

Brown's diagonalizability theorem is, in fact, about elements of left regular band algebras over any field. For the case of probability distributions it suffices to work over $\mathbb{R}$, but there is no added difficulty in tackling the general case. So let $\Bbbk$ be a field and let

$$w = \sum_{m \in M} w_m m \in \Bbbk M. \tag{15.5}$$

For $X \in \Lambda(M)$, define

$$\lambda_X = \sum_{\sigma(m) \geq X} w_m. \tag{15.6}$$

Brown [Bro00, Bro04] showed that the commutative algebra $\Bbbk[w]$ is *split semisimple* (i.e., the minimal polynomial of $w$ splits over $\Bbbk$ into distinct linear factors) provided that $X > Y$ implies $\lambda_X \neq \lambda_Y$. We prove this by showing that if $\lambda_1, \ldots, \lambda_k$ are the distinct elements of $\{\lambda_X \mid X \in \Lambda(M)\}$, then

$$0 = \prod_{i=1}^{k}(w - \lambda_i). \tag{15.7}$$

This immediately implies that the minimal polynomial of $w$ has distinct roots and hence $\Bbbk[w]$ is split semisimple. Everything is based on the following formula for $mw$.

**Lemma 15.13.** *Let $m \in M$. Then*

$$mw = \lambda_{\sigma(m)} m + \sum_{\sigma(n) \ngeq \sigma(m)} w_n mn$$

*and moreover, $\sigma(m) > \sigma(mn)$ for all $n$ with $\sigma(n) \ngeq \sigma(m)$.*

*Proof.* Using that $\sigma(n) \geq \sigma(m)$ implies $mn = m$, we compute

$$\begin{aligned} mw &= \sum_{\sigma(n) \geq \sigma(m)} w_n mn + \sum_{\sigma(n) \ngeq \sigma(m)} w_n mn \\ &= \sum_{\sigma(n) \geq \sigma(m)} w_n m + \sum_{\sigma(n) \ngeq \sigma(m)} w_n mn \\ &= \lambda_{\sigma(m)} m + \sum_{\sigma(n) \ngeq \sigma(m)} w_n mn. \end{aligned}$$

It remains to observe that $\sigma(n) \ngeq \sigma(m)$ implies that $\sigma(mn) = \sigma(m)\sigma(n) < \sigma(m)$. $\square$

The proof of (15.7) proceeds via an induction along $\Lambda(M)$. Let us write $\widehat{0}$ for the minimal ideal of $M$ and $\widehat{1}$ for the ideal $M$, itself. If $X \in \Lambda(M)$, put

$$\Phi_X = \{\lambda_Y \mid Y \leq X\} \text{ and } \Phi'_X = \{\lambda_Y \mid Y < X\}.$$

Our hypothesis says exactly that $\Phi_X = \{\lambda_X\} \mathbin{\dot\cup} \Phi'_X$ (disjoint union). Define polynomials $p_X(z)$ and $q_X(z)$, for $X \in \Lambda(M)$, by

$$\begin{aligned} p_X(z) &= \prod_{\lambda_i \in \Phi_X} (z - \lambda_i) \\ q_X(z) &= \prod_{\lambda_i \in \Phi'_X} (z - \lambda_i) = \frac{p_X(z)}{z - \lambda_X}. \end{aligned}$$

Notice that, for $X > Y$, we have $\Phi_Y \subseteq \Phi'_X$, and hence $p_Y(z)$ divides $q_X(z)$, because $\lambda_X \notin \Phi_Y$ by assumption. Also observe that

$$p_{\widehat{1}}(z) = \prod_{i=1}^{k}(z - \lambda_i)$$

and hence establishing (15.7) is equivalent to proving $p_{\widehat{1}}(w) = 0$.

**Lemma 15.14.** *If $m \in M$, then $m \cdot p_{\sigma(m)}(w) = 0$.*

*Proof.* The proof is by induction on $\sigma(m)$ in the lattice $\Lambda(M)$. Suppose first $\sigma(m) = \widehat{0}$. Observe that $p_{\widehat{0}}(z) = z - \lambda_{\widehat{0}}$. Then, as $\sigma(n) \geq \sigma(m)$ for all $n \in M$, Lemma 15.13 immediately yields $m(w - \lambda_{\sigma(m)}) = 0$. In general, assume that the lemma holds for all $m' \in M$ with $\sigma(m') < \sigma(m)$. Then by Lemma 15.13

$$m \cdot p_{\sigma(m)}(w) = m \cdot (w - \lambda_{\sigma(m)}) \cdot q_{\sigma(m)}(w) = \sum_{\sigma(n) \ngeq \sigma(m)} w_n mn \cdot q_{\sigma(m)}(w) = 0.$$

The last equality follows because $\sigma(n) \ngeq \sigma(m)$ implies $\sigma(m) > \sigma(mn)$ and so $p_{\sigma(mn)}(z)$ divides $q_{\sigma(m)}(z)$, whence induction yields $mn \cdot q_{\sigma(m)}(w) = 0$. □

Applying the lemma with $m$ the identity element of $M$ yields $p_{\widehat{1}}(w) = 0$ and hence we have proved the following theorem.

**Theorem 15.15.** *Let $M$ be a left regular band and $\Bbbk$ a field. Suppose that $w \in \Bbbk M$ is as in* (15.5) *and $\lambda_X$ is as in* (15.6) *for $X \in \Lambda(M)$. If $X > Y$ implies $\lambda_X \neq \lambda_Y$, then $\Bbbk[w]$ is split semisimple, that is, the minimal polynomial of $w$ splits into distinct linear factors over $\Bbbk$.*

As a corollary, we may deduce that transition matrices of left regular band random walks are diagonalizable.

**Theorem 15.16.** *Let $M$ be a left regular band, $\Omega$ a left $M$-set and $P$ a probability on $M$. Then the transition matrix $T$ of the random walk of $M$ on $\Omega$ driven by $P$ is diagonalizable over $\mathbb{R}$.*

*Proof.* First note that if we replace $M$ by the submonoid generated by the support of $P$, we obtain the same Markov chain $(T, \Omega)$. Thus we may assume without loss of generality that the support of $P$ generates $M$. The transition matrix $T$ is the matrix of the operator $P$ on $\mathbb{R}\Omega$ with respect to the basis $\Omega$ by Proposition 15.2. Therefore, the algebra $\mathbb{R}[T]$ is a homomorphic image of the algebra $\mathbb{R}[P]$ and hence the minimal polynomial of $T$ divides the minimal polynomial of $P$. It thus suffices to prove that $\mathbb{R}[P]$ is split semisimple. We shall apply the criterion of Theorem 15.15.

Suppose that $X, Y \in \Lambda(M)$ with $X > Y$. Then we have that

$$\lambda_Y = \sum_{\sigma(m) \geq Y} P(m) \geq \sum_{\sigma(m) \geq X} P(m) = \lambda_X.$$

Moreover, the inequality is strict because $M$ is generated by the support of $P$, $\{m \in M \mid \sigma(m) \geq X\}$ is a submonoid disjoint from $\sigma^{-1}(Y)$ and elements of $\sigma^{-1}(Y)$ can only be expressed as products of elements $m \in \operatorname{supp}(P)$ with $\sigma(m) \geq Y$. Thus there must exist $m \in M$ with $P(m) > 0$, $\sigma(m) \geq Y$ and $\sigma(m) \ngeq X$. Therefore, $X > Y$ implies $\lambda_Y > \lambda_X$ and so $\mathbb{R}[P]$ is split semisimple by Theorem 15.15. □

## 15.6 Examples: revisited

It is now time to apply the results of the previous sections to the Markov chains from Section 15.3 in order to provide some idea of their scope. We shall retain throughout the notation from Section 15.3 and it is suggested that the reader review that section before reading this one.

### 15.6.1 The Tsetlin library

Let $B$ be a finite set (of books). We identify $\Lambda(F(B))$ with $(\mathscr{P}(B), \supseteq)$ via the mapping $c$ as per Section 15.3.1. One then has that $\sigma(w) \geq MmM$ if and only if $c(w) \subseteq c(m)$.

To describe the multiplicities of the eigenvalues of the transition matrix for the Tsetlin library Markov chain, we need to recall the notion of derangement numbers. A *derangement* is a fixed-point-free permutation. Let $d_n$ be the number of derangements of an $n$-element set. Note that $d_0 = 1$ and $d_1 = 0$.

**Theorem 15.17.** *Let $T$ be the transition matrix of the Tsetlin library Markov chain with set of books $B$ and probability $P$ on $B$. Then there is an eigenvalue*

$$\lambda_X = \sum_{b \in X} P(b)$$

*of $T$ for each $X \subseteq B$. The multiplicity of $\lambda_X$ is given by the derangement number $d_{|B|-|X|}$. Moreover, $T$ is diagonalizable.*

*Proof.* We view $(T, \Omega)$ as the random walk of $F(B)$ on $\Omega_B$ driven by $P$. One has that $c(b) \subseteq X$ if and only if $b \in X$ for $X \subseteq B$. Thus the eigenvalue associated to the element of $\Lambda(F(B))$ corresponding to $X$ is $\lambda_X$ by Theorem 15.12. It remains to compute the multiplicity of $\lambda_X$.

Let $|B| = n$. Fix a word $w_X$ with content $X$ for each $X \subseteq B$. Note that $w_X \Omega_B$ consists of all words of content $B$ with $w_X$ as a prefix. Hence $|w_X \Omega_B| = (n - |X|)! = |S_{B \setminus X}|$. We can identify $S_{B \setminus X}$ with the subgroup $G_X$ of $S_n$ fixing $X$ pointwise. If we count elements of $G_X$ by their fixed-point sets, then we obtain

$$|w_X \Omega_B| = |G_X| = \sum_{Y \supseteq X} d_{n-|Y|}$$

for all $X$. Applying Möbius inversion (Theorem 7.6), we conclude that

$$d_{n-|X|} = \sum_{Y \supseteq X} |w_Y \Omega_B| \cdot \mu(Y, X) \tag{15.8}$$

where $\mu$ is the Möbius function of $(\mathscr{P}(B), \supseteq)$. But the right hand side of (15.8) is the multiplicity of $\lambda_X$ by Theorem 15.12.

The diagonalizability of $T$ follows from Theorem 15.16. This completes the proof. $\square$

Specializing to the case where $P$ is the uniform distribution, and hence the Tsetlin library chain is the time reversal of the top-to-random shuffle, we obtain the following result.

**Corollary 15.18.** *Let $T$ be the transition matrix of the top-to-random shuffle Markov chain for a deck of $n$ cards. Then the eigenvalues for $T$ are $\lambda_j = j/n$ with $j = 0, \ldots, n$. The multiplicity of $\lambda_j$ is $\binom{n}{j} d_{n-j}$. Furthermore, $T$ is diagonalizable.*

*Proof.* We retain the notation of Theorem 15.17 with $B = [n]$. Note that if $X \subseteq [n]$, then $\lambda_X = |X|/n = \lambda_{|X|}$. Since there are $\binom{n}{j}$ subsets of size $j$, the result follows. $\square$

We remark that since $d_1 = 0$, it follows that $(n-1)/n$ is not actually an eigenvalue in Corollary 15.18.

*Remark 15.19.* One can model the top-to-random shuffle as a symmetric group random walk. From that point of view, it is not at all obvious, *a priori*, that the eigenvalues should be non-negative, while this is immediate from the monoid viewpoint.

### 15.6.2 The inverse riffle shuffle

We identify $\Lambda(\Sigma_n)$ with $\Pi_n$ as in Section 15.3.2. We shall need the following lemma. Let us fix some notation. If $\pi \in \Pi_n$, then let $O_\pi$ be the set of all permutations in $S_n$ with orbit partition $\pi$ of $[n]$.

**Lemma 15.20.** *Let $\pi = \{P_1, \ldots P_r\}$ be a set partition of $[n]$. Put $\pi! = |P_1|! \cdots |P_r|!$. Then the equality*

$$\pi! = \sum_{\pi' \leq \pi} |O_{\pi'}|$$

*holds.*

*Proof.* The left hand side is the cardinality of the subgroup

$$S_\pi \cong S_{P_1} \times \cdots \times S_{P_r}$$

of $S_n$ that leaves the blocks of $\pi$ invariant. The orbit partition of any element of $S_\pi$ must refine $\pi$ and hence the desired equality follows by counting elements of $S_\pi$ by their orbit partition. $\square$

Now we can prove the main result. We remind the reader that the transition matrix of the riffle shuffle is the transpose of the transition matrix of the inverse riffle shuffle and so they have the same eigenvalues and diagonalizability properties (cf. [BD92, LPW09]).

**Theorem 15.21.** *Let $T$ be the transition matrix of the inverse riffle shuffle Markov chain for a deck of $n$ cards. Then the eigenvalues of $T$ are*

$$\lambda_k = \frac{1}{2^{n-k}}$$

*with $1 \leq k \leq n$. The multiplicity of $\lambda_k$ is the number of permutations of $[n]$ with $k$ orbits. Moreover, $T$ is diagonalizable.*

*Proof.* Let $\pi \in \Pi_n$ with $\pi = \{P_1, \ldots, P_k\}$ and $A \subseteq [n]$. Put $B = [n] \setminus A$. Then one has that $c((A, B)^\wedge) \geq \pi$ if and only if $A$ is the union of some subset of $\{P_1, \ldots, P_k\}$. Thus there are $2^k$ choices for $A$. It follows from Theorem 15.12 that the eigenvalue $\lambda_\pi$ associated to the element of $\Lambda(\Sigma_n)$ corresponding to $\pi$ is $2^k/2^n = 2^{n-k} = \lambda_k$. Notice that all set partitions with $k$ blocks give the eigenvalue $\lambda_k$.

We now compute the multiplicity of $\lambda_\pi$ for $\pi \in \Pi_n$. Fix a preimage $e_\pi \in \Sigma_n$ of each element $\pi \in \Pi_n$. We claim $|e_\pi \Omega_n| = \pi!$. Indeed, if $e_\pi = (P_1, \ldots, P_r)$, then $e_\pi \Omega_n$ can be identified with those linear orderings on $[n]$ in which the elements of $P_1$ appear before those of $P_2$, and so forth. Thus there are $|P_1|! \cdots |P_r|! = \pi!$ such elements. By Lemma 15.20 and Möbius inversion (Theorem 7.6) we have that

$$|O_\pi| = \sum_{\pi' \leq \pi} \pi'! \cdot \mu(\pi', \pi) = \sum_{\pi' \leq \pi} |e_{\pi'} \Omega_n| \cdot \mu(\pi', \pi)$$

where $\mu$ is the Möbius function of $\Pi_n$. Therefore, $\lambda_\pi$ has multiplicity $|O_\pi|$ by Theorem 15.12. As all partitions $\pi$ with $k$ blocks provide the same eigenvalue $\lambda_k$, we deduce that the multiplicity of $\lambda_k$ is the number of permutations of $[n]$ with $k$ orbits.

The diagonalizability of $T$ is immediate from Theorem 15.16. This establishes the theorem. □

We remark that permutations with $k$ orbits are those permutations with $k$ cycles in their cycle decomposition, including cycles of length 1.

### 15.6.3 The Ehrenfest urn model

We identify $\Lambda(L^n)$ with $(\mathscr{P}([n], \supseteq)$ via $c$ as per Section 15.3.3.

**Theorem 15.22.** *Let $T$ be the transition matrix for the Ehrenfest urn model with probability $Q$ on $\{A, B\} \times [n]$. Then $T$ has an eigenvalue*

$$\lambda_S = \sum_{i \in S} (Q(A, i) + Q(B, i))$$

*for each $S \subseteq [n]$ with multiplicity one. Furthermore, $T$ is diagonalizable.*

*Proof.* Put $\Omega = \{A, B\}^n$. As before, let $e_{X,i}$ be the element of $L^n$ with $X \in \{A, B\}$ in position $i \in [n]$ and 1 in all other positions. Then $c(e_{X,i}) \subseteq S$ if and only if $i \in S$. Therefore, $\lambda_S$ is the eigenvalue associated by Theorem 15.12 to the element of $\Lambda(L^n)$ corresponding to $S$.

In order to compute the multiplicity of $\lambda_S$, let $e_S = (x_1, \ldots, x_n)$ with

$$x_i = \begin{cases} A, & \text{if } i \in S \\ 1, & \text{else.} \end{cases}$$

Then $c(e_S) = S$ and we have that $e_S\Omega$ consists of all elements of $\Omega$ with $A$ in each position belonging to $S$. Thus

$$|e_S\Omega| = 2^{n-|S|} = \sum_{T \supseteq S} 1.$$

Möbius inversion (Theorem 7.6) then yields

$$1 = \sum_{T \supseteq S} |e_T\Omega| \cdot \mu(T, S)$$

where $\mu$ is the Möbius function of $\mathscr{P}([n])$. It follows that $\lambda_S$ has multiplicity one by Theorem 15.12.

The diagonalizability of $T$ again is a consequence of Theorem 15.16. $\square$

Specializing to the case where $Q$ is the uniform distribution, we obtain the following result for the classical Ehrenfest urn model or, equivalently, for the lazy hypercube random walk.

**Corollary 15.23.** *Let $T$ be the transition matrix of the classical Ehrenfest urn model with $n$ balls or, equivalently, of the lazy $n$-hypercube random walk. Then $T$ has an eigenvalue $\lambda_k = k/n$ with multiplicity $\binom{n}{k}$ for $0 \leq k \leq n$. Moreover, $T$ is diagonalizable.*

*Proof.* This follows easily from Theorem 15.22. In this case,

$$\lambda_S = \sum_{i \in S} \left( \frac{1}{2n} + \frac{1}{2n} \right) = \frac{|S|}{n} = \lambda_{|S|}.$$

As $\lambda_S$ has multiplicity one and there are $\binom{n}{k}$ subsets of $[n]$ of cardinality $k$, we deduce that $\lambda_k$ has multiplicity $\binom{n}{k}$, as required. $\square$

We remark that Corollary 15.23 can also be proved by modelling the lazy $n$-hypercube random walk as a random walk of $(\mathbb{Z}/2\mathbb{Z})^n$ on itself and using the character theory of abelian groups [Dia88, CSST08].

## 15.7 Exercises

**15.1.** Prove that if $T\colon \Omega \times \Omega \longrightarrow \mathbb{R}$ is the transition matrix of a Markov chain, then $T\nu$ is a probability for any probability $\nu$ on $\Omega$.

**15.2.** Compute the transition matrix for the Tsetlin library with 3 books with respect to the uniform distribution on the set of books.

**15.3.** Prove that if $A$ is a set, $M$ is a left regular band and $\varphi\colon A \longrightarrow M$ is a mapping, then there is a unique homomorphism $\Phi\colon F(A) \longrightarrow M$ such that $\Phi|_A = \varphi$, that is, $F(A)$ is, indeed, a free left regular band.

**15.4.** Verify that $\Sigma_n$ is a monoid.

**15.5.** A coupon collector wishes to collect all the coupons from a set $B$ of coupons. Let $P$ be a probability on $B$. The probability that the next coupon that he draws is $b \in B$ is $P(b)$. So the state set consists of all possible sets of coupons. If we are in state $X \subseteq B$, then with probability $P(b)$ we transition to state $X \cup \{b\}$.

(a) Model the coupon collector Markov chain as a random walk on $(\mathscr{P}(B), \cup)$.
(b) Compute the eigenvalues of the transition matrix.
(c) Prove that the transition matrix is diagonalizable.

**15.6.** Let $B$ be a finite set (of say books) with $n \geq 1$ elements and let $k \geq 1$. Let $P$ be a probability on $B$. Imagine that you have $k$ indistinguishable copies of each book. The states of the Markov chain are all possible orderings of the $nk$ books on a shelf. With probability $P(b)$ the last copy of book $b$ on the shelf is moved to the front of the shelf. Model this Markov chain as a random walk of an $\mathscr{R}$-trivial monoid and compute the eigenvalues of the transition matrix with multiplicities.

**15.7.** Let $G$ be a finite abelian group and $P$ a probability on $G$. Let $T$ be the transition matrix of the random walk of $G$ on itself driven by $P$. Prove that there is an eigenvalue $\lambda_\chi$ of $T$ for each character $\chi\colon G \longrightarrow \mathbb{C}$ given by

$$\lambda_\chi = \sum_{g \in G} P(g)\chi(g)$$

with multiplicity one and that $T$ is diagonalizable.

**15.8.** Use Exercise 15.7 to give another proof of Corollary 15.23.

# Part VII

# Advanced Topics

# 16

# Self-injective, Frobenius and Symmetric Algebras

In this chapter we characterize regular monoids with a self-injective algebra. It turns out that the algebra of such a monoid is a product of matrix algebras over group algebras and hence is a symmetric algebra. The results of this chapter are new to the best of our knowledge.

## 16.1 Background on self-injective algebras

The reader is referred to [Ben98, Section 1.6] for the material of this section. Fix a field $\Bbbk$ and a finite dimensional $\Bbbk$-algebra $A$ for the section. The algebra $A$ is said to be *self-injective* if the regular $A$-module $A$ is an injective module. This is equivalent to each projective module being injective and to each projective indecomposable module being an injective indecomposable module. A part of [Ben98, Proposition 1.6.2] is that injective and projective modules coincide for a self-injective finite dimensional algebra $A$.

**Proposition 16.1.** *Let $A$ be a finite dimensional self-injective $\Bbbk$-algebra. Then a finite dimensional $A$-module $V$ is projective if and only if it is injective.*

The algebra $A$ is said to be a *Frobenius algebra* if there is a linear map $\lambda\colon A \longrightarrow \Bbbk$ such that $\ker \lambda$ contains no non-zero left or right ideal of $A$. If, in addition, $\lambda(ab) = \lambda(ba)$ for all $a, b \in A$, then $A$ is called a *symmetric algebra*. The following proposition is Exercise 4 of [Ben98, Section 1.6].

**Proposition 16.2.** *Let $A$ be a symmetric algebra. Then $M_n(A)$ is a symmetric algebra for all $n \geq 1$.*

The reader should also verify that a direct product of symmetric algebras is symmetric (cf. Exercise 16.2). We recall that if $V$ is a right $A$-module, then $D(V) = \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$ is a left $A$-module. See Appendix A.4 for details on the standard duality $D$.

Frobenius algebras are self-injective. The following result combines [Ben98, Proposition 1.6.2] and [Ben98, Theorem 1.6.3].

**Theorem 16.3.** *Let $A$ be a Frobenius algebra. Then $A \cong D(A)$ and hence $A$ is self-injective. Moreover, if $A$ is symmetric, then $P/\operatorname{rad}(P) \cong \operatorname{soc}(P)$ for all projective indecomposable $A$-modules $P$.*

The primary example of a symmetric algebra is a group algebra [Ben98, Proposition 3.1.2].

**Theorem 16.4.** *Let $G$ be a finite group and $\Bbbk$ a field. Define $\lambda\colon \Bbbk G \longrightarrow \Bbbk$ by*

$$\lambda(g) = \begin{cases} 1, & \text{if } g = 1 \\ 0, & \text{else} \end{cases}$$

*for $g \in G$. Then $\Bbbk G$ is a symmetric algebra with respect to the functional $\lambda$.*

## 16.2 Regular monoids with self-injective algebras

Our goal is to generalize Theorem 5.14 to a result about self-injectivity. First we note that Corollary 10.4, Proposition 16.2, Exercise 16.2 and Theorem 16.4 easily imply the following result, which can be viewed as a warmup to our main result.

**Corollary 16.5.** *Let $M$ be a finite inverse monoid and $\Bbbk$ a field. Then $\Bbbk M$ is a symmetric algebra.*

Now we state and prove the main theorem of this chapter.

**Theorem 16.6.** *Let $M$ be a finite regular monoid and $\Bbbk$ a field. Let $e_1, \ldots, e_s$ be a complete set of idempotent representatives of the $\mathscr{J}$-classes of $M$ and let $n_i$ be the number of $\mathscr{L}$-classes in $J_{e_i}$. Then the following are equivalent.*

*(i)* $\Bbbk M$ *is self-injective.*
*(ii)* $\Bbbk M$ *is a Frobenius algebra.*
*(iii)* $\Bbbk M$ *is a symmetric algebra.*
*(iv)* $\Bbbk M \cong \prod_{i=1}^{s} M_{n_i}(\Bbbk G_{e_i})$.
*(v) The sandwich matrix $P(e_i)$ is invertible over $\Bbbk G_{e_i}$ for $1 \leq i \leq s$.*
*(vi) The natural map $\varphi_{\Bbbk G_{e_i}}\colon \operatorname{Ind}_{G_{e_i}}(\Bbbk G_{e_i}) \longrightarrow \operatorname{Coind}_{G_{e_i}}(\Bbbk G_{e_i})$ is an isomorphism for $1 \leq i \leq s$.*
*(vii)* $\operatorname{Ind}_{G_{e_i}}(\Bbbk G_{e_i}) \cong \operatorname{Coind}_{G_{e_i}}(\Bbbk G_{e_i})$ *for $1 \leq i \leq s$.*

*Proof.* The implication (iv) implies (iii) follows from Proposition 16.2, Exercise 16.2 and Theorem 16.4. The implications (iii) implies (ii) implies (i) hold for finite dimensional algebras in general. The equivalence of (v) and (vi) is a consequence of Lemma 5.15. The equivalence of (vi) and (vii) is the content of Exercise 4.4. It remains to show that (vii) implies (iv) and (i) implies (vii).

We precede in a similar fashion to the proof of Theorem 5.14. Recall that, for $e \in E(M)$, we have

$$\mathrm{Ind}_{G_e}(\Bbbk G_e) = \Bbbk L_e \otimes_{\Bbbk G_e} \Bbbk G_e \cong \Bbbk L_e.$$

Consider a principal series

$$\emptyset = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_s = M$$

for $M$ and without loss of generality assume that $e_k \in J_k = I_k \setminus I_{k-1}$, which is a regular $\mathscr{J}$-class (cf. Proposition 1.18), for $k = 1, \ldots, s$. Then we have

$$\Bbbk I_k/\Bbbk I_{k-1} \cong \Bbbk J_k \cong n_k \cdot \Bbbk L_{e_k} \tag{16.1}$$

by Lemma 5.13.

Suppose first that (vii) holds. We claim that

$$\Bbbk M/\Bbbk I_{k-1} \cong \Bbbk J_k \oplus \Bbbk M/\Bbbk I_k \tag{16.2}$$

as a $\Bbbk M$-module for $1 \leq k \leq s$. Indeed, let $A = \Bbbk M/\Bbbk I_{k-1}$. Then we have that $e_k A e_k \cong \Bbbk G_{e_k}$ (by Corollary 1.15), $\mathrm{Ind}_{e_k}(\Bbbk G_{e_k}) = \mathrm{Ind}_{G_{e_k}}(\Bbbk G_{e_k})$ and $\mathrm{Coind}_{e_k}(\Bbbk G_{e_k}) = \mathrm{Coind}_{G_{e_k}}(\Bbbk G_{e_k})$. As $\Bbbk G_{e_k}$ is self-injective, $\mathrm{Coind}_{G_{e_k}}(\Bbbk G_{e_k})$ is an injective $A$-module by Proposition 4.3. Since $\Bbbk L_e \cong \mathrm{Ind}_{G_{e_k}}(\Bbbk G_{e_k}) \cong \mathrm{Coind}_{G_{e_k}}(\Bbbk G_e)$, we conclude that $\Bbbk L_{e_k}$ is an injective $A$-module. From (16.1) we then conclude that $\Bbbk J_k$ is an injective $A$-module. Thus the exact sequence of $A$-modules

$$0 \longrightarrow \Bbbk J_k \longrightarrow \Bbbk A \longrightarrow \Bbbk M/\Bbbk I_k \longrightarrow 0$$

splits, establishing (16.2).

Applying (16.1) and (16.2) repeatedly yields

$$\Bbbk M \cong \bigoplus_{k=1}^{s} \Bbbk J_k \cong \bigoplus_{k=1}^{s} n_k \cdot \Bbbk L_{e_k}. \tag{16.3}$$

We claim that

$$\mathrm{Hom}_{\Bbbk M}(\Bbbk L_{e_i}, \Bbbk L_{e_j}) \cong \begin{cases} \Bbbk G_{e_i}^{op}, & \text{if } i = j \\ 0, & \text{else.} \end{cases}$$

Then (iv) will follow from (16.3) and the isomorphism $M^{op} \cong \mathrm{End}_{\Bbbk M}(\Bbbk M)$.

Assume first that $Me_jM \nsubseteq Me_iM$ and set $I = Me_iM \setminus J_i$. Putting $A = \Bbbk M/\Bbbk I$, we have that $\Bbbk L_{e_i}$ and $\Bbbk L_{e_j}$ are both $A$ modules, $e_i A e_i \cong \Bbbk G_{e_i}$ and $Ae_i \cong \Bbbk L_{e_i}$ by Corollary 1.15 and Theorem 1.12. Therefore, we have using Proposition 4.1

$$\begin{aligned} \mathrm{Hom}_{\Bbbk M}(\Bbbk L_{e_i}, \Bbbk L_{e_j}) &= \mathrm{Hom}_A(\mathrm{Ind}_{e_i}(\Bbbk G_{e_i}), \Bbbk L_{e_j}) \\ &\cong \mathrm{Hom}_{\Bbbk G_{e_i}}(\Bbbk G_{e_i}, e_i \Bbbk L_{e_j}) = 0 \end{aligned}$$

where the last equality uses that $e_i\Bbbk L_{e_j} = 0$ because $e_i \in I(e_j)$.

Next assume that $Me_jM \subseteq Me_iM$. Let $A = \Bbbk M/\Bbbk I(e_j)$, and so $e_jAe_j \cong \Bbbk G_{e_j}$. Then $\Bbbk L_{e_i}$ and $\Bbbk L_{e_j}$ are both $A$-modules and $\Bbbk L_{e_j} \cong \mathrm{Ind}_{\Bbbk G_{e_j}}(\Bbbk G_{e_j}) \cong \mathrm{Coind}_{\Bbbk G_{e_j}}(\Bbbk G_{e_j}) = \mathrm{Coind}_{e_j}(\Bbbk G_{e_j})$. Proposition 4.1 then yields

$$\begin{aligned}\mathrm{Hom}_{\Bbbk M}(\Bbbk L_{e_i}, \Bbbk L_{e_j}) &= \mathrm{Hom}_A(\Bbbk L_{e_i}, \mathrm{Coind}_{e_j}(\Bbbk G_{e_j}))\\ &\cong \mathrm{Hom}_{\Bbbk G_{e_i}}(e_j\Bbbk L_{e_i}, \Bbbk G_{e_i})\\ &\cong \begin{cases}\mathrm{End}_{\Bbbk G_{e_i}}(\Bbbk G_{e_i}), & \text{if } i = j\\ 0, & \text{else}\end{cases}\end{aligned}$$

where the last isomorphism uses that $e_j\Bbbk L_{e_i} = 0$ if $Me_jM \subsetneq Me_iM$ and $e_i\Bbbk L_{e_i}e_i \cong e_iAe_i \cong \Bbbk G_{e_i}$. As $\mathrm{End}_{\Bbbk G_{e_i}} \cong \Bbbk G_{e_i}^{op}$, this completes the proof of the claim. We conclude that (vii) implies (iv).

