Koszul algebras associated to zonotopes and CAT(0) cube complexes

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the City College of New York

December 7, 2013
Winter CMS Meeting

Outline

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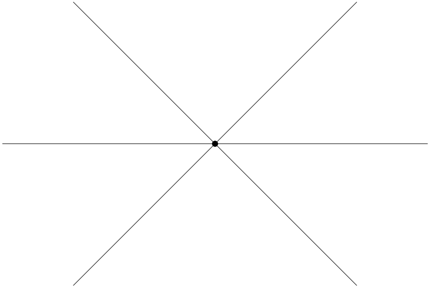
- The original motivation for studying these algebras came from analyzing Markov chains.
- All these algebras are semigroup algebras.

Faces of a hyperplane arrangement

A set of hyperplanes partitions $\mathbb{R}^n$ into *faces*:



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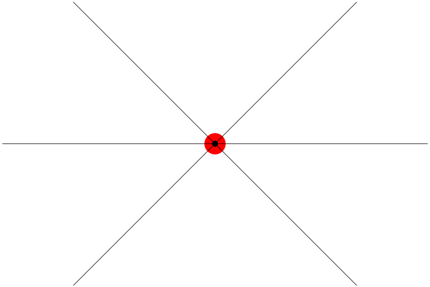

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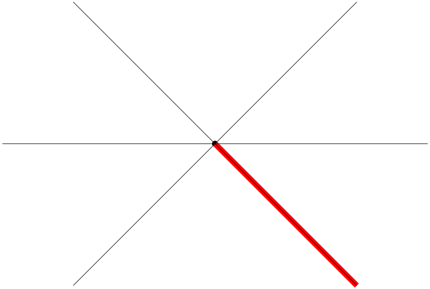

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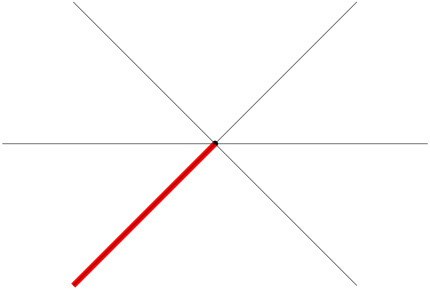

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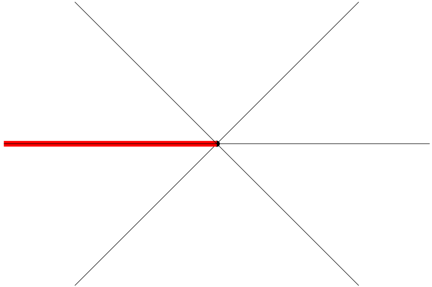

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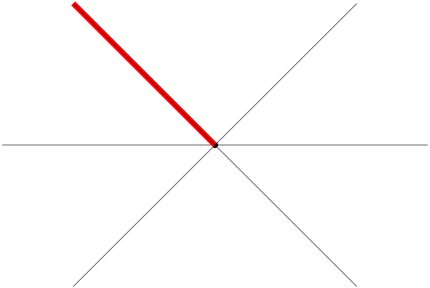

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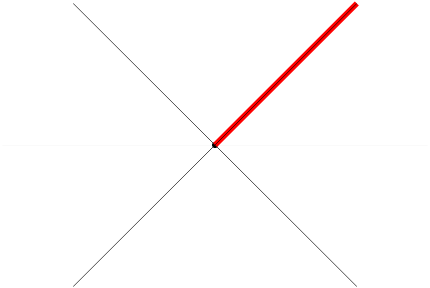

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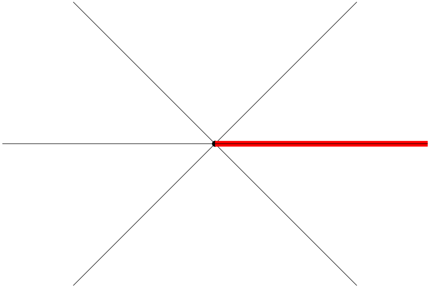

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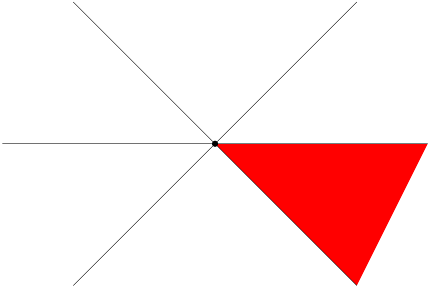

