Étale groupoids and inverse semigroups

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The Exel paradigm

- Ruy Exel suggested the following paradigm for the study of "combinatorial C^*-algebras."
  - Start with a combinatorial object (e.g., a graph or a tiling).
  - Construct an inverse semigroup.
  - From the semigroup construct an étale groupoid.
  - From the groupoid construct a C^*-algebra.
- One can hope to use the algebra to understand the combinatorial object (as in the case of tilings);
- Or use the combinatorial object to understand the algebra (as in the case of graphs).

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The dictionary

- It is by now well known that natural notions for étale groupoids translate into natural notions for C^*-algebras.
- For instance, Morita equivalent groupoids have strongly Morita equivalent algebras.
- The (partial action) semidirect product of a group with a space yields a (partial action) cross product of the group with a commutative C^*-algebra.
- The algebra of a groupoid of germs of an inverse semigroup action on a space translates to an inverse semigroup cross product algebra.

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The dictionary II

- It seems to be less well known what happens when passing from an inverse semigroup to a groupoid.
- Does Morita equivalence pass through to the groupoids?
- Do (partial action) semidirect products become (partial action) semidirect products?
- Functoriality seems not to have been addressed.
- It is also not universally agreed upon which groupoid to assign to an inverse semigroup.
- There are Paterson's universal groupoid, Kellendonk's groupoid and Exel's tight groupoid.
- Here we will focus on Paterson's groupoid, as the others are reductions. All of our results so far are valid for any of these groupoids.

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A primer on inverse semigroups

- Inverse semigroups are the abstract counterparts of pseudogroups of partial homeomorphisms of a space.
- If X is a topological space, then I_X is the pseudogroup of all homeomorphisms between open subsets of X.
- Composition is defined where it makes sense: if $f\colon U \longrightarrow V$ and $g\colon U' \longrightarrow V'$, then

$$f \circ g\colon g^{-1}(U \cap V') \longrightarrow f(U \cap V').$$

- Inverse semigroups can also be viewed as $*$-semigroups of partial isometries of a Hilbert space.

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More on inverse semigroups

- Formally, an inverse semigroup is a semigroup S such that for all $s \in S$, there is a unique element s^* such that

$$ss^*s = s, \ s^*ss^* = s^*$$

.

- Munn and Penrose proved that the idempotents of S commute and hence $E(S)$ is a subsemigroup.
- $E(S)$ is ordered by $e \leq f$ iff $ef = e$ and then $ef = e \wedge f$.
- So $E(S)$ is a meet semilattice.
- The order extends to S by putting $s \leq t$ if $s = te$ for some idempotent e.
- For I_X, the ordering is restriction.

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The maximal group image

- Identifying two elements of S with a common lower bound (i.e., the same germ) yields a congruence.
- The quotient is a group $G(S)$ called the maximal group image.
- The canonical projection $\sigma\colon S \longrightarrow G(S)$ is universal to groups.
- Clearly $E(S) \subseteq \sigma^{-1}(1)$.
- S is called E-unitary if $E(S) = \sigma^{-1}(1)$.
- S is called F-inverse if $eSf \cap \sigma^{-1}(g)$ has a maximum element for all $e, f \in E(S)$ and $g \in G(S)$.
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Extreme examples

- If E is a meet semilattice, then E is an inverse semigroup of "projections."
- Meet semilattices are to inverse semigroups as spaces are to étale groupoids.
- A group is an inverse semigroup.
- If a group G acts on a semilattice E, the semidirect product $E \rtimes G$ is an F-inverse semigroup.
- If G acts partially on E, then $E \rtimes G$ is E-unitary and by a theorem of McAlister all E-unitary inverse semigroups arise in this way.

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More interesting examples

- The bicyclic monoid is the inverse semigroup B generated by a unilateral shift and its adjoint.
- Its semilattice of idempotents is $(\mathbb{N}, \geq)$.
- It is E-unitary with maximal group image is $\mathbb{Z}$ (given by the Fredholm index).
- The polycyclic (or Cuntz) inverse semigroup is given by the presentation $P_X = \langle X \mid x^*y = \delta_{x,y} \rangle$ (as an inverse semigroup with 0).
- It is a (congruence) simple inverse semigroup and can be identified with the inverse semigroup generated by the generators of the Cuntz algebra.
- Similarly, there is a graph inverse semigroup associated to any directed graph which is analogous to the graph C^*-algebra.

