19

Inventory Theory

"Sorry, we're out of that item." How often have you heard that during shopping trips? In many of these cases, what you have encountered are stores that aren't doing a very good job of managing their *inventories* (stocks of goods being held for future use or sale). They aren't placing orders to replenish inventories soon enough to avoid shortages. These stores could benefit from the kinds of techniques of scientific inventory management that are described in this chapter.

It isn't just retail stores that must manage inventories. In fact, inventories pervade the business world. Maintaining inventories is necessary for any company dealing with physical products, including manufacturers, wholesalers, and retailers. For example, manufacturers need inventories of the materials required to make their products. They also need inventories of the finished products awaiting shipment. Similarly, both wholesalers and retailers need to maintain inventories of goods to be available for purchase by customers.

The total value of all inventory—including finished goods, partially finished goods, and raw materials—in the United States is more than a *trillion* dollars. This is more than $4,000 each for every man, woman, and child in the country.

The costs associated with storing ("carrying") inventory are also very large, perhaps a quarter of the value of the inventory. Therefore, the costs being incurred for the storage of inventory in the United States run into the hundreds of billions of dollars annually. Reducing storage costs by avoiding unnecessarily large inventories can enhance any firm's competitiveness.

Some Japanese companies were pioneers in introducing the *just-in-time inventory system*—a system that emphasizes planning and scheduling so that the needed materials arrive "just-in-time" for their use. Huge savings are thereby achieved by reducing inventory levels to a bare minimum.

Many companies in other parts of the world also have been revamping the way in which they manage their inventories. The application of operations research techniques in this area (sometimes called *scientific inventory management*) is providing a powerful tool for gaining a competitive edge.

How do companies use operations research to improve their **inventory policy** for when and how much to replenish their inventory? They use **scientific inventory management** comprising the following steps:

1. Formulate a *mathematical model* describing the behavior of the inventory system.
2. Seek an *optimal* inventory policy with respect to this model.
3. Use a computerized *information processing system* to maintain a record of the current inventory levels.
4. Using this record of current inventory levels, apply the optimal inventory policy to signal when and how much to replenish inventory.

The mathematical inventory models used with this approach can be divided into two broad categories—deterministic models and stochastic models—according to the *predictability of demand* involved. The **demand** for a product in inventory is the number of units that will need to be withdrawn from inventory for some use (e.g., sales) during a specific period. If the demand in future periods can be forecast with considerable precision, it is reasonable to use an inventory policy that assumes that all forecasts will always be completely accurate. This is the case of *known demand* where a *deterministic* inventory model would be used. However, when demand cannot be predicted very well, it becomes necessary to use a *stochastic* inventory model where the demand in any period is a random variable rather than a known constant.

There are several basic considerations involved in determining an inventory policy that must be reflected in the mathematical inventory model. These are illustrated in the examples presented in the first section and then are described in general terms in Sec. 19.2. Section 19.3 develops and analyzes deterministic inventory models for situations where the inventory level is under continuous review. Section 19.4 does the same for situations where the planning is being done for a series of periods rather than continuously. The following three sections present stochastic models, first under continuous review, then for a single period, and finally for a series of periods. The chapter concludes with a discussion of how scientific inventory management is being used in practice to deal with very large inventory systems, as illustrated by case studies at IBM and Hewlett-Packard.

19.1 EXAMPLES

We present two examples in rather different contexts (a manufacturer and a wholesaler) where an inventory policy needs to be developed.

EXAMPLE 1 Manufacturing Speakers for TV Sets

A television manufacturing company produces its own speakers, which are used in the production of its television sets. The television sets are assembled on a continuous production line at a rate of 8,000 per month, with one speaker needed per set. The speakers are produced in batches because they do not warrant setting up a continuous production line, and relatively large quantities can be produced in a short time. Therefore, the speakers are placed into inventory until they are needed for assembly into television sets on the production line. The company is interested in determining when to produce a batch of speakers and how many speakers to produce in each batch. Several costs must be considered:

1. Each time a batch is produced, a **setup cost** of $12,000 is incurred. This cost includes the cost of "tooling up," administrative costs, record keeping, and so forth. Note that the existence of this cost argues for producing speakers in large batches.
2. The **unit production cost** of a single speaker (excluding the setup cost) is $10, independent of the batch size produced. (In general, however, the unit production cost need not be constant and may decrease with batch size.)
3. The production of speakers in large batches leads to a large inventory. The estimated **holding cost** of keeping a speaker in stock is $0.30 per month. This cost includes the cost of capital tied up in inventory. Since the money invested in inventory cannot be used in other productive ways, this cost of capital consists of the lost return (referred to as the *opportunity cost*) because alternative uses of the money must be forgone. Other components of the holding cost include the cost of leasing the storage space, the cost of insurance against loss of inventory by fire, theft, or vandalism, taxes based on the value of the inventory, and the cost of personnel who oversee and protect the inventory.
4. Company policy prohibits deliberately planning for shortages of any of its components. However, a shortage of speakers occasionally crops up, and it has been estimated that each speaker that is not available when required costs $1.10 per month. This **shortage cost** includes the extra cost of installing speakers after the television set is fully assembled otherwise, the interest lost because of the delay in receiving sales revenue, the cost of extra record keeping, and so forth.

We will develop the inventory policy for this example with the help of the first inventory model presented in Sec. 19.3.

EXAMPLE 2 Wholesale Distribution of Bicycles

A wholesale distributor of bicycles is having trouble with shortages of a popular model (a small, one-speed girl's bicycle) and is currently reviewing the inventory policy for this model. The distributor purchases this model bicycle from the manufacturer monthly and then supplies it to various bicycle shops in the western United States in response to purchase orders. What the total demand from bicycle shops will be in any given month is quite uncertain. Therefore, the question is, How many bicycles should be ordered from the manufacturer for any given month, given the stock level leading into that month?

The distributor has analyzed her costs and has determined that the following are important:

1. The **ordering cost**, i.e., the cost of placing an order plus the cost of the bicycles being purchased, has two components: The administrative cost involved in placing an order is estimated as $200, and the actual cost of each bicycle is $35 for this wholesaler.
2. The *holding cost,* i.e., the cost of maintaining an inventory, is $1 per bicycle remaining at the end of the month. This cost represents the costs of capital tied up, warehouse space, insurance, taxes, and so on.
3. The *shortage cost* is the cost of not having a bicycle on hand when needed. This particular model is easily reordered from the manufacturer, and stores usually accept a

delay in delivery. Still, although shortages are permissible, the distributor feels that she incurs a loss, which she estimates to be \$15 per bicycle per month of shortage. This estimated cost takes into account the possible loss of future sales because of the loss of customer goodwill. Other components of this cost include lost interest on delayed sales revenue, and additional administrative costs associated with shortages. If some stores were to cancel orders because of delays, the lost revenues from these lost sales would need to be included in the shortage cost. Fortunately, such cancellations normally do not occur for this model.

We will return to this example again in Sec. 19.6.

These examples illustrate that there are two possibilities for how a firm *replenishes inventory,* depending on the situation. One possibility is that the firm *produces* the needed units itself (like the television manufacturer producing speakers). The other is that the firm *orders* the units from a supplier (like the bicycle distributor ordering bicycles from the manufacturer). Inventory models do not need to distinguish between these two ways of replenishing inventory, so we will use such terms as *producing* and *ordering* interchangeably.

Both examples deal with one specific product (speakers for a certain kind of television set or a certain bicycle model). In most inventory models, just one product is being considered at a time. Except in Sec. 19.8, all the inventory models presented in this chapter assume a single product.

Both examples indicate that there exists a trade-off between the costs involved. The next section discusses the basic cost components of inventory models for determining the optimal trade-off between these costs.

19.2 COMPONENTS OF INVENTORY MODELS

Because inventory policies affect profitability, the choice among policies depends upon their relative profitability. As already seen in Examples 1 and 2, some of the costs that determine this profitability are (1) the ordering costs, (2) holding costs, and (3) shortage costs. Other relevant factors include (4) revenues, (5) salvage costs, and (6) discount rates. These six factors are described in turn below.

The **cost of ordering** an amount z (either through *purchasing* or *producing this amount*) can be represented by a function $c(z)$. The simplest form of this function is one that is directly proportional to the amount ordered, that is, $c \cdot z$, where c represents the unit price paid. Another common assumption is that $c(z)$ is composed of two parts: a term that is directly proportional to the amount ordered and a term that is a constant K for z positive and is 0 for $z = 0$. For this case,

$$\begin{aligned} c(z) &= \text{cost of ordering } z \text{ units} \\ &= \begin{cases} 0 & \text{if } z = 0 \\ K + cz & \text{if } z > 0, \end{cases} \end{aligned}$$

where K = setup cost and c = unit cost.

The constant K includes the administrative cost of ordering or, when producing, the costs involved in setting up to start a production run.

There are other assumptions that can be made about the cost of ordering, but this chapter is restricted to the cases just described.

In Example 1, the speakers are produced and the setup cost for a production run is \$12,000. Furthermore, each speaker costs \$10, so that the *production* cost when ordering a production run of z speakers is given by

$$c(z) = 12{,}000 + 10z, \qquad \text{for } z > 0.$$

In Example 2, the distributor orders bicycles from the manufacturer and the *ordering* cost is given by

$$c(z) = 200 + 35z, \qquad \text{for } z > 0.$$

The **holding cost** (sometimes called the *storage cost*) represents all the costs associated with the storage of the inventory until it is sold or used. Included are the cost of capital tied up, space, insurance, protection, and taxes attributed to storage. The holding cost can be assessed either continuously or on a period-by-period basis. In the latter case, the cost may be a function of the maximum quantity held during a period, the average amount held, or the quantity in inventory at the end of the period. The last viewpoint is usually taken in this chapter.

In the bicycle example, the holding cost is \$1 per bicycle remaining at the end of the month. In the TV speakers example, the holding cost is assessed continuously as \$0.30 per speaker in inventory per month, so the average holding cost per month is \$0.30 times the average number of speakers in inventory.

The **shortage cost** (sometimes called the *unsatisfied demand cost*) is incurred when the amount of the commodity required (demand) exceeds the available stock. This cost depends upon which of the following two cases applies.

In one case, called **backlogging,** the excess demand is not lost, but instead is held until it can be satisfied when the next normal delivery replenishes the inventory. For a firm incurring a temporary shortage in supplying its customers (as for the bicycle example), the shortage cost then can be interpreted as the loss of customers' goodwill and the subsequent reluctance to do business with the firm, the cost of delayed revenue, and the extra administrative costs. For a manufacturer incurring a temporary shortage in materials needed for production (such as a shortage of speakers for assembly into television sets), the shortage cost becomes the cost associated with delaying the completion of the production process.

In the second case, called **no backlogging,** if any excess of demand over available stock occurs, the firm cannot wait for the next normal delivery to meet the excess demand. Either (1) the excess demand is met by a priority shipment, or (2) it is not met at all because the orders are canceled. For situation 1, the shortage cost can be viewed as the cost of the priority shipment. For situation 2, the shortage cost is the loss of current revenue from not meeting the demand plus the cost of losing future business because of lost goodwill.

Revenue may or may not be included in the model. If both the price and the demand for the product are established by the market and so are outside the control of the company, the revenue from sales (assuming demand is met) is independent of the firm's inventory policy and may be neglected. However, if revenue is neglected in the model, the *loss in revenue* must then be included in the shortage cost whenever the firm cannot meet the demand and the sale is lost. Furthermore, even in the case where demand is backlogged, the cost of the delay in revenue must also be included in the shortage cost. With these interpretations, revenue will not be considered explicitly in the remainder of this chapter.

The **salvage value** of an item is the value of a leftover item when no further inventory is desired. The salvage value represents the disposal value of the item to the firm, perhaps through a discounted sale. The negative of the salvage value is called the **salvage cost.** If there is a cost associated with the disposal of an item, the salvage cost may be positive. We assume hereafter that any salvage cost is incorporated into the *holding cost*.

Finally, the **discount rate** takes into account the time value of money. When a firm ties up capital in inventory, the firm is prevented from using this money for alternative purposes. For example, it could invest this money in secure investments, say, government bonds, and have a return on investment 1 year hence of, say, 7 percent. Thus, \$1 invested today would be worth \$1.07 in year 1, or alternatively, a \$1 profit 1 year hence is equivalent to $\alpha = \$1/\$1.07$ today. The quantity α is known as the **discount factor.** Thus, in adding up the total profit from an inventory policy, the profit or costs 1 year hence should be multiplied by α; in 2 years hence by α^2; and so on. (Units of time other than 1 year also can be used.) The total profit calculated in this way normally is referred to as the *net present value.*

In problems having short time horizons, α may be assumed to be 1 (and thereby neglected) because the current value of \$1 delivered during this short time horizon does not change very much. However, in problems having long time horizons, the discount factor must be included.

In using quantitative techniques to seek optimal inventory policies, we use the criterion of minimizing the total (expected) discounted cost. Under the assumptions that the price and demand for the product are not under the control of the company and that the lost or delayed revenue is included in the shortage penalty cost, minimizing cost is equivalent to maximizing net income. Another useful criterion is to keep the inventory policy simple, i.e., keep the rule for indicating *when to order* and *how much to order* both understandable and easy to implement. Most of the policies considered in this chapter possess this property.

As mentioned at the beginning of the chapter, inventory models are usually classified as either *deterministic* or *stochastic* according to whether the demand for a period is known or is a random variable having a known probability distribution. The production of batches of speakers in Example 1 of Sec. 19.1 illustrates deterministic demand because the speakers are used in television assemblies at a fixed rate of 8,000 per month. The bicycle shops' purchases of bicycles from the wholesale distributor in Example 2 of Sec. 19.1 illustrates random demand because the total monthly demand varies from month to month according to some probability distribution. Another component of an inventory model is the **lead time,** which is the amount of time between the placement of an order to replenish inventory (through either purchasing or producing) and the receipt of the goods into inventory. If the lead time always is the same (a *fixed* lead time), then the replenishment can be scheduled just when desired. Most models in this chapter assume that each replenishment occurs just when desired, either because the delivery is nearly instantaneous or because it is known when the replenishment will be needed and there is a fixed lead time.

Another classification refers to whether the current inventory level is being monitored continuously or periodically. In **continuous review,** an order is placed as soon as the stock level falls down to the prescribed reorder point. In **periodic review,** the inventory level is checked at discrete intervals, e.g., at the end of each week, and ordering decisions are made only at these times even if the inventory level dips below the reorder point between the preceding and current review times. (In practice, a periodic review policy can be used to approximate a continuous review policy by making the time interval sufficiently small.)

19.3 DETERMINISTIC CONTINUOUS-REVIEW MODELS

The most common inventory situation faced by manufacturers, retailers, and wholesalers is that stock levels are depleted over time and then are replenished by the arrival of a batch of new units. A simple model representing this situation is the following **economic order quantity model** or, for short, the **EOQ model.** (It sometimes is also referred to as the *economic lot-size model.*)

Units of the product under consideration are assumed to be withdrawn from inventory continuously at a *known constant rate,* denoted by a; that is, the demand is a units per unit time. It is further assumed that inventory is replenished when needed by ordering (through either purchasing or producing) a batch of fixed size (Q units), where all Q units arrive simultaneously at the desired time. For the *basic EOQ model* to be presented first, the only costs to be considered are

K = setup cost for ordering one batch,

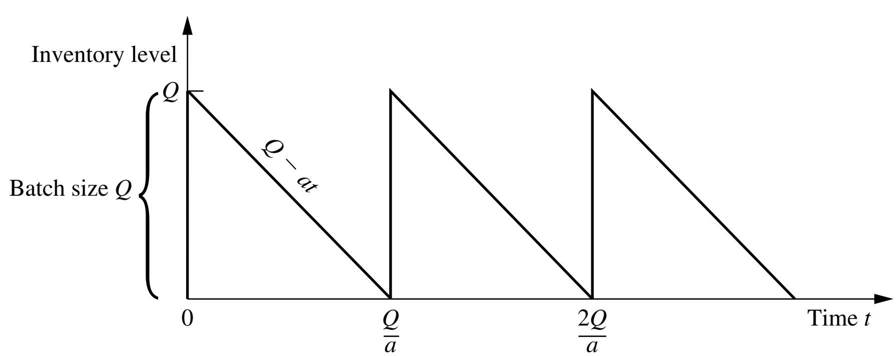
c = unit cost for producing or purchasing each unit,

h = holding cost per unit per unit of time held in inventory.

The objective is to determine when and by how much to replenish inventory so as to minimize the sum of these costs per unit time.

We assume *continuous review,* so that inventory can be replenished whenever the inventory level drops sufficiently low. We shall first assume that shortages are not allowed (but later we will relax this assumption). With the fixed demand rate, shortages can be avoided by replenishing inventory each time the inventory level drops to zero, and this also will minimize the holding cost. Figure 19.1 depicts the resulting pattern of inventory levels over time when we start at time 0 by ordering a batch of Q units in order to increase the initial inventory level from 0 to Q and then repeat this process each time the inventory level drops back down to 0.

FIGURE 19.1
Diagram of inventory level as a function of time for the basic EOQ model.



Example 1 in Sec. 19.1 (manufacturing speakers for TV sets) fits this model and will be used to illustrate the following discussion.

The Basic EOQ Model

To summarize, in addition to the costs specified above, the basic EOQ model makes the following assumptions.

Assumptions (Basic EOQ Model).

1. A known constant demand rate of a units per unit time.
2. The order quantity (Q) to replenish inventory arrives all at once just when desired, namely, when the inventory level drops to 0.
3. Planned shortages are not allowed.

In regard to assumption 2, there usually is a lag between when an order is placed and when it arrives in inventory. As indicated in Sec. 19.2, the amount of time between the placement of an order and its receipt is referred to as the *lead time.* The inventory level at which the order is placed is called the **reorder point.** To satisfy assumption 2, this reorder point needs to be set at the *product* of the demand rate and the lead time. Thus, assumption 2 is implicitly assuming a *constant* lead time.

The time between consecutive replenishments of inventory (the vertical line segments in Fig. 19.1) is referred to as a *cycle.* For the speaker example, a cycle can be viewed as the time between production runs. Thus, if 24,000 speakers are produced in each production run and are used at the rate of 8,000 per month, then the cycle length is $24{,}000/8{,}000 = 3$ months. In general, the cycle length is Q/a.

The total cost per unit time T is obtained from the following components.

$$\text{Production or ordering cost per cycle} = K + cQ.$$

The average inventory level during a cycle is $(Q + 0)/2 = Q/2$ units, and the corresponding cost is $hQ/2$ per unit time. Because the cycle length is Q/a,

$$\text{Holding cost per cycle} = \frac{hQ^2}{2a}.$$

Therefore,

$$\text{Total cost per cycle} = K + cQ + \frac{hQ^2}{2a},$$

so the total cost per unit time is

$$T = \frac{K + cQ + hQ^2/(2a)}{Q/a} = \frac{aK}{Q} + ac + \frac{hQ}{2}.$$

The value of Q, say Q^*, that minimizes T is found by setting the first derivative to zero (and noting that the second derivative is positive).

$$\frac{dT}{dQ} = -\frac{aK}{Q^2} + \frac{h}{2} = 0,$$

so that

$$Q^* = \sqrt{\frac{2aK}{h}},$$

which is the well-known *EOQ formula.*[1] (It also is sometimes referred to as the *square root formula.*) The corresponding *cycle time,* say t^*, is

$$t^* = \frac{Q^*}{a} = \sqrt{\frac{2K}{ah}}.$$

It is interesting to observe that Q^* and t^* change in intuitively plausible ways when a change is made in K, h, or a. As the setup cost K increases, both Q^* and t^* increase (fewer setups). When the unit holding cost h increases, both Q^* and t^* decrease (smaller inventory levels). As the demand rate a increases, Q^* increases (larger batches) but t^* decreases (more frequent setups).

These formulas for Q^* and t^* will now be applied to the speaker example. The appropriate parameter values from Sec. 19.1 are

$$K = 12{,}000, \qquad h = 0.30, \qquad a = 8{,}000,$$

so that

$$Q^* = \sqrt{\frac{(2)(8{,}000)(12{,}000)}{0.30}} = 25{,}298$$

and

$$t^* = \frac{25{,}298}{8{,}000} = 3.2 \text{ months.}$$

Hence, the optimal solution is to set up the production facilities to produce speakers once every 3.2 months and to produce 25,298 speakers each time. (The total cost curve is rather flat near this optimal value, so any similar production run that might be more convenient, say 24,000 speakers every 3 months, would be nearly optimal.)

The EOQ Model with Planned Shortages

One of the banes of any inventory manager is the occurrence of an inventory shortage (sometimes referred to as a *stockout*)—demand that cannot be met currently because the inventory is depleted. This causes a variety of headaches, including dealing with unhappy customers and having extra record keeping to arrange for filling the demand later (*backorders*) when the inventory can be replenished. By assuming that planned shortages are not allowed, the basic EOQ model presented above satisfies the common desire of managers to avoid shortages as much as possible. (Nevertheless, unplanned shortages can still occur if the demand rate and deliveries do not stay on schedule.)

[1]An interesting historical account of this model and formula, including a reprint of a 1913 paper that started it all, is given by D. Erlenkotter, "Ford Whitman Harris and the Economic Order Quantity Model," *Operations Research,* **38:** 937–950, 1990.

However, there are situations where permitting limited planned shortages makes sense from a managerial perspective. The most important requirement is that the customers generally are able and willing to accept a reasonable delay in filling their orders if need be. If so, the costs of incurring shortages described in Secs. 19.1 and 19.2 (including lost future business) should not be exorbitant. If the cost of holding inventory is high relative to these shortage costs, then lowering the average inventory level by permitting occasional brief shortages may be a sound business decision.

The **EOQ model with planned shortages** addresses this kind of situation by replacing only the third assumption of the basic EOQ model by the following new assumption.

> Planned shortages now are allowed. When a shortage occurs, the affected customers will wait for the product to become available again. Their backorders are filled immediately when the order quantity arrives to replenish inventory.

Under these assumptions, the pattern of inventory levels over time has the appearance shown in Fig. 19.2. The saw-toothed appearance is the same as in Fig. 19.1. However, now the inventory levels extend down to negative values that reflect the number of units of the product that are backordered.

Let

p = shortage cost per unit short per unit of time short,

S = inventory level just after a batch of Q units is added to inventory,

$Q - S$ = shortage in inventory just before a batch of Q units is added.

The total cost per unit time now is obtained from the following components.

Production or ordering cost per cycle $= K + cQ$.

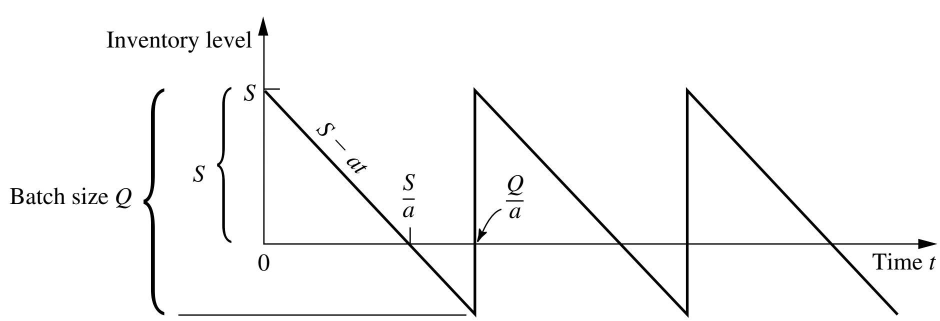
During each cycle, the inventory level is positive for a time S/a. The average inventory level *during this time* is $(S + 0)/2 = S/2$ units, and the corresponding cost is $hS/2$ per unit time. Hence,

$$\text{Holding cost per cycle} = \frac{hS}{2}\frac{S}{a} = \frac{hS^2}{2a}.$$

Similarly, shortages occur for a time $(Q - S)/a$. The average amount of shortages *during this time* is $(0 + Q - S)/2 = (Q - S)/2$ units, and the corresponding cost is $p(Q - S)/2$ per unit time. Hence,

$$\text{Shortage cost per cycle} = \frac{p(Q-S)}{2}\frac{Q-S}{a} = \frac{p(Q-S)^2}{2a}.$$

FIGURE 19.2
Diagram of inventory level as a function of time for the EOQ model with planned shortages.



