Poset cohomology, Leray numbers and the global dimension of left regular bands

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XXII Escola de Álgebra

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- Then we try to put it altogether and state our main results.

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- $|K|$ is the union of the simplices spanned by sets of coordinate vectors corresponding to an element of $\mathcal{F}$.

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- The reduced homology depends only on $|K|$.

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- The nerve of an open cover is fundamental to Čech cohomology.

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- The modern way to formulate his result is via Leray numbers.

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- $\mathrm{Ler}_{\Bbbk}(K) = 0$ iff K is a simplex.

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- The clique complex $\mathsf{Cliq}(G)$ is the flag complex with vertex set V and simplices the cliques of G.

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Cliq(G) is 1-representable iff G is chordal and $\overline{G}$ is a comparability graph (Lekkerkerker, Boland).

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- If K is a flag complex, then $I(K)$ is generated by products $x_i x_j$ with $\{x_i, x_j\}$ a non-edge of K^1.
- Such ideals are often called edge ideals since they correspond to edges of the complementary graph of K^1.

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- I believe Fröberg independently discovered the connection between chordal graphs and Leray number 1 in this context.

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- Markov chains on these objects can be analyzed via LRB representation theory.

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Others:

Björner, Athanasiadis-Diaconis, Chung-Graham, ...

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- Tsetlin Library: shelf of books
  "use a book, then put it at the front"

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This construction generalizes to matroids and interval greedoids.

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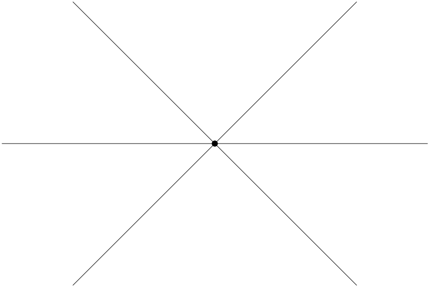
A set of hyperplanes partitions $\mathbb{R}^n$ into *faces*:



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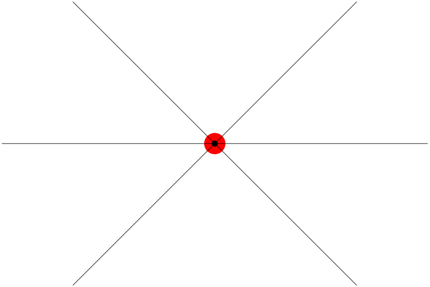

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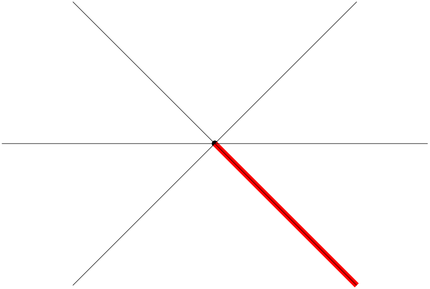

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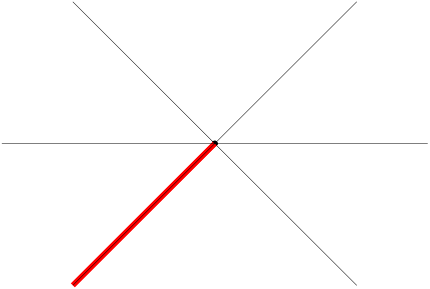

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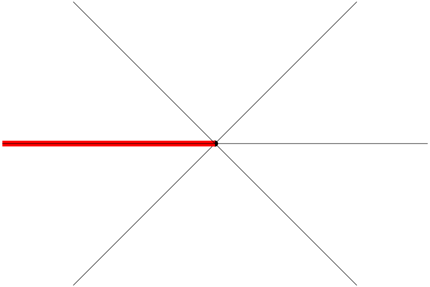