Next assume that (i) holds and we prove (vii). First we prove by induction on $k$ that $\Bbbk M/\Bbbk I_k$ is a projective $\Bbbk M$-module. The base case that $k = 0$ is trivial because $\Bbbk M/\Bbbk I_0 \cong \Bbbk M$. Assume that $\Bbbk M/\Bbbk I_{k-1}$ is a projective $\Bbbk M$-module with $k > 0$ and let $A = \Bbbk M/\Bbbk I_{k-1}$. Then since $A$ is projective as a $\Bbbk M$-module, it follows that every projective $A$-module is a projective $\Bbbk M$-module. But $\Bbbk L_{e_k} \cong Ae_k$ and hence is a projective $A$-module. We conclude that $\Bbbk J_k \cong n_k \cdot \Bbbk L_{e_k}$ (by (16.1)) is a projective $A$-module and hence a projective $\Bbbk M$-module. Therefore, $\Bbbk J_k \cong \Bbbk I_k/\Bbbk I_{k-1}$ is an injective $\Bbbk M$-module by self-injectivity of $\Bbbk M$ and so the exact sequence of $\Bbbk M$-modules

$$0 \longrightarrow \Bbbk I_k/I_{k-1} \longrightarrow \Bbbk M/\Bbbk I_{k-1} \longrightarrow \Bbbk M/\Bbbk I_k \longrightarrow 0$$

splits. We deduce that

$$\Bbbk M/\Bbbk I_{k-1} \cong \Bbbk I_k/I_{k-1} \oplus \Bbbk M/\Bbbk I_k \cong \Bbbk J_k \oplus \Bbbk M/\Bbbk I_k \tag{16.4}$$

and hence $\Bbbk M/\Bbbk I_k$ is a projective $\Bbbk M$-module, being a direct summand in the projective $\Bbbk M$-module $\Bbbk M/\Bbbk I_{k-1}$. This completes the induction.

Repeated application of (16.4) shows that

$$\Bbbk M = \bigoplus_{k\geq 1}^{s} \Bbbk J_k \cong \bigoplus_{k=1}^{s} n_k \cdot \Bbbk L_{e_k} \tag{16.5}$$

and hence each $\Bbbk L_{e_k}$ is a projective $\Bbbk M$-module for $1 \leq k \leq s$.

Let $\Bbbk G_{e_k} = \bigoplus_{r=1}^{m_k} P_{k,r}$ be the decomposition of $\Bbbk G_{e_k}$ into projective indecomposable modules. Then observe that

$$\Bbbk L_{e_k} \cong \mathrm{Ind}_{G_{e_k}}(\Bbbk G_{e_k}) \cong \bigoplus_{r=1}^{m_k} \mathrm{Ind}_{G_{e_k}}(P_{k,r})$$

is a direct sum decomposition into indecomposable modules, necessarily projective, by Corollary 4.8. Thus each projective indecomposable $\Bbbk M$-module is isomorphic to one of the form $\mathrm{Ind}_{G_{e_k}}(P)$ where $P$ is a projective indecomposable $\Bbbk G_{e_k}$-module and $1 \leq k \leq s$ by (16.5).

Now let $A = \Bbbk M/\Bbbk I_k$ (and so $e_k A e_k \cong \Bbbk G_{e_k}$). Then $A$ is projective, and hence injective, as a $\Bbbk M$-module and therefore is injective as an $A$-module. Thus $A$ is self-injective. Let $P$ be a projective indecomposable $\Bbbk G_{e_k}$-module. Then $P$ is injective because $\Bbbk G_{e_k}$ is self-injective by Theorem 16.4. Thus $\mathrm{Coind}_{e_k}(P) = \mathrm{Coind}_{G_{e_k}}(P)$ is an injective indecomposable $A$-module by Proposition 4.3 and Corollary 4.8, and hence is a projective indecomposable $A$-module by Proposition 16.1. Therefore, since $A$ is a projective $\Bbbk M$-module, we conclude that $\mathrm{Coind}_{G_{e_k}}(P)$ is a projective indecomposable $\Bbbk M$-module. We then deduce that $\mathrm{Coind}_{G_{e_k}}(P) \cong \mathrm{Ind}_{G_{e_i}}(Q)$ for some $1 \leq i \leq s$ and projective indecomposable $\Bbbk G_{e_i}$-module $Q$. But since $I(e_k)$ annihilates $\mathrm{Coind}_{G_{e_k}}(P)$ and $I(e_i)$ annihilates $\mathrm{Ind}_{G_{e_i}}(Q)$ we deduce that $i = k$ and $P \cong e_k \mathrm{Coind}_{G_{e_k}}(P) \cong e_k \mathrm{Ind}_{G_{e_k}}(Q) \cong Q$. Thus $\mathrm{Coind}_{G_{e_k}}(P) \cong \mathrm{Ind}_{G_{e_k}}(P)$ for each projective indecomposable $\Bbbk G_{e_k}$-module $P$. It follows that

$$\mathrm{Ind}_{G_{e_k}}(\Bbbk G_{e_k}) \cong \bigoplus_{r=1}^{m_k} \mathrm{Ind}_{G_{e_k}}(P_{k,r}) \cong \bigoplus_{r=1}^{m_k} \mathrm{Coind}_{G_{e_k}}(P_{k,r}) \cong \mathrm{Coind}_{G_{e_k}}(\Bbbk G_{e_k})$$

completing the proof that (i) implies (vii). This establishes the theorem. □

Unlike the case of semisimplicity, a monoid algebra $\Bbbk M$ can be a symmetric algebra without $M$ being regular.

*Example 16.7.* Let $M = \{1, x, x^2\}$ where $x^2 = x^3$. Notice that $M$, and hence $\Bbbk M$, is commutative. Also $M$ is not regular. We claim that $\Bbbk M$ is a symmetric algebra for any field $\Bbbk$. Indeed, define $\lambda\colon \Bbbk M \longrightarrow \Bbbk$ by $\lambda(1) = 2 = \lambda(x)$ and $\lambda(x^2) = 1$. Then $\ker \lambda$ has basis by $1 - x$ and $1 - 2x^2$. It is routine to verify that $\ker \lambda$ contains no non-zero ideal of $\Bbbk M$.

## 16.3 Exercises

**16.1.** Prove Proposition 16.2.

**16.2.** Prove that a direct product of symmetric algebras is symmetric.

**16.3.** Prove Theorem 16.4.

**16.4.** Verify that $\ker \lambda$ from Example 16.7 contains no non-zero ideals.

**16.5.** Let $\mathbb{F}$ be a finite field of characteristic $p$ and let $\Bbbk$ be any field of characteristic $p$. Prove that $\Bbbk M_n(\mathbb{F})$ is not self-injective. (Hint: prove the sandwich matrix for the $\mathscr{J}$-class of rank 1 matrices is not invertible over the group algebra of the group of non-zero elements of $\mathbb{F}$.)

**16.6.** Prove that $\Bbbk M_2(\mathbb{F}_2)$ is self-injective if the characteristic of $\Bbbk$ is different than 2 and that $\mathbb{F}_3 M_2(\mathbb{F}_2)$ is self-injective but not semisimple.

m(A/I)

# 17

# Global Dimension

An important homological invariant of a finite dimensional algebra $A$ is its global dimension. It measures the maximum length of a minimal projective resolution of an $A$-module. Nico obtained a bound on the global dimension of $\Bbbk M$ for a regular monoid $M$ in "good" characteristic [Nic71, Nic72]. Essentially, Nico discovered that $\Bbbk M$ is a quasi-hereditary algebra in the sense of Cline, Parshall and Scott [CPS88], more than 15 years before the notion was defined, and proved the corresponding bound on global dimension. Putcha was the first to observe that $\Bbbk M$ is quasi-hereditary [Put98]. Here we provide a direct approach to Nico's theorem, without introducing the machinery of quasi-hereditary algebras, although it is hidden under the surface (compare with [DR89]). Our approach follows the ideas in [MS11] and uses the results of Auslander, Platzeck and Todorov [APT92] on idempotent ideals in algebras. In this chapter, we do not hesitate to assume familiarity with notions from homological algebra, in particular, with properties of the Ext-functor.

## 17.1 Idempotent ideals and homological algebra

In this section $\Bbbk$ is a field and $A$ is a finite dimensional $\Bbbk$-algebra. The *global dimension* of $A$ is defined by

$$\mathrm{gl.\,dim}\, A = \sup\{n \mid \mathrm{Ext}^n(M,N) \neq 0, \text{for some } M,N \in A\text{-mod}\}. \qquad (17.1)$$

Note that $\mathrm{gl.\,dim}\, A = \infty$ is possible. For example, $\mathrm{gl.\,dim}\, A = 0$ if and only if each finite dimensional $A$-module is injective, if and only if $A$ is semisimple. Indeed, every $A$-module is injective if and only if every submodule has a complement. An algebra $A$ has global dimension at most 1 if and only if each submodule of a finite dimensional projective module is projective. One can readily show that this is equivalent to each left ideal of $A$ being a projective module. An algebra of global dimension at most 1 is called a *hereditary algebra*.

**Lemma 17.1.** *Let $M, N \in A$-mod. Then $\operatorname{Ext}^n_A(M,N) = 0$ if either of the following two conditions hold.*

*(i)* $\operatorname{Ext}^n_A(M,S) = 0$ *for each composition factor $S$ of $N$.*
*(ii)* $\operatorname{Ext}^n_A(S',N) = 0$ *for each composition factor $S'$ of $M$.*

*Proof.* We just handle the case of (i) as the other argument is dual. We proceed by induction on the number of composition factors (i.e., length) of $N$. The case that $N$ has one composition factor is just $N = S$ and so there is nothing to prove. Assume that it is true for $A$-modules with $k$ composition factors and suppose that $N$ has $k+1$ composition factors. Let $L$ be a maximal submodule of $N$ and $S = N/L$. Notice that $L$ has $k$ composition factors, all of which are composition factors of $N$. Thus $\operatorname{Ext}^n_A(M,L) = 0$ by induction. By assumption, $\operatorname{Ext}^n_A(M,S) = 0$. From the exact sequence

$$0 \longrightarrow L \longrightarrow N \longrightarrow S \longrightarrow 0$$

and the long exact sequence for Ext, we obtain an exact sequence

$$0 = \operatorname{Ext}^n_A(M,L) \longrightarrow \operatorname{Ext}^n_A(M,N) \longrightarrow \operatorname{Ext}^n_A(M,S) = 0$$

and so $\operatorname{Ext}^n_A(M,N) = 0$, as required. □

As a consequence, it follows that the sup in (17.1) is achieved by a pair of simple modules.

**Corollary 17.2.** *Let $A$ be a finite dimensional $\Bbbk$-algebra. Then*

$$\text{gl. dim}\, A = \sup\{n \mid \operatorname{Ext}^n(S,S') \neq 0, \text{ for some } S, S' \text{ simple}\}$$

*holds.*

*Proof.* Suppose that $\operatorname{Ext}^n_A(S,S') = 0$ for all finite dimensional simple $A$-modules. Let $M, N$ be finite dimensional $A$-modules. Then $\operatorname{Ext}^n_A(S',N) = 0$ for all simple $A$-modules $S'$ by Lemma 17.1(i). Therefore, $\operatorname{Ext}^n_A(M,N) = 0$ by Lemma 17.1(ii). The corollary follows. □

The next result is a special case of a very general result of Adams and Rieffel [AR67].

**Theorem 17.3.** *Let $A$ be a finite dimensional $\Bbbk$-algebra and let $e \in A$ be an idempotent. Assume that $Ae$ is a flat right $eAe$-module, respectively, $eA$ is a projective left $eAe$-module. Then there are isomorphisms*

$$\operatorname{Ext}^n_A(\operatorname{Ind}_e(V), W) \cong \operatorname{Ext}^n_{eAe}(V, \operatorname{Res}_e(W)) \tag{17.2}$$

$$\operatorname{Ext}^n_A(W, \operatorname{Coind}_e(V)) \cong \operatorname{Ext}^n_{eAe}(\operatorname{Res}_w(W), V) \tag{17.3}$$

*respectively, for all $n \geq 0$, $V \in eAe$-mod and $W \in A$-mod.*

*Proof.* We just handle (17.2), as the other isomorphism is proved in a dual fashion. Let $P_\bullet \longrightarrow V$ be a projective resolution by finite dimensional $eAe$-modules. Then $\mathrm{Ind}_e(P_n)$ is a projective $A$-module by Proposition 4.3. Also the functor $\mathrm{Ind}_e$ is exact by Proposition 4.2. Therefore, $\mathrm{Ind}_e(P_\bullet) \longrightarrow \mathrm{Ind}_e(V)$ is a projective resolution. We conclude using the adjointness of $\mathrm{Ind}_e$ and $\mathrm{Res}_e$ (Proposition 4.1) that

$$\begin{aligned}\mathrm{Ext}^n_A(\mathrm{Ind}_e(V), W) &\cong H^n(\mathrm{Hom}_A(\mathrm{Ind}_e(P_\bullet), W))\\ &\cong H^n(\mathrm{Hom}_{eAe}(P_\bullet, \mathrm{Res}_e(W)))\\ &\cong \mathrm{Ext}^n_{eAe}(V, \mathrm{Res}_e(W))\end{aligned}$$

as required. □

An ideal $I$ of $A$ is said to be *idempotent* if $I^2 = I$. Idempotent ideals enjoy a number of nice homological properties.

**Lemma 17.4.** *Let $I$ be an idempotent ideal of $A$. Then $\mathrm{Ext}^1_A(A/I, N) = 0$ for all $A/I$-modules $N$. If, in addition, $I$ is a projective left $A$-module, then $\mathrm{Ext}^n_A(A/I, N) = 0$ for all $n > 0$.*

*Proof.* Since $A$ is a projective module, the exact sequence

$$0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0 \tag{17.4}$$

gives rise to an exact sequence

$$\mathrm{Hom}_A(I, N) \longrightarrow \mathrm{Ext}^1_A(A/I, N) \longrightarrow \mathrm{Ext}^1_A(A, N) = 0.$$

But $IN = 0$ and so if $\varphi \in \mathrm{Hom}_A(I, N)$, then $\varphi(I) = \varphi(II) = I\varphi(I) = 0$. Therefore, $\mathrm{Hom}_A(I, N) = 0$ and so we conclude that $\mathrm{Ext}^1_A(A/I, N) = 0$.

Assume now that $I$ is projective and $n > 1$. Then the exact sequence (17.4) gives rise to an exact sequence

$$0 = \mathrm{Ext}^{n-1}_A(I, N) \longrightarrow \mathrm{Ext}^n_A(A/I, N) \longrightarrow \mathrm{Ext}^n_A(A, N) = 0$$

and so $\mathrm{Ext}^n_A(A/I, N) = 0$, as required. □

As a corollary, we show that under suitable hypotheses on the idempotent ideal $I$, we can compute the Ext-functor between $A/I$-modules over either $A$ or over $A/I$ and obtain the same result.

**Theorem 17.5.** *Let $A$ be a finite dimensional $\Bbbk$-algebra and let $I$ be an idempotent ideal. Let $M, N$ be finite dimensional $A/I$-modules. Then*

$$\mathrm{Ext}^1_A(M, N) \cong \mathrm{Ext}^1_{A/I}(M, N)$$

*holds. If, in addition, $I$ is a projective left $A$-module, then*

$$\mathrm{Ext}^n_A(M,N) \cong \mathrm{Ext}^n_{A/I}(M,N) \tag{17.5}$$

*for all $n \geq 0$.*

*Proof.* Choose a presentation of $M$ as an $A/I$-module of the form

$$0 \longrightarrow L \xrightarrow{f} m(A/I) \longrightarrow M \longrightarrow 0 \quad .$$

Since $\mathrm{Ext}^1_{A/I}(m(A/I),N) = 0$ by freeness of $m(A/I)$ and $\mathrm{Ext}^1_A(m(A/I),N) = 0$ by Lemma 17.4 we have the following two exact sequences

$$\mathrm{Hom}_{A/I}(m(A/I),N) \xrightarrow{f^*} \mathrm{Hom}_{A/I}(L,N) \longrightarrow \mathrm{Ext}^1_{A/I}(M,N) \longrightarrow 0$$

and

$$\mathrm{Hom}_A(m(A/I),N) \xrightarrow{f^*} \mathrm{Hom}_A(L,N) \longrightarrow \mathrm{Ext}^1_A(M,N) \longrightarrow 0$$

where $f^*$ is the map induced by $f$. But the functors $\mathrm{Hom}_A(-,N)$ and $\mathrm{Hom}_{A/I}(-,N)$ on $A/I$-mod are isomorphic. Thus both $\mathrm{Ext}^1_{A/I}(M,N)$ and $\mathrm{Ext}^1_A(M,N)$ can be identified with the cokernel of the morphism

$$\mathrm{Hom}_{A/I}(m(A/I),N) \xrightarrow{f^*} \mathrm{Hom}_{A/I}(L,N)$$

thereby establishing the first isomorphism.

The second isomorphism is proved simultaneously for all finite dimensional $A/I$-modules $M$ by induction on $n$. The case $n = 0$ is clear and the case $n = 1$ has already been handled. So assume that (17.5) is true for all $M$ and we prove the corresponding statement for $n+1$ with $n \geq 1$. We retain the above notation. Then, for all $k \geq 1$, we have $\mathrm{Ext}^k_{A/I}(m(A/I),N) = 0$ by projectivity of $m(A/I)$ and $\mathrm{Ext}^k_A(m(A/I),N) = 0$ by Lemma 17.4. Therefore, we have the following two exact sequences

$$0 \longrightarrow \mathrm{Ext}^n_{A/I}(L,N) \longrightarrow \mathrm{Ext}^{n+1}_{A/I}(M,N) \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{Ext}^n_A(L,N) \longrightarrow \mathrm{Ext}^{n+1}_A(M,N) \longrightarrow 0$$

By the inductive hypothesis, we have $\mathrm{Ext}^n_{A/I}(L,N) \cong \mathrm{Ext}^n_A(L,N)$ and so $\mathrm{Ext}^{n+1}_{A/I}(M,N) \cong \mathrm{Ext}^{n+1}_A(M,N)$. This completes the proof. $\square$

## 17.2 Global dimension and homological properties of regular monoids

In this section we prove Nico's result on the global dimension of the algebra of a regular monoid. The key ingredient is that we can always work modulo an ideal of the monoid. The following lemma is from [MS11].

**Lemma 17.6.** *Let $M$ be a regular monoid and $\Bbbk$ a field. Let $I$ be an ideal of $M$. Then*

$$\mathrm{Ext}^n_{\Bbbk M}(V,W) \cong \mathrm{Ext}^n_{\Bbbk M/\Bbbk I}(V,W)$$

*for any $\Bbbk M/\Bbbk I$-modules $V,W$ and all $n \geq 0$.*

*Proof.* For the purposes of this proof, we allow $I = \emptyset$ and we proceed by induction on the number of $\mathscr{J}$-classes of $M$ contained in $I$. If $I = \emptyset$, then there is nothing to prove. So assume that the lemma is true for ideals with $k-1$ $\mathscr{J}$-classes and that $I$ has $k \geq 1$ $\mathscr{J}$-classes. Let $J$ be a maximal $\mathscr{J}$-class of $I$, that is, $MJM$ is not contained in $MJ'M$ for any $\mathscr{J}$-class $J' \subseteq I$. Then $I' = I \setminus J$ is an ideal of $M$ with $k-1$ $\mathscr{J}$-classes. Clearly, $V,W$ are $\Bbbk M/\Bbbk I'$-modules and so

$$\mathrm{Ext}^n_{\Bbbk M}(V,W) \cong \mathrm{Ext}^n_{\Bbbk M/\Bbbk I'}(V,W) \tag{17.6}$$

for all $n \geq 0$ by induction.

Note that $\Bbbk I/\Bbbk I'$ is an idempotent ideal of $\Bbbk M/\Bbbk I'$ (idempotent as a consequence of Proposition 1.21) and $\Bbbk M/\Bbbk I \cong (\Bbbk M/\Bbbk I')/(\Bbbk I/\Bbbk I')$ . Let $e \in E(J)$. If $r$ is the number of $\mathscr{L}$-classes contained in $J$, then we have

$$\Bbbk I/\Bbbk I' \cong \Bbbk J \cong r\Bbbk L_e \cong r\,[(\Bbbk M/\Bbbk I')e]$$

by Lemma 5.13 and Theorem 1.12. Thus $\Bbbk I/\Bbbk I'$ is projective as a $\Bbbk M/\Bbbk I'$-module. Applying Theorem 17.5 and (17.6) yields

$$\mathrm{Ext}^n_{\Bbbk M}(V,W) \cong \mathrm{Ext}^n_{\Bbbk M/\Bbbk I'}(V,W) \cong \mathrm{Ext}^n_{\Bbbk M/\Bbbk I}(V,W)$$

completing the induction and the proof of the lemma. □

If $M$ is a regular monoid and $e \in E(M)$, we define the *height* of $e$ by

$$\mathrm{ht}(e) = \max\{k \mid MeM = Me_0M \subsetneq Me_1M \subsetneq \cdots \subsetneq Me_kM = M\}.$$

Notice that $\mathrm{ht}(e) = 0$ if and only if $e = 1$. The maximum height is achieved in the minimal ideal. The key step to proving Nico's theorem is the following lemma bounding the vanishing degree of Ext between simples in terms of the heights of their apexes.

**Lemma 17.7.** *Let $M$ be a regular monoid and $\Bbbk$ a field whose characteristic is $0$ or does not divide the order of any maximal subgroup of $M$. Let $S,S'$ be simple $\Bbbk M$-modules with respective apexes $e,e'$. Then $\mathrm{Ext}^r_{\Bbbk M}(S,S') = 0$ for all $r > \mathrm{ht}(e) + \mathrm{ht}(e')$.*

*Proof.* We induct on the quantity $\mathrm{ht}(e)+\mathrm{ht}(e')$. If $0=\mathrm{ht}(e)+\mathrm{ht}(e')$, then $e=1=e'$. Then $S,S'$ are modules over $\Bbbk M/\Bbbk I(1)\cong \Bbbk G_1$. Lemma 17.6 then implies

$$\mathrm{Ext}^r_{\Bbbk M}(S,S')\cong \mathrm{Ext}^r_{\Bbbk G_1}(S,S')=0$$

for $r>0$ because $\Bbbk G_1$ is a semisimple algebra. Assume now that the result holds for all smaller values of $\mathrm{ht}(e)+\mathrm{ht}(e')$ and that $\mathrm{ht}(e)+\mathrm{ht}(e')>0$. Note that $r>1$ from now on.

We handle two cases. Assume first that $Me'M\nsubseteq MeM$. Let $I=MeM\setminus J_e$; this is an ideal. Let $A=\Bbbk M/\Bbbk I$. Then $eAe\cong \Bbbk G_e$ by Corollary 1.15 and $Ae\cong \Bbbk L_e$ by Theorem 1.12. It follows that $\mathrm{Ind}_e(V)=\mathrm{Ind}_{G_e}(V)$ for any $\Bbbk G_e$-module $V$, where we view $\mathrm{Ind}_e\colon eAe\text{-mod}\longrightarrow A\text{-mod}$. Let us assume that $S=V^\sharp$ with $V$ a simple $\Bbbk G_e$-module. Note that $S'$ is an $A$-module because $e\in I(e')$ and hence $I\subseteq I(e')$. Also, we have that $\mathrm{Ind}_e(V)$ is a projective $A$-module by Proposition 4.3 because $V$ is a projective $\Bbbk G_e$-module by semisimplicity of $\Bbbk G_e$. Therefore, the exact sequence of $A$-modules

$$0\longrightarrow \mathrm{rad}(\mathrm{Ind}_e(V))\longrightarrow \mathrm{Ind}_e(V)\longrightarrow S\longrightarrow 0$$

yields an exact sequence

$$0\longrightarrow \mathrm{Ext}^{r-1}_A(\mathrm{rad}(\mathrm{Ind}_e(V)),S')\longrightarrow \mathrm{Ext}^r_A(S,S')\longrightarrow 0.$$

Lemma 17.6 then implies that $\mathrm{Ext}^{r-1}_{\Bbbk M}(\mathrm{rad}(\mathrm{Ind}_e(V)),S')\cong \mathrm{Ext}^r_{\Bbbk M}(S,S')$. By Theorem 5.5 every composition factor $L$ of $\mathrm{rad}(\mathrm{Ind}_e(V))=\mathrm{rad}(\mathrm{Ind}_{G_e}(V))$ has apex $f$ with $MeM\subsetneq MfM$. Therefore, by induction $\mathrm{Ext}^{r-1}_{\Bbbk M}(L,S')=0$. We conclude that $\mathrm{Ext}^{r-1}_{\Bbbk M}(\mathrm{rad}(\mathrm{Ind}_e(V)),S')=0$ by Lemma 17.1. Thus $\mathrm{Ext}^r_{\Bbbk M}(S,S')=0$, as was required.

Next we assume that $Me'M\subseteq MeM$. Let $A=\Bbbk M/\Bbbk I(e')$. Then both $S,S'$ are $A$-modules. Let $S'=W^\sharp$ with $W$ a simple $\Bbbk G_{e'}$-module. Then $W$ is an injective $\Bbbk G_{e'}$-module by semisimplicty of $\Bbbk G_{e'}$, whence $\mathrm{Coind}_{G_{e'}}(W)=\mathrm{Coind}_{e'}(W)$ is an injective $A$-module by Proposition 4.3, and $S'=\mathrm{soc}(\mathrm{Coind}_{G_{e'}}(W))$. Put $N=\mathrm{Coind}_{G_e}(W)/S'$. Then the exact sequence of $A$-modules

$$0\longrightarrow S'\longrightarrow \mathrm{Coind}_{G_{e'}}(W)\longrightarrow N\longrightarrow 0$$

leads to an exact sequence

$$0\longrightarrow \mathrm{Ext}^{r-1}_A(S,N)\longrightarrow \mathrm{Ext}^r_A(S,S')\longrightarrow 0.$$

Lemma 17.6 then yields that $\mathrm{Ext}^{r-1}_{\Bbbk M}(S,N)\cong \mathrm{Ext}^r_{\Bbbk M}(S,S')$. But each composition factor $L$ of $N$ has apex $f$ with $Me'M\subsetneq MfM$ by Theorem 5.5. Induction then implies that $\mathrm{Ext}^{r-1}_{\Bbbk M}(S,L)=0$. It follows that $\mathrm{Ext}^{r-1}_{\Bbbk M}(S,N)=0$

by Lemma 17.1 and hence $\mathrm{Ext}^r_{\Bbbk M}(S,S')=0$, as desired. This completes the proof of the lemma. □

We now prove Nico's theorem [Nic71, Nic72].

**Theorem 17.8.** *Let $M$ be a finite regular monoid and $\Bbbk$ a field. Then* $\mathrm{gl.\,dim}\,\Bbbk M$ *is finite if and only if the characteristic of $\Bbbk$ is $0$ or does not divide the order of any maximal subgroup of $M$. If these conditions hold, then* $\mathrm{gl.\,dim}\,\Bbbk M\leq 2k$ *where $k$ is the length of the longest chain*

$$Mm_0M\subsetneq Mm_1M\subsetneq\cdots\subsetneq Mm_kM=M$$

*of principal ideals in $M$.*

*Proof.* Suppose first that the characteristic of $\Bbbk$ divides the order of the maximal subgroup $G_e$. Then it is well known that $\mathrm{gl.\,dim}\,\Bbbk G_e=\infty$ (cf. Exercise 17.6). Therefore, we can find, for any $n\geq 0$, a pair $V,W$ of finite dimensional $\Bbbk G_e$-modules with $\mathrm{Ext}^n_{\Bbbk G_e}(V,W)\neq 0$. Let $A=\Bbbk M/\Bbbk I(e)$. Note that $Ae\cong \Bbbk L_e$ is a free right module over $eAe\cong \Bbbk G_e$ by Proposition 1.9. Therefore, Lemma 17.6 and Theorem 17.3 yield

$$\begin{aligned}\mathrm{Ext}^n_{\Bbbk M}(\mathrm{Ind}_{G_e}(V),\mathrm{Coind}_{G_e}(W)) &\cong \mathrm{Ext}^n_A(\mathrm{Ind}_{G_e}(V),\mathrm{Coind}_{G_e}(W))\\ &\cong \mathrm{Ext}^n_{eAe}(V,\mathrm{Res}_e(\mathrm{Coind}_e(W)))\\ &\cong \mathrm{Ext}^n_{\Bbbk G_e}(V,W)\neq 0\end{aligned}$$

where we have used that $\mathrm{Res}_e\circ\mathrm{Coind}_e$ is isomorphic to the identity functor by Proposition 4.4. We conclude that $\mathrm{gl.\,dim}\,\Bbbk M=\infty$.

If $\Bbbk$ has characteristic $0$ or its characteristic divides the order of no maximal subgroup of $M$, then $\mathrm{gl.\,dim}\,\Bbbk M\leq 2k$ by Lemma 17.7 as each idempotent has height at most $k$. □

Computing $\mathrm{Ext}^1$ is important for computing the quiver of an algebra. The following proposition is then helpful.

