Quivers of monoids with basic algebras

Stuart Margolis[1] Benjamin Steinberg[2]

[1]Bar-Ilan University

[2]Carleton University

bsteinbg@math.carleton.ca

http://www.mathstat.carleton.ca/~bsteinbg

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Outline

Group Representation Theory and its Applications

- Group representation theory originated in the late 1800s with the work of Frobenius.
- It took some time before its usefulness was seen.
- The preface of the first edition of Burnside's book on finite groups asserts:

  > *"it would be difficult to find a result that could be most directly obtained by the consideration of groups of linear transformations."*

- Nowadays, group representation theory is routinely applied to such diverse areas as physics, probability, data analysis, graph theory and signal processing.

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Semigroup Representation Theory and Markov Chains

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- The first applications appeared in 1998 in the work of Bidigare, Hanlon and Rockmore (Duke J. Math).
- This was followed by work of Brown and Diaconis, Björner and others.
- Many combinatorial objects, such as real and complex hyperplane arrangements, matroids, oriented matroids and interval greedoids have the structure of a semigroup.
- Semigroup representation theory can be used to analyze random walks on these combinatorial objects.
- Important examples, like card shuffling, can be *more* easily analyzed as semigroup walks than as group walks.

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Further Applications

- Bidigare also established a strong connection between hyperplane face semigroups and Solomon's descent algebra.
- The descent algebra is at the heart of the non-commutative character theory of the symmetric group.
- Saliola used the semigroup approach in his computation of the quiver of the descent algebra in types A and B.
- Semigroups play an important role in the study of the 0-Hecke algebra of a Coxeter group and related algebras (Bergeron, Denton, Hivert, Thiéry, Schilling and others).
- Malandro and Rockmore have used fast Fourier transforms for semigroup algebras to study partially ranked data (Trans. AMS).

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Basic Algebras

- Fix an algebraically closed field k of characteristic 0.
- All algebras are assumed finite dimensional over k and unital.
- A is basic if $A/\operatorname{rad}(A) \cong k^n$.
- In other words, all irreducible representations of A are one-dimensional.
- Equivalently, A is an algebra of upper triangular matrices.
- Each algebra is Morita equivalent to a unique basic algebra.
- So for much of representation theory, it is enough to consider basic algebras.
- Caveat: This viewpoint says nothing about group algebras and so one must analyze one's goals before passing to Morita equivalence.

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Quivers

- A quiver Q is a directed graph.
- The path algebra kQ has basis all paths in Q and multiplication extending concatenation.

Example

The path algebra of A_n is the algebra of $n \times n$ upper triangular matrices.

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \quad \cdots \quad \bullet \longrightarrow \bullet$$

- Gabriel proved every basic algebra is a quotient of a path algebra by an admissible ideal.

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The Quiver of an Algebra

- The set of isoclasses of simple left A-modules is denoted $\mathrm{Irr}(A)$.
- The quiver $Q(A)$ of A has vertex set $\mathrm{Irr}(A)$.
- The number of edges from S to S' is $\dim \mathrm{Ext}^1_A(S, S')$.
- The basic algebra of A is a quotient $kQ(A)$ by an admissible ideal.
- $Q(A)$ encodes the category of projective indecomposable A-modules.
- When A is basic, then $Q(A)$ encodes the two-dimensional representation theory of A.
- In this talk we are interested in the representation theory of monoids with basic algebras.

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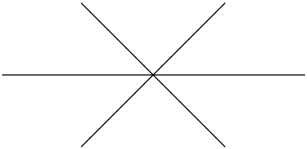
Hyperplane Face Semigroups

- Consider a central hyperplane arrangement $\mathscr{H}$, e.g.,



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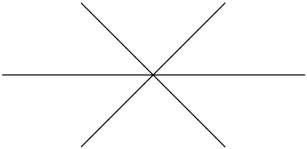

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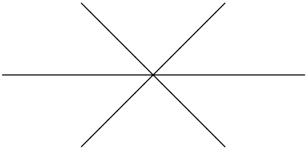

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Left Regular Bands

- $F(\mathscr{H})$ satisfies the identities $x^2 = x$ and $xyx = xy$.
- Semigroups satisfying these identities are called left regular bands (LRBs).
- Semigroups satisfying just $x^2 = x$ are called bands.
- All bands have basic algebras.
- Brown developed a theory of random walks on LRBs.
- These walks include card shuffling, random walks on bases of matroids and the Tsetlin library.
- Using the structure of the semigroup, he showed all LRB random walks are diagonalizable.
- This gave a uniform explanation for diagonalizability of a number of classical Markov chains.