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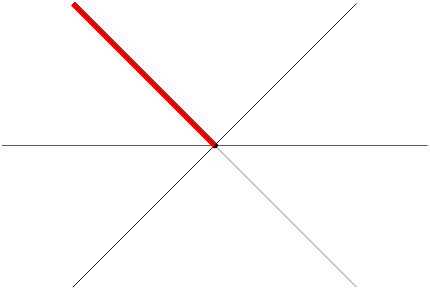

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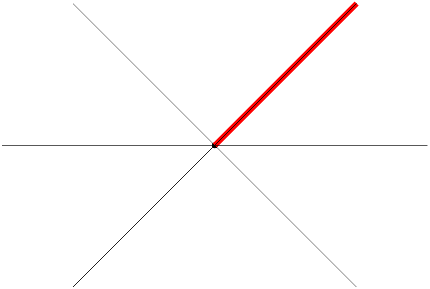

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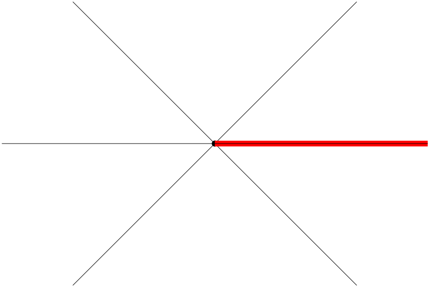

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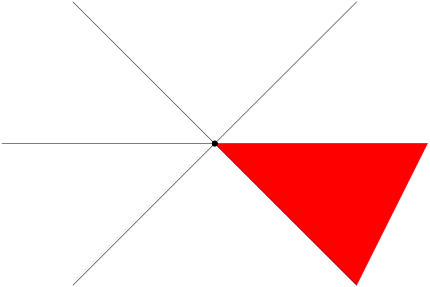

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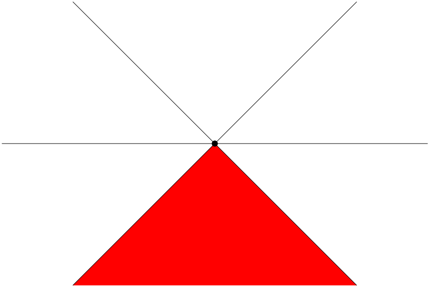

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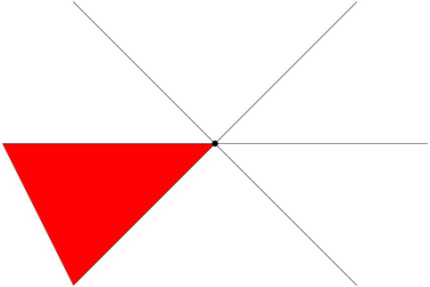

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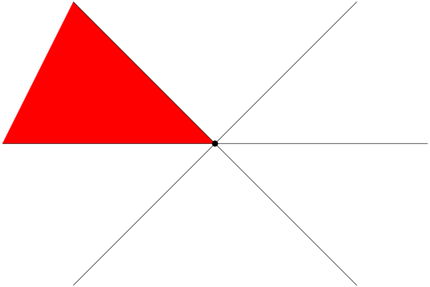

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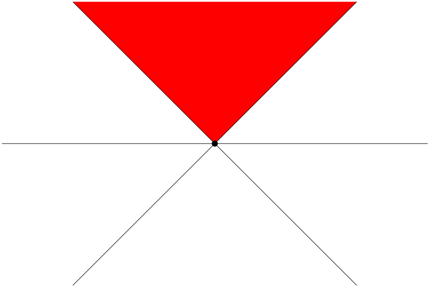

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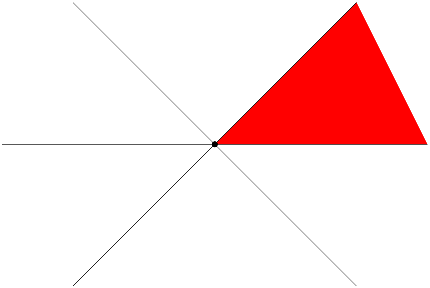