**Proposition 17.9.** *Let $M$ be a regular monoid and $\Bbbk$ a field such that the characteristic of $\Bbbk$ is $0$ or does not divide the order of any maximal subgroup of $M$. Let $S,S'$ be simple modules with apexes $e,e'$ respectively. Then one has*

$$\mathrm{Ext}^1_{\Bbbk M}(S,S')=0$$

*if $MeM=Me'M$ or if $MeM$ and $Me'M$ are incomparable.*

*Proof.* Let $I=MeM\setminus J_e$ and $A=\Bbbk M/\Bbbk I$. Then $S,S'$ are $A$-modules. Suppose that $S=V^\sharp$ with $V$ a simple $\Bbbk G_e$-module. Then $V$ is projective by semisimplicity of $\Bbbk G_e$. Therefore, $\mathrm{Ind}_{G_e}(V)=\mathrm{Ind}_e(V)$ is a projective $A$-module by Proposition 4.3. The exact sequence

$$0\longrightarrow \mathrm{rad}(\mathrm{Ind}_e(V))\longrightarrow \mathrm{Ind}_e(V)\longrightarrow S\longrightarrow 0$$

then yields an exact sequence

$$\mathrm{Hom}_A(\mathrm{rad}(\mathrm{Ind}_e(V)), S') \longrightarrow \mathrm{Ext}^1_A(S, S') \longrightarrow \mathrm{Ext}^1_A(\mathrm{Ind}_e(V), S') = 0.$$

But every composition factor of $\mathrm{rad}(\mathrm{Ind}_e(V)) = \mathrm{rad}(\mathrm{Ind}_{G_e}(V))$ has apex $f$ with $MfM \supsetneq MeM$ by Theorem 5.5. By assumption on $e'$, we conclude that $\mathrm{Hom}_A(\mathrm{rad}(\mathrm{Ind}_e(V)), S') = 0$ and hence $\mathrm{Ext}^1_A(S, S') = 0$. But $\mathrm{Ext}^1_{\Bbbk M}(S, S') = \mathrm{Ext}^1_A(S, S') = 0$ by Lemma 17.6. This completes the proof. $\square$

We now provide a homological proof of the sufficiency of the conditions for semisimplicity in Theorem 5.14. This proof is novel to the text.

**Corollary 17.10.** *Let $M$ be a finite regular monoid and $\Bbbk$ a field. Suppose that the characteristic of $\Bbbk$ does not divide the order of $G_e$ for any $e \in E(M)$ and that the natural homomorphism*

$$\varphi_{\Bbbk G_e} \colon \mathrm{Ind}_{G_e}(\Bbbk G_e) \longrightarrow \mathrm{Coind}_{G_e}(\Bbbk G_e)$$

*is an isomorphism for all $e \in E(M)$. Then $\Bbbk M$ is semisimple.*

*Proof.* The algebra $\Bbbk M$ is semisimple if and only if $\mathrm{gl.\,dim}\, \Bbbk M = 0$. We just are required to show, by Corollary 17.2 and Exercise 17.3, that $\mathrm{Ext}^1_{\Bbbk M}(S, S') = 0$ for all simple $\Bbbk M$-modules $S, S'$. Let $S$ have apex $e$ and $S'$ have apex $f$. By Proposition 17.9 it suffices to consider the cases that $MeM \subsetneq MfM$ and $MfM \subsetneq MeM$. The algebras $\Bbbk G_e$ and $\Bbbk G_f$ are semisimple under the hypotheses on $\Bbbk$ by Maschke's theorem.

Assume first that $MeM \subsetneq MfM$. Let $A = \Bbbk M/\Bbbk I(e)$ and note that $S, S'$ are $A$-modules, $eAe \cong \Bbbk G_e$, which is semisimple, and $\mathrm{Ind}_{G_e}(V) = \mathrm{Ind}_e(V)$ for any $\Bbbk G_e$-module $V$. By Corollary 4.19, we have that $\mathrm{Ind}_e(\Bbbk G_e)$ is a semisimple $A$-module. On the other hand, it is a projective $A$-module by Proposition 4.3. Suppose that $S = V^\sharp$ with $V \in \mathrm{Irr}_\Bbbk(G_e)$. Then $V$ is a direct summand in $\Bbbk G_e$ (as $\Bbbk G_e$ is a semisimple algebra) and hence $\mathrm{Ind}_e(V)$ is a direct summand in $\mathrm{Ind}_e(\Bbbk G_e)$. It follows that $\mathrm{Ind}_e(V)$ is both semisimple and projective as an $A$-module. But then $\mathrm{Ind}_e(V)$ is, in fact, simple by Theorem 4.20 and $S = \mathrm{Ind}_e(V)$. Therefore, $\mathrm{Ext}^1_{\Bbbk M}(S, S') \cong \mathrm{Ext}^1_A(S, S') = 0$, where the first isomorphism follows from Lemma 17.6 and the second because $S$ is a projective $A$-module.

Next assume that $MfM \subsetneq MeM$ and let $A = \Bbbk M/\Bbbk I(f)$. Then $S, S'$ are $A$-modules, $fAf \cong \Bbbk G_f$, which is semisimple, and $\mathrm{Coind}_{G_f}(W) = \mathrm{Coind}_f(W)$ for any $\Bbbk G_f$-module $W$. Corollary 4.19 implies that $\mathrm{Coind}_f(\Bbbk G_f)$ is a semisimple $A$-module. But $\mathrm{Coind}_f(\Bbbk G_f)$ is an injective $A$-module by Proposition 4.3 (because $\Bbbk G_f$ is an injective $\Bbbk G_f$-module as the algebra $\Bbbk G_f$ is semisimple). Suppose that $S' = W^\sharp$ with $W \in \mathrm{Irr}_\Bbbk(G_f)$. Then $W$ is a direct summand in $\Bbbk G_f$ (because $\Bbbk G_f$ is a semisimple algebra) and, therefore, $\mathrm{Coind}_f(W)$ is a direct summand in $\mathrm{Coind}_f(\Bbbk G_f)$. We conclude that $\mathrm{Coind}_f(W)$ is both a semisimple and injective $A$-module. But then

$\mathrm{Coind}_f(V)$ is, in fact, simple by Theorem 4.20 and $S' = \mathrm{Coind}_f(V)$. Therefore, $\mathrm{Ext}^1_{\Bbbk M}(S, S') \cong \mathrm{Ext}^1_A(S, S') = 0$, where the first isomorphism follows from Lemma 17.6 and the second because $S'$ is an injective $A$-module. This completes the proof. □

## 17.3 Exercises

**17.1.** Prove that the following are equivalent for a finite dimensional $\Bbbk$-algebra $A$.

(i) gl. dim $A \leq 1$.
(ii) Each submodule of a finite dimensional projective $A$-module is projective.
(iii) Each left ideal of $A$ is projective.
(iv) $\mathrm{rad}(A)$ is projective.
(v) $\mathrm{rad}(P)$ is projective for each projective indecomposable $A$-module $P$.

**17.2.** Let $A$ be a finite dimensional $\Bbbk$-algebra and $V, W$ be finite dimensional $A$-modules. Prove that $\mathrm{Ext}^n_A(V, W) \cong \mathrm{Ext}^n_{A^{op}}(D(W), D(V))$. Deduce that gl. dim $A =$ gl. dim $A^{op}$.

**17.3.** Assume that $\mathrm{Ext}^{n+1}_A(V, W) = 0$ for all finite dimensional $A$-modules $V, W$ for a finite dimensional algebra $A$. Prove that gl. dim $A \leq n$.

**17.4.** Suppose that $A$ is a finite dimensional $\Bbbk$-algebra and that

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{d_0} V \longrightarrow 0$$

is a projective resolution of a finite dimensional $A$-module $V$. Prove that $\mathrm{Ext}^{n+k}_A(V, W) \cong \mathrm{Ext}^k_A(d_n(P_n), W)$ for all $n, k \geq 0$ and finite dimensional $A$-modules $W$.

**17.5.** Let $V$ be a finite dimensional $A$-module for a finite dimensional algebra $A$. Prove that $\mathrm{Ext}^{n+1}_A(V, W) = 0$ for all finite dimensional $A$-modules $W$ if and only if $V$ admits a projective resolution

$$0 \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{d_0} V \longrightarrow 0.$$

(Hint: use Exercise 17.4 to cut down an arbitrary projective resolution to one of this length.)

**17.6.** Let $A$ be a finite dimensional self-injective $\Bbbk$-algebra and let $V$ be a finite dimensional $A$-module. Prove that if $V$ is not projective, then there is no finite projective resolution of $V$. Deduce that $A$ is either semisimple or gl. dim $A = \infty$. Deduce, in particular, that if $G$ is a finite group and the characteristic of $\Bbbk$ divides $|G|$, then gl. dim $\Bbbk G = \infty$. (Hint: if $V$ is not projective and has a finite projective resolution, then consider a minimal length one and derive a contradiction using that projectives are injectives.)

**17.7.** Let $M$ be a finite regular monoid and $\Bbbk$ a field whose characteristic divides the order of no maximal subgroup of $M$. Prove that if $e \in E(M)$, $V$ is a $\Bbbk G_e$-module and $S$ is a simple $\Bbbk M$-module with apex $f$, then $\mathrm{Ext}^n_{\Bbbk M}(\mathrm{Ind}_{G_e}(V), S) = 0$ unless $MfM \subseteq MeM$ and $n \leq \mathrm{ht}(e) - \mathrm{ht}(f)$. Deduce that if $\mathrm{Ind}_{G_e}(V)$ is simple for each $e \in E(M)$ and simple $\Bbbk G_e$-module $V$, then gl. dim $\Bbbk M$ is bounded by the length of the longest chain

$$Mm_0M \subsetneq Mm_1M \subsetneq \cdots \subsetneq Mm_kM = M$$

of principal ideals in $M$. Formulate and prove a corresponding result for $\mathrm{Coind}_{G_e}(V)$.

**17.8.** Let $M$ be a finite regular monoid and $\Bbbk$ a field whose characteristic divides the order of no maximal subgroup of $M$. Prove that the following are equivalent.

(i) $\mathrm{Coind}_{G_e}(V)$ is simple for each simple $\Bbbk G_e$-module and $e \in E(M)$.
(ii) $\mathrm{Ind}_{G_e}(V)$ is a projective indecomposable module for each simple $\Bbbk G_e$-module and $e \in E(M)$.
(iii) The natural homomorphism $\varphi_{\Bbbk G_e}\colon \mathrm{Ind}_{G_e}(\Bbbk G_e) \longrightarrow \mathrm{Coind}_e(\Bbbk G_e)$ is surjective for each $e \in E(M)$.

(Hint: use Exercise 17.7.)

**17.9.** Let $M$ be a finite regular monoid and $\Bbbk$ a field whose characteristic divides the order of no maximal subgroup of $M$. Prove that the following are equivalent.

(i) $\mathrm{Ind}_{G_e}(V)$ is simple for each simple $\Bbbk G_e$-module and $e \in E(M)$.
(ii) $\mathrm{Coind}_{G_e}(V)$ is an injective indecomposable module for each simple $\Bbbk G_e$-module and $e \in E(M)$.
(iii) The natural homomorphism $\varphi_{\Bbbk G_e}\colon \mathrm{Ind}_{G_e}(\Bbbk G_e) \longrightarrow \mathrm{Coind}_e(\Bbbk G_e)$ is injective for each $e \in E(M)$.

(Hint: use Exercise 17.7.)

**17.10.** Let $G \leq S_n$ and let $M = G \cup C$ where $C$ is the set of constant mappings on $[n]$. Let $\Bbbk$ be a field of characteristic 0. Prove that $\Bbbk M$ is hereditary.

# 18

# Quivers of Monoid Algebras

In this chapter we provide a computation of the quiver of a left regular band algebra, a result of Saliola [Sal07], and of a $\mathscr{J}$-trivial monoid algebra, a result of Denton, Hivert, Schilling and Thiéry [DHST11]. These are special cases of the results of Margolis and the author [MS12a], computing the quiver of an arbitrary rectangular monoid algebra. However, the latter result is much more technical and beyond the scope of this text. We also describe the projective indecomposable modules for $\mathscr{R}$-trivial monoid algebras as partial transformation modules. This result is from the paper of Margolis and the author [MS12a] and again generalizes earlier results of Saliola [Sal07] for left regular bands and of Denton, Hivert, Schilling and Thiéry [DHST11] for $\mathscr{J}$-trivial monoids.

## 18.1 Quivers of algebras

We define in this section the Ext-equiver (or Gabriel quiver) of a finite dimensional algebra over an algebraically closed field. This notion was introduced by Gabriel [Gab72]. Good references for this material are [DK94, ARS97, Ben98, ASS06].

If $A$ is a finite dimensional algebra over an algebraically closed field $\Bbbk$, then the *Ext-quiver* or, more simply, *quiver* of $A$ is the directed graph $Q(A)$ with vertex set $Q(A)_0$ the set of isomorphism classes of simple $A$-modules and with edge set $Q(A)_1$ described as follows. The number of arrows from $[S]$ to $[S']$ is in bijection with $\dim \operatorname{Ext}^1_A(S, S')$. There are a number of alternative descriptions of $Q(A)$ that can be found in the literature [DK94, ARS97, Ben98, ASS06], but we shall not use them here. It is well known that $\operatorname{Ext}^1_A(S, S')$ classifies short exact sequences (or *extensions*, whence the name) of the form

$$0 \longrightarrow S' \longrightarrow E \longrightarrow S \longrightarrow 0 \tag{18.1}$$

where (18.1) is *equivalent* to an extension

$$0 \longrightarrow S' \longrightarrow E' \longrightarrow S \longrightarrow 0$$

if there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & S' \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & S \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & S' \arrow[r] & E' \arrow[r] & S \arrow[r] & 0
\end{tikzcd}$$

where we note that the middle arrow must be an isomorphism by the Five Lemma. The reader should consult [Ben98, CE99] or the appendix of [ASS06] for details.

To explain the importance of the quiver of an algebra, we need to recall the definition of an admissible ideal. Let $Q$ be a finite quiver. The *arrow ideal* $J$ of the path algebra $\Bbbk Q$ is the ideal with basis the set of paths of length at least 1. An ideal $I \subseteq \Bbbk Q$ is said to be an *admissible ideal* if there exists $n \geq 2$ such that $J^n \subseteq I \subseteq J^2$. The following theorem is due to Gabriel [Gab80] and can be found in [Ben98, Proposition 4.1.7] and [ASS06, Theorem II.3.7], for example.

**Theorem 18.1.** *Let $A$ be a basic finite dimensional algebra over an algebraically closed field $\Bbbk$. Then $A \cong \Bbbk Q(A)/I$ for some admissible ideal $I$. Conversely, if $I$ is an admissible ideal of $\Bbbk Q$, then $A = \Bbbk Q/I$ is a basic finite dimensional algebra and $Q(A) = Q$.*

Note that if $A$ is a finite dimensional algebra and $B$ is the unique basic algebra that is Morita equivalent to $A$ (i.e., such that $A$-mod and $B$-mod are equivalent), then $Q(A) = Q(B)$. Representing a basic algebra as a quotient of a path algebra by an admissible ideal is sometimes called finding a *quiver presentation* of the algebra. The next theorem is also due to Gabriel [Gab72]. See [Ben98, Proposition 4.2.4] and [ASS06, Theorem VII.1.7] for a proof.

**Theorem 18.2.** *Let $\Bbbk$ be an algebraically closed field and $Q$ an acyclic quiver. Then $\Bbbk Q$ is hereditary. Moreover, a basic finite dimensional $\Bbbk$-algebra $A$ is hereditary if and only if $Q(A)$ is acyclic and $A \cong \Bbbk Q(A)$.*

In other words, up to Morita equivalence, the basic finite dimensional algebras over an algebraically closed field are the path algebras of acyclic quivers.

## 18.2 Projective indecomposable modules for $\mathscr{R}$-trivial monoid algebras

In this section we give an explicit description of the projective indecomposable modules for the algebra $\Bbbk M$ of an $\mathscr{R}$-trivial monoid $M$. The results here are from the paper of Margolis and the author [MS12a], simultaneously generalizing earlier results of Saliola for left regular bands [Sal07] and Denton, Hivert, Schilling and Thiéry [DHST11] for $\mathscr{J}$-trivial monoids. Fix for this section a field $\Bbbk$ and a finite monoid $M$.

One defines an equivalence relation $\widetilde{\mathscr{L}}$ on $M$, following Fountain, Gomes and Gould [FGG99], by putting $m \mathrel{\widetilde{\mathscr{L}}} n$ if $me = m$ if and only if $ne = n$, for all idempotents $e \in E(M)$, that is, $m \mathrel{\widetilde{\mathscr{L}}} n$ if and only if they have the same idempotent right identities. The $\widetilde{\mathscr{L}}$-class of an element $m$ will be denoted $\widetilde{L}_m$. One can dually define the $\widetilde{\mathscr{R}}$-relation on a monoid by $m \mathrel{\widetilde{\mathscr{R}}} n$ if $em = m$ if and only if $en = n$ for all $e \in E(M)$. We write $\widetilde{R}_m$ for the $\widetilde{\mathscr{R}}$-class of $m \in M$. The reason for the notation $\widetilde{\mathscr{L}}$ is furnished by the following proposition.

**Proposition 18.3.** *Let $m, n \in M$. Then $m \mathrel{\mathscr{L}} n$ implies $m \mathrel{\widetilde{\mathscr{L}}} n$. The converse holds if $m, n$ are regular.*

*Proof.* If $m \mathrel{\mathscr{L}} n$, then $n = um$ with $u \in M$. Thus we have that if $e \in E(M)$ with $me = m$, then $ne = ume = um = n$. Dually, we have that $ne = n$ implies $me = m$. Thus $m \mathrel{\widetilde{\mathscr{L}}} n$. Next assume that $m, n$ are regular and that $m \mathrel{\widetilde{\mathscr{L}}} n$. Then $m = mam$ for some $a \in M$. But $(am)^2 = am$ and so $nam = n$ because $m \mathrel{\widetilde{\mathscr{L}}} n$. Therefore $n \in Mm$. A dual argument shows that $m \in Mn$ and so $m \mathrel{\mathscr{L}} n$, as required. $\square$

Thus $\mathscr{L}$ and $\widetilde{\mathscr{L}}$ coincide for regular monoids. It is not in general true that every $\widetilde{\mathscr{L}}$-class of a monoid contains an idempotent. However, this is the case for $\mathscr{R}$-trivial monoids.

**Proposition 18.4.** *Let $M$ be a finite $\mathscr{R}$-trivial monoid. Then each $\widetilde{\mathscr{L}}$-class of $M$ contains an idempotent (unique up to $\mathscr{L}$-equivalence).*

*Proof.* The uniqueness statement follows from Proposition 18.3. Let $m \in M$ and let $N = \{n \in M \mid mn = m\}$. Then $N$ is a submonoid of $M$. Let $e$ be an idempotent of the minimal ideal of $N$. We claim that $m \mathrel{\widetilde{\mathscr{L}}} e$. Indeed, if $f \in E(M)$ and $ef = e$, then trivially $mf = mef = me = m$. Conversely, if $mf = m$, then $f \in N$ and $ef$ is in the minimal ideal of $N$. By stability (Theorem 1.12), it follows that $eN = efN$. But $N$ is an $\mathscr{R}$-trivial monoid because $M$ is one. Thus $e = ef$. We conclude that $m \mathrel{\widetilde{\mathscr{L}}} e$, as required. $\square$

We now wish to put a $\Bbbk M$-module structure on $\Bbbk \widetilde{L}_m$ for $m \in M$.

**Proposition 18.5.** *Let $e \in E(M)$ and $X = Me \setminus \widetilde{L}_e$. Then $X$ is a left ideal. Therefore, $\Bbbk Me/\Bbbk X \cong \Bbbk \widetilde{L}_e$ has a natural $\Bbbk M$-module structure given by*

$$m \odot z = \begin{cases} mz, & \text{if } mz \in \widetilde{L}_e \\ 0, & \text{else} \end{cases}$$

*for $m \in M$ and $z \in \widetilde{L}_e$.*

*Proof.* First note that because $ee = e$, we have that $me = m$ for all $m \in \widetilde{L}_e$. Thus $\widetilde{L}_e \subseteq Me$. Trivially, if $f \in E(M)$ and $ef = e$, then $mf = m$ for all $m \in Me$. Hence if $x \in Me \setminus \widetilde{L}_e$, then there exists $f \in E(M)$ with $xf = x$ and $ef \neq e$. But then $mxf = mx$ for all $m \in M$ and so $mx \notin \widetilde{L}_e$. We conclude that $X$ is a left ideal. The remainder of the proposition is immediate. $\square$

We drop from now on the notation "$\odot$".

Let $M$ be an $\mathscr{R}$-trivial monoid. In the proof of Corollary 2.6 it was shown that each regular $\mathscr{J}$-class of $M$ is an $\mathscr{L}$-class. In light of this, and Corollary 5.7, the simple $\Bbbk M$-modules are all one-dimensional and in bijection with $\mathscr{L}$-classes of idempotents. To each such $\mathscr{L}$-class $L_e$ (with $e \in E(M)$), we have the corresponding simple $S_{L_e}$ with underlying vector space $\Bbbk$ and with action

$$mc = \begin{cases} c, & \text{if } MeM \subseteq MmM \\ 0, & \text{else} \end{cases}$$

for $m \in M$ and $c \in \Bbbk$. Our goal is to show that $\Bbbk\widetilde{L}_e$ is a projective indecomposable module with simple quotient $S_{L_e}$.

**Proposition 18.6.** *Let $M$ be an $\mathscr{R}$-trivial module and $e \in E(M)$. Then there is a natural $\Bbbk M$-module homomorphism $\eta_e \colon \Bbbk\widetilde{L}_e \longrightarrow S_{L_e}$ given by*

$$\eta_e(m) = \begin{cases} 1, & \text{if } m \in L_e \\ 0, & \text{if } m \in I(e) \end{cases}$$

*for $m \in \widetilde{L}_e$. Moreover,* $\ker \eta_e = \operatorname{rad}(\Bbbk\widetilde{L}_e)$.

*Proof.* If $n \in M$ with $MeM \subseteq MnM$, then $nm \in L_e$ for all $m \in L_e$ by Corollary 2.6. Otherwise, we have that $nL_e \subseteq Me \setminus L_e \subseteq I(e)$. As $\widetilde{L}_e \setminus L_e$ is contained in the left ideal $Me \setminus L_e$, it follows that $\eta_e$ is a $\Bbbk M$-module homomorphism. Because $S_{L_e}$ is simple, trivially $\operatorname{rad}(\Bbbk\widetilde{L}_e) \subseteq \ker \eta_e$. For the converse, it suffices by Theorem A.5 to prove that $\ker \eta_e \subseteq \operatorname{rad}(\Bbbk M)\Bbbk\widetilde{L}_e$.

Note that $\ker \eta_e$ is spanned by the elements of the form $x - x'$ with $x, x' \in L_e$ and the elements $y \in \widetilde{L}_e \setminus L_e$. First let $y \in \widetilde{L}_e \setminus L_e$. By Corollary 12.6, we have that $y - y^\omega \in \operatorname{rad}(\Bbbk M)$. Observe that $y^\omega \in Me \setminus L_e$. Indeed, we have that $\widetilde{L}_e \subseteq Me$ and so $My^\omega \subseteq My \subsetneq Me$. By Proposition 18.3 it follows that $y^\omega \notin \widetilde{L}_e$. So we have in the module $\Bbbk\widetilde{L}_e$ that $(y - y^\omega)e = ye = y$ because $y^\omega e = y^\omega \notin \widetilde{L}_e$. We conclude that $y \in \operatorname{rad}(\Bbbk M)\Bbbk\widetilde{L}_e$.

Next suppose that $x, x' \in L_e$. Then $x - x' \in \operatorname{rad}(\Bbbk M)$ by Corollary 12.6. But $(x - x')e = x - x'$ because $x, x' \in Me$. Thus $x - x' \in \operatorname{rad}(\Bbbk M)\Bbbk\widetilde{L}_e$. This completes the proof that $\ker \eta_e \subseteq \operatorname{rad}(\Bbbk\widetilde{L}_e)$. $\square$

As a corollary, we may deduce that $\eta_e \colon \Bbbk\widetilde{L}_e \longrightarrow S_{L_e}$ is a projective cover.

**Theorem 18.7.** *Let $M$ be a finite $\mathscr{R}$-trivial monoid and $\Bbbk$ a field. Let $e_1, \ldots, e_s$ be a complete set of representatives of the $\mathscr{L}$-classes of idempotents of $M$. Then the modules $\Bbbk\widetilde{L}_{e_1}, \ldots, \Bbbk\widetilde{L}_{e_s}$ form a complete set of representatives of the isomorphism classes of projective indecomposable $\Bbbk M$-modules,*

$$\Bbbk M \cong \bigoplus_{i=1}^{s} \Bbbk\widetilde{L}_{e_i}$$

*and $\eta_{e_i} \colon \Bbbk\widetilde{L}_{e_i} \longrightarrow S_{L_{e_i}}$ is a projective cover for $i = 1, \ldots, s$.*

*Proof.* One has $\Bbbk M/\operatorname{rad}(\Bbbk M) \cong \Bbbk\Lambda(M) \cong \Bbbk^{|\Lambda(M)|} = \Bbbk^s$ by Corollary 12.6 and Corollary 10.5. It follows that $\Bbbk M/\operatorname{rad}(\Bbbk M) \cong \bigoplus_{i=1}^{s} S_{L_{e_i}}$ as a $\Bbbk M$-module. Let $V = \bigoplus_{i=1}^{s} \Bbbk\widetilde{L}_{e_i}$. Then $V/\operatorname{rad}(V) \cong \bigoplus_{i=1}^{s} S_{L_{e_i}}$ by Proposition 18.6. Thus there is a projective cover $\pi \colon \Bbbk M \longrightarrow V$ by Theorem A.17. The homomorphism $\pi$ is an epimorphism. On the other hand, $M = \bigcup_{i=1}^{s} \widetilde{L}_{e_i}$ by Proposition 18.4, and the union is disjoint. Thus we have that

$$|M| = \sum_{i=1}^{s} |\widetilde{L}_{e_i}| = \dim V$$

and so the epimorphism $\pi$ is an isomorphism by dimension considerations. We conclude that $V \cong \Bbbk M$ and hence each $\Bbbk\widetilde{L}_{e_i}$ is a projective $\Bbbk M$-module. It now follows from Proposition 18.6 that

$$\eta_{e_i} \colon \Bbbk\widetilde{L}_{e_i} \longrightarrow S_{L_{e_i}}$$

is a projective cover and hence $\Bbbk\widetilde{L}_{e_i}$ is a projective indecomposable $\Bbbk M$-module, for $i = 1, \ldots, s$, by Theorem A.17. This completes the proof. $\square$

A complete set of orthogonal primitive idempotents for the algebra of an $\mathscr{R}$-trivial monoid was constructed in [BBBS11].

The following result will be used in the next two sections.

**Proposition 18.8.** *Let $M$ be an $\mathscr{R}$-trivial monoid and $\Bbbk$ a field. Let $e, f \in E(M)$. Then $\operatorname{Ext}^1_{\Bbbk M}(S_{L_e}, S_{L_f}) \cong \operatorname{Hom}_{\Bbbk M}(\operatorname{rad}(\Bbbk\widetilde{L}_e), S_{L_f})$.*

*Proof.* Let $\widetilde{L}_e$ be the $\widetilde{\mathscr{L}}$-class of $e \in E(M)$. Then we have the exact sequence

$$0 \longrightarrow \operatorname{rad}(\Bbbk\widetilde{L}_e) \longrightarrow \Bbbk\widetilde{L}_e \longrightarrow S_{L_e} \longrightarrow 0$$

and hence an exact sequence

$$0 \longrightarrow \operatorname{Hom}_{\Bbbk M}(S_{L_e}, S_{L_f}) \longrightarrow \operatorname{Hom}_{\Bbbk M}(\Bbbk\widetilde{L}_e, S_{L_f})$$

$$\longrightarrow \operatorname{Hom}_{\Bbbk M}(\operatorname{rad}(\Bbbk\widetilde{L}_e), S_{L_f}) \longrightarrow \operatorname{Ext}^1_{\Bbbk M}(S_{L_e}, S_{L_f}) \longrightarrow 0$$

by the projectivity of $\Bbbk\widetilde{L}_e$. The map

$$\mathrm{Hom}_{\Bbbk M}(S_{L_e}, S_{L_f}) \longrightarrow \mathrm{Hom}_{\Bbbk M}(\Bbbk\widetilde{L}_e, S_{L_f})$$

is an isomorphism because $\mathrm{rad}(\Bbbk\widetilde{L}_e)$ is contained in the kernel of any homomorphism from $\Bbbk\widetilde{L}_e$ to the simple module $S_{L_f}$ and hence $\mathrm{Ext}^1_{\Bbbk M}(S_{L_e}, S_{L_f}) \cong \mathrm{Hom}_{\Bbbk M}(\mathrm{rad}(\Bbbk\widetilde{L}_e), S_{L_f})$. $\square$

Let us end this section with an explicit computation of the Cartan matrix of the algebra of an $\mathscr{R}$-trivial monoid from [MS12a], generalizing earlier results of Saliola [Sal07] and Denton *et al.* [DHST11].

**Theorem 18.9.** *Let $M$ be a finite $\mathscr{R}$-trivial monoid and $\Bbbk$ an algebraically closed field. Let $e_1, \ldots, e_s$ be a complete set of representatives of the $\mathscr{J}$-classes of idempotents of $M$. Then the Cartan matrix of $M$ is given by*

$$C_{ij} = \sum_{Me_kM \subseteq Me_iM} |e_kM \cap \widetilde{L}_{e_j}| \cdot \mu(Me_kM, Me_iM)$$

*where $\mu$ is the Möbius function of the lattice $\Lambda(M) = \{MeM \mid e \in E(M)\}$.*

*Proof.* Recall that $C_{ij}$ is $[\Bbbk\widetilde{L}_{e_j} : S_{L_{e_i}}]$ because $\Bbbk\widetilde{L}_{e_j} \longrightarrow S_{L_{e_j}}$ is a projective cover. The character $\chi$ of $\Bbbk\widetilde{L}_{e_j}$ is given by

$$\chi(m) = \left|\{x \in \widetilde{L}_{e_j} \mid mx = x\}\right|.$$

Therefore, we have $\chi(e) = |eM \cap \widetilde{L}_{e_j}|$ for $e \in E(M)$. The result now follows from Corollary 8.21. $\square$

Let us specialize this result to the case of a $\mathscr{J}$-trivial monoid $M$. Since each regular $\mathscr{J}$-class of $M$, and hence each regular $\mathscr{L}$-class, contains a single idempotent, it follows that each $\widetilde{\mathscr{L}}$-class of $M$ contains a unique idempotent by Proposition 18.3. If $m \in M$, denote by $m^-$ the unique idempotent in $\widetilde{L}_m$. Each $\widetilde{\mathscr{R}}$-class will also contain a unique idempotent by the dual of Proposition 18.3. The unique idempotent in $\widetilde{R}_m$ will be denoted by $m^+$. One then has that $m^+mm^- = m$ and if $e, f \in E(M)$ with $emf = m$, then $em^+ = m^+$ and $m^-f = m^-$. The following result is from the paper [DHST11].