Therefore,

$$\text{Total cost per cycle} = K + cQ + \frac{hS^2}{2a} + \frac{p(Q-S)^2}{2a},$$

and the *total cost per unit time* is

$$T = \frac{K + cQ + hS^2/(2a) + p(Q-S)^2/(2a)}{Q/a}$$

$$= \frac{aK}{Q} + ac + \frac{hS^2}{2Q} + \frac{p(Q-S)^2}{2Q}.$$

In this model, there are two decision variables (S and Q), so the optimal values (S^* and Q^*) are found by setting the partial derivatives $\partial T/\partial S$ and $\partial T/\partial Q$ equal to zero. Thus,

$$\frac{\partial T}{\partial S} = \frac{hS}{Q} - \frac{p(Q-S)}{Q} = 0.$$

$$\frac{\partial T}{\partial Q} = -\frac{aK}{Q^2} - \frac{hS^2}{2Q^2} + \frac{p(Q-S)}{Q} - \frac{p(Q-S)^2}{2Q^2} = 0.$$

Solving these equations simultaneously leads to

$$S^* = \sqrt{\frac{2aK}{h}}\sqrt{\frac{p}{p+h}}, \qquad Q^* = \sqrt{\frac{2aK}{h}}\sqrt{\frac{p+h}{p}}.$$

The optimal cycle length t^* is given by

$$t^* = \frac{Q^*}{a} = \sqrt{\frac{2K}{ah}}\sqrt{\frac{p+h}{p}}.$$

The maximum shortage is

$$Q^* - S^* = \sqrt{\frac{2aK}{p}}\sqrt{\frac{h}{p+h}}.$$

In addition, from Fig. 19.2, the fraction of time that no shortage exists is given by

$$\frac{S^*/a}{Q^*/a} = \frac{p}{p+h},$$

which is independent of K.

When either p or h is made much larger than the other, the above quantities behave in intuitive ways. In particular, when $p \to \infty$ with h constant (so shortage costs dominate holding costs), $Q^* - S^* \to 0$ whereas both Q^* and t^* converge to their values for the basic EOQ model. Even though the current model permits shortages, $p \to \infty$ implies that having them is not worthwhile.

On the other hand, when $h \to \infty$ with p constant (so holding costs dominate shortage costs), $S^* \to 0$. Thus, having $h \to \infty$ makes it uneconomical to have positive inventory levels, so each new batch of Q^* units goes no further than removing the current shortage in inventory.

If planned shortages are permitted in the speaker example, the *shortage cost* is estimated in Sec. 19.1 as

$$p = 1.10.$$

As before,

$$K = 12{,}000, \qquad h = 0.30, \qquad a = 8{,}000,$$

so now

$$S^* = \sqrt{\frac{(2)(8{,}000)(12{,}000)}{0.30}} \sqrt{\frac{1.1}{1.1 + 0.3}} = 22{,}424,$$

$$Q^* = \sqrt{\frac{(2)(8{,}000)(12{,}000)}{0.30}} \sqrt{\frac{1.1 + 0.3}{1.1}} = 28{,}540,$$

and

$$t^* = \frac{28{,}540}{8{,}000} = 3.6 \text{ months}.$$

Hence, the production facilities are to be set up every 3.6 months to produce 28,540 speakers. The maximum shortage is 6,116 speakers. Note that Q^* and t^* are not very different from the no-shortage case. The reason is that p is much larger than h.

The EOQ Model with Quantity Discounts

When specifying their cost components, the preceding models have assumed that the unit cost of an item is the same regardless of the quantity in the batch. In fact, this assumption resulted in the optimal solutions being independent of this unit cost. The *EOQ model with quantity discounts* replaces this assumption by the following new assumption.

> The unit cost of an item now depends on the quantity in the batch. In particular, an incentive is provided to place a large order by replacing the unit cost for a small quantity by a smaller unit cost for every item in a larger batch, and perhaps by even smaller unit costs for even larger batches.

Otherwise, the assumptions are the same as for the basic EOQ model.

To illustrate this model, consider the TV speakers example introduced in Sec. 19.1. Suppose now that the unit cost for *every* speaker is $c_1 = \$11$ if less than 10,000 speakers are produced, $c_2 = \$10$ if production falls between 10,000 and 80,000 speakers, and $c_3 = \$9.50$ if production exceeds 80,000 speakers. What is the optimal policy? The solution to this specific problem will reveal the general method.

From the results for the basic EOQ model, the total cost per unit time T_j if the unit cost is c_j is given by

$$T_j = \frac{aK}{Q} + ac_j + \frac{hQ}{2}, \qquad \text{for } j = 1, 2, 3.$$

(This expression assumes that h is independent of the unit cost of the items, but a common small refinement would be to make h proportional to the unit cost to reflect the fact that the cost of capital tied up in inventory varies in this way.) A plot of T_j versus Q is shown in Fig. 19.3 for each j, where the solid part of each curve extends over the feasible range of values of Q for that discount category.

For each curve, the value of Q that minimizes T_j is found just as for the basic EOQ model. For $K = 12{,}000$, $h = 0.30$, and $a = 8{,}000$, this value is

$$\sqrt{\frac{(2)(8{,}000)(12{,}000)}{0.30}} = 25{,}298.$$

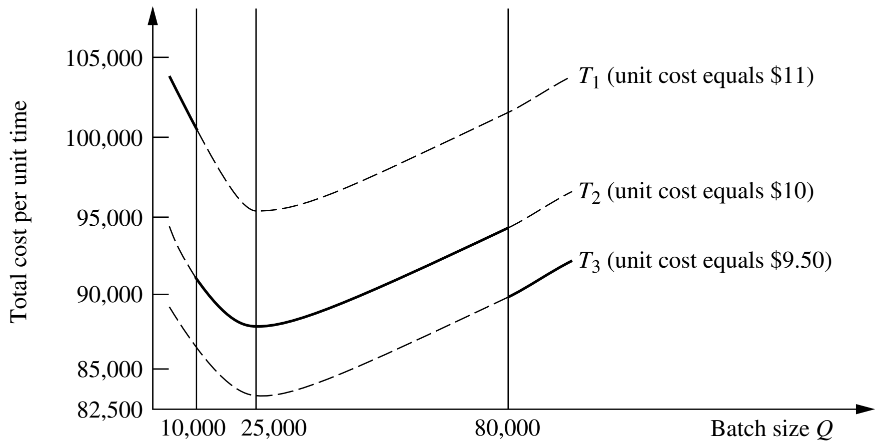


FIGURE 19.3
Total cost per unit time for the speaker example with quantity discounts.

(If h were not independent of the unit cost of the items, then the minimizing value of Q would be slightly different for the different curves.) This minimizing value of Q is a feasible value for the cost function T_2. For any fixed Q, $T_2 < T_1$, so T_1 can be eliminated from further consideration. However, T_3 cannot be immediately discarded. Its minimum feasible value (which occurs at $Q = 80{,}000$) must be compared to T_2 evaluated at 25,298 (which is \$87,589). Because T_3 evaluated at 80,000 equals \$89,200, it is better to produce in quantities of 25,298, so this quantity is the optimal value for this set of quantity discounts.

If the quantity discount led to a unit cost of \$9 (instead of \$9.50) when production exceeded 80,000, then T_3 evaluated at 80,000 would equal 85,200, and the optimal production quantity would become 80,000.

Although this analysis concerned a specific problem, the same approach is applicable to any similar problem. Here is a summary of the general procedure.

1. For each available unit cost c_j, use the EOQ formula for the EOQ model to calculate its optimal order quantity Q_j^*.
2. For each c_j where Q_j^* is within the feasible range of order quantities for c_j, calculate the corresponding total cost per unit time T_j.
3. For each c_j where Q_j^* is not within this feasible range, determine the order quantity Q_j that is at the endpoint of this feasible range that is closest to Q_j^*. Calculate the total cost per unit time T_j for Q_j and c_j.
4. Compare the T_j obtained for all the c_j and choose the minimum T_j. Then choose the order quantity Q_j obtained in step 2 or 3 that gives this minimum T_j.

A similar analysis can be used for other types of quantity discounts, such as incremental quantity discounts where a cost c_0 is incurred for the first q_0 units, c_1 for the next q_1 units, and so on.

Some Useful Excel Templates

For your convenience, we have included five Excel templates for the EOQ models in this chapter's Excel file on the CD-ROM. Two of these templates are for the basic EOQ model. In both cases, you enter basic data (a, K, and h), as well as the lead time for the deliveries and the number of working days per year for the firm. The template then calculates the firm's total annual expenditures for setups and for holding costs, as well as the sum of these two costs (the *total variable cost*). It also calculates the *reorder point*—the inventory level at which the order needs to be placed to replenish inventory so the replenishment will arrive when the inventory level drops to 0. One template (the *Solver version*) enables you to enter any order quantity you want and then see what the annual costs and reorder point would be. This version also enables you to use the Excel Solver to solve for the optimal order quantity. The second template (the *analytical version*) uses the EOQ formula to obtain the optimal order quantity.

The corresponding pair of templates also is provided for the EOQ model with planned shortages. After entering the data (including the unit shortage cost p), each of these templates will obtain the various annual costs (including the annual shortage cost). With the Solver version, you can either enter trial values of the order quantity Q and maximum shortage $Q - S$ or solve for the optimal values, whereas the analytical version uses the formulas for Q^* and $Q^* - S^*$ to obtain the optimal values. The corresponding maximum inventory level S^* also is included in the results.

The final template is an analytical version for the EOQ model with quantity discounts. This template includes the refinement that the unit holding cost h is proportional to the unit cost c, so

$$h = Ic,$$

where the proportionality factor I is referred to as the *inventory holding cost rate.* Thus, the data entered includes I along with a and K. You also need to enter the number of discount categories (where the lowest-quantity category with no discount counts as one of these), as well as the unit price and range of order quantities for each of the categories. The template then finds the feasible order quantity that minimizes the total annual cost for each category, and also shows the individual annual costs (including the annual purchase cost) that would result. Using this information, the template identifies the overall optimal order quantity and the resulting total annual cost.

All these templates can be helpful for calculating a lot of information quickly after entering the basic data for the problem. However, perhaps a more important use is for performing sensitivity analysis on these data. You can immediately see how the results would change for any specific change in the data by entering the new data values in the spreadsheet. Doing this repeatedly for a variety of changes in the data is a convenient way to perform sensitivity analysis.

Observations about EOQ Models

1. If it is assumed that the unit cost of an item is constant throughout time independent of the batch size (as with the first two EOQ models), the unit cost does not appear in the optimal solution for the batch size. This result occurs because no matter what inventory policy is used, the same number of units is required per unit time, so this cost per unit time is fixed.
2. The analysis of the EOQ models assumed that the batch size Q is constant from cycle to cycle. The resulting *optimal* batch size Q^* actually minimizes the total cost per unit time for any cycle, so the analysis shows that this constant batch size should be used from cycle to cycle even if a constant batch size is not assumed.

3. The optimal inventory level at which inventory should be replenished can never be greater than zero under these models. Waiting until the inventory level drops to zero (or less than zero when planned shortages are permitted) reduces both holding costs and the frequency of incurring the setup cost K. However, if the assumptions of *a known constant demand rate* and *the order quantity will arrive just when desired* (because of a constant lead time) are not completely satisfied, it may become prudent to plan to have some "safety stock" left when the inventory is scheduled to be replenished. This is accomplished by increasing the reorder point above that implied by the model.

4. The basic assumptions of the EOQ models are rather demanding ones. They seldom are satisfied completely in practice. For example, even when a constant demand rate is planned (as with the production line in the TV speakers example in Sec. 19.1), interruptions and variations in the demand rate still are likely to occur. It also is very difficult to satisfy the assumption that the order quantity to replenish inventory arrives just when desired. Although the schedule may call for a constant lead time, variations in the actual lead times often will occur. Fortunately, the EOQ models have been found to be robust in the sense that they generally still provide nearly optimal results even when their assumptions are only rough approximations of reality. This is a key reason why these models are so widely used in practice. However, in those cases where the assumptions are significantly violated, it is important to do some preliminary analysis to evaluate the adequacy of an EOQ model before it is used. This preliminary analysis should focus on calculating the total cost per unit time provided by the model for various order quantities and then assessing how this cost curve would change under more realistic assumptions.

A Broader Perspective of the Speaker Example

Example 2 (wholesale distribution of bicycles) introduced in Sec. 19.1 focused on managing the inventory of one model of bicycle. The demand for this product is generated by the wholesaler's customers (various retailers) who purchase these bicycles to replenish their inventories according to their own schedules. The wholesaler has no control over this demand. Because this model is sold separately from other models, its demand does not even depend on the demand for any of the company's other products. Such demand is referred to as **independent demand.**

The situation is different for the speaker example introduced in Sec. 19.1. Here, the product under consideration—television speakers—is just one component being assembled into the company's final product—television sets. Consequently, the demand for the speakers depends on the demand for the television set. The pattern of this demand for the speakers is determined internally by the production schedule that the company establishes for the television sets by adjusting the production rate for the production line producing the sets. Such demand is referred to as **dependent demand.**

The television manufacturing company produces a considerable number of products—various parts and subassemblies—that become components of the television sets. Like the speakers, these various products also are **dependent-demand products.**

Because of the dependencies and interrelationships involved, managing the inventories of dependent-demand products can be considerably more complicated than for independent-demand products. A popular technique for assisting in this task is **material re-**

quirements planning, abbreviated as **MRP.** MRP is a computer-based system for planning, scheduling, and controlling the production of all the components of a final product. The system begins by "exploding" the product by breaking it down into all its subassemblies and then into all its individual component parts. A production schedule is then developed, using the demand and lead time for each component to determine the demand and lead time for the subsequent component in the process. In addition to a *master production schedule* for the final product, a *bill of materials* provides detailed information about all its components. Inventory status records give the current inventory levels, number of units on order, etc., for all the components. When more units of a component need to be ordered, the MRP system automatically generates either a purchase order to the vendor or a work order to the internal department that produces the component.

When the basic EOQ model was used to calculate the optimal production lot size for the speaker example, a very large quantity (25,298 speakers) was obtained. This enables having relatively infrequent setups to initiate production runs (only once every 3.2 months). However, it also causes large average inventory levels (12,649 speakers), which leads to a large total holding cost per year of over \$45,000.

The basic reason for this large cost is the high setup cost of $K = \$12{,}000$ for each production run. The setup cost is so sizable because the production facilities need to be set up again from scratch each time. Consequently, even with less than four production runs per year, the annual setup cost is over \$45,000, just like the annual holding costs.

Rather than continuing to tolerate a \$12,000 setup cost each time in the future, another option for the company is to seek ways to reduce this setup cost. One possibility is to develop methods for quickly transferring machines from one use to another. Another is to dedicate a group of production facilities to the production of speakers so they would remain set up between production runs in preparation for beginning another run whenever needed.

Suppose the setup cost could be drastically reduced from \$12,000 all the way down to $K = \$120$. This would reduce the optimal production lot size from 25,298 speakers down to $Q^* = 2{,}530$ speakers, so a new production run lasting only a brief time would be initiated more than 3 times per month. This also would reduce both the annual setup cost and the annual holding cost from over \$45,000 down to only slightly over \$4,500 each. By having such frequent (but inexpensive) production runs, the speakers would be produced essentially *just in time* for their assembly into television sets.

Just in time actually is a well-developed philosophy for managing inventories. A **just-in-time (JIT)** inventory system places great emphasis on reducing inventory levels to a bare minimum, and so providing the items just in time as they are needed. This philosophy was first developed in Japan, beginning with the Toyota Company in the late 1950s, and is given part of the credit for the remarkable gains in Japanese productivity through much of the late 20th century. The philosophy also has become popular in other parts of the world, including the United States, in more recent years.

Although the just-in-time philosophy sometimes is misinterpreted as being incompatible with using an EOQ model (since the latter gives a large order quantity when the setup cost is large), they actually are complementary. A JIT inventory system focuses on finding ways to greatly reduce the setup costs so that the optimal order quantity will be small. Such a system also seeks ways to reduce the lead time for the delivery of an order, since this reduces the uncertainty about the number of units that will be needed when the delivery occurs. Another emphasis is on improving preventive maintenance so that the required production facilities will be available to produce the units when they are needed. Still another emphasis is on improving the production process to guarantee good quality. Providing just the right number of units just in time does not provide any leeway for including defective units.

In more general terms, the focus of the just-in-time philosophy is on *avoiding waste* wherever it might occur in the production process. One form of waste is unnecessary inventory. Others are unnecessarily large setup costs, unnecessarily long lead times, production facilities that are not operational when they are needed, and defective items. Minimizing these forms of waste is a key component of superior inventory management.

19.4 A DETERMINISTIC PERIODIC-REVIEW MODEL

The preceding section explored the basic EOQ model and some of its variations. The results were dependent upon the assumption of a constant demand rate. When this assumption is relaxed, i.e., when the amounts that need to be withdrawn from inventory are allowed to vary from period to period, the *EOQ formula* no longer ensures a minimum-cost solution.

Consider the following periodic-review model. Planning is to be done for the next n periods regarding how much (if any) to produce or order to replenish inventory at the beginning of each of the periods. (The order to replenish inventory can involve either *purchasing* the units or *producing* them, but the latter case is far more common with applications of this model, so we mainly will use the terminology of *producing* the units.) The demands for the respective periods are *known* (but *not* the same in every period) and are denoted by

$$r_i = \text{demand in period } i, \qquad \text{for } i = 1, 2, \ldots, n.$$

These demands must be met on time. There is no stock on hand initially, but there is still time for a delivery at the beginning of period 1.

The costs included in this model are similar to those for the basic EOQ model:

K = setup cost for producing or purchasing any units to replenish inventory at beginning of period,

c = unit cost for producing or purchasing each unit,

h = holding cost for each unit left in inventory at end of period.

Note that this holding cost h is assessed only on inventory left at the end of a period. There also are holding costs for units that are in inventory for a portion of the period before being withdrawn to satisfy demand. However, these are *fixed* costs that are independent of the inventory policy and so are not relevant to the analysis. Only the *variable* costs that are affected by which inventory policy is chosen, such as the extra holding costs that are incurred by carrying inventory over from one period to the next, are relevant for selecting the inventory policy.

By the same reasoning, the unit cost c is an irrelevant fixed cost because, over all the time periods, all inventory policies produce the same number of units at the same cost. Therefore, c will be dropped from the analysis hereafter.

The objective is to minimize the total cost over the n periods. This is accomplished by ignoring the fixed costs and minimizing the total variable cost over the n periods, as illustrated by the following example.

Example. An airplane manufacturer specializes in producing small airplanes. It has just received an order from a major corporation for 10 customized executive jet airplanes for the use of the corporation's upper management. The order calls for three of the airplanes to be delivered (and paid for) during the upcoming winter months (period 1), two more to be delivered during the spring (period 2), three more during the summer (period 3), and the final two during the fall (period 4).

Setting up the production facilities to meet the corporation's specifications for these airplanes requires a setup cost of \$2 million. The manufacturer has the capacity to produce all 10 airplanes within a couple of months, when the winter season will be under way. However, this would necessitate holding seven of the airplanes in inventory, at a cost of \$200,000 per airplane per period, until their scheduled delivery times. To reduce or eliminate these substantial holding costs, it may be worthwhile to produce a smaller number of these airplanes now and then to repeat the setup (again incurring the cost of \$2 million) in some or all of the subsequent periods to produce additional small numbers. Management would like to determine the least costly production schedule for filling this order.

Thus, using the notation of the model, the demands for this particular airplane during the four upcoming periods (seasons) are

$$r_1 = 3, \qquad r_2 = 2, \qquad r_3 = 3, \qquad r_4 = 2.$$

Using units of millions of dollars, the relevant costs are

$$K = 2, \qquad h = 0.2.$$

The problem is to determine how many airplanes to produce (if any) during the beginning of each of the four periods in order to minimize the total variable cost.

The high setup cost K gives a strong incentive not to produce airplanes every period and preferably just once. However, the significant holding cost h makes it undesirable to carry a large inventory by producing the entire demand for all four periods (10 airplanes) at the beginning. Perhaps the best approach would be an intermediate strategy where airplanes are produced more than once but less than four times. For example, one such feasible solution (but not an optimal one) is depicted in Fig. 19.4, which shows the evolution of the inventory level over the next year that results from producing three airplanes at the beginning of the first period, six airplanes at the beginning of the second period, and one airplane at the beginning of the fourth period. The dots give the inventory levels after any production at the beginning of the four periods.

How can the optimal production schedule be found? For this model in general, production (or purchasing) is automatic in period 1, but a decision on whether to produce must be made for each of the other $n - 1$ periods. Therefore, one approach to solving this model is to enumerate, for each of the 2^{n-1} combinations of production decisions, the possible quantities that can be produced in each period where production is to occur. This approach is rather cumbersome, even for moderate-sized n, so a more efficient method is desirable. Such a method is described next in general terms, and then we will return to

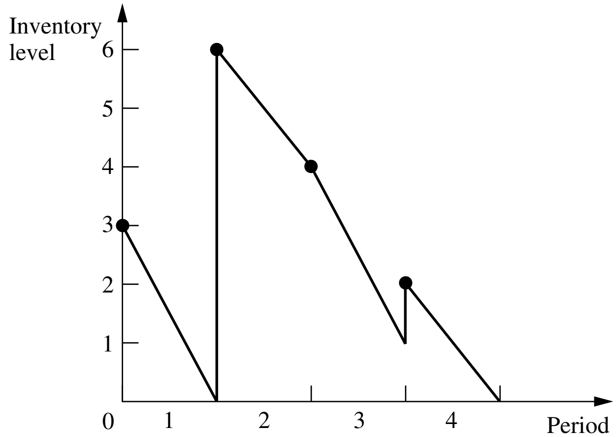


FIGURE 19.4
The inventory levels that result from one sample production schedule for the airplane example.

finding the optimal production schedule for the example. Although the general method can be used when either producing or purchasing to replenish inventory, we now will only use the terminology of producing for definiteness.

An Algorithm

The key to developing an efficient algorithm for finding an *optimal inventory policy* (or equivalently, an *optimal production schedule*) for the above model is the following insight into the nature of an optimal policy.

> An optimal policy (production schedule) produces *only* when the inventory level is *zero.*

To illustrate why this result is true, consider the policy shown in Fig. 19.4 for the example. (Call it policy *A*.) Policy *A* violates the above characterization of an optimal policy because production occurs at the beginning of period 4 when the inventory level is *greater than zero* (namely, one airplane). However, this policy can easily be adjusted to satisfy the above characterization by simply producing one less airplane in period 2 and one more airplane in period 4. This adjusted policy (call it *B*) is shown by the dashed line in Fig. 19.5 wherever *B* differs from *A* (the solid line). Now note that policy *B must* have less total cost than policy *A*. The setup costs (and the production costs) for both policies are the same. However, the holding cost is smaller for *B* than for *A* because *B* has less inventory than *A* in periods 2 and 3 (and the same inventory in the other periods). Therefore, *B* is better than *A*, so *A* cannot be optimal.

This characterization of optimal policies can be used to identify policies that are not optimal. In addition, because it implies that the only choices for the amount produced at the beginning of the ith period are 0, r_i, $r_i + r_{i+1}$, . . . , or $r_i + r_{i+1} + \cdots + r_n$, it can be exploited to obtain an efficient algorithm that is related to the *deterministic dynamic programming* approach described in Sec. 11.3.

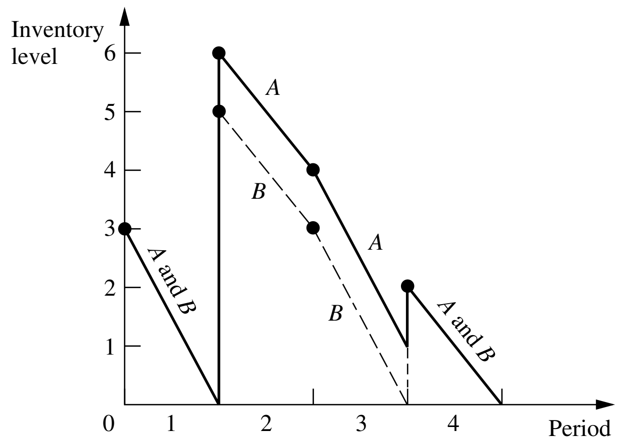


FIGURE 19.5
Comparison of two inventory policies (production schedules) for the airplane example.

In particular, define

C_i = total variable cost of an optimal policy for periods $i, i + 1, \ldots, n$ when period i starts with zero inventory (before producing), for $i = 1, 2, \ldots, n$.

By using the dynamic programming approach of solving *backward* period by period, these C_i values can be found by first finding C_n, then finding C_{n-1}, and so on. Thus, after C_n, $C_{n-1}, \ldots, C_{i+1}$ are found, then C_i can be found from the *recursive relationship*

$$C_i = \underset{j=i,\, i+1,\, \ldots,\, n}{\text{minimum}} \{C_{j+1} + K + h[r_{i+1} + 2r_{i+2} + 3r_{i+3} + \cdots + (j - i)r_j]\},$$

where j can be viewed as an index that denotes the (end of the) period when the inventory reaches a zero level for the first time after production at the beginning of period i. In the time interval from period i through period j, the term with coefficient h represents the total *holding cost* over this interval. When $j = n$, the term $C_{n+1} = 0$. The *minimizing value* of j indicates that if the inventory level does indeed drop to zero upon entering period i, then the production in period i should cover all demand from period i through this period j.

