Self-similar semigroups and finite state Markov chains

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Groups Acting on Rooted Trees and Around

Outline

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- One can try to use self-similarity to recursively compute eigenvalues and then obtain spectral data of the original random walk by a limiting process.
- If the action is essentially free, one recovers in the limit the random walk on the Cayley graph of the group.

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- We study families of finite state Markov chains by realizing them as the levels of a monoid acting on a rooted tree.
- More generally, we use ideas from the theory of iterated wreath products and self-similar groups to analyze families of finite state Markov chains.
- We have managed to do this with some success but are just at the beginning.

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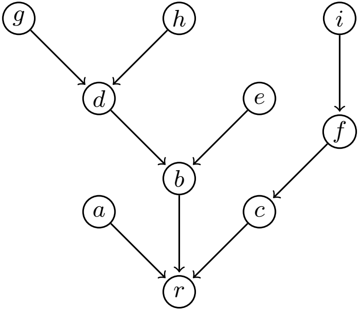
- Prototypical model for the phenomenon of self-organized criticality, like a heap of sand.

Non-abelian sandpile models on routed trees

- One has a finite rooted tree $\mathcal{T}$.



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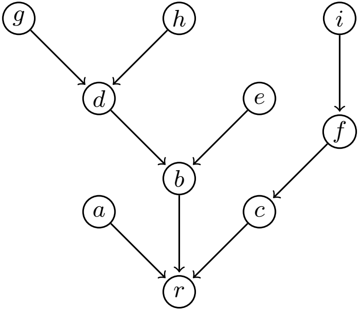

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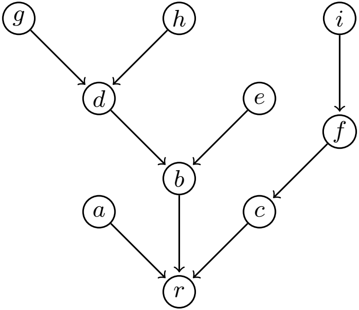

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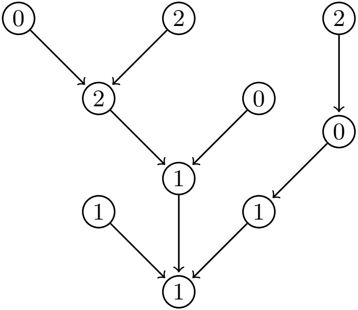
A configuration of the sandpile model



A configuration when all thresholds are 2.

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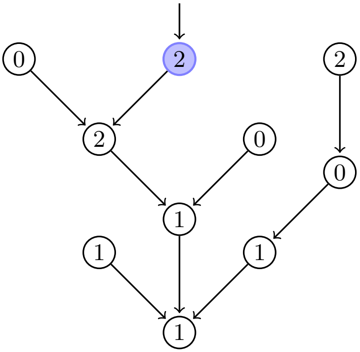
- If all vertices on the geodesic are full, the sand particle leaves the system.

Grain of sand entering at a leaf



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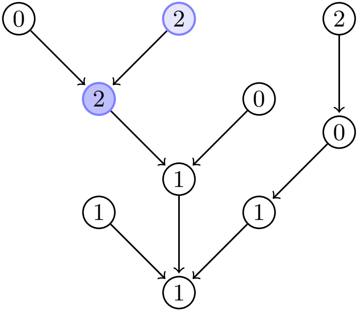

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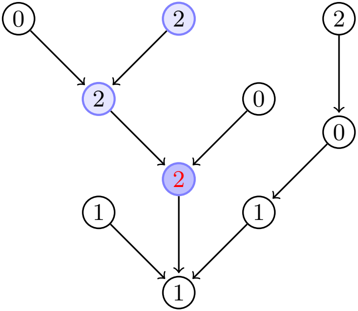

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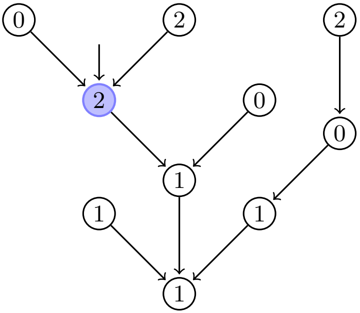
- If all thresholds are 1, the two models coincide.