**Theorem 18.10.** *Let $M$ be a finite $\mathscr{J}$-trivial monoid and $\Bbbk$ an algebraically closed field. Let $e_1, \ldots, e_s$ be the idempotents of $M$. Then the Cartan matrix of $M$ is given by*

$$C_{ij} = \left|\{m \in M \mid m^+ = e_i,\ m^- = e_j\}\right|$$

*for $1 \leq i, k \leq s$.*

*Proof.* Observe that $m \in e_iM \cap \widetilde{L}_{e_j}$ if and only if $m^- = e_j$ and $Mm^+M \subseteq Me_iM$ because in a $\mathscr{J}$-trivial monoid $MeM \subseteq MfM$ if and only if $fe = e$ for all $e, f \in E(M)$ by the dual of Corollary 2.6. Thus we have that

$$|e_iM \cap \widetilde{L}_{e_j}| = \sum_{Me_kM \subseteq Me_iM} \left|\{m \in M \mid m^+ = e_k,\ m^- = e_j\}\right|. \qquad (18.2)$$

An application of Möbius inversion (Theorem 7.6) to (18.2) yields

$$\left|\{m \in M \mid m^+ = e_i,\ m^- = e_j\}\right| = \sum_{Me_kM \subseteq Me_iM} |e_kM \cap \widetilde{L}_{e_j}| \mu(Me_kM, Me_iM)$$

from which the result follows in conjunction with Theorem 18.9. □

## 18.3 The quiver of a left regular band

Fix for this section an algebraically closed field $\Bbbk$. Recall that a left regular band is a regular $\mathscr{R}$-trivial monoid and that each element of a left regular band is idempotent. By Proposition 18.3, we have that $\widetilde{L}_m = L_m$, for $m \in M$, and hence the projective cover of $S_{L_e}$ is $\Bbbk L_e$ by Theorem 18.7.

The following theorem is due to Saliola [Sal07]. A more conceptual, topological proof is given in [MSS]. We retain the notation of the previous section.

**Theorem 18.11.** *Let $M$ be a left regular band and $\Bbbk$ an algebraically closed field. Let $Q(\Bbbk M)$ be the quiver of $\Bbbk M$. Then the vertex set of $Q(\Bbbk M)$ is the set $\Lambda(M) = \{MmM \mid m \in M\}$. There are no arrows from $MmM$ to $MnM$ unless $MmM \subsetneq MnM$. If $MmM \subsetneq MnM$, then the number of arrows from $MmM$ to $MnM$ is one less than the number of connected components of the graph $\Gamma(MmM, MnM)$ with vertex set $nM \cap L_m$ and adjacency relation $v \sim w$ if there exists $x \in nM \setminus \{n\}$ with $xv = v$ and $xw = w$.*

*Proof.* We already know that there is a bijection between $\Lambda(M)$ and simple modules via $MmM \mapsto S_{L_m}$. By Proposition 18.8, we have $\mathrm{Ext}^1_{\Bbbk M}(S_{L_m}, S_{L_n}) \cong \mathrm{Hom}_{\Bbbk M}(\mathrm{rad}(\Bbbk L_m), S_{L_n})$ as $\widetilde{L}_m = L_m$ for $m \in M = E(M)$.

Since the mapping $\Bbbk L_m \longrightarrow S_{L_m}$ sends each basis element $x \in L_m$ to 1, its kernel $\mathrm{rad}(\Bbbk L_m)$ is spanned by all differences $x - y$ with $x, y \in L_m$ and has basis $x - m$ with $x \in L_m \setminus \{m\}$. Suppose that it is not true that $MmM \subsetneq MnM$. Let $\varphi \in \mathrm{Hom}_{\Bbbk M}(\mathrm{rad}(\Bbbk L_m), S_{L_n})$. Then $\varphi(x - m) = n\varphi(x - m) = \varphi(nx - nm) = 0$ because if $MmM = MnM$, then $nx = n = nm$ and if $MmM \nsubseteq MnM$, then $n$ annihilates $\Bbbk L_m$ and hence its submodule $\mathrm{rad}(\Bbbk L_m)$. We conclude that $\varphi = 0$ and hence $\dim \mathrm{Hom}_{\Bbbk M}(\mathrm{rad}(\Bbbk L_m), S_{L_n}) = 0$. Thus there are no arrows from $MmM$ to $MnM$.

So we are left with the case $MmM \subsetneq MnM$. Then $L_{nm} = L_m$ and so without loss of generality we may assume that $nm = m$. Fix, for each connected component $C$ of $\Gamma(MmM, MnM)$, a vertex $\overline{C}$. We write $C_x$ for the component of $x \in nM \cap L_m$. Assume that $m = \overline{C_m}$. Let $V$ be the subspace of $\mathrm{rad}(\Bbbk L_m)$ spanned by all differences $\overline{C} - m$ with $C$ a connected component different than $C_m$. It is easy to see that the elements $\overline{C_x} - m$ with $C_x \neq C_m$ form a basis for $V$ and so $\dim V$ is one less than the number of connected components of $\Gamma(MmM, MnM)$. Recall that $S_{L_n} = \Bbbk$ as a vector space. Hence there is a $\Bbbk$-vector space homomorphism

$$\Psi\colon \operatorname{Hom}_{\Bbbk M}(\mathrm{rad}(\Bbbk L_m), S_{L_n}) \longrightarrow \operatorname{Hom}_{\Bbbk}(V, \Bbbk)$$

given by $\varphi \mapsto \varphi|_V$. We claim that $\Psi$ is an isomorphism.

Suppose that $\Psi(\varphi) = 0$. Let $x \in L_m$. We need to show that $\varphi(x - m) = 0$. Because $\varphi(x-m) = n\varphi(x-m) = \varphi(nx - nm) = \varphi(nx - m)$, we may assume without loss of generality that $x \in nM \cap L_m$. We claim that if $a \sim b$ are adjacent vertices of $\Gamma(MmM, MnM)$, then $\varphi(a - m) = \varphi(b - m)$. Indeed, there exists $c \in nM \setminus \{n\}$ with $ca = a$ and $cb = b$. Therefore, $\varphi(a - b) = \varphi(c(a - b)) = c\varphi(a - b) = 0$ because $c$ annihilates $S_{L_n}$. It is then clear that $\varphi(a-m) = \varphi(b-m)$. It follows that $\varphi(a-m) = \varphi(b-m)$ whenever $a, b$ are in the same connected component and hence $\varphi(x - m) = \varphi(\overline{C_x} - m) = 0$. This concludes the proof that $\Psi$ is injective.

Next suppose that $\eta\colon V \longrightarrow \Bbbk$ is a linear map. Define a $\Bbbk$-linear map

$$\varphi\colon \mathrm{rad}(\Bbbk L_m) \longrightarrow S_{L_n}$$

by

$$\varphi(x - m) = \eta(\overline{C_{nx}} - m) \tag{18.3}$$

for $x \in L_m \setminus \{m\}$. Note that $\varphi(m - m) = 0 = \eta(\overline{C_m} - m)$ because $nm = m$ and $\overline{C_m} = m$. Thus (18.3) is also valid for $x = m$.

If $MnM \subseteq MzM$, then $nz = n$ and so

$$\begin{aligned}\varphi(z(x-m)) &= \varphi(zx - m + m - zm)\\ &= \eta(\overline{C_{nzx}} - m) - \eta(\overline{C_{nzm}} - m)\\ &= \eta(\overline{C_{nx}} - m) = \varphi(x - m) = z\varphi(x - m)\end{aligned}$$

since $nzx = nx$, $nzm = nm = m$ and $\overline{C_m} = m$.

If $MmM \nsubseteq MzM$, then $\varphi(z(x - m)) = \varphi(0) = z\varphi(x - m)$. If $MmM \subseteq MzM \subsetneq MnM$, then

$$\varphi(z(x-m)) = \varphi(zx - m + m - zm) = \eta(\overline{C_{nzx}} - m) - \eta(\overline{C_{nzm}} - m).$$

But $nz \in nM \setminus \{n\}$ implies that $nzx \sim nzm$ and hence $\overline{C_{nzx}} = \overline{C_{nzm}}$. Thus $\varphi(z(x - m)) = 0 = z\varphi(x - m)$. This completes the proof that $\varphi$ is a $\Bbbk M$-module homomorphism. By construction, $\Psi(\varphi) = \eta$. We conclude that $\operatorname{Hom}_{\Bbbk M}(\mathrm{rad}(\Bbbk L_m), S_{L_n}) \cong \operatorname{Hom}_{\Bbbk}(V, \Bbbk)$. As $V$ has dimension one less than the number of components of $\Gamma(MmM, MnM)$, the result follows. $\square$

## 18.4 The quiver of a $\mathscr{J}$-trivial monoid

Fix for this section a finite $\mathscr{J}$-trivial monoid $M$ and an algebraically closed field $\Bbbk$. Recall that if $m \in M$, then $m^+$ and $m^-$ denote the unique idempotents in $\widetilde{R}_m$ and $\widetilde{L}_m$, respectively.

Let us say that $m \in M \setminus E(M)$ is *weakly irreducible* if $m = uv$ with $u^+ = m^+$ and $v^- = m^-$ implies that $u = m^+$ or $v = m^-$. The following theorem is due to Denton, Hivert, Schilling and Thiéry [DHST11]. Our proof is novel.

**Theorem 18.12.** *Let $M$ be a $\mathscr{J}$-trivial monoid and $\Bbbk$ an algebraically closed field. Let $Q(\Bbbk M)$ be the quiver of $\Bbbk M$. Then the vertex set of $Q(\Bbbk M)$ is $E(M)$. The number of arrows from $e$ to $f$ is the number of weakly irreducible elements with $m^+ = f$ and $m^- = e$.*

*Proof.* As $M$ is $\mathscr{J}$-trivial, there is a bijection between $E(M)$ and simple modules via $e \mapsto S_{L_e}$. We have that

$$\mathrm{Ext}^1_{\Bbbk M}(S_{L_e}, S_{L_f}) \cong \mathrm{Hom}_{\Bbbk M}(\mathrm{rad}(\Bbbk \widetilde{L}_e), S_{L_f})$$

by Proposition 18.8.

The unique regular element of $\widetilde{L}_e$ is $\{e\}$ and hence

$$\mathrm{rad}(\widetilde{L}_e) = \widetilde{L}_e \setminus \{e\} = \{m \in Me \setminus \{e\} \mid m^- = e\}$$

by Proposition 18.6. Let $V$ be the subspace of $\mathrm{rad}(\Bbbk \widetilde{L}_e)$ with basis the set $X$ of weakly irreducible elements $m \in M$ with $m^- = e$ and $m^+ = f$. We claim the restriction map

$$\Psi\colon \mathrm{Hom}_{\Bbbk M}(\mathrm{rad}(\Bbbk \widetilde{L}_e), S_{L_f}) \longrightarrow \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$$

is an isomorphism where we recall that $S_{L_f} \cong \Bbbk$ as a $\Bbbk$-vector space.

Suppose that $\Psi(\varphi) = 0$ and $x \in \widetilde{L}_e \setminus \{e\}$. We must show that $\varphi(x) = 0$. Observe that

$$\varphi(x) = f\varphi(x) = \varphi(fx). \tag{18.4}$$

First note that if $fx \notin \widetilde{L}_e$, then $fx = 0$ in $\Bbbk \widetilde{L}_e$ and so $\varphi(x) = 0$ by (18.4). If $fx \in \widetilde{L}_e$, i.e., $(fx)^- = e$, then replacing $x$ by $fx$ and appealing to (18.4), we may assume without loss of generality that $x = fx$. But $fx = x$ implies that $Mx^+M \subseteq MfM$. If the containment is strict, then $\varphi(x) = \varphi(x^+x) = x^+\varphi(x) = 0$. We are thus left with the case $x^+ = f$. If $x$ is not weakly irreducible, then $x = uv$ with $v^- = e$ and $u^+ = f$ and $u \neq f$, $v \neq e$. Then $v \in \mathrm{rad}(\Bbbk \widetilde{L}_e)$ and $\varphi(x) = \varphi(uv) = u\varphi(v) = 0$ because $MuM \subsetneq MfM$. The remaining case is that $x^+ = f$, $x^- = e$ and $x$ is weakly irreducible. But then $x \in V$ and so $\varphi(x) = 0$ by assumption. It follows that $\Psi$ is injective.

Next assume that $\eta\colon V \longrightarrow \Bbbk$ is a linear map and define a mapping

$$\varphi\colon \operatorname{rad}(\Bbbk\widetilde{L}_e) \longrightarrow S_{L_f},$$

for $x \in \widetilde{L}_e \setminus \{e\}$, by

$$\varphi(x) = \begin{cases} \eta(fx), & \text{if } fx \in X \\ 0, & \text{else.} \end{cases}$$

We check that $\varphi$ is a $\Bbbk M$-module homomorphism. It clearly restricts on $V$ to $\eta$. Let $x \in \widetilde{L}_e \setminus \{e\}$. If $MfM \subseteq MzM$, then $fz = f$ and so $fx = fzx$. It follows that $\varphi(zx) = \varphi(x) = z\varphi(x)$. Next suppose that $MfM \nsubseteq MzM$. Then $MfzM \subsetneq MfM$. If $(fzx)^+ \neq f$, then $fzx \notin X$ and so $\varphi(zx) = 0 = z\varphi(x)$. So suppose that $(fzx)^+ = f$. If $(fzx)^- \neq e$, then again $fzx \notin X$ and so $\varphi(zx) = 0 = z\varphi(x)$. So assume that $(fzx)^- = e$. Then since $(fz)^+(fzx) = fzx$ we obtain $(fz)^+ f = f$. But $f(fz) = fz$ implies that $f(fz)^+ = (fz)^+$ and so $f = (fz)^+$. But since $fz \neq f$, $x^- = e$ and $x \neq e$, this shows that $fzx = (fz)x$ is not weakly irreducible and so $fxz \notin X$. Therefore, $\varphi(zx) = 0 = z\varphi(x)$. This completes the proof that $\varphi$ is a $\Bbbk M$-module homomorphism. It follows that there are $\dim V = |X|$ arrows from $MeM$ to $MfM$. This completes the proof of the theorem. □

## 18.5 Sample quiver computations

In this section, we compute the quivers of several left regular band and $\mathscr{J}$-trivial monoid algebras. Throughout this section, $\Bbbk$ is an algebraically closed field.

### 18.5.1 Left regular bands

Let $B$ be a finite set and let $F(B)$ be the free left regular band on $B$. The reader is referred to Section 15.3.1 for details and notation. In particular, there is an isomorphism of $\Lambda(F(B))$ with $(\mathscr{P}(B), \supseteq)$ sending $F(B)wF(B)$ to $c(w)$ where $c(w)$ is the content of $w$, that is, the set of letters appearing in $w$. Denote by $L_X$ the $\mathscr{L}$-class of elements with content $X$ and fix $w_X \in F(B)$ with content $X$ for each $X \subseteq B$.

**Lemma 18.13.** *Let $X \supsetneq Y$ and consider the graph*

$$\Gamma = \Gamma(F(B)w_X F(B), F(B)w_Y F(B))$$

*(as per Theorem 18.11). Then the vertex set of $\Gamma$ is the set of words of content $X$ having $w_Y$ as a prefix. Two words are connected by an edge if and only if they have the same prefix of length $|w_Y|+1$. Thus each connected component of $\Gamma$ is a complete graph and there is one connected component for each element of $X \setminus Y$.*

*Proof.* As $w_Y F(B)$ consists of all words with $w_Y$ as a prefix, the description of the vertex set is clear. Suppose that $u, v \in w_Y F(B) \cap L_X$ are words connected by an edge. Then there exists $x \in w_Y F(B) \setminus \{w_Y\}$ with $x \odot u = u$ and $x \odot v = v$. But then $x$ is a common prefix of $u$ and $v$. Since $x \neq w_Y$, we conclude that $|x| > |w_Y|$ and so $u, v$ have the same prefix of length $|w_Y| + 1$. Conversely, if $u, v$ have the same prefix $x$ of length $|w_Y| + 1$, then $x \odot u = u$ and $x \odot v = v$ and so $u$ and $v$ are adjacent as $x \in w_Y F(A) \setminus \{w_Y\}$. This proves that $\Gamma$ is as described. It follows that $\Gamma$ has a connected component for each $a \in X \setminus Y$ consisting of all words with content $X$ and prefix $w_Y a$ and each component is a complete graph. This completes the proof. □

An immediate consequence of Lemma 18.13 and Theorem 18.11 is the following unpublished result of K. Brown, which first appeared in [Sal07].

**Theorem 18.14.** *The quiver $Q(\Bbbk F(B))$ of the algebra of the free left regular band on a finite set $B$ has vertex set the power set $\mathscr{P}(B)$. There are no edges from $X$ to $Y$ unless $X \supsetneq Y$. If $X \supsetneq Y$, then there are $|X| - |Y| - 1$ edges from $X$ to $Y$.*

K. Brown used Theorem 18.14 and Gabriel's theory of quivers to prove that $\Bbbk F(B)$ is hereditary; the result was first published in [Sal07]. A homological proof appears in [MSS].

Next we compute the quiver of the algebra of $L^n$ where we recall that $L = \{1, A, B\}$ with 1 as the identity and $xy = x$ for all $x, y \in \{A, B\}$. See Section 15.3.3 for details. Recall that $\Lambda(L^n)$ is isomorphic to $(\mathscr{P}([n]), \supseteq)$ via the map $L^n x L^n \mapsto c(x)$ where $c(x) = \{i \mid x_i \neq 1\}$ is the support of $x = (x_1, \ldots, x_n)$. For $S \subseteq [n]$, let $L_S$ be the set of elements $x$ with $c(x) = S$ and let $e_S$ be the vector with $x_i = A$ if $i \in S$ and 1, otherwise.

**Lemma 18.15.** *Let $S \supsetneq T$. Then $\Gamma = \Gamma(L^n e_S L^n, L^n e_T L^n)$ (defined as per Theorem 18.11) has vertex set all $x \in L^n$ with $c(x) = S$ and with $x_i = A$ for all $i \in T$. If $|S| = |T| + 1$, then $\Gamma$ consists of two isolated vertices. Otherwise, $\Gamma$ is connected.*

*Proof.* First note that $e_T L^n$ consists of all elements $x \in L^n$ with $x_i = A$ for $i \in T$. Thus $\Gamma$ has vertex set as in the statement of the lemma. Suppose first that $|S| = |T| + 1$, say, $S = T \cup \{i\}$. Then the only two vertices of $\Gamma$ are $x, y$ where $x$ is obtained by replacing the 1 in position $i$ of $e_T$ by $A$ and $y$ is obtained by replacing it with $B$. Moreover, the only elements of $e_T L^n \setminus \{e_T\}$ that are not in $I(e_S)$ are $x, y$. As $xy = x$ and $yx = y$, we conclude that there are no edges in $\Gamma$ in this case.

Next assume that $|S| > |T| + 1$. Notice that all vertices of $\Gamma$ agree in each coordinate outside of $S \setminus T$. We claim that if two distinct vertices $x, y$ of $\Gamma$ satisfy $x_j = y_j$ for some $j \in S \setminus T$, then they are adjacent. Suppose that $x_j = y_j = X \in \{A, B\}$. Let $z$ be obtained from $e_T$ by changing coordinate $j$ to $X$. Then $z \in e_T L^n \setminus \{e_T\}$ and $zx = x$, $zy = y$. Thus $x, y$ are connected by an edge.

Now let $x, y$ be arbitrary vertices. If they are not adjacent, then by the above $x_k \neq y_k$ for all $k \in S \setminus T$. As $|S| > |T| + 1$, we can find distinct $i, j \in S \setminus T$. Let $z$ be the vertex of $\Gamma$ given by

$$z_k = \begin{cases} x_i, & \text{if } k = i \\ y_k, & \text{else.} \end{cases}$$

Then $z_i = x_i$ implies that $z$ is adjacent to $x$ and $z_j = y_j$ implies that $z$ is adjacent to $y$. We conclude that $\Gamma$ is connected. $\square$

From Lemma 18.15 and Theorem 18.11 we obtain the following result (which is a special case of a result of Saliola [Sal09]).

**Theorem 18.16.** *The quiver $Q(\Bbbk L^n)$ is the Hasse diagram of $(\mathscr{P}([n]), \supseteq)$. That is, it has vertex set $\mathscr{P}([n])$ and there is an edge from $S$ to $T$ if and only if $S \supseteq T$ and $|S| = |T| + 1$.*

### 18.5.2 $\mathscr{J}$-trivial monoids

We next compute the quiver of the algebra of the Catalan monoid. The *Catalan monoid $C_n$* is the monoid of all mappings $f\colon [n] \longrightarrow [n]$ such that:

(i) $i \leq f(i)$ for all $i \in [n]$;
(ii) $i \leq j$ implies $f(i) \leq f(j)$.

In other words, $C_n$ is the monoid of all order-preserving and non-decreasing maps on $[n]$. The reason for the name is that it is well known that the cardinality of $C_n$ is the $n^{th}$ Catalan number $\frac{1}{n+1}\binom{2n}{n}$ (cf. [Sta99, Exercise 6.19(s)]).

**Proposition 18.17.** *The Catalan monoid $C_n$ is $\mathscr{J}$-trivial.*

*Proof.* We claim that $C_n f C_n \subseteq C_n g C_n$ implies that $g$ is pointwise below $f$. From the claim it is immediate that $C_n$ is $\mathscr{J}$-trivial. Indeed, if $f = hgk$ and $i \in [n]$, then $i \leq k(i)$ and $g(i) \leq h(g(i))$ imply that $f(i) = h(g(k(i))) \geq h(g(i)) \geq g(i)$ as $h, g$ are order-preserving. $\square$

A deep theorem of Simon [Sim75] implies that if $M$ is a finite $\mathscr{J}$-trivial monoid, then there exists $n \geq 1$ and a submonoid $N$ of $C_n$ such that $M$ is a homomorphic image of $N$. See [Pin86] for details. The Catalan monoid has been studied by several authors, under different names, in the context of representation theory [HT09, DHST11, Gre12, GM14].

In order to further understand $C_n$, we must compute its set of idempotents. Let $P_n = \{S \subseteq [n] \mid n \in S\}$ be the collection of subsets of $[n]$ containing $n$. Notice that $f(n) = n$ for all $f \in C_n$ because $n \leq f(n) \in [n]$. Therefore, the image of each element of $C_n$ belongs to $P_n$. Recall that the *rank* of a mapping is the cardinality of its image.

**Proposition 18.18.** *For each $S \in P_n$, there is a unique idempotent $e_S \in C_n$ with image $S$. Moreover, one has*

$$e_S(i) = \min\{s \in S \mid i \leq s\}$$

*for $S \in P_n$.*

*Proof.* First suppose that $e, f \in E(C_n)$ have the same image set $S$. As $e$ fixes $S$, we conclude that $ef = f$ and similarly $fe = e$. Hence $e = f$ by $\mathscr{J}$-triviality. Thus there is at most one idempotent with image set $S$. Let us verify that $e_S$ is an idempotent of $C_n$ with image set $S$.

Trivially, $e_S([n]) \subseteq S$. Moreover, if $s \in S$, then $e_S(s) = s$ by definition. Thus $e_S$ is an idempotent mapping with image $S$. It remains to prove that $e_S \in C_n$. If $i \in [n]$, then $i \leq e_S(i)$ from the definition of $e_S$. If $j \leq i$, then $j \leq i \leq e_S(i) \in S$. Thus $e_S(j) \leq e_S(i)$ by definition of $e_S$. This completes the proof. $\square$

If $M$ is a $\mathscr{J}$-trivial monoid, then, for idempotents $e, f \in E(M)$, one has that $e \leq f$ if and only if $MeM \subseteq MfM$ by Corollary 2.6 and its dual.

**Proposition 18.19.** *Let $S, T \in P_n$. Then $C_n e_S C_n \subseteq C_n e_T C_n$ if and only if $S \subseteq T$. This occurs if and only if $e_S e_T = e_S = e_T e_S$.*

*Proof.* As discussed before the proposition, we have $C_n e_S C_n \subseteq C_n e_T C_n$ if and only if $e_S \leq e_T$ because $C_n$ is $\mathscr{J}$-trivial. So suppose that $C_n e_S C_n \subseteq C_n e_T C_n$. Then $e_T e_S = e_S$ and so the image $S$ of $e_S$ is contained in the image $T$ of $e_T$. Conversely, if $S \subseteq T$, then $e_T$ fixes the image $S$ of $e_S$ and so $e_T e_S = e_S$, whence $C_n e_S C_n \subseteq C_n e_T C_n$. This completes the proof. $\square$

Our next goal is to compute $f^+$ and $f^-$ for $f \in C_n$. It will turn out that $f$ is, in fact, uniquely determined by the pair $(f^+, f^-)$. If $S, T \in P_n$, let us write $S \preceq T$ if $|S| = |T|$ and if

$$S = \{i_1 < \ldots < i_k = n\} \quad \text{and} \quad T = \{j_1 < \ldots < j_k = n\}, \tag{18.5}$$

with $i_r \leq j_r$ for all $1 \leq r \leq k$. It is routine to verify that this is a partial order on $P_n$.

**Proposition 18.20.** *Let $f \in C_n$. Let $T$ be the image of $f$ and let*

$$S = \{s \in [n] \mid s = \max f^{-1}(f(s))\}.$$

*Then $f^- = e_S$ and $f^+ = e_T$. Moreover, $S \preceq T$ and if $S$ and $T$ are as in* (18.5)*, then $f(x) = j_r$ if and only if $i_{r-1} < x \leq i_r$ where we take $i_0 = 0$. Hence $f$ is uniquely determined by $f^-$ and $f^+$. Conversely, if $S \preceq T$ and are as in* (18.5)*, then $f$, defined as above, belongs to $C_n$ and satisfies $f^- = e_S$ and $f^+ = e_T$.*

*Proof.* Clearly, $e_X f = f$ if and only if $X$ contains the range of $f$ and so $e_T = f^+$ by Proposition 18.19. Suppose that $fe_X = f$. For $s \in S$, we have that $s \leq e_X(s)$. But also $f(e_X(s)) = f(s)$ and so by definition of $S$ we must have $e_X(s) \leq s$. Thus $e_X$ fixes $S$, that is, $S \subseteq X$. It follows that $fe_X = f$ implies that $e_S \leq e_X$ by Proposition 18.19. Thus to prove that $f^- = e_S$, it remains to show that $fe_S = f$.

If $i \in [n]$ and $s = \max f^{-1}(f(i))$, then $s \in S$ and $i \leq s$. Thus $i \leq e_S(i) \leq s$ by Proposition 18.18. Therefore, $f(s) = f(i) \leq f(e_S(i)) \leq f(s)$ and so $f(i) = f(e_S(i))$. This proves the $f = fe_S$.

From the definition of $S$, it easily follows that $f|_S$ is injective with image $T$. Thus $|S| = |T|$ and hence, since $f$ is order-preserving and non-decreasing, we have that $f(x) = j_r$ if and only if $i_{r-1} < x \leq i_r$, where we take $i_0 = 0$, and $i_r \leq j_r$. In particular, we have $S \preceq T$.

For the converse, if $S \preceq T$ are as in (18.5) and we define $f$ as above, then $f$ is trivially order-preserving. If $i_{r-1} < x \leq i_r$, then $f(x) = j_r \geq i_r \geq x$ and so $f$ is non-decreasing. □

As a consequence of Proposition 18.20 and Theorem 18.10, we may deduce that the Cartan matrix of $\Bbbk C_n$ is the zeta function of the poset $(P_n, \succeq)$.

**Corollary 18.21.** *Let $\Bbbk$ be an algebraically closed field. The Cartan matrix of $\Bbbk C_n$ is the zeta function of the poset $(P_n, \succeq)$ (viewed as a $P_n \times P_n$-matrix).*

We now aim to characterize the weakly irreducible elements of $C_n$. First we need a lemma.

**Lemma 18.22.** *Suppose that $g, h \in C_n$ with $g^- = h^+$. Then $(gh)^+ = g^+$ and $(gh)^- = h^-$.*

*Proof.* Let us put $g^- = e_U = h^+$, $g^+ = e_T$ and $h^- = e_S$. Then $h(S) = U$, $g(U) = T = g([n])$ and $|S| = |U| = |T|$ by Proposition 18.20. Therefore, $gh(S) = T = g([n])$. We deduce that $T$ is the image of $gh$ and hence $(gh)^+ = e_T$ by Proposition 18.20. It follows that $gh$ has rank $|T|$. Trivially, $ghe_S = gh$ and so if $(gh)^- = e_R$, then $R \subseteq S$ by Proposition 18.19. But $|R|$ is the rank of $gh$, which is $|T| = |S|$ and so $R = S$. Thus $(gh)^- = e_S$. This completes the proof. □

If $S, T \subseteq [n]$, then their symmetric difference is $S \triangle T = (S \setminus T) \cup (T \setminus S)$.

**Proposition 18.23.** *One has that $f \in C_n$ is weakly irreducible if and only if $f^- = e_S$ and $f^+ = e_T$ where $T$ covers $S$ in the partial order $\preceq$, that is, $S \triangle T = \{i, i+1\}$ with $i \in S$ and $i+1 \in T$.*

*Proof.* From the definition of $\preceq$, it is straightforward to verify that $T$ covers $S$ if and only if $|S| = |T|$ and $S \triangle T = \{i, i+1\}$ with $i \in S$ and $i+1 \in T$. It follows from Proposition 18.20 that $f \notin E(C_n)$ if and only if $S \neq T$ and that in this case $S \prec T$. So assume that $f \notin E(C_n)$.

Suppose first that $f = gh$ with $g^+ = e_T$, $h^- = e_S$, $g \neq e_T$ and $h \neq e_S$. Then $h^+ = e_U$ with $S \prec U$ and $g^- = e_V$ with $V \prec T$. Notice that $|U| = |S| = |T| = |V|$ is the rank of $f, g, h$ and $U = h([n])$. Therefore, we must have that $g|_U$ is injective. Since $g$ is order-preserving and non-decreasing, we deduce using Proposition 18.20 that $U \preceq V$ and hence $S \prec U \preceq V \prec T$. Thus $T$ does not cover $S$.

