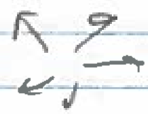
What did we see in the polarizing filters?

Unpolarized $\rightarrow$ Polarized



|||

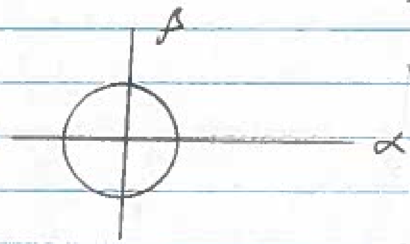
unpolarized light when prepared by //// is in a superposition

$$\alpha\,|\updownarrow\rangle + \beta\,|\leftrightarrow\rangle$$

For now, α & β are real

and we require $\alpha^2 + \beta^2 = 1$

energy or probability



$\alpha^2+\beta^2=1$ describes the unit circle

α + β are probability amplitudes

For the ||| filter above, we expect

$\alpha = \beta = 1/\sqrt{2}$ } easier to leave in this form

The light that comes through is 1/2

With respect to ||| filter, the light is in a superposition of

$$\frac{1}{\sqrt{2}} |\updownarrow\rangle + \frac{1}{\sqrt{2}} |\leftrightarrow\rangle$$

Can confirm by measuring light

||| see ½ the light

Subsequent re measure

|||

|||

Still see ½ the original light

Measure ≡ from original, ½
after |||, 0

||| ≡ one basis

/// \\\ another basis

Measurement prepares the Quantum system with respect to some basis

If we prepare ≡ what do we see subsequently if measure at some angle θ?

$\angle\theta$ orig			$\cos\theta$	$\cos^2\theta$
—	same	1.0		
$\llcorner\theta$	$\pi/2$	0	0	0
—·—	π	1.0	1	1
$\angle\pi/4$		0.5	$1/\sqrt{2}$	$1/2$

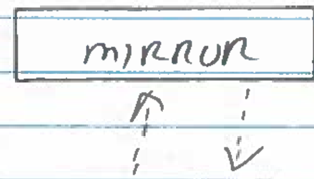
$\cos\theta$ is amplitude
$\cos^2\theta$ is measurement

Back to laptop screen

||| measure /// $1/2$

measure ≡ $1/4$

Level 4 – what happens in that quantum game?
Look at transitions & Java Console
Ignore polarization

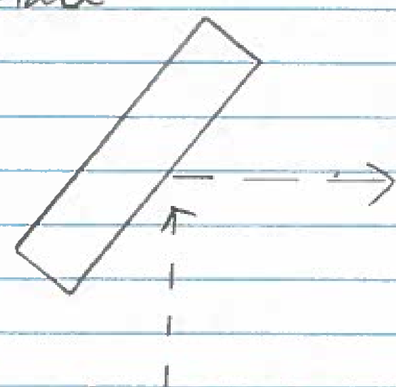


1 + 0i light ≡
-1 + 0i light |||

game uses this!

Same on the other side

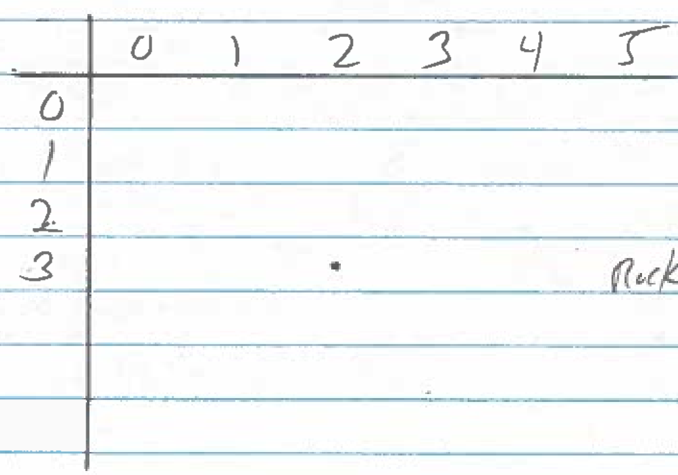
If we rotate



-1 + 0i

Same on other sides

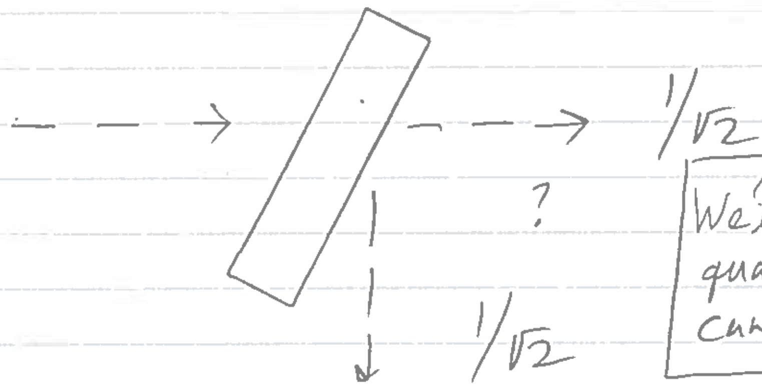
Go back to level 1 see JavaScript console