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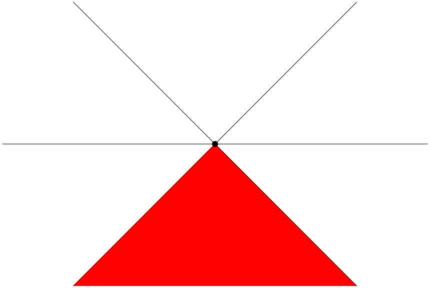

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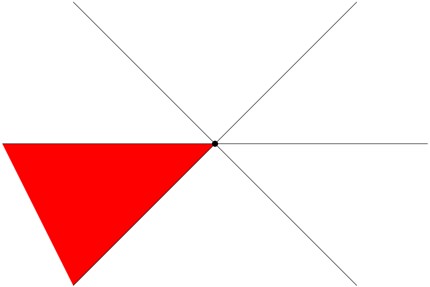

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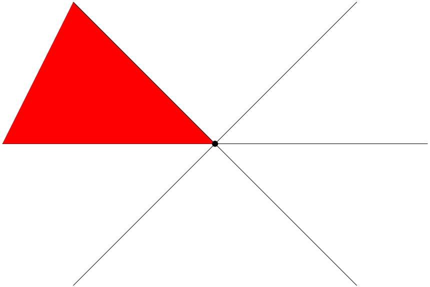

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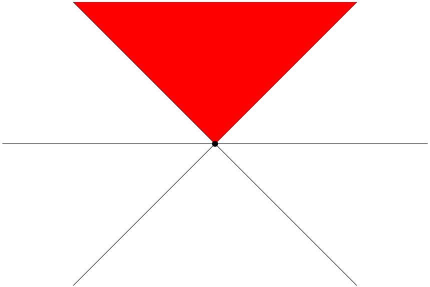

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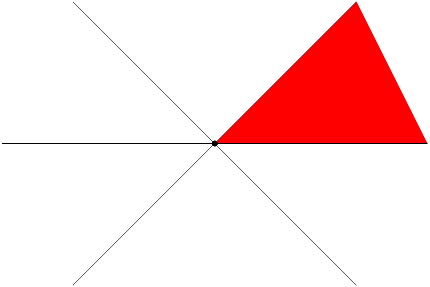

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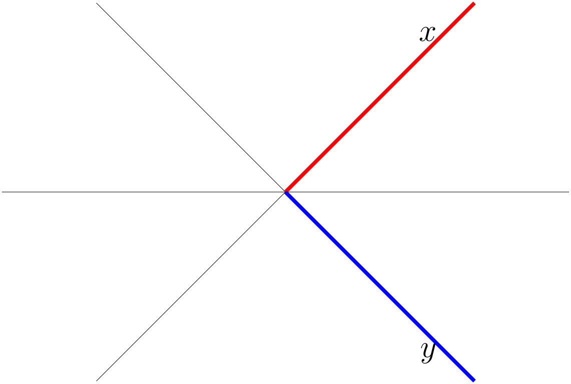

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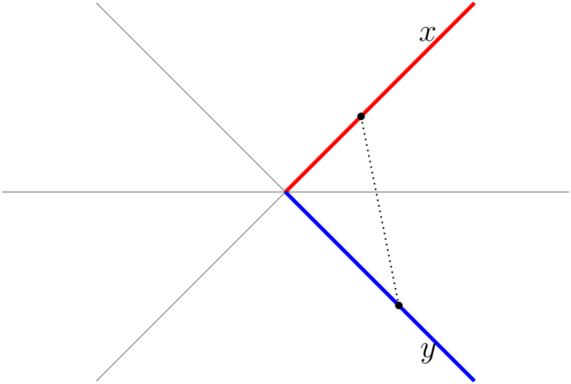

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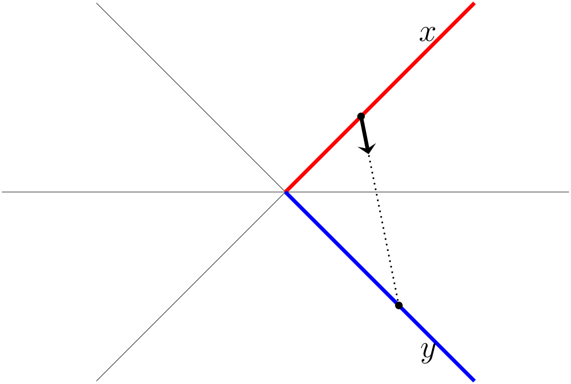