Conversely, if $T$ does not cover $S$, then we can find $U$ with $S \prec U \prec T$. By Proposition 18.20, we can find $g, h$ with $g^- = U$, $g^+ = T$, $h^- = S$ and $h^+ = U$. Moreover, note that $g, h \notin E(C_n)$. Then Lemma 18.22 yields that $(gh)^+ = e_T$ and $(gh)^- = e_S$. Therefore, $gh = f$ by Proposition 18.20 and hence $f$ is not weakly irreducible. □

Putting together Proposition 18.20 and Proposition 18.23 with Theorem 18.12, we obtain the following result, which was proved in Denton *et al.* [DHST11] in a different formulation using 0-Hecke monoids.

**Theorem 18.24.** *The quiver $Q(\Bbbk C_n)$ of the algebra of the Catalan monoid $C_n$ is the Hasse diagram of $(P_n, \preceq)$. That is, it has vertex set the subsets of $[n]$ containing $n$ and there is an edge from $S$ to $T$ if and only their symmetric difference is $S \triangle T = \{i, i+1\}$ with $i \in S$ and $i+1 \in T$.*

In fact, the algebra $\Bbbk C_n$ is isomorphic to the algebra of the category associated to the poset $P_n$ (cf. Example 9.2) or, equivalently, isomorphic to the opposite of the incidence algebra of this poset. This was first proved by Hivert and Thiéry [HT09]. An alternative proof was given in Grensing [Gre12]. Our proof here is new to the best of our knowledge.

**Theorem 18.25.** *Let $C_n$ be the Catalan monoid of degree $n$ and $\Bbbk$ a field. Then $\Bbbk C_n$ is isomorphic to the category algebra of the poset $P_n$. Equivalently, $\Bbbk C_n$ is isomorphic to the incidence algebra of $(P_n, \succeq)$.*

*Proof.* Denote also by $P_n$ the category associated to the poset $(P_n, \preceq)$ as per Example 9.2. Let $A = \Bbbk C_n$ and $B = \Bbbk P_n$. Note that $\dim B = |C_n| = \dim A$ because there is a bijection between the arrow set of $P_n$ and $C_n$ taking $(S,T)$ with $S \preceq T$ to the unique element $g_{S,T} \in C_n$ with $g^-_{S,T} = e_S$ and $g^+_{S,T} = e_T$ as per Proposition 18.20. The strategy for the proof is as follows. We shall define an injective homomorphism $\rho\colon B^{op} \longrightarrow \mathrm{End}_A(A) \cong A^{op}$ (the isomorphism is by Proposition A.20). Then since $\dim A = \dim B$, we will be able to conclude that $\rho$ is an isomorphism and hence $A \cong B$. To implement this scheme, we shall use that $A \cong \bigoplus_{S \in P_n} \Bbbk \widetilde{L}_{e_S}$ by Theorem 18.7.

So let $(S,T)$ be an arrow of $P$. Then we can define a $\Bbbk$-linear map $\rho_{S,T}\colon \Bbbk\widetilde{L}_{e_T} \longrightarrow \Bbbk\widetilde{L}_{e_S}$ on the basis by $\rho_{S,T}(f) = fg_{S,T}$. Note that $(fg_{S,T})^- = g^-_{S,T} = e_S$ by Lemma 18.22 because $f^- = e_T = g^+_{S,T}$. In fact, $\rho_{S,T}$ is a left $A$-module homomorphism. Indeed, if $f \in \widetilde{L}_{e_T}$ and $h \in C_n$ with $hf \in \widetilde{L}_{e_T}$, then $hfg_{S,T} \in \widetilde{L}_{e_S}$ by Lemma 18.22 and $\rho_{S,T}(hf) = (hf)g_{S,T} = h(fg_{S,T}) = h\rho_{S,T}(f)$. On the other hand, if $hf \notin \widetilde{L}_{e_T}$, then since $hfe_T = hf$, we must have $(hf)^- = e_U$ with $U \subsetneq T$ by Proposition 18.19. Therefore, $hf$ has rank $|U| < |T|$. It follows that $hfg_{S,T}$ has rank at most $|U|$, which is less than $|S|$. We conclude that $h(fg_{S,T}) \notin \widetilde{L}_{e_S}$ and thus $\rho_{S,T}(hf) = \rho_{S,T}(0) = 0 = h(fg_{S,T}) = h\rho_{S,T}(f)$. This proves that $\rho_{S,T}$ is an $A$-module homomorphism.

Using the vector space decomposition

$$\mathrm{End}_A\left(\bigoplus_{S\in P_n} \Bbbk\widetilde{L}_{e_S}\right) = \bigoplus_{S,T\in P_n} \mathrm{Hom}_A(\Bbbk\widetilde{L}_{e_T}, \Bbbk\widetilde{L}_{e_S}) \tag{18.6}$$

we can view $\rho_{S,T}$ as an element of $\mathrm{End}_A\left(\bigoplus_{S\in P_n} \Bbbk\widetilde{L}_{e_S}\right)$. We check that

$$\rho\colon B^{op} \longrightarrow \mathrm{End}_A\left(\bigoplus_{S\in P_n} \Bbbk\widetilde{L}_{e_S}\right)$$

given by $\rho((S,T)) = \rho_{S,T}$, for $S \preceq T$, is a homomorphism. It is immediate that if $T = U$, then $\rho_{S,T}\rho_{U,V} = \rho_{S,V}$ because $g_{T,V}g_{S,T} = g_{S,V}$ by a combination of Proposition 18.20 and Lemma 18.22. On the other hand, if $T \neq U$, then $\rho_{S,T}\rho_{U,V} = 0$ because $\rho_{U,V}$ has image contained in the summand $\Bbbk\widetilde{L}_{e_U}$, which is annihilated by $\rho_{S,T}$ whenever $T \neq U$. Thus $\rho$ is a homomorphism.

To see that $\rho$ is injective, note that it takes each basis element $(S,T)$ of $B^{op}$ with $S \preceq T$ to a distinct summand $\mathrm{Hom}_A(\Bbbk\widetilde{L}_{e_T}, \Bbbk\widetilde{L}_{e_S})$ of (18.6). Thus $\rho$ takes the basis of $B^{op}$ to a linearly independent set and hence is injective. The theorem now follows via the argument given in the first paragraph of the proof. $\square$

As the quiver of the category algebra of a poset $P$ is always the Hasse diagram of $P$ (cf. [ASS06]), this leads to another proof of Theorem 18.24.

## 18.6 Exercises

**18.1.** Compute the quiver $Q$ of $\Bbbk M$ where $M = \{1, x_1, \ldots, x_n\}$ with 1 the identity and $x_ix_j = x_i$ for $i = 1, \ldots, n$ and prove that $\Bbbk M \cong \Bbbk Q$.

**18.2.** Prove Theorem 18.9 over $\mathbb{C}$ using Theorem 8.33.

**18.3.** Let $M$ be a finite $\mathscr{L}$-trivial monoid and $\Bbbk$ a field. Let $e \in E(M)$ and let $S_{J_e}$ be the simple $\Bbbk M$-module with apex $e$. Let $V$ be the $\Bbbk$-vector space of mappings $f\colon eM \longrightarrow \Bbbk$ vanishing on $eM \setminus \widetilde{R}_e$ with $\Bbbk M$-module structure given by $(mf)(x) = f(xm)$ for $x \in eM$ and $m \in M$. Prove that $V$ is an injective indecomposable module with simple socle isomorphic to $S_{J_e}$ consisting of those mappings $f \in V$ vanishing on $\widetilde{R}_e \setminus R_e$ and constant on $R_e$.

**18.4.** Prove that $\Bbbk L^n \cong I(\mathscr{P}([n]), \Bbbk)$.

**18.5.** Prove that the quiver of $\Bbbk\Sigma_n$ is the Hasse diagram of the lattice $\Pi_n$.

**18.6.** Compute the quiver of $\mathbb{C}M$ for $M$ the monoid in Exercise 17.10.

**18.7.** Let $M$ be a finite $\mathscr{J}$-trivial monoid whose idempotents are central (i.e., commute with each element of $M$). Prove that each edge of the quiver of $\Bbbk M$ is a loop for any algebraically closed field $\Bbbk$.

**18.8.** Let $M$ be a finite $\mathscr{J}$-trivial monoid with central idempotents. Construct a complete set of orthogonal primitive idempotents for $M$. (Hint: use that $E(M)$ is a lattice.)

**18.9.** Consider the monoid $HD_n$ given by the presentation

$$HD_n = \langle a, b \mid a^2 = a, b^2 = b, w_n(a,b) = w_n(b,a)\rangle$$

for $n \geq 1$ where $w_n(x,y) = xyxy\cdots$ is the alternating word in $x, y$ starting with $x$ of length $n$.

(a) Prove that $HD_n$ is a finite $\mathscr{J}$-trivial monoid with $2n$ elements.
(b) Prove that $E(HD_n) = \{a, b, w_n(a,b)\}$.
(c) Let $\Bbbk$ be an algebraically closed field. Compute the quiver of $\Bbbk HD_n$.

**18.10.** Let $G$ be a finite group. Let $\mathscr{P}_1(G) = \{X \subseteq G \mid 1 \in X\}$ equipped with the product $AB = \{ab \mid a \in A, b \in B\}$. Prove that $\mathscr{P}_1(G)$ is a $\mathscr{J}$-trivial monoid, $E(\mathscr{P}_1(G))$ is the set of subgroups of $G$ and that the natural partial order on $E(\mathscr{P}_1(G))$ is reverse inclusion.

**18.11.** Compute the quiver of $\mathbb{C}\mathscr{P}_1(\mathbb{Z}_6)$. See Exercise 18.10 for the definition.

**18.12.** Compute the quiver of $\mathbb{C}\mathscr{P}_1(S_3)$. See Exercise 18.10 for the definition.

**18.13.** Let $A$ be a finite dimensional algebra over an algebraically closed field $\Bbbk$. Let $e, f$ be primitive idempotents of $A$ and put $S = Ae/\operatorname{rad}(A)e$ and $S' = Af/\operatorname{rad}(A)f$. Prove that $\operatorname{Ext}^1_A(S, S') \cong f[\operatorname{rad}(A)/\operatorname{rad}^2(A)]e$.

**18.14.** Compute the quiver of the algebra $\Bbbk M$ for $M$ the left regular band $\{1, a_i, b_i \mid 1 \leq i \leq n\}$ with 1 the identity and product given by

$$x_i y_j = \begin{cases} y_j, & \text{if } i < j \\ x_i, & \text{else} \end{cases}$$

for $x, y \in \{a, b\}$.

**18.15.** Let $M$ be the left regular band $\{1, a, b, x, y, z, z'\}$ with multiplication table

| | $1$ | $a$ | $b$ | $x$ | $y$ | $z$ | $z'$ |
|---|---|---|---|---|---|---|---|
| $1$ | $1$ | $a$ | $b$ | $x$ | $y$ | $z$ | $z'$ |
| $a$ | $a$ | $a$ | $a$ | $x$ | $x$ | $z$ | $z'$ |
| $b$ | $b$ | $b$ | $b$ | $x$ | $x$ | $z$ | $z'$ |
| $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $z'$ | $z'$ |
| $y$ | $y$ | $y$ | $y$ | $y$ | $y$ | $z'$ | $z'$ |
| $z$ | $z$ | $z$ | $z$ | $z'$ | $z'$ | $z$ | $z'$ |
| $z'$ | $z'$ | $z'$ | $z'$ | $z'$ | $z'$ | $z'$ | $z'$ |

(a) Compute the quiver of $\Bbbk M$.
(b) Prove that $\Bbbk L_x \otimes \Bbbk L_z \cong S_{L_a} \oplus S_{L_1}$, which is not projective, and hence the tensor product of projective modules need not be projective.

# 19

# Further Developments

This chapter highlights some further developments in the representation theory of finite monoids that are beyond the scope of this text. No proofs are presented.

## 19.1 Monoids of Lie type

There is a well-developed theory of linear algebraic monoids, due principally to Putcha and Renner [Put88, Ren05]. It is a beautiful interplay between the theory of algebraic groups and semigroup theory. Putcha introduced the finite analogues of linear algebraic monoids, called *finite monoids of Lie type*, see [Put89, Put95] or [Ren05, Chapter 10]. We do not attempt to give the definition here, but the example to keep in mind is the monoid $M_n(\mathbb{F}_q)$ of $n \times n$ matrices over the field $\mathbb{F}_q$ of $q$ elements.

An exciting theorem, due to Okniński and Putcha [OP91], asserts the semisimplicity of the complex algebra of a finite monoid of Lie type.

**Theorem 19.1.** *Let $M$ be a finite monoid of Lie type. Then $\mathbb{C}M$ is semisimple. In particular, $\mathbb{C}M_n(\mathbb{F}_q)$ is semisimple.*

This result was improved later by Putcha [Put99] who proved that $\Bbbk M$ is semisimple as long as the characteristic of $\Bbbk$ does not divide the order of the group of units of $M$.

Independently, Kovács gave an elementary (but ingenious) direct approach to the study of the monoid algebra of $M_n(\mathbb{F}_q)$ and proved the following remarkable result [Kov92].

**Theorem 19.2.** *Let $q = p^m$ with $p$ prime and $m \geq 1$ and let $\Bbbk$ be a commutative ring such that $p$ is invertible in $\Bbbk$. Then $\Bbbk M_n(\mathbb{F}_q)$ is isomorphic to a direct product of matrix algebras over the group algebras $\Bbbk GL_r(\mathbb{F}_q)$ of the general linear groups of degree $0 \leq r \leq n$. In particular, if $\Bbbk$ is a field, then $\Bbbk M_n(\mathbb{F}_q)$ is semisimple whenever the characteristic of $\Bbbk$ does not divide the order of $GL_n(\mathbb{F}_q)$.*

The modular representation theory of finite monoids of Lie type has also been studied. Motivated by problems in stable homotopy theory, Harris and Kuhn proved the following delightful result [HK88].

**Theorem 19.3.** *Let $q = p^m$ with $p$ prime and $m \geq 1$. Let $\rho$ be an irreducible representation of $M_n(\mathbb{F}_q)$ over the algebraic closure of $\mathbb{F}_p$. Then the restriction of $\rho$ to the general linear group $GL_n(\mathbb{F}_q)$ is also irreducible.*

This theorem was extended by Putcha and Renner to arbitrary finite monoids of Lie type [PR93], cf. [Ren05, Theorem 10.10].

## 19.2 The representation theory of the full transformation monoid

In Section 5.3 we presented Putcha's results on the irreducible representations of the full transformation monoid. In this section, we discuss further aspects of the representation theory of $T_n$.

Recall that a finite dimensional algebra $A$ has finite representation type if there are only finitely many isomorphism classes of finite dimensional indecomposable $A$-modules. Let $T_n$ denote the monoid of all mappings on an $n$-element set. Ponizovskii proved that $\mathbb{C}T_n$ has finite representation type for $n \leq 3$ and conjectured that this was true for all $n$ [Pon87]. Putcha disproved Ponizovskii's conjecture by showing that $\mathbb{C}T_n$ does not have finite representation type for $n \geq 5$ [Put98]. He also computed the quiver of $\mathbb{C}T_4$. Ringel [Rin00] computed a quiver presentation for $\mathbb{C}T_4$ and proved that it is of finite representation type and has global dimension 3. (It is easy to check that $\mathbb{C}T_n$ also has global dimension $n-1$ for $n = 1, 2, 3$.) We summarize this discussion in the following theorem.

**Theorem 19.4.** *The algebra $\mathbb{C}T_n$ of the full transformation monoid of degree $n$ has finite representation type if and only if $n \leq 4$.*

It is an open question to compute the quiver of $\mathbb{C}T_n$ in full generality. The author [Ste15] has recently proved that the quiver of $\mathbb{C}T_n$ is acyclic, for all $n \geq 1$, and has computed the global dimension of $\mathbb{C}T_n$.

**Theorem 19.5 (Steinberg).** *The global dimension of $\mathbb{C}T_n$ is $n-1$ for all $n \geq 1$.*

The author and V. Mazorchuk have computed the characteristic tilting module and the Ringel dual of $\mathbb{C}T_n$ with respect to its natural structure of a quasihereditary algebra (unpublished).

## 19.3 The representation theory of left regular bands

The representation theory of left regular bands was put to good effect in the analysis of Markov chains [BD98, BHR99, Bro00, Bro04] and in the representation theory of Coxeter groups and Solomon's descent algebra [Bro00, Bro04, AM06, Sch06, Sal08, Sal10]. It was therefore natural to delve more deeply into their representation theory. This has turned out to be a treasure trove of fascinating interconnections between semigroup theory, combinatorics and topology.

The initial steps were taken by Saliola [Sal07], who computed a complete set of orthogonal primitive idempotents, the projective indecomposable modules, the Cartan matrix and the quiver for a left regular band algebra. For hyperplane face monoids [BHR99], Saliola computed a quiver presentation, calculated all Ext-spaces between simple modules and proved that the monoid algebra is the Koszul dual of the incidence algebra of the intersection lattice of the arrangement [Sal09]. Key to his approach were resolutions of the simple modules obtained by Brown and Diaconis [BD98] using the cellular chain complexes of associated zonotopes. Saliola proved these resolutions are, in fact, the minimal projective resolutions. The reader is referred to [Ben98, ASS06] for the definition of a minimal projective resolution.

A more detailed analysis of left regular band algebras was then undertaken by the author together with Margolis and Saliola. The first set of results can be found in [MSS]; a self-contained monograph describing further results is forthcoming. The remainder of this section will require some familiarity with algebraic topology.

Fix a field $\Bbbk$ and a left regular band $M$. We continue to denote the lattice of principal left ideals by $\Lambda(M) = \{Mm \mid m \in M\}$. Let $\sigma\colon M \longrightarrow \Lambda(M)$ be the natural homomorphism given by $\sigma(m) = Mm$. Inspired by hyperplane theory and the theory of oriented matroids [BLVS+99], for $X \in \Lambda(M)$, the submonoid

$$M_{\geq X} = \{m \in M \mid \sigma(m) \geq X\}$$

is called the *contraction* of $M$ to $X$. There is a natural surjective homomorphism $\rho_X\colon \Bbbk M \longrightarrow \Bbbk M_{\geq X}$ given on $m \in M$ by

$$\rho_X(m) = \begin{cases} m, & \text{if } m \in M_{\geq X} \\ 0, & \text{else.} \end{cases}$$

Moreover, the simple $\Bbbk M$-module $S_X$ associated to $X$ is just the trivial $\Bbbk M_{\geq X}$-module, viewed as a $\Bbbk M$-module via $\rho_X$.

If $P$ is a finite poset, then the *order complex* $\Delta(P)$ is the simplicial complex with vertex set $P$ and whose simplices are the chains (or totally ordered subsets) of $P$. Let us view $M$ as a poset via the natural order. Because $m \leq n$ if and only if $mM \subseteq nM$ by Proposition 2.8, the action of $M$ on itself by left multiplication is order-preserving. Therefore, $M$ acts by simplicial maps on $\Delta(M)$. Moreover, $\Delta(M)$ is a contractible simplicial complex because 1 is a maximum element of $M$ and hence a cone point of $\Delta(M)$. It follows that the augmented simplicial chain complex for $\Delta(M)$ is a resolution of the trivial module by $\Bbbk M$-modules. More generally, the augmented simplicial chain complex of $\Delta(M_{\geq X})$ is a resolution of the simple module $S_X$ by $\Bbbk M$-modules via $\rho_X$. In forthcoming work with Margolis and Saliola, we prove that this is a projective resolution.

**Theorem 19.6.** *Let $M$ be a left regular band and $\Bbbk$ a field. Let $X \in \Lambda(M)$. Then the augmented simplicial chain complex of $\Delta(M_{\geq X})$ yields a finite projective resolution of the simple module $S_X$ associated to $X$.*

Let us say that $M$ is a *CW left regular band* if $M_{\geq X}$ is the face poset of a regular CW complex (a CW complex whose attaching maps are homeomorphisms) for all $X \in \Lambda(M)$. This regular CW complex is unique up to cellular isomorphism (cf. [Bjö84, BLVS$^+$99]). Note that $M$ is the contraction of $M$ to the minimum element of $\Lambda(M)$ and so $M$ itself is the face poset of a regular CW complex. Examples of CW left regular bands include hyperplane face monoids [BHR99], oriented matroids [BLVS$^+$99] and complex hyperplane face monoids [BZ92, Bjö08]. In this case, one has that $M_{\geq X}$ acts by cellular maps on the corresponding regular CW complex (which must, in fact, be homeomorphic to a closed ball). In our forthcoming work with Margolis and Saliola, the following theorem is proved. It generalizes Saliola's results for hyperplane face monoids [Sal09].

**Theorem 19.7.** *Let $M$ be a CW left regular band and $\Bbbk$ a field.*

*(i) The augmented cellular chain complex of the regular CW complex associated to $M_{\geq X}$ provides the minimal projective resolution of the simple module $S_X$ corresponding to $X \in \Lambda(M)$.*
*(ii) The quiver $Q$ of $\Bbbk M$ is the Hasse diagram of $\Lambda(M)$.*
*(iii) $\Bbbk M \cong \Bbbk Q/I$ where $I$ is the (admissible) ideal generated by the sum of all paths in $Q$ of length 2.*
*(iv) $\Bbbk M$ is a (graded) Koszul algebra with Koszul dual the incidence algebra of $\Lambda(M)$.*
*(v) The global dimension of $\Bbbk M$ is the dimension of the regular CW complex whose face poset is $M$.*

Finding a quiver presentation for the algebra of an arbitrary left regular band is an outstanding open question.

The author, with Margolis and Saliola, computed the Ext-spaces between simple modules for left regular band algebras in [MSS]. As a consequence, we were able to compute the global dimension of the algebras of all the examples appearing in the literature, as well as new ones. The main result of [MSS] is as follows.

**Theorem 19.8.** *Let $M$ be a left regular band and $\Bbbk$ a field. Let $X, Y \in \Lambda(M)$. Fix $e_Y$ with $Me_Y = Y$. Then*

$$\mathrm{Ext}^n_{\Bbbk M}(S_X, S_Y) = \begin{cases} \Bbbk, & \text{if } X = Y,\ n = 0 \\ \widetilde{H}^{n-1}(\Delta(e_Y M_{\geq X} \setminus \{e_Y\}); \Bbbk), & \text{if } Y > X,\ n \geq 1 \\ 0, & \text{else} \end{cases}$$

*where $\widetilde{H}^q(K; \Bbbk)$ denotes the reduced cohomology in dimension $q$ of the simplicial complex $K$ with coefficients in $\Bbbk$.*

As a consequence, we obtain an alternative description of the quiver of a left regular band.

**Corollary 19.9.** *Let $M$ be a left regular band and $\Bbbk$ a field. Then the quiver of $\Bbbk M$ has vertex set $\Lambda(M)$. There are no arrows $X \longrightarrow Y$ unless $X < Y$, in which case the number of arrows $X \longrightarrow Y$ is one fewer than the number of connected components of $\Delta(e_Y M_{\geq X} \setminus \{e_Y\})$ where $Y = Me_Y$.*

The reader is invited to prove directly that Theorem 18.11 and Corollary 19.9 give the same answer.

The proof of Theorem 19.8 in [MSS] goes through classifying spaces of small categories and Quillen's celebrated Theorem A [Qui73]. In our forthcoming monograph with Margolis and Saliola, we give a more direct proof using Theorem 19.6.

Let us consider a sample application. It is well known, and fairly easy to show, that if the Hasse diagram of a poset $P$ is a forest, then each connected component of $\Delta(P)$ is contractible (a proof can be found in [MSS], for example). If $M$ is a left regular band whose Hasse diagram is a tree rooted at 1, then the Hasse diagam of $e_Y M_{\geq X} \setminus \{e_Y\}$ will be a forest for all $X < Y$ in $\Lambda(M)$. Theorem 19.8 then implies that $\Bbbk M$ has global dimension at most 1, that is, $\Bbbk M$ is hereditary. For instance, if $F(B)$ is the free left regular band on a set $B$, then the natural partial order is the opposite of the prefix order. Therefore, the Hasse diagram of $F(B)$ is a tree rooted at 1. We can thus deduce K. Brown's result that the algebra $\Bbbk F(B)$ is hereditary from Theorem 19.8.

In [MSS] we proved the following cute result.

**Theorem 19.10.** *Let $Q$ be a finite acyclic quiver and $\Bbbk$ a field. Then the path algebra $\Bbbk Q$ is isomorphic to the algebra of a left regular band.*

In light of Gabriel's theorem [Gab72], it follows that each hereditary finite dimensional algebra over an algebraically closed field is Morita equivalent to the algebra of a left regular band. In other words, the representation theory of left regular bands is at least as rich as the representation theory of quivers!

Another interesting open question is to classify left regular band algebras of finite representation type. Path algebras of finite representation type were famously classified by Gabriel [Gab73] (cf. [Ben98, ASS06]).

## 19.4 The Burnside problem for linear monoids

In this section, we lift the restriction that all monoids are finite. A monoid $M$ is said to be *periodic* if, for all $m \in M$, there exist $c, d > 0$ such that $m^c = m^{c+d}$. Of course, finite monoids are periodic. The Burnside problem concerns hypotheses that guarantee that a finitely generated periodic monoid must be finite. A classical theorem of Schur [CR88] states that any finitely generated periodic group of matrices over a field is finite. This was extended to monoids by McNaughton and Zalcstein [MZ75].

**Theorem 19.11 (McNaughton-Zalcstein).** *Let $M \leq M_n(\Bbbk)$ be a finitely generated periodic monoid of matrices. Then $M$ is finite.*

A number of proofs of Theorem 19.11 have since appeared, cf. [Str83, FGG97, Okn98, Ste12b]. The key difference between the group case and the monoid case lies in the process of passing to the completely reducible case because monoid homomorphisms do not have a kernel. The completely reducible case was handled for groups by Burnside [CR88] and applies *mutatis mutandis* to monoids.

One elegant approach is based on an induction on the number of composition factors for $\Bbbk^n$ as a module over the subalgebra $A$ spanned by $M \leq M_n(\Bbbk)$, where the base case of a simple module is handled by Burnside's theorem. For the inductive step, one puts $M$ in a $2 \times 2$-block upper triangular form such that the diagonal block monoids are finite by induction. Then one applies to the projection to the block diagonal the following highly non-trivial theorem of Brown [Bro71] (see also [RS09, Theorem 4.2.4]). A semigroup $S$ is said to be *locally finite* if each finitely generated subsemigroup of $S$ is finite.

**Theorem 19.12 (Brown).** *Let $\varphi\colon S \longrightarrow T$ be a semigroup homomorphism such that $T$ is locally finite and $\varphi^{-1}(e)$ is locally finite for each $e \in E(T)$. Then $S$ is locally finite.*

The reader should note that the special case of Brown's theorem where $S$ and $T$ are groups is trivial.

# A

# Finite Dimensional Algebras

This appendix reviews the necessary background material from the theory of finite dimensional algebras. Standard texts covering most of this subject matter are [CR88, Lam91, Ben98, ASS06]. Readers familiar with this material are urged to skim this appendix or skip it entirely. Very few proofs will be given here, as the results can be found in the references. Let us remark that, unlike the case of group algebras, monoid algebras are seldom semisimple, even over the complex numbers. This forces us to use more of the theory of finite dimensional algebras than would be encountered in a first course on the ordinary representation theory of finite groups. Some of the more elaborate tools, like projective covers, are not used except for in the final chapters of the text. Most of the text uses nothing beyond Wedderburn theory.

Let us fix for the rest of this appendix a field $\Bbbk$ and a finite dimensional $\Bbbk$-algebra $A$.

## A.1 Semisimple modules and algebras

An $A$-module $S$ is *simple* if $S \neq 0$ and the only submodules of $S$ are $0$ and $S$. Equivalently, a non-zero module $S$ is simple if $Av = S$ for all non-zero vectors $v \in S$. Notice that $S = Av$, for $v \neq 0$, implies that $S$ is a cyclic module and hence a quotient of $A$. Thus every simple $A$-module is finite dimensional. Schur's lemma asserts that there are very few homomorphisms between simple modules. See [CR88, Lemma 27.3].

**Lemma A.1 (Schur).** *If $S, S'$ are simple $A$-modules, then every non-zero homomorphism $\varphi\colon S \longrightarrow S'$ is an isomorphism. In particular,* $\mathrm{End}_A(S)$ *is a finite dimensional division algebra over* $\Bbbk$. *If* $\Bbbk$ *is algebraically closed, then* $\mathrm{End}_A(S) = \Bbbk \cdot 1_S \cong \Bbbk$.

An $A$-module $M$ is *semisimple* if $A = \bigoplus_{\alpha \in F} S_\alpha$ for some family of simple submodules $\{S_\alpha \mid \alpha \in F\}$. The following proposition is the content of [CR88, Theorem 15.3].

**Proposition A.2.** *Let $M$ be an $A$-module. Then the following are equivalent.*

*(i)* $M$ *is semisimple.*
*(ii)* $M = \sum_{\alpha \in F} S_\alpha$ *with* $S_\alpha \leq M$ *simple for all* $\alpha \in F$.
*(iii) For each submodule* $N \leq M$, *there is a submodule* $N' \leq M$ *such that* $M = N \oplus N'$.

It follows that every $A$-module $M$ has a unique maximal semisimple submodule $\mathrm{soc}(M)$, called the *socle* of $M$. Indeed, $\mathrm{soc}(M)$ is the sum of all the simple submodules of $M$; it is semisimple by Proposition A.2.

**Proposition A.3.** *The subcategory of semisimple $A$-modules is closed under taking submodules, quotient modules and direct sums.*

*Proof.* Closure under direct sum is clear from the definition. Closure under quotients is immediate from Proposition A.2(ii) and the fact that a quotient of a simple module is either 0 or simple. Closure under submodules follows from closure under quotients because each submodule is a direct summand by Proposition A.2(iii) and hence a quotient. □

If $V$ is an $A$-module, we define $\mathrm{rad}(V)$ to be the intersection of all maximal submodules of $V$. The next proposition is essentially [ASS06, Corollary I.3.8].

**Proposition A.4.** *Let $V$ be a finite dimensional $A$-module.*

*(i)* $V$ *is semisimple if and only if* $\mathrm{rad}(V) = 0$.
*(ii)* $V/\mathrm{rad}(V)$ *is semisimple.*
*(iii) If* $W \leq V$, *then* $V/W$ *is semisimple if and only if* $\mathrm{rad}(V) \subseteq W$.

