[Maybe teach earlier?]

Measurement operators
In the standard computational basis

$$M_0 = |0\rangle\langle 0|$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$M_1 = |1\rangle\langle 1|$$

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

they collapse a QCircuit

These are not unitary, and we use them differently than gates:

$$p(m) = \langle \psi | M_m^* M_m | \psi \rangle$$

Say $\psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ $M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

outcome

$$p(|0\rangle) = (\alpha^* \ \beta^*) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$= (\alpha^* \ 0) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha^* \alpha = |\alpha|^2$$

Similarly $p(1 \text{ outcome}) = |\beta|^2$

Measurement ops in a given basis are complete

$$\sum_m M_m^* M = I$$

Check: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = I$

The state after measurement M_m on $|\psi\rangle$ is

$$M_m |\psi\rangle, \text{ but}$$

normalized as follows

$$\frac{M_m |\psi\rangle}{\sqrt{\langle\psi| M_m^* M_m |\psi\rangle}}$$

So after M_0 on $|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

we get state
$$\frac{\begin{pmatrix} \alpha \\ 0 \end{pmatrix}}{|\alpha|}$$

$$= \alpha/|\alpha| \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

say $\alpha = re^{i\theta}$

then $|\alpha| = r$ so

$\alpha/|\alpha| = e^{i\theta}$, a global phase

applied to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & can be

ignored

After M_0 on $|\psi\rangle$ we get $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ w/ prob $|\alpha|^2$

Consider

$$M_+ = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$= 1/2 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$M_- = 1/2 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$M_+^* M_+ + M_-^* M_- = I$ as required

$$p(+\text{ outcom}) \text{ on } |\psi\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \end{pmatrix} \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \tfrac{1}{2} \begin{pmatrix} 1 & 1 \end{pmatrix} \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \tfrac{1}{4} \begin{pmatrix} 2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \tfrac{1}{2}$$

leaving system in the state

$$\underbrace{\tfrac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}}_{1/\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} = \underbrace{\tfrac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}}_{1/\sqrt{2}}$$

$$= \tfrac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = |+\rangle \quad \checkmark$$

Distinguishing orthonormal states
(1-qubit antipodal on Bloch Sphere)

Suppose $|\psi\rangle = |0\rangle$ is prepared

then

$$p(0\text{ seen}) = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 1$$

$$p(1\text{ seen}) = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 0$$

This is generally true

There is a measurement that distinguishes $|\psi_1\rangle$ & $|\psi_2\rangle$ if these are orthogonal

Thm [this has come up as a question]
Non orthogonal states cannot reliably be distinguished (by measurement)

Sketch

Alice performs measurement M_j with outcome j (success)

Alice computes $i = f(j)$ to say for certain that we were in state $|\psi_i\rangle$

Elitzur- Vaidman bomb

[better- cmu version next page]

photon ⟶ 

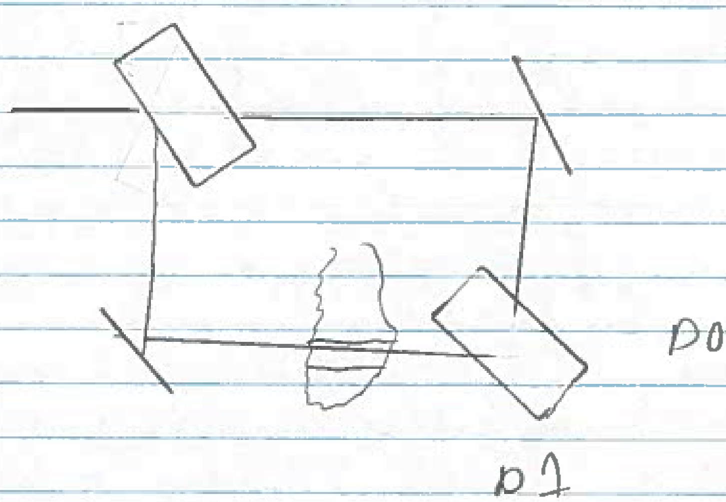
if detects, bomb explodes

<u>but</u> photon may pass thru undisturbed w/ no explosion

Can we detect a bomb that would explode without exploding it

Can we certify the bomb? Not classically

Mach Zender Interferometer

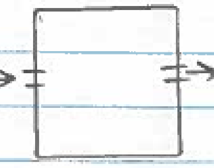


Quantum Game
Level 4

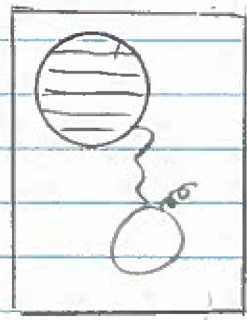
If bomb defective
destr. interference
1.0 prob detect at D0
No way photon reaches D1