The algorithm for solving the model consists basically of solving for $C_n, C_{n-1}, \ldots,$ C_1 in turn. For $i = 1$, the minimizing value of j then indicates that the production in period 1 should cover the demand through period j, so the second production will be in period $j + 1$. For $i = j + 1$, the new minimizing value of j identifies the time interval covered by the second production, and so forth to the end. We will illustrate this approach with the example.

The application of this algorithm is much quicker than the full dynamic programming approach.[1] As in dynamic programming, $C_n, C_{n-1}, \ldots, C_2$ must be found before C_1 is obtained. However, the number of calculations is much smaller, and the number of possible production quantities is greatly reduced.

[1]The full dynamic programming approach is useful, however, for solving *generalizations* of the model (e.g., *nonlinear* production cost and holding cost functions) where the above algorithm is no longer applicable. (See Probs. 19.4-3 and 19.4-4 for examples where dynamic programming would be used to deal with generalizations of the model.)

Application of the Algorithm to the Example

Returning to the airplane example, first we consider the case of finding C_4, the cost of the optimal policy from the beginning of period 4 to the end of the planning horizon:

$$C_4 = C_5 + 2 = 0 + 2 = 2.$$

To find C_3, we must consider two cases, namely, the first time after period 3 when the inventory reaches a zero level occurs at (1) the end of the third period or (2) the end of the fourth period. In the recursive relationship for C_3, these two cases correspond to (1) $j = 3$ and (2) $j = 4$. Denote the corresponding costs (the right-hand side of the recursive relationship with this j) by $C_3^{(3)}$ and $C_3^{(4)}$, respectively. The policy associated with $C_3^{(3)}$ calls for producing only for period 3 and then following the optimal policy for period 4, whereas the policy associated with $C_3^{(4)}$ calls for producing for periods 3 and 4. The cost C_3 is then the minimum of $C_3^{(3)}$ and $C_3^{(4)}$. These cases are reflected by the policies given in Fig. 19.6.

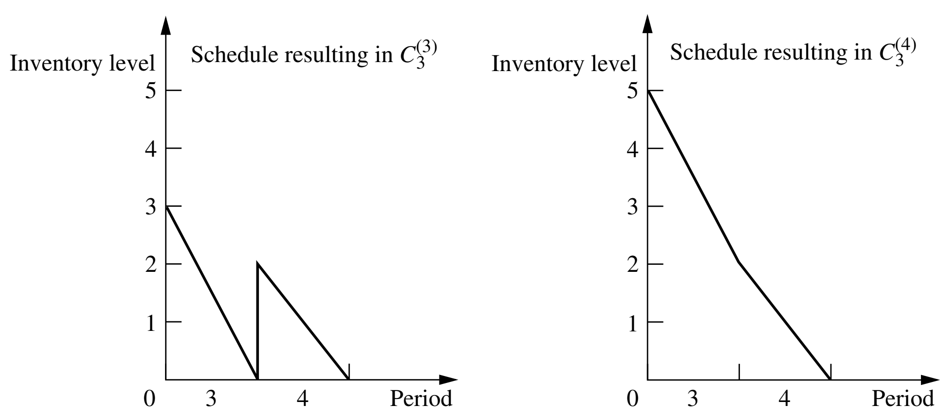
$$C_3^{(3)} = C_4 + 2 = 2 + 2 = 4.$$
$$C_3^{(4)} = C_5 + 2 + 0.2(2) = 0 + 2 + 0.4 = 2.4.$$
$$C_3 = \min\{4, 2.4\} = 2.4.$$

Therefore, if the inventory level drops to zero upon entering period 3 (so production should occur then), the production in period 3 should cover the demand for both periods 3 and 4.

To find C_2, we must consider three cases, namely, the first time after period 2 when the inventory reaches a zero level occurs at (1) the end of the second period, (2) the end of the third period, or (3) the end of the fourth period. In the recursive relationship for C_2, these cases correspond to (1) $j = 2$, (2) $j = 3$, and (3) $j = 4$, where the corresponding costs are $C_2^{(2)}$, $C_2^{(3)}$, and $C_2^{(4)}$, respectively. The cost C_2 is then the minimum of $C_2^{(2)}$, $C_2^{(3)}$, and $C_2^{(4)}$.

$$C_2^{(2)} = C_3 + 2 = 2.4 + 2 = 4.4.$$
$$C_2^{(3)} = C_4 + 2 + 0.2(3) = 2 + 2 + 0.6 = 4.6.$$
$$C_2^{(4)} = C_5 + 2 + 0.2[3 + 2(2)] = 0 + 2 + 1.4 = 3.4.$$
$$C_2 = \min\{4.4, 4.6, 3.4\} = 3.4.$$

FIGURE 19.6
Alternative production schedules when production is required at the beginning of period 3 for the airplane example.



Consequently, if production occurs in period 2 (because the inventory level drops to zero), this production should cover the demand for all the remaining periods.

Finally, to find C_1, we must consider four cases, namely, the first time after period 1 when the inventory reaches zero occurs at the end of (1) the first period, (2) the second period, (3) the third period, or (4) the fourth period. These cases correspond to $j = 1, 2, 3, 4$ and to the costs $C_1^{(1)}, C_1^{(2)}, C_1^{(3)}, C_1^{(4)}$, respectively. The cost C_1 is then the minimum of $C_1^{(1)}, C_1^{(2)}, C_1^{(3)}$, and $C_1^{(4)}$.

$$C_1^{(1)} = C_2 + 2 = 3.4 + 2 = 5.4.$$
$$C_1^{(2)} = C_3 + 2 + 0.2(2) = 2.4 + 2 + 0.4 = 4.8.$$
$$C_1^{(3)} = C_4 + 2 + 0.2[2 + 2(3)] = 2 + 2 + 1.6 = 5.6.$$
$$C_1^{(4)} = C_5 + 2 + 0.2[2 + 2(3) + 3(2)] = 0 + 2 + 2.8 = 4.8.$$
$$C_1 = \min\{5.4, 4.8, 5.6, 4.8\} = 4.8.$$

Note that $C_1^{(2)}$ and $C_1^{(4)}$ tie as the minimum, giving C_1. This means that the policies corresponding to $C_1^{(2)}$ and $C_1^{(4)}$ tie as being the optimal policies. The $C_1^{(4)}$ policy says to produce enough in period 1 to cover the demand for all four periods. The $C_1^{(2)}$ policy covers only the demand through period 2. Since the latter policy has the inventory level drop to zero at the end of period 2, the C_3 result is used next, namely, produce enough in period 3 to cover the demand for periods 3 and 4. The resulting production schedules are summarized below.

Optimal Production Schedules.

1. Produce 10 airplanes in period 1.

Total variable cost = \$4.8 million.

2. Produce 5 airplanes in period 1 and 5 airplanes in period 3.

Total variable cost = \$4.8 million.

19.5 A STOCHASTIC CONTINUOUS-REVIEW MODEL

We now turn to *stochastic* inventory models, which are designed for analyzing inventory systems where there is considerable uncertainty about future demands. In this section, we consider a *continuous-review* inventory system. Thus, the inventory level is being monitored on a continuous basis so that a new order can be placed as soon as the inventory level drops to the reorder point.

The traditional method of implementing a *continuous-review* inventory system was to use a **two-bin system.** All the units for a particular product would be held in two bins. The capacity of one bin would equal the reorder point. The units would first be withdrawn from the other bin. Therefore, the emptying of this second bin would trigger placing a new order. During the lead time until this order is received, units would then be withdrawn from the first bin.

In more recent years, two-bin systems have been largely replaced by **computerized inventory systems.** Each addition to inventory and each sale causing a withdrawal are recorded electronically, so that the current inventory level always is in the computer. (For example, the modern scanning devices at retail store checkout stands may both itemize your purchases and record the sales of stable products for purposes of adjusting the current inventory levels.) Therefore, the computer will trigger a new order as soon as the inventory level has dropped to the reorder point. Several excellent software packages are available from software companies for implementing such a system.

Because of the extensive use of computers for modern inventory management, continuous-review inventory systems have become increasingly prevalent for products that are sufficiently important to warrant a formal inventory policy.

A continuous-review inventory system for a particular product normally will be based on two critical numbers:

R = reorder point.
Q = order quantity.

For a manufacturer managing its finished products inventory, the order will be for a *production run* of size Q. For a wholesaler or retailer (or a manufacturer replenishing its raw materials inventory from a supplier), the order will be a *purchase order* for Q units of the product.

An inventory policy based on these two critical numbers is a simple one.

Inventory policy: Whenever the inventory level of the product drops to R units, place an order for Q more units to replenish the inventory.

Such a policy is often called a *reorder-point, order-quantity policy,* or **(R, Q) policy** for short. [Consequently, the overall model might be referred to as the (R, Q) model. Other variations of these names, such as (Q, R) policy, (Q, R) model, etc., also are sometimes used.]

After summarizing the model's assumptions, we will outline how R and Q can be determined.

The Assumptions of the Model

1. Each application involves a single product.
2. The inventory level is under *continuous review,* so its current value always is known.
3. An (R, Q) policy is to be used, so the only decisions to be made are to choose R and Q.
4. There is a *lead time* between when the order is placed and when the order quantity is received. This lead time can be either fixed or variable.
5. The *demand* for withdrawing units from inventory to sell them (or for any other purpose) during this lead time is uncertain. However, the probability distribution of demand is known (or at least estimated).
6. If a stockout occurs before the order is received, the excess demand is *backlogged,* so that the backorders are filled once the order arrives.
7. A fixed *setup cost* (denoted by K) is incurred each time an order is placed.
8. Except for this setup cost, the cost of the order is proportional to the order quantity Q.
9. A certain holding cost (denoted by h) is incurred for each unit in inventory per unit time.
10. When a stockout occurs, a certain shortage cost (denoted by p) is incurred for each unit backordered per unit time until the backorder is filled.

This model is closely related to the *EOQ model with planned shortages* presented in Sec. 19.3. In fact, all these assumptions also are consistent with that model, with the one key exception of assumption 5. Rather than having uncertain demand, that model assumed *known demand* with a fixed rate.

Because of the close relationship between these two models, their results should be fairly similar. The main difference is that, because of the uncertain demand for the current model, some safety stock needs to be added when setting the reorder point to provide some cushion for having well-above-average demand during the lead time. Otherwise, the trade-offs between the various cost factors are basically the same, so the order quantities from the two models should be similar.

Choosing the Order Quantity *Q*

The most straightforward approach to choosing Q for the current model is to simply use the formula given in Sec. 19.3 for the EOQ model with planned shortages. This formula is

$$Q = \sqrt{\frac{2AK}{h}}\sqrt{\frac{p+h}{p}},$$

where A now is the *average* demand per unit time, and where K, h, and p are defined in assumptions 7, 9, and 10, respectively.

This Q will be only an approximation of the optimal order quantity for the current model. However, no formula is available for the exact value of the optimal order quantity, so an approximation is needed. Fortunately, the approximation given above is a fairly good one.[1]

Choosing the Reorder Point *R*

A common approach to choosing the reorder point R is to base it on management's desired level of service to customers. Thus, the starting point is to obtain a managerial decision on service level. (Problem 19.5-3 analyzes the factors involved in this managerial decision.)

Service level can be defined in a number of different ways in this context, as outlined below.

Alternative Measures of Service Level.

1. The probability that a stockout will not occur between the time an order is placed and the order quantity is received.
2. The average number of stockouts per year.
3. The average percentage of annual demand that can be satisfied immediately (no stockout).
4. The average delay in filling backorders when a stockout occurs.
5. The overall average delay in filling orders (where the delay without a stockout is 0).

Measures 1 and 2 are closely related. For example, suppose that the order quantity Q has been set at 10 percent of the annual demand, so an average of 10 orders are placed

[1]For further information about the quality of this approximation, see S. Axsäter, "Using the Deterministic EOQ Formula in Stochastic Inventory Control," *Management Science,* **42:** 830–834, 1996. Also see Y.-S. Zheng, "On Properties of Stochastic Systems," *Management Science,* **38:** 87–103, 1992.

per year. If the probability is 0.2 that a stockout *will* occur during the lead time until an order is received, then the average number of stockouts per year would be $10(0.2) = 2$.

Measures 2 and 3 also are related. For example, suppose an average of 2 stockouts occur per year and the average length of a stockout is 9 days. Since $2(9) = 18$ days of stockout per year are essentially 5 percent of the year, the average percentage of annual demand that can be satisfied immediately would be 95 percent.

In addition, measures 3, 4, and 5 are related. For example, suppose that the average percentage of annual demand that can be satisfied immediately is 95 percent and the average delay in filling backorders when a stockout occurs is 5 days. Since only 5 percent of the customers incur this delay, the overall average delay in filling orders then would be $0.05(5) = 0.25$ day per order.

A managerial decision needs to be made on the desired value of at least one of these measures of service level. After selecting one of these measures on which to focus primary attention, it is useful to explore the implications of several alternative values of this measure on some of the other measures before choosing the best alternative.

Measure 1 probably is the most convenient one to use as the primary measure, so we now will focus on this case. We will denote the desired level of service under this measure by L, so

L = management's desired probability that a stockout will not occur between the time an order quantity is placed and the order quantity is received.

Using measure 1 involves working with the estimated probability distribution of the following random variable.

D = demand during the lead time in filling an order.

For example, with a uniform distribution, the formula for choosing the reorder point R is a simple one.

If the probability distribution of D is a *uniform distribution* over the interval from a to b, set

$$R = a + L(b - a),$$

because then

$$P(D \leq R) = L.$$

Since the mean of this distribution is

$$E(D) = \frac{a + b}{2},$$

the amount of **safety stock** (the expected inventory level *just* before the order quantity is received) provided by the reorder point R is

$$\text{Safety stock} = R - E(D) = a + L(b - a) - \frac{a + b}{2}$$
$$= \left(L - \frac{1}{2}\right)(b - a).$$

When the demand distribution is something other than a uniform distribution, the procedure for choosing R is similar.

General Procedure for Choosing R under Service Level Measure 1.

1. Choose L.
2. Solve for R such that

$$P(D \leq R) = L.$$

For example, suppose that D has a normal distribution with mean μ and variance σ^2, as shown in Fig. 19.7. Given the value of L, the table for the normal distribution given in Appendix 5 then can be used to determine the value of R. In particular, you just need to find the value of K_{1-L} in this table and then plug into the following formula to find R.

$$R = \mu + K_{1-L}\sigma.$$

The resulting amount of safety stock is

$$\text{Safety stock} = R - \mu = K_{1-L}\sigma.$$

To illustrate, if $L = 0.75$, then $K_{1-L} = 0.675$, so

$$R = \mu + 0.675\sigma,$$

as shown in Fig. 19.7. This provides

$$\text{Safety stock} = 0.675\sigma.$$

Your OR Courseware also includes an Excel template that will calculate both the order quantity Q and the reorder point R for you. You need to enter the average demand per unit time (A), the costs (K, h, and p), and the service level based on measure 1. You also indicate whether the probability distribution of the demand during the lead time is a uniform distribution or a normal distribution. For a uniform distribution, you specify the interval over which the distribution extends by entering the lower endpoint and upper endpoint of this interval. For a normal distribution, you instead enter the mean μ and standard deviation σ of the distribution. After you provide all this information, the template immediately calculates Q and R and displays these results on the right side.

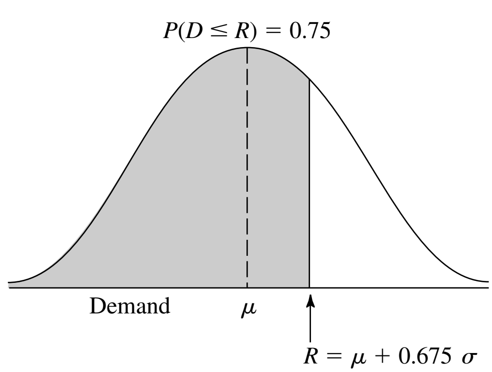


FIGURE 19.7
Calculation of the reorder point R for the stochastic continuous-review model when $L = 0.75$ and the probability distribution of the demand over the lead time is a normal distribution with mean μ and standard deviation σ.

An Example. Consider once again Example 1 (manufacturing speakers for TV sets) presented in Sec. 19.1. Recall that the setup cost to produce the speakers is $K = \$12{,}000$, the unit holding cost is $h = \$0.30$ per speaker per month, and the unit shortage cost is $p = \$1.10$ per speaker per month.

Originally, there was a fixed demand rate of 8,000 speakers per month to be assembled into television sets being produced on a production line at this fixed rate. However, sales of the TV sets have been quite variable, so the inventory level of finished sets has fluctuated widely. To reduce inventory holding costs for finished sets, management has decided to adjust the production rate for the sets on a daily basis to better match the output with the incoming orders.

Consequently, the demand for the speakers now is quite variable. There is a *lead time* of 1 month between ordering a production run to produce speakers and having speakers ready for assembly into television sets. The demand for speakers during this lead time is a random variable D that has a normal distribution with a mean of 8,000 and a standard deviation of 2,000. To minimize the risk of disrupting the production line producing the TV sets, management has decided that the safety stock for speakers should be large enough to avoid a stockout during this lead time 95 percent of the time.

To apply the model, the order quantity for each production run of speakers should be

$$Q = \sqrt{\frac{2AK}{h}}\sqrt{\frac{p+h}{p}} = \sqrt{\frac{2(8{,}000)(12{,}000)}{0.30}}\sqrt{\frac{1.1+0.3}{1.1}} = 28{,}540.$$

This is the same order quantity that was found by the EOQ model with planned shortages in Sec. 19.3 for the previous version of this example where there was a *constant* (rather than average) demand rate of 8,000 speakers per month and planned shortages were allowed. However, the key difference from before is that safety stock now needs to be provided to counteract the variable demand. Management has chosen a service level of $L = 0.95$, so the normal table in Appendix 5 gives $K_{1-L} = 1.645$. Therefore, the reorder point should be

$$R = \mu + K_{1-L}\sigma = 8{,}000 + 1.645(2{,}000) = 11{,}290.$$

The resulting amount of safety stock is

$$\text{Safety stock} = R - \mu = 3{,}290.$$

19.6 A STOCHASTIC SINGLE-PERIOD MODEL FOR PERISHABLE PRODUCTS

When choosing the inventory model to use for a particular product, a distinction should be made between two types of products. One type is a **stable product,** which will remain sellable indefinitely so there is no deadline for disposing of its inventory. This is the kind of product considered in the preceding sections (as well as the next section). The other type, by contrast, is a **perishable product,** which can be carried in inventory for only a very limited period of time before it can no longer be sold. This is the kind of product for which the single-period model (and its variations) presented in this section is designed. In particular, the single period in the model is the very limited period before the product can no longer be sold.

One example of a perishable product is a daily newspaper being sold at a newsstand. A particular day's newspaper can be carried in inventory for only a single day before it becomes outdated and needs to be replaced by the next day's newspaper. When the demand for the newspaper is a random variable (as assumed in this section), the owner of the newsstand needs to choose a daily order quantity that provides an appropriate trade-off between the potential cost of overordering (the wasted expense of ordering more newspapers than can be sold) and the potential cost of underordering (the lost profit from ordering fewer newspapers than can be sold). This section's model enables solving for the daily order quantity that would maximize the expected profit.

Because the general problem being analyzed fits this example so well, the problem has traditionally been called the **newsboy problem.**[1] However, it has always been recognized that the model being used is just as applicable to other perishable products as to newspapers. In fact, most of the applications have been to perishable products other than newspapers, including the examples of perishable products listed below.

Some Types of Perishable Products

As you read through the list below of various types of perishable products, think about how the inventory management of such products is analogous to a newsstand dealing with a daily newspaper since these products also cannot be sold after a single time period. All that may differ is that the length of this time period may be a week, a month, or even several months rather than just one day.

1. Periodicals, such as newspapers and magazines.
2. Flowers being sold by a florist.
3. The makings of fresh food to be prepared in a restaurant.
4. Produce, including fresh fruits and vegetables, to be sold in a grocery store.
5. Christmas trees.
6. Seasonal clothing, such as winter coats, where any goods remaining at the end of the season must be sold at highly discounted prices to clear space for the next season.
7. Seasonal greeting cards.
8. Fashion goods that will be out of style soon.
9. New cars at the end of a model year.
10. Any product that will be obsolete soon.
11. Vital spare parts that must be produced during the last production run of a certain model of a product (e.g., an airplane) for use as needed throughout the lengthy field life of that model.
12. Reservations provided by an airline for a particular flight. Reservations provided in excess of the number of seats available (overbooking) can be viewed as the inventory of a perishable product (they cannot be sold after the flight has occurred), where the demand then is the number of no-shows. With this interpretation, the cost of underordering (too little overbooking) would be the lost profit from empty seats and the cost of overordering (too much overbooking) would be the cost of compensating bumped customers.

[1]Recently, some writers have been substituting the name *newsvendor problem.* Other names include the *single-period probabilistic model* and *single-period stochastic model.*

This last type is a particularly interesting one because major airlines (and various other companies involved with transporting passengers) now are making extensive use of this section's model to analyze how much overbooking to do. For example, an article in the January–February 1992 issue of *Interfaces* describes how *American Airlines* is dealing with overbooking in this way. In addition, the article describes how the company is also using operations research to address some related issues (such as the fare structure). These particular OR applications (commonly called *revenue management*) are credited with increasing American Airline's annual revenues by over $500 million. The total impact on annual profits throughout the passenger transportation industry would run into the billions of dollars.

When managing the inventory of these various types of perishable products, it is occasionally necessary to deal with some considerations beyond those that will be discussed in this section. Extensive research has been conducted to extend the model to encompass these considerations, and considerable progress has been made. Further information is available in the footnoted references.[1]

An Example

Refer back to Example 2 in Sec. 19.1, which involves the wholesale distribution of a particular bicycle model (a small one-speed girl's bicycle). There now has been a new development. The manufacturer has just informed the distributor that this model is being discontinued. To help clear out its stock, the manufacturer is offering the distributor the opportunity to make one final purchase at very favorable terms, namely, a *unit cost* of only $20 per bicycle. With these special arrangements, the distributor also would incur *no setup cost* to place this order.

The distributor feels that this offer provides an ideal opportunity to make one final round of sales to its customers (bicycle shops) for the upcoming Christmas season for a reduced price of only $45 per bicycle, thereby making a profit of $25 per bicycle. This will need to be a one-time sale only because this model soon will be replaced by a new model that will make it obsolete. Therefore, any bicycles not sold during this sale will become almost worthless. However, the distributor believes that she will be able to dispose of any remaining bicycles after Christmas by selling them for the nominal price of $10 each (the *salvage value*), thereby recovering half of her purchase cost. Considering this loss if she orders more than she can sell, as well as the lost profit if she orders fewer than can be sold, the distributor needs to decide what order quantity to submit to the manufacturer.

Another relevant expense is the cost of maintaining unsold bicycles in inventory until they can be disposed of after Christmas. Combining the cost of capital tied up in inventory and other storage costs, this inventory cost is estimated to be $1 per bicycle remaining in inventory after Christmas. Thus, considering the salvage value of $10 as well, the *unit holding cost* is $-\$9$ per bicycle left in inventory at the end.

Two remaining cost components still require discussion, the shortage cost and the revenue. If the demand exceeds the supply, those customers who fail to purchase a bicy-

[1]See H.-S. Lau and A. H.-L. Lau, "The Newsstand Problem: A Capacitated Multiple Product Single-Period Inventory Problem," *European Journal of Operational Research,* **94:** 29–42, Oct. 11, 1996, and its references. Also see pp. 610–628 in E. L. Porteus, "Stochastic Inventory Theory," in D. P. Heyman and M. J. Sobel (eds.), *Stochastic Models,* North Holland, Amsterdam, 1990.

cle may bear some ill will, thereby resulting in a "cost" to the distributor. This cost is the per-item quantification of the loss of goodwill times the unsatisfied demand whenever a shortage occurs. The distributor considers this cost to be negligible.

If we adopt the criterion of maximizing profit, we must include revenue in the model. Indeed, the total profit is equal to total revenue minus the costs incurred (the ordering, holding, and shortage costs). Assuming no initial inventory, this profit for the distributor is

$$\begin{aligned}\text{Profit} = {} & \$45 \times \text{number sold by distributor}\\ & - \$20 \times \text{number purchased by distributor}\\ & + \$9 \times \text{number unsold and so disposed of for salvage value.}\end{aligned}$$

Let

$$y = \text{number purchased by distributor}$$

and

$$D = \text{demand by bicycle shops (a random variable),}$$

so that

$$\begin{aligned}\min\{D, y\} &= \text{number sold,}\\ \max\{0, y - D\} &= \text{number unsold.}\end{aligned}$$

Then

$$\text{Profit} = 45 \min\{D, y\} - 20y + 9 \max\{0, y - D\}.$$

The first term also can be written as

$$45 \min\{D, y\} = 45D - 45 \max\{0, D - y\}.$$

The term $45 \max\{0, D - y\}$ represents the *lost revenue from unsatisfied demand.* This lost revenue, plus any cost of the loss of customer goodwill due to unsatisfied demand (assumed negligible in this example), will be interpreted as the *shortage cost* throughout this section.