We can view $A$, itself, as a finite dimensional $A$-module called the *regular module*. Maximal submodules of $A$ are just maximal left ideals and so $\mathrm{rad}(A)$ is the intersection of all maximal left ideals of $A$. However, $\mathrm{rad}(A)$ enjoys a number of additional properties. Recall that an ideal $I$ of an algebra $A$ is *nilpotent* if $I^n = 0$ for some $n \geq 1$.

The following theorem can be extracted from [ASS06, Corollary 1.4], [Ben98, Proposition 1.2.5 and Theorem 1.2.7] and [CR88, Theorem 25.24] with a little effort.

**Theorem A.5.** *Let $A$ be a finite dimensional $\Bbbk$-algebra.*

*(i)* $\mathrm{rad}(A)$ *is the intersection of all maximal right ideals of* $A$.
*(ii)* $\mathrm{rad}(A)$ *is a two-sided ideal.*
*(iii)* $\mathrm{rad}(A)$ *is the intersection of the annihilators of the simple $A$-modules.*
*(iv) If* $V$ *is a finite dimensional $A$-module, then* $\mathrm{rad}(V) = \mathrm{rad}(A) \cdot V$.
*(v) An $A$-module $M$ is semisimple if and only if* $\mathrm{rad}(A) \cdot M = 0$.
*(vi)* $\mathrm{rad}(A)$ *is nilpotent.*
*(vii)* $\mathrm{rad}(A)$ *is the largest nilpotent ideal of* $A$.

One also has Nakayama's lemma (cf. [Ben98, Lemma 1.2.3]).

**Lemma A.6 (Nakayama).** *Let $V$ be a finite dimensional $A$-module. Then one has that $\operatorname{rad}(A) \cdot V = V$ implies $V = 0$.*

A finite dimensional $\Bbbk$-algebra is *semisimple* if the regular module $A$ is a semisimple module. Wedderburn famously characterized semisimple algebras. See [Ben98, Theorem 1.3.4 and Theorem 1.3.5].

**Theorem A.7 (Wedderburn).** *The following are equivalent for a finite dimensional $\Bbbk$-algebra $A$.*

*(i)* *$A$ is semisimple.*
*(ii)* *Each $A$-module is semisimple.*
*(iii)* $\operatorname{rad}(A) = 0$.
*(iv)* $A \cong \prod_{i=1}^{r} M_{n_i}(D_i)$ *where the $D_i$ are finite dimensional division algebras over $\Bbbk$.*

*Moreover, if $A$ is semisimple, then $A$ has finitely many simple modules $S_1, \ldots, S_r$ up to isomorphism and in (iv) (after reordering) one has $D_i \cong \operatorname{End}_A(S_i)^{op}$ and $n_i = \dim S_i$. Furthermore, there is an $A$-module isomorphism*

$$A \cong \bigoplus_{i=1}^{r} n_i \cdot S_i.$$

*If, in addition, $\Bbbk$ is algebraically closed, then $D_i = \Bbbk$ for all $i = 1, \ldots, r$.*

The class of semisimple algebras is closed under quotients.

**Proposition A.8.** *Quotients of a semisimple finite dimensional algebra are semisimple.*

*Proof.* If $A$ is semisimple and $I$ is an ideal, then $A/I$ is a direct sum of simple submodules as an $A$-module and hence as an $A/I$-module. □

Factoring a finite dimensional algebra by its radical results in a semisimple algebra. More precisely, we have the following, which combines elements of [ASS06, Corollary 1.4] and [CR88, Theorem 25.24].

**Theorem A.9.** *Let $A$ be a finite dimensional $\Bbbk$-algebra.*

*(i)* $A/\operatorname{rad}(A)$ *is semisimple.*
*(ii)* $A/\operatorname{rad}(A)$-mod *can be identified with the full subcategory of* $A$-mod *whose objects are the semisimple $A$-modules.*
*(iii)* *If $I$ is an ideal of $A$ with $A/I$ semisimple, then* $\operatorname{rad}(A) \subseteq I$.

*In particular, the simple $A$-modules are the same as the simple $A/\operatorname{rad}(A)$-modules and hence $A$ has only finitely many isomorphism classes of simple $A$-modules.*

A useful corollary of Theorem A.9, which is exploited in Chapter 12, is the following.

**Corollary A.10.** *Suppose that $A$ is a finite dimensional $\Bbbk$-algebra and $I$ is a nilpotent ideal such that $A/I$ is semisimple. Then $I = \mathrm{rad}(A)$.*

*Proof.* The inclusion $I \subseteq \mathrm{rad}(A)$ follows from Theorem A.5 and the inclusion $\mathrm{rad}(A) \subseteq I$ follows from Theorem A.9. □

Another theorem of Wedderburn, which is employed elsewhere in the text, gives a criterion for nilpotency of an ideal. An element $a \in A$ is *nilpotent* if $a^n = 0$ for some $n \geq 1$. Clearly, each element of a nilpotent ideal is nilpotent. The following strong converse is due to Wedderburn, cf. [CR88, Theorem 27.27] (where we note that the hypothesis that the field is algebraically closed is unnecessary since having a basis of nilpotent elements is preserved under extension of scalars).

**Theorem A.11.** *Let $A$ be a finite dimensional $\Bbbk$-algebra. Then an ideal $I$ of $A$ is nilpotent if and only if it spanned by a set of nilpotent elements.*

For one of the exercises, we shall need the following proposition (cf. [Lam91, Proposition 11.7]). Recall that an $A$-module $V$ is *faithful* if its annihilator is 0, that is, $aV = 0$ implies $a = 0$.

**Proposition A.12.** *Let $A$ be a finite dimensional $\Bbbk$-algebra with a faithful simple module. Then $A$ is simple, i.e., has no proper non-zero ideals.*

Next we discuss the Jordan-Hölder theorem for finite dimensional modules. For a proof, see [CR88, Theorem 13.7]. If $V$ is a finite dimensional $A$-module, a *composition series* for $V$ is an unrefinable chain of submodules

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = V.$$

We call $n$ the *length* of the composition series and the simple modules $V_i/V_{i-1}$ the *composition factors*.

**Theorem A.13 (Jordan-Hölder).** *Let $V$ be a finite dimensional $A$-module and let*

$$\begin{aligned} 0 &= V_0 \subsetneq \cdots \subsetneq V_n = V \\ 0 &= W_0 \subsetneq \cdots \subsetneq W_m = V \end{aligned}$$

*be composition series for $V$. Then $m = n$ and there exists $\sigma \in S_n$ such that $V_i/V_{i-1} \cong W_{\sigma(i)}/W_{\sigma(i)-1}$ for $i = 1, \ldots, n$.*

It follows from the Jordan-Hölder theorem that we can unambiguously define the *length* of $V$ to be the length of a composition series and we can define, for a simple $A$-module $S$, its *multiplicity* as a composition factor in $V$ to be the number $[V : S]$ of composition factors in some composition series that are isomorphic to $S$.

The following theorem was originally proved by Frobenius and Schur in the context of complex group algebras and can be found as [CR88, Theorem 27.8].

**Theorem A.14.** *Let $\Bbbk$ be an algebraically closed field and $A$ a finite dimensional $\Bbbk$-algebra. Suppose that $S_1, \ldots, S_r$ form a complete set of representatives of the isomorphism classes of simple $A$-modules. Fix a basis for each $S_i$ and let $\varphi^{(k)}\colon A \longrightarrow M_{n_k}(\Bbbk)$ be the $\Bbbk$-algebra homomorphism given by sending $a \in A$ to the matrix*

$$\varphi^{(k)}(a) = \left(\varphi_{ij}^{(k)}(a)\right)$$

*of the operator $v \mapsto av$. Then the linear functionals*

$$\varphi_{ij}^{(k)}\colon A \longrightarrow \Bbbk$$

*with $1 \leq i, j \leq n_k$ and $1 \leq k \leq r$ are linearly independent over $\Bbbk$.*

## A.2 Indecomposable modules

A non-zero $A$-module $M$ is *indecomposable* if $M = M' \oplus M''$ implies $M' = 0$ or $M'' = 0$. Every simple module is indecomposable, but the converse only holds when $A$ is semisimple. The Krull-Schmidt theorem [CR88, Theorem 14.5] asserts that each finite dimensional module admits an essentially unique decomposition into a direct sum of indecomposable modules.

**Theorem A.15 (Krull-Schmidt).** *If $A$ is a finite dimensional $\Bbbk$-algebra and $V$ is a finite dimensional $A$-module, then $V = \bigoplus_{i=1}^{s} M_i$ with the $M_i$ indecomposable submodules. Moreover, if $V \cong \bigoplus_{i=1}^{r} N_i$ with the $N_i$ indecomposable, then $r = s$ and there is $\sigma \in S_r$ such that $M_i \cong N_{\sigma(i)}$ for $i = 1, \ldots, r$.*

A finite dimensional algebra is said to be of *finite representation type* if it has only finitely many isomorphism classes of finite dimensional indecomposable modules.

Applying the Krull-Schmidt theorem to the regular module $A$, we have that $A = \bigoplus_{i=1}^{s} P_i$ where the $P_i$ are indecomposable and, moreover, projective. The following theorem combines aspects of [CR88, Theorem 54.11, Corollary 54.13 and Corollary 54.14] and the discussion on Page 14 of [Ben98].

**Theorem A.16.** *Let $A$ be a finite dimensional $\Bbbk$-algebra and suppose that*

$$A = \bigoplus_{i=1}^{s} P_i$$

*is a decomposition of $A$ into indecomposable submodules.*

*(i) Every projective indecomposable module is isomorphic to $P_i$ for some $i = 1, \ldots, s$.*
*(ii) $P_i / \operatorname{rad}(P_i)$ is simple.*
*(iii) $P_i \cong P_j$ if and only if $P_i / \operatorname{rad}(P_i) \cong P_j / \operatorname{rad}(P_j)$.*

*(iv) One has that*

$$A/\operatorname{rad}(A) = \bigoplus_{i=1}^{s} P_i/\operatorname{rad}(P_i).$$

*In particular, each simple $A$-module is isomorphic to one the form $P/\operatorname{rad}(P)$ for some projective indecomposable (unique up to isomorphism) and the multiplicity of $P/\operatorname{rad}(P)$ as a direct summand in $A/\operatorname{rad}(A)$ coincides with the multiplicity of $P$ as a direct summand in $A$.*

Projective indecomposable modules are also called *principal indecomposable modules* in the literature.

If $V$ is a finite dimensional $A$-module and $P$ a finite dimensional projective $A$-module, then an epimorphism $\varphi\colon P \longrightarrow V$ is called a *projective cover* if $W + \ker\varphi = P$ implies $W = P$ for $W$ a submodule of $P$. Using Nakayama's lemma, one can show that this is equivalent to $\ker\varphi \subseteq \operatorname{rad}(P)$. The following is the content of [ASS06, Theorem I.5.8] (which assumes throughout that the ground field is algebraically closed, but does not need it for this result).

**Theorem A.17.** *Let $S_1, \dots, S_r$ be a complete set of representatives of the isomorphism classes of simple $A$-modules and let $P_1, \dots, P_r$ be a complete set of representatives of the isomorphism classes of projective indecomposable $A$-modules, ordered so that $P_i/\operatorname{rad}(P_i) \cong S_i$.*

*(i) The canonical epimorphism $\eta\colon P_i \longrightarrow P_i/\operatorname{rad}(P_i)$ is a projective cover.*
*(ii) If $V$ is a finite dimensional $A$-module with*

$$V/\operatorname{rad}(V) \cong \bigoplus_{i=1}^{r} n_i \cdot S_i,$$

*then there is a projective cover*

$$\psi\colon \bigoplus_{i=1}^{r} n_i \cdot P_i \longrightarrow V.$$

*(iii) If $\varphi\colon P \longrightarrow V$ and $\psi\colon Q \longrightarrow V$ are projective covers. Then there is an isomorphism $\tau\colon P \longrightarrow Q$ such that*

$$\begin{tikzcd} P \arrow[rr, "\tau"] \arrow[dr, "\varphi"'] & & Q \arrow[dl, "\psi"] \\ & V & \end{tikzcd}$$

*commutes.*

In other words, each finite dimensional module has a projective cover and the corresponding projective module is unique up to isomorphism. It is not the case that the isomorphism $\tau$ in Theorem A.17(iii) is unique.

## A.3 Idempotents

The set $E(A)$ of idempotents of a finite dimensional algebra $A$ plays a crucial role in the theory. Two idempotents $e, f \in E(A)$ are *orthogonal* if $ef = 0 = fe$.

**Proposition A.18.** *Let $e, f \in E(A)$ be orthogonal idempotents. Then $e + f$ is an idempotent and $A(e + f) = Ae \oplus Af$.*

*Proof.* We compute $(e+f)^2 = e^2+ef+fe+f^2 = e+f$ since $e, f$ are orthogonal idempotents. As $a(e+f) = ae+af$, trivially $A(e+f) = Ae+Af$. If $a \in Ae\cap Af$, then $ae = a = af$ and so $a = af = aef = 0$. Thus $A(e + f) = Ae \oplus Af$. $\square$

If $e \in E(A)$, then $(1 - e)^2 = 1 - 2e + e^2 = 1 - e$ and so $1 - e \in E(A)$. Trivially, $e, 1-e$ are orthogonal and so Proposition A.18 admits the following corollary.

**Corollary A.19.** *If $e \in E(A)$, then $A = Ae \oplus A(1 - e)$ and hence $Ae$ is projective.*

A collection $e_1, \ldots, e_n$ of pairwise orthogonal idempotents of $A$ is called a *complete set of orthogonal idempotents* if $1 = e_1 + \cdots + e_n$. In this case $A \cong Ae_1 \oplus \cdots \oplus Ae_n$ by repeated application of Proposition A.18.

Note that if $A$ is finite dimensioinal and $e \in E(A)$, then $\mathrm{rad}(Ae) = \mathrm{rad}(A)Ae = \mathrm{rad}(A)e$. Proposition 1.8 has the following analogue for algebras [Ben98, Lemma 1.3.3].

**Proposition A.20.** *Let $e \in E(A)$ and let $M$ be an $A$-module.*

*(i)* $\mathrm{Hom}_A(Ae, M) \cong eM$ *via* $\varphi \mapsto \varphi(e)$.
*(ii)* $\mathrm{End}_A(Ae) \cong (eAe)^{op}$.

*In particular,* $\mathrm{End}_A(A) \cong A^{op}$.

If $L$ is a direct summand in the regular module $A$, then there is a projection $\pi\colon A \longrightarrow L$. If $\iota\colon L \longrightarrow A$ is the inclusion, then $\iota\pi \in \mathrm{End}_A(A)$ is an idempotent and so by Proposition A.20 there exists $e \in E(A)$ with $\iota\pi(1) = e$. Therefore, $L = \iota\pi(A) = A\iota\pi(1) = Ae$ and we have proved the following.

**Proposition A.21.** *Every direct summand of the regular module $A$ is of the form $Ae$ with $e \in E(A)$.*

A non-zero idempotent $e \in E(A)$ is said to be *primitive* if $e = e_1 + e_2$ with $e_1, e_2$ orthogonal implies that $e_1 = 0$ or $e_2 = 0$.

**Proposition A.22.** *Let $e \in E(A) \setminus \{0\}$. Then the following are equivalent.*

*(i) $e$ is primitive.*
*(ii) $Ae$ is indecomposable.*
*(iii) $E(eAe) = \{0, e\}$.*

*Proof.* Suppose that $e$ is primitive and let $f \in E(eAe)$. Then $f, e-f$ are orthogonal idempotents and $e = f + (e-f)$. Thus $f = 0$ or $e - f = 0$ by primitivity of $e$. This shows that (i) implies (iii). Assume that (iii) holds. If $Ae = V \oplus W$, then there is an idempotent $\pi \in \mathrm{End}_A(Ae)$ with $\pi(Ae) = V$. But $\mathrm{End}(Ae) \cong (eAe)^{op}$ by Proposition A.20 and so $\pi = 0$ or $\pi = 1_{Ae}$. Thus $V = 0$ or $W = 0$ and so $Ae$ is indecomposable. This proves that (iii) implies (ii). Assume that (ii) holds. If $e = e_1 + e_2$ with $e_1, e_2$ orthogonal idempotents, then $Ae = Ae_1 \oplus Ae_2$ by Proposition A.18. Thus $Ae_1 = 0$ or $Ae_2 = 0$, that is, we have $e_1 = 0$ or $e_2 = 0$. This establishes that $e$ is primitive and completes the proof. □

We say that a complete set of orthogonal idempotents $e_1, \ldots, e_s$ is a *complete set of orthogonal primitive idempotents* if each $e_i$ is primitive. The next theorem is essentially [Ben98, Corollary 1.7.4] and the discussion following it.

**Theorem A.23.** *Let $A$ be a finite dimensional $\Bbbk$-algebra.*

*(i) $A = \bigoplus_{i=1}^{s} Ae_i$ with $e_i \in E(A)$ is a direct sum decomposition into projective indecomposable modules if and only if $e_1, \ldots, e_s$ is a complete set of orthogonal primitive idempotents.*
*(ii) If $\{e_1, \ldots, e_s\}$ is a complete set of orthogonal primitive idempotents for $A$, then $e_1 + \mathrm{rad}(A), \ldots, e_s + \mathrm{rad}(A)$ is a complete set of orthogonal primitive idempotents for $A/\mathrm{rad}(A)$.*
*(iii) Every complete set of orthogonal primitive idempotents for $A/\mathrm{rad}(A)$ is of the form $e_1 + \mathrm{rad}(A), \ldots, e_s + \mathrm{rad}(A)$ with $e_1, \ldots, e_s$ a complete set of orthogonal primitive idempotents for $A$.*

It follows that the projective indecomposable $A$-modules are of the form $Ae$ and the simple $A$-modules are of the form $Ae/\mathrm{rad}(A)e$ with $e$ a primitive idempotent.

Another proposition concerning idempotents that is used in the text is the following, which combines [CR88, Theorem 54.12] and [CR88, Theorem 54.16].

**Proposition A.24.** *Let $A$ be a finite dimensional algebra over a field $\Bbbk$ and let $e$ be a primitive idempotent of $A$. Let $S = Ae/\mathrm{rad}(A)e$ be the corresponding simple module. If $V$ is a finite dimensional $A$-module, then $\mathrm{Hom}_A(Ae, V) \cong eV \neq 0$ if and only if $S$ is a composition factor of $V$. Moreover, if $\Bbbk$ is algebraically closed, then $\dim eV = [V : S]$.*

The *center* $Z(A)$ of $A$ consists of those elements of $A$ that commute with all elements of $A$. A *central idempotent* is an idempotent $e$ of $Z(A)$. Notice that $eAe = Ae = AeA$ in this case. If $e_1, \ldots, e_n$ is a complete set of orthogonal central idempotents, then the direct sum decomposition $A = Ae_1 \oplus \cdots \oplus Ae_n$ is actually a direct product decomposition $A = Ae_1 \times \cdots \times Ae_n$ of $\Bbbk$-algebras. It is known that $(E(Z(A)), \leq)$ is a finite boolean algebra. The atoms of $E(Z(A))$ are called the *central primitive idempotents*. Equivalently, a central primitive idempotent is a central idempotent $e$ such that $e = e_1 + e_2$ with $e_1, e_2$ central and orthogonal implies that $e_1 = 0$ or $e_2 = 0$ or, equivalently, such that $E(Z(Ae)) = \{0, e\}$. The central primitive idempotents form a complete set of orthogonal idempotents. If $e$ is a central primitive idempotent, then $Ae$ is an *indecomposable algebra*, i.e., cannot be expressed as a direct product of two non-zero algebras, and is called a *block* of $A$. The reader is referred to [Ben98, Section 1.8] for details.

If $e$ is a central idempotent of $A$ and $V$ is a simple $A$-module, then $eV$ is a submodule and hence $eV = 0$ or $eV = V$. Consequently, if $e_1, \ldots, e_n$ is a complete set of orthogonal central idempotents of $A$ and $V$ is a simple $A$-module, then there is a unique $i \in \{1, \ldots, n\}$ with $e_iV = V$ and $e_jV = 0$ for $j \neq i$. The $A$-module structure on $V$ then comes from the $Ae_i$-module structure on $V$ and the projection $A \longrightarrow Ae_i$. A similar remark holds for indecomposable modules. Thus simple modules, and more generally indecomposable modules, belong to a block.

## A.4 Duality and Morita equivalence

Let $A$ be a finite dimensional $\Bbbk$-algebra. Using the vector space dual, we can turn left $A$-modules into right $A$-modules, or equivalently left $A^{op}$-modules. More precisely, if $V$ is a left $A$-module, then $D(V) = \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$ is a right $A$-module where $(\varphi a)(v) = \varphi(av)$ for $a \in A$, $v \in V$ and $\varphi \in \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$. The contravariant functor $D\colon A\text{-mod} \longrightarrow A^{op}\text{-mod}$ is called the *standard duality*. We write $D_A$ if the algebra $A$ is not clear from context. The standard duality enjoys the following properties; see [ASS06, Section I.2.9].

**Theorem A.25.** *Let $A$ be a finite dimensional $\Bbbk$-algebra.*

*(i)* $D_{A^{op}} \circ D_A \cong 1_{A\text{-mod}}$.
*(ii) $D$ is a contravariant equivalence of categories.*
*(iii) $D$ sends projective modules to injective modules.*
*(iv) $D$ send injective modules to projective modules.*
*(v) $D$ sends simple modules to simple modules.*
*(vi) $D$ preserves indecomposability.*

We recall here that a functor $F\colon C \longrightarrow D$ is an *equivalence* of categories if there is a functor $G\colon D \longrightarrow C$, called a *quasi-inverse*, such that the functors $G \circ F$ and $F \circ G$ are naturally isomorphic to the identity functor on $C$ and $D$, respectively. Equivalently, one has that $F$ is an equivalence if and only if it is fully faithful and essentially surjective. A functor $F\colon C \longrightarrow D$ is *fully faithful* if, for $c, c'$ objects of $C$, one has that $F\colon \mathrm{Hom}(c, c') \longrightarrow \mathrm{Hom}(F(c), F(c'))$ is a bijection and $F$ is *essentially surjective* if each object of $D$ is isomorphic to an object of the form $F(c)$ with $c$ an object of $C$. The reader is referred to [Mac98] for details.

As a consequence we obtain a description of the injective indecomposable modules (cf. [ASS06, Corollary I.5.17]).

**Corollary A.26.** *Let $A$ be a finite dimensional $\Bbbk$-algebra and let $e_1, \ldots, e_s$ be a complete set of orthogonal primitive idempotents.*

*(i) The injective indecomposable $A$-modules are precisely those of the form $D(e_iA)$ for $i = 1, \ldots, s$ (up to isomorphism).*
*(ii) $Ae_i/\operatorname{rad}(A)e_i \cong \operatorname{soc}(D(e_iA))$ for $i = 1, \ldots, s$.*
*(iii) $D(e_iA) \cong D(e_jA)$ if and only if $Ae_i \cong Ae_j$, if and only if $Ae_i/\operatorname{rad}(A)e_i \cong Ae_j/\operatorname{rad}(A)e_j$.*

Two finite dimensional $\Bbbk$-algebras $A, B$ are said to be *Morita equivalent* if the categories $A$-mod and $B$-mod are equivalent.

*Example A.27.* If $A$ is a finite dimensional $\Bbbk$-algebra and $n \geq 1$, then $A$ is Morita equivalent to $M_n(A)$. The equivalence sends an $A$-module $V$ to $V^n$ where

$$(a_{ij})(v_1, \ldots, v_n) = \left(\sum_{j=1}^{n} a_{1j}v_j, \ldots, \sum_{j=1}^{n} a_{nj}v_j\right)$$

gives the module structure.

Notice that if $A$ is semisimple and $B$ is Morita equivalent to $A$, then $B$ is also semisimple. Indeed, $A$ is semisimple if and only if each finite dimensional $A$-module is injective by Proposition A.2. But an equivalence of categories sends injective modules to injective modules and so we conclude that each finite dimensional $B$-module is injective. Therefore, $B$ is semisimple.

Chapter 4 proves a special case of the following well-known theorem, cf. [Ben98, Theorem 2.2.6].

**Theorem A.28.** *Let $A$ and $B$ be finite dimensional $\Bbbk$-algebras. Then $A$ is Morita equivalent to $B$ if and only if there is a finite dimensional projective $A$-module $P$ such that $\operatorname{End}_A(P) \cong B^{op}$ and each projective indecomposable $A$-module is isomorphic to a direct summand in $P$.*

# B

# Group Representation Theory

This appendix surveys those aspects of the representation theory of finite groups that are used throughout the text. The final section provides a brief review of the representation theory of the symmetric group. Monoid representation theory very much builds on group representation theory and so one should have a solid foundation in the latter subject before attempting to master the former. Good references for the representation theory of finite groups are [Isa76, Ser77, CR88]. See also [Ste12a]. We mostly omit proofs, although we occasionally do provide a sketch or complete proof for convenience of the reader.

## B.1 Group algebras

The first fundamental theorem about group algebras is Maschke's characterization of semisimplicity (cf. [CR88, Theorem 15.6]).

**Theorem B.1 (Maschke).** *Let $G$ be a finite group and $\Bbbk$ a field. Then $\Bbbk G$ is semisimple if and only if either the characteristic of $\Bbbk$ is zero or it does not divide the order of $G$.*

Let $\Bbbk$ be an algebraically closed field of characteristic zero. Then $G$ is abelian if and only if $\Bbbk G$ is commutative. Since $\Bbbk G$ is semisimple and $\Bbbk$ is algebraically closed, we deduce from Wedderburn's theorem that this will be the case if and only if each simple $\Bbbk G$-module is one-dimensional. We have thus proved the following result.

**Proposition B.2.** *Let $\Bbbk$ be an algebraically closed field of characteristic zero and let $G$ be a finite group. Then $G$ is abelian if and only if $\Bbbk G \cong \Bbbk^G$, which is equivalent to each simple $\Bbbk G$-module being one-dimensional.*

## B.2 Group character theory

Let us now specialize to the case of the field of complex numbers, although much of what we say holds over any algebraically closed field of characteristic zero. Fix a finite group $G$ for the section.

If $\rho\colon G \longrightarrow M_n(\mathbb{C})$ is a representation, then the *character* of $\rho$ is the mapping $\chi_\rho\colon G \longrightarrow \mathbb{C}$ defined by $\chi_\rho(g) = \mathrm{Tr}(\rho(g))$ where $\mathrm{Tr}(A)$ is the trace of a matrix $A$. It is clear that equivalent representations have the same character and hence if $V$ is a finite dimensional $\mathbb{C}G$-module, then we can define the *character* $\chi_V$ of $V$ by putting $\chi_V = \chi_\rho$ for any representation $\rho$ afforded by $V$. We sometimes say that $V$ affords the character $\chi$. Note that $\chi_V(1) = \dim V$. A character is called *irreducible* if it is the character of a simple module. One useful property of characters is the following (a proof of which can be found on Page 221 of [CR88]).

**Proposition B.3.** *Let $\chi$ be a character of a finite group $G$. Then $\chi(g^{-1}) = \overline{\chi(g)}$ for $g \in G$.*

Isomorphic modules are easily seen to have the same character. In fact, the character determines the module up to isomorphism. Let us enter into a bit of detail. A mapping $f\colon G \longrightarrow \mathbb{C}$ is called a *class function* if it is constant on conjugacy classes. The set $\mathrm{Cl}(G)$ of class functions is a subspace of $\mathbb{C}^G$ whose dimension is the number of conjugacy classes of $G$ and, in fact, it is a subalgebra where $\mathbb{C}^G$ is made a $\mathbb{C}$-algebra via pointwise operations. We can define an inner product on $\mathbb{C}^G$ by putting

$$\langle f, h\rangle_G = \frac{1}{|G|}\sum_{g\in G} f(g)\overline{h(g)}.$$

If $G$ is understood, we just write $\langle f, h\rangle$. Clearly, each character is a class function because the trace is constant on similarity classes of matrices. Our statement of the first orthogonality relations combines Theorem 8.16 and Corollary 8.17 of [Lam91].

**Theorem B.4 (First orthogonality relations).** *Let $S_1, \ldots, S_r$ be a complete set of representatives of the isomorphism classes of simple $\mathbb{C}G$-modules. Then the irreducible characters $\chi_{S_1}, \ldots, \chi_{S_r}$ form an orthonormal basis for the space $\mathrm{Cl}(G)$ of class functions.*

As an immediate consequence, one can compute the number of isomorphism classes of simple $\mathbb{C}G$-modules.

**Corollary B.5.** *Let $G$ be a finite group. The number of isomorphism classes of simple $\mathbb{C}G$-modules coincides with the number of conjugacy classes of $G$.*

Retaining the notation of Theorem B.4, if $V$ is a $\mathbb{C}G$-module with

$$V \cong \bigoplus_{i=1}^{r} m_i \cdot S_i,$$

then $\chi_V = m_1\chi_{S_1} + \cdots + m_r\chi_{S_r}$ and so we deduce from Theorem B.4 that $m_i = \langle \chi_V, \chi_{S_i} \rangle$. Therefore, $V$ is determined up to isomorphism by its character. We summarize this discussion in the following corollary.

**Corollary B.6.** *Let $S_1, \ldots, S_r$ be a complete set of representatives of the isomorphism classes of simple $\mathbb{C}G$-modules and let $V$ be a finite dimensional $\mathbb{C}G$-module. Then*

$$V \cong \bigoplus_{i=1}^{r} m_i \cdot S_i$$

*where $m_i = \langle \chi_V, \chi_{S_i} \rangle$. Thus two finite dimensional $\mathbb{C}G$-modules $V$ and $W$ are isomorphic if and only if $\chi_V = \chi_W$.*

It is for this reason that group representation theory over $\mathbb{C}$ reduces to character theory.

Theorem B.4 and Corollary B.6 lead to the following criterion for simplicity.

**Corollary B.7.** *Let $V$ be a finite dimensional $\mathbb{C}G$-module. Then $V$ is simple if and only if $\langle \chi_V, \chi_V \rangle = 1$.*

*Proof.* Necessity is clear from Theorem B.4. For sufficiency, let $S_1, \ldots, S_r$ be a complete set of non-isomorphic simple $\mathbb{C}G$-module and observe that if

$$V \cong \bigoplus_{i=1}^{r} m_i \cdot S_i,$$

then $\chi_V = m_1\chi_{S_1} + \cdots + m_r\chi_{S_r}$. By Theorem B.4, we then have that $1 = \langle \chi_V, \chi_V \rangle = m_1^2 + \cdots + m_r^2$ and so there exists $i$ such that $m_i = 1$ and $m_j = 0$ for $j \neq i$. We conclude that $V \cong S_i$ is simple. $\square$

The next theorem summarizes some further properties of group representations that we shall not require in the sequel, but which give some flavor of the subject. Proofs can be found in [CR88, (27.21), Theorem 33.7 and Theorem 33.8].

