Symbolic Dynamics, Profinite Groups and Profinite Monoids

Benjamin Steinberg

Carleton University

University of Leipzig

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An ultrametric completion of the free monoid

- A^* denotes the free monoid on A.
- We always assume $2 \leq |A| < \infty$.
- An ultrametric can be defined on A^* by putting

$$d(u, v) = |A|^{-|u \wedge v|}$$

  where $u \wedge v$ is the longest common prefix of u and v.
- The completion is $A^* \cup A^\omega$, which can be viewed as a regular rooted tree together with its boundary.
- The boundary A^ω is the realm of symbolic dynamics.

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Symbolic dynamics

- The shift map $\sigma\colon A^\omega \to A^\omega$ is given by

$$\sigma(a_0a_1\cdots) = a_1a_2\cdots .$$

- A subshift is a closed, non-empty, shift-invariant subspace of A^ω.
- Subshifts $\mathscr{X} \subseteq A^\omega$ and $\mathscr{Y} \subseteq B^\omega$ are conjugate if there is a homeomorphism $\psi\colon \mathscr{X} \to \mathscr{Y}$ so that

$$\begin{tikzcd} \mathscr{X} \arrow[r, "\sigma"] \arrow[d, "\psi"'] & \mathscr{X} \arrow[d, "\psi"] \\ \mathscr{Y} \arrow[r, "\sigma"] & \mathscr{Y} \end{tikzcd}$$

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commutes.

Irreducible subshifts

- Let $\mathscr{X} \subseteq A^\omega$ be a subshift.
- $\mathscr{X}$ is said to be irreducible if it has a dense orbit.
- Let $L(\mathscr{X}) \subseteq A^*$ denote the language of all finite factors of elements of $\mathscr{X}$.
- It turns out $\mathscr{X}$ is irreducible if and only if, for all $u, v \in L(\mathscr{X})$, there exists $w \in A^*$ so that $uwv \in L(\mathscr{X})$.
- In this talk we consider only irreducible subshifts.
- The map $\mathscr{X} \mapsto L(\mathscr{X})$ is injective.
- Indeed, $\mathscr{X} = \partial\overline{L(\mathscr{X})}$ (it is the boundary of the subtree spanned by $L(\mathscr{X})$).

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Minimal subshifts

- A minimal subshift must be the closure of the orbit of any of its elements under the shift.
- It follows immediately that minimal subshifts are irreducible.
- A word $w \in A^\omega$ generates a minimal subshift if and only if w is uniformly recurrent.
- This means that if v is a finite factor of w, then there exists $N > 0$ so that each factor of w of length N contains v as a factor: the "bounded gaps property."
- If $u \in A^*$, then the subshift generated by

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Uniform recurrence and primitive substitutions

- The famous Morse-Thue cube-free word is uniformly recurrent. It is the fixed point obtained by iterating the substitution

$$a \mapsto ab, b \mapsto ba$$

starting from a.

  - a
  - ab
  - $abba$
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  - $abbabaabbaababba\cdots$

- A substitution $f\colon A^* \to A^*$ is primitive if there exists $N > 0$ so that each letter of A appears in $f^N(a)$, all $a \in A$.
- If f is a primitive substitution with a the first letter of $f(a)$, then $\lim f^n(a)$ is a uniformly recurrent word.

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Sofic shifts

- A subshift $\mathscr{X} \subseteq A^\omega$ is said to be of finite type if there is a finite set F of forbidden factors defining $\mathscr{X}$. That is, $L(\mathscr{X}) = A^* \setminus A^*FA^*$.
- In this case, $L(\mathscr{X})$ is a regular language, i.e., recognized by a finite automaton.
- Weiss defined $\mathscr{X}$ to be a sofic shift if $L(\mathscr{X})$ is regular.
- Sofic shifts are precisely the quotients (in the appropriate category) of subshifts of finite type.
- Irreducible sofic shifts can always be recognized by a strongly connected automaton all of whose states are initial and final.
- A minimal sofic shift must be periodic (follow a cycle in the automaton).

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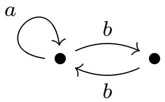
The even shift

- An automaton recognizing the even shift:



- The even shift consists of all infinite words with an even number of occurrences of b between consecutive occurrences of a.

