

# Recent progress on the structure of free profinite monoids

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# Free profinite monoids

- Equational descriptions of classes of regular languages rely on the expressive power of the free profinite monoid  $\widehat{A^*}$ .  
(Throughout this talk  $A$  is a finite alphabet.)
- Over the last ten years we have begun to obtain a much clearer picture of the structure of this object.
- My goal here is to touch on the following topics:
  - Free clopen submonoids;
  - Ideal structure;
  - Maximal subgroups;
  - Finite subsemigroups.

# The profinite completion of the free monoid

- For  $u \neq v \in A^*$ , define  $s(u, v)$  to be the minimum size of a finite monoid separating  $u$  from  $v$ . Put  $s(u, u) = \infty$ .
- $A^*$  is residually finite, so  $s(u, v)$  is well defined.
- The profinite ultrametric on  $A^*$  is defined by

$$d(u, v) = 2^{-s(u, v)}.$$

- The completion is the free profinite monoid  $\widehat{A^*}$ .
- It is a compact, totally disconnected (i.e., profinite) monoid.
- Elements of  $\widehat{A^*}$  are called *profinite words*.

# Stone duality and the free profinite monoid

- $\text{Reg}(A^*)$  is a boolean ring with the operations of symmetric difference and intersection as addition and multiplication.

- There is a natural comultiplication

$\Delta: \text{Reg}(A^*) \rightarrow \text{Reg}(A^*) \otimes_{\mathbb{F}_2} \text{Reg}(A^*)$  given by

$$\Delta(L) = \sum_{ab \in \eta_L(L)} \eta_L^{-1}(a) \otimes \eta_L^{-1}(b)$$

where  $\eta_L: A^* \rightarrow M_L$  is the syntactic morphism.

- There is a counit  $\lambda: \text{Reg}(A^*) \rightarrow \mathbb{F}_2$  given by

$$\lambda(L) = \begin{cases} 1 & \varepsilon \in L \\ 0 & \text{else.} \end{cases}$$

- So  $\text{Reg}(A^*)$  is a bialgebra and hence its Zariski spectrum  $\text{Spec}(\text{Reg}(A^*))$  is a profinite monoid by Stone duality.

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# Stone duality and the free profinite monoid II

## Theorem (Almeida;Rhodes-BS)

$$\widehat{A^*} \cong \text{Spec}(\text{Reg}(A^*)).$$

- The isomorphism at the level of topological spaces is due to Almeida; the algebraic part to us.
- As a consequence of Almeida's part, clopen subsets of  $\widehat{A^*}$  are in bijection with regular languages.
- $L \in \text{Reg}(A^*)$  corresponds to  $\overline{L} \subseteq \widehat{A^*}$ .
- Conversely, if  $K \subseteq \widehat{A^*}$  is clopen, then  $K \cap A^*$  is regular.
- In particular, clopen submonoids of  $\widehat{A^*}$  are in bijection with regular submonoids of  $A^*$ .
- In summary,  $\widehat{A^*}$  is the geometric object corresponding to  $\text{Reg}(A^*)$ .



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- In summary,  $\widehat{A^*}$  is the geometric object corresponding to  $\text{Reg}(A^*)$ .

# Stone duality and varieties of languages

- $\text{Reg}$  is a contravariant functor from the category of free monoids to the category of (boolean) bialgebras.
- A **variety of languages** is precisely a **subfunctor** of  $\text{Reg}$ .
- Stone duality says that the Zariski spectrum functor gives a duality between the categories of boolean bialgebras and profinite monoids.
- If  $\mathcal{V}$  is a variety of languages (viewed as a functor), then the composition  $A^* \mapsto \text{Spec}(\mathcal{V}(A^*))$  produces the free pro- $\mathbf{V}$  monoid on  $A$ , where  $\mathbf{V}$  is the pseudovariety of monoids corresponding to  $\mathcal{V}$ .
- Duality eases proofs: every finite image of  $\varprojlim T_i$  factors through a  $T_i$  dualizes to the trivial statement every finite subbialgebra of  $\varinjlim B_i$  factors through a  $B_i$ .

# Free clopen submonoids

- It is known that clopen subgroups of free profinite groups are free.
- Clopen submonoids of  $\widehat{A}^*$  need not be free: e.g.  $\overline{\{x^2, x^3\}^*}$ .
- Almeida asked in his book: does a free profinite monoid on  $n$  generators embed as a closed submonoid of a free profinite monoid on 2 generators.
- Koryakov showed in 1995 the code  $C_n = \{y, xy, \dots, x^{n-1}y\}$  freely generates a free clopen submonoid of  $\widehat{\{x, y\}^*}$  of rank  $n$ .