Now note that $45D$ is independent of the inventory policy (the value of y chosen) and so can be deleted from the objective function, which leaves

$$\text{Relevant profit} = -45 \max\{0, D - y\} - 20y + 9 \max\{0, y - D\}$$

to be maximized. All the terms on the right are the *negative* of *costs,* where these costs are the *shortage cost,* the *ordering cost,* and the *holding cost* (which has a negative value here), respectively. Rather than *maximizing* the *negative* of *total cost,* we instead will do the equivalent of *minimizing*

$$\text{Total cost} = 45 \max\{0, D - y\} + 20y - 9 \max\{0, y - D\}.$$

More precisely, since total cost is a random variable (because D is a random variable), the objective adopted for the model is to *minimize the expected total cost.*

In the discussion about the interpretation of the shortage cost, we assumed that the unsatisfied demand was lost (no backlogging). If the unsatisfied demand could be met by a priority shipment, similar reasoning applies. The revenue component of net income would become the sales price of a bicycle ($45) times the demand *minus* the unit cost of the priority shipment times the unsatisfied demand whenever a shortage occurs. If our wholesale distributor could be forced to meet the unsatisfied demand by purchasing bicycles from the manufacturer for \$35 each plus an air freight charge of, say, \$2 each, then the appropriate shortage cost would be \$37 per bicycle. (If there were any costs associated with loss of goodwill, these also would be added to this amount.)

The distributor does not know what the demand for these bicycles will be; i.e., demand D is a random variable. However, an optimal inventory policy can be obtained if information about the probability distribution of D is available. Let

$$P_D(d) = P\{D = d\}.$$

It will be assumed that $P_D(d)$ is known for all values of d.

We now are in a position to summarize the model in general terms.

The Assumptions of the Model

1. Each application involves a single perishable product.
2. Each application involves a single time period because the product cannot be sold later.
3. However, it will be possible to dispose of any units of the product remaining at the end of the period, perhaps even receiving a *salvage value* for the units.
4. There is no initial inventory on hand.
5. The only decision to be made is the value of y, the number of units to order (either through purchasing or producing) so they can be placed into inventory at the beginning of the period.
6. The *demand* for withdrawing units from inventory to sell them (or for any other purpose) during the period is a random variable D. However, the probability distribution of D is known (or at least estimated).
7. After deleting the revenue if the demand were satisfied (since this is independent of the decision y), the objective becomes to minimize the expected total cost, where the cost components are

c = unit cost for purchasing or producing each unit,

h = holding cost per unit remaining at end of period (includes storage cost minus salvage value),

p = shortage cost per unit of unsatisfied demand (includes lost revenue and cost of loss of customer goodwill).

Analysis of the Model

The decision on the value of y, the amount of inventory to acquire, depends heavily on the probability distribution of demand D. More than the expected demand may be desirable, but probably less than the maximum possible demand. A trade-off is needed between (1) the risk of being short and thereby incurring shortage costs and (2) the risk of having an excess and thereby incurring wasted costs of ordering and holding excess units. This is accomplished by minimizing the expected value (in the statistical sense) of the sum of these costs.

The amount sold is given by

$$\min\{D, y\} = \begin{cases} D & \text{if } D < y \\ y & \text{if } D \geq y. \end{cases}$$

Hence, the cost incurred if the demand is D and y is stocked is given by

$$C(D, y) = cy + p \max\{0, D - y\} + h \max\{0, y - D\}.$$

Because the demand is a random variable [with probability distribution $P_D(d)$], this cost is also a random variable. The expected cost is then given by $C(y)$, where

$$C(y) = E[C(D, y)] = \sum_{d=0}^{\infty} (cy + p \max\{0, d - y\} + h \max\{0, y - d\})P_D(d)$$

$$= cy + \sum_{d=y}^{\infty} p(d - y)P_D(d) + \sum_{d=0}^{y-1} h(y - d)P_D(d).$$

The function $C(y)$ depends upon the probability distribution of D. Frequently, a representation of this probability distribution is difficult to find, particularly when the demand ranges over a large number of possible values. Hence, this *discrete random variable* is often approximated by a *continuous random variable.* Furthermore, when demand ranges over a large number of possible values, this approximation will generally yield a nearly exact value of the optimal amount of inventory to stock. In addition, when discrete demand is used, the resulting expressions may become slightly more difficult to solve analytically. Therefore, unless otherwise stated, *continuous demand* is assumed throughout the remainder of this chapter.

For this continuous random variable D, let

$\varphi_D(\xi)$ = probability density function of D

and

$\Phi(a)$ = cumulative distribution function (CDF) of D,

so

$$\Phi(a) = \int_0^a \varphi_D(\xi)\, d\xi.$$

When choosing an order quantity y, the CDF $\Phi(y)$ becomes the probability that a shortage will *not* occur before the period ends. As in the preceding section, this probability is referred to as the **service level** being provided by the order quantity. The corresponding expected cost $C(y)$ is expressed as

$$C(y) = E[C(D, y)] = \int_0^{\infty} C(\xi, y)\varphi_D(\xi)\, d\xi$$

$$= \int_0^{\infty} (cy + p \max\{0, \xi - y\} + h \max\{0, y - \xi\})\varphi_D(\xi)\, d\xi$$

$$= cy + \int_y^{\infty} p(\xi - y)\varphi_D(\xi)\, d\xi + \int_0^{y} h(y - \xi)\varphi_D(\xi)\, d\xi$$

$$= cy + L(y),$$

where $L(y)$ is often called the *expected shortage plus holding cost.* It then becomes necessary to find the value of y, say y^0, which minimizes $C(y)$. First we give the answer, and then we will show the derivation a little later.

The optimal quantity to order y^0 is that value which satisfies

$$\Phi(y^0) = \frac{p - c}{p + h}.$$

Thus, $\Phi(y^0)$ is the *optimal service level* and the corresponding order quantity y^0 can be obtained either by solving this equation algebraically or by plotting the CDF and then identifying y^0 graphically. To interpret the right-hand side of this equation, the numerator can be viewed as

$$\begin{aligned} p - c &= \text{unit cost of underordering} \\ &= \text{decrease in profit that results from failing to order a unit that could have been sold during the period.} \end{aligned}$$

Similarly,

$$\begin{aligned} c + h &= \text{unit cost of overordering} \\ &= \text{decrease in profit that results from ordering a unit that could not be sold during the period.} \end{aligned}$$

Therefore, denoting the unit cost of underordering and of overordering by C_{under} and C_{over}, respectively, this equation is specifying that

$$\text{Optimal service level} = \frac{C_{\text{under}}}{C_{\text{under}} + C_{\text{over}}}.$$

When the demand has either a uniform or an exponential distribution, an Excel template is available in your OR Courseware for calculating y^0.

If D is assumed to be a discrete random variable having the CDF

$$F_D(b) = \sum_{d=0}^{b} P_D(d),$$

a similar result for the optimal order quantity is obtained. In particular, the optimal quantity to order y^0 is the smallest integer such that

$$F_D(y^0) \geq \frac{p - c}{p + h}.$$

Application to the Example

Returning to the bicycle example described at the beginning of this section, we assume that the demand has an exponential distribution with a mean of 10,000, so that its probability density function is

$$\varphi_D(\xi) = \begin{cases} \dfrac{1}{10{,}000} e^{-\xi/10{,}000} & \text{if } \xi \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

and the CDF is

$$\Phi(a) = \int_0^a \frac{1}{10{,}000} e^{-\xi/10{,}000}\, d\xi = 1 - e^{-a/10{,}000}.$$

From the data given,

$$c = 20, \qquad p = 45, \qquad h = -9.$$

Consequently, the optimal quantity to order y^0 is that value which satisfies

$$1 - e^{-y^0/10{,}000} = \frac{45 - 20}{45 - 9} = 0.69444.$$

By using the natural logarithm (denoted by ln), this equation can be solved as follows:

$$\begin{aligned} e^{-y^0/10{,}000} &= 0.30556, \\ \ln e^{-y^0/10{,}000} &= \ln 0.30556, \\ \frac{-y_0}{10{,}000} &= -1.1856, \\ y^0 &= 11{,}856. \end{aligned}$$

Therefore, the distributor should stock 11,856 bicycles in the Christmas season. Note that this number is slightly more than the expected demand of 10,000.

Whenever the demand is exponential with expectation λ, then y^0 can be obtained from the relation

$$y^0 = -\lambda \ln \frac{c + h}{p + h}.$$

The Model with Initial Stock Level

In the above model we assume that there is no initial inventory. As a slight variation, suppose now that the distributor begins with 500 bicycles on hand. How does this stock influence the optimal inventory policy?

In general terms, suppose that the initial stock level is given by x, and the decision to be made is the value of y, the inventory level *after replenishment* by ordering (or producing) additional units. Thus, $y - x$ is to be ordered, so that

$$\text{Amount available } (y) = \text{initial stock } (x) + \text{amount ordered } (y - x).$$

The cost equation presented earlier remains identical except for the term that was previously cy. This term now becomes $c(y - x)$, so that minimizing the expected cost is given by

$$\min_{y \ge x} \left[c(y - x) + \int_y^{\infty} p(\xi - y)\varphi_D(\xi)\, d\xi + \int_0^y h(y - \xi)\varphi_D(\xi)\, d\xi \right].$$

The constraint $y \ge x$ must be added because the inventory level y after replenishing cannot be less than the initial inventory level x.

The optimal inventory policy is the following:

$$\text{If} \quad x \begin{cases} < y^0 & \text{order } y^0 - x \text{ to bring inventory level up to } y^0 \\ \ge y^0 & \text{do not order,} \end{cases}$$

where y^0 satisfies

$$\Phi(y^0) = \frac{p - c}{p + h}.$$

Thus, in the bicycle example, if there are 500 bicycles on hand, the optimal policy is to bring the inventory level up to 11,856 bicycles (which implies ordering 11,356 additional bicycles). On the other hand, if there were 12,000 bicycles already on hand, the optimal policy would be not to order.

Derivation of the Optimal Policy.[1] We start by assuming that the initial stock level is zero.

For any positive constants c_1 and c_2, define $g(\xi, y)$ as

$$g(\xi, y) = \begin{cases} c_1(y - \xi) & \text{if } y > \xi \\ c_2(\xi - y) & \text{if } y \le \xi, \end{cases}$$

and let

$$G(y) = \int_0^\infty g(\xi, y)\varphi_D(\xi)\, d\xi + cy,$$

where $c > 0$. Then $G(y)$ is minimized at $y = y^0$, where y^0 is the solution to

$$\Phi(y^0) = \frac{c_2 - c}{c_2 + c_1}.$$

To see why this value of y^0 minimizes $G(y)$, note that, by definition,

$$G(y) = c_1 \int_0^y (y - \xi)\varphi_D(\xi)\, d\xi + c_2 \int_y^\infty (\xi - y)\varphi_D(\xi)\, d\xi + cy.$$

Taking the derivative (see the end of Appendix 3) and setting it equal to zero lead to

$$\frac{dG(y)}{dy} = c_1 \int_0^y \varphi_D(\xi)\, d\xi - c_2 \int_y^\infty \varphi_D(\xi)\, d\xi + c = 0.$$

This expression implies that

$$c_1\Phi(y^0) - c_2[1 - \Phi(y^0)] + c = 0,$$

because

$$\int_0^\infty \varphi_D(\xi)\, d\xi = 1.$$

Solving this expression results in

$$\Phi(y^0) = \frac{c_2 - c}{c_2 + c_1}.$$

[1]This subsection may be omitted by the less mathematically inclined reader.

The solution of this equation minimizes $G(y)$ because

$$\frac{d^2G(y)}{dy^2} = (c_1 + c_2)\varphi_D(y) \geq 0$$

for all y.

To apply this result, it is sufficient to show that

$$C(y) = cy + \int_y^\infty p(\xi - y)\varphi_D(\xi)\, d\xi + \int_0^y h(y - \xi)\varphi_D(\xi)\, d\xi$$

has the form of $G(y)$. Clearly, $c_1 = h$, $c_2 = p$, and $c = c$, so that the optimal quantity to order y^0 is that value which satisfies

$$\Phi(y^0) = \frac{p - c}{p + h}.$$

To derive the results for the case where the initial stock level is $x > 0$, recall that it is necessary to solve the relationship

$$\min_{y \geq x} \left\{ -cx + \left[\int_y^\infty p(\xi - y)\varphi_D(\xi)\, d\xi + \int_0^y h(y - \xi)\varphi_D(\xi)\, d\xi + cy \right] \right\}.$$

Note that the expression in brackets has the form of $G(y)$, with $c_1 = h$, $c_2 = p$, and $c = c$. Hence, the cost function to be minimized can be written as

$$\min_{y \geq x} \{-cx + G(y)\}.$$

It is clear that $-cx$ is a constant, so that it is sufficient to find the y that satisfies the expression

$$\min_{y \geq x} G(y).$$

Therefore, the value of y^0 that minimizes $G(y)$ satisfies

$$\Phi(y^0) = \frac{p - c}{p + h}.$$

Furthermore, $G(y)$ must be a convex function, because

$$\frac{d^2G(y)}{dy^2} \geq 0.$$

Also,

$$\lim_{y \to 0} \frac{dG(y)}{dy} = c - p,$$

which is negative,[1] and

$$\lim_{y \to \infty} \frac{dG(y)}{dy} = h + c,$$

[1]If $c - p$ is nonnegative, $G(y)$ will be a monotone increasing function. This implies that the item should not be stocked, that is, $y^0 = 0$.

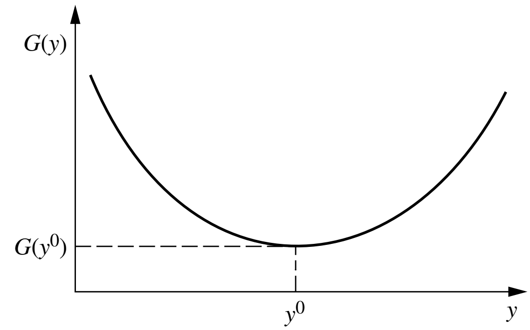


FIGURE 19.8
Graph of $G(y)$ for the stochastic single-period model.

which is positive. Hence, $G(y)$ must be as shown in Fig. 19.8. Thus, the optimal policy must be given by the following:

> If $x < y^0$, order $y^0 - x$ to bring the inventory level up to y^0, because y^0 can be achieved together with the minimum value $G(y^0)$. If $x \geq y^0$, do not order because any $G(y)$ with $y > x$ must exceed $G(x)$.

A similar argument can be constructed for obtaining optimal policies for the following model with nonlinear costs.

Model with Nonlinear Costs. Similar results for these models can be obtained for other than linear holding and shortage costs. Denote the holding cost by

$$\begin{array}{ll} h[y - D] & \text{if } y \geq D, \\ 0 & \text{if } y < D, \end{array}$$

where $h[\cdot]$ is a mathematical function, not necessarily linear.

Similarly, the shortage cost can be denoted by

$$\begin{array}{ll} p[D - y] & \text{if } D \geq y, \\ 0 & \text{if } D < y, \end{array}$$

where $p[\cdot]$ is also a function, not necessarily linear.

Thus, the total expected cost is given by

$$c(y - x) + \int_y^\infty p[\xi - y]\varphi_D(\xi)\, d\xi + \int_0^y h[y - \xi]\varphi_D(\xi)\, d\xi,$$

where x is the amount on hand.

If $L(y)$ is defined as the *expected shortage plus holding cost,* i.e.,

$$L(y) = \int_y^\infty p[\xi - y]\varphi_D(\xi)\, d\xi + \int_0^y h[y - \xi]\varphi_D(\xi)\, d\xi,$$

then the total expected cost can be written as

$$c(y - x) + L(y).$$

The optimal policy is obtained by minimizing this expression, subject to the constraint that $y \geq x$, that is,

$$\min_{y \geq x} \{c(y - x) + L(y)\}.$$

If $L(y)$ is strictly convex[1] [a sufficient condition being that the shortage and holding costs each are convex and $\varphi_D(\xi) > 0$], then the optimal policy is the following:

$$\text{If} \quad x \begin{cases} < y^0 & \text{order } y^0 - x \text{ to bring inventory level up to } y^0 \\ \geq y^0 & \text{do not order,} \end{cases}$$

where y^0 is the value of y that satisfies the expression

$$\frac{dL(y)}{dy} + c = 0.$$

A Single-Period Model with a Setup Cost

In discussing the bicycle example previously in this section, we assumed that there was no setup cost incurred in ordering the bicycles for the Christmas season. Suppose now that the cost of placing this special order is \$800, so this cost should be included in the analysis of the model. In fact, inclusion of the setup cost generally causes major changes in the results.

In general, the setup cost will be denoted by K. To begin, the shortage and holding costs will each be assumed to be linear. Their resulting effect is then given by $L(y)$, where

$$L(y) = p\int_y^{\infty} (\xi - y)\varphi_D(\xi)\, d\xi + h\int_0^y (y - \xi)\varphi_D(\xi)\, d\xi.$$

Thus, the total expected cost incurred by bringing the inventory level up to y is given by

$$\begin{array}{ll} K + c(y - x) + L(y) & \text{if } y > x, \\ L(x) & \text{if } y = x. \end{array}$$

Note that $cy + L(y)$ is the same expected cost considered earlier when the setup cost was omitted. If $cy + L(y)$ is drawn as a function of y, it will appear as shown in Fig. 19.9.[2] Define S as the value of y that minimizes $cy + L(y)$, and define s as the smallest value of y for which $cs + L(s) = K + cS + L(S)$. From Fig. 19.9, it can be seen that

$$\text{If} \quad x > S, \quad \text{then} \quad K + cy + L(y) > cx + L(x), \quad \text{for all } y > x,$$

so that

$$K + c(y - x) + L(y) > L(x).$$

The left-hand side of the last inequality represents the expected total cost of ordering $y - x$ to bring the inventory level up to y, and the right-hand side of this inequality

[1]See Appendix 2 for the definition of a strictly convex function.

[2]In the derivation of the optimal policy for the single-period model with no setup cost, $cy + L(y)$ was denoted by $G(y)$ and was rigorously shown to be a convex function of the form plotted in Fig. 19.9.

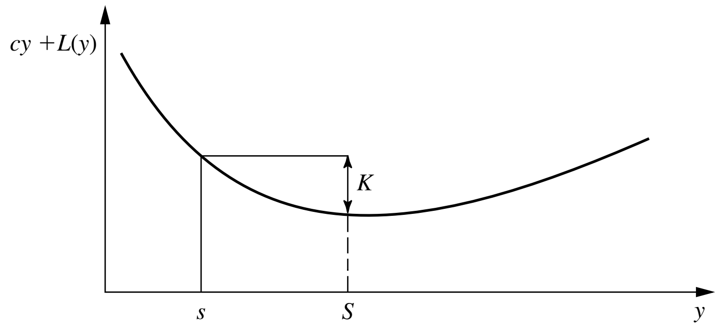


FIGURE 19.9
Graph of $cy + L(y)$ for the stochastic single-period model with a setup cost.

represents the expected total cost if no ordering occurs. Hence, the optimal policy indicates that if $x > S$, do not order.

If $s \leq x \leq S$, it can also be seen from Fig. 19.9 that

$$K + cy + L(y) \geq cx + L(x), \qquad \text{for all } y > x,$$

so that

$$K + c(y - x) + L(y) \geq L(x).$$

Again, no ordering is less expensive than ordering.

Finally, if $x < s$, it follows from Fig. 19.9 that

$$\min_{y \geq x} \{K + cy + L(y)\} = K + cS + L(S) < cx + L(x),$$

or

$$\min_{y \geq x} \{K + c(y - x) + L(y)\} = K + c(S - x) + L(S) < L(x),$$

so that it pays to order. The minimum cost is incurred by bringing the inventory level up to S.

The optimal inventory policy is the following:

$$\text{If} \qquad x\begin{cases} < s & \text{order } S - x \text{ to bring inventory level up to } S \\ \geq s & \text{do not order.} \end{cases}$$

The value of S is obtained from

$$\Phi(S) = \frac{p - c}{p + h},$$

and s is the smallest value that satisfies the expression

$$cs + L(s) = K + cS + L(S).$$

When the demand has either a uniform or an exponential distribution, an Excel template is available in your OR Courseware for calculating s and S.

This kind of policy is referred to as an **(*s*, *S*) policy.** It has had extensive use in industry.

Example. Referring to the bicycle example, we found earlier that

$$y^0 = S = 11{,}856.$$

If $K = 800$, $c = 20$, $p = 45$, and $h = -9$, then s is obtained from

$$20s + 45\int_s^{\infty} (\xi - s)\frac{1}{10{,}000}\, e^{-\xi/10{,}000}\, d\xi - 9\int_0^s (s - \xi)\frac{1}{10{,}000}e^{-\xi/10{,}000}\, d\xi$$

$$= 800 + 20(11{,}856) + 45\int_{11{,}856}^{\infty} (\xi - 11{,}856)\frac{1}{10{,}000}e^{-\xi/10{,}000}\, d\xi$$

$$-9\int_0^{11{,}856} (11{,}856 - \xi)\frac{1}{10{,}000}e^{-\xi/10{,}000}\, d\xi,$$

which leads to

$$s = 10{,}674.$$

Hence, the optimal policy calls for bringing the inventory level up to $S = 11{,}856$ bicycles if the amount on hand is less than $s = 10{,}674$. Otherwise, no order is placed.

Solution When the Demand Distribution Is Exponential. Now consider the special case where the distribution of demand D is exponential, i.e.,

$$\varphi_D(\xi) = \alpha e^{-\alpha\xi}, \qquad \text{for } \xi \geq 0.$$

From the no-setup-cost results,

$$1 - e^{-\alpha S} = \frac{p - c}{p + h},$$

so

$$S = \frac{1}{\alpha} \ln \frac{h + p}{h + c}.$$

For any y,

$$cy + L(y) = cy + h\int_0^y (y - \xi)\alpha e^{-\alpha\xi}\, d\xi + p\int_y^{\infty} (\xi - y)\alpha e^{-\alpha\xi}\, d\xi$$

$$= (c + h)y + \frac{1}{\alpha}(h + p)e^{-\alpha y} - \frac{h}{\alpha}.$$

Evaluating $cy + L(y)$ at the points $y = s$ and $y = S$ leads to

$$(c + h)s + \frac{1}{\alpha}(h + p)e^{-\alpha s} - \frac{h}{\alpha} = K + (c + h)S + \frac{1}{\alpha}(h + p)e^{-\alpha S} - \frac{h}{\alpha},$$

or

$$(c + h)s + \frac{1}{\alpha}(h + p)e^{-\alpha s} = K + (c + h)S + \frac{1}{\alpha}(c + h).$$

Although this last equation does not have a closed-form solution, it can be solved numerically quite easily. An approximate analytical solution can be obtained as follows. By letting

$$\Delta = S - s,$$

the last equation yields

$$e^{\alpha\Delta} = \frac{\alpha K}{c + h} + \alpha\Delta + 1.$$

If $\alpha\Delta$ is close to zero, $e^{\alpha\Delta}$ can be expanded into a Taylor series around zero. If the terms beyond the quadratic term are neglected, the result becomes

$$1 + \alpha\Delta + \frac{\alpha^2\Delta^2}{2} \cong \frac{\alpha K}{c + h} + \alpha\Delta + 1,$$

so that

$$\Delta = \sqrt{\frac{2K}{\alpha(c + h)}}.$$

Using this approximation in the bicycle example results in

$$\Delta = \sqrt{\frac{(2)(10{,}000)(800)}{20 - 9}} = 1{,}206,$$

which is quite close to the exact value of $\Delta = 1{,}182$.

Model with Nonlinear Costs. These results can be extended to the case where the expected shortage plus holding cost $L(y)$ is a *strictly convex* function. This extension results in a strictly convex $cy + L(y)$, similar to Fig. 19.9.

For this model, the optimal inventory policy has the following form:

$$\text{If} \quad x \begin{cases} < s & \text{order } S - x \text{ to bring inventory level up to } S \\ \geq s & \text{do not order,} \end{cases}$$

where S is the value of y that satisfies

$$c + \frac{dL(y)}{dy} = 0$$

and s is the smallest value that satisfies the expression

$$cs + L(s) = K + cS + L(S).$$

19.7 STOCHASTIC PERIODIC-REVIEW MODELS

The preceding section presented a stochastic single-period model that is designed for dealing with *perishable* products. We now return to considering *stable* products that will remain sellable indefinitely, as in the first five sections of the chapter. We again assume that the demand is uncertain so that a stochastic model is needed. However, in contrast to the continuous-review inventory system considered in Sec. 19.5, we now assume that the system is only being monitored periodically. At the end of each period, when the current inventory level is determined, a decision is made on how much to order (if any) to replenish inventory for the next period. Each of these decisions takes into account the planning for multiple periods into the future.

