Recent progress on the structure of free profinite monoids

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Free profinite monoids

- Equational descriptions of classes of regular languages rely on the expressive power of the free profinite monoid $\widehat{A^*}$. (Throughout this talk A is a finite alphabet.)
- Over the last ten years we have begun to obtain a much clearer picture of the structure of this object.
- My goal here is to touch on the following topics:
  - Free clopen submonoids;
  - Ideal structure;
  - Maximal subgroups;
  - Finite subsemigroups.

The profinite completion of the free monoid

- For $u \neq v \in A^*$, define $s(u,v)$ to be the minimum size of a finite monoid separating u from v. Put $s(u,u) = \infty$.
- A^* is residually finite, so $s(u,v)$ is well defined.
- The profinite ultrametric on A^* is defined by

$$d(u,v) = 2^{-s(u,v)}.$$

- The completion is the free profinite monoid $\widehat{A^*}$.
- It is a compact, totally disconnected (i.e., profinite) monoid.
- Elements of $\widehat{A^*}$ are called *profinite words*.

Stone duality and the free profinite monoid

- $\mathrm{Reg}(A^*)$ is a boolean ring with the operations of symmetric difference and intersection as addition and multiplication.
- There is a natural comultiplication $\Delta\colon \mathrm{Reg}(A^*) \to \mathrm{Reg}(A^*) \otimes_{\mathbb{F}_2} \mathrm{Reg}(A^*)$ given by

$$\Delta(L) = \sum_{ab \in \eta_L(L)} \eta_L^{-1}(a) \otimes \eta_L^{-1}(b)$$

where $\eta_L\colon A^* \to M_L$ is the syntactic morphism.
- There is a counit $\lambda\colon \mathrm{Reg}(A^*) \to \mathbb{F}_2$ given by

$$\lambda(L) = \begin{cases} 1 & \varepsilon \in L \\ 0 & \text{else.} \end{cases}$$

- So $\mathrm{Reg}(A^*)$ is a bialgebra and hence its Zariski spectrum $\mathrm{Spec}(\mathrm{Reg}(A^*))$ is a profinite monoid by Stone duality.

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Stone duality and the free profinite monoid II

Theorem (Almeida;Rhodes-BS)

$\widehat{A^*} \cong \mathrm{Spec}(\mathrm{Reg}(A^*))$.

- The isomorphism at the level of topological spaces is due to Almeida; the algebraic part to us.
- As a consequence of Almeida's part, clopen subsets of $\widehat{A^*}$ are in bijection with regular languages.
- $L \in \mathrm{Reg}(A^*)$ corresponds to $\overline{L} \subseteq \widehat{A^*}$.
- Conversely, if $K \subseteq \widehat{A^*}$ is clopen, then $K \cap A^*$ is regular.
- In particular, clopen submonoids of $\widehat{A^*}$ are in bijection with regular submonoids of A^*.
- In summary, $\widehat{A^*}$ is the geometric object corresponding to $\mathrm{Reg}(A^*)$.

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Stone duality and varieties of languages

- Reg is a contravariant functor from the category of free monoids to the category of (boolean) bialgebras.
- A variety of languages is precisely a subfunctor of Reg.
- Stone duality says that the Zariski spectrum functor gives a duality between the categories of boolean bialgebras and profinite monoids.
- If $\mathscr{V}$ is a variety of languages (viewed as a functor), then the composition $A^* \mapsto \mathrm{Spec}(\mathscr{V}(A^*))$ produces the free pro-$\mathbf{V}$ monoid on A, where $\mathbf{V}$ is the pseudovariety of monoids corresponding to $\mathscr{V}$.
- Duality eases proofs: every finite image of $\varprojlim T_i$ factors through a T_i dualizes to the trivial statement every finite subbialgebra of $\varinjlim B_i$ factors through a B_i.

Free clopen submonoids

- It is known that clopen subgroups of free profinite groups are free.
- Clopen submonoids of $\widehat{A^*}$ need not be free: e.g. $\overline{\{x^2, x^3\}^*}$.
- Almeida asked in his book: does a free profinite monoid on n generators embed as a closed submonoid of a free profinite monoid on 2 generators.
- Koryakov showed in 1995 the code $C_n = \{y, xy, \ldots, x^{n-1}y\}$ freely generates a free clopen submonoid of $\widehat{\{x, y\}^*}$ of rank n.