Theorem (Margolis, Sapir, Weil 98)

*Any finite code  $C \subseteq A^*$  freely generates a free clopen profinite submonoid of  $\widehat{A}^*$ .*

- Recall:  $C \subseteq A^*$  is a **code** if  $C$  freely generates  $C^*$ .

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## Free clopen submonoids II

- If one takes an infinite regular code, like  $x^*y$ , then it generates a clopen submonoid of  $\widehat{A^*}$ .
- Does it freely generate a free profinite monoid?
- **No!** The free profinite monoid on a discrete set  $X$  contains its Stone-Czech compactification  $\beta X$ .
- $\beta X$  is not metrizable if  $X$  is infinite, but  $\widehat{A^*}$  is metrizable when  $A$  is finite. So  $\widehat{A^*}$  does not contain a free profinite monoid on an infinite set.
- If  $X$  is a topological space, the free profinite monoid  $\widehat{X^*}$  on  $X$  is defined via the usual universal property:

$$\begin{array}{ccc}
 X & \longrightarrow & \widehat{X^*} \\
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## Free clopen submonoids III

- This topological obstruction is the only obstacle to generalizing the result of Margolis, Sapir and Weil.

### Theorem (Almeida, BS)

*The free clopen submonoids of  $\widehat{A^*}$  are precisely the closures of regular free submonoids of  $A^*$ . Moreover, if  $C$  is a regular code, then  $\overline{C}$  is the unique closed (and in fact clopen) basis for  $\overline{C^*}$ .*

- The proof uses unambiguous automata and wreath products.
- It is in the same spirit as the case of finite codes, but the topology makes the proof more technical.

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# The ideal structure of free profinite monoids

- Henckell, Rhodes and I — and independently Almeida and Costa — observed that overlap lemmas for free monoids also work to a large extent for free profinite monoids.
- This led me to the **Prime Ideal Theorem**.
- An ideal  $I$  in a semigroup is called *prime* if  $ab \in I$  implies  $a \in I$  or  $b \in I$ .
- An ideal  $I$  in a semigroup is *idempotent* if  $I^2 = I$ .
- For example, an element  $x$  generates an idempotent ideal if and only if  $x$  is regular.
- The minimal ideal of a profinite semigroup is an idempotent ideal.

# The ideal structure of free profinite monoids II

## Theorem (Prime Ideal Theorem, BS)

*Every idempotent ideal of  $\widehat{A}^*$  is prime.*

- The case of the minimal ideal had been obtained earlier by Almeida and Volkov using symbolic dynamics and entropy.
- This theorem admits a number of important consequences.

## Corollary

*Suppose  $x \in \widehat{A}^*$  and  $x^n$  is a group element for some  $n \geq 1$ . Then  $x$  is a group element. In particular, all elements of finite order in  $\widehat{A}^*$  are group elements.*

- The case of finite order was obtained earlier by me and Rhodes.

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# The ideal structure of free profinite monoids III

## Corollary

*If  $B \subseteq \widehat{A}^*$  is a band, then the principal ideals of  $B$  form a chain.*

## Proof.

- Let  $e, f \in B$ . Then  $ef \in B$  and so idempotent.
- Hence  $ef$  generates a prime ideal so  $e \mathcal{J} ef$  or  $f \mathcal{J} ef$ .
- Then  $e \mathcal{R} ef$  or  $f \mathcal{L} ef$ .
- Since these are idempotents, this holds in  $B$ .
- Thus  $e, f$  are comparable in the  $\mathcal{J}$ -order on  $B$ .



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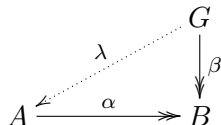
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# Free and projective profinite groups

- The free profinite group on a topological space is defined in the same way as for monoids.
- A profinite group  $G$  is called *projective* if:



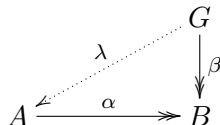
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## Theorem

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# Maximal subgroups of $\widehat{A}^*$

- If  $e \in \widehat{A}^*$  is an idempotent,  $G_e$  denotes the maximal subgroup at  $e$ .
- Let  $\widehat{F}_A$  be a free profinite group on  $A$ .
- There is a natural surjective map  $\varphi: \widehat{A}^* \rightarrow \widehat{F}_A$ .
- If  $e$  is an idempotent of the minimal ideal  $I$ , then  $\varphi(G_e) = \widehat{F}_A$ .
- So  $\varphi$  splits and hence all projective profinite groups can embed in a free profinite monoid (observation of Almeida and Volkov).
- Margolis and I observed that the maximal subgroup of the minimal ideal maps onto any metrizable profinite group.

