On the Irreducible Representations of Inverse Semigroups

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A passion of Douglas

- Throughout Douglas's career, he turned time and time again to the subject of inverse semigroup algebras.
- In this thesis work he described all irreducible representations of finite inverse semigroups as being induced from maximal subgroups.
- He later generalized this result to inverse semigroups satisfying the descending chain condition on idempotents.
- However, notice the bicyclic monoid B only has trivial maximal subgroups. Since $\mathbb{Z}$ is a homomorphic image of B, clearly any irreducible representation of $\mathbb{Z}$ can be lifted to B.
- Thus in general maximal subgroups do not suffice.
- To deal with the general case, Douglas pursued an alternate tactic.

Douglas's approach

- Douglas first classified finite dimensional irreducible representations of 0-simple semigroups.
- He then reduced the general case to the 0-simple case.
- Douglas had already shown that any finite dimensional irreducible representation of an ideal extends to the whole semigroup.
- He then showed that finite dimensional irreps of an inverse semigroup S are parameterized by irreducible representations of 0-simple inverse semigroups I/J where I, J are ideals of S.
- It took Douglas a bit of work to deduce his earlier result for inverse semigroups with dcc on idempotents from this result.

A different approach

- We give a new approach to the irreducible representations of inverse semigroups in the spirit of Douglas's original approach via maximal subgroups.
- The idea that this could be done first arose in unpublished work of the speaker with Haatja and Margolis in 2002.
- C^*-algebraist Paterson had introduced the universal groupoid of an inverse semigroup and shown that the groupoid and the semigroup have isomorphic C^*-algebras.
- We noticed the universal groupoid of S is the underlying groupoid (in the sense of Lawson's book) of a certain inverse subsemigroup of Schein's coset semigroup $K(S)$.
- Our idea was to use this to study irreducible representations, but we never realized this approach.

A groupoid approach

- Instead, I've generalized groupoid algebras to arbitrary fields.
- I have the described the finite dimensional irreducible representations of any groupoid algebra.
- The algebra of the universal groupoid is still isomorphic to the inverse semigroup algebra in this setting.
- So the desired results for inverse semigroups are obtained by specialization.
- The groupoid approach also gives an easy description of the center of an inverse semigroup algebra.
- There are interesting groupoid algebras that are not inverse semigroups algebras.
- For instance, the quotient of the polycyclic algebra KP_2 by the relation $xx^* + yy^* = 1$ is a groupoid algebra.

The spectrum of a semilattice

- If E is a semilattice, a character of E is a non-zero homomorphism $\varphi\colon E \to \{0,1\}$.
- The character space (or spectrum) $\widehat{E}$ of E is topologized as a subspace of $\{0,1\}^E$.
- Thus $\widehat{E}$ has a basis of compact open subsets (and so is totally disconnected).
- For $e \in E$, put $D(e) = \{\varphi \in \widehat{E} \mid \varphi(e) = 1\}$. This is a compact open set and such sets generate the boolean ring of compact open sets.
- If $e \in E$, the principal character associated to e is given by:

$$\chi_e(f) = \begin{cases} 1 & f \geq e \\ 0 & \text{else.} \end{cases}$$

- The principal characters are dense, so $\widehat{E}$ is a completion of E.

The spectrum of a semilattice

- Let S be an inverse semigroup with idempotent set E.
- The Munn representation dualizes to an action on $\widehat{E}$ (due to Paterson).
- $s\cdot\colon D(s^*s) \to D(ss^*)$ is given by $s\varphi(e) = \varphi(s^*es)$.
- If $e \in E$, then $e \in D(s^*s)$ iff $e \leq s^*s$, in which case $s\chi_e = \chi_{ses^*}$.
- So the spectral action is also a completion of the Munn representation.
- Notice that the orbit of χ_e is $\{\chi_f \mid e \mathscr{D} f\}$, so orbits generalize $\mathscr{D}$-classes.
- Although we won't use the topology in this talk, it is essential.

Theorem (BS)

The semigroup algebra KS has a unit if and only if $\widehat{E}$ is compact.

