Étale groupoid algebras

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Groups, Rings and Group Rings

Outline

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Representation Theory

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- Groupoids are espoused by Connes as noncommutative models of spaces.

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- Over a field, these are precisely the idempotent-generated commutative algebras.
- Surprising similarities between operator algebras and their algebraic analogues have been known for some time.

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- Groupoid algebras over $\mathbb{C}$ were rediscovered later by L. O. Clark, C. Farthing, A. Sims and M. Tomforde, who have kindly dubbed them "Steinberg algebras."
- Many of my hopes have since been borne out by J. Brown, L. O. Clark, C. Farthing, A. Sims and M. Tomforde, who seem to produce new results faster than I can keep up with.

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- Often $\mathcal{G}^{(0)}$ is called the unit space of $\mathcal{G}$.

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- Renault calls an étale groupoid $\mathcal{G}$ ample if $\mathcal{G}^{(0)}$ has a basis of compact open sets.
- Many important C^*-algebras come from ample groupoids: group algebras, Cuntz-Krieger algebras and inverse semigroup algebras.

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- In fact, discrete groups and groupoids are ample.

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- This guarantees X has enough continuous maps into any non-trivial discrete ring to separate points.

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- Inversion is
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- $(x \xrightarrow{m-n} y)^{-1} = y \xrightarrow{n-m} x$.
- This ample groupoid gives rise to Cuntz-Krieger C^*-algebras and Leavitt path algebras.

Inverse semigroups

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- If G is a group acting on a semilattice E, then the semidirect product $E \rtimes G$ is an inverse semigroup.

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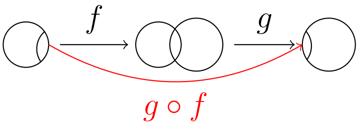

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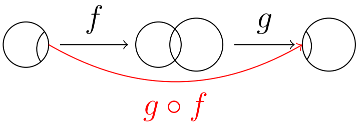

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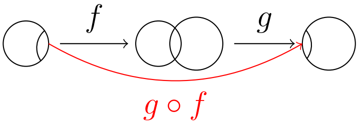

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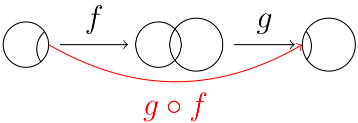

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- Inverse semigroups abstract pseudogroups of partial homeomorphisms from differential geometry.

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- If $s \in S$ and $e \in E(S)$, then $\theta(se)$ is the restriction of $\theta(s)$ to $\operatorname{dom}(\theta(e))$.

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- It is ample if X has a basis of compact open sets and the domains of elements of S are compact open.

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Warning

☡ This definition needs adjustment for the non-Hausdorff case.

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Let S be an inverse semigroup and $\Bbbk$ a comm. ring with 1. Then $\Bbbk S \cong \Bbbk\mathcal{G}(S)$.

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- $C^*(\mathcal{G})$ is the completion of $\mathbb{C}\mathcal{G}$ (Stone-Weierstrass).

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The class functions form the center of $\Bbbk\mathcal{G}$.

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Let $\mathcal{G}$ be an effective, Hausdorff ample groupoid with a dense orbit. Then

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- This generalizes an earlier result of L. O. Clark and C. Edie-Michelle for the minimal case.

Simplicity

Theorem (L. O. Clark, C. Edie-Michelle)

Let $\mathcal{G}$ be a Hausdorff ample groupoid and $\Bbbk$ a field. Then $\Bbbk\mathcal{G}$ is simple if and only if $\mathcal{G}$ is effective and minimal.

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- There is a non-Hausdorff version.

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Proof.

Apply Amitsur '59 with $X = \mathcal{G}^{(0)}$. $\square$

Applications to inverse semigroup algebras

Corollary

Let $\Bbbk$ be a semiprimitive comm. ring with 1 and S an inverse semigroup. If $\Bbbk G_e$ is semiprimitive for every maximal subgroup G_e of S, then $\Bbbk S$ is semiprimitve. The converse holds if $E(S)$ is pseudofinite.

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Let $\mathbb{k}$ be a semiprimitive comm. ring with 1 and S an inverse semigroup. If $\mathbb{k}G_e$ is semiprimitive for every maximal subgroup G_e of S, then $\mathbb{k}S$ is semiprimitve. The converse holds if $E(S)$ is pseudofinite.

- Sufficiency was proved by Domanov (1976).
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- This is a topological reformulation of the original definition.

Unitary modules

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- R and S are Morita equivalent if $R\text{-mod}$ is equivalent to $S\text{-mod}$.
- Morita equivalence can also be formulated in terms of Morita contexts.

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Theorem (BS)

$\Bbbk\mathcal{G}\text{-mod}$ *is equivalent to the category of* $\mathcal{G}$*-sheaves of* $\Bbbk$*-modules.*

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- If Z is a locally compact space and $f\colon Z \to \mathcal{G}^{(0)}$ is continuous, there is a pullback groupoid $\mathcal{G}[Z]$.

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- Groupoids $\mathcal{G}$ and $\mathcal{H}$ are Morita equivalent if there is a locally compact space Z and continuous open surjections $p\colon Z \to \mathcal{G}^{(0)}$ and $q\colon Z \to \mathcal{H}^{(0)}$ such that $\mathcal{G}[Z] \cong \mathcal{H}[Z]$.

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- Gives a uniform explanation for Morita equivalences of graph C^*-algebras and Leavitt path algebras.
- The original proof constructed a Morita context.
- A result of Moerdijk implies if $\mathcal{G}, \mathcal{H}$ are Morita equivalent, then so are their categories of sheaves.

Schützenberger representations

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- If $f \in \Bbbk\mathcal{G}$ and $t \in L_x$, then

$$f \cdot t = \sum_{d(s)=r(t)} f(s)st.$$

Induction and restriction functors

- There is an exact functor $\mathrm{Ind}_x\colon \Bbbk G_x\text{-mod} \to \Bbbk\mathcal{G}\text{-mod}$ given by
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Theorem (BS)

Let $\Bbbk$ be a field and $\mathcal{G}$ an ample groupoid. Then the finite dimensional simple $\Bbbk\mathcal{G}$-modules are the $\mathrm{Ind}_x(M)$ with $|\mathcal{O}_x| < \infty$ and M a finite dimensional simple $\Bbbk G_x$-module.

A question of Exel

Question (Exel)

Let $\Bbbk$ be a field and $\mathcal{G}$ a Hausdorff ample groupoid. Is every primitive ideal of $\Bbbk\mathcal{G}$ a kernel of an induced module $\mathrm{Ind}_x(M)$ with M a simple $\Bbbk G_x$-module?

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- A C^*-algebraic analogue is true if $\mathcal{G}$ is amenable.
- It is true for Leavitt path algebras.

Further work

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The end

Thank you for your attention!