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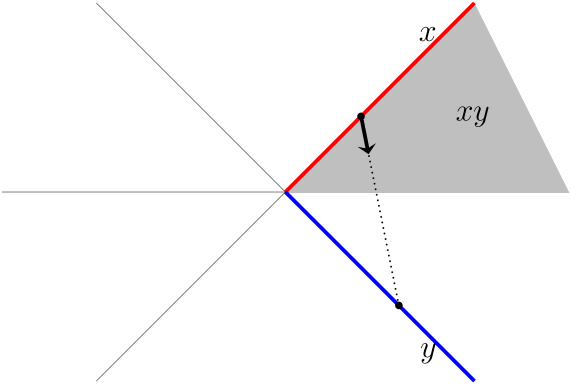

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$\circ$	0	$+$	$-$
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- One easily checks $\{+, -, 0\}$ satisfies the identities $x^2 = x$ and $xyx = xy$.

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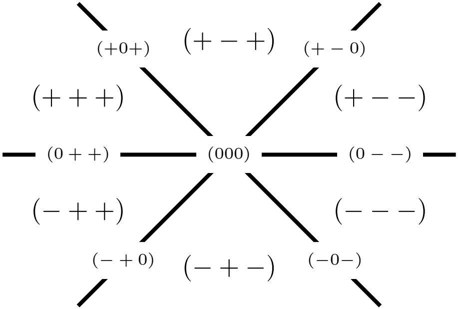
- This whole setup generalizes to oriented matroids.

Examples



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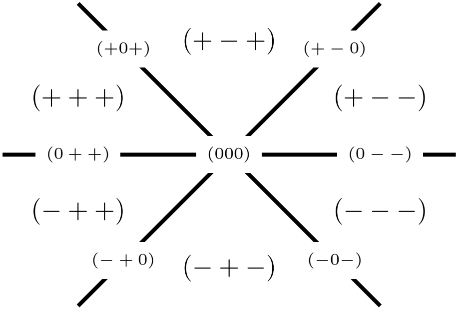

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Example (The coordinate arrangement)

The face monoid of $\{x_i = 0 \mid i = 1, \ldots, n\}$ in $\mathbb{R}^n$ is $\{0, +, -\}^n$. The chambers are the orthants.

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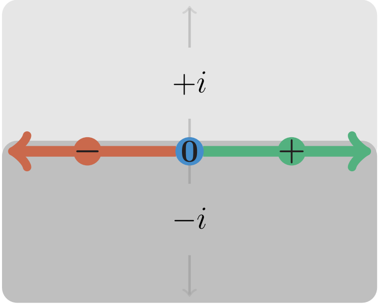

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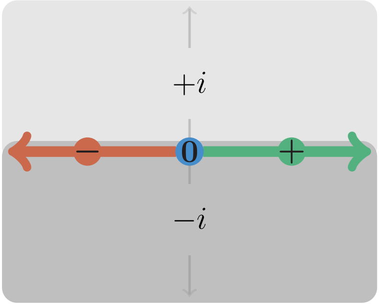

- $\{0, +, -\}$ is a submonoid and $\pm i \circ x = \pm i$, $y \circ \pm i = \pm i$ if $y \in \{0, +, -\}$.

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- Similarly, complex arrangements, affine arrangements and oriented matroids have dual regular cell complexes.

Permutohedron

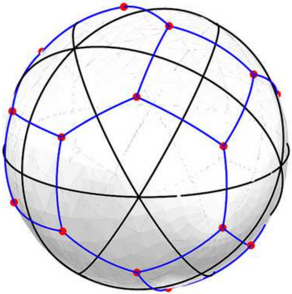

Figure : Permutohedron for S_4 by Mike Zabrocki

Zonohedra

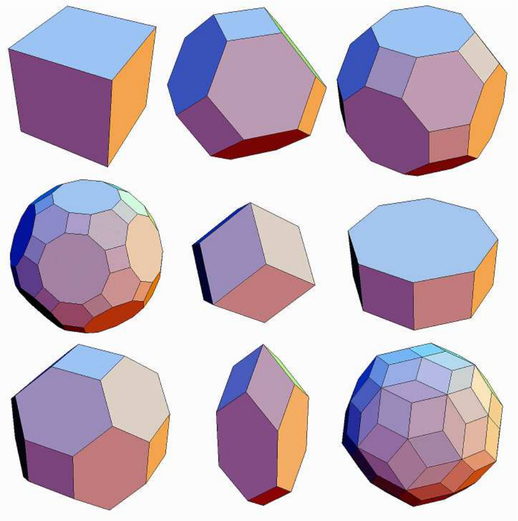

Figure : 3-dimensional zontopes (zonohedra)