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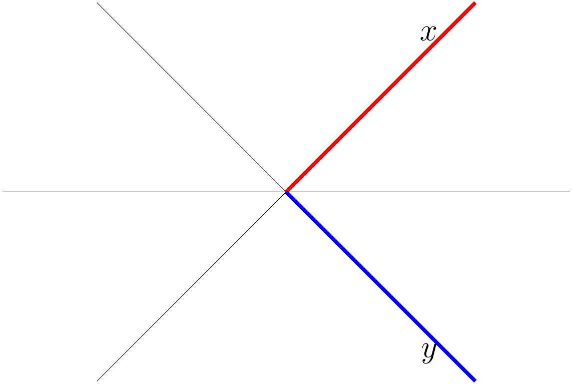

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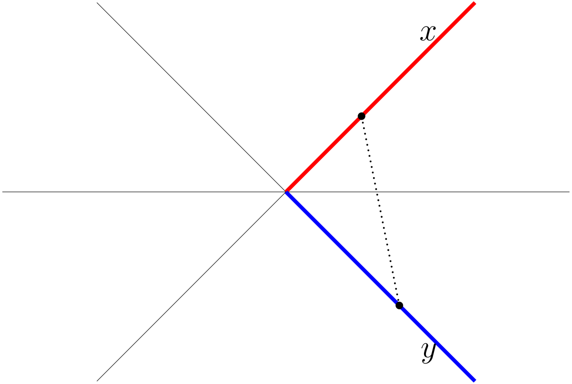

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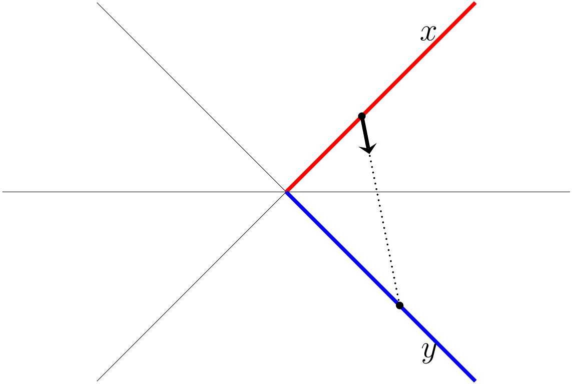

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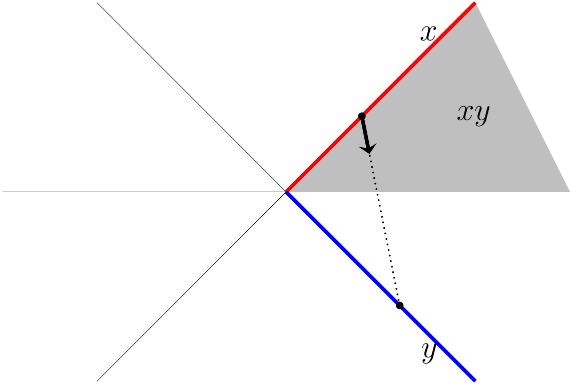

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- For instance, in type A the algebra $\Sigma(W)$ maps onto the character ring with nilpotent kernel.

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- These are LRB-analogues of free partially commutative monoids and groups.

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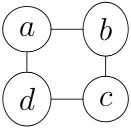
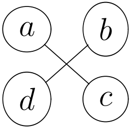
$G =$ 

$\overline{G} =$ 

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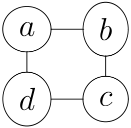
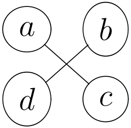
$G =$ 

$\overline{G} =$ 

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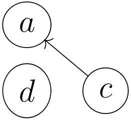

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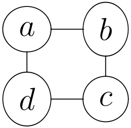
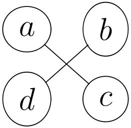
$G =$ 

$\overline{G} =$ 

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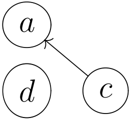

In $B(G)$: $cad = cda = dca$ (c comes before a since $c \to a$)

Random walk on $B(G)$

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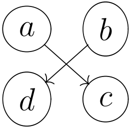

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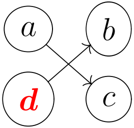

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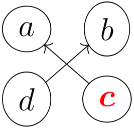

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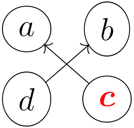

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Athanasiadis-Diaconis (2010): studied this chain using a different LRB (graphical arrangement of $\overline{G}$)

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- Our research was motivated by trying to obtain a conceptual proof.

