Poset cohomology, Leray numbers and the global dimension of left regular bands

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Outline

History

- Over the past 15 years, it has become apparent that a number of combinatorial objects have the structure of a type of semigroup called a left regular band (LRB).
- Examples include:
  - the faces of a real or complex hyperplane arrangement
  - oriented matroids
  - matroids
  - interval greedoids
  - various gadgets associated to graphs.
- The character theory of these semigroups has been successfully used to analyze random walks on these objects.
- An incomplete list of authors who have used this approach is: Bidigare, Hanlon and Rockmore, Brown and Diaconis, Brown, Björner, Athanasiadis and Diaconis, and Chung and Graham.

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Solomon's descent algebra

- There is also an important connection, due to Bidigare, between Solomon's descent algebra and LRBs.
- The descent algebra is a subalgebra of the group algebra of a Coxeter group W.
- It plays the role of a non-commutative analogue of the character ring.
- Bidigare observed that it is the algebra of W-invariants of the algebra of the face LRB of the Coxeter arrangement $\mathscr{H}_W$ associated to W.
- The face LRB of $\mathscr{H}_W$ plays a prominent role in the second edition of Brown's book Buildings and also in the books of Aguiar and Mahajan.
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Tsetlin library

- Imagine you have a bookshelf containing books with the exciting titles $1, \ldots, n$.
- You have no time to organize your books, so you use the following self-organizing system:
  - Each time you remove a book, you replace it at the front of the shelf.
- As time goes on, you expect your favorite books to be located near the front of the shelf, whereas the books you never use will all be toward the back of the shelf.
- This is called the Tsetlin library and was considered by computer scientists in the context of lists.
- The story of LRBs in combinatorics began with this Markov chain.

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Tsetlin library: a Markov chain model

- The states of the Markov chain are the total orderings of $[n] = \{1, \ldots, \mathsf{n}\}$.
- With probability p_i we remove the book entitled i and place it at the front.
- E.g. if $n = 5$, we may be in state 32154 and pull out book 5. The resulting state is 53214.
- One wants to compute stationary distributions, rates of convergence, eigenvalues, diagonalizability, etc.
- Bidigare, Hanlon and Rockmore analyzed this as a hyperplane chamber walk.
- Ken Brown observed that it is more easily modelled as a walk on the free LRB.

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The Tsetlin library: a free left regular band walk

- An LRB is a semigroup satisfying the identities:
  - $x^2 = x$
  - $xyx = xy$
- Informally, these identities say to ignore repetitions.
- We consider only monoids here.
- The free LRB $F(A)$ on a set A consists of all injective words over A.
- Define a product by concatenating and then removing repetitions. E.g., if $A = [5]$, one has $5 \cdot 32154 =$
- Identifying injective words with full support and total orderings, we have:
  - The total orders form the unique minimal left ideal.
  - Left multiplication by generators 'implements' the Tsetlin library.

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A walk of Athanasiadis and Diaconis

- Let $\Gamma = (V, E)$ be a (simple) graph.
- The state space of our walk is all acyclic orientations of Γ.
- Transitions involve randomly choosing a vertex v of Γ and reorienting all edges incident on v away from it.
- For example if we have the acyclically oriented graph

$$\begin{tikzcd} 1 & 2 \arrow[l] \\ 3 \arrow[u] & 4 \arrow[l] \arrow[u] \end{tikzcd}$$

and choose vertex 1, we obtain the orientation

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A walk of Athanasiadis and Diaconis II

- When Γ is complete, then acyclic orientations are in bijection with total orderings of the vertices.
- Edges are directed from smaller to larger elements.
- The walk we obtain is then the Tsetlin library.
- Athanasiadis and Diaconis analyzed this as a walk on the graphic hyperplane arrangement.
- I independently discovered this walk (unpublished) as a walk on a free partially commutative (pc) LRB.
- Just as the free LRB is the 'natural' model of the Tsetlin library, free pc LRBs are the natural models of the Athanasiadis/Diaconis walk.

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- When Γ is complete, then acyclic orientations are in bijection with total orderings of the vertices.
- Edges are directed from smaller to larger elements.
- The walk we obtain is then the Tsetlin library.
- Athanasiadis and Diaconis analyzed this as a walk on the graphic hyperplane arrangement.
- I independently discovered this walk (unpublished) as a walk on a free partially commutative (pc) LRB.
- Just as the free LRB is the 'natural' model of the Tsetlin library, free pc LRBs are the natural models of the Athanasiadis/Diaconis walk.