An oriented matroid

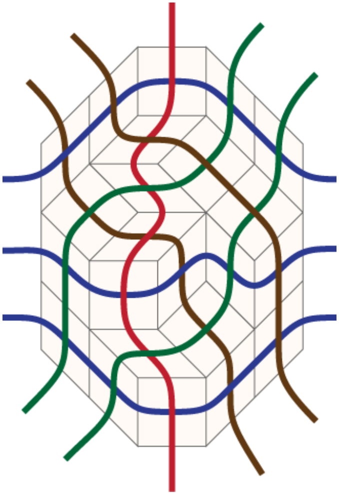

Figure : An oriented matroid corresponding to a zonotopal tiling

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- Every subcomplex of the zonotope is a subsemigroup of $\mathcal{F}(\mathcal{A})$.

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- Billera, Holmes and Vogtmann introduced a space of phylogenetic trees, which is a $\mathrm{CAT}(0)$ cube complex.
- Speyer showed that the space of phylogenetic trees is the tropical analog of the Grassmannian.

A CAT(0) square complex

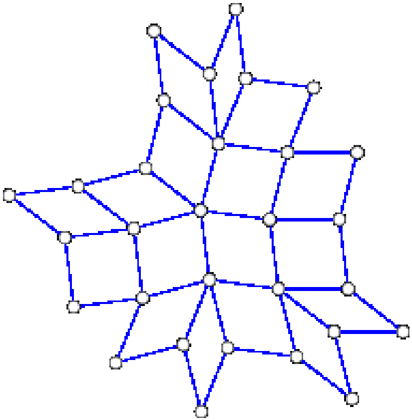

Figure : A CAT(0) square complex

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- Again minimal projective resolutions can be constructed from the augmented cellular chain complex.

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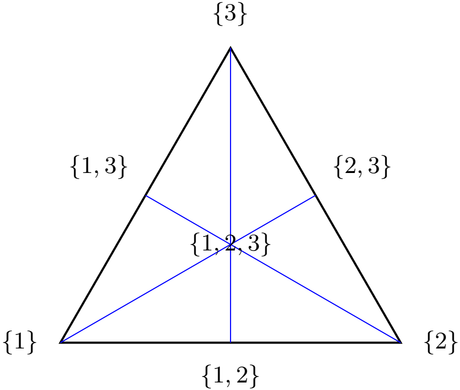

Figure : Order complex of the power set of $\{1, 2, 3\}$

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- All simples arise like this.

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- There are axioms characterizing face posets of regular CW complexes (cf. Björner).

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- This depends only on the homeomorphism type of $\Delta(P)$.

A quiver presentation

Theorem (Margolis,Saliola,BS)

Let B be a CW LRB. Let Q be the Hasse diagram of $\Lambda(B) = \{Ba \mid a \in B\}$. Then $\Bbbk B \cong \Bbbk Q/(r)$ where r is the sum of all paths of length 2.

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- Our approach was inspired by his, but is more conceptual, which in turn led to simplifications.

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- It follows that the global dimension of $\Bbbk B$ is $\dim X$.

An example

- Let $L_n = \{0, \pm 1, \dots, \pm n\}$ with the product given by

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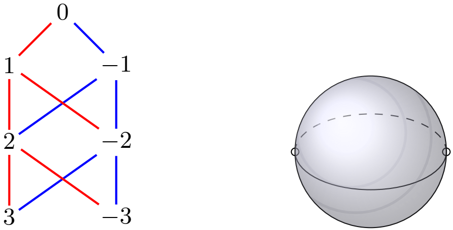

Figure : The Hasse diagram of L_3 and cell decomposition of the 3-ball.

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- It is Koszul with Ext-algebra isomorphic $\Bbbk A_{n+1}$, which is the incidence algebra of $\{0 < 1 < \cdots < n\}$.

The end

Thank you for your attention!