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Mathematics for Statistics

Contents

1 Preliminaries

This document summarizes fundamental mathematical concepts for statistics. The concepts discussed in this document correspond to a German high school level in mathematics. For further details on the concepts, the reader is referred to standard textbooks such as Sydsaeter and Hammond (2016): "Essential mathematics for economic analysis."

Section 1 to Section 4 is necessary for statistics 1, Section 5 for statistics 2. In the following, *italics* indicate definitions.

1.1 Number Sets and Relations

We distinguish the following number sets:

Table 1: Number Sets

Name	Notation	Meaning
Natural Numbers	$\mathbb{N}$	$\{1, 2, 3, \dots\}$
Integers	$\mathbb{Z}$	$\{\dots, -2, -1, 0, 1, 2, \dots\}$
Rational Numbers	$\mathbb{Q}$	$\{\frac{q}{r}$ for which holds: $q, r \in \mathbb{Z}\}$
Real Numbers	$\mathbb{R}$	Rational numbers plus irrational numbers like $\sqrt{2}, \pi$

Remark. $\mathbb{N}$ is a subset of $\mathbb{Z}$, $\mathbb{Z}$ is a subset of $\mathbb{Q}$, and $\mathbb{Q}$ is a subset of $\mathbb{R}$.

Two real numbers, a and b, may have the following relationships:

Table 2: Number Relations

Relation	Meaning
$a > b$ $a < b$	a is greater than b a is smaller than b
$a \geq b$ $a \leq b$	a is greater than or equal to b a is smaller than or equal to b
$a = b$ $a \neq b$	a is equal to b. Meaning: $a \leq b$ **and** $a \geq b$ a and b are unequal. Meaning: $a > b$ **or** $a < b$

1.2 Sums and Products of numbers

The sum of n numbers $a_1, \dots, a_n$ is denoted by:

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_{n-1} + a_n.$$

If $E = \{a_1, \dots, a_n\}$, we may write their sum as $\sum\limits_{a \in E} a$. For instance, the sum of all real numbers is $\sum\limits_{a \in \mathbb{R}} a$.

The product of n numbers $a_1, \dots, a_n$ is denoted by:

$$\prod_{i=1}^{n} a_i = a_1 \cdot a_2 \cdot \ldots \cdot a_{n-1} \cdot a_n.$$

If $E = \{a_1, \dots, a_n\}$, we may write their product as $\prod\limits_{a \in E} a$. For instance, the product of all real numbers is $\prod\limits_{a \in \mathbb{R}} a$.

Numbers and Percent. We convert a number from decimal to percent by multiplying it by 100 and adding a percent sign. For instance:

$$0.1 = 0.1 \cdot 100\% = 10\%$$

2 Sets

2.1 Definition and Notation

Definition (Set)**.** A set is a well-defined and unordered collection of distinct objects.

"Distinct" means that a set does not contain one and the same object several times: each object of a set is different from any other object of the set. Members of a set S are called elements of S. If s is an element of S, we write: $s \in S$. A set is called *finite* if it has finitely many elements, otherwise it is called *infinite*.

Example. The following are sets:

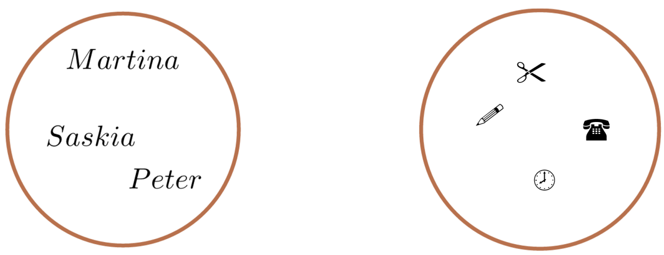


Figure 1: Examples of Sets

Notation (Finite Sets)**.** A finite set is usually denoted by a list of its elements, separated by commas, and surrounded by set brackets.

Example. The sets above are denoted by {Martina,Saskia,Peter} and {⏱, ☎, ✂,✎}.

Notation. (infinite sets). An infinite set is usually denoted by the elements of a larger set and a condition that these elements must meet: elements: condition, where ":" means "for which holds".

Example. The set of all numbers greater than or equal to 5 is denoted by: $\{x \in \mathbb{R} : \; x \geq 5\}$, which translates into all real numbers x for which holds x is greater than or equal to 5.

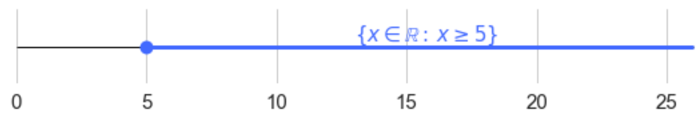


Figure 2: All real numbers greater/equal 5

Example. We can also define a set with the help of a function. Let $f(x) = x^2$, then $\{x \in \mathbb{R} : \; f(x) = 4\} = \{-2, 2\}$.

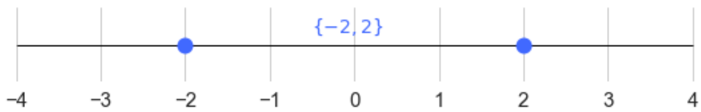


Figure 3: Set $\{-2, 2\}$

Definition (Interval)**.** An interval is a set of real numbers that contains all real numbers lying between two numbers $a < b$ of the set.

Notation. Intervals are denoted in the following way:

(i) Closed intervals: $[a, b] = \{x \in \mathbb{R} : \; a \leq x \leq b\}$

(ii) Open intervals: $(a, b) = \{x \in \mathbb{R} : \; a < x < b\}$

(iii) Half-open intervals: $[a, b) = \{x \in \mathbb{R} : \; a \leq x < b\}$

Example.

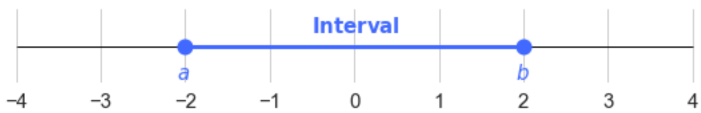


Figure 4: Closed Interval

2.2 Set Operations

Let E, F be sets and S be the universe of all objects of interest (the set containing all objects of interest). The following are the basic set operations:

(i) **Subsets and Supersets.** E is a subset of F (notation: $E \subseteq F$, $E \subset F$) or, equivalently, F is a superset of E if E is contained in F:

$$\text{For all } x: \; x \in E \text{ implies } x \in F.$$

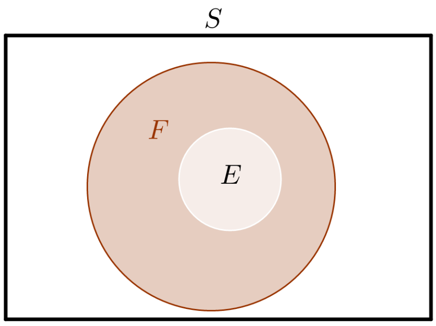


Figure 5: Subset

Remark. $E \subset F$ excludes $E = F$, while $E \subseteq F$ allows for it.