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The Meakin-Margolis-Munn construction

- Let G be a group with generators X and corresponding Cayley graph Γ.
- $M_A(G)$ consists of all equivalence classes of triples $[g, \Lambda, h]$ where Λ is a finite connected subgraph of Γ and g, h are vertices of Λ.
- Equivalence is up to translation by G.
- The product is given by

$$[g, \Lambda, h][g', \Lambda', h'] = [g, \Lambda \cup h\Lambda', hh'].$$

- $M_A(G)$ is E-unitary with maximal group image G.
- If G is free on A, then $M_A(G)$ is a free inverse monoid on A.
- Free inverse monoids classify graph immersions over a bouquet.

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- Let G be a group with generators X and corresponding Cayley graph Γ.
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Tiling semigroups

- Let $\mathcal{T}$ be a tiling of $\mathbb{R}^d$.
- Kellendonk's tiling semigroup has elements 0 and equivalence classes $[t, P, u]$ where P is a pattern and t, u are tiles of P.
- Equivalence is up to translation in $\mathbb{R}^d$.
- The product $[t, P, u][t', P', u']$ is non-zero iff there are $v, v' \in \mathbb{R}^d$ such that $u + v = t' + v'$ and $(P + v) \cup (P' + v')$ is a pattern.
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Some F-inverse monoids

- Let X be an irreducible algebraic variety and S_X the inverse semigroup of isomorphisms between open subvarieties.
- Then S_X is an F-inverse monoid and the maximal group image is the group of birational automorphisms of X.
- When $X = \mathbb{P}^1$, then S_X is the Möbius inverse semigroup and $G(S_X)$ is the group of linear fractional transformations.
- Let $\{a, b\}^*$ be a free monoid on $\{a, b\}$ and S the inverse semigroup of isomorphisms between finitely generated essential right ideals of $\{a, b\}^*$.
- Then S is an F-inverse monoid with maximal group image Thompson's simple group V.
- If G is a group and S is the inverse semigroup of virtual automorphisms of G, then S is F-inverse and the maximal group image is the abstract commensurator of G.

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F-inverse monoids and partial group actions

- Exel associated to every group G an F-inverse monoid $S(G)$ whose actions correspond to partial actions of G.
- The same semigroup was discovered earlier by Birget and Rhodes in a different context.
- Szendrei had shown $S(G)$ is the universal F-inverse cover of G.
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Strongly 0-E-unitary inverse semigroups

- An inverse semigroups with 0 cannot be E-unitary without being a semilattice.
- So one needs an appropriate variation of the concept for this context.
- If S is an inverse semigroup with 0, the universal group $U(S)$ is the group with generators $S \setminus \{0\}$ and relations given $s \cdot t = st$ if $st \neq 0$.
- The natural map $\sigma\colon S \setminus \{0\} \longrightarrow U(S)$ satisfies $\sigma(s)\sigma(t) = \sigma(st)$ whenever $st \neq 0$ and is universal for this property.
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- The polycyclic inverse monoid (or Cuntz semigroup) P_X is strongly 0-E-unitary with universal group the free group on X.
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- More generally, graph inverse semigroups are strongly 0-E-unitary.
- Tiling inverse semigroups are strongly 0-E-unitary (Lawson/Kellendonk).
- It seems most of the inverse semigroups appearing in the C^*-algebra literature are strongly 0-E-unitary.
- A graph immersion over a bouquet of circles can be extended to a regular covering iff the associated inverse semigroup is strongly 0-E-unitary.
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- It was observed independently by McAlister and by me and Margolis that S is strongly 0-E-unitary iff there is:
  - an E-unitary inverse semigroup T (with maximal group image $U(S)$)
  - and an ideal I such that $S \cong T/I$.
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The groupoid of germs

- An action $S \curvearrowright X$ of an inverse semigroup on a space X is a homomorphism $\theta\colon S \longrightarrow I_X$.
- We always assume that $X = \bigcup_{e \in E(S)} X_e$ where X_e is the domain of e.
- Note $\theta(s)\colon X_{s^*s} \longrightarrow X_{ss^*}$.
- Let us call the action special if X_e is clopen for each $e \in E(S)$.
- The groupoid of germs $S \ltimes X$ has arrow space $(S \times X)/{\sim}$ where $(s,x) \sim (t,y)$ iff $x = y$ and $\exists u \leq s,t$ with $x \in X_{u^*u}$.
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The groupoid of germs

