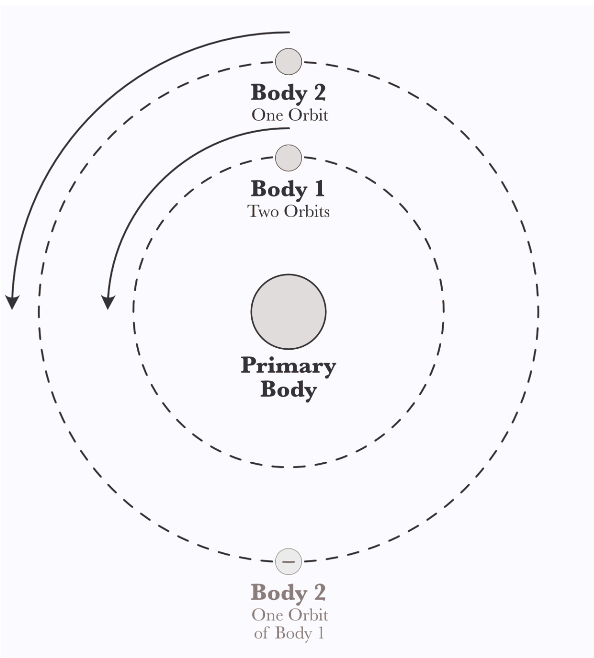
Orbital Resonances 1 — Mean Motion Resonance

Mean Motion Resonance

When two bodies orbit a third body in common, there will be resonances in their orbits. If one body's orbit is an exact integer value different from the other's, the two are said to be in *mean motion resonance*, and the ratio of their orbits is **a : b**, denoted as the longer orbital period followed by the shorter orbital period, separated by a colon (:).



For example, in the figure above, Body 1 orbits twice for every one orbit of Body 2. This is a *first order resonanc*, because the difference between the two values is only 1 unit:

$$2 - 1 = 1$$

Other examples of first-order resonances1 would be 6 : 5, 13 : 12, etc.

Examples of a *second-order resonance* would be those in which the orbital periods were separated by a difference of 2; e.g. 3 : 1 or 5 : 3. The higher the order of the resonance, the less stable the orbits will be over time.

Mean motion resonances can arise between three or more bodies. Jupiter's moons Io, Europa, and Ganymede experience a 4 : 2 : 1 mean motion resonance between them, such that for every orbit Ganymede makes, Europa completes two, and Io completes four. This also means that Io orbits twice per each orbit of Europa.

However, an interesting phenomenon also occurs, such that Ganymede, Europa, and Io *never line up* on the same side of Jupiter. Any two of them may line up at various points in their orbital dance, but all three never do.

> While N-conjunctions are *technically* possible in N-orbiton systems, they are reliant on very specific conditions of orbital eccentricity and coplanarity; mean motion resonances between more than three objects should be treated for all intents and purposes as *impossible*.

Non-Integer Resonances

When two bodies orbit a central body, but their orbital periods are:

1. not integer multiples of the system's base time unit *T*; and,
2. not directly commensurate2 with one another,
   … a more sophisticated approach is required to calculate periodic alignments of the system.

The *base time unit* (T), may be years, months, days, even hours or minutes.

The interval between such instances of alignment is called the *synodic period*; here, we'll refer to it as a *synodion* Σ (pl: *synodia*), for reasons which will become clear presently. The moment of alignment, when the two orbitons are closest to one another, we call a *synodos* (pl: *synodoi*).

Thus, the synodion represents the time it takes for the two orbitons to return to the same relative configuration, such as having their centers fall on a line that includes the center of the central body. While synodia are still predictable, their periodicity is less intuitive that in mean motion resonance systems, requiring more precise calculations.

The best way to grasp these concepts is by working through an example. By appling the rules of synodic periods to a specific pair of orbiting bodies, we can illustrate how to calculate a synodion, and explore the patterns it reveals.

