Matrix mortality and the Pin-Černý Conjecture

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- By an automaton $\mathscr{A} = (Q, \Sigma)$, we understand a complete deterministic automaton with state set Q, input alphabet Σ and no initial or final states.
- If $w \in \Sigma^*$, then the rank of w is $\mathrm{rk}(w) = |Qw|$.
- If $\mathrm{rk}(w) = 1$, then w is called a reset word.
- Define $\mathrm{rk}(\mathscr{A}) = \min\{\mathrm{rk}(w) \mid w \in \Sigma^*\}$ (Pin).
- $\mathscr{A}$ is synchronizing if $\mathrm{rk}(\mathscr{A}) = 1$, i.e., it admits a reset word.

Conjecture (Černý-Pin)

An automaton $\mathscr{A}$ of rank r admits a word w of length at most $(n-r)^2$ with $\mathrm{rk}(w) = r$.

- The case $r = 1$ is due to Černý; the more general conjecture is a variation on an earlier conjecture of Pin.

Synchronizing automata

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Černý's examples

- Černý showed that the shortest length reset word for the n-state synchronizing automaton with transitions

$$a = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 3 & 4 & \cdots & 1 \end{pmatrix}, \ b = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 2 & 3 & \cdots & n \end{pmatrix}$$

  is $(n-1)^2$.
- The Černý automaton for $n = 4$:

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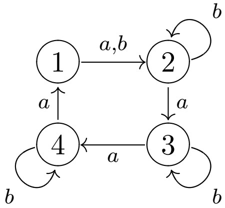

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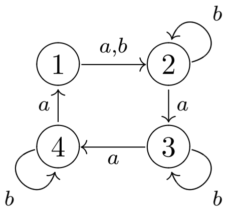

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Pin's bounds

- It is straightforward to obtain a cubic upper bound of $\frac{n^3-n}{3}$ on reset words for synchronizing automata.
- The best known upper bound for the synchronizing case is $\frac{n^3-n}{6}$, which was proved by Pin modulo an extremal set theory result of Frankl.
- Improving a bound by a factor of 2 can be hard work!
- The lower bound of $(n-r)^2$ for rank r is due to Pin.
- Pin also has an analogous cubic upper bound for rank r.
- Probabilistically speaking, all automata are synchronizing with reset word of length at most $2n$.
- The remainder of the Černý literature consists of a vast array of special, but interesting, cases.
- There is no time to survey the whole literature here.

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Some known results

- The special cases treated so far tend to be of two sorts:
  1. Combinatorial restrictions are imposed on the automata;
  2. Algebraic restrictions are imposed on the transition monoid.
- A key example of the first sort is the result of Dubuc that the Černý conjecture holds for circular automata: automata where one of the input letters cyclically permutes the state set.
- Kari proved that if the underlying digraph of the automaton is Eulerian, then the Černý conjecture holds.
- An important algebraic result is that of Trahtman establishing the Černý conjecture for automata with aperiodic transition monoid with an upper bound of $n(n-1)/2$.
- Rystsov showed that if the transition monoid is commutative, then the Černý conjecture holds with an upper bound of $n-1$ (which is sharp).

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Representation theoretic approaches

- Representation theory is the study of algebraic objects using linear algebra.
- Many papers on Černý's conjecture make use in some form or the other of representation theory without using the full strength of the subject.
- For instance, Dubuc's paper on circular automata implicitly relies on properties of representations of cyclic groups.
- An approach using rational power series, pioneered by Béal, also relies on representation theory as representation theory lies in the foundations of weighted automata theory.
- Rystsov has a number of papers that make use of matrix representations to attack cases of the Černý conjecture.
- My goal is to explore representation theoretic approaches to the Černý conjecture.

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Matrix mortality

Theorem (Rystsov)

Suppose that it is true that, given a set $\Sigma \subseteq M_n(K)$ of $n \times n$ matrices over a field K such that

1. $|\langle \Sigma \rangle| < \infty$
2. $0 \in \langle \Sigma \rangle$,

there is a word $w \in \Sigma^*$ of length at most n^2 representing the zero matrix. Then the Černý-Pin conjecture is true.

- Unfortunately, Rystsov's conjecture is false.
- Paterson showed that it is undecidable whether the monoid generated by a finite subset of $M_3(\mathbb{Z})$ contains 0 (The Matrix Mortality Problem).
- If Rystsov's conjecture were true, then by considering reduction modulo primes this problem would be decidable.

