Synchronizing groups: a survey

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Outline

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Example (Černý)

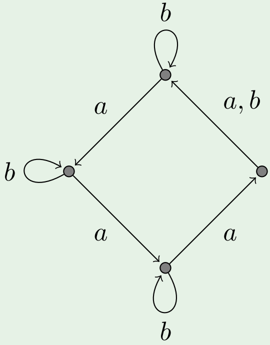

Figure : Unique minimum length reset word is $b(a^3b)^2$.

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An n-state synchronizing automaton has a reset word of length at most $(n-1)^2$.

- The best upper bound to date is $\dfrac{n^3-n}{6}$ (Pin/Frankl '81).

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- Dubuc removed the condition on prime number of states for (2) in 1998; (1) does not generalize.
- Inspired by (1), I introduced (2005) (and indep. Araújo [unpublished]) the notion of a synchronizing group.

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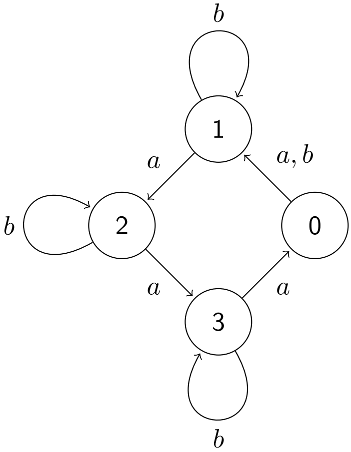
- The automaton is synchronizing iff $\langle A \rangle$ contains a constant map.

Černý's example



$$a = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \end{pmatrix} \qquad b = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 2 & 3 \end{pmatrix}$$

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- So f has rank 1 iff f is a constant map.
- f is singular if $f \in T_\Omega \setminus S_\Omega$.

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- G is primitive if Ω admits no nontrivial G-invariant partition.

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Proof.

Section-regular partitions essentially correspond to minimal rank idempotents that can't be synchronized. □

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Theorem

2-transitive $\implies$ 2-homogeneous $\implies$ synchronizing $\implies$ basic $\implies$ primitive $\implies$ transitive.

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- Thus every primitive group of size $2p$ with p an odd prime is synchronizing.
- Synchronizing groups have a fairly large density among primitive permutation groups using CFSG.

Connections with graph theory

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Theorem (Cameron)

G is synchronizing iff there are no nontrivial G-invariant graphs on Ω with $\omega(X) = \chi(X)$.

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1. *If $k \mid n$, then S_n acting on k-sets is not synchronizing.*
2. *If $n \geq 7$, then S_n acting on 3-sets is synchronizing iff $n \equiv 2, 4, 5 \pmod 6$ and $n \neq 8$.*

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- The graph X of the polar space is the graph on Ω with edges the collinear points.

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- Not everything is known about the existence of these creatures.

Some examples

Example

- $\mathrm{PSp}(2r, q)$, $\mathrm{PSU}(2r+1, q)$ and $\mathrm{P}\Omega^{-}(2r+2, q)$ are synchronizing for all $r \geq 2$, except $\mathrm{PSp}(4, q)$ with q even.
- $\mathrm{P}\Omega(5, q)$ for $q = 3, 5, 7$ is synchronizing.

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The general situation for classical groups is still very much open.

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2. *G is $\mathbb{R}I \iff G$ is 2-homogeneous.*

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Theorem (Arnold, BS '06)

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- Using CFSG, it has been shown that the only remaining examples are $\mathrm{PSL}(2,q)$ and $\mathrm{P}\Gamma L(2,q)$ acting in degree $\frac{1}{2}q(q-1)$ with $q = 2^n \geq 8$ and $q-1$ a Mersenne prime.

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Another class we have considered are separating groups (introduced by P. Neumann).

Neumann's separation lemma

Lemma (P. Neumann)

Let $G \leq S_\Omega$ be transitive and let $A, B \subseteq \Omega$ such that $|A| \cdot |B| < |\Omega|$. Then there exists $g \in G$ with $A \cap Bg = \emptyset$.

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Proof.

If π is a section-regular partition with section S and A is a block of π, then $|A| \cdot |S| = |\Omega|$ and $|A \cap Sg| = 1$ for all $g \in G$. $\square$

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- $\mathrm{P}\Omega(5, q)$ for $q = 3, 5, 7$ is synchronizing but not separating.

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The End

Thanks for your attention!