**Theorem B.8.** *Let $G$ be a finite group and let $d_1, \ldots, d_r$ be the dimensions of a complete set of representatives of the isomorphism classes of simple $\mathbb{C}G$-modules.*

*(i)* $d_1^2 + \cdots + d_r^2 = |G|$.
*(ii)* $d_i$ *divides* $|G|$ *for* $i = 1, \ldots, r$.
*(iii) The number of one-dimensional representations of* $G$ *is* $|G/[G,G]|$.

*(iv) If $\chi_1, \ldots, \chi_r$ are the irreducible characters of $G$, then the*

$$e_i = \frac{\chi(1)}{|G|} \sum_{g \in G} \overline{\chi(g)} \cdot g$$

*for $i = 1, \ldots, r$, form a complete set of orthogonal central primitive idempotents of $\mathbb{C}G$.*

Let $\chi_1, \ldots, \chi_r$ be the distinct irreducible characters of $G$ and let $C_1, \ldots, C_r$ be the conjugacy classes of $G$. The *character table* of $G$ is the $r \times r$ matrix $X(G)$ given by $X(G)_{ij} = \chi_i(C_j)$ where $\chi_i(C_j)$ is the value that $\chi_i$ takes on the conjugacy class $C_j$. Sometime we index the rows of $X(G)$ by the irreducible characters and the columns by the conjugacy classes.

The next theorem is a reformulation of what is often called the second orthogonality relations. Recall that if $A = (a_{ij}) \in M_r(\mathbb{C})$, then $A^* = (\overline{a_{ji}})$ is the conjugate-transpose.

**Theorem B.9 (Second orthogonality relations).** *Let $G$ be a finite group with irreducible characters $\chi_1, \ldots, \chi_r$ and conjugacy classes $C_1, \ldots, C_r$. Put $z_i = |C_i|/|G|$ for $i = 1, \ldots, r$ and let $Z$ be the $r \times r$ diagonal matrix with $Z_{ii} = z_i$. Then $X(G)ZX(G)^* = I$ and hence $X(G)$ is invertible with inverse $ZX(G)^*$.*

*Proof.* Let $A = X(G)ZX(G)^*$. Then we compute that

$$A_{ij} = \frac{1}{|G|} \sum_{k=1}^{r} \chi_i(C_k) \cdot |C_k| \cdot \overline{\chi_j(C_k)} = \frac{1}{|G|} \sum_{g \in G} \chi_i(g)\overline{\chi_j(g)} = \langle \chi_i, \chi_j \rangle = I_{ij}$$

where the last equality uses the first orthogonality relations. □

## B.3 Permutation modules

Let $G$ be a finite group and $\Omega$ a finite $G$-set. Let $\Bbbk$ be a field. Then $\Bbbk\Omega$ is a $\Bbbk G$-module by extending linearly the action of $G$ on $\Omega$, i.e.,

$$g \cdot \sum_{\alpha \in \Omega} c_\alpha \alpha = \sum_{\alpha \in \Omega} c_\alpha g\alpha$$

for $g \in G$. One usually calls $\Bbbk\Omega$ a *permutation module*. Recall that $\Omega$ is a *transitive* $G$-set if $G$ has a single orbit on $\Omega$. In other words, $G$ acts transitively on $\Omega$ if, for all $\alpha, \beta \in \Omega$, there exists $g \in G$ with $g\alpha = \beta$. If $G$ acts transitively on $\Omega$ and if $H \leq G$ is the stabilizer of $\omega \in \Omega$, then $|\Omega| = [G : H]$. Set $[\Omega] = \sum_{\omega \in \Omega} \omega$. This vector will play an important role.

If $V$ is a $\Bbbk G$-module, then one puts

$$V^G = \{v \in V \mid gv = v, \forall g \in G\}.$$

So $V^G$ is the subspace of fixed vectors. Observe that $V^G$ is the submodule of $\mathrm{soc}(V)$ spanned by all copies of the trivial $\Bbbk G$-module and is isomorphic to the direct sum of $\dim V^G$ copies of the trivial module. In particular, if $\Bbbk = \mathbb{C}$ and $\dim V < \infty$, then $\dim V^G = \langle \chi_V, \chi_1 \rangle$ where $\chi_1$ is the character of the trivial module.

**Proposition B.10.** *Let $G$ be a finite group, $\Omega$ a transitive $G$-set and $\Bbbk$ a field of characteristic* 0. *Then $(\Bbbk\Omega)^G = \Bbbk[\Omega]$. Moreover, if*

$$U = \frac{1}{|G|} \sum_{g \in G} g,$$

*then $U$ is an idempotent with $U \cdot \Bbbk\Omega = (\Bbbk\Omega)^G$ and $U\beta = \frac{1}{|\Omega|}[\Omega]$ for $\beta \in \Omega$.*

*Proof.* Put $V = \Bbbk\Omega$. It is clear that $g[\Omega] = [\Omega]$ for all $g \in G$ because $G$ permutes $\Omega$. Suppose that $v = \sum_{\alpha \in \Omega} c_\alpha \alpha$ belongs to $V^G$. Fix $\omega \in \Omega$. If $\alpha \in \Omega$ and $g \in G$ with $g\omega = \alpha$, then the coefficient of $\alpha$ in $gv$ is $c_\omega$. As $gv = v$ for all $g \in G$, we conclude that $v = c_\omega[\Omega]$. This establishes that $V^G = \Bbbk[\Omega]$.

If $h \in G$, then since left multiplication by $h$ permutes $G$, we have $hU = U$. It follows easily that $U^2 = U$ and that $UV \subseteq V^G$. But if $v \in V^G$, then

$$Uv = \frac{1}{|G|} \sum_{g \in G} gv = \frac{1}{|G|} \sum_{g \in G} v = v$$

and so $UV = V^G$ (note that this argument works for any $\Bbbk G$-module $V$).

Let $\beta \in \Omega$ and let $H \leq G$ be the stabilizer of $\beta$. Fix $g \in G$ with $g\beta = \alpha$. Then, for $g' \in G$, one has that $g'\beta = \alpha$ if and only if $g' \in gH$. Therefore, we compute

$$U\beta = \frac{1}{|G|} \sum_{g \in G} g\beta = \frac{|H|}{|G|} \sum_{\alpha \in \Omega} \alpha = \frac{1}{|\Omega|}[\Omega]$$

as $[G : H] = |\Omega|$. This completes the proof. $\square$

An immediate corollary of Proposition B.10 is the following result. For a $G$-set $\Omega$ and $g \in G$, put $\mathrm{Fix}(g) = \{\alpha \in \Omega \mid g\alpha = \alpha\}$. Note that

$$\chi_{\mathbb{C}\Omega}(g) = |\mathrm{Fix}(g)| \tag{B.1}$$

by considering the matrix representation afforded by $\mathbb{C}\Omega$ with respect to the basis $\Omega$ of $\mathbb{C}\Omega$. In particular, taking $\Omega = G$ with the regular action of $G$ yields

$$\chi_{\mathbb{C}G}(g) = \begin{cases} |G|, & \text{if } g = 1 \\ 0, & \text{else} \end{cases} \tag{B.2}$$

a fact used elsewhere in the text.

**Corollary B.11.** *Let $\Omega$ be a finite $G$-set for a finite group $G$ and suppose that $G$ has $s$ orbits on $\Omega$. Let $\chi_1$ be the trivial character of $G$. Then*

$$s = \langle \chi_{\mathbb{C}\Omega}, \chi_1 \rangle = \frac{1}{|G|} \sum_{g \in G} |\mathrm{Fix}(g)|$$

*holds.*

*Proof.* If $\Omega_1, \ldots, \Omega_s$ are the orbits of $G$, then $\mathbb{C}\Omega = \bigoplus_{i=1}^{s} \mathbb{C}\Omega_i$. Therefore, we have that

$$\langle \chi_{\mathbb{C}\Omega}, \chi_1 \rangle = \sum_{i=1}^{s} \langle \chi_{\mathbb{C}\Omega_i}, \chi_1 \rangle = s$$

where the last equality uses that $\langle \chi_{\Omega_i}, \chi_1 \rangle = \dim(\mathbb{C}\Omega_i)^G = 1$ by Proposition B.10. Taking into account (B.1), the corollary follows. $\square$

Corollary B.11 is sometimes called the Cauchy-Frobenius-Burnside lemma.

If $\Omega$ is a $G$-set, let us define $\mathrm{Aug}(\Bbbk\Omega)$ to be the submodule of $\Bbbk\Omega$ consisting of those $v = \sum_{\alpha \in \Omega} c_\alpha \alpha$ with $\sum_{\alpha \in \Omega} c_\alpha = 0$. Note that $\Bbbk\Omega = \mathrm{Aug}(\Bbbk\Omega) \oplus \Bbbk[\Omega]$ if $\Bbbk$ is of characteristic 0.

If $\Omega$ is a $G$-set, then $\Omega^2$ is a $G$-set via $g(\alpha, \beta) = (g\alpha, g\beta)$. The diagonal

$$\Delta = \{(\alpha, \alpha) \in \Omega^2 \mid \alpha \in \Omega\}$$

is obviously $G$-invariant and hence so is its complement $\Omega^2 \setminus \Delta$. One says that $G$ acts 2-*transitively* or *doubly transitively* on $\Omega$ if $G$ is transitive on $\Omega^2 \setminus \Delta$.

**Proposition B.12.** *Let $G$ be a finite group and $\Omega$ a transitive $G$-set. Then $G$ is 2-transitive if and only if* $\mathrm{Aug}(\mathbb{C}\Omega)$ *is simple.*

*Proof.* Let $\theta$ be the character afforded by $\mathbb{C}\Omega$ and let $\chi_1$ be the trivial character of $G$. It is then clear from (B.1) that $\chi_{\mathbb{C}\Omega^2} = \theta^2$. Also $\mathbb{C}\Omega \cong \mathbb{C}\Delta$ since $\Omega$ and $\Delta$ are isomorphic $G$-sets. Let $\Upsilon = \Omega^2 \setminus \Delta$. The direct sum decomposition $\mathbb{C}\Omega^2 = \mathbb{C}\Upsilon \oplus \mathbb{C}\Delta$ shows that $\theta^2 = \chi_{\mathbb{C}\Upsilon} + \theta$. Also note that if $\mu$ is the character afforded by $\mathrm{Aug}(\mathbb{C}\Omega)$, then $\mu = \theta - \chi_1$ because $\mathbb{C}\Omega = \mathrm{Aug}(\mathbb{C}\Omega) \oplus \mathbb{C}[\Omega]$.

We observe that

$$\langle \theta^2, \chi_1 \rangle = \frac{1}{|G|} \sum_{g \in G} \theta^2(g) = \langle \theta, \theta \rangle$$

since $\theta$ is integer-valued. Therefore, we compute

$$\begin{aligned} \langle \mu, \mu \rangle &= \langle \theta, \theta \rangle - 2\langle \theta, \chi_1 \rangle + \langle \chi_1, \chi_1 \rangle = \langle \theta^2, \chi_1 \rangle - \langle \theta, \chi_1 \rangle = \langle \theta^2 - \theta, \chi_1 \rangle \\ &= \langle \chi_{\mathbb{C}\Upsilon}, \chi_1 \rangle \end{aligned}$$

using that $\langle \theta, \chi_1 \rangle = 1 = \langle \chi_1, \chi_1 \rangle$ by Proposition B.10.

It follows from Corollary B.11 that $\langle \mu, \mu \rangle$ is the number of orbits of $G$ on $\Upsilon$ and hence $\mathrm{Aug}(\mathbb{C}\Omega)$ is simple if and only if $G$ is doubly transitive by Corollary B.7. $\square$

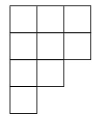


**Fig. B.1.** The Young diagram of $\lambda = (3, 3, 2, 1)$

## B.4 The representation theory of the symmetric group

We review here some elements of the representation theory of the symmetric group $S_n$. Standard references for this material include [Jam78, JK81, CR88, FH91, Mac95, Sta99, Sag01, CSST10]. No proofs are provided. We assume here that $\Bbbk$ is a field of characteristic 0. Put $[n] = \{1, \ldots, n\}$ for $n \geq 0$.

A *partition* of $n$ is a non-increasing sequence $(\lambda_1, \ldots, \lambda_m)$ of positive integers with $\lambda_1 + \cdots + \lambda_m = n$. The $\lambda_i$ are called the *parts* of $\lambda_i$. The set of partitions of $n$ is denoted $\mathcal{P}_n$. We allow an empty partition () of 0. We shall write $i^m$ as short hand for $m$ consecutive occurrences of $i$. For example $(3^2, 2, 1^4)$ is short hand for $(3, 3, 2, 1, 1, 1, 1)$.

It is convenient to represent partitions by Young diagrams. If $\lambda = (\lambda_1, \ldots, \lambda_m)$ is a partition of $n$, then the *Young diagram* of $\lambda$ consists of $n$ boxes placed into $m$ rows where the $i^{th}$ row has $\lambda_i$ boxes. For example, if $\lambda = (3, 3, 2, 1)$, then the Young diagram of $\lambda$ is in Figure B.1. Conversely, any diagram consisting of $n$ boxes arranged into rows such that the number of boxes in each row is non-increasing (going from top to bottom) is the Young diagram of a unique partition of $n$.

The *cycle type* of $f \in S_n$ is the partition $\lambda(f) = (\lambda_1, \ldots, \lambda_m)$ where $m$ is the number of orbits of $f$ and the $\lambda_i$ are the sizes of the orbits of $f$ (with multiplicities) listed in non-increasing order. For example, if $f \in S_{10}$ has cycle decomposition $(1\ 2\ 3)(4\ 5)(6\ 7\ 9)$, then $\lambda(f) = (3^2, 2, 1^2)$. It is well known that two permutations are conjugate if and only if they have the same cycle type. The conjugacy class of permutations of cycle type $\lambda$ will be denoted $C_\lambda$.

The simple $\Bbbk S_n$-modules are indexed by partitions $\lambda$ of $n$. There is a canonical labeling and the simple module associated to a partition $\lambda$ is denoted by $S_\lambda$ and is called the *Specht module* associated to $\lambda$. We recall the construction for completeness, although we shall not require much of its properties.

If $\lambda$ is a partition of $n$, then a *Young tableaux* of shape $\lambda$ is an array $T$ of integers obtained by placing $1, \ldots, n$ into the boxes of the Young diagram for $\lambda$. There are clearly $n!$ tableaux of shape $\lambda$. A *standard Young tableaux* of shape $\lambda$ is a tableaux whose entries are increasing along each row and column. For example, if $\lambda = (3, 2, 1)$, then some standard tableaux are as in Figure B.2

To explicitly construct $S_\lambda$, let $T_\lambda$ be the tableaux of shape $\lambda$ with the numbers $1, \ldots, n$ arranged from in order from left to right starting at the top left entry. For example, $T_{(3,2,1)}$ is the first tableaux in Figure B.2. Let $P_\lambda$ be the subgroup of $S_n$ preserving the rows of $T_\lambda$ and $Q_\lambda$ the subgroup preserving the columns of $T_\lambda$. Consider the elements

$$\begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4&5\\ \cline{1-2} 6\\ \cline{1-1}\end{array}\qquad\begin{array}{|c|c|c|}\hline 1&3&4\\ \hline 2&5\\ \cline{1-2} 6\\ \cline{1-1}\end{array}\qquad\begin{array}{|c|c|c|}\hline 1&2&4\\ \hline 3&6\\ \cline{1-2} 5\\ \cline{1-1}\end{array}\,.$$

**Fig. B.2.** Some standard Young tableaux of shape $(3, 2, 1)$

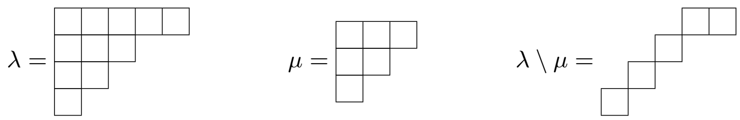


**Fig. B.3.** A horizontal strip

$$a_\lambda = \sum_{f\in P_\lambda} f \quad \text{and} \quad b_\lambda = \sum_{f\in Q_\lambda} \operatorname{sgn}(f)f$$

of $\Bbbk S_n$. The *Young symmetrizer* associated to $\lambda$ is $c_\lambda = a_\lambda b_\lambda$ and the Specht module is $S_\lambda = \Bbbk S_n c_\lambda$.

We remark that $S_{(n)}$ is the trivial $\Bbbk S_n$-module and $S_{(1^n)}$ is the one-dimensional sign representation $\operatorname{sgn}\colon S_n \longrightarrow \Bbbk \setminus \{0\}$ sending a permutation to its sign. Also, as $c_\lambda \in \mathbb{Q}S_n$, it follows that $S_\lambda$ is defined over $\mathbb{Q}$, that is, there is a basis of $S_\lambda$ so that each permutation is sent to a matrix with rational entries. The dimension $f_\lambda$ of $S_\lambda$ is known to be the number of standard Young tableaux of shape $\lambda$; see [FH91, Page 57].

Suppose that $r \leq n$ and that $V$ is a $\Bbbk S_r$-module and $W$ is a $\Bbbk S_{n-r}$-module. Their outer product $V \boxtimes W$ is the $\Bbbk S_n$-module defined as follows. We view $S_r \times S_{n-r}$ as a subgroup of $S_n$ by identifying $S_r$ with the permutations fixing $\{r+1, \ldots, n\}$ pointwise and $S_{n-r}$ with the permutations fixing $[r]$ pointwise. Note that $V \otimes W$ is a $\Bbbk[S_r \times S_{n-r}]$-module via the action $(f, g)(v \otimes w) = fv \otimes gw$ for $f \in S_r$, $g \in S_{n-r}$, $v \in V$ and $w \in W$. The *outer product* of $V$ and $W$ is then the induced representation

$$V \boxtimes W = \Bbbk S_n \otimes_{\Bbbk[S_r \times S_{n-r}]} (V \otimes W).$$

Outer products with the trivial module are easily decomposed thanks to what is known as *Pieri's rule*.

If $\lambda$ is a partition of $n$ and $\mu$ is a partition of $r \leq n$, then we write $\mu \subseteq \lambda$ if the Young diagram of $\mu$ is contained in the Young diagram of $\lambda$. We say that the skew diagram $\lambda \setminus \mu$ is a *horizontal strip* if no two boxes of $\lambda \setminus \mu$ are in the same column. For example, if $\lambda = (5, 3, 2, 1)$ and $\mu = (3, 2, 1)$, then $\mu \subseteq \lambda$ and $\lambda \setminus \mu$ is a horizontal strip; see Figure B.3. The following theorem is [JK81, Corollary 2.8.3].

**Theorem B.13 (Pieri's rule).** *Let $\mu$ be a partition of $r$ with $0 \leq r \leq n$. Then the decomposition*

$$S_\mu \boxtimes S_{(n-r)} = \bigoplus_{\substack{\mu\subseteq\lambda \\ \lambda\setminus\mu \text{ is a horizontal strip}}} S_\lambda$$

*holds.*

Notice that the decomposition in Theorem B.13 is multiplicity-free, i.e, has no isomorphic summands.

## B.5 Exercises

**B.1.** Let $G$ be a group and let $V, W$ be finite dimensional $\mathbb{C}G$-modules. Then $\mathrm{Hom}_{\mathbb{C}}(V, W)$ is a $\mathbb{C}G$-module via $(g\varphi)(v) = g\varphi(g^{-1}v)$.

(a) Prove that $\mathrm{Hom}_{\mathbb{C}}(V, W)^G = \mathrm{Hom}_{\mathbb{C}G}(V, W)$.
(b) Prove that the character $\theta$ of $\mathrm{Hom}_{\mathbb{C}}(V, W)$ is given by $\theta(g) = \chi(g)\overline{\psi(g)}$ where $\psi$ is the character of $V$ and $\chi$ is the character of $W$.
(c) Use the previous parts and that $U = \frac{1}{|G|}\sum_{g\in G} g$ is the projector to the fixed subspace of a $\mathbb{C}G$-module to prove the first orthogonality relations.

**B.2.** Suppose that $G$ is a finite group and $\Bbbk$ is a field of characteristic $p$ dividing $|G|$. Prove that the trivial $\Bbbk G$-module is not projective. (Hint: if it were, there would be a primitive idempotent $e$ such that $\Bbbk Ge \cong \Bbbk$.)

**B.3.** Let $G$ be a finite group and $H \leq G$ a subgroup. Let $V$ be a $\Bbbk H$-module and $W$ a $\Bbbk G$-module. Prove that $\mathrm{Hom}_{\Bbbk G}(\Bbbk G \otimes_{\Bbbk H} V, W) \cong \mathrm{Hom}_{\Bbbk H}(V, W)$ where we view $W$ as a $\Bbbk H$-module via restriction of scalars. This is known as *Frobenius reciprocity*.

**B.4.** Let $G$ be a finite group and $g, h \in G$. Prove that $g$ and $h$ are conjugate in $G$ if and only if, for every representation $\rho\colon G \longrightarrow M_n(\mathbb{C})$ with $n \geq 1$, there is an invertible matrix $A \in M_n(\mathbb{C})$ with $A\rho(g)A^{-1} = \rho(h)$.

# References


[AD10] Christos A. Athanasiadis and Persi Diaconis. Functions of random walks on hyperplane arrangements. *Adv. in Appl. Math.*, 45(3):410–437, 2010.

[AKS14a] Arvind Ayyer, Steven Klee, and Anne Schilling. Combinatorial Markov chains on linear extensions. *J. Algebraic Combin.*, 39(4):853–881, 2014.

[AKS14b] Arvind Ayyer, Steven Klee, and Anne Schilling. Markov chains for promotion operators. *Fields Institute Communications*, (71):285–304, 2014.

[Alm94] Jorge Almeida. *Finite semigroups and universal algebra*, volume 3 of *Series in Algebra*. World Scientific Publishing Co. Inc., River Edge, NJ, 1994. Translated from the 1992 Portuguese original and revised by the author.

[AM06] Marcelo Aguiar and Swapneel Mahajan. *Coxeter groups and Hopf algebras*, volume 23 of *Fields Institute Monographs*. American Mathematical Society, Providence, RI, 2006. With a foreword by Nantel Bergeron.

[AMSV09] Jorge Almeida, Stuart Margolis, Benjamin Steinberg, and Mikhail Volkov. Representation theory of finite semigroups, semigroup radicals and formal language theory. *Trans. Amer. Math. Soc.*, 361(3):1429–1461, 2009.

[APT92] M. Auslander, M. I. Platzeck, and G. Todorov. Homological theory of idempotent ideals. *Trans. Amer. Math. Soc.*, 332(2):667–692, 1992.

[AR67] William W. Adams and Marc A. Rieffel. Adjoint functors and derived functors with an application to the cohomology of semigroups. *J. Algebra*, 7:25–34, 1967.

[ARS97] Maurice Auslander, Idun Reiten, and Sverre O. Smalø. *Representation theory of Artin algebras*, volume 36 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997. Corrected reprint of the 1995 original.

[AS06] Fredrick Arnold and Benjamin Steinberg. Synchronizing groups and automata. *Theoret. Comput. Sci.*, 359(1-3):101–110, 2006.

[AS09] J. Almeida and B. Steinberg. Matrix mortality and the Černý-Pin conjecture. In *Developments in language theory*, volume 5583 of *Lecture Notes in Comput. Sci.*, pages 67–80. Springer, Berlin, 2009.

[ASS06] Ibrahim Assem, Daniel Simson, and Andrzej Skowroński. *Elements of the representation theory of associative algebras. Vol. 1*, volume 65 of *Lon-*

*don Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2006. Techniques of representation theory.

[ASST15a] Arvind Ayyer, Anne Schilling, Benjamin Steinberg, and Nicolas M. Thiéry. Directed Nonabelian Sandpile Models on Trees. *Comm. Math. Phys.*, 335(3):1065–1098, 2015.

[ASST15b] Arvind Ayyer, Anne Schilling, Benjamin Steinberg, and Nicolas M. Thiéry. Markov chains, $\mathscr{R}$-trivial monoids and representation theory. *Internat. J. Algebra Comput.*, 25(1-2):169–231, 2015.

[BBBS11] Chris Berg, Nantel Bergeron, Sandeep Bhargava, and Franco Saliola. Primitive orthogonal idempotents for $R$-trivial monoids. *J. Algebra*, 348:446–461, 2011.

[BBD82] A. A. Beĭlinson, J. Bernstein, and P. Deligne. Faisceaux pervers. In *Analysis and topology on singular spaces, I (Luminy, 1981)*, volume 100 of *Astérisque*, pages 5–171. Soc. Math. France, Paris, 1982.

[BBD99] Louis J. Billera, Kenneth S. Brown, and Persi Diaconis. Random walks and plane arrangements in three dimensions. *Amer. Math. Monthly*, 106(6):502–524, 1999.

[BD92] Dave Bayer and Persi Diaconis. Trailing the dovetail shuffle to its lair. *Ann. Appl. Probab.*, 2(2):294–313, 1992.

[BD98] Kenneth S. Brown and Persi Diaconis. Random walks and hyperplane arrangements. *Ann. Probab.*, 26(4):1813–1854, 1998.

[Ben98] D. J. Benson. *Representations and cohomology. I*, volume 30 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1998. Basic representation theory of finite groups and associative algebras.

[BHR99] Pat Bidigare, Phil Hanlon, and Dan Rockmore. A combinatorial description of the spectrum for the Tsetlin library and its generalization to hyperplane arrangements. *Duke Math. J.*, 99(1):135–174, 1999.

[Bjö84] A. Björner. Posets, regular CW complexes and Bruhat order. *European J. Combin.*, 5(1):7–16, 1984.

[Bjö08] Anders Björner. Random walks, arrangements, cell complexes, greedoids, and self-organizing libraries. In *Building bridges*, volume 19 of *Bolyai Soc. Math. Stud.*, pages 165–203. Springer, Berlin, 2008.

[Bjö09] Anders Björner. Note: Random-to-front shuffles on trees. *Electron. Commun. Probab.*, 14:36–41, 2009.

[BLVS+99] Anders Björner, Michel Las Vergnas, Bernd Sturmfels, Neil White, and Günter M. Ziegler. *Oriented matroids*, volume 46 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second edition, 1999.

[BPR10] Jean Berstel, Dominique Perrin, and Christophe Reutenauer. *Codes and automata*, volume 129 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2010.

[BR90] Jean Berstel and Christophe Reutenauer. Zeta functions of formal languages. *Trans. Amer. Math. Soc.*, 321(2):533–546, 1990.

[BR11] Jean Berstel and Christophe Reutenauer. *Noncommutative rational series with applications*, volume 137 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2011.

[Bra64] Richard Brauer. A note on theorems of Burnside and Blichfeldt. *Proc. Amer. Math. Soc.*, 15:31–34, 1964.

[Bro71] T. C. Brown. An interesting combinatorial method in the theory of locally finite semigroups. *Pacific J. Math.*, 36:285–289, 1971.

[Bro00] Kenneth S. Brown. Semigroups, rings, and Markov chains. *J. Theoret. Probab.*, 13(3):871–938, 2000.

[Bro04] Kenneth S. Brown. Semigroup and ring theoretical methods in probability. In *Representations of finite dimensional algebras and related topics in Lie theory and geometry*, volume 40 of *Fields Inst. Commun.*, pages 3–26. Amer. Math. Soc., Providence, RI, 2004.

[Bur55] W. Burnside. *Theory of groups of finite order.* Dover Publications, Inc., New York, 1955. 2d ed.

[BZ92] Anders Björner and Günter M. Ziegler. Combinatorial stratification of complex arrangements. *J. Amer. Math. Soc.*, 5(1):105–149, 1992.

[CE99] Henri Cartan and Samuel Eilenberg. *Homological algebra*. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1999. With an appendix by David A. Buchsbaum, Reprint of the 1956 original.

[CEFN14] Thomas Church, Jordan S Ellenberg, Benson Farb, and Rohit Nagpal. FI-modules over Noetherian rings. *Geom. Topol.*, 18(5):2951–2984, 2014.

[CG12] Fan Chung and Ron Graham. Edge flipping in graphs. *Adv. in Appl. Math.*, 48(1):37–63, 2012.

[Cli41] A. H. Clifford. Semigroups admitting relative inverses. *Ann. of Math. (2)*, 42:1037–1049, 1941.

[Cli42] A. H. Clifford. Matrix representations of completely simple semigroups. *Amer. J. Math.*, 64:327–342, 1942.

[CP61] A. H. Clifford and G. B. Preston. *The algebraic theory of semigroups. Vol. I.* Mathematical Surveys, No. 7. American Mathematical Society, Providence, R.I., 1961.

[CP67] A. H. Clifford and G. B. Preston. *The algebraic theory of semigroups. Vol. II.* Mathematical Surveys, No. 7. American Mathematical Society, Providence, R.I., 1967.

[CPS88] E. Cline, B. Parshall, and L. Scott. Finite-dimensional algebras and highest weight categories. *J. Reine Angew. Math.*, 391:85–99, 1988.

[CPS96] Edward Cline, Brian Parshall, and Leonard Scott. Stratifying endomorphism algebras. *Mem. Amer. Math. Soc.*, 124(591):viii+119, 1996.

[CR88] Charles W. Curtis and Irving Reiner. *Representation theory of finite groups and associative algebras.* Wiley Classics Library. John Wiley & Sons Inc., New York, 1988. Reprint of the 1962 original, A Wiley-Interscience Publication.

[CSST08] Tullio Ceccherini-Silberstein, Fabio Scarabotti, and Filippo Tolli. *Harmonic analysis on finite groups*, volume 108 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2008. Representation theory, Gelfand pairs and Markov chains.

[CSST10] Tullio Ceccherini-Silberstein, Fabio Scarabotti, and Filippo Tolli. *Representation theory of the symmetric groups*, volume 121 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2010. The Okounkov-Vershik approach, character formulas, and partition algebras.

[DF95] Robert P. Dobrow and James Allen Fill. On the Markov chain for the move-to-root rule for binary search trees. *Ann. Appl. Probab.*, 5(1):1–19, 1995.