(ii) **Set Difference.** The difference between E and F (notation: $E \setminus F$) is the set of all elements that are contained in E but not in F:

$$E \setminus F = \{x \in S: \; x \in E \text{ and } x \notin F\}.$$

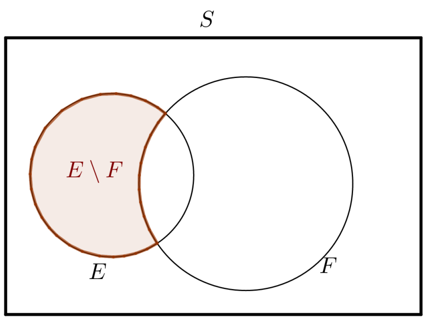


Figure 6: Set Difference

(iii) **Complementary Set.** The complement of E (notation: E^c) is the set that contains all elements that are not elements of E:

$$E^c = \{x \in S: \; x \notin E\} = S \setminus E.$$

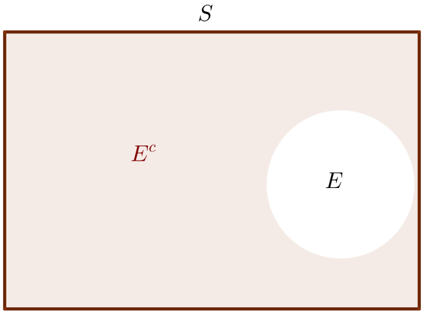


Figure 7: Complementary Set

(iv) **Set Union.** The union of E and F (notation: $E \cup F$) is the set of all elements that are contained in E, or F, or both:

$$E \cup F = \{x \in S : x \in E \text{ and/or } x \in F\}.$$

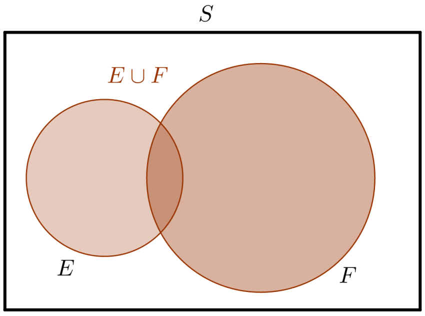


Figure 8: Set Union

Remark. A union of n sets, $E_1, \ldots, E_n$ can be written as $\bigcup_{i=1}^{n} E_i = E_1 \cup \ldots \cup E_n$. If $\mathcal{E} = \{E_1, \ldots, E_n\}$, we can also write $\bigcup_{E \in \mathcal{E}} E$.

(v) **Set Intersection.** The intersection of E and F (notation: $E \cap F$) is the set of all elements that are simultaneously contained in E and F:

$$E \cap F = \{x \in S : x \in E \text{ and } x \in F\}.$$

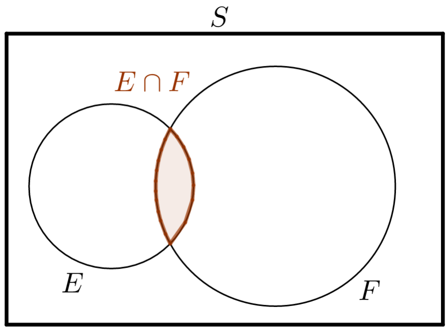


Figure 9: Set Intersection

Remark. An intersection of n sets, $E_1, \ldots, E_n$ can be written as $\bigcap_{i=1}^{n} E_i = E_1 \cap \ldots \cap E_n$. If $\mathcal{E} = \{E_1, \ldots, E_n\}$, we can also write $\bigcap_{E \in \mathcal{E}} E$.

(vi) **Cartesian Product.** The Cartesian product of two sets E and F (notation: $E \times F$) is the set of all ordered pairs (e, f) where $e \in E$ and $f \in F$:

$$E \times F = \{(e, f) : e \in E \text{ and } f \in F\}.$$

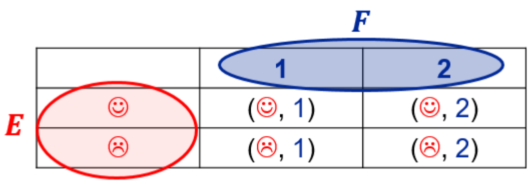


Figure 10: Cartesian Product

Remark. The Cartesian product of n sets, $E_1 \times \ldots \times E_n$ can be written as $\prod_{i=1}^{n} E_i$. If $\mathcal{E} = \{E_1, \ldots, E_n\}$, we can also write $\prod_{E \in \mathcal{E}} E$. Elements of a Cartesian product of dimension n are called *n-tuples*.

Definition (Empty Set)**.** Let E be a set. The empty set $\varnothing$ is a set that contains literally nothing: $\varnothing = E \setminus E = S \setminus S$.

Definition (Disjoint Sets)**.** Two sets E and F are called *disjoint* if their intersection is empty: $E \cap F = \varnothing$.

2.3 Collection of Sets

A *collection of sets* is a set whose elements are sets. We denote collection of sets by script letters.

Definition (Power Set)**.** The *power set* of a set S, $\mathcal{P}(S)$, is the set that contains all subsets of S: $\mathcal{P}(S) = \{E : E \subseteq S\}$.

Example. If $S = \{1, 2, 3\}$, then

$$\mathcal{P}(S) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, S\}.$$

Definition (Partition)**.** A collection of sets $\mathcal{T}$ is called *partition* of S if

(i) The elements of $\mathcal{T}$ are disjoint: $E \cap F = \varnothing$ for all $E, F \in \mathcal{T}$.

(ii) The union of all elements of $\mathcal{T}$ is S: $\bigcup_{E \in \mathcal{T}} E = S$.

Example. Let $S = \{1, 2, 3\}$. The finest partition of S is S itself. Another partition is, e.g.:

$$\mathcal{T} = \{\{1\}, \{2, 3\}\}.$$

Definition (σ-Algebra)**.** Given a set S. A *σ-algebra* (also known as σ-field) $\mathcal{A}$ on S is a collection of subsets of S, $\mathcal{A} \subseteq \mathcal{P}(S)$ that satisfies

(i) $S \in \mathcal{A}$

(ii) Closed under complementation. If E is in $\mathcal{A}$, then so is E^c.

(iii) Closed under countable unions. If E, F are in $\mathcal{A}$, then so is $E \cup F$.

Example. Let $S = \{1, 2, 3, 4\}$.

– The largest σ-algebra is the *discrete σ-algebra*: $\mathcal{P}(S)$

– The smallest σ-algebra is the *minimal or trivial σ-algebra*: $\{\varnothing, S\}$.

– Each σ-algebra of the form $\{\varnothing, E, E^c, S\}$ is called *simple σ-algebra generated by E*. For instance: $\{\varnothing, \{1\}, \{2, 3, 4\}, S\}$.

– The smallest σ-algebra that contains all the open subsets (open intervals) of $\mathbb{R}$ is called *Borel σ-Algebra.*

Remarks.

(i) In the statistical nomenclature, elements of S are called *simple events* and S is called *sample space*, elements of $\mathcal{A}$ are called *events* and $\mathcal{A}$ is the *event space.*

(ii) A tuple $(S, \mathcal{A})$ is called *measurable space.* A triple $(S, \mathcal{A}, P)$, where P is a measure, is called *measure space.* If P is a *probability measure* (satisfies the probability axioms), we call $(S, \mathcal{A}, P)$ a *probability space.* The formal definitions of these concepts are in Appendix A.1.

2.4 Exercises

2.4.1 Which of the following sets is finite?

a) $[4, 8]$

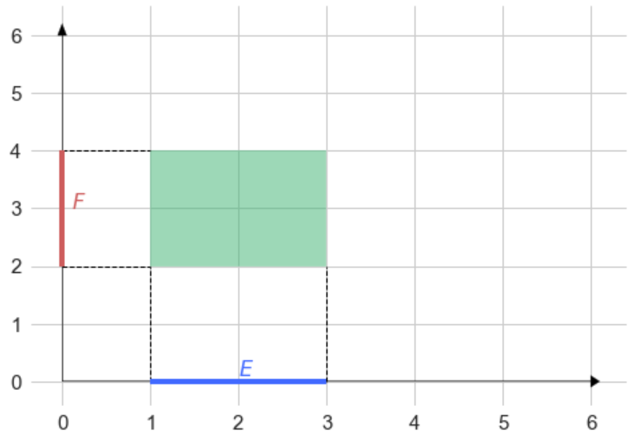
b) $\{3, 4, 10\}$

c) $\mathcal{P}(S)$ where $S = \{Cat, Dog\}$

d) $\mathbb{N}$

2.4.2 Write down the set of all positive real numbers x that are less than 10 and where $f(x) = 3 + 4x$ is greater than 2.