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Mortality functions

- The proof of Rystsov's result uses only the field $\mathbb{F}_2$.
- We find it more convenient to work with the field $\mathbb{Q}$ of rational numbers.
- A monoid homomorphism $\rho\colon M \to M_n(\mathbb{Q})$ is called a representation of degree n.
- A function $f\colon \mathbb{N} \to \mathbb{N}$ is a mortality function for the monoid M if, for all representations $\rho\colon M \to M_n(\mathbb{Q})$ with $0 \in \rho(M)$ and all generating sets Σ for M, there exists $w \in \Sigma^*$ of length at most $f(n)$ such that $\rho(w) = 0$.
- Of course $f(n) = |M| - 1$ is a mortality function for a finite monoid M.
- We say f is a mortality function for a class of monoids $\mathscr{C}$ if it is a mortality function for all monoids in $\mathscr{C}$.
- By a universal mortality function, we mean a mortality function for the class of all finite monoids.

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Let $\mathscr{A}$ be an n-state automaton of rank r with transition monoid M and suppose that f is a superadditive mortality function for M. Then there is a word of length at most $f(n-r)$ having rank r.

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- We outline a proof of the theorem for the synchronizing case, as it is much easier.
- Let $\mathscr{A} = (Q, \Sigma)$ be an n-state automaton with transition monoid M and assume $Q = \{1, \ldots, n\}$.
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- To each $a \in \Sigma$, associate the linear transformation $\rho(a)$ given by $e_i\rho(a) = e_{i \cdot a}$.
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- We claim that $w \in \Sigma^*$ is a synchronizing word iff $\rho(w)|_{V_0} = 0$.
- Indeed, $(e_i - e_j)\rho(w) = e_{i \cdot w} - e_{j \cdot w}$.
- So $\rho(w)$ annihilates V_0 iff $i \cdot w = j \cdot w$ for all $1 \leq i < j \leq n$.
- But this occurs iff $|Q \cdot w| = 1$, i.e., w is a reset word.
- It now follows that if f is a mortality function for M, then there is a reset word w for $\mathscr{A}$ of length at most $f(n-1)$.
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- For example, V_0 from the above proof is an M-invariant subspace of V.
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- Every representation can be 'built up' from irreducible representations in much the same way that every finite group can be 'built up' from finite simple groups.
- This allowed us to prove the following result by induction on the dimension.

Theorem (Almeida,BS)

Let f be a superadditive function. Then f is a mortality function for M iff f is a mortality function for each irreducible representation of M.

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Irreducible representations

- There is a well-developed theory of irreducible representations of finite monoids due to Munn-Ponizovsky and further elaborated by Rhodes and Zalcstein.
- In particular, the irreducible representations of a finite monoid M can be constructed from the irreducible representations of its maximal subgroups.
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The case of DS

- A finite monoid belongs to the class DS if $e \in MaM \cap MbM$ implies $e \in MabM$ for all idempotents $e \in M$.
- Recall that e is idempotent if $e^2 = e$.
- Equivalently, $M \in$ DS iff $M \times M$ cannot recognize the language $(ab)^*$.
- This class was introduced independently by Putcha and Schützenberger.
- Examples of monoids in DS include:
  - commutative monoids (obvious);
  - monoids satisfying an identity $x^m = x$ (Clifford);
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- The variety DS appears frequently in the algebraic theory of automata.

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- In previous work, we showed with Almeida, Margolis and Volkov that the Černý conjecture holds for automata with transition monoid in DS using representation theory.
- Volkov asked in LATA 2008 whether the bound of $(n-1)^2$ was tight for this class.
- The main ingredient in our proof was the following result.

Lemma (AMSV)

If $\rho\colon M \to M_n(\mathbb{Q})$ is an irreducible representation of a monoid in DS *such that $0 \in \rho(M)$ and Σ is a generating set for M, then there is a letter $a \in \Sigma$ with $\rho(a) = 0$.*

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Theorem (Almeida, BS)

If $\mathscr{A}$ is an n-state, rank r automaton with transition monoid in DS*, then there is a word w of length at most $n - r$ with $\mathrm{rk}(w) = r$.*

- This bound is easily seen to be sharp by considering automata over unary alphabets.
- For instance,

$$\begin{tikzcd} 1 \arrow[r, "a"] & 2 \arrow[r, "a"] & \cdots \arrow[r, "a"] & n \arrow[loop right, "a"] \end{tikzcd}$$

is synchronizing with minimum length reset word a^{n-1} and the transition monoid is commutative.