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Solomon's Descent Algebra

- If W is a finite Coxeter group, there is an associated reflection arrangement $\mathscr{H}_W$.
- W acts by automorphisms on $F(\mathscr{H}_W)$ and hence on $kF(\mathscr{H}_W)$.
- Bidigare proved that the algebra $kF(\mathscr{H}_W)^W$ of W-invariants is isomorphic to Solomon's descent algebra.
- The descent algebra is a subalgebra of kW.
- It is a basic algebra with semisimple quotient a subalgebra of the character ring of W.
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Solomon's Descent Algebra

- If W is a finite Coxeter group, there is an associated reflection arrangement $\mathscr{H}_W$.
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- Saliola developed in his thesis the representation theory of LRBs.
- He computed the quiver, the Cartan invariants and a complete set of primitive idempotents.
- He also gave a semigroup theoretic model of the projective indecomposables.
- That is, he gave a basis B for the projective indecomposables such that $B \cup \{0\}$ is invariant under the action of the semigroup.
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Hereditary Algebras and Left Regular Bands

- An algebra A is hereditary if each left ideal of A is projective.
- Equivalently, each submodule of a projective A-module is projective.
- Gabriel proved that a basic algebra is hereditary iff it is the path algebra of an acyclic quiver.
- Recently, Margolis, Saliola and I proved that if M is an LRB such that the Hasse diagram of the poset of principal right ideals aM of M is a tree, then kM is hereditary.
- This includes the algebras of nearly all LRBs considered by Brown except the hyperplane face semigroups!
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The Mantaci-Reutenauer Descent Algebra

- Mantaci and Reutenauer introduced a generalization of the descent algebra for wreath products $G \wr S_n$ with G abelian.
- Generalizations have since been considered for G non-abelian, independently, by Baumann and Hohlweg, and by Novelli and Thibon.
- Again the descent algebra can be viewed as a non-commutative ring of characters.
- Hsiao introduced a certain subsemigroup M of $F(\mathscr{H}_{S_n}) \times G^n$.
- S_n acts naturally on kM and the algebra of S_n-invariants is the Mantaci-Reutenauer descent algebra.
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0-Hecke Monoids

- Let W be a finite Coxeter group with Coxeter generators S.
- The Hecke algebra $\mathcal{H}(W,S)$ has generators S and relations the braid relations of W and the relations

$$s^2 = (q-1)s + 1, \text{ for } s \in S.$$

- When $q = 1$, one recovers the group algebra kW.
- When $q = 0$, one obtains the monoid algebra of the 0-Hecke monoid $M(W)$.
- $M(W)$ is the monoid generated by S subject to the braid relations of W and the idempotent relations $s^2 = s$ for $s \in S$.

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0-Hecke Monoids Continued

- Tits's solution to the word problem in Coxeter groups also solves the word problem in $M(W)$.
- In particular, two reduced words are equal in W iff they are equal in $M(W)$.
- So $M(W)$ is in bijection with W.
- The strong Bruhat ordering is compatible on $M(W)$.
- From this one deduces $M(W)$ is $\mathscr{J}$-trivial.
- A monoid M is $\mathscr{J}$-trivial if $MaM = MbM \implies a = b$.
- The representation theory of 0-Hecke algebras was first studied by Norton in the 70s.
- Denton, Hivert, Thiéry and Schilling obtained the classical results, and new ones, as a special case of their theory of representations of $\mathscr{J}$-trivial monoids.

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$\mathscr{J}$-trivial Semigroups

- Path algebras and truncated path algebras are semigroup algebras of $\mathscr{J}$-trivial semigroups.
- So are incidence algebras of posets.
- Bautista, Gabriel, Roiter and Salmerón have shown that every basic algebra of finite representation type is the algebra of a $\mathscr{J}$-trivial semigroup.
- The Catalan monoid C_n is $\mathscr{J}$-trivial.
- It consists of all maps $f\colon [n] \to [n]$ such that f is order-preserving and non-increasing.
- $|C_n|$ is the n^{th}-Catalan number.
- Denton *et al.* have computed quivers, primitive idempotents, Cartan invariants and projective indecomposables for $\mathscr{J}$-trivial monoids.

