CSE 468T Introduction to Quantum Computing	**Spring 2022**
Exam I	
Given: 9 March 2022	Due: End of class

READ THIS before starting! This exam is open-book, open-notes, open-Internet, but you must do this work on your own without contact or conversations with any person. Because this exam is given in a somewhat distributed manner, no questions will be answered, and no clarifications will be given. State your assumptions and count on us to be fair and flexible, especially if we have been unclear.

Your work must be legible. Work that is difficult to read will receive no credit. There is a blank page at the end if you want to show extra work there.

There are 118 points available for this exam, but it will only be scored out of 100. Extra points earned here will count toward your total exam grade, including Exam II.

You must sign the pledge below for your exam to count. Any cheating will cause the students involved to receive an F for this course. Other unpleasant actions may be taken.

You must fill in your identifying information correctly.

Print clearly the following information:
Name (print clearly):
Student 6-digit ID (print *really* clearly):

Pledge: On my honor, I have neither given nor received any unauthorized aid on this exam.

Signed: ______________________________

(Be sure you filled in your information in the box above!)

1. (30 points) For the **true/false** questions below, indicate your response by marking an **x** in the appropriate box, like this: [x] **true** [ ] **false** or [ ] **true** [x] **false**.

   Each response is worth 3 points. You can miss any two responses and still get full credit for this portion.

   - Every unitary matrix has an inverse, which is that matrix's conjugate transpose. [ ] **true** [ ] **false**
   - The states $\alpha|0\rangle + \beta|1\rangle$ and $\alpha|0\rangle - \beta|1\rangle$ differ only by a global phase. [ ] **true** [ ] **false**
   - The states $\alpha|0\rangle - \beta|1\rangle$ and $-\alpha|0\rangle + \beta|1\rangle$ differ only by a global phase. [ ] **true** [ ] **false**
   - The states $i\begin{pmatrix}0\\1\end{pmatrix}$ and $\begin{pmatrix}0\\-i\end{pmatrix}$ differ only by a global phase. [ ] **true** [ ] **false**
   - Consider a state $\psi = \alpha|0\rangle + \beta|1\rangle$, and let $\psi^\star$ denote the conjugate transpose of ψ. Then $\psi^\star \times \psi = 0$. [ ] **true** [ ] **false**
   - The conjugate transpose of $\begin{pmatrix}1\\i\end{pmatrix}$ is $(1 \;\; i)$. [ ] **true** [ ] **false**
   - The actions of any sequence of unitary gates applied to a single qubit can be represented by a single, unitary matrix. [ ] **true** [ ] **false**
   - If the gate A is applied to ψ and the resulting state is then processed by gate B, the result is $A(B(\psi))$ [ ] **true** [ ] **false**
   - For the Pauli matrices, $\mathbf{XYZZY} = \mathbf{X}$ [ ] **true** [ ] **false**
   - The column vector for representing the state of an n-qubit system has n entries. [ ] **true** [ ] **false**
   - Given the state $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$, measuring either qubit determines the value of the other qubit. [ ] **true** [ ] **false**
   - The state $\frac{|01\rangle+|10\rangle}{\sqrt{2}}$ is an entangled state. [ ] **true** [ ] **false**

2. (30 points) For each question below, fill in the blank. Your answer must appear in the provided blank for proper credit. Write each response in the provided blank space, fully above the dark line. For example, to express $\frac{i}{\sqrt{3}}$ you would write $\underline{\quad i/\sqrt{3}\quad}$. Each response is worth 2 points. You can miss one response and get full credit for this portion.

   - As evident on the Bloch sphere, the number of real parameters necessary to describe an arbitrary state of a single qubit is __________.
   - As reflected in qiskit's **U** gate, the number of real parameters necessary to describe an arbitrary, single-qubit, unitary gate is __________.
   - On the Bloch sphere, $|0\rangle$ is the __________ pole.
   - The states $|+\rangle$ and $|-\rangle$ are found on which axis (x, y, or z) of the Bloch sphere? __________
   - Consider a state ψ on the Bloch sphere which is an eigenstate of some unitary gate U. The minimum (angular) distance between ψ and $U(\psi)$ on the Bloch sphere is __________ radians.
   - $\mathbf{X}\,|0\rangle =$ _______ $|0\rangle +$ _______ $|1\rangle$
   - $\mathbf{Z}\,|+\rangle =$ _______ $|0\rangle +$ _______ $|1\rangle$
   - $\mathbf{H}|0\rangle =$ _______ $|0\rangle +$ _______ $|1\rangle$
   - Suppose a single-qubit quantum system is measured "using" an operator M. Then the quantum system collapses into a state ψ, where ψ is an __________ of M.
   - Complete the matrix below so that it is unitary:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \underline{\quad} \\ -i & 1 \end{pmatrix}$$