A Worked Example

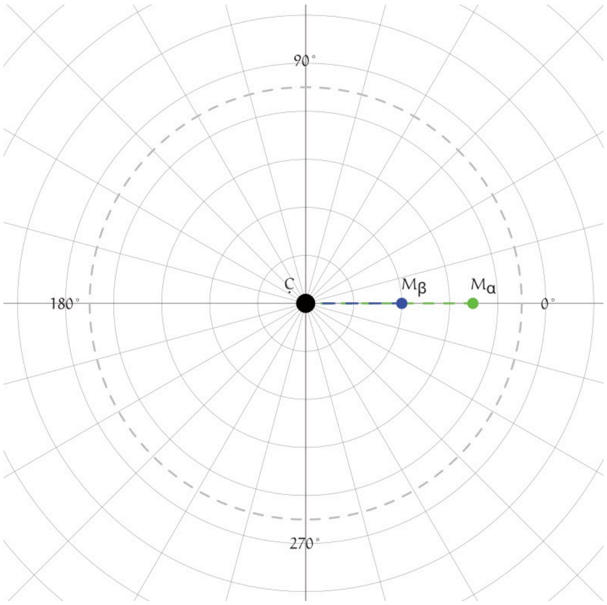
Let's imagine a system of three bodies, the central body (*centron* $\dot{C}$) and two less massive bodies orbiting it (the *orbitons* M_α and M_β). M_α has the longer orbital period (P_α; the time it takes M_α to orbit the centron) and M_β has the shorter orbital period (P_β; the time it takes M_β to orbit the centron).

> While the masses of the bodies have no bearing on the following equations, they *do* matter from a nomenclature perspective. The more massive body is always the *primary*, the second-most massive body is the *secondary*; the third most is the *tertiary*, etc. Their *radii* are not a factor in determining their hierarchy.
> For instance, Mercury is more massive than Ganymede, but Ganymede has a larger radius.

> If the two of them were to form a two-body system, *Ganymede would orbit Mercury*, even though it is larger in radius and volume than Mercury.

For this example, we will define:

$$P_\alpha = 24.36 \text{ days}$$
$$P_\beta = 11.72 \text{ days}$$



The above figure illustrates the starting configuration of our system (T_0). The starting angle at which all three bodies are lined up we call the *base angle*, denoted by B^θ.

We can already make some determinations about our system:

$$P_\delta = P_\alpha - P_\beta = 24.36 - 11.72 = 12.64 \text{ days}$$
$$P_Q = \frac{P_\beta}{P_\alpha} = \frac{11.72}{24.36} = 0.481 \text{ days}$$

The ratio of their orbits (P_Q) is not an integer, so these orbitons are not in a mean motion resonance; for every orbit of M_β, M_α only completes 0.481 of its orbit, because M_α takes 12.64 days longer to orbit the centron than does M_β.

Since a full orbit for *either* orbiton is $360°$, we can figure out how many degrees of their orbit each orbiton completes within the system base time unit, which in this case would be 1 day.

$$\theta_\alpha = \frac{360°}{24.36} = 14.778° \qquad \text{The } \textit{minor unit angle}$$
$$\theta_\beta = \frac{360°}{11.72} = 30.717° \qquad \text{The } \textit{major unit angle}$$

This tells us how many degrees M_α "lags behind" M_β per day:

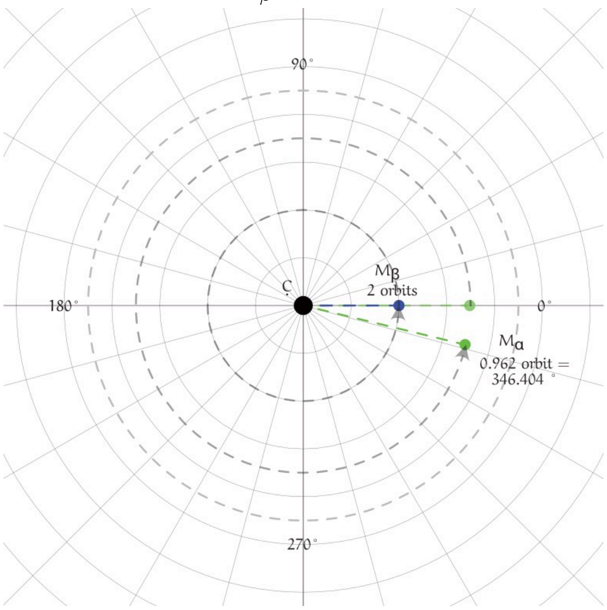
$$\theta_\Delta = \theta_\beta - \theta_\alpha = 30.717° - 14.778° = 15.939° \qquad \text{The } \textit{orbiton difference angle}$$

So, every day M_β moves almost $16°$ farther in its orbit than M_α does in its orbit.

