Profinite groups associated to symbolic dynamical systems

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- There is a natural action of $\mathbb{Z}$ on $A^{\mathbb{Z}}$ via the shift map $\sigma\colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ given by

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- A symbolic dynamical system, or subshift, or simply shift, is a closed, non-empty, shift-invariant subspace of $A^{\mathbb{Z}}$.

Conjugacy

- Subshifts $\mathscr{X} \subseteq A^{\mathbb{Z}}$ and $\mathscr{Y} \subseteq B^{\mathbb{Z}}$ are conjugate if there is a $\mathbb{Z}$-equivariant homeomorphism $\psi\colon \mathscr{X} \to \mathscr{Y}$, i.e.,

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  commutes.
- Usually one wants to classify shifts up to conjugacy, although sometimes weaker equivalence relations are considered.

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- Symbolic encodings of dynamical systems on manifolds.

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- Every shift contains a minimal subshift.
- A word $w \in A^{\mathbb{Z}}$ generates a minimal subshift iff it is uniformly recurrent.
- This means that if v is a finite factor of w, then there exists $N > 0$ so that each factor of w of length N contains v as a factor: the "bounded gaps property."

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- The general case is obtained by replacing ab with an arbitrary finite word.

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- We will only be interested in irreducible subshifts.

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- $\mathscr{X}$ is irreducible iff, for all $u, v \in L(\mathscr{X})$, there exists $w \in A^*$ so that $uwv \in L(\mathscr{X})$.

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- Up to conjugacy one can assume f is proper.

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- An endomorphism of A^* that extends to an automorphism of F_A is called a positive automorphism.

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- Much attention has been devoted to classifying shifts of finite type up to conjugacy.

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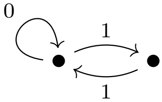
An example: the even shift

- The even shift consists of all labels of a bi-infinite path in the automaton



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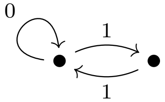

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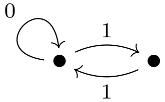

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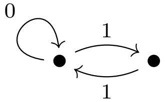

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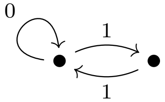

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- In the above example all vertices are initial and terminal.

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- Irreducible sofic shifts can always be recognized by a strongly connected automaton all of whose states are initial and final.
- A minimal sofic shift must be periodic (follow a cycle in the automaton).

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- The regular languages form a boolean algebra.

Bigger and better boundaries

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- J. Almeida had the idea of relating symbolic dynamical systems to the universal compact totally disconnected boundary of a factorial prolongable language.
- Namely, he considered the boundaries of these languages inside the profinite completion of the free monoid.

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- $\widehat{A^*}$ can also be described as the inverse limit of all finite A-generated monoids.
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- If C is any closed subset of $\widehat{A^*}$, then the boundary of C is

$$\partial C = C \cap \partial\widehat{A^*}.$$

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- In particular, the ideal $I(\mathscr{X})$ associated to an irreducible subshift $\mathscr{X}$ is prime.

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- So there is a profinite group invariant associated to an irreducible subshift via the free profinite monoid!

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- A profinite group G is projective if given a diagram

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- They are also the profinite groups of cohomological dimension one.

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- To state their results we need some notation.
- If M is a finitely generated profinite monoid, then $\mathrm{End}(M)$ is a profinite monoid in the compact-open topology.
- So, if $f \in \mathrm{End}(M)$, then $f^{n!}$ converges to an idempotent f^{ω}.

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Theorem (Almeida, Costa)

Let $f\colon A^* \to A^*$ *be a proper aperiodic primitive endomorphism. Then*

$$G(\mathscr{X}_f) = \langle A \mid f^\omega(a) = a, a \in A \rangle$$

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- This answers negatively a part of Margolis's question.

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- I can now prove it using cohomology of profinite monoids.

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- A result of myself and Rhodes shows the above theorem applies to free profinite monoids.
- Since $\operatorname{cd} \widehat{A^*} = 1$, the theorem implies projectivity of maximal subgroups of $\widehat{A^*}$.

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- But projective profinite groups are torsion-free so $C = \{s^k\}$.

Torsion in free profinite monoids

- Any element s of finite order in $\widehat{A^*}$ must satisfy $s^n = s^{n+m}$ for some $n, m \geq 1$.
- $C = \{s^n, \ldots, s^{n+m-1}\}$ is a finite cyclic subgroup with identity s^k where $n \leq k \leq n+m-1$ is divisible by m.
- As idempotents generate prime ideals, it follows that s and s^k generate the same ideal.
- Standard profinite semigroup theory then implies $s \in C$.
- But projective profinite groups are torsion-free so $C = \{s^k\}$.

Theorem (Rhodes, BS)

Every element of finite order in $\widehat{A^*}$ is an idempotent.

The free profinite group on a set converging to 1

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- The cardinality of Y is called the rank of $\widehat{F}_Y$.
- Every metrizable profinite group has a countable set of generators converging to 1.

The maximal subgroup of the minimal ideal is free

Theorem (BS)

The maximal subgroup $G(A^{\mathbb{Z}})$ *of the minimal ideal of* $\widehat{A^*}$ *is a free profinite group of countable rank.*

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- The proof relies on Iwasawa's criterion: a countably based profinite group G is free of countable rank iff given a diagram

$$\begin{tikzcd} & G \arrow[d, two heads, "\varphi"] \\ A \arrow[r, two heads, "\alpha"] & B \end{tikzcd}$$

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  of epimorphisms (A and B are finite), there exists an epimorphism $\lambda \colon G \twoheadrightarrow A$ so that the diagram commutes.

Profinite groups associated to irreducible sofic subshifts

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- It is then natural to ask whether the result for the full shift extends to all irreducible sofic shifts.
- Minimal sofic shifts are periodic and hence have free procyclic associated groups.
- So the interesting case is the non-minimal case.

A topological characterization of sofic shifts

- The fact $\widehat{A^*}$ is the Stone dual of the boolean algebra of regular languages has a topological consequence.

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A subshift $\mathscr{X} \subseteq A^{\mathbb{Z}}$ is sofic iff $\overline{L(\mathscr{X})}$ is a clopen subset of $\widehat{A^*}$.

- This is almost enough to generalize my proof from the full shift to an arbitrary non-periodic sofic shift.
- But technically my proof scheme only works for irreducible sofic shifts containing a periodic subshift defined over a strictly smaller alphabet.

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Theorem (Costa, BS)

The profinite group associated to a non-periodic irreducible sofic shift is a free profinite group of countable rank.

Idempotent generators of the ideal of a sofic shift

- It is natural to ask where the idempotents corresponding to irreducible sofic shifts 'sit' in $\widehat{A^*}$.

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- So there is a dense set of idempotents whose associated maximal subgroups are free profinite.
- Given a strongly connected automaton accepting an irreducible sofic shift $\mathscr{X}$, we can effectively construct an idempotent generator of $I(\mathscr{X})$.

Open questions

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- Can a free pro-p group be a maximal subgroup of $\widehat{A^*}$ (Zalesskii)?
- Is the profinite group associated to a minimal shift finitely generated?
- What are the possible finite subsemigroups of a free profinite monoid?

The end

Thank you for your Attention!