col, row, dir

Rock 2, 3, →

Beam splitter



$\boxed{\text{We'll see no quantum gate can do this}}$

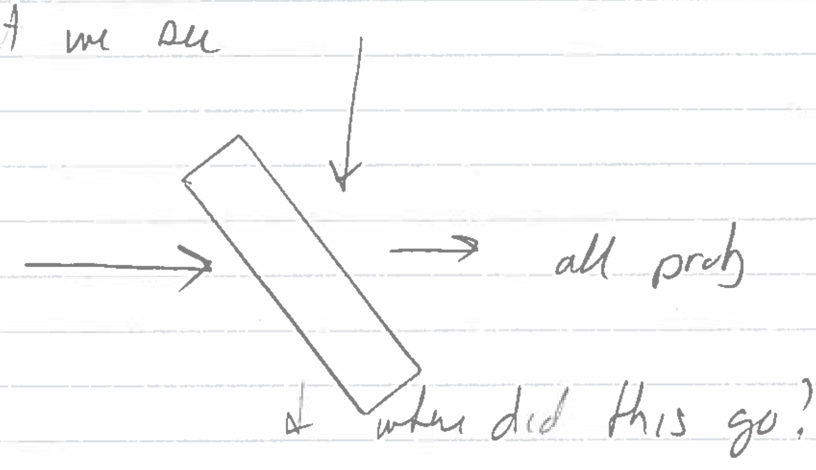
One photon in superposition of two places

If we measure we expect 1/2 + 1/2 see same

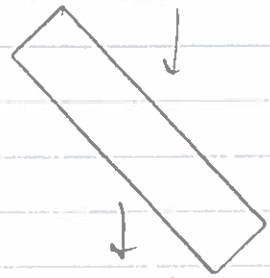
If we split again 1/2 1/4 1/4

We expect level 4 bottom right to split

But we see

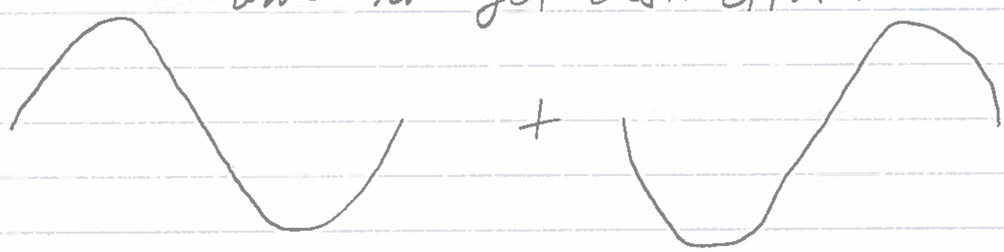


How?



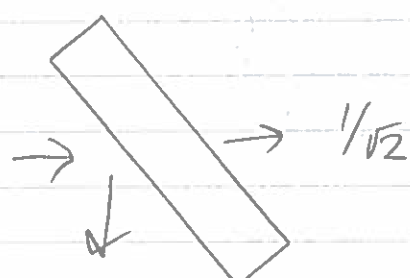
cancel   Wave nature of photons

There must be some destructive interference! How much to shift a wave to get destruction?