We begin with the simplest case where the planning is only being done for the next two periods and no setup cost is incurred when placing an order to replenish inventory.

A Stochastic Two-Period Model with No Setup Cost

One option with a stochastic periodic-review inventory system is to plan ahead only one period at a time, using the stochastic single-period model from the preceding section to make the ordering decision each time. However, this approach would only provide a relatively crude approximation. If the probability distribution of demand in each period can be forecasted multiple periods into the future, better decisions can be made by coordinating the plans for all these periods than by planning ahead just one period at a time. This can be quite difficult for many periods, but is considerably less difficult when considering only two periods at a time.

Even for a planning horizon of two periods, using the optimal one-period solution twice is not generally the optimal policy for the two-period problem. Smaller costs can usually be achieved by viewing the problem from a two-period viewpoint and then using the methods of probabilistic dynamic programming introduced in Sec. 11.4 to obtain the best inventory policy.

Assumptions. Except for having two periods, the assumptions for this model are basically the same as for the one-period model presented in the preceding section, as summarized below.

1. Each application involves a single stable product.
2. Planning is being done for two periods, where unsatisfied demand in period 1 is backlogged to be met in period 2, but there is no backlogging of unsatisfied demand in period 2.
3. The demands D_1 and D_2 for periods 1 and 2 are *independent and identically distributed* random variables. Their common probability distribution has probability density function $\phi_D(\xi)$ and cumulative distribution function $\Phi(\xi)$.
4. The initial inventory level (before replenishing) at the beginning of period 1 is x_1 ($x_1 \geq 0$).
5. The decisions to be made are y_1 and y_2, the inventory levels to reach by replenishing (if needed) at the beginning of period 1 and period 2, respectively.
6. The objective is to *minimize the expected total cost for both periods,* where the cost components for each period are

$$c = \text{unit cost for purchasing or producing each unit,}$$

$$h = \text{holding cost per unit remaining at end of each period,}$$

$$p = \text{shortage cost per unit of unsatisfied demand at end of each period.}$$

For simplicity, we are assuming that the demand distributions for the two periods are the same and that the values of the above cost components also are the same for the two periods. In many applications, there will be differences between the periods that should be incorporated into the analysis. For example, because of assumption 2, the value of p

may well be different for the two periods. Such extensions of the model can be incorporated into the dynamic programming analysis presented below, but we will not delve into these extensions.

Analysis. To begin the analysis, let

y_i^0 = optimal value of y_i, for $i = 1, 2$,
$C_1(x_1)$ = expected total cost for both periods when following an optimal policy given that x_1 is initial inventory level (before replenishing) at beginning of period 1,
$C_2(x_2)$ = expected total cost for just period 2 when following an optimal policy given that x_2 is inventory level (before replenishing) at beginning of period 2.

To use the dynamic programming approach, we begin by solving for $C_2(x_2)$ and y_2^0, where there is just one period to go. Then we will use these results to find $C_1(x_1)$ and y_1^0.

From the results for the single-period model, y_2^0 is found by solving the equation

$$\Phi(y_2^0) = \frac{p - c}{p + h}.$$

Given x_2, the resulting optimal policy then is the following:

$$\text{If} \quad x_2 \begin{cases} < y_2^0 & \text{order } y_2^0 - x_2 \text{ to bring inventory level up to } y_2^0 \\ \geq y_2^0 & \text{do not order.} \end{cases}$$

The cost of this optimal policy can be expressed as

$$C_2(x_2) = \begin{cases} L(x_2) & \text{if } x_2 \geq y_2^0 \\ c(y_2^0 - x_2) + L(y_2^0) & \text{if } x_2 < y_2^0, \end{cases}$$

where $L(z)$ is the expected shortage plus holding cost for a single period when the inventory level (after replenishing) is z. Now $L(z)$ can be expressed as

$$L(z) = \int_z^\infty p(\xi - z)\varphi_D(\xi)\, d\xi + \int_0^z h(z - \xi)\varphi_D(\xi)\, d\xi.$$

When both periods 1 and 2 are considered, the costs incurred consist of the ordering cost $c(y_1 - x_1)$, the expected shortage plus holding cost $L(y_1)$, and the costs associated with following an optimal policy during the second period. Thus, the expected cost of following the optimal policy for two periods is given by

$$C_1(x_1) = \min_{y_1 \geq x_1} \{c(y_1 - x_1) + L(y_1) + E[C_2(x_2)]\},$$

where $E[C_2(x_2)]$ is obtained as follows. Note that

$$x_2 = y_1 - D_1,$$

so x_2 is a random variable when beginning period 1. Thus,

$$C_2(x_2) = C_2(y_1 - D_1) = \begin{cases} L(y_1 - D_1) & \text{if } y_1 - D_1 \geq y_2^0 \\ c(y_2^0 - y_1 + D_1) + L(y_2^0) & \text{if } y_1 - D_1 < y_2^0. \end{cases}$$

Hence, $C_2(x_2)$ is a random variable, and its expected value is given by

$$
\begin{aligned}
E[C_2(x_2)] &= \int_0^{\infty} C_2(y_1 - \xi)\varphi_D(\xi)\, d\xi \\
&= \int_0^{y_1 - y_2^0} L(y_1 - \xi)\varphi_D(\xi)\, d\xi \\
&\quad + \int_{y_1 - y_2^0}^{\infty} [c(y_2^0 - y_1 + \xi) + L(y_2^0)]\varphi_D(\xi)\, d\xi.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
C_1(x_1) = \min_{y_1 \geq x_1} \Big\{ & c(y_1 - x_1) + L(y_1) + \int_0^{y_1 - y_2^0} L(y_1 - \xi)\varphi_D(\xi)\, d\xi \\
& + \int_{y_1 - y_2^0}^{\infty} [(y_2^0 - y_1 + \xi) + L(y_2^0)]\varphi_D(\xi)\, d\xi \Big\}.
\end{aligned}
$$

It can be shown that $C_1(x_1)$ has a unique minimum and that the optimal value of y_1, denoted by y_1^0, satisfies the equation

$$
\begin{aligned}
-p + (p + h)\Phi(y_1^0) &+ (c - p)\Phi(y_1^0 - y_2^0) \\
&+ (p + h) \int_0^{y_1^0 - y_2^0} \Phi(y_1^0 - \xi)\varphi_D(\xi)\, d\xi = 0.
\end{aligned}
$$

The resulting optimal policy for period 1 then is the following:

$$
\text{If} \quad x_1 \begin{cases} < y_1^0 & \text{order } y_1^0 - x_1 \text{ to bring inventory level up to } y_1^0 \\ \geq y_1^0 & \text{do not order.} \end{cases}
$$

The procedure for finding y_1^0 reduces to a simpler result for certain demand distributions. We summarize two such cases next.

Suppose that the demand in each period has a *uniform distribution* over the range 0 to t, that is,

$$
\varphi_D(\xi) = \begin{cases} \dfrac{1}{t} & \text{if } 0 \leq \xi \leq t \\ 0 & \text{otherwise.} \end{cases}
$$

Then y_1^0 can be obtained from the expression

$$
y_1^0 = \sqrt{(y_2^0)^2 + \frac{2t(c - p)}{p + h} y_2^0 + \frac{t^2[2p(p + h) + (h + c)^2]}{(p + h)^2}} - \frac{t(h + c)}{p + h}.
$$

Now suppose that the demand in each period has an *exponential distribution,* i.e.,

$$
\phi(\xi) = \alpha e^{-\alpha\xi}, \qquad \text{for } \xi \geq 0.
$$

Then y_1^0 satisfies the relationship

$$(h + c)e^{-\alpha(y_1^0 - y_2^0)} + (p + h)e^{-\alpha y_1^0} + \alpha(p + h)(y_1^0 - y_2^0)e^{-\alpha y_1^0} = 2h + c.$$

An alternative way of finding y_1^0 is to let z^0 denote $\alpha(y_1^0 - y_2^0)$. Then z^0 satisfies the relationship

$$e^{-z^0}[(h + c) + (p + h)e^{-\alpha y_2^0} + z^0(p + h)e^{-\alpha y_2^0}] = 2h + c,$$

and

$$y_1^0 = \frac{1}{\alpha} z^0 + y_2^0.$$

When the demand has either a uniform or an exponential distribution, an Excel template is available in your OR Courseware for calculating y_1^0 and y_2^0.

Example. Consider a two-period problem where

$$c = 10, \qquad h = 10, \qquad p = 15,$$

and where the probability density function of the demand in each period is given by

$$\varphi_D(\xi) = \begin{cases} \dfrac{1}{10} & \text{if } 0 \le \xi \le 10 \\ 0 & \text{otherwise,} \end{cases}$$

so that the cumulative distribution function of demand is

$$\Phi(\xi) = \begin{cases} 0 & \text{if } \xi < 0 \\ \dfrac{\xi}{10} & \text{if } 0 \le \xi \le 10 \\ 1 & \text{if } \xi > 10. \end{cases}$$

We find y_2^0 from the equation

$$\Phi(y_2^0) = \frac{p - c}{p + h} = \frac{15 - 10}{15 + 10} = \frac{1}{5},$$

so that

$$y_2^0 = 2.$$

To find y_1^0, we plug into the expression given for y_1^0 for the case of a *uniform* demand distribution, and we obtain

$$y_1^0 = \sqrt{2^2 + \frac{2(10)(10 - 15)}{15 + 10}(2) + 10^2\, \frac{2(15)(15 + 10) + (10 + 10)^2}{(15 + 10)^2}} - \frac{10(10 + 10)}{15 + 10}$$
$$= \sqrt{4 - 8 + 184} - 8 = 13.42 - 8 = 5.42.$$

Substituting $y_1^0 = 5$ and $y_1^0 = 6$ into $C_1(x_1)$ leads to a smaller value with $y_1^0 = 5$. Thus, the optimal policy can be described as follows:

If	$x_1 < 5$,	order $5 - x_1$ to bring inventory level up to 5.
If	$x_1 \geq 5$,	do not order in period 1.
If	$x_2 < 2$,	order $2 - x_2$ to bring inventory level up to 2.
If	$x_2 \geq 2$,	do not order in period 2.

Since unsatisfied demand in period 1 is backlogged to be met in period 2, $x_2 = 5 - D$ can turn out to be either positive or negative.

Stochastic Multiperiod Models—An Overview

The two-period model can be extended to several periods or to an infinite number of periods. This section presents a summary of multiperiod results that have practical importance.

Multiperiod Model with No Setup Cost. Consider the direct extension of the above two-period model to n periods ($n > 2$) with the identical assumptions. The only difference is that a *discount factor* α (described in Sec. 19.2), with $0 < \alpha < 1$, now will be used in calculating the expected total cost for n periods. The problem still is to find the critical numbers $y_1^0, y_2^0, \ldots, y_n^0$ that describe the optimal inventory policy. As in the two-period model, these values are difficult to obtain numerically, but it can be shown[1] that the optimal policy has the following form.

For each period i ($i = 1, 2, \ldots, n$), with x_i as the inventory level entering that period (before replenishing), do the following

$$\text{If} \quad x_i \begin{cases} < y_i^0 & \text{order } y_i^0 - x_i \text{ to bring inventory level up to } y_i^0 \\ \geq y_i^0 & \text{do not order in period } i. \end{cases}$$

Furthermore,

$$y_n^0 \leq y_{n-1}^0 \leq \cdots \leq y_2^0 \leq y_1^0.$$

For the *infinite-period* case (where $n = \infty$), all these critical numbers $y_1^0, y_2^0, \ldots$ are *equal*. Let y^0 denote this constant value. It can be shown that y^0 satisfies the equation

$$\Phi(y^0) = \frac{p - c(1 - \alpha)}{p + h}.$$

When the demand has either a uniform or an exponential distribution, an Excel template is available in your OR Courseware for calculating y^0.

A Variation of the Multiperiod Inventory Model with No Setup Cost. These results for the infinite-period case (all the critical numbers equal the same value y^0 and y^0 satisfies the above equation) also apply when n is finite if two new assumptions are

[1]See Theorem 4 in R. Bellman, I. Glicksberg, and O. Gross, "On the Optimal Inventory Equation," *Management Science,* **2:** 83–104, 1955. Also see p. 163 in K. J. Arrow, S. Karlin, and H. Scarf (eds.), *Studies in the Mathematical Theory of Inventory and Production,* Stanford University Press, Stanford, CA, 1958.

made about what happens at the end of the last period. One new assumption is that each unit left over at the end of the final period can be salvaged with a return of the initial purchase cost c. Similarly, if there is a shortage at this time, assume that the shortage is met by an emergency shipment with the same unit purchase cost c.

Example. Consider again the bicycle example as it was introduced in Example 2 of Sec. 19.1. The cost estimates given there imply that

$$c = 35, \qquad h = 1, \qquad p = 15.$$

Suppose now that the distributor places an order with the manufacturer for various bicycle models on the first working day of each month. Because of this routine, she is willing to assume that the marginal setup cost is zero for including an order for the bicycle model under consideration. The appropriate discount factor is $\alpha = 0.995$. From past history, the distribution of demand can be approximated by a uniform distribution with the probability density function

$$\varphi_D(\xi) = \begin{cases} \dfrac{1}{800} & \text{if } 0 \le \xi \le 800 \\ 0 & \text{otherwise,} \end{cases}$$

so the cumulative distribution function over this interval is

$$\Phi(\xi) = \frac{1}{800}\xi, \qquad \text{if } 0 \le \xi \le 800.$$

The distributor expects to stock this model indefinitely, so the *infinite-period model with no setup cost* is appropriate.

For this model, the critical number y^0 for every period satisfies the equation

$$\Phi(y^0) = \frac{p - c(1-\alpha)}{p + h},$$

so

$$\frac{y^0}{800} = \frac{15 - 35(1 - 0.995)}{15 + 1} = 0.9266,$$

which yields $y^0 = 741$. Thus, if the number of bicycles on hand x at the first of each month is fewer than 741, the optimal policy calls for bringing the inventory level up to 741 (ordering $741 - x$ bicycles). Otherwise, no order is placed.

Multiperiod Model with Setup Cost. The introduction of a fixed setup cost K that is incurred when ordering (whether through purchasing or producing) often adds more realism to the model. For the *single-period model with a setup cost* described in Sec. 19.6, we found that an (s, S) policy is optimal, so that the two critical numbers s and S indicate *when* to order (namely, if the inventory level is less than s) and *how much* to order (bring the inventory level up to S). Now with multiple periods, an (s, S) policy again is optimal, but the value of each critical number may be different in different periods. Let

s_i and S_i denote these critical numbers for period i, and again let x_i be the inventory level (before replenishing) at the beginning of period i.

The optimal policy is to do the following at the beginning of each period i ($i = 1, 2, \ldots, n$):

$$\text{If } x_i \begin{cases} < s_i & \text{order } S_i - x_i \text{ to bring inventory level up to } S_i \\ \geq s_i & \text{do not order.} \end{cases}$$

Unfortunately, computing exact values of the s_i and S_i is extremely difficult.

A Multiperiod Model with Batch Orders and No Setup Cost. In the preceding models, *any quantity* could be ordered (or produced) at the beginning of each period. However, in some applications, the product may come in a standard batch size, e.g., a case or a truckload. Let Q be the number of units in each batch. In our next model we assume that the number of units ordered must be a *nonnegative integer multiple* of Q.

This model makes the same assumptions about what happens at the end of the last period as the variation of the multiperiod model with no setup cost presented earlier. Thus, we assume that each unit left over at the end of the final period can be salvaged with a return of the initial purchase cost c. Similarly, if there is a shortage at this time, we assume that the shortage is met by an emergency shipment with the same unit purchase cost c.

Otherwise, the assumptions are the same as for our standard multiperiod model with no setup cost.

The optimal policy for this model is known as a **(k, Q) policy** because it uses a critical number k and the quantity Q as described below.

If at the beginning of a period the inventory level (before replenishing) is less than k, an order should be placed for the smallest integer multiple of Q that will bring the inventory level up to at least k (and probably higher). Otherwise, an order should not be placed. The same critical number k is used in each period.

The critical number k is chosen as follows. Plot the function

$$G(y) = (1 - \alpha)cy + h \int_0^y (y - \xi)\varphi_D(\xi)\, d\xi + p \int_y^\infty (\xi - y)\varphi_D(\xi)\, d\xi,$$

as shown in Fig. 19.10. This function necessarily has the convex shape shown in the figure. As before, the minimizing value y^0 satisfies the equation

$$\Phi(y^0) = \frac{p - c(1 - \alpha)}{p + h}.$$

As shown in this figure, if a "ruler" of length Q is placed horizontally into the "valley," k is that value of the abscissa to the left of y^0 where the ruler intersects the valley. If the inventory level lies in R_1, then Q is ordered; if it lies in R_2, then $2Q$ is ordered; and so on. However, if the inventory level is at least k, then no order should be placed.

These results hold regardless of whether the number of periods n is finite or infinite.

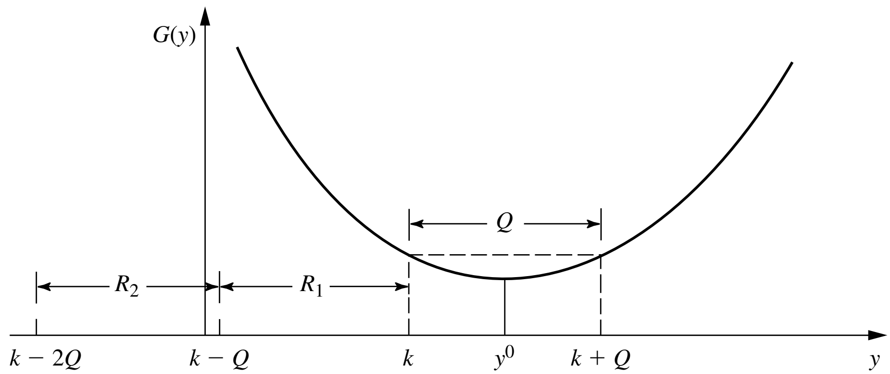


FIGURE 19.10
Plot of the $G(y)$ function for the stochastic multiperiod model with batch orders and no setup cost.

19.8 LARGER INVENTORY SYSTEMS IN PRACTICE

All the inventory models presented in this chapter have been concerned with the management of the inventory of a single product at a single geographical location. Such models provide the basic building blocks of scientific inventory management.

Multiproduct Inventory Systems

However, it is important to recognize that many inventory systems must deal simultaneously with many products, sometimes even hundreds or thousands of products. Furthermore, the inventory of each product often is dispersed geographically, perhaps even globally.

With multiple products, it commonly is possible to apply the appropriate single-product model to each of the products individually. However, companies may not bother to do this for the less important products because of the costs involved in regularly monitoring the inventory level to implement such a model. One popular approach in practice is the **ABC control method.** This involves dividing the products into three groups called the *A* group, *B* group, and *C* group. The products in the *A* group are the particularly important ones that are to be carefully monitored according to a formal inventory model. Products in the *C* group are the least important, so they are only monitored informally on a very occasional basis. Group *B* products receive an intermediate treatment.

It occasionally is not appropriate to apply a single-product inventory model because of interactions between the products. Various interactions are possible. Perhaps similar products can be substituted for each other as needed. For a manufacturer, perhaps its products must compete for production time when ordering production runs. For a wholesaler or retailer, perhaps its setup cost for ordering a product can be reduced by placing a joint order for a number of products simultaneously. Perhaps there also are joint budget limitations involving all the products. Perhaps the products need to compete for limited storage space.

It is common in practice to have a little bit of such interactions between products and still apply a single-product inventory model as a reasonable approximation. However, when an interaction is playing a major role, further analysis is needed. Some research has been conducted already to develop *multiproduct inventory models* to deal with some of these interactions.

Multiechelon Inventory Systems

Our growing global economy has caused a dramatic shift in inventory management entering the 21st century. Now, as never before, the inventory of many manufacturers is scattered throughout the world. Even the inventory of an individual product may be dispersed globally.

This inventory may be stored initially at the point or points of manufacture (one *echelon* of the inventory system), then at national or regional warehouses (a second echelon), then at field distribution centers (a third echelon), etc. Such a system with multiple echelons of inventory is referred to as a **multiechelon inventory system.** In the case of a fully integrated corporation that both manufactures its products and sells them at the retail level, its echelons will extend all the way down to its retail outlets.

Some coordination is needed between the inventories of any particular product at the different echelons. Since the inventory at each echelon (except the top one) is replenished from the next higher echelon, the inventory level currently needed at the higher echelon is affected by how soon replenishment will be needed at the various locations for the lower echelon.

Considerable research (with roots tracing back to the middle of the 20th century) is being conducted to develop multiechelon inventory models.

Now let us see how one major corporation has been managing one of its multiechelon inventory systems.

Multiechelon Inventory Management at IBM[1]

IBM has roughly 1,000 products in service. Therefore, it employs over 15,000 customer engineers who are trained to repair and maintain all the installed computer systems sold or leased by IBM throughout the United States.

To support this effort, IBM maintains a huge multiechelon inventory system of spare parts. This system controls over 200,000 part numbers, with the total inventory valued in the *billions of dollars.* Millions of parts transactions are processed annually.

The echelons of this system start with the manufacture of the parts, then national or regional warehouses, then field distribution centers, then parts stations, and finally many thousand outside locations (including customer stock locations and the car trunks or tool chests of the company's customer engineers).

To coordinate and control all these inventories at the different echelons, a huge computerized system called *Optimizer* was developed. Optimizer consists of four major modules. A forecasting system module contains a few programs for estimating the failure rates of individual types of parts. A data delivery system module consists of approximately 100 programs that process over 15 gigabytes of data to provide the needed input into Optimizer. A decision system module then optimizes control of the inventories on a weekly basis. The fourth module includes six programs that integrate Optimizer into IBM's Parts Inventory Management System (PIMS). PIMS is a sophisticated information and control system that contains millions of lines of code.

Optimizer tracks the inventory level for each part number at all stocking locations (except at the outside locations, where only parts costing more than a certain threshold

[1]M. Cohen, P. V. Kamesam, P. Kleindorfer, H. Lee, and A. Tekerian, "Optimizer: IBM's Multi-Echelon Inventory Systems for Managing Service Logistics," *Interfaces,* **20:** 65–82, Jan.–Feb., 1990.

are tracked). An (R, Q) type of inventory policy is used for each part at each location and echelon in the system.

Careful planning was required to *implement* such a complex system after it had been designed. Three factors proved to be especially important in achieving a successful implementation. The first was the inclusion of a *user team* (consisting of operational managers) as advisers to the project team throughout the study. By the time of the implementation phase, these operational managers had a strong sense of ownership and so had become ardent supporters for installing Optimizer in their functional areas. A second success factor was a very extensive *user acceptance test* whereby users could identify problem areas that needed rectifying prior to full implementation. The third key was that the new system was phased in gradually, with careful testing at each phase, so the major bugs would be eliminated before the system went live nationally.

This new multiechelon inventory system proved to be extremely successful. It provided savings of about \$20 million per year through improved operational efficiency. It also gave even larger annual savings in holding costs (including the cost of capital tied up in inventory) by reducing the value of IBM's inventories by over \$250 million. Despite this large reduction in inventories, the improved inventory management still enabled providing better service to IBM's customers. Specifically, the new system yielded a 10 percent improvement in the parts availability at the lower echelons (where the customers are affected) while maintaining the parts availability levels at the higher echelons.

Supply Chain Management

Another key concept that has emerged in this global economy is that of supply chain management. This concept pushes the management of a multiechelon inventory system one step further by also considering what needs to happen to bring a product into the inventory system in the first place. However, as with inventory management, a main purpose still is to win the competitive battle against other companies in bringing the product to the customers as promptly as possible.

A **supply chain** is a network of facilities that procure raw materials, transform them into intermediate goods and then final products, and finally deliver the products to customers through a distribution system that includes a (probably multiechelon) inventory system. Thus, it spans procurement, manufacturing, and distribution, with effective inventory management as one key element. To fill orders efficiently, it is necessary to understand the linkages and interrelationships of all the key elements of the supply chain. Therefore, integrated management of the supply chain has become a key success factor for some of today's leading companies.

We summarize below the experience of one of the companies that have led the way in making supply chain management part of their corporate culture.

Supply Chain Management at Hewlett-Packard[1]

Hewlett-Packard (HP) is one of today's leading high-technology companies. Its scope is truly global. Nearly half of its employees are outside the United States. In 1993, it had

[1]H. L. Lee and C. Billington, "The Evolution of Supply-Chain-Management Models and Practices at Hewlett-Packard," *Interfaces,* **25:** 42–63, Sept.–Oct., 1995.

manufacturing or research and development sites in 16 countries, as well as sales and service offices in 110 countries. Its total number of catalog products exceeded 22,000.