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Generalizing maximal subgroups and $\mathscr{L}$-classes

- Let $\varphi \in \widehat{E}$ and let S_φ be the stabilizer of φ.
- S_φ is an inverse semigroup and so has a maximal group image G_φ called the isotropy group of S at φ.
- $G_{\chi_e} = G_e$ for $e \in E$.
- Let $\tilde{L}_\varphi$ be the set of $s \in S$ defined at φ. Note: $S_\varphi \subseteq \tilde{L}_\varphi$ and acts on the right of it.
- Define L_φ to be the quotient of $\tilde{L}_\varphi$ by the equivalence relation that identifies two elements if they have a common lower bound in $\tilde{L}_\varphi$. It is the set of germs of S at φ.
- Then $G_\varphi \subseteq L_\varphi$ and acts freely on the right of it.
- L_φ / G_φ is in bijection with the orbit of φ.
- Note $L_{\chi_e} = L_e$ for $e \in E$.

The Schützenberger representation

- If $\varphi \in \widehat{E}$, then S acts by partial bijections on the left of L_φ and the action of G_φ on the right is by automorphisms of this action.
- One puts $s[s',\varphi] = [ss',\varphi]$ if $ss' \in \tilde{L}_\varphi$ and undefined otherwise.
- If φ is a principal character this reduces to the usual Schützenberger representation.
- Standard semigroup results generalize.
- For example if S is finitely generated (presented) and the orbit of φ is finite, then G_φ is finitely generated (presented).

An example: the bicyclic monoid

- Let B be the bicyclic monoid.
- $E = (\mathbb{N}, \geq)$.
- $\widehat{E} = \mathbb{N} \cup \{\infty\}$ with ∞ as a one-point compactification.
- The character corresponding to ∞ is the trivial one sending all idempotents to 1.
- One has $\tilde{L}_\varphi = B_\varphi = B$ and $L_\varphi = G_\varphi = \mathbb{Z}$, the maximal group image of B.
- The Schützenberger representation is the natural action of B on $\mathbb{Z}$.
- More generally, if S is any inverse semigroup and φ is the trivial character, then G_φ is the maximal group image of S.

An example: the free inverse monoid

- Let FI be the free inverse monoid on X.
- Then E consists of all finite subtrees of the Cayley graph of the free group F on X ordered by $\supseteq$.
- Then $\widehat{E}$ can be viewed as the space of all subtrees of the Cayley graph of F containing 1 with the usual topology on marked subgraphs.
- The character associated to T sends a finite subtree to 1 iff it is contained in T.
- There are many new isotropy subgroups, all of which are free.
- E.g. if $H \leq F$ is a subgroup and T is the subtree spanned by the elements of H, then the isotropy group of FI at T is H.

Inverse semigroups with dcc on idempotents

- Every character of a semilattice E is principal iff it satisfies dcc.
- However, the topology is discrete iff each principal downset of E is finite.
- For example, $(\mathbb{N}, \leq)$ satisfies dcc.
- However, one can verify that the topology makes 0 the one-point compactification of the positive integers.

The setup

- Let S be an inverse semigroup with idempotent set E.
- We fix a field K.
- The semigroup algebra KS is the K-vector space with basis S and multiplication linearly extending the product in S.
- We only consider KS-modules V so that $KS \cdot V = V$.
- There is a bijection between simple KS-modules V and irreducible representations $\varphi\colon S \to \mathrm{End}_K(V)$.
- Our approach is based on my interpretation of Munn's approach for finite semigroups via results of Green for algebras.

The restriction functor

- Let $\varphi \in \widehat{E}$ be a character.
- We define a pair of adjoint functors between $KS\text{-mod}$ and $KG_\varphi\text{-mod}$ that depend only on the orbit of φ.
- The restriction functor $\mathrm{Res}_\varphi \colon KS\text{-mod} \to KG_\varphi\text{-mod}$ takes a KS-module V to

$$\mathrm{Res}_\varphi(V) = \bigcap_{e \in E(S_\varphi)} eV.$$

- One can verify that $\mathrm{Res}_\varphi(V)$ is S_φ-invariant. Moreover, since $E(S_\varphi)$ acts trivially on $\mathrm{Res}_\varphi(V)$, it is actually a KG_φ-module.
- That is, the action of S_φ factors through its maximal group image G_φ.
- If χ_e is a principal character, $\mathrm{Res}_{\chi_e}(V) = eV$.