would cancel   shift by π

Look up the beam splitter

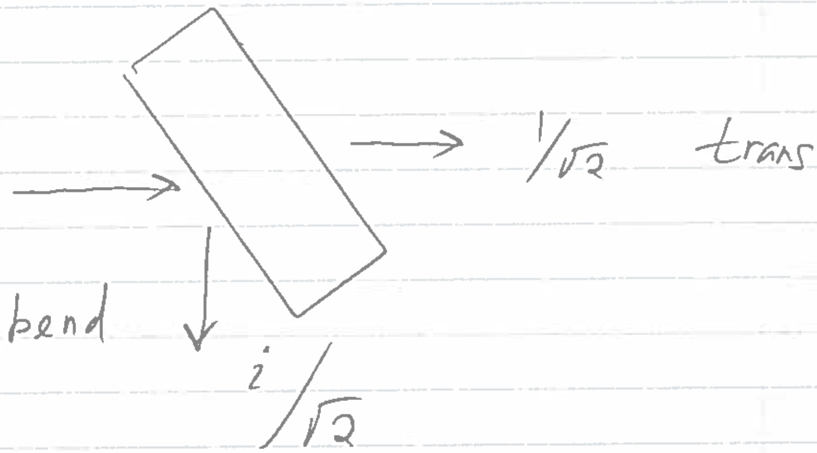


$1/\sqrt{2}$ shift by $\pi/4$

$\times 2$ so shift of $\pi/2$

Why? p. 18 Kaye text

Spec of the beam splitter



→

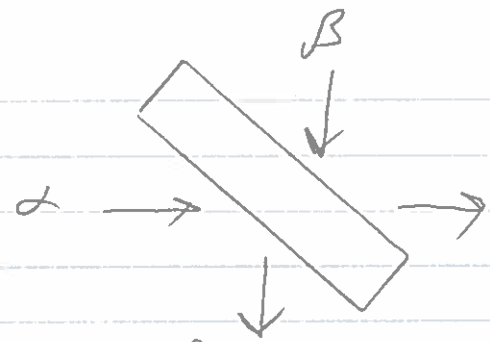
↓

Both not just one or the other!
We can't use a bit
We need to say how much → pass thru
how much ↓ bent

$\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ subject to $|\alpha|^2 + |\beta|^2 = 1$
SHOW ket version too

$\rightarrow \begin{pmatrix}1\\ 0\end{pmatrix}$ $\quad\downarrow \begin{pmatrix}0\\ 1\end{pmatrix}$

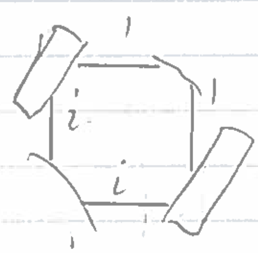
Beam split $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}$ thru
bent



$\left(\frac{1}{\sqrt{2}}\alpha + \frac{i}{\sqrt{2}}\beta\right)|\rightarrow\rangle$

$\left(\frac{i}{\sqrt{2}}\alpha + \frac{1}{\sqrt{2}}\beta\right)|\downarrow\rangle$

$$\underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}}_{\text{effect of beam splitter}}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$



2nd splitter

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}i\\ 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2i\\ i^2+1\end{pmatrix}$$

$$= \begin{pmatrix}i\\ 0\end{pmatrix}$$ all right

GATE MATRIX

$$\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

We require such matrices to be UNITARY

Given input

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \qquad |\alpha|^2 + |\beta|^2 = 1$$

the UNITARY matrix produces outputs with the same property – preserves the norms of vectors

No Probability is lost

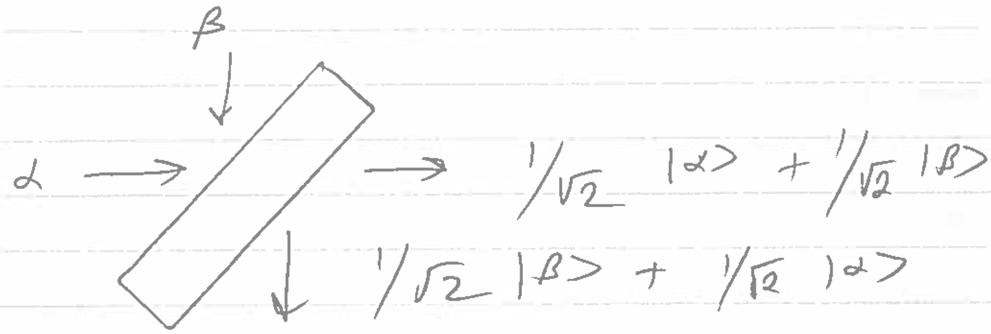
U is unitary then

$$U^* U = I \quad \text{identity matrix}$$

Verify

$$\underbrace{\frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}}_{\text{conj. trans.}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \checkmark$$

Recall we thought



$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$ IS NOT Unitary
Show

In these games a photon can be in some number of superpositions
Each mutually exclusive to the others (orthogonal)