- An action $S \curvearrowright X$ of an inverse semigroup on a space X is a homomorphism $\theta\colon S \longrightarrow I_X$.
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- Note $\theta(s)\colon X_{s^*s} \longrightarrow X_{ss^*}$.
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Paterson's universal groupoid

- Fix an inverse semigroup S with semilattice of idempotents E.
- Let $\widehat{E}$ be the character space of E, that is, the space of non-zero homomorphisms $E \longrightarrow \{0,1\}$ equipped with the topology of pointwise convergence.
- When working in the category of inverse semigroups with 0, we assume characters preserve 0.
- Elements of $\widehat{E}$ can also be identified with filters on E.
- Then $S \curvearrowright \widehat{E}$ by putting $\widehat{E}_e = \{\varphi \mid \varphi(e) = 1\}$ and $s\varphi(f) = \varphi(s^*fs)$ for $\varphi \in \widehat{E}_{s^*s}$.
- This is a special action.
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A result of Khoshkam and Skandalis

- Khoshkam and Skandalis proved the following result.

Theorem

Let S be an E-unitary inverse semigroup such that $eSf \cap \sigma^{-1}(g)$ is finitely generated as a downset for all $e, f \in E(S)$ and $g \in G(S)$. Then there is an action $G(S) \curvearrowright X$ such that $\mathscr{G}(S)$ is Morita equivalent to $G(S) \ltimes X$.

- F-inverse semigroups satisfy the Khoshkam-Skandalis condition since $eSf \cap \sigma^{-1}(g)$ has a maximum.
- The proof uses a generalization of the enveloping action of a partial group action.
- An inverse semigroup S is locally E-unitary if eSe is E-unitary for all $e \in E(S)$.
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Khoshkam-Skandalis from a partial action viewpoint

- There is another way to view the Khoshkam-Skandalis result.

Theorem (Milan, BS)

Let S be an E-unitary inverse semigroup. Then there is a partial action of $G(S)$ on $\widehat{E(S)}$ such that $\mathscr{G}(S) \cong G(S) \ltimes \widehat{E(S)}$.

- Here the partial action semidirect product is as in the sense of Abadie.
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The strongly 0-E-unitary case

- Let S be an inverse semigroup and I an ideal.
- Then $\widehat{E(S/I)}$ can be identified with those characters of $E(S)$ vanishing on I.
- If S is E-unitary, then $\widehat{E(S/I)}$ is invariant under the partial action of $G(S)$.

Theorem (Milan, BS)

If S is strongly 0-E-unitary with universal group $U(S)$, then $\mathscr{G}(S) \cong U(S) \ltimes \widehat{E}(S)$ for an appropriate partial action of $U(S)$.

- A similar result holds for Exel's tight groupoid and explains a number of partial action cross product results in the literature.
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Beyond groups

- The Khoshkam-Skandalis result gives a condition that guarantees $C^*(S)$ is strongly Morita equivalent to $C_0(X) \rtimes G(S)$ for some action $G(S) \curvearrowright X$.
- What are conditions on a homomorphism $S \xrightarrow{\varphi} T$ that guarantee that $C^*(S)$ is strongly Morita equivalent to a cross product $C_0(X) \rtimes T$?
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Functoriality

- If $E \xrightarrow{\varphi} F$ is a semilattice homomorphism, then there is a natural cts map $\widehat{F} \longrightarrow \widehat{E}$ given by "pulling back."
- But we want a map the other way — and there is one.
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Functoriality II

- The Alexandrov topology on a poset takes the downsets as the open sets. (It is T_0.)
- The quasi-compact open sets are the finitely generated downsets.
- A map of spaces is coherent if the inverse image of a quasi-compact open is quasi-compact open.
- It is a folklore fact from the theory of locales that φ_* is cts iff φ is locally coherent.

Theorem (Milan,BS)

The universal groupoid is functorial on the category of inverse semigroups with maps that are locally coherent on the semilattices of idempotents.