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- This gives a large family of algebras of global dimension 2.

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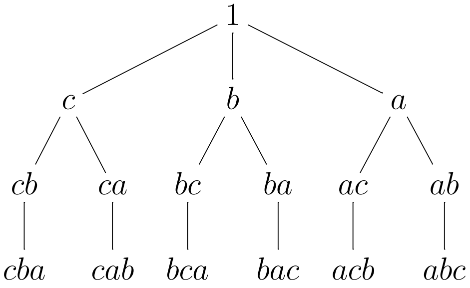
Example: $F(\{a, b, c\})$



The poset for a hyperplane arrangement

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Simple $\Bbbk B$-modules

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- So the simple $\Bbbk B$-modules S_X are 1-dimensional, indexed by $X \in \Lambda(B)$.

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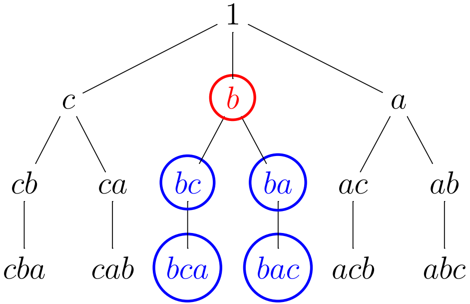
Example: $Babc \subseteq Bb$



$$B_{[Babc,b)} = \{bc, ba, bca, bac\}$$

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- I will outline here a new approach that we have just discovered.

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- If B is the face poset of a regular CW complex K, then the augmented cellular chain complex of K yields the minimal projective resolution of $\Bbbk$.
- This happens for real and complex hyperplane arrangements and for oriented matroids.

Ext as relative simplicial cohomology

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- Contractibility of $\Delta(B)$ and the exact sequence for relative cohomology yield $\mathrm{Ext}^n_{\Bbbk B}(\Bbbk, S_{\widehat{1}}) = \widetilde{H}^{n-1}(\Delta(B \setminus \{1\});\Bbbk)$.

Computation of Ext

Theorem (Margolis-Saliola-BS)

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$$\operatorname{Ext}^n_{\Bbbk B}(S_X, S_Y)$$

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All of our previous results follow from this and the duality between simplicial homology and cohomology over a field.

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- It encodes those modules over A with a composition series of length at most 2.

Quiver of $\Bbbk B$

Corollary

The quiver of $\Bbbk B$ *has vertex set* $\Lambda(B)$*. The number of arrows* $X \to Y$ *is* 0 *if* $X \not< Y$*; otherwise, it is one less than the number of connected components of* $\Delta(B_{[X,Y)})$*.*

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For $X < Y$:

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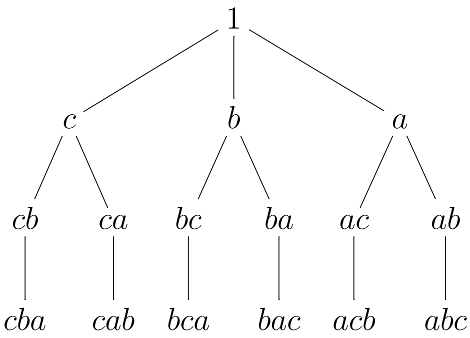
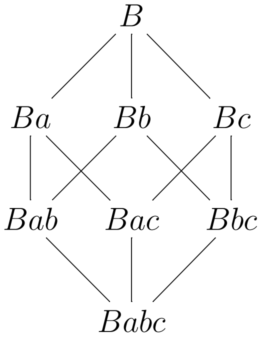
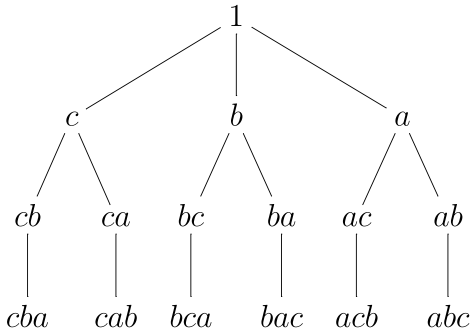
The connected components of $\Delta(P)$ and the Hasse diagram coincide for a poset P.