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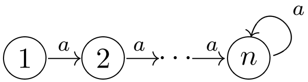

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- A finite monoid belongs to the class EDS if its idempotents generate a submonoid in DS.
- DS $\subseteq$ EDS.
- Monoids with commuting idempotents (such as inverse monoids) belong to EDS.
- More generally, monoids whose idempotents form a submonoid belong to EDS.
- It is known that a monoid M belongs to EDS iff it cannot recognize the language $\{a, b\}^* ab \{a, b\}^*$.
- Equivalently, EDS is the largest variety of monoids that cannot recognize all 2-testable languages.

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The case of EDS

- Suppose that M is a Σ-generated finite monoid.
- The Rhodes-Zalcstein theory allows us to associate to each irreducible representation $\rho\colon M \to M_n(\mathbb{Q})$ of M a finite automaton $\mathscr{A}(\rho)$ over Σ.
- $0 \in \rho(M)$ iff $\mathscr{A}(\rho)$ is synchronizing with a sink state.
- The 0-words for the representation are precisely the reset words for the automaton.
- When $M \in \mathrm{EDS}$, Almeida and I showed that $\mathscr{A}(\rho)$ has at most $n+1$ states.
- Rystsov showed that, for m-state synchronizing automata with sink state, there is a synchronizing word of length at most $m(m-1)/2$ (and this bound is sharp).
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Theorem (Almeida, BS)

If $\mathscr{A}$ is an n-state, rank r automaton with transition monoid in EDS, *then there is a word w of length at most*

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- Given the undecidability of the Matrix Mortality Problem for 3×3 integer matrices, it is not altogether clear that a universal mortality function exists.
- On the other hand, Simon/Mandel and Jacob independently proved that there is a recursive bound on the order of a finite k-generated submonoid of $M_n(\mathbb{Q})$.
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A universal mortality function

- Given the undecidability of the Matrix Mortality Problem for 3×3 integer matrices, it is not altogether clear that a universal mortality function exists.
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A universal mortality function II

Theorem (Almeida, BS)

The function

$$f(n) = \begin{cases} 1 & n = 1 \\ (2n-1)^{n^2} - 1 & n > 1 \end{cases}$$

is a superadditive universal mortality function.

- We know this upper bound is not tight.
- The best lower bound we have is n^2.
- For aperiodic monoids, we can now prove $2^n - 1$ is a mortality function (the article in the Proceedings has $2^{n^2} - 1$).

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Theorem (Pin '78)

Let $\mathscr{A} = (Q, \Sigma)$ be a synchronizing automaton so that $|Q|$ is a prime p and some element of Σ cyclically permutes Q. Then:

1. *$\mathscr{A}$ is synchronizing if and only if Σ contains a non-permutation;*
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Synchronizing groups

- Motivated by the first part of Pin's Theorem, I defined in 2004 the notion of a synchronizing group.
- A permutation group $G \subseteq S_n$ is called synchronizing if, for all non-permutations t of $\{1, \ldots, n\}$, the automaton $(\{1, \ldots, n\}, G \cup \{t\})$ is synchronizing.
- Pin's Theorem implies that cyclic groups of prime order are synchronizing.
- It is easy to see that 2-transitive groups are synchronizing.
- With Arnold, I proved synchronizing groups are primitive and gave a sufficient condition for a group to be synchronizing in terms of representation theory that covers the above results.
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- Dubuc extended the second part of Pin's Theorem to arbitrary automata containing a cyclic permutation via an ingenious linear algebraic argument.

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Let $\mathscr{A} = (Q, \Sigma)$ be a synchronizing automaton on n states such that Σ contains a cyclic permutation of the states. Then $\mathscr{A}$ has a reset word of length at most $(n-1)^2$.

