Möbius Functions and Semigroup Representation Theory

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Inverse Semigroups

- Just as groups abstract permutations, inverse semigroups abstract partial permutations.
- A semigroup S is an inverse semigroup if, for all $s \in S$, there exists unique $s' \in S$ such that $ss's = s$ and $s'ss' = s'$.
- One writes s^{-1} for s'.
- The motivating example is the symmetric inverse monoid I_n of all partial permutations of an n-element set.
- The Preston-Wagner theorem says every inverse semigroup of order n embeds in I_n.

Rook Monoid

- A typical element of I_4 is

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ - & 1 & - & 3 \end{pmatrix}.$$

- Of course

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & - & 4 & - \end{pmatrix}.$$

- Alternatively, σ can be represented by the rook matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

- The monoid R_n of all $n \times n$ rook matrices, called the rook monoid, is isomorphic to I_n.
- The inverse in this context is the transpose matrix.

Examples

- The signed symmetric inverse monoid consists of all rook matrices with entries in $\{\pm 1\}$.
- It can be identified with the wreath product $\mathbb{Z}/2\mathbb{Z} \wr I_n$.
- The monoid of uniform block permutations UB_n consists of all bijections between partitions of n preserving sizes of blocks.
- A typical example from UB_5 is

$$\sigma = \begin{pmatrix} \{1,3\} & \{2\} & \{4,5\} \\ \{1,5\} & \{3\} & \{2,4\} \end{pmatrix}$$

- UB_n can be identified with the semigroup of partial permutations of the support lattice of the Coxeter complex for S_n generated by the partial identities of the supports and the action of S_n.

Renner monoids

- Let M be a reductive algebraic monoid (e.g. $M_n(K)$).
- Let G be the unit group of M; it is reductive (e.g. $GL_n(K)$).
- Let T be a maximal torus and B a Borel subgroup.
- Let $W = N(T)/T$ be the Weyl group.

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Theorem (Renner)

$M = \bigsqcup_{r \in R} BrB$ where $R = \overline{N(T)}/T$ is a finite inverse monoid with unit group W.

- R is called the Renner monoid of M. For $M_n(K)$, the Renner monoid is the rook monoid R_n.

Goals

1. Find an alternative basis for an inverse semigroup algebra.
2. Use this basis to identify the algebra as a direct product of matrix algebras over group algebras.
3. Compute explicitly the irreducible representations.
4. Give a combinatorial method to compute multiplicities of irreducible constituents in an arbitrary representation.
5. Discuss applications to random walks.

The Structure of Inverse Semigroups via an Example

- The idempotents of I_n are the partial identities 1_X with $X \subseteq [n]$.
- They form a lattice isomorphic to 2^X as $1_X \cdot 1_Y = 1_{X\cap Y} = 1_Y \cdot 1_X$.

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- I_n can be ordered by $\sigma \leq \tau$ if σ is a restriction of τ.
- The set of elements $\sigma \in I_n$ with domain and range X is a group isomorphic to S_X.
- This structure is common to all inverse semigroups.

The Structure of Inverse Semigroups

- Let S be an inverse semigroup and $E(S)$ its set of idempotents.
- $E(S)$ is a commutative semigroup.
- $E(S)$ is a meet-semilattice ordered by $e \leq f$ if $e = ef$.
- The ordering on $E(S)$ extends to S by $s \leq t$ if $s \in tE(S)$.
- If $e \in E(S)$, then $G_e = \{s \in S \mid ss^{-1} = e = s^{-1}s\}$ is a group called the maximal subgroup at e.
- It is the unit group of eSe.

Isomorphism of Idempotents

- Idempotents $e, f \in E(S)$ are isomorphic if $\exists s \in S$ such that $s^{-1}s = e$ and $ss^{-1} = f$.
- We represent this by an arrow $s^{-1}s \xrightarrow{s} ss^{-1}$ and write $\mathrm{dom}(s) = s^{-1}s$ and $\mathrm{ran}(s) = ss^{-1}$.
- If $e \cong f$, then $G_e \cong G_f$.

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- If $e \cong f$, then $G_e \cong G_f$.
- One can in fact form a groupoid with objects $E(S)$ and arrows $s^{-1}s \xrightarrow{s} ss^{-1}$.
- Composition is given by

$$e \xrightarrow{s} f \xrightarrow{t} e' = e \xrightarrow{st} e'.$$

The Groupoid Basis

- Let K be a field and S a finite inverse semigroup.
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Theorem (BS)

The basis $\{\overline{s} \mid s \in S\}$ satisfies

$$\overline{s} \cdot \overline{t} = \begin{cases} \overline{st} & \mathrm{dom}(s) = \mathrm{ran}(t) \\ 0 & \textit{otherwise.} \end{cases}$$

Orthogonal Idempotents and a Decomposition

- From the theorem, it follows $\{\overline{e} \mid e \in E(S)\}$ is a set of orthogonal idempotents summing to 1.
- Moreover, $\overline{e}KS \cong \overline{f}KS$ if and only if $e \cong f$.
- $\overline{e}KS\overline{e} \cong KG_e$.
- Let $e_1, \ldots, e_r$ be a transversal to the set of isomorphism classes of idempotents of S.
- Let n_i be the number of idempotents isomorphic to e_i.

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Proof.