2.4.3 Write down the set below in set notation.



2.4.4 Write down the set of all Latin letters.

2.4.5 Let $E = [10, 20]$. Write down E^c.

2.4.6 Let $E = \{Apple, Pie\}$ and $F = \{Peter, Saskia, Francis\}$. Write down the union of E and F.

2.4.7 Let $E = \{x \in \mathbb{R} : x \leq 23\}$ and $F = \mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$. Write down the intersection of E and F.

2.4.8 Let $E = \{Apple, 3, Monkey\}$ and $F = \{0, 3, 10\}$. Write down the set differences $E \setminus F$ and $F \setminus E$.

2.4.9 Which of the following sets are empty?

a) $\{HumanBeings\} \cap \{Animals\}$

b) $\{x \in \mathbb{R} : x < 0\} \cap \{x \in \mathbb{R} : x^2 = 0\}$

c) $\{x \in \mathbb{R} : x < 0\} \cup \{x \in \mathbb{R} : x^2 = 0\}$

d) $\{x \in \mathbb{R} : x^2 = 0\} \setminus \{0\}$

2.4.10 Find the power sets of the sets $E = \{a, b, c\}$ and $F = \{\{2, 3\}, 6\}$.

2.4.11 Let $S = \{\epsdice{1}, \epsdice{2}, \epsdice{3}, \epsdice{4}, \epsdice{5}, \epsdice{6}\}$. Write down 3 partitions of S.

2.4.12 Let $S = \{\epsdice{1}, \epsdice{2}, \epsdice{3}, \epsdice{4}, \epsdice{5}, \epsdice{6}\}$. Which of the following are event spaces?

a) $\{\varnothing, S\}$

b) $\{\varnothing, \{\epsdice{1}, \epsdice{2}\}, \{\epsdice{3}, \epsdice{4}, \epsdice{5}, \epsdice{6}\}, S\}$

c) $\{\varnothing, \epsdice{1}, \epsdice{2}, \{\epsdice{1}, \epsdice{2}\}, \{\epsdice{3}, \epsdice{4}, \epsdice{5}, \epsdice{6}\}, \{\epsdice{1}, \epsdice{3}, \epsdice{4}, \epsdice{5}, \epsdice{6}\}, \{\epsdice{2}, \epsdice{3}, \epsdice{4}, \epsdice{5}, \epsdice{6}\}, S\}$

d) $\{\varnothing, \{\epsdice{1}, \epsdice{2}\}, \{\epsdice{3}, \epsdice{4}\}, \{\epsdice{5}, \epsdice{6}\}, S\}$

2.4.13 From 24 students in a class, 15 play volleyball, 15 handball and 10 basketball, 5 students play volleyball and basketball, 7 play handball and basketball, 3 play only handball and 4 practice all three sports. How many students only play basketball? How many students play volleyball and handball? How many students do not play any of the three sports?

3 Combinatorics

3.1 Basic Operations

Definition (Factorial)**.** The factorial of a positive integer n, denoted by $n!$, is the product of all positive integers less than or equal to n:

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1 = \prod_{i=1}^{n} i.$$

Example. $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$.

Interpretation. There are $n!$ different ways of arranging n distinct objects into a sequence, the permutations of those objects. Hence, there are, e.g., 24 possibilities to arrange 4 objects into a sequence.

Definition (Binomial Coefficient)**.** Definition (binomial coefficient). The binomial coefficient $\binom{n}{x}$ (read: "n choose x") is defined as:

$$\binom{n}{x} = \frac{n!}{x! \cdot (n-x)!}.$$

Example. $\binom{6}{2} = \frac{6!}{2! \cdot 4!} = \frac{6 \cdot 5 \cdot 4!}{2! \cdot 4!} = 15$.

Definition (Combination)**.** Every group of x elements chosen from a set of n elements in which the ordering of the chosen elements is unimportant (i.e., groups and are equivalent) is called a combination of the x-th order of n elements. The number of combinations, $C(n,x)$, without repetition is

$$C(n,x) = \binom{n}{x}.$$

Definition (Variation)**.** Every group of x elements chosen from a set of n elements in which the ordering of the chosen elements is important (i.e., groups and are not equivalent) is called a variation of the x-th order of n elements. The number of variations, $V(n,x)$, without repetition is

$$V(n,x) = \binom{n}{x} \cdot x!.$$

Example. There are $C(6,2) = \binom{6}{2} = 15$ combinations for choosing 2 elements from a set of 6 elements and $C(6,2) = \binom{6}{2} \cdot 2! = 30$ variations.

3.2 Exercises

3.2.1 How many possibilities are there to arrange 6 objects into a sequence?

3.2.2 How many possibilities are there to choose 5 elements from 10 elements without order and repetition?

3.2.3 How many possibilities are there to choose 3 elements from 15 elements with order and without repetition?

4 Calculus

4.1 Functions

Definition (Function)**.** A *function* f is a mapping that associates each element of a set A to a single element of a set B: $f : A \to B$. The set A is called *domain* and the set B is called *target space* of f.

Example. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$.

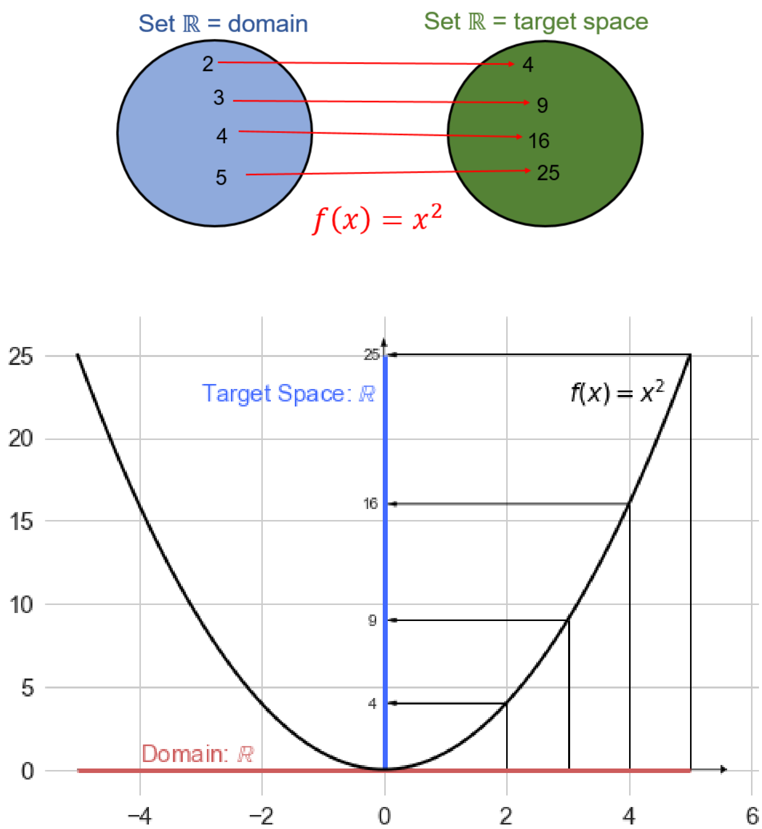


Figure 11: Domain and Target Space

Definition (Image)**.** Given a function, $f : A \to B$. The *image* of a set $E \subseteq A$ under f is the set

$$f(E) = \{f(s) : \ s \in E\}.$$

Definition (Preimage)**.** Given a function, $f : A \to B$. The *Preimage* of a set $E \subseteq B$ under f is the set

$$f^{-1}(E) = \{s \in A : \ f(s) \in E\}.$$

Definition (Range)**.** The *range* of a function, $f : A \to B$ is the image of its domain:

$$range(f) = f(A).$$

Example. Let $f(x) = x^2$. The preimage of $[4, 36]$ is: $[-6, -2] \cup [2, 6]$.

