The submonoid membership problem for groups

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Integer programming

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- The submonoid membership problem for arbitrary groups is a non-commutative analogue of integer programming.

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- The (Uniform) Rational Subset Membership Problem:
  - Given $w \in \Sigma^*$ and a finite automaton $\mathscr{A}$ over Σ, is $\pi(w) \in \pi(L(\mathscr{A}))$?
- Decidability of these problems is independent of Σ.

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- Compare: all finitely generated metabelian groups have decidable generalized word problem (Romanovskii).
- The Rips construction produces hyperbolic groups with undecidable generalized word problem.

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  - The language of geodesic words belonging to a quasiconvex subgroup of a hyperbolic group.

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  - finitely generated submonoids;
  - double cosets of finitely generated subgroups.

Examples

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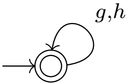

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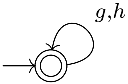

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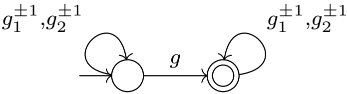

recognizes the double coset $\langle g_1, g_2\rangle g \langle g_1, g_2\rangle$.

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- If $\mathrm{Rat}(G)$ is closed under intersection, then G is a Howson group.

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- Dahmani and Guirardel solved equations over hyperbolic groups with special rational constraints.
- Dahmani and Groves use rational subsets in their solution to the isomorphism problem for toral relatively hyperbolic groups.
- The order of g is finite if and only if $g^{-1} \in g^*$, so decidability of submonoid membership gives decidability of order.

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- It reduces to INTEGER PROGRAMMING.

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Theorem (Kambites, Silva, BS (2007))

Decidability of rational subset membership is preserved by free products with amalgamation and HNN-extensions with finite edge groups.

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Theorem (Lohrey, BS (2008))

Every group in the class $\mathscr{C}$ has decidable rational subset membership problem.

Graph groups: the generalized word problem

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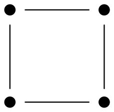

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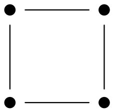
C4 $=$ 

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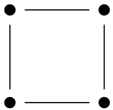

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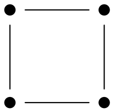

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Theorem (Kapovich, Myasnikov, Weidmann (2005))

The generalized word problem is decidable for chordal graph groups.

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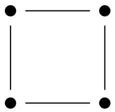
Let P4 $= \bullet \text{———} \bullet \text{———} \bullet \text{———} \bullet$ and C4 $=$ 

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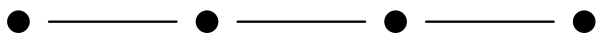
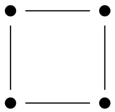
2. *submonoid membership is decidable for $\mathscr{G}(\Gamma)$;*
3. *Γ contains neither an induced* C4 *nor* P4*.*

Graph groups: the rational subset problem

Let P4 =  and C4 = 

Theorem (Lohrey, BS (2008))

Let Γ be a graph. Then the following are equivalent:

1. *rational subset membership is decidable for $\mathscr{G}(\Gamma)$;*
2. *submonoid membership is decidable for $\mathscr{G}(\Gamma)$;*
3. *Γ contains neither an induced* C4 *nor* P4*.*

Since P4 is chordal, we obtain the first example of a group with decidable generalized word problem but undecidable submonoid membership problem.

The direct product of two free monoids

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- This is a simple encoding of the Post correspondence problem.

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Theorem (Lohrey, BS (2010))

The submonoid and rational subset membership problems are equivalent for groups with two or more ends.

- Recall: a group has 2 or more ends iff it splits over a finite subgroup.

Metabelian groups

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- This latter membership problem is decidable.

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- Recall: Integer Programming is NP-complete.

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- The proof encodes an undecidable problem on planar tilings.

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- Choosing m large enough, we can encode our subsemimodule with undecidable membership as a finitely generated submonoid of M_2.

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Theorem (Lohrey, BS (2011))

The 2-dimensional lamplighter group $\mathbb{Z}/2\mathbb{Z} \wr (\mathbb{Z} \times \mathbb{Z})$ has undecidable rational subset membership problem.

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Question

Is submonoid membership decidable for nilpotent groups?

The end

THANK YOU FOR YOUR ATTENTION!