# A question of Margolis

## Question (Margolis 97)

- ❶ *Is every maximal subgroup of  $\widehat{A}^*$  a free profinite group, or at least projective?*
  - ❷ *Is the maximal subgroup of the minimal ideal of  $\widehat{A}^*$  a free profinite group?*
- Free profinite groups (and hence projective profinite groups) are torsion-free.
  - Is  $\widehat{A}^*$  torsion-free? That is, are all elements of finite order in  $\widehat{A}^*$  idempotent?
  - We saw earlier that all finite order elements of  $\widehat{A}^*$  are group elements.
  - So if every maximal subgroup of  $\widehat{A}^*$  is projective, then all elements of finite order must be idempotents.

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# Symbolic dynamics

- Almeida and his co-workers were the first to make progress on these questions. Their approach used symbolic dynamics.
- The *shift* map  $\sigma: A^\omega \rightarrow A^\omega$  is given by

$$\sigma(a_0a_1\cdots) = a_1a_2\cdots.$$

- A *subshift* is a closed subspace of  $A^\omega$  closed under the shift.
- A minimal subshift must be the closure of the orbit of an infinite word under the shift.
- A word  $w \in A^\omega$  generates a minimal subshift if and only if  $w$  is *uniformly recurrent*.
- This means that if  $v$  is a finite factor of  $w$ , then there exists  $N > 0$  so that each factor of  $w$  of length  $N$  contains  $v$  as a factor: the “bounded gaps property.”

# Uniform recurrence and substitutions

- The famous Morse-Thue cube-free word is uniformly recurrent. It is the fixed point obtained by iterating the substitution  $a \mapsto ab, b \mapsto ba$  starting from  $a$ .
- A substitution  $f: A^* \rightarrow A^*$  is called *primitive* if there exists  $N > 0$  so that each letter of  $A$  appears in  $f^N(a)$ , all  $a \in A$ .
- If  $f$  is a primitive substitution with  $a$  the first letter of  $f(a)$ , then  $\lim f^n(a)$  is a uniformly recurrent word.
- Set  $\partial \widehat{A}^* = \widehat{A}^* \setminus A^*$ .
- There is a natural continuous surjection  $\pi: \partial \widehat{A}^* \rightarrow A^\omega$  since regular languages can “remember” prefixes.

# Minimal subshifts and maximal principal ideals

- Almeida defined a profinite word  $w$  to be *uniformly recurrent* if given a finite factor  $v$  of  $w$ , there exists  $N > 0$  so that every factor of  $w$  of length  $N$  contains  $v$ .

## Theorem (Almeida)

- $w \in \hat{A}^*$  is uniformly recurrent iff  $\hat{A}^* w \hat{A}^*$  is a maximal principal ideal of  $\partial \hat{A}^*$ .
  - $\pi: \partial \hat{A}^* \rightarrow A^\omega$  sends uniformly recurrent profinite words onto uniformly recurrent infinite words.
  - $\pi$  induces a bijection between minimal subshifts and maximal principal ideals of  $\partial \hat{A}^*$ .
- So there is a principal ideal associated to each minimal subshift.

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# The profinite group associated to a minimal subshift

- The ideal associated to a minimal subshift can be generated by an idempotent.
- Thus there is a unique (up to isomorphism) maximal subgroup generating this ideal.
- In other words, there is a profinite group associated to each minimal subshift.
- Almeida showed that this group is a conjugacy invariant of the subshift.
- What can be said about these maximal subgroups of  $\widehat{A}^*$ ?

## The profinite group associated to a minimal subshift II

- Almeida showed that the groups corresponding to minimal subshifts arising from certain primitive substitutions are free profinite.
- For instance the groups associated to Sturmian and Arnoux-Rauzy subshifts are free profinite groups.
- Almeida showed if  $f$  is the substitution  $a \mapsto a^3b, b \mapsto ab$ , then the group associated to  $\lim f^n(a)$  is projective but not free.
- Almeida presented this work at the Fields workshop on profinite groups organized by me and Ribes in 2005.
- Lubotzky asked after Almeida's talk whether these groups must always be projective.
- Recently, Almeida and Costa showed that the profinite group associated to the Morse-Thue infinite word is not free profinite.