Poset and $\Lambda(B)$ for $B = F(\{a, b, c\})$





Computing the quiver of $B = F(\{a, b, c\})$



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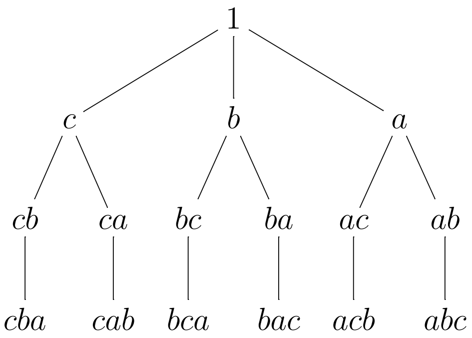

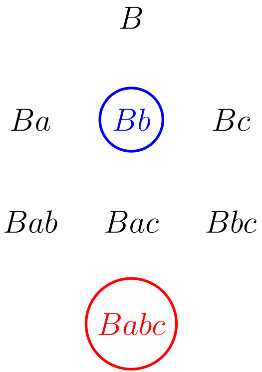

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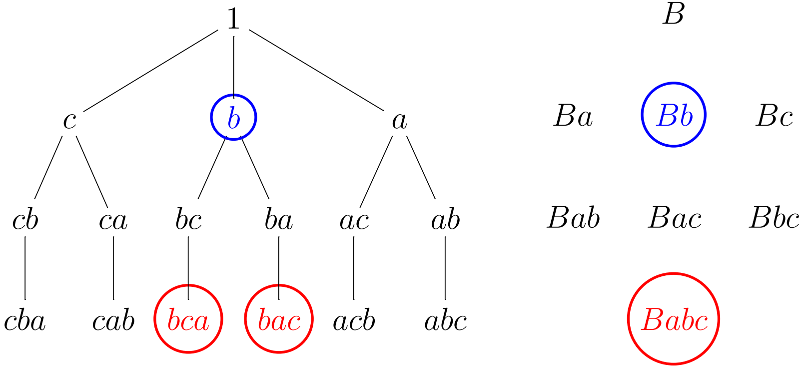

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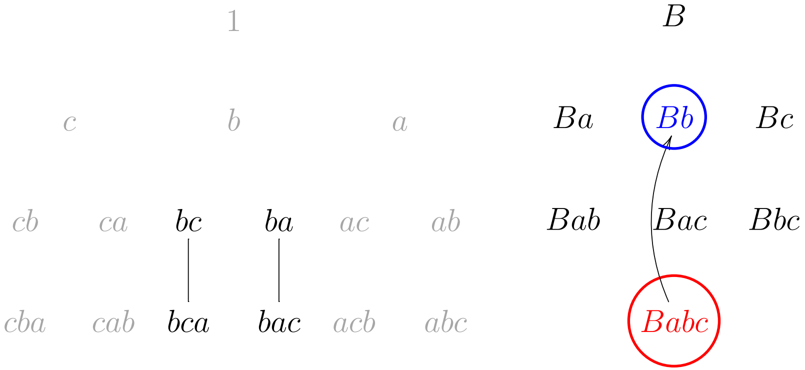