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$\mathscr{R}$-trivial Monoids

- A monoid M is $\mathscr{R}$-trivial if $aM = bM \implies aM = bM$.
- LRBs are precisely the $\mathscr{R}$-trivial bands.
- $\mathscr{J}$-trivial implies $\mathscr{R}$-trivial.
- A typical example is E_n, the monoid of maps $f\colon [n] \to [n]$ such that $xf \leq x$ (with action on the right).
- $|E_n| = n!$.
- Each $\mathscr{R}$-trivial monoid of size n embeds in E_n.
- Berg, N. Bergeron, Bhargava and Saliola have computed primitive idempotents for $\mathscr{R}$-trivial monoids.
- We have computed quivers, Cartan invariants and semigroup theoretic models of the projective indecomposables.
- Our results are independent of Berg *et al.*

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- A monoid M is $\mathscr{R}$-trivial if $aM = bM \implies aM = bM$.
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Finite Monoids

- Let M be a finite monoid.
- $E(M)$ is the set of idempotents of M.

Theorem (Standard Facts)

For $e, f \in E(M)$, the following are equivalent:

1. $Me \cong Mf$ *as left* M*-sets*
2. $eM \cong fM$ *as right* M*-sets*
3. $\exists a, b \in M$ *such that* $ab = e$ *and* $ba = f$
4. $MeM = MfM$

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Maximal Subgroups

- If $e \in E(M)$, then $\mathrm{End}_M(Me) = eMe$.
- Let G_e be the group of units of eMe.
- $G_e = \mathrm{Aut}_M(Me)$ is called the maximal subgroup at e.
- Notice e, f conjugate implies $G_e \cong G_f$.
- Much of finite semigroup theory is about using maximal subgroups to understand the whole semigroup.
- For instance, the maximal subgroups control the irreducible representations of the semigroup.

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Rectangular Monoids

- We say M is a rectangular monoid if each conjugacy class of idempotents is a subsemigroup.
- For example, all groups are examples of rectangular monoids.
- All semigroups discussed in this talk are rectangular monoids.
- The class of rectangular monoids was introduced by Schützenberger and his school under the name $\mathbf{DO}$.

Theorem (Almeida, Margolis, BS, Volkov)

The algebra kM of a finite monoid M is basic if and only if:

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An Example

- Our goal is to "compute" the quiver of a rectangular monoid.
- Let us give an example to explain what this means.
- Let G be a group and X a $G \times G$-set.
- We view X as having commuting left and right actions by G.
- Let $M = G \cup X \cup \{0\}$ where $X^2 = 0$.
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An Example Continued

- Let $V_1, \ldots, V_s$ be the simple G-modules.
- Each is a simple M-module by letting $X \cup \{0\}$ act as 0.
- There is one additional simple M-module: the trivial module.
- The vertex of $Q(kM)$ corresponding to the trivial module is isolated.
- The number of arrows from V_i to V_j is the multiplicity of $V_j \otimes_k V_i^*$ as a composition factor of the $G \times G$-module kX.
- Since every simple $G \times G$-module is of this form, $Q(kM)$ encodes the composition factors of kX.
- So our goal is to compute the quivers of rectangular monoids modulo the character theory of finite groups.

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The Simple Modules

- Let M be rectangular monoid.
- Let $\Lambda(M)$ be the set of conjugacy classes of idempotents.
- For each $X \in \Lambda(M)$, fix a maximal subgroup G_X.
- There is a surjective homomorphism $\rho_X \colon kM \to kG_X$.
- Each simple M-module is lifted via ρ_X from a simple G_X-module for some X.
- So
$$\operatorname{Irr}(kM) = \coprod_{X \in \Lambda(M)} \operatorname{Irr}(kG_X).$$
- In most of our examples, e.g., bands and $\mathscr{R}$-trivial monoids, the maximal subgroups are all trivial.

