The Karoubi envelope and the classification of Markov-Dyck shifts

Alfredo Costa, Universidade de Coimbra
Benjamin Steinberg, City College of New York

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Outline

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- We will show how it classifies Markov-Dyck shifts up to flow equivalence under mild hypotheses on the graphs.
- I will sketch a simple proof of our main result using results of Nasu.

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- This is analogous to von Neumann-Murray equivalence of projections.

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- You should think of eS as a projective indecomposable.
- Hence if $\mathbb{K}(S)$ is equivalent to $\mathbb{K}(T)$, we should think of S and T as Morita equivalent in some sense.

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- From the construction, $\mathbb{K}(S) = \mathbb{K}(LU(S))$.

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- Lawson first explicitly proved S is Morita equivalent to T iff $\mathbb{K}(S)$ is equivalent to $\mathbb{K}(T)$.
- If S, T are any semigroups, we say S and T are Morita equivalent up to local units if $\mathbb{K}(S)$ is equivalent to $\mathbb{K}(T)$.

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- They have isomorphic lattices of ideals.
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- Our syntactic invariants of shifts mostly come from Morita invariant properties of semigroups with local units.

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- It is a nonempty, factorial, prolongable language and all such languages come from shifts.
- Many syntactic invariants of $\mathscr{X}$ have been constructed from $L(\mathscr{X})$ via its syntactic semigroup.

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- Convention: We put $S(A^{\mathbb{Z}}) = S(A^+) \cup \{0\}$.

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- These last three properties are all Morita invariants up to local units.

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- In other words, the Morita equivalence class of $S(\mathscr{X})$ (up to local units) is a flow equivalence invariant of $\mathscr{X}$.
- Alfredo will explain in his talk how a number of syntactic invariants of shifts in the literature can be deduced from this result.

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- Each nonzero element of P_G can be uniquely written $s_p s_q^*$.

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Theorem (BS)

Morita equivalent inverse semigroups have Morita equivalent universal, reduced and tight C^*-algebras.

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- Let $D_G \subseteq (\Sigma_G)^{\mathbb{Z}}$ be the corresponding subshift.

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- Dyck shifts were classified by Matsumoto up to flow equivalence.

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- It follows that if S and T have zeroes, then $\mathbb{K}(S)$ is equivalent to $\mathbb{K}(T)$ iff their proper parts are equivalent.

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- Two free categories are equivalent iff the graphs are isomorphic.
- So $\mathbb{K}(P_G)$ equivalent to $\mathbb{K}(P_H)$ implies $G \cong H$.

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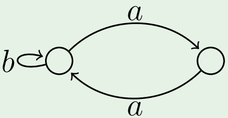

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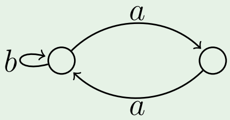

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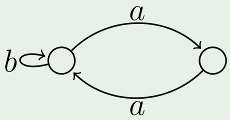

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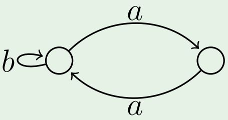

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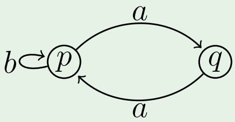

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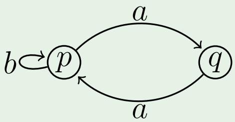

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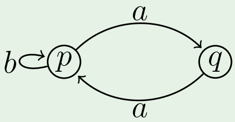

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- The semigroup $S(\mathcal{A}) = \langle \{ f_a \mid a \in A \} \rangle$ is called the transition semigroup of $\mathcal{A}$.

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- If $\mathscr{X}$ is flow equivalent to $\mathscr{Y}$, then this functor is 'remembered' by the equivalence of $\mathbb{K}(\mathscr{X})$ with $\mathbb{K}(\mathscr{Y})$ (see Alfredo's talk).

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- We call $\mathcal{A}_1, \mathcal{A}_2$ the components of $\mathcal{A}$.

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- If so, the following is a simpler proof of our main result.

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Theorem (Lawson)

Two semigroups with local units are Morita equivalent iff they have a common enlargement.

Completion of the proof sketch

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- The equivalence takes the $LU(S_1)$-set $Q_1 \otimes LU(S_1)$ to the $LU(S_2)$-set $Q_2 \otimes LU(S_2)$.

The end

Thank you for your attention!