Free partially commutative left regular bands

- Let $\Gamma = (V, E)$ be a graph.
- The free pc LRB $F(\Gamma)$ has presentation
$$\langle V \mid xy = yx, \forall \{x, y\} \in E\rangle$$
as an LRB.
- If $E = \emptyset$, $F(\Gamma) = F(V)$ (the free LRB).
- If Γ is complete, $F(\Gamma) \cong (P(V), \cup)$ (the free commutative LRB).
- The general case is an interpolation.
- For example, $F(C_4) \cong F([2]) \times F([2])$.
- In general, $F(\Gamma * \Omega) = F(\Gamma) \times F(\Omega)$.

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A brief history of partial commutativity

- Free partially commutative monoids were introduced by Cartier and Foata in 1969 to give a conceptual approach to proving combinatorial identities.
- Today they are studied by computer scientists under the name trace monoids as models of concurrent systems.
- The corresponding group theoretic objects were first studied around 1978 and are called either graph groups or right-angled Artin groups.
- They are an extremely hot topic in geometric group theory today.
- The corresponding associative algebras have been studied under the name graph algebras.
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The word problem for $F(\Gamma)$

- To solve the word problem for $F(\Gamma)$, we must say when two injective words over V are equivalent.
- The short answer is when they are equivalent in the corresponding free pc monoid.
- Let w be a word with support W.
- Define an acyclic orientation of $\overline{\Gamma}[W]$ by orienting an edge from x to y if x occurs before y in w.
- Two words are equivalent in $F(\Gamma)$ iff they give the same acyclically oriented induced subgraph of $\overline{\Gamma}$.
- The topologically sortings of an acyclically oriented induced subgraph of $\overline{\Gamma}$ recover the corresponding class of words.
- So $F(\Gamma)$ can be identified with the set of acyclically oriented induced subgraphs of $\overline{\Gamma}$.

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The Athanasiadis and Diaconis walk: revisited

- For example, if Γ is $\overline{C_4}$, then the acyclically oriented graph

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & 2 \\ 3 & 4 \arrow[l] \arrow[u] \end{tikzcd}$$

  corresponds to the words 1423, 1432, 4123 and 4132.
- The minimal left ideal of $F(\Gamma)$ consists of the acyclic orientations of $\overline{\Gamma}$.
- Multiplying on the left by a vertex v reorients all edges incident on v away from it.
- Thus the Athanasiadis and Diaconis walk on $\overline{\Gamma}$ is a random walk of $F(\Gamma)$.
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Matroid left regular bands

- As a last example, we consider the LRB associated to a matroid by Brown.
- Let $\mathscr{M}$ be a matroid with ground set E.
- $B(\mathscr{M})$ is the set of all ordered independent subsets of E.
- The product is concatenation followed by removal of elements that are dependent on previous elements.
- If all subsets of E are independent, then one gets the free LRB.
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Hereditary algebras

- From now on fix a field k.
- All k-algebras A are assumed unital and finite dimensional.
- A is hereditary if each left ideal of A is projective.
- An equivalent condition is that submodules of projective modules are projective.
- Modern representation theory is based on the case of hereditary algebras.
- Brown showed using a non-bijective counting argument and quiver theory that the algebra of a free left regular band is hereditary.
- This work arose from trying to obtain a conceptual understanding of this result.

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Global dimension

- The global dimension of A is

$$\operatorname{gl.dim} A = \sup\{n \mid \operatorname{Ext}^n_A(M,N) \neq 0\}$$

where M, N run over all A-modules.

- One can in fact restrict to M and N simple.
- $\operatorname{gl.dim} A = 0$ iff A is semisimple.
- $\operatorname{gl.dim} A \leq 1$ iff A is hereditary.
- If B is an LRB, then $\operatorname{gl.dim} kB = 0$ iff B is commutative.
- Commutative LRBs are just lattices made into monoids via one of their two operations.
- A result of Nico implies $\operatorname{gl.dim} kB$ is bounded by the length of the longest chain of principal left ideals in B.

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Leray numbers of simplicial complexes

- Let K be a finite simplicial complex with vertex set V.
- The (k-)Leray number of K is

$$L_k(K) = \min\{d \mid \widetilde{H}^d(K[W], k) = 0\}$$

for all subsets $W \subseteq V$.

- $L_k(K)$ turns out to be the Castelnuovo-Mumford regularity of the Stanley-Reisner ring of K.