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- Let $\Lambda(M)$ be the set of conjugacy classes of idempotents.
- For each $X \in \Lambda(M)$, fix a maximal subgroup G_X.
- There is a surjective homomorphism $\rho_X \colon kM \to kG_X$.
- Each simple M-module is lifted via ρ_X from a simple G_X-module for some X.
- So
$$\mathrm{Irr}(kM) = \coprod_{X \in \Lambda(M)} \mathrm{Irr}(kG_X).$$
- In most of our examples, e.g., bands and $\mathscr{R}$-trivial monoids, the maximal subgroups are all trivial.

The Quiver of kM

- The vertex set of $Q(kM)$ is $\coprod_{X\in\Lambda(M)} \mathrm{Irr}(kG_X)$.
- For each pair $(X,Y)\in\Lambda(M)^2$, we explicitly construct a certain $G_Y\times G_X$-module $V_{X,Y}$.
- Suppose $U\in\mathrm{Irr}(kG_X)$ and $V\in\mathrm{Irr}(kG_Y)$.
- The number of edges $U\longrightarrow V$ is the multiplicity of the simple $G_Y\times G_X$-module $V\otimes_k U^*$ as a composition factor in $V_{X,Y}$.
- Thus computing $Q(kM)$ boils down to decomposing the $G_Y\times G_X$-modules $V_{X,Y}$.
- When the maximal subgroups of M are trivial, the vertex set is $\Lambda(M)$.
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Hochschild Cohomology

- Let me give an idea of our techniques.
- Recall the number of edges from U to V is $\dim \mathrm{Ext}^1_{kM}(U,V)$.
- There is an alternative formulation in terms of primitive idempotents.
- We use the description of Ext in terms of Hochschild cohomology.
- Namely, $\mathrm{Hom}_k(U,V)$ is an M-bimodule in a natural way.
- $\mathrm{Ext}^n_{kM}(U,V) \cong H^n(M, \mathrm{Hom}_k(U,V))$.
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Derivations

- Let A be an M-bimodule.
- A derivation of M in A is a mapping $d\colon M \to A$ satisfying

$$d(mn) = md(n) + d(m)n.$$

- A derivation is inner if there exists $a \in A$ with $d(m) = ma - am$.
- $H^1(M, A)$ is the space of outer derivations, i.e., derivations modulo inner derivations.

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Representability

- If $U \in \mathrm{Irr}(G_X)$ and $V \in \mathrm{Irr}(G_Y)$, then $\mathrm{Hom}_k(U, V)$ is in fact a $G_Y \times G_X$-module.
- More generally, any $G_Y \times G_X$-module is naturally an M-bimodule.
- For each $X, Y \in \Lambda(M)$, we show $H^1(M, A)$ is representable as a functor from $G_Y \times G_X$-modules to k-vector spaces.
- That is, there is a $G_Y \times G_X$-module $V_{X,Y}$ such that $H^1(M, A) \cong \mathrm{Hom}_{k[G_Y \times G_X]}(V_{X,Y}, A)$.
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The Case of Bands

- Suppose M is a band (i.e., all elements of M are idempotent).
- The vertex set of $Q(kM)$ is $\Lambda(kM)$.
- If MXM, MYM are incomparable or $X = Y$, there are no edges $X \to Y$.
- If $MXM \subsetneq MYM$, then we may suppose $e_X = e_Ye_Xe_Y$.
- Let $L_X = \{m \in M \mid Mm = Me_X\}$.
- Let $\equiv$ be the smallest equivalence relation on e_YL_X with:
  1. $e_Ymx \equiv e_Yme_Yx$ for $x \in L_X$ and $MmM \supseteq MXM$;
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- If $MYM \subsetneq MXM$, then the number of edges $X \to Y$ equals the number of edges $Y \to X$ in $Q(kM^{op})$.

Quiver of the Free Band

- There is a free band $FB(A)$ on any set A.
- J. A. Green and D. Rees solved the word problem for $FB(A)$.
- They proved A finite implies $FB(A)$ is finite.
- The quiver $Q(kFB(A))$ for A finite is as follows.
- The vertex set is 2^A.
- If $Y \subsetneq X$, then there are $|X \setminus Y| - 1$ edges $X \to Y$.
- If $Y \supsetneq X$, then there are $|Y \setminus X| - 1$ edges from $X \to Y$.
- There are no other edges.

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The End

Thank you for your Attention!