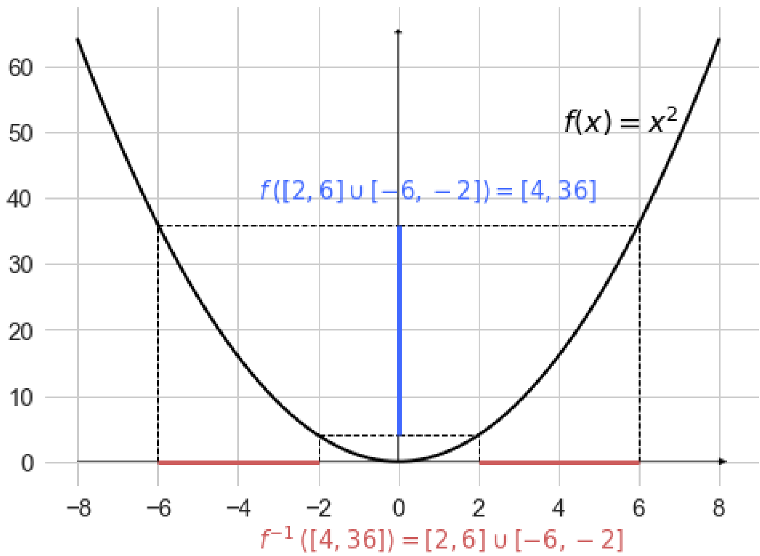


Figure 12: Image and Preimage

Functions of one variable are functions whose domain is one-dimensional. Functions of several variables have a multidimensional domain.

Example. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is a function of one variable:

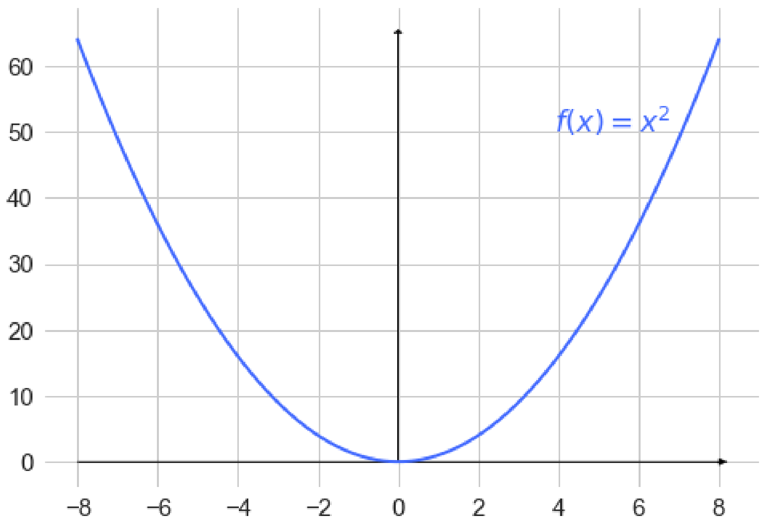


Figure 13: Graph of the function $f(x) = x^2$

Example. The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x^2 + y^2$ is a function of several variables:

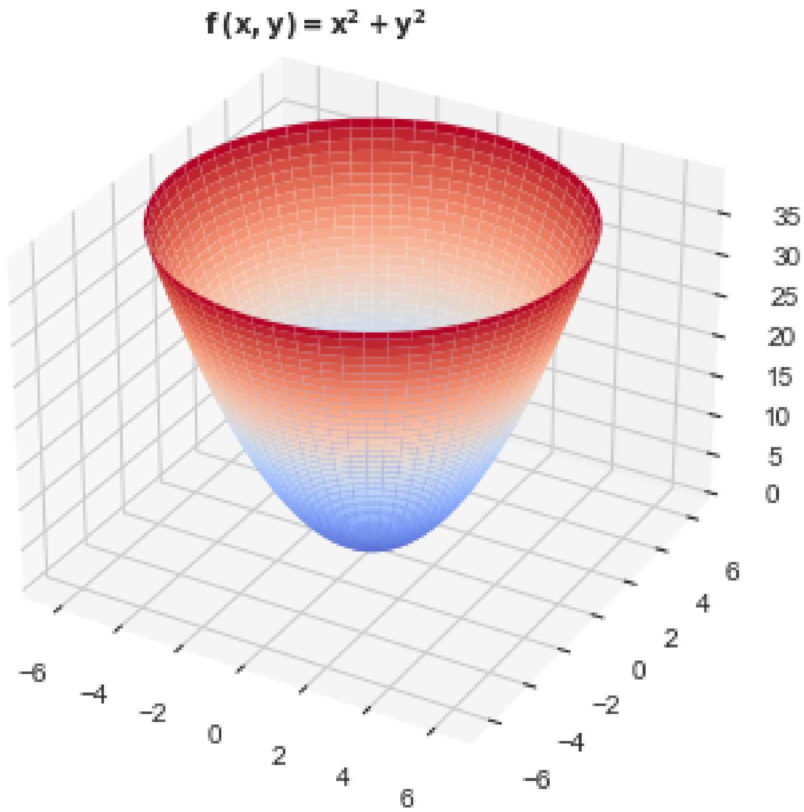


Figure 14: Graph of the function $f(x, y) = x^2 + y^2$

4.2 Derivatives and Integrals

4.2.1 Derivatives

The derivative of a real-valued function with respect to one of its variables measures the sensitivity of the function to the changes in the direction of the variable.

Definition (Derivative)**.** The *derivative* of a real-valued function of one variable is:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Example. Let $f(x) = x^2$, then $f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} 2x + h = 2x$.

Graphically, a derivative represents the slope of the tangent line on the function at the point:

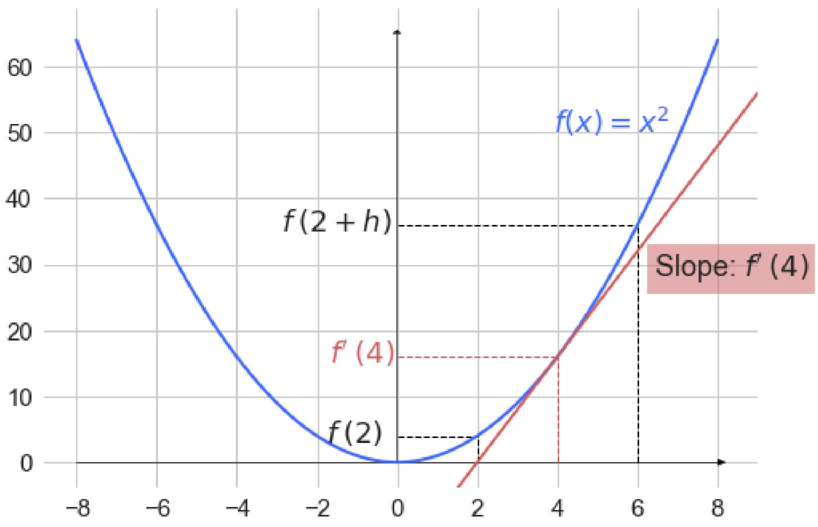


Figure 15: A derivative of the function $f(x) = x^2$

Computation Rules. One of the most prominent computation rules for derivatives is:

$$f(x) = x^b \text{ implies } f'(x) = b \cdot x^{b-1},$$

where b is a real number.