Theorem (Folklore)

1. $L_k(K) = 0$ *iff* K *is a simplex.*
2. $L_k(K) \leq 1$ *iff* K *is the clique complex of a chordal graph.*

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1. $L_k(K) = 0$ *iff* K *is a simplex.*
2. $L_k(K) \leq 1$ *iff* K *is the clique complex of a chordal graph.*

Leray numbers of simplicial complexes

- Let K be a finite simplicial complex with vertex set V.
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Clique complexes

- Recall the clique complex $K(\Gamma)$ of a graph $\Gamma = (V, E)$ is the simplicial complex on V whose simplices are the cliques of Γ.
- A graph is chordal if it contains no induced cycle C_n with $n \geq 4$.
- If P is a poset, the comparability graph $\Gamma(P)$ of P has vertex set P and edges the comparable pairs.
- $\Delta(P) = K(\Gamma(P))$ is called the order complex of P.
- If the Hasse diagram of P is a rooted tree, then it is easy to see that $\Gamma(P)$ is chordal (in fact has no induced P_4 or C_4).

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The poset of a left regular band

- Let B be an LRB.
- Then $aB = bB$ iff $a = b$.
- Thus B is partially ordered by $a \leq b$ iff $aB \subseteq bB$.
- Equivalently, $a \leq b$ iff $ba = a$. (Brown uses the reverse ordering.)
- The identity is the maximum element in this order.
- B is right hereditary if the Hasse diagram of B is a tree.
- Equivalently, all right ideals of B are projective right B-sets.
- The free LRB is right hereditary.
- LRBs associated to matroids and interval greedoids are right hereditary.

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A simplified form of our results

- Let us begin with a weak formulation of our results.

Theorem (Margolis,Saliola,BS)

1. *If B is an LRB,* $\mathrm{gl.\,dim}\, kB \leq L_k(\Delta(B))$.
2. *If Γ is a graph,* $\mathrm{gl.\,dim}\, kF(\Gamma) = L_k(K(\Gamma))$.

- Thus if B is right hereditary, then kB is hereditary.
- In particular $kF(A)$ is hereditary.
- $kF(\Gamma)$ is hereditary iff Γ is chordal.
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- The bound in (1) is tight for a lattice only when it is a chain.
- (2) gives a new algebraic interpretation of the Leray number of a clique complex.

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The support lattice

- To give the precise formulation of our results, we need to consider another poset.
- Let B be an LRB.
- Let $\Lambda(B)$ be the poset of principal left ideals of B.
- A result of Clifford implies $Ba \cap Bb = Bab$.
- Thus $\Lambda(B)$ is a lattice ordered by inclusion.
- We view it as a monoid via intersection.
- $\sigma \colon B \to \Lambda(B)$ given by $\sigma(a) = Ba$ is a homomorphism.
- Following Brown we call $\Lambda(B)$ the support lattice and σ the support map.
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The simple modules

- $kB/\mathrm{rad}(kB) \cong k\Lambda(B)$.
- Thus the irreducible representations of B are of degree 1 and are parameterized by $\Lambda(B)$.
- For each $X \in \Lambda(B)$, there is a representation $\theta_X \colon B \to k$ given by

$$\theta_X(a) = \begin{cases} 1 & \sigma(a) \geq X \\ 0 & \text{else.} \end{cases}$$

- Denote by S_X the corresponding simple module.
- For example if $\Gamma = (V, E)$ is a graph, the support lattice of $F(\Gamma)$ can be identified with $P(V)$.
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Some important subposets

- To compute $\mathrm{Ext}^n_{kB}(S_X, S_Y)$ we need to consider certain subposets of B.
- Fix, for each $X \in \Lambda(X)$, an element e_X with $X = Be_X$.
- For $X < Y$, define

$$B(X,Y) = \{a \in B \mid X \leq \sigma(a),\ a < e_Y\}.$$

- Up to isomorphism, it doesn't depend on the choice of e_Y.
- Let $\Delta(X,Y)$ be the order complex of $B(X,Y)$.
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- We are now ready to state our main result.

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- To compute $\mathrm{Ext}^n_{kB}(S_X, S_Y)$ we need to consider certain subposets of B.
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The computation of global dimension

Theorem (Margolis,Saliola,BS)

Let B be an LRB and $X, Y \in \Lambda(B)$. Then

$$\mathrm{Ext}^n_{kB}(S_X, S_Y) = \begin{cases} k & \text{for } X = Y,\ n = 0 \\ \widetilde{H}^{n-1}(\Delta(X,Y), k) & \text{for } X < Y,\ n > 0 \\ 0 & \text{else.} \end{cases}$$

- Consequently, $\mathrm{gl.\,dim}\, kB \leq L_k(\Delta(B))$.

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The quiver of kB

- The quiver of an algebra A is the digraph $Q(A)$ with vertex set the simple A-modules.
- There are $\dim \operatorname{Ext}^1_A(S_1, S_2)$ edges from $S_1 \to S_2$.

Corollary

$Q(kB)$ has vertex set $\Lambda(B)$. The number of edges $X \to Y$ is 0 unless $X < Y$, in which case it is one less than the number of connected components of $\Delta(X, Y)$.