Qudit - a quantum superposition of d states $d \geq 2$ Split + Split

Qutrit - Qudit $d=3$

Qubit - $d=2$

Notation

Dirac's BRA-KET

convenient notation for inner products & q-states

$|\psi\rangle$ is a "ket"

column vector, like we have seen

Single qubit

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

α, β complex

$|\alpha|^2 + |\beta^2| = 1$

$\langle\psi|$ bra of ψ

is row vector conjugate of ψ

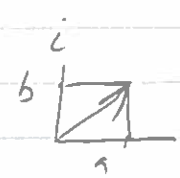
$$= (\alpha^* \ \beta^*)$$

$\langle\psi|\psi\rangle$ is $(\alpha^* \ \beta^*)\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

$$= \alpha^*\alpha + \beta^*\beta$$

Recall if

$$x = a + bi$$
$$x^* = a - bi$$



$$x^* x = a^2 + abi - abi - b^2 i^2$$
$$= a^2 + b^2 = |x|^2$$

We require

$$|\alpha|^2 + |\beta|^2 = 1$$

which is to say

$$\langle\psi|\psi\rangle = \underline{1}$$

Works in general for qudits

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}$$

$$\langle\psi|\psi\rangle = \underline{1} = \alpha^*\alpha + \beta^*\beta + \gamma^*\gamma + \delta^*\delta$$

Single Qubit systems

Photon after 1 beam splitter
Polarization of light
Electron in 2 states

State of such a system

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ + $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ basis vectors

orthogonal : inner prod = 0

α + β are amplitudes, complex

State Space Postulate

The state of a quantum system is a unit vector in a HILBERT (complex Euclidean) space

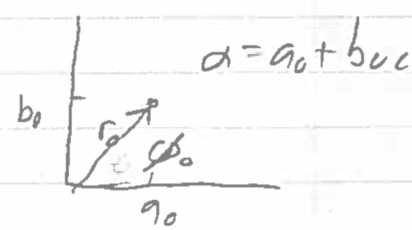
$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

with $|\alpha|^2 + |\beta|^2 = 1$

Consider

$$\alpha = a_0 + b_0 i$$

$$\alpha = r_0(\cos\phi_0 + i\sin\phi_0)$$



$$= r_0 e^{i\phi_0}$$

r real ≥ 0

$$\beta = r_1 e^{i\phi_1}$$

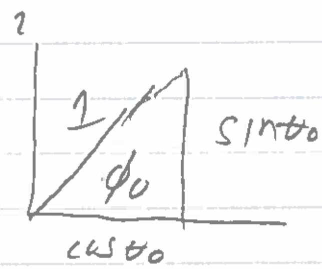
$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$= \underbrace{r_0 e^{i\phi_0}}_{\text{pull out}}|0\rangle + r_1 e^{i\phi_1}|1\rangle$$

$$= \underbrace{e^{i\phi_0}}_{\text{global phase}}\left[r_0|0\rangle + r_1 e^{i(\phi_1-\phi_0)}|1\rangle\right]$$

global phase affects no measurement

$e^{i\phi_0}$



Measuring $e^{i\phi_0}$ is always 1

So

$$|\psi\rangle = r_0|0\rangle + r_1 e^{i(\underbrace{\phi_1-\phi_0}_{\phi})}|1\rangle$$

$$= r_0|0\rangle + \underbrace{r_1 e^{i\phi}}_{1 \text{ measured}}|1\rangle$$

We need $r_0^2 + r_1^2 = 1 \rightarrow r_0, r_1 \le 1$

Let $r_0 = \cos\theta/2$ arbitrary for now

$r_1^2 = 1 - \cos^2\theta/2$

$\rightarrow r_1 = \sin\theta/2$

$$\boxed{|\psi\rangle = \cos\theta/2\,|0\rangle + e^{i\phi}\sin\theta/2\,|1\rangle}$$

You'd think $\alpha|0\rangle + \beta|1\rangle$ has 4 parameters
complex parts of α & β

But

we see here, there are only 2 real parameters θ ϕ
BLOCH SPHERE

Bloch Sphere

5:46 uz Video

Qubit state sits on
surface of UNIT Sphere

θ ϕ

$|z\rangle$ is $\theta = 0$ $\phi = 0$

is $1\,|0\rangle + 0\,|1\rangle$

$-|z\rangle$ $\theta = \pi$ $\phi = 0$

is $|1\rangle$ [this is why $\theta/2$]