Further computation rules are

(i) Chain Rule. $(f(g))' = f'(g) \cdot g'$

(ii) Product Rule. $(f \cdot g)' = f' \cdot g + f \cdot g'$

(iii) Quotient Rule. $(f/g)' = (f' \cdot g - f \cdot g')/g^2$

Example. Let $f(x) = x^2$ and $g(x) = 3 + 4x$. Find the derivatives of the functions $f(g)$, $f \cdot g$, and f/g:

– $(f(g))' = f'(g) \cdot g' = 2(3 + 4x) \cdot 4 = 24 + 32x$

– $(f \cdot g)' = f' \cdot g + f \cdot g' = 2x \cdot (3 + 4x) + x^2 \cdot 4 = 12x^2 + 6x$

– $(f/g)' = (f' \cdot g - f \cdot g')/g^2 = \frac{2x \cdot (3+4x) - x^2 \cdot 4}{(3+4x)^2} = \frac{4x^2+6x}{(3+4x)^2}$

Definition (Partial Derivative)**.** A *partial derivative* refers to the derivative of a function of several variables with respect to one of its variables. The partial derivative of function f with respect to variable x is denoted by f_x or $\frac{\partial f}{\partial x}$.

Example. Let $f(x, y) = x^2 \cdot \sqrt{y}$. Function f has two partial derivatives: $f_x = 2x \cdot \sqrt{y}$ and $f_y = x^2 \cdot 1/2 \cdot y^{-1/2} = \frac{x^2}{2\sqrt{y}}$.

4.2.2 Integrals

The integral $\int_a^b f(x)\,\mathrm{d}x$ of a function $f(x)$ for a given interval $[a, b]$ represents the area under the curve of f:

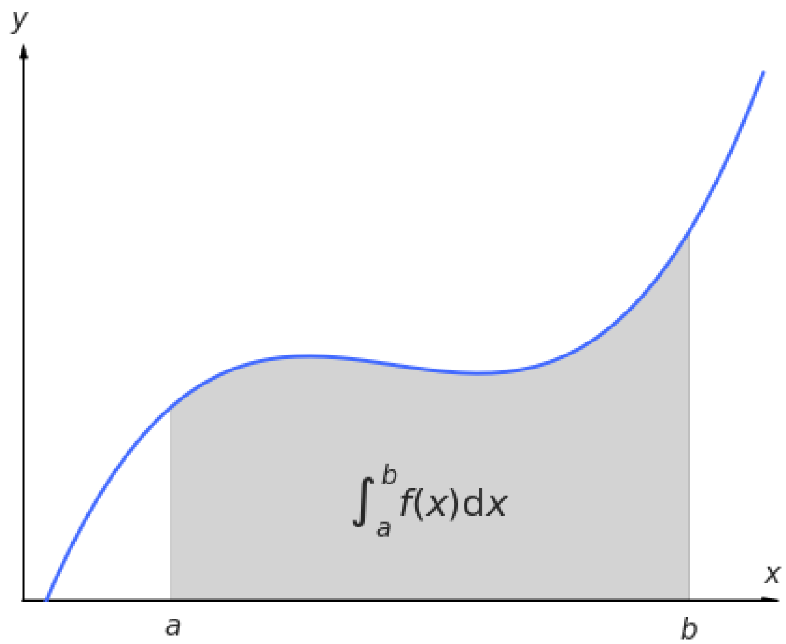


Figure 16: Integral

We calculate the integral of a function f by determining the antiderivative F of f:

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a).$$

4.3 Exercises

4.4.1 Find the derivatives of the following functions:

a) $f(x) = 2x^3 - 6x^2$
b) $f(x) = \sqrt[6]{x^2}$
c) $f(x) = -(x^3 - 3)^3$
d) $f(x) = \frac{e^x}{x^2}$
e) $f(x) = \frac{4}{\sqrt{x}} \cdot 3x^2$
f) $f(x) = (2x^2 - 5x + 6)/(-x + 2)$
g) $f(x) = a^x$

4.4.2 Find the partial derivatives of the following functions:

a) $f(x, y) = x^3 - 3x^2y + xy^2 + 6y^3$
b) $f(x, y) = \frac{x}{y} - \frac{y}{x}$
c) $f(x, y) = e^{\frac{x}{y}}$
d) $f(x, y) = \ln(x^2 + y^2)$

4.4.3 Find the preimage of the set $E = \{1, 3\}$ under the function $f(x) = \sqrt{x}$ and the preimage of the set $F = [2, 4]$ under the function $g(x) = 4x^2$.

5 Vectors and Matrices

5.1 Vectors

In general, a *vector space* (also called a *linear space*) is a set of objects equipped with a structure so that we may add the objects together and multiply ("scale") them by scalars (which are often real numbers).

We restrict our attention to Euclidean Vector Spaces:

Definition (Euclidean Vector Space)**.** A *Euclidean vector space* is a quadruple $(\mathbb{R}^n, +, \cdot, \bullet)$ that consists of:

(i) A n-ary Cartesian product of the real numbers:

$$\mathbb{R}^n = \mathbb{R} \times \ldots \times \mathbb{R} = \{(x_1, \ldots, x_n) : x_1, \ldots, x_n \in \mathbb{R}\}.$$

(ii) An operation + (called *addition*) defined by

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n).$$

(iii) An operation $\cdot$ (called *scalar multiplication*) defined by

$$a \cdot (x_1, \ldots, x_n) = (a \cdot x_1, \ldots, a \cdot x_n),$$

where $a \in \mathbb{R}$.

(iv) An operation $\bullet$ (called *scalar product*) defined by

$$(x_1, \ldots, x_n) \bullet (y_1, \ldots, y_n) = x_1 \cdot y_1 + \ldots + x_n \cdot y_n = \sum_{i=1}^{n} x_i \cdot y_i.$$

Definition (Vector)**.** Elements of a Euclidean vector space $(\mathbb{R}^n, +, \cdot, \bullet)$ are called *vectors*.

Remark. Technically, elements of $\mathbb{R}^n$ are called *n-tuples* or *points*. Vectors can be added, multiplied, and multiplied by numbers, points cannot. For the sake of simplicity, we will not distinguish between vectors and tuples.

A vector is an ordered list of numbers, represented in row or column form.

– row vector ($1 \times n$ vector): $(x_1, \ldots, x_n)$

– column vector ($n \times 1$ vector): $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

Notation. To simplify notation, we denote both $\cdot$ and $\bullet$ by $\cdot$. Similarly, we denote both $\mathbb{R}^n$ and $(\mathbb{R}^n, +, \cdot, \bullet)$ by $\mathbb{R}^n$. A vector is denoted by $\mathbf{x}$ or $\vec{x}$ or x.

Example.