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- Khoshkam and Skandalis proved their result along the following lines.
- First they gave a general condition on a functor $\mathscr{G} \longrightarrow G$ from a groupoid to a group (a cocycle) that guarantees $\mathscr{G}$ is Morita equivalent to $G \ltimes X$ for an appropriate action $G \curvearrowright X$.
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Let $\varphi\colon \mathscr{G} \longrightarrow \mathscr{H}$ be a cts functor with $\mathscr{G}$ locally compact and $\mathscr{H}$ étale. Suppose that:

- *the map $\mathscr{G} \longrightarrow \mathscr{G}^0 \times_{\mathscr{H}^0} \mathscr{H} \times_{\mathscr{H}^0} \mathscr{G}^0$ given by*

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Groupoids of germs as semidirect products

- To apply the previous situation, we need to identify groupoids of germs as semidirect products.

Theorem (Milan, BS)

1. *There is a bijection between special actions $T \curvearrowright X$ and actions $\mathscr{G}(T) \curvearrowright X$.*
2. *Moreover, the groupoid of germs $T \ltimes X$ is isomorphic to the semidirect product $\mathscr{G}(T) \ltimes X$.*

- For instance, if $\mathscr{G}(T)$ acts on X with $X \xrightarrow{p} \widehat{E(T)}$, then we put $X_e = p^{-1}(\widehat{E}_e)$.
- For $x \in X_{t^*t}$, we define $tx = [t, p(x)]x$.
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The Khoshkam-Skandalis condition: revisited

- To prove our main result, one verifies that if $S \xrightarrow{\varphi} T$ is a locally idempotent-pure morphism satisfying the KS condition, then the induced map $\mathscr{G}(S) \longrightarrow \mathscr{G}(T)$ satisfies the conditions of our Morita theorem.
- Thus $\mathscr{G}(S)$ is Morita equivalent to $\mathscr{G}(T) \ltimes X \cong T \ltimes X$.
- Therefore, $C^*(S)$ is strongly Morita equivalent to $C_0(X) \rtimes T$.

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Morita equivalence

- Let S be an inverse semigroup.
- Let S-$\mathbf{Set}$ be the category of sets X equipped with an action $S \times X \longrightarrow X$ by total functions such that $SX = X$.
- Talwar defined two inverse semigroups S and T to be Morita equivalent if S-$\mathbf{Set}$ is equivalent to T-$\mathbf{Set}$.
- For example, if S is an inverse monoid, then S and T are Morita equivalent iff there is an idempotent e with $T = TeT$ and $S \cong eTe$.
- An inverse semigroup is F-inverse iff it is Morita equivalent to a semidirect product of a group and a semilattice.
- Lawson calls S an enlargement of a subsemigroup T if $S = STS$ and $T = TST$.
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Strong Morita equivalence

- I defined inverse semigroups S and T to be strongly Morita equivalent if there is an equivalence bimodule for S and T.
- By definition, this consists of a set X, which is an (S,T)-biset equipped with surjective "inner products"

$$\langle -,-\rangle\colon\ X\times X\longrightarrow S\ ,\ \text{and}\ [-,-]\colon\ X\times X\longrightarrow T$$

  such that the following axioms hold, where $x,y,z\in X$, $s\in S$, and $t\in T$:
  - $\langle sx,y\rangle = s\langle x,y\rangle$ $\qquad [x,yt]=[x,y]t$
  - $\langle y,x\rangle = \langle x,y\rangle^*$ $\qquad [x,y]=[y,x]^*$
  - $\langle x,x\rangle x = x$ $\qquad x[x,x]=x$
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Strong Morita equivalence and groupoids

Theorem (BS)

If S and T are strongly Morita equivalent, then $\mathscr{G}(S)$ and $\mathscr{G}(T)$ are Morita equivalent.

- Every F-inverse semigroup S has an enlargement $E \rtimes G(S)$.
- Thus $C^*(S)$ is strongly Morita equivalent to $C_0(\widehat{E}) \rtimes G(S)$.
- This yields an alternative proof to that of Khoshkam and Skandalis.
- It is easy to see that strong Morita equivalence implies Morita equivalence.
- I had conjectured the converse is false.

Theorem (Funk, Lawson, BS)

Morita equivalence and strong Morita equivalence are the same.

- The proof uses topos theoretic ideas.

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Thanks for your attention!
Je vous remercie de votre attention!
Obrigado pela sua atenção!