If bomb is good
Bomb explodes $p = 1/2$
Photon takes D0 $p = 1/4$
Photon takes D1 $p = 1/4$

Bomb is good but doesn't explode

CMU version You get a box, empty 

OR Bomb!

photon → 

if horizontal $|0\rangle$
photon goes through
else bomb explode if $|1\rangle$.

Box effectively measures in $\{|0\rangle, |1\rangle\}$

Which box do you have? Empty or not
Classically — nothing you can do cleverly
send $|0\rangle$ no info, empty or no explode
$|1\rangle$ bomb will explode if it's there

Idea

Send in $1/\sqrt{2}\,(|0\rangle + |1\rangle) = |+\rangle$

Measure photon in +/− basis

1) Empty? $|+\rangle$ in, $|+\rangle$ out, see $|+\rangle$ 100% prob
2) Bomb

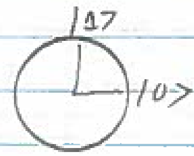
$|+\rangle \longrightarrow$  → 50% $|0\rangle$
→ 50% $|1\rangle$ Boom

if $|0\rangle$ makes it thru, measure
$|+\rangle$ 50%
$|-\rangle$ 50% — bomb must be good

25% good bomb
25% inconclusive
50% explode

Better than classic!

Better version



Start with $|0\rangle$ and rotate slightly

Rotate by θ $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

where $|0\rangle$ goes   where $|1\rangle$ goes

R_ϵ rotate $|0\rangle$ counterclockwise by ϵ

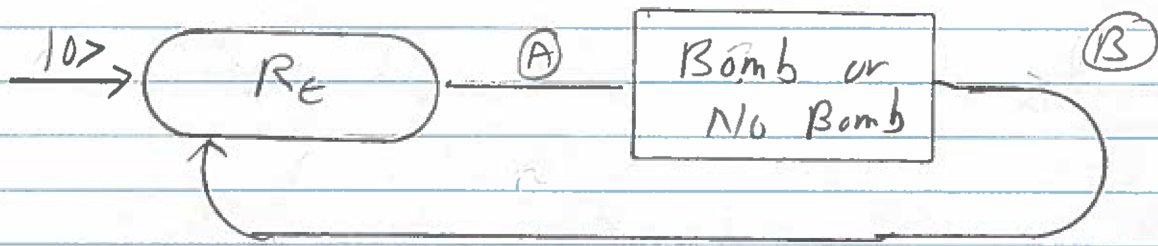
Send into box

No Bomb – state still angle ϵ

Bomb? state either $|0\rangle$ or $|1\rangle$

Prob $|0\rangle$ $\cos^2\epsilon$ close to 1

Prob $|1\rangle$ $\sin^2\epsilon$ small $\sim \epsilon^2$



Pick large n, say 100
$\rightarrow \epsilon = 90°/n$ or $\pi/2n$

Run the loop above n times

At B we see either

A if Box empty
$|0\rangle$ if Bomb there, no explode

Choose n as large as you like
so ϵ is as small as you like

After n iterations, at B
No bomb, particle is at $90°$
$\rightarrow$ measures $|1\rangle$

Bomb, particle at $|0\rangle$ measures
$|0\rangle$

Claim the bomb is arbitrarily unlikely
to explode

Why?

Bomb present

$P[\text{see } |0\rangle] = \cos^2 \epsilon$ close to 1

$P[\text{explode}] = \sin^2 \epsilon$

In radians $\sin \epsilon \leq \epsilon$ small ϵ

$\sin^2 \epsilon \leq \epsilon^2$

ϵ is $1/1000$, ϵ^2 is $1/1,000,000$

Very low chance of explosion

$$Pr[\text{explode}] = \underbrace{\epsilon^2 + \epsilon^2 + \dots + \epsilon^2}_{n}$$

$$= n\epsilon^2 \qquad \epsilon = \pi/2n$$

$$= \frac{n\pi^2}{4n^2} = \frac{\pi^2}{4n} \leq \frac{2.5}{n}$$

Can be made arbitrarily small