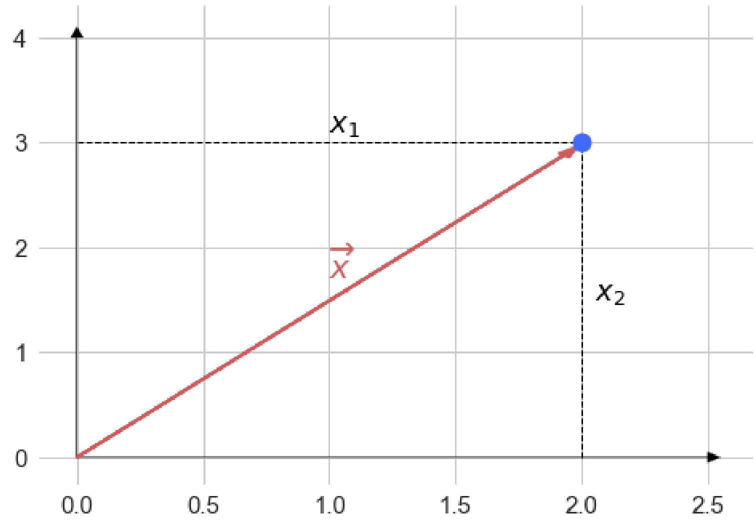


Figure 17: A vector in $\mathbb{R}^2$

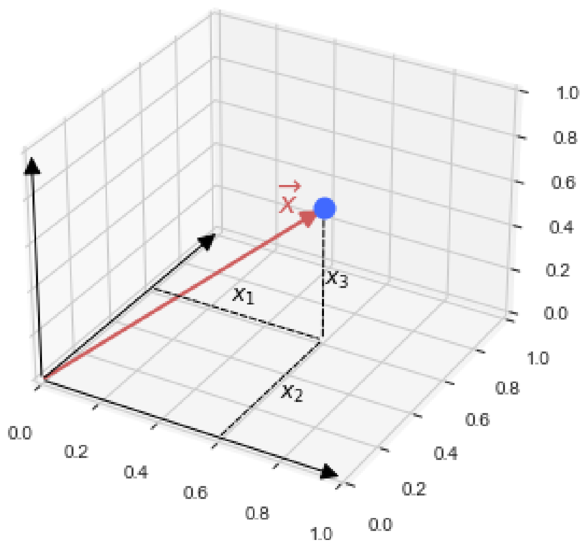


Figure 18: A vector in $\mathbb{R}^3$

Example. Let $\vec{x}_1 = (1,2,3)$ and $\vec{x}_2 = (4,5,6)$, then:

– Vector Addition:

$$\vec{x}_1 + \vec{x}_2 = (1+4, 2+5, 3+6) = (5,7,9)$$

– Vector Multiplication (Scalar Product):

$$\vec{x}_1 \cdot \vec{x}_2 = (1 \cdot 4) + (2 \cdot 5) + (3 \cdot 6) = 32$$

– Scalar Multiplication:

$$4 \cdot \vec{x}_1 = 4 \cdot (1, 2, 3) = (4 \cdot 1, 4 \cdot 2, 4 \cdot 3) = (4, 8, 12)$$

We can use vectors to describe systems of linear equations.

Example. The system:

$$\begin{aligned} 3x_1 + 2x_2 &= 1 \\ 2x_1 - 2x_2 &= -2 \end{aligned}$$

can be written as:

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} \cdot x_1 + \begin{pmatrix} 2 \\ -2 \end{pmatrix} \cdot x_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Definition (Linear Dependence)**.** A sequence of vectors $\vec{x}_1, \ldots, \vec{x}_k$ is called *linearly dependent* if there exists real numbers $a_1, \ldots, a_k$, not all zero, such that:

$$a_1 \cdot \vec{x}_1 + \ldots + a_k \cdot \vec{x}_k = \vec{0},$$

where $\vec{0} = (0, \ldots, 0)$.

If the vectors are linearly independent, then there exists at least one $a_i \neq 0$. Suppose $a_1 \neq 0$, then we may write $\vec{x}_1$ as a linear combination of the other vectors:

$$\vec{x}_1 = \frac{-a_2}{a_1} \vec{x}_2 + \ldots + \frac{-a_k}{a_1} \cdot \vec{x}_k.$$

Remark. If a sequence of vectors is not linearly dependent, then it is called *linearly independent.*

Example. Consider the vectors: $\vec{x}_1 = (1, 1), \vec{x}_2 = (-3, 2)$, and $\vec{x}_3 = (2, 4)$.

First, check if $\vec{x}_1$ and $\vec{x}_2$ are linearly dependent: are there numbers a_1 and a_2, not both zero, such that $a_1\vec{x}_1 + a_2\vec{x}_2 = \vec{0}$? Notice that this is a system of linear equations:

$$a_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Solving this system w.r.t. (a_1, a_2) yields the unique solution $(a_1^*, a_2^*) = (0, 0) = \vec{0}$. Hence, the vectors $\vec{x}_1$ and $\vec{x}_2$ are linearly independent.

Now, check if all 3 vectors are linearly independent. This translates into the following system of linear equations:

$$a_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} -3 \\ 2 \end{pmatrix} + a_3 \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Solving this system w.r.t. (a_1, a_2) yields the set of solutions $\{(a_1^*, a_2^*) = (\frac{-3a_3}{2}, \frac{a_3}{2}) : a_3 \in \mathbb{R}\}$, which contains non-zero solutions. For instance, for $a_3 = 1$, we have $(a_1^*, a_2^*) = (\frac{-3}{2}, \frac{1}{2})$. Hence, the 3 vectors are linearly dependent.

5.2 Matrices

5.2.1 Definition

Definition (Matrix)**.** A *matrix* is a rectangular array or table of items, arranged in rows and columns.

Remark. Formally, a real-valued $m \times n$ *matrix* $A = (a_{ij}) \in \mathbb{R}^{m \times n}$ is a function that assigns a number $a_{ij} \in \mathbb{R}$ to each index pair $(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$:

$$A : \{1, \ldots, m\} \times \{1, \ldots, n\} \to \mathbb{R}.$$

$$\begin{array}{cc} & \xrightarrow{\text{n columns}} \\ \text{m rows} \downarrow & \begin{pmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{pmatrix} \end{array}$$

Figure 19: $m \times n$ matrix

Example.

$$A = \begin{pmatrix} 2 & 4 \\ 5 & 7 \end{pmatrix}$$

5.2.2 Matrix Operations

The following are the basic matrix operations:

(i) **Addition.** Let $A, B \in \mathbb{R}^{m \times n}$ be two matrices of the same size, then:

$$A + B = \begin{pmatrix} a_{11} + b_{11} & \ldots & a_{1n} + b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & \ldots & a_{mn} + b_{mn} \end{pmatrix}$$

Example.

$$\begin{pmatrix} 2 & 4 \\ 5 & 7 \end{pmatrix} + \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 8 \\ 7 & 10 \end{pmatrix}$$

(ii) **Scalar Multiplication.** Let $A \in \mathbb{R}^{m\times n}$, then for $c \in \mathbb{R}$:

$$c \cdot A = \begin{pmatrix} c \cdot a_{11} & \dots & c \cdot a_{1n} \\ \vdots & \ddots & \vdots \\ c \cdot a_{m1} & \dots & c \cdot a_{mn} \end{pmatrix}$$

Example.

$$3 \cdot \begin{pmatrix} 2 & 4 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 6 & 12 \\ 15 & 21 \end{pmatrix}$$

(iii) **Matrix Multiplication.** We can multiply two matrices A and B if the number of columns of the left matrix equals the numbers of rows of the right matrix. Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times l}$, then we multiply each row of A by the columns of B:

$$A \cdot B = \begin{pmatrix} a_{11} \cdot b_{11} + \ldots + a_{1n} \cdot b_{n1} & \dots & a_{11} \cdot b_{1l} + \ldots + a_{1n} \cdot b_{nl} \\ \vdots & \ddots & \vdots \\ a_{m1} \cdot b_{11} + \ldots + a_{mn} \cdot b_{n1} & \dots & a_{m1} \cdot b_{1l} + \ldots + a_{mn} \cdot b_{nl} \end{pmatrix}$$