We can also calculate ho many degrees of its orbit M_α completes (subtends) per each full orbit of M_β:

$$\alpha^\circ = P_Q \times 360 = 0.481 \times 360 = 173.202^\circ$$

So, after *two* of M_β's orbits:



$$2a^\circ = (2)(173.202) = 346.404^\circ$$

… M_α still hasn't completed one of its orbits, it has only completed

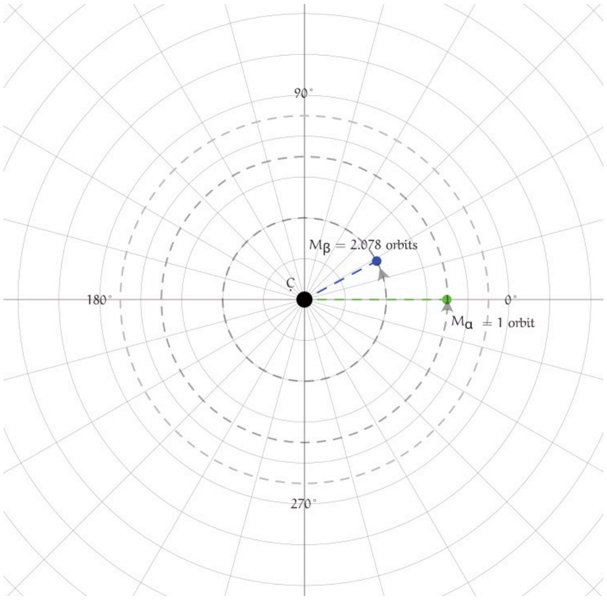
$$\frac{346.404}{360} = 0.962$$

orbits of the centron.

Conversely, by the time M_α completes one full orbit of the centron, M_β will have completed

$$P_R = \frac{24.36}{11.72} = 2.078$$

orbits of the centron.



It is a curious feature of circular motion that if you get far enough ahead of the other body, you eventually come up behind it. Think of a 5000-meter Olympic footrace. All of the competitors start out on the same line and travel in the same direction. Let's say there are two runners, Number 11 and Number 24.

Number 11 quickly pulls out ahead of Number 24 and for a time it is quite obvious that she is "way out ahead". But if she maintains her higher running speed, eventually she will be *so far ahead* of Number 24 that *she will appear to be running behind them*. Anyone looking in on the race at just that moment could be forgiven for thinking that Number 11 is losing, perhaps badly.

However, eventually Number 11 will catch up to Number 24 and pass them again; but for a moment they will be side-by-side, just as they were at the start of the race *but not back on the starting line*; their meeting point will be some angular distance around the track from where they began. Where and when this moment occurs is entirely a matter of how fast each runner is running.

Returning to our orbitons, it becomes obvious that at some point while they are orbiting, they will "meet up" at their closest approach to one another, just as our runners did in the Olympic race — they will have achieved synodos. The first synodos to occur after T_0 we label T_1; the second we label T_2, and so on.

Here's the really cool thing: we can calculate exactly *where* and *when* T_1, T_2, etc. take place!

The When

The synodic period (mentioned above) is the period of time that transpires between any two synodoi:

$$\begin{aligned}\Sigma &= \frac{P_\alpha \times P_\beta}{|P_\alpha - P_\beta|} \\ &= \frac{24.36 \times 11.72}{|24.26 - 11.72|} \\ &= \frac{285.499}{12.64} \\ \Sigma &= 22.587 \text{ days} \checkmark\end{aligned}$$

So, each synodos occurs just over $22\frac{1}{2}$ days after the preceding one.

> Note that if we divide 360° by the difference angle θ_Δ
>
> $$\Sigma = \frac{360}{\theta_\Delta} = \frac{360}{15.939} = 22.587 \text{ days}$$

$\ldots wealsogetthesynodicperiod. Noticethatthesynodion,$ \$22.587\$ $daysismorethantwoorbitsof$ \$ M_β \$ and

\begin{align}
F\alpha &= \dfrac{P\alpha}{\Sigma} = \dfrac{24.36}{22.587} = 0.9272; \text{orbit}\
F\beta &= \dfrac{P\beta}{\Sigma} = \dfrac{11.72}{22.587} = 1.9272; \text{orbits}
\end{align}

$Doyouseeit? The decimal fraction of both of these quotients is the same$: \$0.9272\$. $Wedesignatethisasthe$

\Theta = F_\Sigma \times 360^\circ = (0.9272)(333.7975^\circ) \qquad \text{The \emph{synodial angle}}

$Subtractingthesynodialanglefrom$ \$ 360° \$:

\Delta_\Theta = 360^\circ - \Theta = 360^\circ - 333.7975^\circ = 26.203^\circ \qquad \text{The \emph{synodial lag angle}}

$\ldots wenowknowthateachsuccessivesynodosoccurs$ \$22.587\$ $daysafter_when_and$ \$ 26.203° \$"short"$ofwher$