A hidden profinite monoid

- Multiplication on A^* is uniformly continuous in the ultrametric defining the completion $A^* \cup A^\omega$.
- Thus $A^* \cup A^\omega$ has the structure of an A-generated profinite monoid.
- The product is given by

$$u \cdot v = \begin{cases} uv & u \in A^* \\ u & u \in A^\omega. \end{cases}$$

- It follows that $A^* \cup A^\omega$ is a continuous homomorphic image of the profinite completion $\widehat{A^*}$ of A^*, i.e., of the free profinite monoid on A.
- Almeida used this to "lift" symbolic dynamics to the free profinite monoid.

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The profinite completion of the free monoid

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- A^* is residually finite, so $\sigma(u,v)$ is well defined.
- The profinite ultrametric on A^* is defined by

$$d(u,v) = |A|^{-\sigma(u,v)}.$$

- The completion is the free profinite monoid $\widehat{A^*}$.
- Set $\partial\widehat{A^*} = \widehat{A^*} \setminus A^*$; it is an ideal of $\widehat{A^*}$.
- $\widehat{A^*}$ plays a crucial role in automata theory because it is the Zariski spectrum of the Boolean ring of regular languages.

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- For $u \neq v \in A^*$, define $\sigma(u,v)$ to be the minimum size of a finite monoid separating u from v.
- A^* is residually finite, so $\sigma(u,v)$ is well defined.
- The profinite ultrametric on A^* is defined by

$$d(u,v) = |A|^{-\sigma(u,v)}.$$

- The completion is the free profinite monoid $\widehat{A^*}$.
- Set $\partial\widehat{A^*} = \widehat{A^*} \setminus A^*$; it is an ideal of $\widehat{A^*}$.
- $\widehat{A^*}$ plays a crucial role in automata theory because it is the Zariski spectrum of the Boolean ring of regular languages.

The Almeida dictionary

- Let $\mathscr{X} \subseteq A^\omega$ be an irreducible subshift.
- Almeida established the following result:

Theorem (Almeida)

- *The map $\mathscr{X} \mapsto \overline{L(\mathscr{X})} \cap \partial\widehat{A^*}$ is injective.*
- *Among all principal ideals $\widehat{A^*}u\widehat{A^*}$ intersecting $\overline{L(\mathscr{X})}$ there is a unique minimal one, denoted $I(\mathscr{X})$.*
- *$I(\mathscr{X})$ can be generated by an idempotent $e = e^2$.*
- *The subshift $\mathscr{X}$ is minimal if and only if $I(\mathscr{X})$ is maximal among principal ideals in $\partial\widehat{A^*}$.*
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Maximal subgroups of a profinite monoid

- Suppose M is a profinite monoid and $e \in M$ is an idempotent.
- Then eMe is a profinite monoid with identity e.
- The group of units G_e of eMe is a profinite group known as the maximal subgroup at e.
- If e, f are idempotents, then $MeM = MfM$ implies $eMe \cong fMf$ and hence $G_e \cong G_f$.
- Thus to each principal ideal generated by an idempotent, we can associate a unique maximal subgroup (up to isomorphism).
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- Let $\widehat{F}_A$ be a free profinite group on A.
- There is a natural surjective map $\varphi\colon \widehat{A^*} \to \widehat{F}_A$.
- If e is an idempotent of the minimal ideal I, then $e\widehat{A^*}e = G_e$ and $\varphi(G_e) = \widehat{F}_A$.
- So φ splits and hence all projective profinite groups embed in free profinite monoids (observation of Almeida and Volkov).
- Margolis and I observed that the maximal subgroup of the minimal ideal maps onto any countably based profinite group.

Question (Margolis 97)

1. *Is every maximal subgroup of $\widehat{A^*}$ a free profinite group, or at least projective?*
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- A profinite group G is projective if given a diagram

$$\begin{tikzcd} & G \arrow[d, two heads, "\varphi"] \\ A \arrow[r, two heads, "\alpha"] & B \end{tikzcd}$$

  of epimorphisms of profinite groups, there exists a homomorphism $\lambda\colon G \to A$ so that the diagram commutes.
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Profinite groups associated to minimal subshifts

- Almeida and Volkov showed that the group associated to a periodic subshift is free procyclic.
- Almeida used techniques from symbolic dynamics to study the group associated to a primitive substitution.