The induction functor

- The commuting left/right actions of S and G_φ on L_φ give KL_φ the structure of a KS-KG_φ-bimodule.
- So there is induction functor $\mathrm{Ind}_\varphi\colon KG_\varphi\text{-mod} \to KS\text{-mod}$:

$$V \longmapsto KL_\varphi \otimes_{KG_\varphi} V$$

  for a KG_φ-module V.
- The functor Ind_φ is exact.

Theorem (BS)

Let $\varphi \in \widehat{E}$. Then:

1. Ind_φ *is left adjoint to* Res_φ*;*
2. $\mathrm{Res}_\varphi \mathrm{Ind}_\varphi$ *is naturally isomorphic to the identity functor on* $KG_\varphi\text{-mod}$.

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Induction, restriction and simple modules

- Induction and restriction link the simple modules of KS with the irreducible representations of the isotropy groups.

Theorem (BS)

Let $\varphi \in \widehat{E}$.

1. *If V is a simple KG_φ-module, then $\mathrm{Ind}_\varphi(V)$ is a simple KS-module.*
2. *If V is a simple KS-module, then $\mathrm{Res}_\varphi(V) = 0$ or $\mathrm{Res}_\varphi(V)$ is a simple KG_φ-module.*

- Let us call a KS-module V spectral if $\mathrm{Res}_\varphi(V) \neq 0$ for some $\varphi \in \widehat{E}$, i.e., V is detected by the spectrum of E.

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The structure of spectral simple modules

Theorem (BS)

There is a bijection between isomorphism classes of spectral simple KS-modules and pairs (φ, V) where $\varphi \in \widehat{E}$ and $V \in \mathrm{Irr}(KG_\varphi)$ (up to the orbit of φ and the isomorphism class of V).

Proof.

- Suppose V is a spectral simple module and $W = \mathrm{Res}_\varphi(V) \neq 0$.
- The identity map $W \to W = \mathrm{Res}_\varphi(V)$ yields a non-zero morphism $\mathrm{Ind}_\varphi(W) \to V$ via the adjunction.
- But $\mathrm{Ind}_\varphi(W)$ is simple, so this morphism is an isomorphism by Schur's Lemma.

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$\square$

Conditions that guarantee spectrality

- But which modules are spectral?

Lemma (BS)

Let $\varphi\colon S \to \mathrm{End}_K(V)$ be an irreducible representation such that $\varphi(S)$ contains a primitive idempotent. Then V is a spectral simple KS-module.

- So if S satisfies dcc on idempotents, then every simple KS-module is spectral and we recover Munn's results.
- One can also show that every simple module for a semilattice of groups or for an ω-inverse semigroup is spectral.
- The regular representation of S is spectral if and only if $\widehat{E}$ contains an isolated point.

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Finite dimensional irreducible representations

- It is well known that any subsemilattice of $M_n(K)$ has at most 2^n elements and hence contains a primitive idempotent.
- One can easily check that $\mathrm{Ind}_\varphi(V)$ is finite dimensional iff the orbit of φ is finite and V is finite dimensional.
- Thus we get the following generalization of Douglas's original approach to the general case.

Theorem (BS)

There is a bijection between:

- *Isomorphism classes of finite dimensional simple KS-modules;*
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Some consequences

- Using this theorem, we can give necessary and sufficient conditions for an inverse semigroup to have enough finite dimensional irreducible representations to separate points.
- If S has enough finite dimensional irreps to separate points, then clearly each maximal subgroup also does.
- This is false for isotropy groups.
- The Birget-Rhodes expansion of any finitely generated infinite simple group (say Thompson's group V) is residually finite and hence has enough finite dimensional irreps over $\mathbb{C}$ to separate points.
- By a result of Malcev any finitely generated linear group is residually finite. Thus any finite dimensional irrep of a finitely generated infinite simple group is trivial.
- Since V is an isotropy group of its Birget-Rhodes expansion, this shows the isotropy groups may have no non-trivial irreps.

Separating points over $\mathbb{C}$

- It is straightforward to generalize Malcev's result to inverse semigroups.
- Consequently, a finitely generated inverse semigroup has enough finite dimensional irreducible representations over $\mathbb{C}$ to separate points if and only if it is residually finite.

The end

Thank you for your Attention!