$|0\rangle$ + $|1\rangle$ are orthogonal
logically but on
Bloch Sphere antipodal

$|x\rangle$ $\theta = \pi/2$ (90°) $\phi = 0$

$\cos \pi/4\ |0\rangle + 1 \sin \pi/4\ |1\rangle$

$= 1/\sqrt{2}\ |0\rangle + 1/\sqrt{2}\ |1\rangle$

We call this $|+\rangle$

$-|x\rangle$ $\theta = \pi/2$ $\phi = \pi$

$\cos \pi/4 \, |0\rangle + e^{i\pi} \sin \pi/4 \, |1\rangle$

$= 1/\sqrt{2} \, |0\rangle + [\quad -1 \quad] \, 1/\sqrt{2} \, |1\rangle$

$= 1/\sqrt{2} \, |0\rangle - \frac{1}{\sqrt{2}} \, |1\rangle$

We call this $|-\rangle$

$|+\rangle$ and $|-\rangle$ are antipodal

Orthogonal

Test dot prod. $= \left(1/\sqrt{2} \langle 0| + 1/\sqrt{2} \langle 1|\right)$

$\cdot \left(1/\sqrt{2} |0\rangle - 1/\sqrt{2} |1\rangle\right)$

$= 1/2 \langle 0|0\rangle - 1/2 \langle 0|1\rangle = 0$

$+ 1/2 \langle 1|0\rangle - 1/2 \langle 1|1\rangle$

1

$= 1/2 - 1/2 = 0$

EASIER

$$\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = 1/2 - 1/2 = 0$$

$|+\rangle$ & $|-\rangle$ are also basis vectors for a qubit

To obtain $|+\rangle$ & $|-\rangle$ from $\alpha|0\rangle + \beta|1\rangle$ we need

$$1/\sqrt{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Hadamard gate

$$|0\rangle \quad H \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = |+\rangle$$

$$|1\rangle \quad H \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = |-\rangle$$

Hadamard takes classical $|0\rangle$

& puts it in a superposn

$$\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

classic measurment sees

$|0\rangle$ ½ time
$|1\rangle$ ½ time

Truly random

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\text{inverse}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$a+b=1 \qquad a=1$
$a-b=0 \qquad b=1$

$c+d=0 \qquad c=1$
$c-d=1 \qquad d=-1$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Wow H is its own inverse
(really H^*) $H^*H=I$

38

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= |0\rangle$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= |1\rangle$$

Back to Bloch Sphere

$|y\rangle$ $\quad \theta = \pi/2 \quad \phi = \pi/2$

$$1/\sqrt{2}\,|0\rangle + i/\sqrt{2}\,|1\rangle$$

$-|y\rangle$ $\quad \theta = \pi/2 \quad \phi = -\pi/2$

$$1/\sqrt{2}\,|0\rangle - i/\sqrt{2}\,|1\rangle$$

We can think of the Bloch Sphere as a model of the 3 dimensions of electron spin – x, y, z

Measurement of one dimension of spin should leave the other dimensions randomized with equal likelihood of observing either antipodal in those dimensions

If we measure $|z\rangle$ we observe either $|0\rangle$ or $|1\rangle$

Say we observe $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$|x\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$-|x\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

We see $|x\rangle$ or $-|x\rangle$ w/ prob 1/2

same for $|y\rangle$ and $-|y\rangle$

* Can you express $|0\rangle$ and $|1\rangle$ in terms of $|+\rangle$ and $|-\rangle$?

Show that measuring $|+\rangle$ and then $|0\rangle$ has $\frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle$ outcome

Back to the Bloch sphere

A qubit is some point on the surface

Measurement produces $|0\rangle$ or $|1\rangle$

The outcome depends on θ — is the point more N or S hemisphere?

ϕ has NO EFFECT on measurement

If we rotate around X axis by π radians,

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \Longrightarrow \begin{matrix} \beta \\ \alpha \end{matrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

X-gate

UNITARY?

Notes p.33 $|\psi\rangle$

Z followed from wanting to measure 1, -1 for $|0\rangle$ & $|1\rangle$

Z is also Bloch sphere rotation by π around z axis

Phase change only

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ -\alpha_1 \end{pmatrix}$$

$$\text{Prob} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \alpha_0^2$$

$$\text{Prob} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (-\alpha_1)^2 = \alpha_1^2$$

If we prepare spin along $|x\rangle$ or $-|x\rangle$ we measure +1 or -1 subsequently but WRT z we see

$|0\rangle$ +1 1/2 time

$|1\rangle$ -1 1/2 time

We had $|+\rangle = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ $|-\rangle = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$

X

$$\begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = 1 \cdot \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$\begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = -1 \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ NOT gate