Theorem (Margolis,Sapir,Weil 98)

Any finite code $C \subseteq A^*$ freely generates a free clopen profinite submonoid of $\widehat{A^*}$.

- Recall: $C \subseteq A^*$ is a *code* if C freely generates C^*.

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Free clopen submonoids II

- If one takes an infinite regular code, like x^*y, then it generates a clopen submonoid of $\widehat{A^*}$.
- Does it freely generate a free profinite monoid?
- No! The free profinite monoid on a discrete set X contains its Stone-Czech compactification βX.
- βX is not metrizable if X is infinite, but $\widehat{A^*}$ is metrizable when A is finite. So $\widehat{A^*}$ does not contain a free profinite monoid on an infinite set.
- If X is a topological space, the free profinite monoid $\widehat{X^*}$ on X is defined via the usual universal property:

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & \widehat{X^*} \arrow[d, dotted] \\ & M \end{tikzcd} \qquad \text{(profinite)}$$

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Free clopen submonoids III

- This topological obstruction is the only obstacle to generalizing the result of Margolis, Sapir and Weil.

Theorem (Almeida, BS)

The free clopen submonoids of $\widehat{A^*}$ are precisely the closures of regular free submonoids of A^*. Moreover, if C is a regular code, then $\overline{C}$ is the unique closed (and in fact clopen) basis for $\overline{C^*}$.

- The proof uses unambiguous automata and wreath products.
- It is in the same spirit as the case of finite codes, but the topology makes the proof more technical.

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The ideal structure of free profinite monoids

- Henckell, Rhodes and I — and independently Almeida and Costa — observed that overlap lemmas for free monoids also work to a large extent for free profinite monoids.
- This led me to the Prime Ideal Theorem.
- An ideal I in a semigroup is called *prime* if $ab \in I$ implies $a \in I$ or $b \in I$.
- An ideal I in a semigroup is *idempotent* if $I^2 = I$.
- For example, an element x generates an idempotent ideal if and only if x is regular.
- The minimal ideal of a profinite semigroup is an idempotent ideal.

The ideal structure of free profinite monoids II

Theorem (Prime Ideal Theorem, BS)

Every idempotent ideal of $\widehat{A^*}$ is prime.

- The case of the minimal ideal had been obtained earlier by Almeida and Volkov using symbolic dynamics and entropy.
- This theorem admits a number of important consequences.

Corollary

Suppose $x \in \widehat{A^*}$ and x^n is a group element for some $n \geq 1$. Then x is a group element. In particular, all elements of finite order in $\widehat{A^*}$ are group elements.

- The case of finite order was obtained earlier by me and Rhodes.

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The ideal structure of free profinite monoids III

Corollary

If $B \subseteq \widehat{A^*}$ is a band, then the principal ideals of B form a chain.

Proof.

- Let $e, f \in B$. Then $ef \in B$ and so idempotent.
- Hence ef generates a prime ideal so $e \mathscr{J} ef$ or $f \mathscr{J} ef$.
- Then $e \mathscr{R} ef$ or $f \mathscr{L} ef$.
- Since these are idempotents, this holds in B.
- Thus e, f are comparable in the $\mathscr{J}$-order on B.

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$\square$

Free and projective profinite groups

- The free profinite group on a topological space is defined in the same way as for monoids.
- A profinite group G is called *projective* if:

$$\begin{tikzcd} & G \arrow[dl, dotted, "\lambda"'] \arrow[d, two heads, "\beta"] \\ A \arrow[r, two heads, "\alpha"] & B \end{tikzcd}$$

- It is enough to consider the case A, B are finite.

Theorem

The projective profinite groups are precisely the closed subgroups of free profinite groups.

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Maximal subgroups of $\widehat{A^*}$

- If $e \in \widehat{A^*}$ is an idempotent, G_e denotes the maximal subgroup at e.
- Let $\widehat{F}_A$ be a free profinite group on A.
- There is a natural surjective map $\varphi\colon \widehat{A^*} \to \widehat{F}_A$.
- If e is an idempotent of the minimal ideal I, then $\varphi(G_e) = \widehat{F}_A$.
- So φ splits and hence all projective profinite groups can embed in a free profinite monoid (observation of Almeida and Volkov).
- Margolis and I observed that the maximal subgroup of the minimal ideal maps onto any metrizable profinite group.