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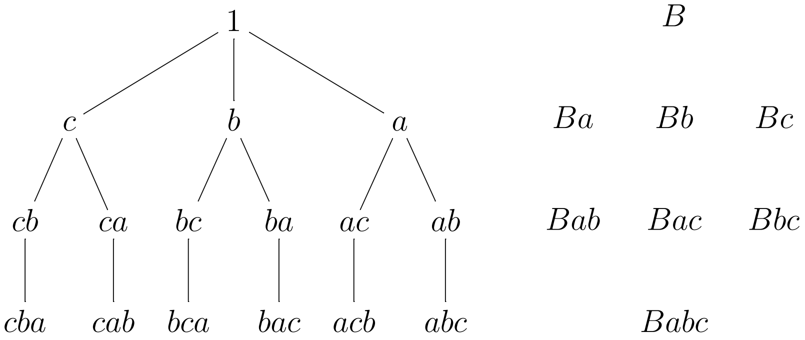

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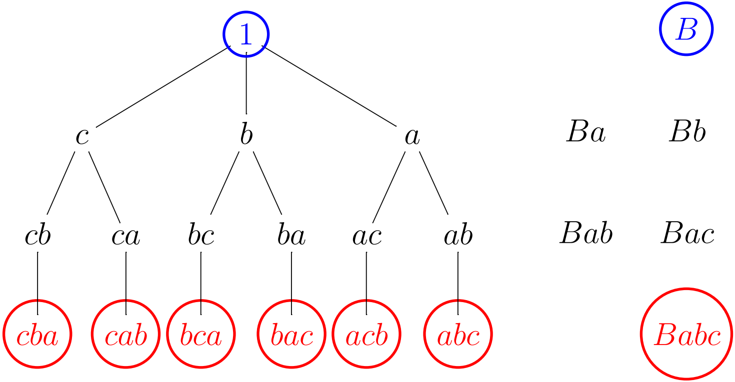

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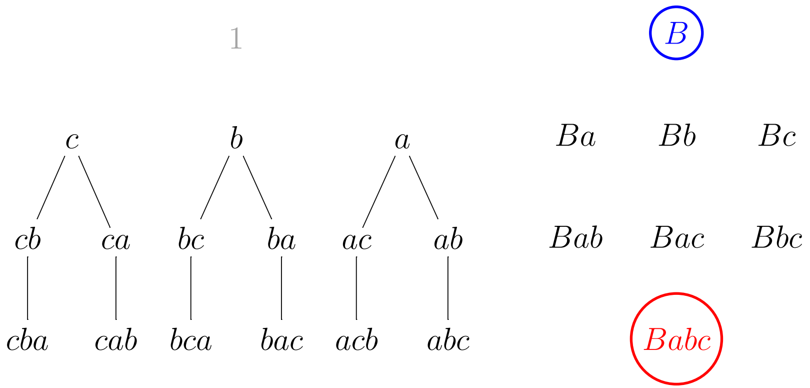

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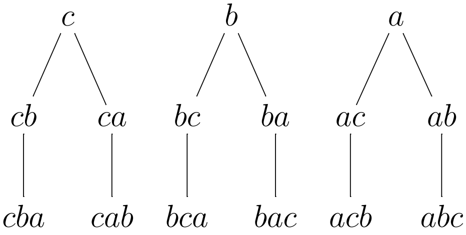

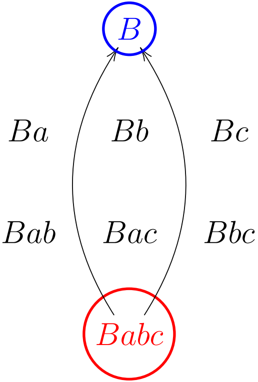

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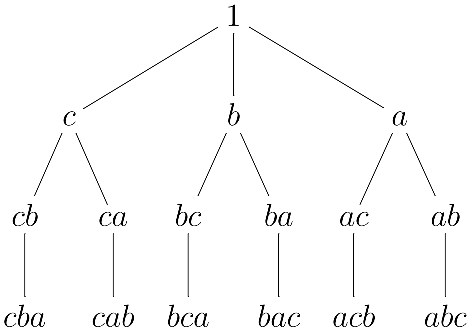