Late in the 1980s, HP faced inventories mounting into the billions of dollars and alarming customer dissatisfaction with its order fulfillment process. Management was very concerned, since order fulfillment was becoming a major battlefield in the high-technology industries. Recognizing the need for OR models to support top management decision making, HP formed a group known as Strategic Planning and Modeling (SPaM) in 1988. Management charged the group with developing and introducing innovations in OR and industrial engineering.

In 1989, SPaM began bringing supply chain management concepts into HP. HP's supply chain includes manufacturing integrated circuits, board assembly, final assembly, and delivery to customers on a global basis. With such diverse and complex products, grappling with supply chain issues can be very challenging. Variabilities and uncertainties are prevalent all along the chain. Suppliers can be late in their shipments, or the incoming materials may be flawed. The production process may break down, or the production yield may be imperfect. Finally, product demands also are highly uncertain.

Much of SPaM's initial focus was on inventory modeling. This effort led to the development of HP's *Worldwide Inventory Network Optimizer (WINO).* Like IBM's Optimizer described earlier in this section, WINO manages a multiechelon inventory system. However, rather than dealing just with inventories of finished products, WINO also considers the inventories of incoming goods and departing goods at each site along the supply chain.

WINO uses a discrete-review inventory model to determine the reorder point and order quantities for each of these inventories. By introducing more frequent reviews of inventories, better balancing of related inventories, elimination of redundant safety stocks, etc., inventory reductions of 10 to 30 percent typically were obtained.

WINO was even extended to include the inventory systems of some key dealers. This enabled reducing the inventories of finished products at both HP's distribution centers and the dealers while maintaining the same service target for the customers.

SPaM's initial focus on inventory modeling soon broadened to dealing with distribution strategy issues. For example, its realignment of the distribution network in Europe reduced the total distribution cost there by $18 million per year.

SPaM's work also evolved into other functional areas, including design and engineering, finance, and marketing.

The importance of supply chain management now is recognized throughout HP. Several key divisions have formalized such positions as supply chain project managers, supply chain analysts, and supply chain coordinators. These individuals work closely with SPaM to ensure that supply chain models are used effectively and to identify new problems that feed SPaM's research and development effort.

The work of SPaM in applying OR to integrate supply chain management into HP has paid tremendous dividends. SPaM has often identified cost savings of $10 million to $40 million per year from just a single project. Therefore, total cost savings now run into the hundreds of millions of dollars annually. There have been key intangible benefits as well, including enhancing HP's reputation as a progressive company that can be counted on by its customers to fill their orders promptly.

19.9 CONCLUSIONS

We have introduced only rather basic kinds of inventory models here, but they serve the purpose of introducing the general nature of inventory models. Furthermore, they are sufficiently accurate representations of many actual inventory situations that they frequently are useful in practice. For example, the EOQ models have been particularly widely used. These models are sometimes modified to include some type of stochastic demand, such as the stochastic continuous-review model does. The stochastic single-period model is a very convenient one for perishable products. The stochastic multiperiod models have been important in characterizing the types of policies to follow, for example, (s, S) policies, even though the optimal values of s and S may be difficult to obtain.

Nevertheless, many inventory situations possess complications that are not taken into account by the models in this chapter, e.g., interactions between products or multiple echelons of a supply system. More complex models have been formulated in an attempt to fit such situations, but it is difficult to achieve both adequate realism and sufficient tractability to be useful in practice. The development of useful models for supply chain management currently is a particularly active area of research.

Continued growth is occurring in the computerization of inventory data processing, along with an accompanying growth in scientific inventory management.

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LEARNING AIDS FOR THIS CHAPTER IN YOUR OR COURSEWARE

"Ch. 19—Inventory Theory" Excel File:

Templates for the Basic EOQ Model (a Solver Version and an Analytical Version)
Templates for the EOQ Model with Planned Shortages (a Solver Version and an Analytical Version)
Template for the EOQ Model with Quantity Discounts (Analytical Version Only)

Template for the Stochastic Continuous-Review Model
Template for the Stochastic Single-Period Model for Perishable Products, No Setup Cost
Template for the Stochastic Single-Period Model for Perishable Products, with Setup Cost
Template for the Stochastic Two-Period Model, No Setup Cost
Template for the Stochastic Infinite-Period Model, No Setup Cost

"Ch. 19—Inventory Theory" LINGO File for Selected Examples

PROBLEMS

To the left of each of the following problems (or their parts), we have inserted a T whenever one of the templates listed above can be useful. An asterisk on the problem number indicates that at least a partial answer is given in the back of the book.

T **19.3-1.*** Suppose that the demand for a product is 30 units per month and the items are withdrawn at a constant rate. The setup cost each time a production run is undertaken to replenish inventory is $15. The production cost is $1 per item, and the inventory holding cost is $0.30 per item per month.
(a) Assuming shortages are not allowed, determine how often to make a production run and what size it should be.
(b) If shortages are allowed but cost $3 per item per month, determine how often to make a production run and what size it should be.

T **19.3-2.** The demand for a product is 600 units per week, and the items are withdrawn at a constant rate. The setup cost for placing an order to replenish inventory is $25. The unit cost of each item is $3, and the inventory holding cost is $0.05 per item per week.
(a) Assuming shortages are not allowed, determine how often to order and what size the order should be.
(b) If shortages are allowed but cost $2 per item per week, determine how often to order and what size the order should be.

19.3-3.* Tim Madsen is the purchasing agent for Computer Center, a large discount computer store. He has recently added the hottest new computer, the Power model, to the store's stock of goods. Sales of this model now are running at about 13 per week. Tim purchases these customers directly from the manufacturer at a unit cost of $3,000, where each shipment takes half a week to arrive.

Tim routinely uses the basic EOQ model to determine the store's inventory policy for each of its more important products. For this purpose, he estimates that the annual cost of holding items in inventory is 20 percent of their purchase cost. He also estimates that the administrative cost associated with placing each order is $75.
T **(a)** Tim currently is using the policy of ordering 5 Power model computers at a time, where each order is timed to have the shipment arrive just about when the inventory of these computers is being depleted. Use the Solver version of the Excel template for the basic EOQ model to determine the various annual costs being incurred with this policy.
T **(b)** Use this same spreadsheet to generate a table that shows how these costs would change if the order quantity were changed to the following values: 5, 7, 9, . . . , 25.
T **(c)** Use the Solver to find the optimal order quantity.
T **(d)** Now use the analytical version of the Excel template for the basic EOQ model (which applies the EOQ formula directly) to find the optimal quantity. Compare the results (including the various costs) with those obtained in part (*c*).
(e) Verify your answer for the optimal order quantity obtained in part (*d*) by applying the EOQ formula by hand.
(f) With the optimal order quantity obtained above, how frequently will orders need to be placed on the average? What should the approximate inventory level be when each order is placed?
(g) How much does the optimal inventory policy reduce the total variable inventory cost per year (holding costs plus administrative costs for placing orders) for Power model computers from that for the policy described in part (*a*)? What is the percentage reduction?

19.3-4. The Blue Cab Company is the primary taxi company in the city of Maintown. It uses gasoline at the rate of 8,500 gallons per month. Because this is such a major cost, the company has made a special arrangement with the Amicable Petroleum Company to purchase a huge quantity of gasoline at a reduced price of $1.05 per gallon every few months. The cost of arranging for each order, including placing the gasoline into storage, is $1,000. The cost of holding the gasoline in storage is estimated to be $0.01 per gallon per month.
T **(a)** Use the Solver version of the Excel template for the basic EOQ model to determine the costs that would be incurred annually if the gasoline were to be ordered monthly.
T **(b)** Use this same spreadsheet to generate a table that shows how these costs would change if the number of months between orders were to be changed to the following values: 1, 2, 3, . . . , 10.

T **(c)** Use the Solver to find the optimal order quantity.

T **(d)** Now use the analytical version of the Excel template for the basic EOQ model to find the optimal order quantity. Compare the results (including the various costs) with those obtained in part (*c*).

(e) Verify your answer for the optimal order quantity obtained in part (*d*) by applying the EOQ formula by hand.

T **19.3-5.** Computronics is a manufacturer of calculators, currently producing 200 per week. One component for every calculator is a liquid crystal display (LCD), which the company purchases from Displays, Inc. (DI) for $1 per LCD. Computronics management wants to avoid any shortage of LCDs, since this would disrupt production, so DI guarantees a delivery time of $\frac{1}{2}$ week on each order. The placement of each order is estimated to require 1 hour of clerical time, with a direct cost of $15 per hour plus overhead costs of another $5 per hour. A rough estimate has been made that the annual cost of capital tied up in Computronics' inventory is 15 percent of the value (measured by purchase cost) of the inventory. Other costs associated with storing and protecting the LCDs in inventory amount to 5 cents per LCD per year.

(a) What should the order quantity and reorder point be for the LCDs? What is the corresponding total variable inventory cost per year (holding costs plus administrative costs for placing orders)?

(b) Suppose the true annual cost of capital tied up in Computronics' inventory actually is 10 percent of the value of the inventory. Then what should the order quantity be? What is the difference between this order quantity and the one obtained in part (*a*)? What would the total variable inventory cost per year (TVC) be? How much more would TVC be if the order quantity obtained in part (*a*) still were used here because of the incorrect estimate of the cost of capital tied up in inventory?

(c) Repeat part (*b*) if the true annual cost of capital tied up in Computronics' inventory actually is 20 percent of the value of the inventory.

(d) Perform sensitivity analysis systematically on the unit holding cost by generating a table that shows what the optimal order quantity would be if the true annual cost of capital tied up in Computronics' inventory were each of the following percentages of the value of the inventory: 10, 12, 14, 16, 18, 20.

(e) Assuming that the rough estimate of 15 percent is correct for the cost of capital, perform sensitivity analysis on the setup cost by generating a table that shows what the optimal order quantity would be if the true number of hours of clerical time required to place each order were each of the following: 0.5, 0.75, 1, 1.25, 1.5.

(f) Perform sensitivity analysis simultaneously on the unit holding cost and the setup cost by generating a table that shows the optimal order quantity for the various combinations of values considered in parts (*d*) and (*e*).

19.3-6. For the basic EOQ model, use the square root formula to determine how Q^* would change for each of the following changes in the costs or the demand rate. (Unless otherwise noted, consider each change by itself.)

(a) The setup cost is reduced to 25 percent of its original value.

(b) The annual demand rate becomes four times as large as its original value.

(c) Both changes in parts (*a*) and (*b*).

(d) The unit holding cost is reduced to 25 percent of its original value.

(e) Both changes in parts (*a*) and (*d*).

19.3-7.* Kris Lee, the owner and manager of the Quality Hardware Store, is reassessing his inventory policy for hammers. He sells an average of 50 hammers per month, so he has been placing an order to purchase 50 hammers from a wholesaler at a cost of $20 per hammer at the end of each month. However, Kris does all the ordering for the store himself and finds that this is taking a great deal of his time. He estimates that the value of his time spent in placing each order for hammers is $75.

(a) What would the unit holding cost for hammers need to be for Kris' current inventory policy to be optimal according to the basic EOQ model? What is this unit holding cost as a percentage of the unit acquisition cost?

T **(b)** What is the optimal order quantity if the unit holding cost actually is 20 percent of the unit acquisition cost? What is the corresponding value of TVC = total variable inventory cost per year (holding costs plus the administrative costs for placing orders)? What is TVC for the current inventory policy?

T **(c)** If the wholesaler typically delivers an order of hammers in 5 working days (out of 25 working days in an average month), what should the reorder point be (according to the basic EOQ model)?

(d) Kris doesn't like to incur inventory shortages of important items. Therefore, he has decided to add a safety stock of 5 hammers to safeguard against late deliveries and larger-than-usual sales. What is his new reorder point? How much does this safety stock add to TVC?

19.3-8. Cindy Stewart and Misty Whitworth graduated from business school together. They now are inventory managers for competing wholesale distributors, making use of the scientific inventory management techniques they learned in school. Both of them are purchasing 85-horsepower speedboat engines for their inventories from the same manufacturer. Cindy has found that the setup cost for initiating each order is $200 and the unit holding cost is $400.

Cindy has learned that Misty is ordering 10 engines each time. Cindy assumes that Misty is using the basic EOQ model and has the same setup cost and unit holding cost as Cindy. Show how Cindy can use this information to deduce what the annual demand rate must be for Misty's company for these engines.

19.3-9.* Consider Example 1 (manufacturing speakers for TV sets) introduced in Sec. 19.1 and used in Sec. 19.3 to illustrate the EOQ models. Use the EOQ model with planned shortages to solve this example when the unit shortage cost is changed to \$5 per speaker short per month.

19.3-10. Speedy Wheels is a wholesale distributor of bicycles. Its Inventory Manager, Ricky Sapolo, is currently reviewing the inventory policy for one popular model that is selling at the rate of 250 per month. The administrative cost for placing an order for this model from the manufacturer is \$200 and the purchase price is \$70 per bicycle. The annual cost of the capital tied up in inventory is 20 percent of the value (based on purchase price) of these bicycles. The additional cost of storing the bicycles—including leasing warehouse space, insurance, taxes, and so on—is \$6 per bicycle per year.

T **(a)** Use the basic EOQ model to determine the optimal order quantity and the total variable inventory cost per year.

T **(b)** Speedy Wheel's customers (retail outlets) generally do not object to short delays in having their orders filled. Therefore, management has agreed to a new policy of having small planned shortages occasionally to reduce the variable inventory cost. After consultations with management, Ricky estimates that the annual shortage cost (including lost future business) would be \$30 times the average number of bicycles short throughout the year. Use the EOQ model with planned shortages to determine the new optimal inventory policy.

T **19.3-11.** Reconsider Prob. 19.3-3. Because of the popularity of the Power model computer, Tim Madsen has found that customers are willing to purchase a computer even when none are currently in stock as long as they can be assured that their order will be filled in a reasonable period of time. Therefore, Tim has decided to switch from the basic EOQ model to the EOQ model with planned shortages, using a shortage cost of \$200 per computer short per year.

(a) Use the Solver version of the Excel template for the EOQ model with planned shortages (with constraints added in the Solver dialogue box that C10:C11 = integer) to find the new optimal inventory policy and its total variable inventory cost per year (TVC). What is the reduction in the value of TVC found for Prob. 19.3-3 (and given in the back of the book) when planned shortages were not allowed?

(b) Use this same spreadsheet to generate a table that shows how TVC and its components would change if the maximum shortage were kept the same as found in part (*a*) but the order quantity were changed to the following values: 15, 17, 19, . . . , 35.

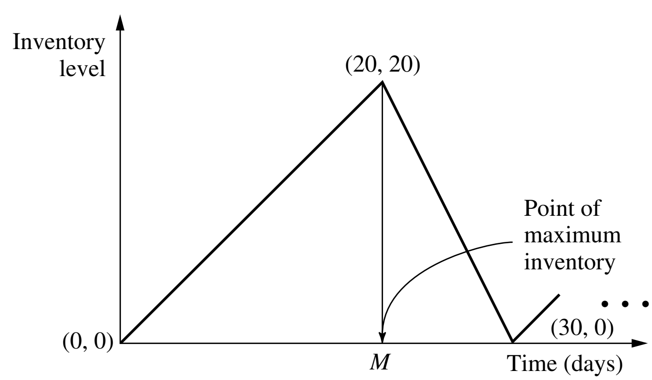
(c) Use this same spreadsheet to generate a table that shows how TVC and its components would change if the order quantity were kept the same as found in part (*a*) but the maximum shortage were changed to the following values: 10, 12, 14, . . . , 30.

19.3-12. You have been hired as an operations research consultant by a company to reevaluate the inventory policy for one of its products. The company currently uses the basic EOQ model. Under this model, the optimal order quantity for this product is 1,000 units, so the maximum inventory level also is 1,000 units and the maximum shortage is 0.

You have decided to recommend that the company switch to using the EOQ model with planned shortages instead after determining how large the unit shortage cost (p) is compared to the unit holding cost (h). Prepare a table for management that shows what the optimal order quantity, maximum inventory level, and maximum shortage would be under this model for each of the following ratios of p to h: $\frac{1}{3}$, 1, 2, 3, 5, 10.

19.3-13. Consider the EOQ model with planned shortages, as presented in Sec. 19.3. Suppose, however, that the constraint $S/Q = 0.8$ is added to the model. Derive the expression for the optimal value of Q.

19.3-14. In the basic EOQ model, suppose the stock is replenished uniformly (rather than instantaneously) at the rate of b items per unit time until the order quantity Q is fulfilled. Withdrawals from the inventory are made at the rate of a items per unit time, where $a < b$. Replenishments and withdrawals of the inventory are made simultaneously. For example, if Q is 60, b is 3 per day, and a is 2 per day, then 3 units of stock arrive each day for days 1 to 20, 31 to 50, and so on, whereas units are withdrawn at the rate of 2 per day every day. The diagram of inventory level versus time is given below for this example.



(a) Find the total cost per unit time in terms of the setup cost K, production quantity Q, unit cost c, holding cost h, withdrawal rate a, and replenishment rate b.

(b) Determine the economic order quantity Q^*.

19.3-15.* MBI is a manufacturer of personal computers. All its personal computers use a 3.5-inch high-density floppy disk drive which it purchases from Ynos. MBI operates its factory 52 weeks per year, which requires assembling 100 of these floppy disk drives into computers per week. MBI's annual holding cost rate is 20 percent of the value (based on purchase cost) of the inventory. Regardless of order size, the administrative cost of placing an order with Ynos has been estimated to be $50. A quantity discount is offered by Ynos for large orders as shown below, where the price for each category applies to *every* disk drive purchased.

Discount Category	Quantity Purchased	Price (per Disk Drive)
1	1 to 99	$100
2	100 to 499	$ 95
3	500 or more	$ 90

T **(a)** Determine the optimal order quantity according to the EOQ model with quantity discounts. What is the resulting total cost per year?

(b) With this order quantity, how many orders need to be placed per year? What is the time interval between orders?

19.3-16. The Gilbreth family drinks a case of Royal Cola every day, 365 days a year. Fortunately, a local distributor offers quantity discounts for large orders as shown in the table below, where the price for each category applies to *every* case purchased. Considering the cost of gasoline, Mr. Gilbreth estimates it costs him about $5 to go pick up an order of Royal Cola. Mr. Gilbreth also is an investor in the stock market, where he has been earning a 20 percent average annual return. He considers the return lost by buying the Royal Cola instead of stock to be the only holding cost for the Royal Cola.

Discount Category	Quantity Purchased	Price (per Case)
1	1 to 49	$4.00
2	50 to 99	$3.90
3	100 or more	$3.80

T **(a)** Determine the optimal order quantity according to the EOQ model with quantity discounts. What is the resulting total cost per year?

(b) With this order quantity, how many orders need to be placed per year? What is the time interval between orders?

19.3-17. Kenichi Kaneko is the manager of a production department which uses 400 boxes of rivets per year. To hold down his inventory level, Kenichi has been ordering only 50 boxes each time. However, the supplier of rivets now is offering a discount for higher-quantity orders according to the following price schedule, where the price for each category applies to *every* box purchased.

Discount Category	Quantity	Price (per Box)
1	1 to 99	$8.50
2	100 to 999	$8.00
3	1,000 or more	$7.50

The company uses an annual holding cost rate of 20 percent of the price of the item. The total cost associated with placing an order is $80 per order.

Kenichi has decided to use the EOQ model with quantity discounts to determine his optimal inventory policy for rivets.

(a) For each discount category, write an expression for the total cost per year (TC) as a function of the order quantity Q.

T **(b)** For each discount category, use the EOQ formula for the basic EOQ model to calculate the value of Q (feasible or infeasible) that gives the minimum value of TC. (You may use the analytical version of the Excel template for the basic EOQ model to perform this calculation if you wish.)

(c) For each discount category, use the results from parts (*a*) and (*b*) to determine the *feasible* value of Q that gives the *feasible* minimum value of TC and to calculate this value of TC.

(d) Draw rough hand curves of TC versus Q for each of the discount categories. Use the same format as in Fig. 19.3 (a solid curve where feasible and a dashed curve where infeasible). Show the points found in parts (*b*) and (*c*). However, you don't need to perform any additional calculations to make the curves particularly accurate at other points.

(e) Use the results from parts (*c*) and (*d*) to determine the optimal order quantity and the corresponding value of TC.

T **(f)** Use the Excel template for the EOQ model with quantity discounts to check your answers in parts (*b*), (*c*), and (*e*).

(g) For discount category 2, the value of Q that minimizes TC turns out to be feasible. Explain why learning this fact would allow you to rule out discount category 1 as a candidate for providing the optimal order quantity without even performing the calculations for this category that were done in parts (*b*) and (*c*).

(h) Given the optimal order quantity from parts (*e*) and (*f*), how many orders need to be placed per year? What is the time interval between orders?

19.3-18. Sarah operates a concession stand at a downtown location throughout the year. One of her most popular items is circus peanuts, selling about 200 bags per month.

Sarah purchases the circus peanuts from Peter's Peanut Shop. She has been purchasing 100 bags at a time. However, to encourage larger purchases, Peter now is offering her discounts for larger order sizes according to the following price schedule, where the price for each category applies to *every* bag purchased.

Discount Category	Order Quantity	Price (per Bag)
1	1 to 199	\$1.00
2	200 to 499	\$0.95
3	500 or more	\$0.90

Sarah wants to use the EOQ model with quantity discounts to determine what her order quantity should be. For this purpose, she estimates an annual holding cost rate of 17 percent of the value (based on purchase price) of the peanuts. She also estimates a setup cost of \$4 for placing each order.

Follow the instructions of Prob. 19.3-17 to analyze Sarah's problem.

19.4-1. Suppose that production planning is to be done for the next 5 months, where the respective demands are $r_1 = 2$, $r_2 = 4$, $r_3 = 2$, $r_4 = 2$, and $r_5 = 3$. The setup cost is \$4,000, the unit production cost is \$1,000, and the unit holding cost is \$300. Use the deterministic periodic-review model to determine the optimal production schedule that satisfies the monthly requirements.

19.4-2. Reconsider the example used to illustrate the deterministic periodic-review model in Sec. 19.4. Solve this problem when the demands are increased by 1 airplane in each period.

19.4-3. Reconsider the example used to illustrate the deterministic periodic-review model in Sec. 19.4. Suppose that the following single change is made in the example. The cost of producing each airplane now varies from period to period. In particular, in addition to the setup cost of \$2 million, the cost of producing airplanes in either period 1 or period 3 is \$1.4 million per airplane, whereas it is only \$1 million per airplane in either period 2 or period 4.

Use dynamic programming to determine how many airplanes (if any) should be produced in each of the four periods to minimize the total cost.

19.4-4.* Consider a situation where a particular product is produced and placed in in-process inventory until it is needed in a subsequent production process. The number of units required in each of the next 3 months, the setup cost, and the regular-time unit production cost (in units of thousands of dollars) that would be incurred in each month are as follows:

Month	Requirement	Setup Cost	Regular-Time Unit Cost
1	1	5	8
2	3	10	10
3	2	5	9

There currently is 1 unit in inventory, and we want to have 2 units in inventory at the end of 3 months. A maximum of 3 units can be produced on regular-time production in each month, although 1 additional unit can be produced on overtime at a cost that is 2 larger than the regular-time unit production cost. The holding cost is 2 per unit for each extra month that it is stored.

Use dynamic programming to determine how many units should be produced in each month to minimize the total cost.

19.4-5. Consider a situation where a particular product is produced and placed in in-process inventory until it is needed in a subsequent production process. No units currently are in inventory, but three units will be needed in the coming month and an additional four units will be needed in the following month. The unit production cost is the same in either month. The setup cost to produce in either month is \$5,000. The holding cost for each unit left in inventory at the end of a month is \$300.

Determine the optimal schedule that satisfies the monthly requirements by using the algorithm presented in Sec. 19.4.

19.5-1. Henry Edsel is the owner of Honest Henry's, the largest car dealership in its part of the country. His most popular car model is the Triton, so his largest costs are those associated with ordering these cars from the factory and maintaining an inventory of Tritons on the lot. Therefore, Henry has asked his general manager, Ruby Willis, who once took a course in operations research, to use this background to develop a cost-effective policy for when to place these orders for Tritons and how many to order each time.

Ruby decides to use the stochastic continuous-review model presented in Sec. 19.5 to determine an (R, Q) policy. After some investigation, she estimates that the administrative cost for placing each order is \$1,500 (a lot of paperwork is needed for ordering cars), the holding cost for each car is \$3,000 per year (15 percent of the agency's purchase price of \$20,000), and the shortage cost per car short is \$1,000 per year (an estimated probability of $\frac{1}{3}$ of losing a car sale and its profit of about \$3,000). After considering both the seriousness of incurring shortages and the high holding cost, Ruby and Henry agree to use a 75 percent service level (a probability of 0.75 of not incurring a shortage between the time an order is placed and the delivery of the cars ordered). Based on previous experience, they also estimate that the Tritons sell at a relatively uniform rate of about 900 per year.