Theorem (Almeida)

Let $f\colon A^* \to A^*$ be a primitive substitution which is invertible over the free group F_A. Then the profinite group associated to the corresponding minimal subshift is a free profinite group on A.

- For example, the Fibonacci word is the fixed point of the invertible substitution $a \mapsto ab, b \mapsto a$. The associated maximal subgroup is then free of rank 2.
- More generally, the group associated to any Sturmian system is free of rank 2.

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Theorem (Rhodes, BS)

The closed subgroups of $\widehat{A^*}$ are precisely the projective profinite groups.

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Torsion in free profinite monoids

- Any element s of finite order in $\widehat{A^*}$ must satisfy $s^n = s^{n+m}$ for some $n, m \geq 1$.
- But then $C = \{s^n, \ldots, s^{n+m-1}\}$ is a finite cyclic subgroup with identity s^k some $n \leq k \leq n+m-1$.
- But then, since idempotents generate a prime ideal, it follows that s and s^k generate the same ideal.
- Standard profinite semigroup theory then implies $s \in C$.
- But projective profinite groups are torsion-free so $C = \{s^k\}$.

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Torsion in free profinite monoids

- Any element s of finite order in $\widehat{A^*}$ must satisfy $s^n = s^{n+m}$ for some $n, m \geq 1$.
- But then $C = \{s^n, \ldots, s^{n+m-1}\}$ is a finite cyclic subgroup with identity s^k some $n \leq k \leq n+m-1$.
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  - $\overline{\langle Y\rangle} = G$;
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Theorem (BS)

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- The proof relies on Iwasawa's criterion: a countably based profinite group G is free of countable rank if and only if given a diagram

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- Again wreath products play a role: this time iterated wreath products.
- The idea is based on Bernhard Neumann's proof that every countable semigroup embeds in a 2-generated semigroup, and variations on this theme.
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Another free profinite subgroup

- Recall that we have a canonical projection $\varphi\colon \widehat{A^*} \twoheadrightarrow \widehat{F}_A$ where $\widehat{F}_A$ is the free profinite group generated by A.
- Moreover, φ restricts to an epimorphism $\varphi\colon G \twoheadrightarrow \widehat{F}_A$ where G is the maximal subgroup of the minimal ideal I.
- Let $K = \ker\varphi$.

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Profinite groups associated to irreducible sofic subshifts

- The minimal ideal of $\widehat{A^*}$ is the principal ideal associated to the full shift A^ω, which is an irreducible sofic shift.
- It is then natural to ask whether the result for the minimal ideal extends to all irreducible sofic shifts.
- Minimal sofic shifts are periodic and hence have free procyclic associated groups.
- So the interesting case is the non-minimal case.

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The profinite group associated to a non-minimal irreducible sofic shift is a free profinite group of countable rank.

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- A key observation in the proof is that the following objects are the same:
  - Syntactic monoids of languages of the form $L(\mathscr{X})$ with $\mathscr{X}$ an irreducible sofic shift
  - Generalized group mapping monoids with an aperiodic 0-minimal ideal.
- A semigroup S is generalized group mapping if it has a (necessarily unique and regular) 0-minimal ideal I on which it acts faithfully on both the left and right.
- The so-called Fischer cover of an irreducible sofic shift is just the Schützenberger graph of the aperiodic 0-minimal ideal.
- Generalized group mapping semigroups were introduced by Rhodes in his work on the complexity of finite semigroups.
- The theory of lifting regular $\mathscr{J}$-classes, again due to Rhodes, plays a salient role in the proof of our theorem.

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Open questions

- Which projective profinite groups can be maximal subgroups of a free profinite monoid (Zalesskii)?
- Are there any trivial maximal subgroups of $\widehat{A^*}$ other than the group of units?
- Can a free pro-p group be a maximal subgroup of $\widehat{A^*}$ (Zalesskii)?
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