\widehat{\Theta} = 360^\circ - MOD((S \times \Delta_\Theta), 360)

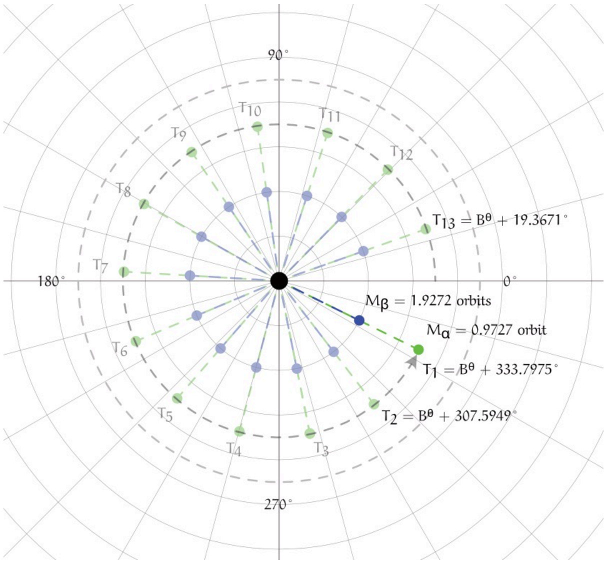
$whereSisthenumberofthesynodosafter$ \$ T_0 \$; $e.g. forsynodos$ \$ $T_6, S = 6$ \$. $Thisleadsusnaturallytotheq$

\widehat{\Theta} = 0^\circ

$\ldots evertrueafter$ \$ T_0 \$? $Theansweris$"yes"$andwecancalculatethat, too.$

Orbital Resonances 2 — The Synodion

The Synodial Epoch



The *synodial epoch* (Y) is the time interval after which a synodion occurs at the base angle B^θ agin, completing a full cycle.

The *epochal aggregate* (Y_0) is the total number of synodia that transpire within a synodial epoch.

Whereas we used the synodic formula:

$$\Sigma = \frac{P_\alpha \times P_\beta}{|P_\alpha - P_\beta|}$$

to calculate the synodic period, we need to use a different method to calculate the Epochal Aggregate; we employ both the *Least Common Multiple* (LCM) and the *Greatest Common Divisor* (GCD):

$$LCM(a, b) = \frac{a \times b}{GCD(a, b)}$$

> ❗ Important ❗
>
> 1. Most modern calculators, spreadsheets, and programming languages have built-in functions for calculating both LCM and GCD, but be aware that *both of these can only be correctly calculated for integer inputs*. If something like LCM(1.5, 3.3) returns a value rather than an error, it will likely be the LCM of 1 and 3, not of 1.5 and 3.3.
> 2. The GCD *can* be calculated by hand using the *Euclidean Algorithm* (detailed elsewhere), calculators, spreadsheets, or programmed scripts are much faster.

We need to calculate the LCM of $P_\alpha = 24.36$ and $P_\beta = 11.72$, but neither of these are integers, so we *normalize* them by multiplying by the factor of ten which will render the more precise of the two into an integer value. In this case, both have two decimal places of precision, so multiplying by $10^2 = 100$ will be sufficient:

$$P'_\alpha = 24.36 \times 100 = 2436 \quad \text{and} \quad P'_\beta = 11.72 \times 100 = 1172$$

Now we can compute the LCM by:

$$\begin{aligned} LCM(a,b) &= \frac{a \times b}{GCD(a,b)} \\ &= \frac{2436 \times 1172}{GCD(2436, 1172)} \\ &= \frac{2854992}{4} \\ &= 713748 \checkmark \end{aligned}$$

Now, *since we normalized our inputs* by scaling up by a factor of $10^2 = 100$, we need to scale this result *down* by the same factor:

$$Y_0 = \frac{713748}{100} = 7137.48 \text{ days} \checkmark$$

The *epochal interval* (Ψ) is calculated by dividing the Epochal Aggregate (Y_0) by the synodic period (Σ):

$$\begin{aligned} \Psi &= \frac{Y_0}{\Sigma} \\ &= \frac{7137.48}{22.587} \\ &= 316 \text{ synodia} \end{aligned}$$

This reveals that T_{316} occurs in the same configuration at the base angle B^θ. We can double-check by supplying $S = 316$ to our synodian instance angle ($\widehat{\Theta}$) equation:

$$\begin{aligned} \widehat{\Theta} &= 360^\circ - MOD((S \times \Lambda_\Theta), 360^\circ) \\ &= 360^\circ - MOD((316 \times 26.203^\circ), 360^\circ) \\ &= 360^\circ - 360^\circ \\ &= 0 \checkmark \end{aligned}$$

To convert the epochal aggregate $\Psi = 7137.48$ days into something more useful, we can divide by the number of days in a sidereal year (≈ 365.2564):

$$Y_0^y = \frac{7137.48}{365.2564} = 19.54 \text{ years} \checkmark$$

… which is about

$$19^y\ 197^d\ 11^h\ 16^m\ 20.724^s$$

give-or-take a millisecond.