Toppling in the Landslide model



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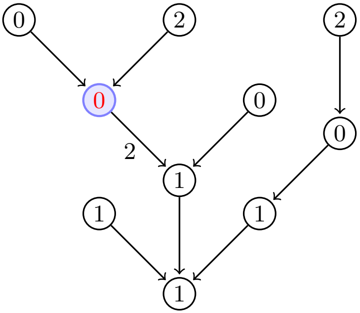

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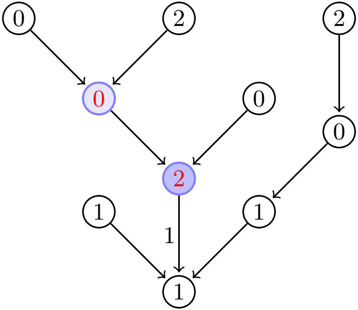

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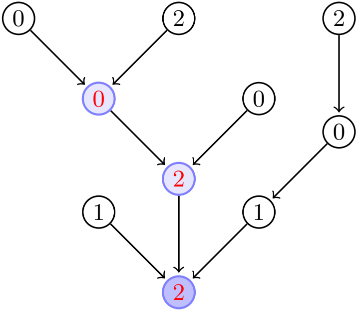

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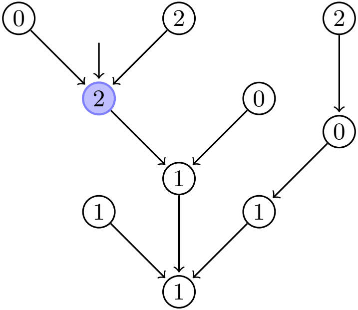

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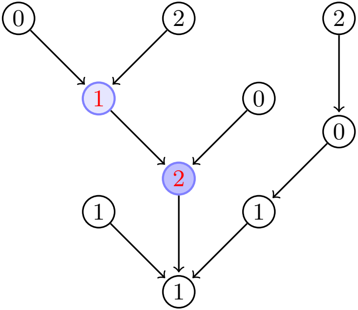

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- This is only a random walk on S_n when all books are equally likely to be chosen: random-to-top shuffle.

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$\mathscr{R}$-trivial monoids

- A monoid M is $\mathscr{R}$-trivial if $mM = nM \implies m = n$.
- All simple $\mathbb{C}M$-modules are 1-dimensional.
- The characters take on only values $0, 1$.
- The character table is the incidence matrix (ζ-function) of a certain lattice.
- The inverse is the the Möbius function.
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- Computing multiplicities amounts to computing the number of fixed points of each idempotent and performing a Möbius inversion.

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$$\theta_\ell(t_\ell, t) = \begin{cases} (t_\ell - 1, \sigma_{\ell'} t) & \text{if } t_\ell > 0 \\ (0, t) & \text{if } t_\ell = 0 \end{cases}$$

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    3. to bound the mixing time.

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$$\pi((t_v)) = \prod_{v \in V} \frac{Q_v^{t_v} p_v^{T_v - t_v}}{\sum_{i=0}^{T_v} Q_v^i p_v^{T_v - i}}$$
where T_v is the threshold at v.

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- The multiplicity for λ_S is $\prod_{v \notin S} T_v$.

Mixing time: Landslide model

Theorem (ASST 2013)

The rate of convergence to stationarity is bounded by

$$\|P^k - \pi\|_{TV} \leq \exp\left(-\frac{(kp - (|V| - 1))^2}{2kp}\right)$$

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So the mixing time is bounded by $2|V|/p$.

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- Similar, but messier results, hold for variant 2.

The end

Thank you for your attention!

Merci de votre attention!