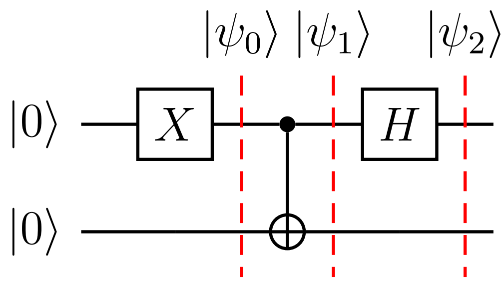
   - Complete the matrix below so that it yields the result shown and so the matrix is unitary:

$$\begin{pmatrix} 1 & \underline{\quad} \\ 0 & \underline{\quad} \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

   - For all θ, $\left|e^{i\,\theta}\right| =$ __________

3. (20 points)

- (10 points) Consider this circuit:



Fill in the column vectors to show your analysis of the states at the indicated positions of the circuit. The rest of this page is scratch space: **the parts to fill in are on the next page**.

Continued on next page. . .

$$|\psi_0\rangle = \begin{pmatrix} \_\_\_ \\ \_\_\_ \\ \_\_\_ \\ \_\_\_ \end{pmatrix} \quad |\psi_1\rangle = \begin{pmatrix} \_\_\_ \\ \_\_\_ \\ \_\_\_ \\ \_\_\_ \end{pmatrix} \quad |\psi_2\rangle = \begin{pmatrix} \_\_\_ \\ \_\_\_ \\ \_\_\_ \\ \_\_\_ \end{pmatrix}$$

- Either Express $|\psi_2\rangle$ as the product of two 1-quibit states:

$$|\psi_2\rangle = \begin{pmatrix} \_\_\_ \\ \_\_\_ \end{pmatrix} \otimes \begin{pmatrix} \_\_\_ \\ \_\_\_ \end{pmatrix}$$

or complete the proof below to show that it cannot be factored:

$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} \_\_\_ \\ \_\_\_ \\ \_\_\_ \\ \_\_\_ \end{pmatrix} \text{ (copy this from your } \psi_2 \text{ answer top of page)}$$

$$a \cdot c = \_\_\_$$

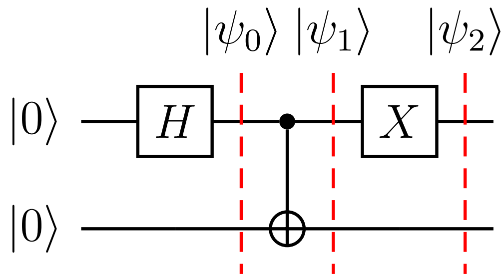
$$a \cdot d = \_\_\_$$

$$b \cdot c = \_\_\_$$

$$b \cdot d = \_\_\_$$

What is the contradiction, if any?

- (10 points) Consider this circuit:



Fill in the column vectors to show your analysis of the states at the indicated positions of the circuit. The rest of this page is scratch space: **the parts to fill in are on the next page**.

Continued on next page...

$$|\psi_0\rangle = \begin{pmatrix} \rule{2em}{1pt} \\ \rule{2em}{1pt} \\ \rule{2em}{1pt} \\ \rule{2em}{1pt} \end{pmatrix} \quad |\psi_1\rangle = \begin{pmatrix} \rule{2em}{1pt} \\ \rule{2em}{1pt} \\ \rule{2em}{1pt} \\ \rule{2em}{1pt} \end{pmatrix} \quad |\psi_2\rangle = \begin{pmatrix} \rule{2em}{1pt} \\ \rule{2em}{1pt} \\ \rule{2em}{1pt} \\ \rule{2em}{1pt} \end{pmatrix}$$

- Either Express $|\psi_2\rangle$ as the product of two 1-quibit states:

$$|\psi_2\rangle = \begin{pmatrix} \rule{2em}{1pt} \\ \rule{2em}{1pt} \end{pmatrix} \otimes \begin{pmatrix} \rule{2em}{1pt} \\ \rule{2em}{1pt} \end{pmatrix}$$

or complete the proof below to show that it cannot be factored:

$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} \rule{2em}{1pt} \\ \rule{2em}{1pt} \\ \rule{2em}{1pt} \\ \rule{2em}{1pt} \end{pmatrix}$$ (copy this from your ψ_2 answer top of page)

$a \cdot c =$ ______

$a \cdot d =$ ______

$b \cdot c =$ ______

$b \cdot d =$ ______

What is the contradiction, if any?