$KS \cong \prod_{i=1}^{r} \mathrm{End}(n_i\overline{e}_iKS) \cong \prod_{i=1}^{r} M_{n_i}(\overline{e}_iKS\overline{e}_i) \cong \prod_{i=1}^{r} M_{n_i}(KG_{e_i})$. $\square$

The Algebra of I_n

- For I_n, we can take as a transversal $\{1_{[i]} \mid i = 0, \ldots, n\}$.
- Then $G_{1_{[i]}} \cong S_i$ and $n_i = \binom{n}{i}$.
- So $KI_n \cong \prod_{i=0}^{n} M_{\binom{n}{i}}(KS_i)$.
- The corresponding central idempotents are

$$e_i = \sum_{|X|=i} \sum_{Y \subseteq X} (-1)^{|X|-|Y|} 1_Y$$

- This explicit decomposition for KI_n was first discovered by Solomon.
- In general, the idempotents of a Renner monoid form the face lattice of a rational polytope. Hence the Möbius function is particularly nice in this context.

Some History

- Munn and Ponizovskii showed in the fifties that the algebra of an inverse semigroup has an ideal series whose successive quotients are the $M_{n_i}(KG_{e_i})$. This implies our decomposition.
- But it is not good enough to compute multiplicities of irreducible constituents.
- Solomon obtained the explicit decomposition, but did not use it to compute multiplicities.
- After I introduced the groupoid basis, it was exploited by Rockmore and Malandro to develop Fast Fourier Transforms for the symmetric inverse monoid.

Multiplicities

- We retain our previous notation.
- $\mathbb{C}S$ is Morita equivalent to $\mathbb{C}G_1 \times \cdots \times \mathbb{C}G_r$.
- So $\mathrm{Irr}(S) \cong \coprod_{i=1}^{r} \mathrm{Irr}(G_i)$.
- Let θ be a character of S and let χ be an irreducible character of G_i.
- The associated irreducible character of S is denoted χ^*.
- For $f \leq E(S)$, define $\theta_f(s) = \theta(sf)$.

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Theorem (BS)

The multiplicity of χ^* in θ is given by

$$\sum_{f \leq e} \langle \chi, \theta_f \rangle \mu(f, e).$$

Tensor Powers

- Let G be a finite group.
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Theorem (BS)

Let $\chi \in \mathrm{Irr}(G \wr S_r)$. Then the multiplicity of χ^* in θ^p is

$$\frac{1}{|G|^{r-p}} \deg(\chi) S(p,r)$$

where $S(p,r)$ is the Stirling number of the second kind.

- This generalizes a result of Solomon for $|G| = 1$, but even in this case our proof is easier as Solomon used a more complicated method to compute multiplicities.

Triangularizable Semigroups

Theorem (AMSV)

The following are equivalent for a finite semigroup S:

1. $\mathbb{C}S$ *is basic;*
2. *All irreducible representations of S have degree 1;*
3. S *admits a faithful representation by upper triangular matrices;*
4. *There is a morphism $\varphi : S \to T$ with T a commutative inverse semigroup such that the induced map $\tilde{\varphi} : \mathbb{C}S \to \mathbb{C}T$ is the semisimple quotient;*
5. *All subgroups of S are abelian and there exists $n > 0$ such that regular elements satisfy $x^n = x$ and products of idempotents satisfy $x^n = x^{n+1}$;*

Semigroups satisfying these conditions are called triangularizable.

Random Walks on Monoids

- Let $\pi = \sum_{m \in M} \pi_m m$ be a probability measure on a finite monoid M.
- Fix a minimal right ideal R of M.
- For $r_1, r_2 \in R$, let $T_{r_1 r_2}$ be the probability $r_1 m = r_2$ if $m \in M$ is chosen according to π.
- Let $T = (T_{r_1 r_2})$ be the transition matrix. Then $T^n_{r_1 r_2}$ is the probability of going from r_1 to r_2 on the n^{th}-step of the walk.

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- What is the spectrum of T?
- Can you compute the stationary distribution (the probability vector with eigenvalue 1)?
- Diaconis did this for abelian groups, Brown did this for idempotent semigroups (bands), following earlier work of Bidigare, Hanlon and Rockmore for face semigroups of hyperplane arrangements.

Diaconis-Brown trick

- $\mathbb{C}R$ is a right ideal in $\mathbb{C}M$. Let $\rho : M \to M_{|R|}(\mathbb{C})$ be the associated matrix representation with respect to the basis R.
- Key observation: $T = \rho(\pi)$.
- Suppose now M is triangularizable. Taking a basis adapted to a composition series for $\mathbb{C}R$ yields

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$$\rho \sim \begin{pmatrix} \chi_1 & 0 & \cdots & 0 \\ * & \chi_2 & 0 & \vdots \\ \vdots & * & \ddots & 0 \\ * & \cdots & * & \chi_{|R|} \end{pmatrix}$$

- T has an eigenvalue $\lambda_\chi = \sum_{m \in M} \pi_m \chi(m)$ associated to each $\chi \in \operatorname{Irr} M$.
- The multiplicity of λ_χ is the multiplicity of χ in ρ.
- Our work on inverse semigroups allows us to explicitly determine these.

The Spectrum

- Choose an idempotent transversal $\mathscr{E} = \{e_1, \ldots, e_r\}$ to the set of isomorphism classes of $E(M)$.
- $\operatorname{Irr} M = \coprod \operatorname{Irr} G_{e_i}$ ($G_{e_i} =$ unit group of $e_i M e_i$).
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$$\lambda_\chi = \sum_{e_i \in MmM} \pi_m \chi(e_i m e_i).$$

- The set $\mathscr{E}$ is a meet-semilattice with respect to the ordering $e_j \leq_{\mathscr{J}} e_i$ if $Me_jM \subseteq Me_iM$. Let μ be its Möbius function.
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- The multiplicity of λ_χ is given by

$$\sum_{e_j \leq_{\mathscr{J}} e_i} \langle \chi, \varphi_j \rangle \mu(e_j, e_i)$$

  where $\varphi_j(g)$ is the number of fixed points of $e_j g e_j$ on R.
- This recovers the results of Diaconis on abelian groups and of Brown on bands.