Example.

$$\begin{pmatrix} 3 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 4 & 0 \end{pmatrix}$$

Step 1:

$$\begin{pmatrix} \color{red}{3} & \color{red}{2} & \color{red}{1} \\ 1 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} \color{blue}{1} & 2 \\ \color{blue}{0} & 1 \\ \color{blue}{4} & 0 \end{pmatrix} = \begin{pmatrix} \color{red}{3} \cdot \color{blue}{1} + \color{red}{2} \cdot \color{blue}{0} + \color{red}{1} \cdot \color{blue}{4} & * \\ * & * \end{pmatrix} = \begin{pmatrix} 7 & * \\ * & * \end{pmatrix}$$

Step 2:

$$\begin{pmatrix} \color{red}{3} & \color{red}{2} & \color{red}{1} \\ 1 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & \color{blue}{2} \\ 0 & \color{blue}{1} \\ 4 & \color{blue}{0} \end{pmatrix} = \begin{pmatrix} 7 & \color{red}{3} \cdot \color{blue}{2} + \color{red}{2} \cdot \color{blue}{1} + \color{red}{1} \cdot \color{blue}{0} \\ * & * \end{pmatrix} = \begin{pmatrix} 7 & 8 \\ * & * \end{pmatrix}$$

Step 3:

$$\begin{pmatrix} 3 & 2 & 1 \\ \color{red}{1} & \color{red}{0} & \color{red}{2} \end{pmatrix} \cdot \begin{pmatrix} \color{blue}{1} & 2 \\ \color{blue}{0} & 1 \\ \color{blue}{4} & 0 \end{pmatrix} = \begin{pmatrix} 7 & 8 \\ \color{red}{1} \cdot \color{blue}{1} + \color{red}{0} \cdot \color{blue}{0} + \color{red}{2} \cdot \color{blue}{4} & * \end{pmatrix} = \begin{pmatrix} 7 & 8 \\ 9 & * \end{pmatrix}$$

Step 4:

$$\begin{pmatrix} 3 & 2 & 1 \\ \color{red}{1} & \color{red}{0} & \color{red}{2} \end{pmatrix} \cdot \begin{pmatrix} 1 & \color{blue}{2} \\ 0 & \color{blue}{1} \\ 4 & \color{blue}{0} \end{pmatrix} = \begin{pmatrix} 7 & 8 \\ 9 & \color{red}{1} \cdot \color{blue}{2} + \color{red}{0} \cdot \color{blue}{1} + \color{red}{2} \cdot \color{blue}{0} \end{pmatrix} = \begin{pmatrix} 7 & 8 \\ 9 & 2 \end{pmatrix}$$

(iv) **Transpose.** The transpose of a matrix $A = (a_{ij}) \in \mathbb{R}^{m\times n}$ is a matrix $A^T = (a_{ji}) \in \mathbb{R}^{n\times m}$.
Example.

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix} \Longrightarrow A^T = \begin{pmatrix} 3 & 1 \\ 2 & 0 \\ 1 & 2 \end{pmatrix}$$

(v) **Inverse.** We can only determine the inverse A^{-1} of a squared matrix $A \in \mathbb{R}^{n\times n}$. Furthermore, the matrix needs to be invertible.

$$A \cdot A^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} = I.$$

The matrix I is called *identity matrix*.
Example.

$$\text{The inverse of } = \begin{pmatrix} 2 & 1 \\ 6 & 4 \end{pmatrix} \text{ is } A^{-1} = \begin{pmatrix} 2 & -0,5 \\ -3 & 1 \end{pmatrix},$$

because:

$$A \cdot A^{-1} = \begin{pmatrix} 2 & 1 \\ 6 & 4 \end{pmatrix} \cdot \begin{pmatrix} 2 & -0,5 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 4-3 & -1+1 \\ 12-12 & -3+4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We can use matrices to describe systems of linear equations.

Example. The system:

$$\begin{aligned} 3x_1 + 2x_2 &= 1 \\ 2x_1 - 2x_2 &= -2 \end{aligned}$$

can be written as:

$$\begin{pmatrix} 3 & 2 \\ 1 & -2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

5.3 Exercises

5.3.1 Compute the following:

a) $a + 4 \cdot \begin{pmatrix} 8 \\ 3 \\ 5 \end{pmatrix}$

b) $\begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix} + \begin{pmatrix} 7 \\ 10 \\ 3 \end{pmatrix}$

c) $\begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 7 \\ 10 \\ 3 \end{pmatrix}$

5.3.2 Write down the following system of linear equations by using vectors:

$$\begin{aligned} 20x_1 - 20x_2 + 15x_3 &= 1 \\ 20x_1 - 11x_2 + 18x_3 &= -11 \\ 17x_1 - 8x_2 - 10x_3 &= 0 \end{aligned}$$

5.3.3 Check if the following vectors are linearly independent.

a) $(9, -6)$ and $(-6, 1)$ and $(5, 10)$

b) $(9, 7, 4)$ and $(-3, 5, 0)$

c) $(6, 8)$ and $(-3, 5)$

d) $(2, 1)$ and $(4, 2)$

5.3.4 Compute the following:

a) $7 + 5 \cdot \begin{pmatrix} 2 & 0 & 10 \\ 10 & 1 & 10 \end{pmatrix}$

b) $(3, 8, 3) \cdot \begin{pmatrix} 4 \\ 5 \\ 10 \end{pmatrix}$

c) $(1, 10, 3) \cdot \begin{pmatrix} 4 & 1 \\ 7 & 2 \\ 4 & 1 \end{pmatrix}$

d) $\begin{pmatrix} 6 & 1 & 2 \\ 3 & 2 & 9 \end{pmatrix} \cdot \begin{pmatrix} 8 & 3 \\ 8 & 4 \\ 10 & 6 \end{pmatrix}$

5.3.5 Find the transpose of the matrix $A = \begin{pmatrix} 6 & 1 & 2 \\ 3 & 2 & 9 \end{pmatrix}$.

5.3.6 Write down the following system of linear equations in matrix form:

$$\begin{aligned} 20x_1 - 20x_2 + 15x_3 &= 1 \\ 20x_1 - 11x_2 + 18x_3 &= -11 \\ 17x_1 - 8x_2 - 10x_3 &= 0 \end{aligned}$$

6 Solutions to the Exercises

6.1 Section 2

2.4.1 b) and c)

2.4.2 $\{x \in [0, 10] : f(x) > 2\}$

2.4.3 $\{(x, y) : x \in [1, 3], y \in [2, 4]\} = [1, 3] \times [2, 4]$

2.4.4 { l:l is a Latin letter }

2.4.5 $E^c = (-\infty, 10) \cup (20, \infty)$

2.4.6 $E \cup F = \{Apple, Pie, Peter, Saskia, Francis\}$

2.4.7 $E \cap F = [0, 23]$

2.4.8 $E \setminus F = \{Apple, Monkey\}$ and $F \setminus E = \{0, 10\}$

2.4.9 a), b), and d)

2.4.10 $\mathcal{P}(E) = \{\varnothing, a, b, c, \{a, b\}, \{a, c\}, \{b, c\}, E\}$ and $\mathcal{P}(F) = \{\varnothing, \{2, 3\}, 6, \{2, 3, 6\}, F\}$.

