On a conjecture of Karrass and Solitar

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Winter CMS Meeting

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Conjecture (Karrass/Solitar)

Let $G = A * B$ be a free product of nontrivial groups. Then a finitely generated subgroup of G has finite index iff it intersects nontrivially each nontrivial normal subgroup of G.

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Let $G = A * B$ be a free product of nontrivial groups and H a subgroup of finite Kurosh rank. Then H is of finite index iff it intersects nontrivially each nontrivial normal subgroup of G.

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- Finite Kurosh rank means roughly that the Kurosh decomposition has finitely many free factors and the free part is of finite rank.
- This is the "right" analog of finite generation for free products.

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- Both proofs use Stallings graphs and small cancellation.

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- Finitely generated subgroups have finite Kurosh rank.
- Any subgroup of a conjugate of A or B has finite Kurosh rank.

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- $\mathrm{Core}(H)$ is the induced subcomplex on the core vertices.

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- A related graph has been considered by Ivanov, but seems to correspond to representing G as the fundamental group of the space where K_A and K_B are connected by an edge.

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- Ivanov also does these constructions with his graph, but his approach somehow is more technical.

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- Essentially start with w and then run off $\mathrm{Core}(H)$.
- Hence there is a cyclically reduced word w that does not label any path in $\mathrm{Core}(H)$ (i.e., has rank 0).

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- Small cancellation theory over free products yields that some cyclic conjugate s of z or z^{-1} contains u as a factor.
- But s labels a loop in $\mathrm{Core}(H)$, contradicting the choices of u, w.

The end

Thank you for your attention!