A question of Margolis

Question (Margolis 97)

1. *Is every maximal subgroup of $\widehat{A^*}$ a free profinite group, or at least projective?*
2. *Is the maximal subgroup of the minimal ideal of $\widehat{A^*}$ a free profinite group?*

- Free profinite groups (and hence projective profinite groups) are torsion-free.
- Is $\widehat{A^*}$ torsion-free? That is, are all elements of finite order in $\widehat{A^*}$ idempotent?
- We saw earlier that all finite order elements of $\widehat{A^*}$ are group elements.
- So if every maximal subgroup of $\widehat{A^*}$ is projective, then all elements of finite order must be idempotents.

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Symbolic dynamics

- Almeida and his co-workers were the first to make progress on these questions. Their approach used symbolic dynamics.
- The *shift* map $\sigma\colon A^\omega \to A^\omega$ is given by

$$\sigma(a_0a_1\cdots) = a_1a_2\cdots .$$

- A *subshift* is a closed subspace of A^ω closed under the shift.
- A minimal subshift must be the closure of the orbit of an infinite word under the shift.
- A word $w \in A^\omega$ generates a minimal subshift if and only if w is *uniformly recurrent*.
- This means that if v is a finite factor of w, then there exists $N > 0$ so that each factor of w of length N contains v as a factor: the "bounded gaps property."

Uniform recurrence and substitutions

- The famous Morse-Thue cube-free word is uniformly recurrent. It is the fixed point obtained by iterating the substitution $a \mapsto ab, b \mapsto ba$ starting from a.
- A substitution $f\colon A^* \to A^*$ is called *primitive* if there exists $N > 0$ so that each letter of A appears in $f^N(a)$, all $a \in A$.
- If f is a primitive substitution with a the first letter of $f(a)$, then $\lim f^n(a)$ is a uniformly recurrent word.
- Set $\partial\widehat{A^*} = \widehat{A^*} \setminus A^*$.
- There is a natural continuous surjection $\pi\colon \partial\widehat{A^*} \to A^\omega$ since regular languages can "remember" prefixes.

Minimal subshifts and maximal principal ideals

- Almeida defined a profinite word w to be *uniformly recurrent* if given a finite factor v of w, there exists $N > 0$ so that every factor of w of length N contains v.

Theorem (Almeida)

1. *$w \in \widehat{A^*}$ is uniformly recurrent iff $\widehat{A^*} w \widehat{A^*}$ is a maximal principal ideal of $\partial\widehat{A^*}$.*
2. *$\pi \colon \partial\widehat{A^*} \to A^\omega$ sends uniformly recurrent profinite words onto uniformly recurrent infinite words.*
3. *π induces a bijection between minimal subshifts and maximal principal ideals of $\partial\widehat{A^*}$.*

- So there is a principal ideal associated to each minimal subshift.

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The profinite group associated to a minimal subshift

- The ideal associated to a minimal subshift can be generated by an idempotent.
- Thus there is a unique (up to isomorphism) maximal subgroup generating this ideal.
- In other words, there is a profinite group associated to each minimal subshift.
- Almeida showed that this group is a conjugacy invariant of the subshift.
- What can be said about these maximal subgroups of $\widehat{A^*}$?

The profinite group associated to a minimal subshift II

- Almeida showed that the groups corresponding to minimal subshifts arising from certain primitive substitutions are free profinite.
- For instance the groups associated to Sturmian and Arnoux-Rauzy subshifts are free profinite groups.
- Almeida showed if f is the substitution $a \mapsto a^3b, b \mapsto ab$, then the group associated to $\lim f^n(a)$ is projective but not free.
- Almeida presented this work at the Fields workshop on profinite groups organized by me and Ribes in 2005.
- Lubotzky asked after Almeida's talk whether these groups must always be projective.
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The projectivity theorem

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The closed subgroups of $\widehat{A^*}$ are precisely the projective profinite groups. Hence $\widehat{A^*}$ is torsion-free.

- The proof uses wreath products and the Schützenberger representation in order to extend maps from a maximal subgroup to the whole free profinite monoid.
- Ribes later pointed out to us a similar proof scheme used by Cossey, Kegel and Kovács for the case of free profinite groups.
- Ribes and I have used the same ideas to give simple algebraic proofs of the Nielsen-Schreier and Kurosh Theorems via wreath products (in both the abstract and profinite settings).