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- Almeida showed if  $f$  is the substitution  $a \mapsto a^3b, b \mapsto ab$ , then the group associated to  $\lim f^n(a)$  is projective but not free.
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# The projectivity theorem

## Theorem (Rhodes, BS)

*The closed subgroups of  $\widehat{A}^*$  are precisely the projective profinite groups. Hence  $\widehat{A}^*$  is torsion-free.*

- The proof uses wreath products and the Schützenberger representation in order to extend maps from a maximal subgroup to the whole free profinite monoid.
- Ribes later pointed out to us a similar proof scheme used by Cossey, Kegel and Kovács for the case of free profinite groups.
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## Finite subsemigroups of $\widehat{A}^*$

- It follows that every finite subsemigroup  $B$  of  $\widehat{A}^*$  is a band which is a  $\mathcal{J}$ -chain.
- If  $B$  has a zero, then it is a chain of idempotents.
- If the minimal ideal of  $B$  is a left (right) zero semigroup, then it is an  $\mathcal{L}$ -chain ( $\mathcal{R}$ -chain).
- Rectangular bands of arbitrary size embed in  $\widehat{A}^*$ .
- Is it decidable which bands embed in  $\widehat{A}^*$ ?

# Free profinite groups on a set converging to 1

- A subset  $Y$  of a profinite group  $G$  is a *set of generators converging to 1* if:
  - $\overline{\langle Y \rangle} = G$ ;
  - Each neighborhood of 1 contains all but finitely many elements of  $Y$ .
- One can define a free profinite group  $\widehat{F}_Y$  on a set  $Y$  of generators converging to 1. The cardinality of  $Y$  is called the *rank* of  $\widehat{F}_Y$ .
- A free profinite group on a topological space  $X$  is also free on a set of generators converging to 1 of the same cardinality as the boolean algebra of clopen subsets of  $X$ .



# The maximal subgroup of the minimal ideal is free

## Theorem (BS)

*The maximal subgroup of the minimal ideal of  $\widehat{A}^*$  is a free profinite group of countable rank.*

- The proof uses **Iwasawa's criterion**: a metrizable profinite group  $G$  is free profinite on a countable set of generators converging to 1 if and only if given a diagram



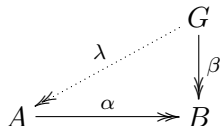
of epimorphisms ( $A$  and  $B$  are finite), there exists an epimorphism  $\lambda: G \twoheadrightarrow A$  so that the diagram commutes.

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# The maximal subgroup of the minimal ideal is free II

- Again wreath products play a role: this time iterated wreath products.
- The idea is based on Bernhard Neumann's proof that every countable semigroup embeds in a 2-generated semigroup, and variations on this theme.
- The most relevant variant for us embeds any countable group as the maximal subgroup of the minimal ideal of a 2-generated monoid with cyclic group of units.
- Ideas from Krohn-Rhodes Theory and the Synthesis Theorem also play a role.

## More on the minimal ideal

- If  $I$  is the minimal ideal of  $\widehat{A}^*$  and  $E(I)$  is its set of idempotents, then  $E(I)$  is a profinite space.
- There is a continuous retraction  $\pi: I \rightarrow E(I)$  so that each fiber of  $\pi$  is the maximal subgroup  $G$  of  $I$ .
- That is to say,  $I$  is a principal  $G$ -bundle with base space  $E(I)$ .
- So our results go a long way towards understanding the structure of  $I$ .

## Another free profinite subgroup

- Recall that we have a canonical projection  $\varphi: \widehat{A}^* \twoheadrightarrow \widehat{F}_A$  where  $\widehat{F}_A$  is the free profinite group generated by  $A$ .
- Moreover,  $\varphi$  restricts to an epimorphism  $\varphi: G \twoheadrightarrow \widehat{F}_A$  where  $G$  is the maximal subgroup of the minimal ideal  $I$ .
- Let  $K = \ker \varphi$ .

### Theorem (BS)

*The subgroup  $K$  is a free profinite group of countable rank.*

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# Open questions

- Which projective profinite groups can be maximal subgroups of a free profinite monoid (Zaleskii)?
  - Can a free pro- $p$  group be a maximal subgroup of  $\widehat{A}^*$ ?
- Let  $I$  be the minimal ideal of  $\widehat{A}^*$  and let  $S = \overline{\langle E(I) \rangle}$  be the closed subsemigroup generated by its idempotents.
  - Is the maximal subgroup  $H$  of  $S$  a free profinite group of countable rank?
  - The subgroup  $K$  is the normal closure of  $H$ .
  - $H$  maps onto every countably based profinite group.
  - We think that our proof should show that  $H$  is free.
- Classify projective profinite monoids.
  - Do the finite projective monoids form a recursive class?



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