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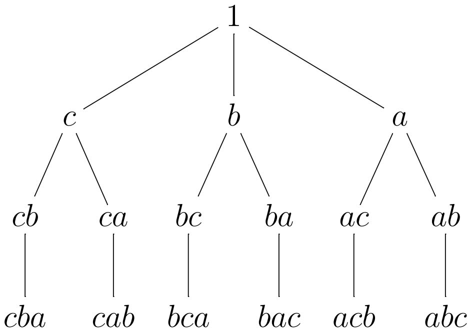

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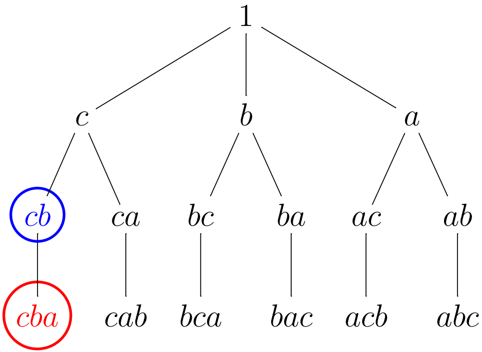

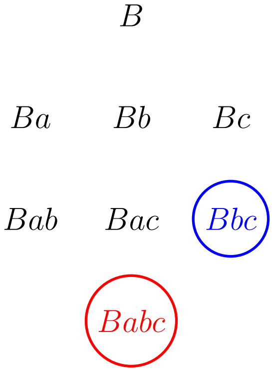

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1

$c \quad b \quad a$

$cb \quad ca \quad bc \quad ba \quad ac \quad ab$

$cba \quad cab \quad bca \quad bac \quad acb \quad abc$

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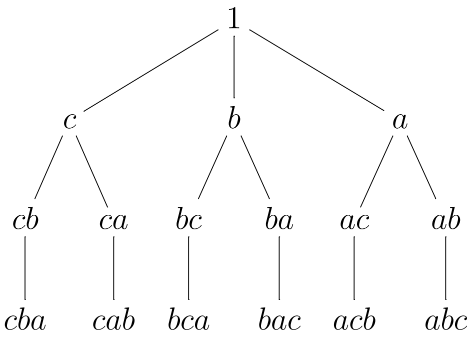
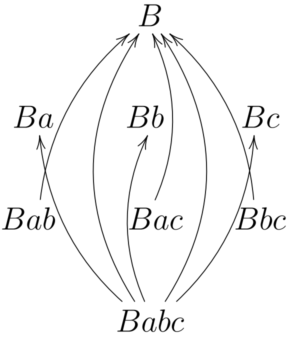
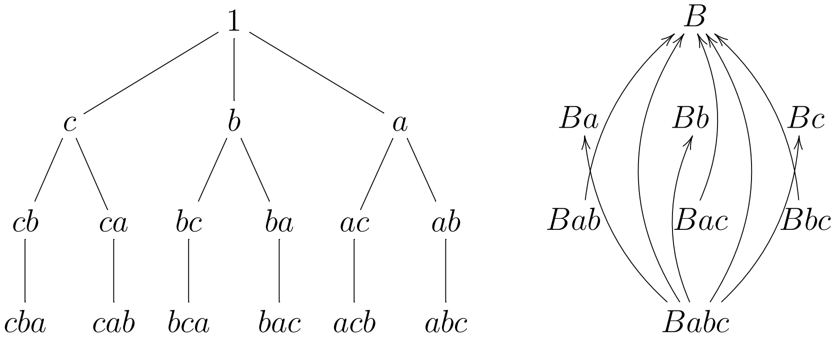
$Babc$

Quiver of $B = F(\{a,b,c\})$





Quiver of $B = F(\{a, b, c\})$



Since $\Bbbk B$ is hereditary it is isomorphic to the path algebra on this quiver.