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Černý Cayley graphs

- Let G be a group of order n and Δ a generating set of G.
- The automaton (G, Δ) is called the Cayley graph of G with respect to Δ. A typical transition is of the form $g \xrightarrow{a} ga$ with $g \in G$, $a \in \Sigma$.
- Let us say that an automaton $\mathscr{A}$ contains the Cayley graph (G, Δ) if $\mathscr{A} = (G, \Sigma)$ where $\Delta \subseteq \Sigma$.
- So $\mathscr{A}$ is obtained from the Cayley graph by adding new transitions but no new states.
- Call (G, Δ) a Černý Cayley graph if every synchronizing automaton containing it has a reset word of length at most $(n-1)^2$.
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- Dubuc's theorem says that $(\mathbb{Z}_n, \{1\})$ is a Černý Cayley graph.
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Rystsov's Theorem

- The above notion was implicitly considered by Rystsov.
- In 1995, he proved a synchronizing automaton containing the Cayley graph of a group of order n admits a reset word of length $\leq 2(n-1)^2$.
- He proved in fact a slightly better result.
- Let (G, Δ) be a Cayley graph with $|G| = n > 1$.
- Define $\mathrm{diam}_\Delta(G)$ to be the least m so that any two states of (G, Δ) can be connected by a word of length at most m.
- $1 \leq \mathrm{diam}_\Delta(G) \leq n-1$.

Theorem (Rystsov '95)

A synchronizing automaton containing the Cayley graph (G, Δ) has a reset word of length at most $1 + (n - 1 + \mathrm{diam}_\Delta(G))(n-2)$.

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- So Rystsov's bound only achieves the Černý bound when the diameter is 1, i.e., all non-trivial elements of G belong to the generating set.
- We aim to improve his bound so that in many cases we achieve the Černý bound.
- Even when we do not achieve the Černý bound with our main result, our techniques often suffice to establish a family of Cayley graphs is Černý.
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- We shall call an irreducible representation of a group an irrep.
- For groups, an arbitrary representation is a direct sum of irreps, which is not the case for monoids.
- If $|G| = n$, then the degree of any irrep is between 1 and $n-1$.

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For a finite group G, define $m(G)$ to be the maximal degree of an irrep of G over $\mathbb{Q}$.

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Let (G, Δ) be a Cayley graph of a group of order n. Then any synchronizing automaton containing (G, Δ) admits a reset word of length at most

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Direct products of cyclic groups of prime order

- Let p be a prime.
- To show that $\mathbb{Z}_p^k$ is a Černý group, it suffices to consider the Cayley graph with respect to a basis Δ.
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Dihedral groups

- Let D_n be the dihedral group of order $2n$ (the symmetry group of a regular n-gon).
- Let Δ consist of a reflection and a rotation by $2\pi/n$.
- Then $\mathrm{diam}_\Delta(D_n) \leq \lceil \frac{n+1}{2} \rceil$.
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- If $n = p^a q^b$ where $p \leq q$ are odd primes, then one verifies that $\lceil \frac{n+1}{2} \rceil \leq \phi(n)$ and so we obtain a Černý Cayley graph.

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Symmetric groups

- It is known that the symmetric group S_n has p_n irreducible representations where p_n is the number of partitions of n.
- The sum of the squares of the degrees of the irreps of S_n is $n!$.
- Thus $m(S_n)^2 p_n \geq n!$, i.e., $m(S_n) \geq \sqrt{n!/p_n}$.
- $p_n \sim \frac{\exp\left(\pi\sqrt{2n/3}\right)}{4n\sqrt{3}}$ and $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.
- Therefore, $m(S_n)$ grows extremely quickly as a function of n.
- With Coxeter-Moore generators $(1\ 2), (2\ 3), \ldots, (n-1\ n)$, the diameter is $\binom{n}{2}$ [think "Bubble Sort"] and so we obtain a Černý Cayley graph for n large enough.
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Symmetric groups

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- Thus $m(S_n)^2 p_n \geq n!$, i.e., $m(S_n) \geq \sqrt{n!/p_n}$.
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Special linear groups

- Let p be a prime.
- $SL(2,p) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a,b,c,d \in \mathbb{Z}_p, ad-bc=1 \right\}$.
- A standard generating set Δ for $SL(2,p)$ consists of the matrices
$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$
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- In the group theory literature there is much more detailed information about representations over $\mathbb{C}$ than over $\mathbb{Q}$.
- Schur index theory allows one to use Galois theory in order to understand irreps over $\mathbb{Q}$ in terms of irreps over $\mathbb{C}$.
- Via these methods, we computed $m(SL(2,p)) \geq \max\left\{(p+1)\frac{\phi(p-1)}{2}, (p-1)\frac{\phi(p+1)}{2}\right\}$.
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Theorem (BS)

Let $p \geq 17$ be a prime. Then the Cayley graph of $SL(2,p)$ with respect to the generators $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ *is a Černý Cayley graph.*

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Open questions

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Vielen Dank für Ihre Aufmerksamkeit!