[DHST11] T. Denton, F. Hivert, A. Schilling, and N. Thiéry. On the representation theory of finite $\mathcal{J}$-trivial monoids. *Sém. Lothar. Combin.*, 64:Art. B64d, 34 pp. (electronic), 2011.

[Dia88] Persi Diaconis. *Group representations in probability and statistics*. Institute of Mathematical Statistics Lecture Notes—Monograph Series, 11. Institute of Mathematical Statistics, Hayward, CA, 1988.

[Dia98] Persi Diaconis. From shuffling cards to walking around the building: an introduction to modern Markov chain theory. In *Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998)*, number Extra Vol. I, pages 187–204, 1998.

[DK94] Yurij A. Drozd and Vladimir V. Kirichenko. *Finite-dimensional algebras*. Springer-Verlag, Berlin, 1994. Translated from the 1980 Russian original and with an appendix by Vlastimil Dlab.

[DR89] Vlastimil Dlab and Claus Michael Ringel. Quasi-hereditary algebras. *Illinois J. Math.*, 33(2):280–291, 1989.

[Dub98] L. Dubuc. Sur les automates circulaires et la conjecture de Černý. *RAIRO Inform. Théor. Appl.*, 32(1-3):21–34, 1998.

[ECH$^+$92] David B. A. Epstein, James W. Cannon, Derek F. Holt, Silvio V. F. Levy, Michael S. Paterson, and William P. Thurston. *Word processing in groups*. Jones and Bartlett Publishers, Boston, MA, 1992.

[Eil74] Samuel Eilenberg. *Automata, languages, and machines. Vol. A*. Academic Press, New York, 1974. Pure and Applied Mathematics, Vol. 58.

[Eil76] Samuel Eilenberg. *Automata, languages, and machines. Vol. B*. Academic Press, New York, 1976. With two chapters ("Depth decomposition theorem" and "Complexity of semigroups and morphisms") by Bret Tilson, Pure and Applied Mathematics, Vol. 59.

[FGG97] A. Freedman, R. N. Gupta, and R. M. Guralnick. Shirshov's theorem and representations of semigroups. *Pacific J. Math.*, (Special Issue):159–176, 1997. Olga Taussky-Todd: in memoriam.

[FGG99] John Fountain, Gracinda M. S. Gomes, and Victoria Gould. Enlargements, semiabundancy and unipotent monoids. *Comm. Algebra*, 27(2):595–614, 1999.

[FH91] William Fulton and Joe Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. A first course, Readings in Mathematics.

[FH96] James Allen Fill and Lars Holst. On the distribution of search cost for the move-to-front rule. *Random Structures Algorithms*, 8(3):179–186, 1996.

[Fil96] James Allen Fill. An exact formula for the move-to-front rule for self-organizing lists. *J. Theoret. Probab.*, 9(1):113–160, 1996.

[Gab72] Peter Gabriel. Unzerlegbare Darstellungen. I. *Manuscripta Math.*, 6:71–103; correction, ibid. 6 (1972), 309, 1972.

[Gab73] Peter Gabriel. Indecomposable representations. II. In *Symposia Mathematica, Vol. XI (Convegno di Algebra Commutativa, INDAM, Rome, 1971)*, pages 81–104. Academic Press, London, 1973.

[Gab80] Peter Gabriel. Auslander-Reiten sequences and representation-finite algebras. In *Representation theory, I (Proc. Workshop, Carleton Univ., Ottawa, Ont., 1979)*, volume 831 of *Lecture Notes in Math.*, pages 1–71. Springer, Berlin, 1980.

[GM14] Anna-Louise Grensing and Volodymyr Mazorchuk. Categorification of the Catalan monoid. *Semigroup Forum*, 89(1):155–168, 2014.

[GMS09] Olexandr Ganyushkin, Volodymyr Mazorchuk, and Benjamin Steinberg. On the irreducible representations of a finite semigroup. *Proc. Amer. Math. Soc.*, 137(11):3585–3592, 2009.

[Gre51] J. A. Green. On the structure of semigroups. *Ann. of Math. (2)*, 54:163–172, 1951.

[Gre80] James A. Green. *Polynomial representations of* $\mathrm{GL}_n$, volume 830 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1980.

[Gre12] Anna-Louise Grensing. Monoid algebras of projection functors. *J. Algebra*, 369:16–41, 2012.

[Hen72] W. J. Hendricks. The stationary distribution of an interesting Markov chain. *J. Appl. Probability*, 9:231–233, 1972.

[Hig92] Peter M. Higgins. *Techniques of semigroup theory*. Oxford Science Publications. The Clarendon Press Oxford University Press, New York, 1992. With a foreword by G. B. Preston.

[HK88] John C. Harris and Nicholas J. Kuhn. Stable decompositions of classifying spaces of finite abelian $p$-groups. *Math. Proc. Cambridge Philos. Soc.*, 103(3):427–449, 1988.

[How95] John M. Howie. *Fundamentals of semigroup theory*, volume 12 of *London Mathematical Society Monographs. New Series*. The Clarendon Press Oxford University Press, New York, 1995. Oxford Science Publications.

[HT09] Florent Hivert and Nicolas M. Thiéry. The Hecke group algebra of a Coxeter group and its representation theory. *J. Algebra*, 321(8):2230–2258, 2009.

[HZ57] Edwin Hewitt and Herbert S. Zuckerman. The irreducible representations of a semi-group related to the symmetric group. *Illinois J. Math.*, 1:188–213, 1957.

[Isa76] I. Martin Isaacs. *Character theory of finite groups*. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1976. Pure and Applied Mathematics, No. 69.

[Jam78] G. D. James. *The representation theory of the symmetric groups*, volume 682 of *Lecture Notes in Mathematics*. Springer, Berlin, 1978.

[JK81] Gordon James and Adalbert Kerber. *The representation theory of the symmetric group*, volume 16 of *Encyclopedia of Mathematics and its Applications*. Addison-Wesley Publishing Co., Reading, Mass., 1981. With a foreword by P. M. Cohn, With an introduction by Gilbert de B. Robinson.

[Kar03] Jarkko Kari. Synchronizing finite automata on Eulerian digraphs. *Theoret. Comput. Sci.*, 295(1-3):223–232, 2003. Mathematical foundations of computer science (Mariánské Lázně, 2001).

[KK94] Piotr Krasoń and Nicholas J. Kuhn. On embedding polynomial functors in symmetric powers. *J. Algebra*, 163(1):281–294, 1994.

[KM09] Ganna Kudryavtseva and Volodymyr Mazorchuk. On three approaches to conjugacy in semigroups. *Semigroup Forum*, 78(1):14–20, 2009.

[Kov92] L. G. Kovács. Semigroup algebras of the full matrix semigroup over a finite field. *Proc. Amer. Math. Soc.*, 116(4):911–919, 1992.

[KRT68] K. Krohn, J. Rhodes, and B. Tilson. *Algebraic theory of machines, languages, and semigroups*. Edited by Michael A. Arbib. With a major contribution by Kenneth Krohn and John L. Rhodes. Academic Press, New York, 1968. Chapters 1, 5–9.

[Kuh94a] Nicholas J. Kuhn. Generic representations of the finite general linear groups and the Steenrod algebra. I. *Amer. J. Math.*, 116(2):327–360, 1994.

[Kuh94b] Nicholas J. Kuhn. Generic representations of the finite general linear groups and the Steenrod algebra. II. *K-Theory*, 8(4):395–428, 1994.

[Lal79] Gérard Lallement. *Semigroups and combinatorial applications.* John Wiley & Sons, New York-Chichester-Brisbane, 1979. Pure and Applied Mathematics, A Wiley-Interscience Publication.

[Lam91] T. Y. Lam. *A first course in noncommutative rings*, volume 131 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1991.

[Law98] Mark V. Lawson. *Inverse semigroups.* World Scientific Publishing Co. Inc., River Edge, NJ, 1998. The theory of partial symmetries.

[Li11] Liping Li. A characterization of finite EI categories with hereditary category algebras. *J. Algebra*, 345:213–241, 2011.

[Li14] Liping Li. On the representation types of category algebras of finite EI categories. *J. Algebra*, 402:178–218, 2014.

[LM95] Douglas Lind and Brian Marcus. *An introduction to symbolic dynamics and coding.* Cambridge University Press, Cambridge, 1995.

[Lot97] M. Lothaire. *Combinatorics on words.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1997. With a foreword by Roger Lyndon and a preface by Dominique Perrin, Corrected reprint of the 1983 original, with a new preface by Perrin.

[Lot02] M. Lothaire. *Algebraic combinatorics on words*, volume 90 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, 2002.

[LP69] Gérard Lallement and Mario Petrich. Irreducible matrix representations of finite semigroups. *Trans. Amer. Math. Soc.*, 139:393–412, 1969.

[LPW09] David A. Levin, Yuval Peres, and Elizabeth L. Wilmer. *Markov chains and mixing times.* American Mathematical Society, Providence, RI, 2009. With a chapter by James G. Propp and David B. Wilson.

[Lüc89] Wolfgang Lück. *Transformation groups and algebraic K-theory*, volume 1408 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 1989. Mathematica Gottingensis.

[Mac95] I. G. Macdonald. *Symmetric functions and Hall polynomials.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.

[Mac98] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 1998.

[Mal10] Martin E. Malandro. Fast Fourier transforms for finite inverse semigroups. *J. Algebra*, 324(2):282–312, 2010.

[Mal13] Martin E. Malandro. Inverse semigroup spectral analysis for partially ranked data. *Appl. Comput. Harmon. Anal.*, 35(1):16–38, 2013.

[McA71] Donald B. McAlister. Representations of semigroups by linear transformations. I, II. *Semigroup Forum 2 (1971), no. 3, 189–263; ibid.*, 2(4):283–320, 1971.

[McA72] D. B. McAlister. Characters of finite semigroups. *J. Algebra*, 22:183–200, 1972.

[Mit72] Barry Mitchell. Rings with several objects. *Advances in Math.*, 8:1–161, 1972.

[MQS15] Ariane M. Masuda, Luciane Quoos, and Benjamin Steinberg. Character theory of monoids over an arbitrary field. *J. Algebra*, 431:107–126, 2015.

[MR10] Martin Malandro and Dan Rockmore. Fast Fourier transforms for the rook monoid. *Trans. Amer. Math. Soc.*, 362(2):1009–1045, 2010.

[MS11] Stuart Margolis and Benjamin Steinberg. The quiver of an algebra associated to the Mantaci-Reutenauer descent algebra and the homology of regular semigroups. *Algebr. Represent. Theory*, 14(1):131–159, 2011.

[MS12a] Stuart Margolis and Benjamin Steinberg. Quivers of monoids with basic algebras. *Compos. Math.*, 148(5):1516–1560, 2012.

[MS12b] Volodymyr Mazorchuk and Benjamin Steinberg. Double Catalan monoids. *J. Algebraic Combin.*, 36(3):333–354, 2012.

[MSS] S. W. Margolis, F. Saliola, and B. Steinberg. Combinatorial topology and the global dimension of algebras arising in combinatorics. *J. Eur. Math. Soc. (JEMS)*. To appear.

[Mun55] W. D. Munn. On semigroup algebras. *Proc. Cambridge Philos. Soc.*, 51:1–15, 1955.

[Mun57a] W. D. Munn. The characters of the symmetric inverse semigroup. *Proc. Cambridge Philos. Soc.*, 53:13–18, 1957.

[Mun57b] W. D. Munn. Matrix representations of semigroups. *Proc. Cambrdige Philos. Soc.*, 53:5–12, 1957.

[Mun60] W. D. Munn. Irreducible matrix representations of semigroups. *Quart. J. Math. Oxford Ser. (2)*, 11:295–309, 1960.

[MZ75] Robert McNaughton and Yechezkel Zalcstein. The Burnside problem for semigroups. *J. Algebra*, 34:292–299, 1975.

[Neu09] Peter M. Neumann. Primitive permutation groups and their section-regular partitions. *Michigan Math. J.*, 58(1):309–322, 2009.

[Nic71] William R. Nico. Homological dimension in semigroup algebras. *J. Algebra*, 18:404–413, 1971.

[Nic72] William R. Nico. An improved upper bound for global dimension of semigroup algebras. *Proc. Amer. Math. Soc.*, 35:34–36, 1972.

[Nor79] P. N. Norton. 0-Hecke algebras. *J. Austral. Math. Soc. Ser. A*, 27(3):337–357, 1979.

[Okn91] Jan Okniński. *Semigroup algebras*, volume 138 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker Inc., New York, 1991.

[Okn98] Jan Okniński. *Semigroups of matrices*, volume 6 of *Series in Algebra*. World Scientific Publishing Co. Inc., River Edge, NJ, 1998.

[OP91] Jan Okniński and Mohan S. Putcha. Complex representations of matrix semigroups. *Trans. Amer. Math. Soc.*, 323(2):563–581, 1991.

[Pas14] D. S. Passman. Elementary bialgebra properties of group rings and enveloping rings: an introduction to Hopf algebras. *Comm. Algebra*, 42(5):2222–2253, 2014.

[Pat99] Alan L. T. Paterson. *Groupoids, inverse semigroups, and their operator algebras*, volume 170 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1999.

[Pet63] Mario Petrich. The maximal semilattice decomposition of a semigroup. *Bull. Amer. Math. Soc.*, 69:342–344, 1963.

[Pet64] Mario Petrich. The maximal semilattice decomposition of a semigroup. *Math. Z.*, 85:68–82, 1964.

[Pet84] Mario Petrich. *Inverse semigroups*. Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, 1984. A Wiley-Interscience Publication.

[Pha91] R. M. Phatarfod. On the matrix occurring in a linear search problem. *J. Appl. Probab.*, 28(2):336–346, 1991.

[Pin78] J.-E. Pin. Sur un cas particulier de la conjecture de Cerny. In *Automata, languages and programming (Fifth Internat. Colloq., Udine, 1978)*, volume 62 of *Lecture Notes in Comput. Sci.*, pages 345–352. Springer, Berlin, 1978.

[Pin81] Jean-Eric Pin. Le problème de la synchronisation et la conjecture de Černý. In *Noncommutative structures in algebra and geometric combinatorics (Naples, 1978)*, volume 109 of *Quad. "Ricerca Sci."*, pages 37–48. CNR, Rome, 1981.

[Pin86] J.-E. Pin. *Varieties of formal languages*. Foundations of Computer Science. Plenum Publishing Corp., New York, 1986. With a preface by M.-P. Schützenberger, Translated from the French by A. Howie.

[Pon56] I. S. Ponizovskiĭ. On matrix representations of associative systems. *Mat. Sb. N.S.*, 38(80):241–260, 1956.

[Pon87] I. S. Ponizovskiĭ. Some examples of semigroup algebras of finite representation type. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 160(Anal. Teor. Chisel i Teor. Funktsii. 8):229–238, 302, 1987.

[PQ95] D. S. Passman and Declan Quinn. Burnside's theorem for Hopf algebras. *Proc. Amer. Math. Soc.*, 123(2):327–333, 1995.

[PR93] Mohan S. Putcha and Lex E. Renner. The canonical compactification of a finite group of Lie type. *Trans. Amer. Math. Soc.*, 337(1):305–319, 1993.

[Put88] Mohan S. Putcha. *Linear algebraic monoids*, volume 133 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1988.

[Put89] Mohan S. Putcha. Monoids on groups with $BN$-pairs. *J. Algebra*, 120(1):139–169, 1989.

[Put95] Mohan S. Putcha. Monoids of Lie type. In *Semigroups, formal languages and groups (York, 1993)*, volume 466 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 353–367. Kluwer Acad. Publ., Dordrecht, 1995.

[Put96] Mohan S. Putcha. Complex representations of finite monoids. *Proc. London Math. Soc. (3)*, 73(3):623–641, 1996.

[Put98] Mohan S. Putcha. Complex representations of finite monoids. II. Highest weight categories and quivers. *J. Algebra*, 205(1):53–76, 1998.

[Put99] Mohan S. Putcha. Hecke algebras and semisimplicity of monoid algebras. *J. Algebra*, 218(2):488–508, 1999.

[Qui73] Daniel Quillen. Higher algebraic $K$-theory. I. In *Algebraic $K$-theory, I: Higher $K$-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 85–147. Lecture Notes in Math., Vol. 341. Springer, Berlin, 1973.

[Ren05] Lex E. Renner. *Linear algebraic monoids*, volume 134 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2005. Invariant Theory and Algebraic Transformation Groups, V.

[Rho69a] John Rhodes. Algebraic theory of finite semigroups. Structure numbers and structure theorems for finite semigroups. In K Folley, editor, *Semigroups (Proc. Sympos., Wayne State Univ., Detroit, Mich., 1968)*, pages 125–162. Academic Press, New York, 1969.

[Rho69b] John Rhodes. Characters and complexity of finite semigroups. *J Combinatorial Theory*, 6:67–85, 1969.

[Rie67] M. A. Rieffel. Burnside's theorem for representations of Hopf algebras. *J. Algebra*, 6:123–130, 1967.

[Rin00] C. M. Ringel. The representation type of the full transformation semigroup $T_4$. *Semigroup Forum*, 61(3):429–434, 2000.

[Rot64] Gian-Carlo Rota. On the foundations of combinatorial theory. I. Theory of Möbius functions. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 2:340–368 (1964), 1964.

[RS09] John Rhodes and Benjamin Steinberg. *The q-theory of finite semigroups*. Springer Monographs in Mathematics. Springer, New York, 2009.

[Ruk78] A. V. Rukolaĭne. The center of the semigroup algebra of a finite inverse semigroup over the field of complex numbers. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 75:154–158, 198, 1978. Rings and linear groups.

[Ruk80] A. V. Rukolaĭne. Semigroup algebras of finite inverse semigroups over arbitrary fields. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 103:117–123, 159, 1980. Modules and linear groups.

[RZ91] John Rhodes and Yechezkel Zalcstein. Elementary representation and character theory of finite semigroups and its application. In *Monoids and semigroups with applications (Berkeley, CA, 1989)*, pages 334–367. World Sci. Publ., River Edge, NJ, 1991.

[Sag01] Bruce E. Sagan. *The symmetric group*, volume 203 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 2001. Representations, combinatorial algorithms, and symmetric functions.

[Sal07] Franco V. Saliola. The quiver of the semigroup algebra of a left regular band. *Internat. J. Algebra Comput.*, 17(8):1593–1610, 2007.

[Sal08] Franco V. Saliola. On the quiver of the descent algebra. *J. Algebra*, 320(11):3866–3894, 2008.

[Sal09] Franco V. Saliola. The face semigroup algebra of a hyperplane arrangement. *Canad. J. Math.*, 61(4):904–929, 2009.

[Sal10] Franco V. Saliola. The Loewy length of the descent algebra of type $D$. *Algebr. Represent. Theory*, 13(2):243–254, 2010.

[Sal12] Franco Saliola. Eigenvectors for a random walk on a left-regular band. *Adv. in Appl. Math.*, 48(2):306–311, 2012.

[Sch58] Marcel-Paul Schützenberger. Sur la représentation monomiale des demi-groupes. *C. R. Acad. Sci. Paris*, 246:865–867, 1958.

[Sch06] Manfred Schocker. The module structure of the Solomon-Tits algebra of the symmetric group. *J. Algebra*, 301(2):554–586, 2006.

[Sch08] M. Schocker. Radical of weakly ordered semigroup algebras. *J. Algebraic Combin.*, 28(1):231–234, 2008. With a foreword by Nantel Bergeron.

[Ser77] Jean-Pierre Serre. *Linear representations of finite groups.* Springer-Verlag, New York, 1977. Translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, Vol. 42.

[Sim75] Imre Simon. Piecewise testable events. In *Automata theory and formal languages (Second GI Conf., Kaiserslautern, 1975)*, pages 214–222. Lecture Notes in Comput. Sci., Vol. 33. Springer, Berlin, 1975.

[Sol67] Louis Solomon. The Burnside algebra of a finite group. *J. Combinatorial Theory*, 2:603–615, 1967.

[Sol02] Louis Solomon. Representations of the rook monoid. *J. Algebra*, 256(2):309–342, 2002.

[Sta97] Richard P. Stanley. *Enumerative combinatorics. Vol. 1*, volume 49 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1997. With a foreword by Gian-Carlo Rota, Corrected reprint of the 1986 original.

[Sta99] Richard P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.

[Ste62] Robert Steinberg. Complete sets of representations of algebras. *Proc. Amer. Math. Soc.*, 13:746–747, 1962.

[Ste06] Benjamin Steinberg. Möbius functions and semigroup representation theory. *J. Combin. Theory Ser. A*, 113(5):866–881, 2006.

[Ste08] Benjamin Steinberg. Möbius functions and semigroup representation theory. II. Character formulas and multiplicities. *Adv. Math.*, 217(4):1521–1557, 2008.

[Ste10a] B. Steinberg. A simple proof of Brown's diagonalizability theorem. http://arxiv.org/abs/1010.0716, October 2010.

[Ste10b] B. Steinberg. A theory of transformation monoids: combinatorics and representation theory. *Electron. J. Combin.*, 17(1):Research Paper 164, 56 pp. (electronic), 2010.

[Ste10c] Benjamin Steinberg. A groupoid approach to discrete inverse semigroup algebras. *Adv. Math.*, 223(2):689–727, 2010.

[Ste12a] Benjamin Steinberg. *Representation theory of finite groups.* Universitext. Springer, New York, 2012. An introductory approach.

[Ste12b] Benjamin Steinberg. Yet another solution to the Burnside problem for matrix semigroups. *Canad. Math. Bull.*, 55(1):188–192, 2012.

[Ste15] B. Steinberg. The global dimension of the full transformation monoid. http://arxiv.org/abs/1502.00959, February 2015.

[Str83] Howard Straubing. The Burnside problem for semigroups of matrices. In *Combinatorics on words (Waterloo, Ont., 1982)*, pages 279–295. Academic Press, Toronto, ON, 1983.

[Str94] Howard Straubing. *Finite automata, formal logic, and circuit complexity.* Progress in Theoretical Computer Science. Birkhäuser Boston Inc., Boston, MA, 1994.

[SY11] Andrzej Skowroński and Kunio Yamagata. *Frobenius algebras. I.* EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2011. Basic representation theory.

[Ter99] Audrey Terras. *Fourier analysis on finite groups and applications*, volume 43 of *London Mathematical Society Student Texts.* Cambridge University Press, Cambridge, 1999.

[Thi12] Nicolas Thiéry. Cartan invariant matrices for finite monoids. In *DMTCS Proceedings, 24th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2012)*, pages 887–898. DMTCS, 2012.

[Til69] Bret R. Tilson. Appendix to "Algebraic theory of finite semigroups". On the $p$-length of $p$-solvable semigroups: Preliminary results. In K. Folley, editor, *Semigroups (Proc. Sympos., Wayne State Univ., Detroit, Mich., 1968)*, pages 163–208. Academic Press, New York, 1969.

[Č64] Ján Černý. A remark on homogeneous experiments with finite automata. *Mat.-Fyz. Časopis Sloven. Akad. Vied*, 14:208–216, 1964.

[Vol08] M. V. Volkov. Synchronizing automata and the Černý conjecture. In Carlos Martín-Vide, F. Otto, and H Fernau, editors, *Language and Automata Theory and Applications Second International Conference, LATA 2008, Tarragona, Spain, March 13-19, 2008.*, volume 5196 of *Lecture Notes in Computer Science*, pages 11–27, Berlin / Heidelberg, 2008. Springer.

[Web07] Peter Webb. An introduction to the representations and cohomology of categories. In *Group representation theory*, pages 149–173. EPFL Press, Lausanne, 2007.

[Web08] Peter Webb. Standard stratifications of EI categories and Alperin's weight conjecture. *J. Algebra*, 320(12):4073–4091, 2008.

[Wei73] Benjamin Weiss. Subshifts of finite type and sofic systems. *Monatsh. Math.*, 77:462–474, 1973.

# Index


2-transitive, 254
$M$-equivariant, 9
$M$-set, 9
$\mathbf{LG}_p$-congruence, 144
$\mathbf{LG}_p$-morphism, 144
**LI**-congruence, 142
**LI**-morphism, 94, 142
$\mathscr{J}$-class
  coprime, 20
  regular, 14
$\mathscr{J}$-trivial, 9
$\mathscr{R}$-trivial, 9, 22

action
  partial transformations, 170
acyclic, 106
adjacency matrix, 159
admissible ideal, 216
affords, 46, 85
algebra
  hereditary, 205
apex, 48, 110
arrow ideal, 216
augmentation ideal, 135
augmentation map, 135, 168
automaton, 164

band, 20
basic, 145
bi-ideal, 135
block, 247

Cartan matrix, 75
Catalan monoid, 226
category algebra, 106
center, 246
central idempotent, 246
central primitive idempotent, 246
character, 84, 85
  group algebra module, 250
  group representation, 250
  irreducible, 250
character table, 86
  group, 252
class function, 81
Clifford-Munn-Ponizovskii theorem, 49
coinduction functor, 36
column monomial, 62
complete set
  orthogonal idempotents, 245
  orthogonal primitive idempotents, 246
completely reducible, 46
composition series, 242
comultiplication, 135
congruence, 6
  $M$-set, 166
conjugate
  words, 155
content, 183
contracted monoid algebra, 47
contraction
  left regular band, 235
coprime
  $\mathscr{J}$-class, 20
  idempotent, 20
coprime $\mathscr{J}$-class, 20

coprime idempotent, 20
CW left regular band, 236
cycle type, 255
cyclic language, 154
cyclic subsemigroup, 7

decomposition matrix, 73
defined over, 55
degree, 45
derangment, 192
direct product, 6
domain
  partial mapping, 25
doubly transitive, 254

edge shift, 159
Ehrenfest urn model, 186
EI-category, 106
enveloping algebra, 99
equivalence
  of categories, 247
equivalent
  matrix representations, 45
  representations, 45
  short exact sequences, 215
essentially surjective, 247
Eulerian, 174
even shift, 160
Ext-quiver, 215
extension
  modules, 215

faithful
  $M$-set, 9
  module, 242
  representation, 45
finite representation type, 234, 243
formal language, 153
free category, 106
free left regular band, 182, 224
free monoid, 153
freely, 10
Frobenius algebra, 199
Frobenius reciprocity, 257
full transformation monoid, 166
fully faithful, 247

Gabriel quiver, 215
generalized conjugacy classes, 82
global dimension, 205
Green's relations, 8
  category, 108
Grothendieck group, 74
Grothendieck ring, 69
group algebra, 46
group of units, 5
groupoid, 111
  of an inverse monoid, 115

Hashimoto's edge adjacency operator, 162
height, 209
hereditary algebra, 205
heredity chain, 66
heredity ideal, 66
homomorphism, 6

ideal, 8
  admissible, 216
  left, 8
  prime, 20
  principal, 8
  right, 8
  two-sided, 8
idempotent, 6
  coprime, 20
idempotent ideal, 207
identity, 5
Ihara zeta function, 155, 162
incidence algebra, 77
indecomposable
  algebra, 247
index, 7, 81
induction functor, 36
inverse, 25
inverse monoid, 25
irreducible character, 250
irreducible representations, 46
isomorphism
  of $M$-sets, 10

join, 19

Karoubi envelope, 114

language, 153
  cyclic, 154
lattice, 19

left ideal, 8
left regular band, 23, 182, 221
  free, 182
length
  composition series, 242
  module, 242
locally finite, 238
locally trivial, 148

Möbius function, 78
Markov chain, 180
matrix representation, 45
maximal subgroup, 6
  category, 109
meet, 19
minimal ideal, 8
monoid, 5
  inverse, 25
  of Lie type, 233
monoid algebra, 46
Morita equivalent, 248
multiplication, 5
multiplicity
  composition factor, 242

natural module
  full transformation monoid, 51
  symmetric inverse monoid, 121, 123
natural partial order, 6, 29
natural transformation, 107
nilpotent
  element, 242
  ideal, 240

opposite
  algebra, 35
opposite semigroup, 6
order complex, 235
orthogonal, 245
outer product, 55, 256

partial mapping, 25
partial transformation, 25
partial transformation module, 170
partition, 51, 125, 255
parts, 255
path algebra, 107
path category, 106
period, 7, 81
periodic, 238
permutation module, 252
Pieri's rule, 256
poset, 19
Preston-Wagner theorem, 30
prime ideal, 20
primitive, 155, 166, 245
primitive root, 155
principal indecomposable modules, 244
principal series, 13
probability, 179
projective cover, 244
pumping lemma, 8

quasi-hereditary, 66
quasi-inverse, 247
quiver, 106, 215
  of an algebra, 215
quiver presentation, 216

random walk, 180
random-to-top shuffle, 182
rank, 9, 26, 165, 226
recollement, 35, 36
rectangular, 146
recurrent, 101
Rees quotient, 15
regular
  $\mathscr{J}$-class, 14
  element, 14
  monoid, 14
regular $\mathscr{J}$-class, 14
regular language, 153
regular module, 240
representation, 45
representation ring, 71
reset word, 171
restriction functor, 36
Rhodes radical, 141
  congruence, 141
riffle shuffling, 183
right ideal, 8
ring of class functions, 81
rook monoid, 26
row monomial, 63

sandwich matrix, 59
Schützenberger representation, 48
  left, 62

right, 63
self-injective alebra, 199
semigroup, 5
locally trivial, 148
semisimple
algebra, 241
module, 239
shift, 154
shift map, 154
simple, 239
small category, 105
socle, 240
sofic shift, 154
Specht module, 255
split semisimple, 189
stability, 11
standard duality, 247
state set, 180
stationary distribution, 180
Stirling number
second kind, 121
subgroup, 5
submonoid, 5
subsemigroup, 5
support, 180
symbolic dynamical system, 154
symmetric inverse monoid, 26
synchronizing, 170
synchronizing group, 169
syntactic congruence, 163
syntactic monoid, 154, 163

tensor product
modules, 47
top-to-random shuffle, 182
transformation module, 166
transformation monoid, 166
transition matrix, 180
transitive, 165, 252
triangular Grothendieck ring, 74
triangularizable, 139
triangularizable module, 74
trivial module, 47
trivial representation, 47
Tsetlin library, 182
two-sided ideal, 8

unit, 5

virtual character, 85

weakly irreducible, 223
wreath product, 32

Young diagram, 255
Young symmetrizer, 256
Young tableaux, 255
standard, 255

zero, 6
zero element, 6
zeta function
Ihara, 155, 162
language, 153
mapping, 156
poset, 78
shift, 154