X also rotates by π around Bloch Sphere

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_0 \end{pmatrix}$$

Probabilities SWAP

Rotate by π around Y AXIS also swaps probability but need a different matrix because Y orth. to X

eigenvectors X $\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ $\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$

$|+\rangle$ & $|-\rangle$

$|x\rangle$ $-|x\rangle$ on sphere

$|y\rangle = \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix}$ $-|y\rangle = \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix}$

Y

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix} = 1 \cdot \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix}$$

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix} = -1 \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix}$$

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = \begin{pmatrix} -i\alpha_1 \\ i\alpha_0 \end{pmatrix}$$

$$\alpha_1 = a_1 + ib_1$$

$$-i\alpha_1 = b - ia_1 \rightarrow |-i\alpha_1| = |\alpha_1|$$

Prob $|0\rangle$ is $|\alpha_1|^2$ SWAP!
$|1\rangle$ is $|\alpha_0|^2$

Pauli $X \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $Y \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$Z \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

Math Interlude

$e^{i\theta A}$ where A is a matrix

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

consider $A \mid A^2 = I$ $\quad I = \begin{bmatrix} 1, & 0 \\ 0, & 1 \end{bmatrix}$

$\rightarrow A^4 = I \ldots$

$$e^{i\theta A} =$$

$$I + \frac{i\theta A}{1!} + \frac{i^2\theta^2 A^2}{2!} + \frac{i^3\theta^3 A^3}{3!} + \frac{i^4\theta^4 A^4}{4!} + \frac{i^5\theta^5 A^5}{5!} \ldots$$

$$= I + \frac{i\theta A}{1!} - \frac{\theta^2 I}{2!} - \frac{i\theta^3 A}{3!} + \frac{\theta^4 I}{4!} + \frac{i\theta^5 A}{5!}$$

$$= I\left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\right] + i\left[\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \ldots\right] A$$

$$= I \cos\theta + i \sin\theta \cdot A$$

$$\boxed{e^{i\theta A} = \cos\theta \cdot I + i \sin\theta \cdot A}$$

41.5

Rotate by θ around X, Y, Z

$$R_X(\theta) = e^{-i\theta/2 X}$$

$$R_Y(\theta) = e^{-i\theta/2 Y}$$

$$R_Z(\theta) = e^{-i\theta/2 Z}$$

where X, Y, Z are the three Pauli matrices

Rotate by θ (any angle) use $e^{-i\frac{\theta}{2}A}$

$$A \begin{cases} \text{X-axis} & X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ & \text{Also called NOT} \\ \text{Y-AXIS} & Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ \text{Z AXIS} & Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{cases}$$

Claim rotations of π around X, Y, Z are the X, Y, Z matrices!

$$X: \quad \underbrace{\cos \pi/2}_{\text{ALWAYS } 0} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i \underbrace{\sin \pi/2}_{\text{ALWAYS } 1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= 0 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad -i \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{X}$$

$-i\,A$ same as A

$\underbrace{-i}_{\text{global phase}}$ any $e^{i\theta}$

because $|e^{i\theta}| = 1$

so

X, Y, Z rotate by π
around Bloch X, Y, Z
axes

*HW prob

1) Consider a point on the Bloch Spere θ ϕ $|\psi\rangle$

Where is its anti podal point?
θ', ϕ' $|\psi'\rangle$

$$\theta' = \pi - \theta$$
$$\phi' = \phi + \pi$$

2) Show $\langle\psi'|\psi\rangle = 0$

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

Three parameters suffice to characterize any unitary matrix for a single qubit

1) Any operator U can be written as

$$e^{i\alpha} R_{\hat{n}}(\theta)$$

rotate about $\hat{n}$ axis

global factor - normalize

2) $R_{\hat{n}}(\theta) =$

$$\cos \theta/2 - i \sin \theta/2 \left[n_x X + n_y Y + n_z Z \right]$$

n_x n_y n_z are
a unit vector
so only 2 of 3 are
freely chosen

Say θ, n_x, n_z

Example $\begin{matrix} n_x & 1/\sqrt{2} \\ n_z & 1/\sqrt{2} \end{matrix} \rightarrow n_y = 0$

$n_x^2 + n_y^2 + n_z^2 = 1$

$\theta = \pi$

$$U = \boxed{?} \cos \pi/2 \; I - i \sin \pi/2 \left[1/\sqrt{2} (X + Z) \right]$$

$$= 0 - i \; 1/\sqrt{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

choose $\alpha = \pi/2$

$$e^{i\pi/2} \left(-i \, 1/\sqrt{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right)$$

$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ H gate!