After an order is placed, the cars are delivered in about two-thirds of a month. Ruby's best estimate of the probability distribution of demand during the lead time before a delivery arrives is a normal distribution with a mean of 50 and a standard deviation of 15.

(a) Solve by hand for the order quantity.
(b) Use a table for the normal distribution (Appendix 5) to solve for the reorder point.
T **(c)** Use the Excel template for this model in your OR Courseware to check your answers in parts (*a*) and (*b*).
(d) Given your previous answers, how much safety stock does this inventory policy provide?
(e) This policy can lead to placing a new order before the delivery from the preceding order arrives. Indicate when this would happen.

19.5-2. One of the largest selling items in J.C. Ward's Department Store is a new model of refrigerator that is highly energy-efficient. About 40 of these refrigerators are being sold per month. It takes about a week for the store to obtain more refrigerators from a wholesaler. The demand during this time has a uniform distribution between 5 and 15. The administrative cost of placing each order is \$40. For each refrigerator, the holding cost per month is \$8 and the shortage cost per month is estimated to be \$1.

The store's inventory manager has decided to use the stochastic continuous-review model presented in Sec. 19.5, with a service level (measure 1) of 0.8, to determine an (R, Q) policy.

(a) Solve by hand for R and Q.
T **(b)** Use the corresponding Excel template to check your answer in part (*a*).
(c) What will be the average number of stockouts per year with this inventory policy?

19.5-3. When using the stochastic continuous-review model presented in Sec. 19.5, a difficult managerial judgment decision needs to be made on the level of service to provide to customers. The purpose of this problem is to enable you to explore the trade-off involved in making this decision.

Assume that the measure of service level being used is $L =$ probability that a stockout will not occur during the lead time. Since management generally places a high priority on providing excellent service to customers, the temptation is to assign a very high value to L. However, this would result in providing a very large amount of safety stock, which runs counter to management's desire to eliminate unnecessary inventory. (Remember the *just-in-time philosophy* discussed in Sec. 19.3 that is heavily influencing managerial thinking today.) What is the best trade-off between providing good service and eliminating unnecessary inventory?

Assume that the probability distribution of demand during the lead time is a normal distribution with mean μ and standard deviation σ. Then the reorder point R is $R = \mu + K_{1-L}\sigma$, where K_{1-L} is obtained from Appendix 5. The amount of safety stock provided by this reorder point is $K_{1-L}\sigma$. Thus, if h denotes the holding cost for each unit held in inventory per year, the *average annual holding cost for safety stock* (denoted by C) is $C = hK_{1-L}\sigma$.

(a) Construct a table with five columns. The first column is the service level L, with values 0.5, 0.75, 0.9, 0.95, 0.99, and 0.999. The next four columns give C for four cases. Case 1 is $h = \$1$ and $\sigma = 1$. Case 2 is $h = \$100$ and $\sigma = 1$. Case 3 is $h = \$1$ and $\sigma = 100$. Case 4 is $h = \$100$ and $\sigma = 100$.
(b) Construct a second table that is based on the table obtained in part (*a*). The new table has five rows and the same five columns as the first table. Each entry in the new table is obtained by subtracting the corresponding entry in the first table from the entry in the next row of the first table. For example, the entries in the first column of the new table are $0.75 - 0.5 = 0.25$, $0.9 - 0.75 = 0.15$, $0.95 - 0.9 = 0.05$, $0.99 - 0.95 = 0.04$, and $0.999 - 0.99 = 0.009$. Since these entries represent increases in the service level L, each entry in the next four columns represents the increase in C that would result from increasing L by the amount shown in the first column.
(c) Based on these two tables, what advice would you give a manager who needs to make a decision on the value of L to use?

19.5-4. The preceding problem describes the factors involved in making a managerial decision on the service level L to use. It also points out that for any given values of L, h (the unit holding cost per year), and σ (the standard deviation when the demand during the lead time has a normal distribution), the average annual holding cost for the safety stock would turn out to be $C = hK_{1-L}\sigma$, where C denotes this holding cost and K_{1-L} is given in Appendix 5. Thus, the amount of variability in the demand, as measured by σ, has a major impact on this holding cost C.

The value of σ is substantially affected by the duration of the lead time. In particular, σ increases as the lead time increases. The purpose of this problem is to enable you to explore this relationship further.

To make this more concrete, suppose that the inventory system under consideration currently has the following values: $L = 0.9$, $h = \$100$, and $\sigma = 100$ with a lead time of 4 days. However, the vendor being used to replenish inventory is proposing a change in the delivery schedule that would change your lead time. You want to determine how this would change σ and C.

We assume for this inventory system (as is commonly the case) that the demands on separate days are statistically independent. In this case, the relationship between σ and the lead time is given by the formula

$$\sigma = \sqrt{d}\sigma_1,$$

where $d =$ number of days in the lead time,
$\sigma_1 =$ standard deviation if $d = 1$.

(a) Calculate C for the current inventory system.
(b) Determine σ_1. Then find how C would change if the lead time were reduced from 4 days to 1 day.
(c) How would C change if the lead time were doubled, from 4 days to 8 days?
(d) How long would the lead time need to be in order for C to double from its current value with a lead time of 4 days?

19.5-5. What is the effect on the amount of safety stock provided by the stochastic continuous-review model presented in Sec. 19.5 when the following change is made in the inventory system. (Consider each change independently.)
(a) The lead time is reduced to 0 (instantaneous delivery).
(b) The service level (measure 1) is decreased.
(c) The unit shortage cost is doubled.
(d) The mean of the probability distribution of demand during the lead time is increased (with no other change to the distribution).
(e) The probability distribution of demand during the lead time is a uniform distribution from a to b, but now $(b - a)$ has been doubled.
(f) The probability distribution of demand during the lead time is a normal distribution with mean μ and standard deviation σ, but now σ has been doubled.

19.5-6.* Jed Walker is the manager of Have a Cow, a hamburger restaurant in the downtown area. Jed has been purchasing all the restaurant's beef from Ground Chuck (a local supplier) but is considering switching to Chuck Wagon (a national warehouse) because its prices are lower.

Weekly demand for beef averages 500 pounds, with some variability from week to week. Jed estimates that the *annual* holding cost is 30 cents per pound of beef. When he runs out of beef, Jed is forced to buy from the grocery store next door. The high purchase cost and the hassle involved are estimated to cost him about \$3 per pound of beef short. To help avoid shortages, Jed has decided to keep enough safety stock to prevent a shortage before the delivery arrives during 95 percent of the order cycles. Placing an order only requires sending a simple fax, so the administrative cost is negligible.

Have a Cow's contract with Ground Chuck is as follows: The purchase price is \$1.49 per pound. A fixed cost of \$25 per order is added for shipping and handling. The shipment is guaranteed to arrive within 2 days. Jed estimates that the demand for beef during this lead time has a uniform distribution from 50 to 150 pounds.

The Chuck Wagon is proposing the following terms: The beef will be priced at \$1.35 per pound. The Chuck Wagon ships via refrigerated truck, and so charges additional shipping costs of \$200 per order plus \$0.10 per pound. The shipment time will be roughly a week, but is guaranteed not to exceed 10 days. Jed estimates that the probability distribution of demand during this lead time will be a normal distribution with a mean of 500 pounds and a standard deviation of 200 pounds.

T **(a)** Use the stochastic continuous-review model presented in Sec. 19.5 to obtain an (R, Q) policy for Have a Cow for each of the two alternatives of which supplier to use.
(b) Show how the reorder point is calculated for each of these two policies.
(c) Determine and compare the amount of safety stock provided by the two policies obtained in part (a).
(d) Determine and compare the average annual holding cost under these two policies.
(e) Determine and compare the average annual acquisition cost (combining purchase price and shipping cost) under these two policies.
(f) Since shortages are very infrequent, the only important costs for comparing the two suppliers are those obtained in parts (d) and (e). Add these costs for each supplier. Which supplier should be selected?
(g) Jed likes to use the beef (which he keeps in a freezer) within a month of receiving it. How would this influence his choice of supplier?

19.5-7. Micro-Apple is a manufacturer of personal computers. It currently manufactures a single model—the MacinDOS—on an assembly line at a steady rate of 500 per week. MicroApple orders the floppy disk drives for the MacinDOS (1 per computer) from an outside supplier at a cost of \$30 each. Additional administrative costs for placing an order total \$30. The annual holding cost is \$6 per drive. If MicroApple stocks out of floppy disk drives, production is halted, costing \$100 per drive short. Because of the seriousness of stockouts, management wants to keep enough safety stock to prevent a shortage before the delivery arrives during 99 percent of the order cycles.

The supplier now is offering two shipping options. With option 1, the lead time would have a normal distribution with a mean of 0.5 week and a standard deviation of 0.1 week. For each order, the shipping cost charged to MicroApple would be \$100 plus \$3 per drive. With option 2, the lead time would have a uniform distribution from 1.0 week to 2.0 weeks. For each order, the shipping cost charged to MicroApple would be \$20 plus \$2 per drive.

T **(a)** Use the stochastic continuous-review model presented in Sec. 19.5 to obtain an (R, Q) policy under each of these two shipping options.
(b) Show how the reorder point is calculated for each of these two policies.
(c) Determine and compare the amount of safety stock provided by these two policies.
(d) Determine and compare the average annual holding cost under these two policies.

(e) Determine and compare the average annual acquisition cost (combining purchase price and shipping cost) under these two policies.

(f) Since shortages are very infrequent (and very small when they do occur), the only important costs for comparing the two shipping options are those obtained in parts (*d*) and (*e*). Add these costs for each option. Which option should be selected?

T **19.6-1.** A newspaper stand purchases newspapers for \$0.36 and sells them for \$0.50. The shortage cost is \$0.50 per newspaper (because the dealer buys papers at retail price to satisfy shortages). The holding cost is \$0.002 per newspaper left at the end of the day. The demand distribution is a uniform distribution between 200 and 300. Find the optimal number of papers to buy.

19.6-2. Freddie the newsboy runs a newstand. Because of a nearby financial services office, one of the newspapers he sells is the daily *Financial Journal.* He purchases copies of this newspaper from its distributor at the beginning of each day for \$1.50 per copy, sells it for \$2.50 each, and then receives a refund of \$0.50 from the distributor the next morning for each unsold copy. The number of requests for this newspaper range from 15 to 18 copies per day. Freddie estimates that there are 15 requests on 40 percent of the days, 16 requests on 20 percent of the days, 17 requests on 30 percent of the days, and 18 requests on the remaining days.

(a) Use Bayes' decision rule presented in Sec. 15.2 to determine what Freddie's new order quantity should be to maximize his expected daily profit.

(b) Apply Bayes' decision rule again, but this time with the criterion of minimizing Freddie's expected daily cost of underordering or overordering.

(c) Use the stochastic single-period model for perishable products to determine Freddie's optimal order quantity.

(d) Draw the cumulative distribution function of demand and then show graphically how the model in part (*c*) finds the optimal order quantity.

19.6-3. Jennifer's Donut House serves a large variety of doughnuts, one of which is a blueberry-filled, chocolate-covered, supersized doughnut supreme with sprinkles. This is an extra large doughnut that is meant to be shared by a whole family. Since the dough requires so long to rise, preparation of these doughnuts begins at 4:00 in the morning, so a decision on how many to prepare must be made long before learning how many will be needed. The cost of the ingredients and labor required to prepare each of these doughnuts is \$1. Their sale price is \$3 each. Any not sold that day are sold to a local discount grocery store for \$0.50. Over the last several weeks, the number of these doughnuts sold for \$3 each day has been tracked. These data are summarized next.

Number Sold	Percentage of Days
0	10%
1	15%
2	20%
3	30%
4	15%
5	10%

(a) What is the unit cost of underordering? The unit cost of overordering?

(b) Use Bayes' decision rule presented in Sec. 15.2 to determine how many of these doughnuts should be prepared each day to minimize the average daily cost of underordering or overordering.

(c) After plotting the cumulative distribution function of demand, apply the stochastic single-period model for perishable products graphically to determine how many of these doughnuts to prepare each day.

(d) Given the answer in part (*c*), what will be the probability of running short of these doughnuts on any given day?

(e) Some families make a special trip to the Donut House just to buy this special doughnut. Therefore, Jennifer thinks that the cost when they run short might be greater than just the lost profit. In particular, there may be a cost for lost customer goodwill each time a customer orders this doughnut but none are available. How high would this cost have to be before they should prepare one more of these doughnuts each day than was found in part (*c*)?

19.6-4.* Swanson's Bakery is well known for producing the best fresh bread in the city, so the sales are very substantial. The daily demand for its fresh bread has a uniform distribution between 300 and 600 loaves. The bread is baked in the early morning, before the bakery opens for business, at a cost of \$2 per loaf. It then is sold that day for \$3 per loaf. Any bread not sold on the day it is baked is relabeled as day-old bread and sold subsequently at a discount price of \$1.50 per loaf.

(a) Apply the stochastic single-period model for perishable products to determine the optimal service level.

(b) Apply this model graphically to determine the optimal number of loaves to bake each morning.

(c) With such a wide range of possible values in the demand distribution, it is difficult to draw the graph in part (*b*) carefully enough to determine the exact value of the optimal number of loaves. Use algebra to calculate this exact value.

(d) Given your answer in part (*a*), what is the probability of incurring a shortage of fresh bread on any given day?

(e) Because the bakery's bread is so popular, its customers are quite disappointed when a shortage occurs. The owner of the bakery, Ken Swanson, places high priority on keeping his customers satisfied, so he doesn't like having shortages. He feels that the analysis also should consider the loss of customer goodwill due to shortages. Since this loss of goodwill can have a negative effect on future sales, he estimates that a cost of \$1.50 per loaf should be assessed each time a customer cannot purchase fresh bread because of a shortage. Determine the new optimal number of loaves to bake each day with this change. What is the new probability of incurring a shortage of fresh bread on any given day?

19.6-5. Reconsider Prob. 19.6-4. The bakery owner, Ken Swanson, now wants you to conduct a financial analysis of various inventory policies. You are to begin with the policy obtained in the first four parts of Prob. 19.6-4 (ignoring any cost for the loss of customer goodwill). As given with the answers in the back of the book, this policy is to bake 500 loaves of bread each morning, which gives a probability of incurring a shortage of $\frac{1}{3}$.

(a) For any day that a shortage *does* occur, calculate the revenue from selling fresh bread.
(b) For those days where shortages do *not* occur, use the probability distribution of demand to determine the expected number of loaves of fresh bread sold. Use this number to calculate the expected daily revenue from selling fresh bread on those days.
(c) Combine your results from parts (*a*) and (*b*) to calculate the expected daily revenue from selling fresh bread when considering *all* days.
(d) Calculate the expected daily revenue from selling day-old bread.
(e) Use the results in parts (*c*) and (*d*) to calculate the expected total daily revenue and then the expected daily profit (excluding overhead).
(f) Now consider the inventory policy of baking 600 loaves each morning, so that shortages never occur. Calculate the expected daily profit (excluding overhead) from this policy.
(g) Consider the inventory policy found in part (*e*) of Prob. 19.6-4. As implied by the answers in the back of the book, this policy is to bake 550 loaves each morning, which gives a probability of incurring a shortage of $\frac{1}{6}$. Since this policy is midway between the policy considered here in parts (*a*) to (*e*) and the one considered in part (*f*), its expected daily profit (excluding overhead and the cost of the loss of customer goodwill) also is midway between the expected daily profit for those two policies. Use this fact to determine its expected daily profit.
(h) Now consider the cost of the loss of customer goodwill for the inventory policy analyzed in part (*g*). Calculate the expected daily cost of the loss of customer goodwill and then the expected daily profit when considering this cost.
(i) Repeat part (*h*) for the inventory policy considered in parts (*a*) to (*e*).

19.6-6. Reconsider Prob. 19.6-4. The bakery owner, Ken Swanson, now has developed a new plan to decrease the size of shortages. The bread will be baked twice a day, once before the bakery opens (as before) and the other during the day after it becomes clearer what the demand for that day will be. The first baking will produce 300 loaves to cover the minimum demand for the day. The size of the second baking will be based on an estimate of the remaining demand for the day. This remaining demand is assumed to have a uniform distribution from a to b, where the values of a and b are chosen each day based on the sales so far. It is anticipated that $(b - a)$ typically will be approximately 75, as opposed to the range of 300 for the distribution of demand in Prob. 19.6-4.

(a) Ignoring any cost of the loss of customer goodwill [as in parts (*a*) to (*d*) of Prob. 19.6-4], write a formula for how many loaves should be produced in the second baking in terms of a and b.
(b) What is the probability of still incurring a shortage of fresh bread on any given day? How should this answer compare to the corresponding probability in Prob. 19.6-4?
(c) When $b - a = 75$, what is the maximum size of a shortage that can occur? What is the maximum number of loaves of fresh bread that will not be sold? How do these answers compare to the corresponding numbers for the situation in Prob. 19.6-4 where only one (early morning) baking occurs per day?
(d) Now consider just the cost of underordering and the cost of overordering. Given your answers in part (*c*), how should the expected total daily cost of underordering and overordering for this new plan compare with that for the situation in Prob. 19.6-4? What does this say in general about the value of obtaining as much information as possible about what the demand will be before placing the final order for a perishable product?
(e) Repeat parts (*a*), (*b*), and (*c*) when including the cost of the loss of customer goodwill as in part (*e*) of Prob. 19.6-4.

19.6-7. Suppose that the demand D for a spare airplane part has an exponential distribution with mean 50, that is,

$$\varphi_D(\xi) = \begin{cases} \dfrac{1}{50}e^{-\xi/50} & \text{for } \xi \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

This airplane will be obsolete in 1 year, so all production of the spare part is to take place at present. The production costs now are \$1,000 per item—that is, $c = 1{,}000$—but they become \$10,000 per item if they must be supplied at later dates—that is, $p = 10{,}000$. The holding costs, charged on the excess after the end of the period, are \$300 per item.

T **(a)** Determine the optimal number of spare parts to produce.
(b) Suppose that the manufacturer has 23 parts already in inventory (from a similar, but now obsolete airplane). Determine the optimal inventory policy.

(c) Suppose that p cannot be determined now, but the manufacturer wishes to order a quantity so that the probability of a shortage equals 0.1. How many units should be ordered?

(d) If the manufacturer were following an optimal policy that resulted in ordering the quantity found in part (c), what is the implied value of p?

19.6-8.* A college student, Stan Ford, recently took a course in operations research. He now enjoys applying what he learned to optimize his personal decisions. He is analyzing one such decision currently, namely, how much money (if any) to take out of his savings account to buy $100 traveler's checks before leaving on a short vacation trip to Europe next summer.

Stan already has used the money he had in his checking account to buy traveler's checks worth $1,200, but this may not be enough. In fact, he has estimated the probability distribution of what he will need as shown in the following table:

Amount needed ($)	1,000	1,100	1,200	1,300	1,400	1,500	1,600	1,700
Probability	0.05	0.10	0.15	0.25	0.20	0.10	0.10	0.05

If he turns out to have less than he needs, then he will have to leave Europe 1 day early for every $100 short. Because he places a value of $150 on each day in Europe, each day lost would thereby represent a net loss of $50 to him. However, every $100 traveler's check costs an extra $1. Furthermore, each such check left over at the end of the trip (which would be redeposited in the savings account) represents a loss of $2 in interest that could have been earned in the savings account during the trip, so he does not want to purchase too many.

(a) Describe how this problem can be interpreted to be an inventory problem with uncertain demand for a perishable product. Also identify the unit cost of underordering and the unit cost of overordering.

(b) Use Bayes' decision rule presented in Sec. 15.2 to determine how many additional $100 travelers' checks Stan should purchase to minimize his expected cost of underordering or overordering.

(c) Use the stochastic single-period model for perishable products and the table of probabilities to make Stan's decision.

(d) Draw a graph of the CDF of demand to show the application of the model in part (c) graphically.

19.6-9. Reconsider Prob. 19.5-1 involving Henry Edsel's car dealership. The current model year is almost over, but the Tritons are selling so well that the current inventory will be depleted before the end-of-year demand can be satisfied. Fortunately, there still is time to place one more order with the factory to replenish the inventory of Tritons just about when the current supply will be gone.

The general manager, Ruby Willis, now needs to decide how many Tritons to order from the factory. Each one costs $20,000. She then is able to sell them at an average price of $23,000, provided they are sold before the end of the model year. However, any of these Tritons left at the end of the model year would then need to be sold at a special sale price of $19,500. Furthermore, Ruby estimates that the extra cost of the capital tied up by holding these cars such an unusually long time would be $500 per car, so the net revenue would be only $19,000. Since she would lose $1,000 on each of these cars left at the end of the model year, Ruby concludes that she needs to be cautious to avoid ordering too many cars, but she also wants to avoid running out of cars to sell before the end of the model year if possible. Therefore, she decides to use the stochastic single-period model for perishable products to select the order quantity. To do this, she estimates that the number of Tritons being ordered now that could be sold before the end of the model year has a normal distribution with a mean of 50 and a standard deviation of 15.

(a) Determine the optimal service level.

(b) Determine the number of Tritons that Ruby should order from the factory.

19.6-10. The management of Quality Airlines has decided to base its overbooking policy on the stochastic single-period model for perishable products, since this will maximize expected profit. This policy now needs to be applied to a new flight from Seattle to Atlanta. The airplane has 125 seats available for a fare of $250. However, since there commonly are a few no-shows, the airline should accept a few more than 125 reservations. On those occasions when more than 125 people arrive to take the flight, the airline will find volunteers who are willing to be put on a later flight in return for being given a certificate worth $150 toward any future travel on this airline.

Based on previous experience with similar flights, it is estimated that the relative frequency of the number of no-shows will be as shown below.

Number of No-Shows	Relative Frequency
0	5%
1	10%
2	15%
3	15%
4	15%
5	15%
6	10%
7	10%
8	5%

(a) When interpreting this problem as an inventory problem, what are the units of a perishable product being placed into inventory?
(b) Identify the unit cost of underordering and the unit cost of overordering.
(c) Use the model with these costs to determine how many overbooked reservations to accept.
(d) Draw a graph of the CDF of demand to show the application of the model graphically.

19.6-11. The campus bookstore must decide how many textbooks to order for a course that will be offered only once. The number of students who will take the course is a random variable D, whose distribution can be approximated by a (continuous) uniform distribution on the interval [40, 60]. After the quarter starts, the value of D becomes known. If D exceeds the number of books available, the known shortfall is made up by placing a rush order at a cost of \$14 plus \$2 per book over the normal ordering cost. If D is less than the stock on hand, the extra books are returned for their original ordering cost less \$1 each. What is the order quantity that minimizes the expected cost?

19.6-12. Consider the following inventory model, which is a single-period model with known density of demand $\varphi_D(\xi) = e^{-\xi}$ for $\xi \geq 0$ and $\varphi_D(\xi) = 0$ elsewhere. There are two costs connected with the model. The first is the purchase cost, given by $c(y - x)$. The second is a cost p that is incurred once if there is *any* unsatisfied demand (independent of the amount of unsatisfied demand).
(a) If x units are available and goods are ordered to bring the inventory level up to y (if $x < y$), write the expression for the expected loss and describe completely the optimal policy.
(b) If a fixed cost K is also incurred whenever an order is placed, describe the optimal policy.

T **19.6-13.** Find the optimal ordering policy for the stochastic single-period model with a setup cost where the demand has the probability density function

$$\varphi_D(\xi) = \begin{cases} \dfrac{1}{20} & \text{for } 0 \leq \xi \leq 20 \\ 0 & \text{otherwise,} \end{cases}$$

and the costs are

$$\begin{aligned} \text{Holding cost} &= \$1 \text{ per item,} \\ \text{Shortage cost} &= \$3 \text{ per item,} \\ \text{Setup cost} &= \$1.50, \\ \text{Production cost} &= \$2 \text{ per item.} \end{aligned}$$

Show your work, and then check your answer by using the corresponding Excel template in your OR Courseware.

T **19.6-14.** Using the approximation for finding the optimal policy for the stochastic single-period model with a setup cost when demand has an exponential distribution, find this policy when

$$\varphi_D(\xi) = \begin{cases} \dfrac{1}{25} e^{-\xi/25} & \text{for } \xi \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

and the costs are

$$\begin{aligned} \text{Holding cost} &= 40 \text{ cents per item,} \\ \text{Shortage cost} &= \$1.50 \text{ per item,} \\ \text{Purchase price} &= \$1 \text{ per item,} \\ \text{Setup cost} &= \$10. \end{aligned}$$

Show your work, and then check your answer by using the corresponding Excel template in your OR Courseware.