Diving Deeper

Above, we learned about the synodic period equation:

$$\Sigma = \frac{P_\alpha \times P_\beta}{|P_\alpha - P_\beta|}$$

Here is a comprehensive listing of the related equations:

$$\text{Given: } P_\alpha > P_\beta$$

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$$\frac{1}{\Sigma} = \left|\frac{1}{P_\alpha} - \frac{1}{P_\beta}\right| \qquad \frac{1}{P_\alpha} = \frac{1}{P_\beta} - \frac{1}{\Sigma} \qquad \frac{1}{P_\beta} = \frac{1}{P_\alpha} + \frac{1}{\Sigma}$$

$$Q = \frac{P_\beta}{P_\alpha} \qquad P_\alpha = P_\beta \times R = \frac{P_\beta}{Q} = \left(\frac{\Sigma}{Q} - \Sigma\right) = \Sigma(R-1)$$

$$R = \frac{P_\alpha}{P_\beta} \qquad P_\beta = P_\alpha \times Q = \frac{P_\alpha}{R} = \left(\Sigma - \frac{\Sigma}{R}\right) = \Sigma(1-Q)$$

Where:

- Σ = synodion
- P_α = period of the outer body, M_α (longer)
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- Q = quotient of the outer body period to the inner body period
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> *Note that the second row of equations perform the same calculations as the first row by using the reciprocals of the orbital periods.*

Quarter Synodial Epoch

A quarter synodial epoch ($Y_{/4}$) is an important quantity, as well; at these intervals, synodoi occur precisely on a cardinal angle ($90^\circ, 180^\circ, 270^\circ$) in succession. We call these the *cardinal synodoi*. They occur, of course every one fourth of the synodion:

$$Y_{/4} = \frac{Y_0}{4}$$

In our worked example above, the quarter synodial epoch is:

$$Y_{/4} = \frac{Y_0}{4} = \frac{7137.48}{4} = 1784.37 \text{ days}$$

$$Y^y_{/4} = \frac{Y_{/4}}{365.2564} = \frac{1784.37}{365.2564} \approx 4.89 \text{ years}$$

Quarter Epochal Interval

Similarly, a *quarter epochal interval* ($\Psi_{/4}$) is the number of *common synodoi* that occur between cardinal synodoi:

$$\Psi_{/4} = \frac{\Psi_0}{4} = \frac{316}{4} = 79 \text{ synodoi}$$

The Epochal Complements

There are two of these, the *epochal major complement* (Y_α) and the *epochal minor complement* (Y_β), which are the number of complete orbits of orbiton M_α and M_β, respectively, per each synodial epoch:

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In our worked example above:

$$Y_\alpha = \frac{Y_0}{P_\alpha} = \frac{7137.48}{24.36} = 293 \text{ orbits} \checkmark$$

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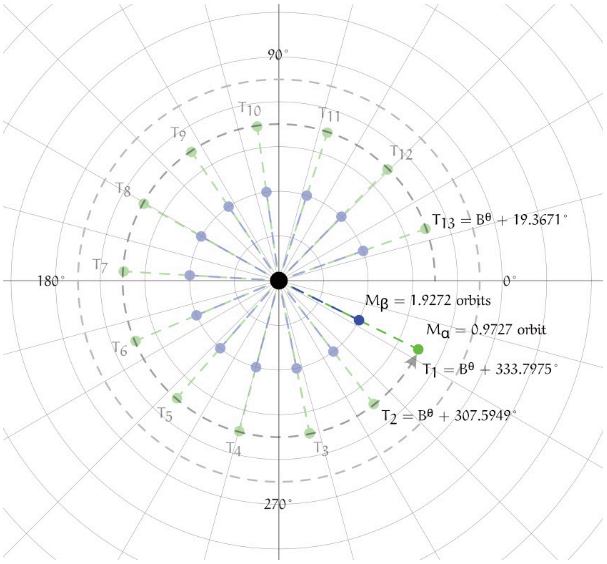
Orbital Resonances 3 — The Synodial Epoch

The Synodial Epoch



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