2.4.11 $\mathcal{T}_1 = \big\{\{\text{⚀}, \text{⚁}, \text{⚂}\}, \{\text{⚃}, \text{⚄}, \text{⚅}\}\big\}$, $\mathcal{T}_2 = \big\{\{\text{⚀}, \text{⚁}, \text{⚂}, \text{⚃}\}, \{\text{⚄}, \text{⚅}\}\big\}$, $\mathcal{T}_3 = \big\{\{\text{⚀}\}, \{\text{⚁}, \text{⚂}, \text{⚃}, \text{⚄}, \text{⚅}\}\big\}$

2.4.12 a), b), and c)

2.4.13 2.4.13. 2 students only play basketball, 9 play volleyball and handball, furthermore 1 doesn't play any of the three sports.

6.2 Section 3

3.2.1 720

3.2.2 252

3.2.3 2730

6.3 Section 4

4.4.1 a) $f'(x) = 6x^2 - 12x$

b) $f'(x) = \frac{1}{3\sqrt[6]{x^4}}$

c) $f'(x) = -9x^2(x^3 - 3)^2$

d) $f'(x) = \frac{(x-2)e^x}{x^3}$

e) $f'(x) = 18\sqrt{x}$

f) $f'(x) = \frac{-2x^2+8x-4}{(-x+2)^2}$

g) $f'(x) = a^x \cdot \ln(a)$

4.4.2 a) $f_x(x,y) = 3x^2 - 6xy + y^2$ and $f_y(x,y) = -3x^2 + 2xy + 18y^2$

b) $f_x(x,y) = \frac{1}{y} - \frac{y}{x^2}$ and $f_y(x,y) = \frac{-x}{y^2} - \frac{-1}{x}$

c) $f_x(x,y) = \frac{1}{y}e^{x/y}$ and $f_y(x,y) = \frac{1}{x}e^{x/y}$

d) $f_x(x,y) = \frac{2x}{x^2+y^2}$ and $f_y(x,y) = \frac{2y}{x^2+y^2}$

4.4.3 $f^{-1}(\{1,3\}) = \{1,9\}$ and $g^{-1}([2,4]) = [\sqrt{\frac{1}{2}}, 1]$

6.4 Section 5

5.3.1 a) $\begin{pmatrix} 32+a \\ 12+a \\ 20+a \end{pmatrix}$

b) $\begin{pmatrix} 9 \\ 10 \\ 9 \end{pmatrix}$

c) 32

5.3.2 $\begin{pmatrix} 20 \\ 20 \\ 17 \end{pmatrix} x_1 + \begin{pmatrix} -20 \\ -11 \\ -8 \end{pmatrix} x_2 + \begin{pmatrix} 15 \\ 18 \\ -10 \end{pmatrix} x_3 = \begin{pmatrix} 1 \\ -11 \\ 0 \end{pmatrix}$

5.3.3 a) Linearly dependent

b) Linearly independent

c) Linearly independent

d) Linearly dependent

5.3.4 a) $\begin{pmatrix} 17 & 7 & 57 \\ 57 & 12 & 57 \end{pmatrix}$

b) 82

c) $(86, 24)$

d) $\begin{pmatrix} 76 & 34 \\ 130 & 71 \end{pmatrix}$

5.3.5 $A^T = \begin{pmatrix} 6 & 3 \\ 1 & 2 \\ 2 & 9 \end{pmatrix}$

5.3.6 $\begin{pmatrix} 20 & -20 & 15 \\ 20 & -11 & 18 \\ 17 & -8 & -10 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -11 \\ 0 \end{pmatrix}$

A Appendix

For interested students only.

A.1 Measures (Section 2)

Definition (Measure)**.** Given a measurable space $(S, \mathcal{A})$. A *measure* is a function $P : \mathcal{A} \to \mathbb{R} \cup \infty$ that satisfies

(i) Non-Negativity. For all $E \in \mathcal{A}$, $P(E) \geq 0$.

(ii) Null Empty Set. $P(\varnothing) = 0$.

(iii) Countable Additivity. For all countable collections $\{E_1, E_2, E_3, \dots\}$ of pairwise disjoint sets in $\mathcal{A}$,

$$P(E_1 \cup E_2 \cup E_3 \cup \dots) = P(E_1) + P(E_2) + P(E_3) + \dots .$$

Example. The following are measures:

– Lebesgue measure P_l on the Borel σ-algebra. For all intervals $[a, b] \in$ Borel,

$$P_l([a, b]) = b - a$$

– Counting measure P_c on $(S, \mathcal{A})$. For all sets $E \in \mathcal{A}$,

$$P_c(E) = \begin{cases} \text{number of elements of } E & \text{, if E is finite} \\ \infty & \text{, else} \end{cases}$$

– Dirac measure $P_{d(s)}$ on $(S, \mathcal{A})$. For a given $s \in S$ and all sets $E \in \mathcal{A}$,

$$P_{d(s)}(E) = \begin{cases} 1 & \text{, if } s \in E \\ 0 & \text{, else} \end{cases}$$

Definition (Probability Measure)**.** A measure that satisfies $P(S) = 1$ (Normalization) is called *probability measure.*

A.2 Measurable Function (Section 4)

Definition (Measurable Function)**.** Let $(S, \mathcal{A})$ and $(S', \mathcal{A}')$ be measurable spaces. A function $f : S \to S'$ is called *measurable* if the preimage of every $E' \in \mathcal{A}'$ under f is an element of $\mathcal{A}$, formally:

$$f^{-1}(E') \in \mathcal{A} \text{ for all } E' \in \mathcal{A}'.$$

Example.

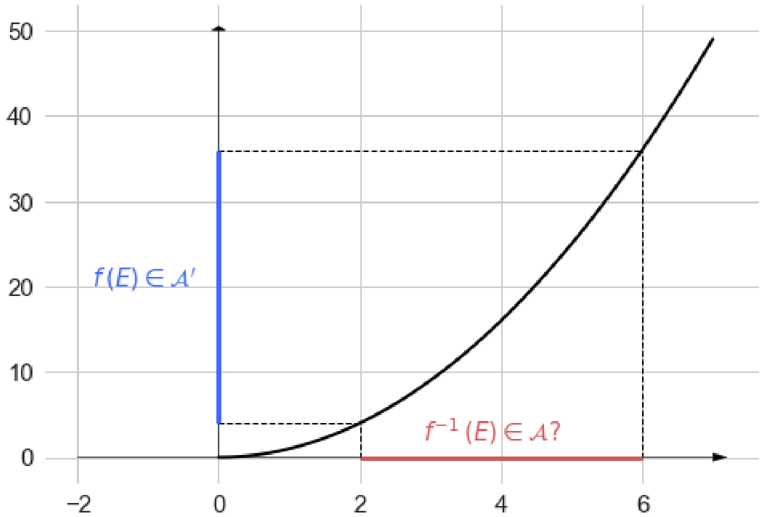


Figure 20: Measurable Function

A.3 Invertible Matrix (Section 5)

Definition (Rank)**.** The *rank* of a matrix A is the dimension of the vector space generated (or spanned) by its columns. This corresponds to the maximal number of linearly independent columns of A.

Remark. A $n \times n$ matrix has *full rank* if $rank(A) = n$.

Definition (Invertible)**.** The following are equivalent:

(i) $n \times n$ matrix A is *invertible.*

(ii) A has full rank.

(iii) The equation $A\vec{x} = \vec{0}$ has only the trivial solution $\vec{x} = \vec{0}$.

(iv) The columns of A are linearly independent.