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Geometric left regular bands

- Recall that $b \geq a$ iff $ba = a$.
- Thus $B_{\geq a} = \{b \mid b \geq a\}$ is a submonoid.
- We say that B is geometric if $B_{\geq a}$ is commutative for all $a \in B$.
- In this case $B_{\geq a}$ is a lattice with meet given by the product.
- Sets with a common lower bound have a meet in a geometric LRB.
- Examples include hyperplane face semigroups, oriented matroids, right hereditary LRBs and free pc LRBs.

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The commutation graph

- Let B be a geometric LRB and $X < Y$ in $\Lambda(B)$.
- Define $\Gamma(X, Y)$ to be the graph whose vertices are the maximal elements of $B(X, Y)$.
- Edges are between elements that commute.
- A set of vertices forms a clique if and only if it has a lower bound.

Theorem (Margolis,Saliola,BS)

$\Delta(X, Y)$ is homotopy equivalent to $K(\Gamma(X, Y))$.

Proof.

Rota's cross-cut theorem. □

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Examples

- For example, if B is right hereditary (its Hasse diagram is a tree), then $\Gamma(X,Y)$ has no edges because elements joined by an edge have a common lower bound.
- This gives another proof that kB is hereditary when B is right hereditary.
- If $\Gamma = (V,E)$ is a graph, then $F(\Gamma)(V,\emptyset) = F(\Gamma) \setminus \{1\}$.
- The maximal elements of $F(\Gamma)$ are the elements of V.
- Two elements of V commute iff they form an edge of Γ.
- So $\Gamma(V,\emptyset) = \Gamma$.
- More generally, if $Y \subsetneq X \subseteq V$, then $\Gamma(X,Y)$ is isomorphic to the induced subgraph $\Gamma[X \setminus Y]$.
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The proof idea

- The interesting case is to compute $\mathrm{Ext}^n_{kB}(S_X, S_Y)$ when $X < Y$ in $\Lambda(B)$.
- An Eckmann-Shapiro lemma argument reduces us to the case $X = \widehat{0}$ and $Y = \widehat{1}$ (however the LRB will change).
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- Let $I = B(\widehat{0}, \widehat{1}) = B \setminus \{1\}$; it is an ideal.
- $S_{\widehat{1}}$ is annihilated by I.
- If Z is a right B-set, then k^Z is a left kB-module.
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- The Eckmann-Shapiro lemma yields if Z is a right B-set

$$H^n(B, k^Z) \cong H^n(B \ltimes Z, k)$$

  where $B \ltimes Z$ is a finite category known as the Grothendieck construction or category of elements of Z.
- One has $H^n(\mathscr{C}, k) = H^n(|\mathscr{C}|, k)$ where $|\mathscr{C}|$ is the classifying space of the category $\mathscr{C}$.
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- If P is a poset, then P is a category with object set P and a unique arrow $p \to q$ if $p \leq q$.
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- One has $H^n(\mathscr{C}, k) = H^n(|\mathscr{C}|, k)$ where $|\mathscr{C}|$ is the classifying space of the category $\mathscr{C}$.
- $|\mathscr{C}|$ has vertices and edges the objects and arrows of $\mathscr{C}$ and simplices commutative diagrams forming the 1-skeleton of a simplex.
- If P is a poset, then P is a category with object set P and a unique arrow $p \to q$ if $p \leq q$.
- $|P| = \Delta(P)$.

Classifying spaces

- The Eckmann-Shapiro lemma yields if Z is a right B-set
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Quillen's Theorem A

- Quillen gave a very general criterion for a functor to induce a homotopy equivalence of classifying spaces.
- The case where the codomain is a poset is easiest to describe.
- Let $F\colon \mathscr{C} \to P$ be a functor from a category $\mathscr{C}$ to a poset P.
- For $p \in P$, define the fiber F/p of F under p to be $F^{-1}(P_{\leq p})$.

Theorem (Quillen's Theorem A)

If $|F/p|$ is contractible for all $p \in P$, then F induces a homotopy equivalence $|\mathscr{C}| \simeq \Delta(P)$.

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Completion of the proof sketch

- If R is a right ideal of an LRB B, then there is a functor $F\colon B \ltimes R \to R$ where we view R as a poset.
- Using Quillen's Theorem A, we show that F induces a homotopy equivalence $|B \ltimes R| \simeq \Delta(R)$.
- Thus $H^n(B \ltimes R, k) \cong H^n(\Delta(R), k)$.
- Applying this to $R = I$ yields our main result:

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\begin{aligned}
\mathrm{Ext}^n_{kB}(S_{\hat{0}}, S_{\hat{1}}) &= H^n(B, S_{\hat{1}}) \\
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