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- Ribes and I have used the same ideas to give simple algebraic proofs of the Nielsen-Schreier and Kurosh Theorems via wreath products (in both the abstract and profinite settings).

The projectivity theorem

Theorem (Rhodes, BS)

The closed subgroups of $\widehat{A^*}$ are precisely the projective profinite groups. Hence $\widehat{A^*}$ is torsion-free.

- The proof uses wreath products and the Schützenberger representation in order to extend maps from a maximal subgroup to the whole free profinite monoid.
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Finite subsemigroups of $\widehat{A^*}$

- It follows that every finite subsemigroup B of $\widehat{A^*}$ is a band which is a $\mathscr{J}$-chain.
- If B has a zero, then it is a chain of idempotents.
- If the minimal ideal of B is a left (right) zero semigroup, then it is an $\mathscr{L}$-chain ($\mathscr{R}$-chain).
- Rectangular bands of arbitrary size embed in $\widehat{A^*}$.
- Is it decidable which bands embed in $\widehat{A^*}$?

Free profinite groups on a set converging to 1

- A subset Y of a profinite group G is a *set of generators converging to 1* if:
  - $\overline{\langle Y \rangle} = G$;
  - Each neighborhood of 1 contains all but finitely many elements of Y.
- One can define a free profinite group $\widehat{F}_Y$ on a set Y of generators converging to 1. The cardinality of Y is called the *rank* of $\widehat{F}_Y$.
- A free profinite group on a topological space X is also free on a set of generators converging to 1 of the same cardinality as the boolean algebra of clopen subsets of X.

The maximal subgroup of the minimal ideal is free

Theorem (BS)

The maximal subgroup of the minimal ideal of $\widehat{A^*}$ is a free profinite group of countable rank.

- The proof uses Iwasawa's criterion: a metrizable profinite group G is free profinite on a countable set of generators converging to 1 if and only if given a diagram

$$\begin{tikzcd} & G \arrow[dl, dotted, two heads, "\lambda"'] \arrow[d, "\beta"] \\ A \arrow[r, two heads, "\alpha"] & B \end{tikzcd}$$

of epimorphisms (A and B are finite), there exists an epimorphism $\lambda\colon G \twoheadrightarrow A$ so that the diagram commutes.

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The maximal subgroup of the minimal ideal is free II

- Again wreath products play a role: this time iterated wreath products.
- The idea is based on Bernhard Neumann's proof that every countable semigroup embeds in a 2-generated semigroup, and variations on this theme.
- The most relevant variant for us embeds any countable group as the maximal subgroup of the minimal ideal of a 2-generated monoid with cyclic group of units.
- Ideas from Krohn-Rhodes Theory and the Synthesis Theorem also play a role.

More on the minimal ideal

- If I is the minimal ideal of $\widehat{A^*}$ and $E(I)$ is its set of idempotents, then $E(I)$ is a profinite space.
- There is a continuous retraction $\pi\colon I \longrightarrow E(I)$ so that each fiber of π is the maximal subgroup G of I.
- That is to say, I is a principal G-bundle with base space $E(I)$.
- So our results go a long way towards understanding the structure of I.

Another free profinite subgroup

- Recall that we have a canonical projection $\varphi\colon \widehat{A^*} \twoheadrightarrow \widehat{F}_A$ where $\widehat{F}_A$ is the free profinite group generated by A.
- Moreover, φ restricts to an epimorphism $\varphi\colon G \twoheadrightarrow \widehat{F}_A$ where G is the maximal subgroup of the minimal ideal I.
- Let $K = \ker \varphi$.

Theorem (BS)

The subgroup K is a free profinite group of countable rank.

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- One just needs to show that every finite group is an image of K.

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Open questions

- Which projective profinite groups can be maximal subgroups of a free profinite monoid (Zalesskii)?
  - Can a free pro-p group be a maximal subgroup of $\widehat{A^*}$?
- Let I be the minimal ideal of $\widehat{A^*}$ and let $S = \overline{\langle E(I) \rangle}$ be the closed subsemigroup generated by its idempotents.
  - Is the maximal subgroup H of S a free profinite group of countable rank?
  - The subgroup K is the normal closure of H.
  - H maps onto every countably based profinite group.
  - We think that our proof should show that H is free.
- Classify projective profinite monoids.
  - Do the finite projective monoids form a recursive class?

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