4. (10 points) Bloch Sphere Orienteering: all answers are in the standard basis. Write each amplitude in the provided blank space, fully above the dark line. For example, to express $\frac{i}{\sqrt{3}}$ you would write $\underline{\quad i/\sqrt{3} \quad}$.
Each response is worth 1 point.

- We begin at the North pole of the Bloch sphere.
- We are in state ______ $|0\rangle$ + ______ $|1\rangle$.
- We experience a **Y** gate. We are now at ______ $|0\rangle$ + ______ $|1\rangle$.
- We return to the North pole.
- We experience an **H** gate. We are now at ______ $|0\rangle$ + ______ $|1\rangle$.
- We then experience a **Z** gate. We are now at state ______ $|0\rangle$ + ______ $|1\rangle$.
- We finally experience an **H** gate. We are now at state ______ $|0\rangle$ + ______ $|1\rangle$.

5. (10 points) Suppose Alice and Bob decide to form a shared key using the BB84 protocol. The table below shows the bases they will publish along with their observed results. Recall the correspondence between observed quantum state and bit values:

Basis	0	1
+	↑	→
×	↗	↘

Fill in the table below and be sure to put your final answer for grading in the space provided at the bottom of this page. Note the following:

Agreed Bit? After publishing their list of bases, what bit (0 or 1), **IF ANY**, would Alice and Bob each believe they share, assuming they do not believe Eve is present.

Eve Detected? Does this particular row allow detection of Eve if Alice's and Bob's bits from this row are published?

Alice Sends		Bob Receives		Agreed bit?		Eve
Basis	Obs	Basis	Obs	Alice	Bob	Detected?
+	→	+	→			☐ **true** ☐ **false**
+	↑	+	↑			☐ **true** ☐ **false**
+	→	+	→			☐ **true** ☐ **false**
+	↑	×	↗			☐ **true** ☐ **false**
+	→	×	↘			☐ **true** ☐ **false**
×	↗	×	↘			☐ **true** ☐ **false**
×	↘	×	↘			☐ **true** ☐ **false**
×	↗	×	↗			☐ **true** ☐ **false**
×	↗	×	↗			☐ **true** ☐ **false**

- Alice believes the shared key is ________________
- Bob believes the shared key is ________________
- Eve could have been detected in how many rows of your table? ________

6. (10 points) Consider the following entangled state of three qubits:

$$\frac{|000\rangle + |111\rangle}{\sqrt{2}}$$

In lecture we studied how to entangle that third qubit with the first two. In this problem we consider *disentangling* the third qubit from the first two, without resorting to measurement.

Recall that we entangle the third new qubit q_3 with those that came before (q_1, q_2) using a **CNOT** gate, controlled by either q_1 or q_2, and applied to q_3 which is initialized to $|0\rangle$. Because **CNOT** is its own inverse, we try using the gate again to disentangle q_3.

(a) (2 points)

$$(\mathbf{I} \otimes \mathbf{CNOT})(|000\rangle) = |\rule{2em}{0.15em}\rangle$$

(b) (2 points)

$$(\mathbf{I} \otimes \mathbf{CNOT})(|111\rangle = |\rule{2em}{0.15em}\rangle$$

(c) (3 points) By linearity,

$$(\mathbf{I} \otimes \mathbf{CNOT})\left(\frac{|000\rangle + |111\rangle}{\sqrt{2}}\right) = \frac{\rule{3em}{0.15em} + \rule{3em}{0.15em}}{\sqrt{2}} = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \otimes |\rule{2em}{0.15em}\rangle$$

(d) (2 points) Qubit q_3 is now disentangled and in its current state it measures as $|\rule{2em}{0.15em}\rangle$.

(e) (1 points) Measuring q_3 causes collapse of the other two qubits? ☐ **true** ☐ **false**