T **19.7-1.** Consider the following inventory situation. Demands in different periods are independent but with a common probability density function given by

$$\varphi_D(\xi) = \begin{cases} \dfrac{e^{-\xi/25}}{25} & \text{for } \xi \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Orders may be placed at the start of each period without setup cost at a unit cost of $c = 10$. There are a holding cost of 6 per unit remaining in stock at the end of each period and a shortage cost of 15 per unit of unsatisfied demand at the end of each period (with backlogging except for the final period).
(a) Find the optimal one-period policy.
(b) Find the optimal two-period policy.

T **19.7-2.** Consider the following inventory situation. Demands in different periods are independent but with a common probability density function $\phi_D(\xi) = \frac{1}{50}$ for $0 \leq \xi \leq 50$. Orders may be placed at the start of each period without setup cost at a unit cost of $c = 10$. There are a holding cost of 8 per unit remaining in stock at the end of each period and a penalty cost of 15 per unit of unsatisfied demand at the end of each period (with backlogging except for the final period).
(a) Find the optimal one-period policy.
(b) Find the optimal two-period policy.

T **19.7-3.*** Find the optimal inventory policy for the following two-period model by using a discount factor of $\alpha = 0.9$. The demand D has the probability density function

$$\varphi_D(\xi) = \begin{cases} \dfrac{1}{25} e^{-\xi/25} & \text{for } \xi \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

and the costs are

$$\text{Holding cost} = \$0.25 \text{ per item},$$
$$\text{Shortage cost} = \$2 \text{ per item},$$
$$\text{Purchase price} = \$1 \text{ per item}.$$

Stock left over at the end of the final period is salvaged for $1 per item, and shortages remaining at this time are met by purchasing the needed items at $1 per item.

T **19.7-4.** Solve Prob. 19.7-3 for a two-period model, assuming no salvage value, no backlogging at the end of the second period, and no discounting.

T **19.7-5.** Solve Prob. 19.7-3 for an infinite-period model.

T **19.7-6.** Determine the optimal inventory policy when the goods are to be ordered at the end of every month from now on. The cost of bringing the inventory level up to y when x already is available is given by $2(y - x)$. Similarly, the cost of having the monthly demand D exceed y is given by $5(D - y)$. The probability density function for D is given by $\phi_D(\xi) = e^{-\xi}$. The holding cost when y exceeds D is given by $y - D$. A monthly discount factor of 0.95 is used.

T **19.7-7.** Solve the inventory problem given in Prob. 19.7-6, but assume that the policy is to be used for only 1 year (a 12-period model). Shortages are backlogged each month, except that any shortages remaining at the end of the year are made up by purchasing similar items at a unit cost of $2. Any remaining inventory at the end of the year can be sold at a unit price of $2.

T **19.7-8.** A supplier of high-fidelity receiver kits is interested in using an optimal inventory policy. The distribution of demand per month is uniform between 2,000 and 3,000 kits. The supplier's cost for each kit is $150. The holding cost is estimated to be $2 per kit remaining at the end of a month, and the shortage cost is $30 per kit of unsatisfied demand at the end of a month. Using a monthly discount factor of $\alpha = 0.99$, find the optimal inventory policy for this infinite-period problem.

T **19.7-9.** The weekly demand for a certain type of electronic calculator is estimated to be

$$\varphi_D(\xi) = \begin{cases} \dfrac{1}{1{,}000} e^{-\xi/1{,}000} & \text{for } \xi \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

The unit cost of these calculators is $80. The holding cost is $0.70 per calculator remaining at the end of a week. The shortage cost is $2 per calculator of unsatisfied demand at the end of a week. Using a weekly discount factor of $\alpha = 0.998$, find the optimal inventory policy for this infinite-period problem.

19.7-10.* Consider a one-period model where the only two costs are the holding cost, given by

$$h(y - D) = \frac{3}{10}(y - D), \qquad \text{for } y \geq D,$$

and the shortage cost, given by

$$p(D - y) = 2.5(D - y), \qquad \text{for } D \geq y.$$

The probability density function for demand is given by

$$\varphi_D(\xi) = \begin{cases} \dfrac{e^{-\xi/25}}{25} & \text{for } \xi \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

If you order, you must order an *integer* number of *batches* of 100 units each, and this quantity is delivered immediately. Let $G(y)$ denote the total expected cost when there are y units available for the period (after ordering).

(a) Write the expression for $G(y)$.
(b) What is the optimal ordering policy?

19.7-11. Find the optimal (k, Q) policy for Prob. 19.7-10 for an infinite-period model with a discount factor of $\alpha = 0.90$.

19.7-12. For the infinite-period model with no setup cost, show that the value of y^0 that satisfies

$$\Phi(y^0) = \frac{p - c(1 - \alpha)}{p + h}$$

is equivalent to the value of y that satisfies

$$\frac{dL(y)}{dy} + c(1 - \alpha) = 0,$$

where $L(y)$, the expected shortage plus holding cost, is given by

$$L(y) = \int_y^\infty p(\xi - y)\varphi_D(\xi)\, d\xi + \int_0^y h(y - \xi)\varphi_D(\xi)\, d\xi.$$

CASE 19.1 BRUSHING UP ON INVENTORY CONTROL

Robert Gates rounds the corner of the street and smiles when he sees his wife pruning rose bushes in their front yard. He slowly pulls his car into the driveway, turns off the engine, and falls into his wife's open arms.

"How was your day?" she asks.

"Great! The drugstore business could not be better!" Robert replies, "Except for the traffic coming home from work! That traffic can drive a sane man crazy! I am so tense right now. I think I will go inside and make myself a relaxing martini."

Robert enters the house and walks directly into the kitchen. He sees the mail on the kitchen counter and begins flipping through the various bills and advertisements until he comes across the new issue of *OR/MS Today.* He prepares his drink, grabs the magazine, treads into the living room, and settles comfortably into his recliner. He has all that he wants—except for one thing. He sees the remote control lying on the top of the television. He sets his drink and magazine on the coffee table and reaches for the remote control. Now, with the remote control in one hand, the magazine in the other, and the drink on the table near him, Robert is finally the master of his domain.

Robert turns on the television and flips the channels until he finds the local news. He then opens the magazine and begins reading an article about scientific inventory management. Occasionally he glances at the television to learn the latest in business, weather, and sports.

As Robert delves deeper into the article, he becomes distracted by a commercial on television about toothbrushes. His pulse quickens slightly in fear because the commercial for Totalee toothbrushes reminds him of the dentist. The commerical concludes that the customer should buy a Totalee toothbrush because the toothbrush is Totalee revolutionary and Totalee effective. It certainly is effective; it is the most popular toothbrush on the market!

At that moment, with the inventory article and the toothbrush commercial fresh in his mind, Robert experiences a flash of brilliance. He knows how to control the inventory of Totalee toothbrushes at Nightingale Drugstore!

As the inventory control manager at Nightingale Drugstore, Robert has been experiencing problems keeping Totalee toothbrushes in stock. He has discovered that customers are very loyal to the Totalee brand name since Totalee holds a patent on the toothbrush endorsed by 9 out of 10 dentists. Customers are willing to wait for the toothbrushes to arrive at Nightingale Drugstore since the drugstore sells the toothbrushes for 20 percent less than other local stores. This demand for the toothbrushes at Nightingale means that the drugstore is often out of Totalee toothbrushes. The store is able to receive a shipment of toothbrushes several hours after an order is placed to the Totalee regional warehouse because the warehouse is only 20 miles away from the store. Nevertheless, the current inventory situation causes problems because numerous emergency orders cost the store unnecessary time and paperwork and because customers become disgruntled when they must return to the store later in the day.

Robert now knows a way to prevent the inventory problems through scientific inventory management! He grabs his coat and car keys and rushes out of the house.

As he runs to the car, his wife yells, "Honey, where are you going?"

"I'm sorry, darling," Robert yells back. "I have just discovered a way to control the inventory of a critical item at the drugstore. I am really excited because I am able to apply my industrial engineering degree to my job! I need to get the data from the store and work out the new inventory policy! I will be back before dinner!"

Because rush hour traffic has dissipated, the drive to the drugstore takes Robert no time at all. He unlocks the darkened store and heads directly to his office where he rummages through file cabinets to find demand and cost data for Totalee toothbrushes over the past year.

Aha! Just as he suspected! The demand data for the toothbrushes is almost constant across the months. Whether in winter or summer, customers have teeth to brush, and they need toothbrushes. Since a toothbrush will wear out after a few months of use, customers will always return to buy another toothbrush. The demand data shows that Nightingale Drugstore customers purchase an average of 250 Totalee toothbrushes per month (30 days).

After examining the demand data, Robert investigates the cost data. Because Nightingale Drugstore is such a good customer, Totalee charges its lowest wholesale price of only $1.25 per toothbrush. Robert spends about 20 minutes to place each order with Totalee. His salary and benefits add up to $18.75 per hour. The annual holding cost for the inventory is 12 percent of the capital tied up in the inventory of Totalee toothbrushes.

(a) Robert decides to create an inventory policy that normally fulfills all demand since he believes that stock-outs are just not worth the hassle of calming customers or the risk of losing future business. He therefore does not allow any planned shortages. Since Nightingale Drugstore receives an order several hours after it is placed, Robert makes the simplifying assumption that delivery is instantaneous. What is the optimal inventory policy under these conditions? How many Totalee toothbrushes should Robert order each time and how frequently? What is the total variable inventory cost per year with this policy?

(b) Totalee has been experiencing financial problems because the company has lost money trying to branch into producing other personal hygiene products, such as hairbrushes and dental floss. The company has therefore decided to close the warehouse located 20 miles from Nightingale Drugstore. The drugstore must now place orders with a warehouse located 350 miles away and must wait 6 days after it places an order to receive the shipment. Given this new lead time, how many Totalee toothbrushes should Robert order each time, and when should he order?

(c) Robert begins to wonder whether he would save money if he allows planned shortages to occur. Customers would wait to buy the toothbrushes from Nightingale since they have high brand loyalty and since Nightingale sells the toothbrushes for less. Even though customers would wait to purchase the Totalee toothbrush from Nightingale, they would become unhappy with the prospect of having to return to the store again for the product. Robert decides that he needs to place a dollar value on the negative ramifications from shortages. He knows that an employee would have to calm each disgruntled customer and track down the delivery date for a new shipment of Totalee toothbrushes. Robert also believes that customers would become upset with the inconvenience of shopping at Nightingale and would perhaps begin looking for another store providing better service. He estimates the costs of dealing with disgruntled customers and losing customer goodwill and future sales as $1.50 per unit short per year. Given the 6-day lead time and the shortage allowance, how many Totalee toothbrushes should Robert order each time, and when should he order? What is the maximum shortage under this optimal inventory policy? What is the total variable inventory cost per year?

(d) Robert realizes that his estimate for the shortage cost is simply that—an estimate. He realizes that employees sometimes must spend several minutes with each customer who wishes to purchase a toothbrush when none is currently available. In addition, he realizes that the cost of losing customer goodwill and future sales could vary within a wide range. He estimates that the cost of dealing with disgruntled customers and losing customer goodwill and future sales could range from 85 cents to $25 per unit short per year. What effect would changing the estimate of the unit shortage cost have on the inventory policy and total variable inventory cost per year found in part (*c*)?

(e) Closing warehouses has not improved Totalee's bottom line significantly, so the company has decided to institute a discount policy to encourage more sales. Totalee will charge $1.25 per toothbrush for any order of up to 500 toothbrushes, $1.15 per toothbrush for orders of more than 500 but less than 1000 toothbrushes, and $1 per toothbrush for orders of 1000 toothbrushes or more. Robert still assumes a 6-day lead time, but he does not want planned shortages to occur. Under the new discount policy, how many Totalee toothbrushes should Robert order each time, and when should he order? What is the total inventory cost (including purchase costs) per year?

CASE 19.2 TNT: TACKLING NEWSBOY'S TEACHINGS

Howie Rogers sits in an isolated booth at his favorite coffee shop completely immersed in the classified ads of the local newspaper. He is searching for his next get-rich-quick venture. As he meticulously reviews each ad, he absent-mindedly sips his lemonade and wonders how he will be able to exploit each opportunity to his advantage.

He is becoming quite disillusioned with his chosen vocation of being an entrepreneur looking for high-flying ventures. These past few years have not dealt him a lucky hand. Every project he has embarked upon has ended in utter disaster, and he is slowly coming to the realization that he just might have to find a real job.

He reads the date at the top of the newspaper. June 18. Ohhhh. No need to look for a real job until the end of the summer.

Each advertisement Howie reviews registers as only a minor blip on his radar screen until the word Corvette jumps out at him. He narrows his eyes and reads:

> WIN A NEW CORVETTE AND EARN CASH AT THE SAME TIME! Fourth of July is fast approaching, and we need YOU to sell firecrackers. Call 1-800-555-3426 to establish a firecracker stand in your neighborhood. Earn fast money AND win the car of your dreams!

Well, certainly not a business that will make him a millionaire, but a worthwhile endeavor nonetheless! Howie tears the advertisement out of the newspaper and heads to the payphone in the back.

A brief—but informative—conversation reveals the details of the operation. Leisure Limited, a large wholesaler that distributes holiday products—Christmas decorations, Easter decorations, firecrackers, etc.—to small independents for resale, is recruiting entrepreneurs to run local firecracker stands for the Fourth of July. The wholesaler is offering to rent wooden shacks to entrepreneurs who will purchase firecrackers from Leisure Limited and will subsequently resell the firecrackers in these shacks on the side of the road to local customers for a higher price. The entrepreneurs will sell firecrackers until the Fourth of July, but after the holiday, customers will no longer want to purchase firecrackers until New Year's Eve. Therefore, the entrepreneurs will return any firecrackers not sold by the Fourth of July while keeping the revenues from all firecrackers sold. Leisure Limited will refund only part of the cost of the returned firecrackers, however, since returned firecrackers must be restocked and since they lose their explosiveness with age. And the Corvette? The individual who sells the greatest number of Leisure Limited firecrackers in the state will win a new Corvette.

Before Howie hangs up the phone, the Leisure Limited representative reveals one hitch—once an entrepreneur places an order for firecrackers, 7 days are required for the delivery of the firecrackers. Howie realizes that he better get started quickly so that he will be able to sell firecrackers during the week preceding the Fourth of July when most of the demand occurs.

People could call Howie many things, but "pokey" they could not. Howie springs to action by reserving a wooden shack and scheduling a delivery 7 days hence. He then places another quarter in the payphone to order firecracker sets, but as he starts dialing the phone, he realizes that he has no idea how many sets he should order.

How should he solve this problem? If he orders too few firecracker sets, he will not have time to place and receive another order before the holiday and will therefore lose valuable sales (not to mention the chance to win the Corvette). If he orders too many firecracker sets, he will simply throw away money since he will not obtain a full refund for the cost of the surplus sets.

Quite a dilemma! He hangs up the phone and bangs his head against the hard concrete wall. After several bangs, he stands up straight with a thought. Of course! His sister would help him. She had graduated from college several years ago with an industrial engineering degree, and he is sure that she will agree to help him.

Howie calls Talia, his sister, at her work and explains his problem. Once she hears the problem, she is confident that she will be able to tell Howie how many sets he should order. Her dedicated operations research teacher in college had taught her well. Talia asks Howie to give her the number for Leisure Limited, and she would then have the answer for him the next day.

Talia calls Leisure Limited and asks to speak to the manager on duty. Buddy Williams, the manager, takes her call, and Talia explains to him that she wants to run a firecracker stand. To decide the number of firecracker sets she should order, however, she needs some information from him. She persuades Buddy that he should not hesitate to give her the information since a more informed order is better for Leisure Limited—the wholesaler will not lose too many sales and will not have to deal with too many returns.

Talia receives the following information from Buddy. Entrepreneurs purchase firecracker sets from Leisure Limited at a cost of \$3.00 per set. Entrepreneurs are able to sell the firecracker sets for any price that they deem reasonable. In addition to the wholesale price of the firecracker sets, entrepreneurs also have to pay administrative and delivery fees for each order they place. These fees average approximately \$20.00 per order. After the Fourth of July, Leisure Limited returns only half of the wholesale cost for each firecracker set returned. To return the unsold firecracker sets, entrepreneurs also have to pay shipping costs that average \$0.50 per firecracker set.

Finally, Talia asks about the demand for firecracker sets. Buddy is not able to give her very specific information, but he is able to give her general information about last year's sales. Data compiled from last year's stand sales throughout the state indicate that stands sold between 120 and 420 firecracker sets. The stands operated any time between June 20 and July 4 and sold the firecracker sets for an average of \$5.00 per set.

Talia thanks Buddy, hangs up the phone, and begins making assumptions to help her overcome the lack of specific data. Even though Howie will operate his stand only during the week preceding the Fourth of July, she decides to use the demands quoted by Buddy for simplicity. She assumes that the demand follows a uniform distribution. She decides to use the average of \$5.00 for the unit sale price.

(a) How many firecracker sets should Howie purchase from Leisure Limited to maximize his expected profit?
(b) How would Howie's order quantity change if Leisure Limited refunds 75 percent of the wholesale price for returned firecracker sets? How would it change if Leisure Limited refunds 25 percent of the wholesale price for returned firecracker sets?
(c) Howie is not happy with selling the firecracker sets for \$5.00 per set. He needs to make some serious dough! Suppose Howie wants to sell the firecracker sets for \$6.00 per set instead. What factors would Talia have to take into account when recalculating the optimal order quantity?
(d) What do you think of Talia's strategy for estimating demand?

CASE 19.3 JETTISONING SURPLUS STOCK

Scarlett Windermere cautiously approaches the expansive gray factory building and experiences a mixture of fear and excitement. The first day of a new consulting assignment always leaves her fighting conflicting emotions. She takes a deep breath, clutches her briefcase, and marches into the small, stuffy reception area of American Aerospace.

"Scarlett Windermere here to see Bryan Zimmerman," she says to the bored security guard behind the reception desk.

The security guard eyes Scarlett suspiciously and says, "Ya don't belong here, do ya? Of course ya don't. Then ya gotta fill out this paperwork for a temporary security pass."

As Scarlett completes the necessary paperwork, Bryan exits through the heavy door leading to the factory floor and enters the reception area. His eyes roam the reception area and rest upon Scarlett. He approaches Scarlett booming, "So you must be the inventory expert—Scarlett Windermere. So glad to finally meet you face to face! They already got you pouring out your life story, huh? Well, there will be enough time for that. Right now, let's get you back to the factory floor to help me solve my inventory problem!"

And with that, Bryan stuffs a pair of safety glasses in Scarlett's right hand, stuffs the incomplete security forms in her left hand, and hustles her through the heavy security door.

As Scarlett walks through the security door, she feels as though she has entered another world. Machines twice the size of humans line the aisles as far as the eye can see. These monsters make high-pitched squeals or low, horrifying rumbles as they cut and grind metal. Surrounding these machines are shelves piled with metal pieces.

As Bryan leads Scarlett down the aisles of the factory, he yells to her over the machines, "As you well know from the proposal stage of this project, this factory produces the stationary parts for the military jet engines American Aerospace sells. Most people think the aerospace industry is real high-tech. Well, not this factory. This factory is as dirty as they come. Jet engines are made out of a lot of solid metal parts, and this factory cuts, grinds, and welds those parts."

"This factory produces over 200 different stationary parts for jet engines. Each jet engine model requires different parts. And each part requires different raw materials. Hence, the factory's current inventory problem."

"We hold all kinds of raw materials—from rivets to steel sheets—here on the factory floor, and we currently mismanage our raw materials inventory. We order enough raw materials to produce a year's worth of some stationary parts, but only enough raw materials to produce a week's worth of others. We waste a ton of money stocking raw materials that are not needed and lose a ton of money dealing with late deliveries of orders. We need you to tell us how to control the inventory—how many raw materials we need to stock for each part, how often we need to order additional raw materials, and how many we should order."

As she walks down the aisle, Scarlett studies the shelves and shelves of inventory. She has quite a mission to accomplish in this factory!

Bryan continues, "Let me tell you how we receive orders for this factory. Whenever the American Aerospace sales department gets an order for a particular jet engine, the order is transferred to its assembly plant here on the site. The assembly plant then submits an order to this factory here for the stationary parts required to assemble the engine. Unfortunately, because this factory is frequently running out of raw materials, it takes us an average of a month between the time we receive an order and the time we deliver the finished order to the assembly plant. The finished order includes all the stationary parts needed to assemble that particular jet engine. BUT—and that's a big but—the delivery time really depends upon which stationary parts are included in the order."

Scarlett interrupts Bryan and says, "Then I guess now would be as good a time as any to start collecting the details of the orders and solving your inventory problem!"

Bryan smiles and says, "That's the attitude I like to see—chomping at the bit to solve the problem! Well, I'll show you to your computer. We just had another consulting firm complete a data warehouse started by American Aerospace three years ago, so you can access any of the data you need right from your desktop!" And with a flurry, Bryan heads back down the aisle.

Scarlett realizes that the inventory system is quite complicated. She remembers a golden rule from her consulting firm: break down a complex system into simple parts. She therefore decides to analyze the control of inventory for each stationary part independently. But with 200 different stationary parts, where should she begin?

She remembers that when the assembly plant receives an order for a particular jet engine, it places an order with the factory for the stationary parts required to assemble the engine. The factory delivers an order to the assembly plant when all stationary parts for that order have been completed. The stationary part that takes the longest to complete in a given order therefore determines the delivery date of the order.

Scarlett decides to begin her analysis with the most time-intensive stationary part required to assemble the most popular jet engine. She types a command into the computer to determine the most popular jet engine. She learns that the MX332 has received the largest number of orders over the past year. She types another command to generate the following printout of the monthly orders for the MX332.

Month	Number of MX332 ordered
June	25
July	31
August	18
September	22
October	40
November	19
December	38
January	21
February	25
March	36
April	34
May	28
June	27

She enters the monthly order quantities for the MX332 into a computerized statistical program to estimate the underlying distribution. She learns that the orders roughly follow a normal distribution. It appears to Scarlett that the number of orders in a particular month does not depend on the number of orders in the previous or following months.

(a) What is the sample mean and sample variance of the set of monthly orders for the MX332?

Scarlett next researches the most time-intensive stationary part required to assemble the MX332. She types a command into the computer to generate a list of parts required to assemble the MX332. She then types a command to list the average delivery time for each part. She learns that part 10003487 typically requires the longest time to complete, and that this part is only used for the MX332. She investigates the pattern for the part further and learns that over the past year, part 10003487 has taken an average of one month to complete once an order is placed. She also learns that the factory can produce the part almost immediately if all the necessary raw materials for the production process are on hand. So the completion time actually depends on how long it takes to obtain these raw materials from the supplier. On those unusual occasions when all the raw materials already are available in inventory, the completion time for the part is essentially zero. But typically the completion time is $1\frac{1}{2}$ months.

Scarlett performs further analysis on the computer and learns that each MX332 jet engine requires two parts numbered 10003487. Each part 10003487 accepts one solid steel part molded into a cylindrical shape as its main raw material input. The data show that several times the delivery of all the stationary parts for the MX332 to the assembly plant got delayed for up to $1\frac{1}{2}$ months only because a part 10003487 was not completed. And why wasn't it completed? The factory had run out of those steel parts and had to wait for another shipment from its supplier! It takes the supplier $1\frac{1}{2}$ months to produce and deliver the steel parts after receiving an order from the factory. Once an order of steel parts arrives, the factory quickly sets up and executes a production run to use all the steel parts for producing parts 10003487. Apparently the production problems in the factory are mainly due to the inventory management for those unassuming steel parts. And that inventory management appears to be completely out of whack. The only good news is that there is no significant administrative cost associated with placing an order for the steel parts with the supplier.

After Scarlett has finished her work on the computer, she heads to Bryan's office to obtain the financials needed to complete her analysis. A short meeting with Bryan yields the following financial information.

Setup cost for a production run to produce part 10003487	\$5,800
Holding cost for machine part 10003487	\$750 per part per year
Shortage cost for part 10003487 (includes outsourcing cost, cost of production delay, and cost of the loss of future orders)	\$3,250 per part per year
Desired probability that a shortage for machine part 10003487 will not occur between the time an order for the steel parts is placed and the time the order is delivered	0.85

Now Scarlett has all of the information necessary to perform her inventory analysis for part 10003487!

(b) What is the inventory policy that American Aerospace should implement for part 10003487?
(c) What are the average annual holding costs and setup costs associated with this inventory policy?

(d) How do the average annual holding costs and setup costs change if the desired probability that a shortage will not occur between the time an order is placed and the time the order is delivered is increased to 0.95?
(e) Do you think Scarlett's independent analysis of each stationary part could generate inaccurate inventory policies? Why or why not?
(f) Scarlett knows that the aerospace industry is very cyclical—the industry experiences several years of high sales, several years of mediocre sales, and several years of low sales. How would you recommend incorporating this fact into the analysis?