Phase Estimation

Suppose the amplitudes of states

$|0\rangle \ldots |2^n - 1\rangle$ are

related by a constant phase difference

Consider an n-qubit state
the integer y here is interpreted
base-2 as the encoding of a basis
state

$n = 3$ $|uuu\rangle$
$y = 5$ $|101\rangle$

$|5\rangle$ is ok but
this is not $|5\rangle$
of a single qudit
but instead one of
the 8 basis states
of a 3-qubit state

$1/\sqrt{2}\,(|0\rangle - |1\rangle)$ Recall is $|-\rangle$

$= 1/\sqrt{2}\left(|0\rangle + -1\,|1\rangle\right)$

$= 1/\sqrt{2}\left(e^{0}\,|0\rangle + e^{i\pi}\,|1\rangle\right)$

$= 1/\sqrt{2}\left(e^{2\pi i(0)}\,|0\rangle\, e^{2\pi i(1/2)}\,|1\rangle\right)$

$$= \frac{1}{\sqrt{2}} \sum_{y=0}^{1} e^{2\pi i w y} |y\rangle$$

where $w = 1/2$

The $|-\rangle$ state is periodic, $w = 1/2$

the $|+\rangle$ state is periodic $w = 0$ or 1

$w \in (0, 1)$ [some sources exclude 1]

State $H^{\otimes n-1}(|0\rangle) \otimes |-\rangle$

$$= \frac{1}{\sqrt{2^n}} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ \vdots \end{pmatrix}$$

$$= \frac{1}{\sqrt{2^n}} \sum_{y=0}^{2^n-1} e^{2\pi i (1/2) y} |y\rangle$$

Given a periodic state

$$|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{y=0}^{2^n-1} e^{2\pi i w y} |y\rangle$$

Can we compute an estimate of w?

[see Fourier video]

Since $0 \le w < 1$

we can write $w = 0.x_1 x_2 x_3 \cdots$
in binary

$w = 1/2 \qquad w = 0.100\ldots$

Then $2^k w$ shifts the binary point <u>right</u> k places

$$2^k w = x_1 x_2 \cdots x_k . x_{k+1} x_{k+2} \cdots$$

Also since $e^{2\pi i k} = 1$ for any integer k

$$e^{2\pi i (2^k w)} = e^{2\pi i (x_1 x_2 \cdots x_k)} \times e^{2\pi i (0.x_{k+1} x_{k+1} \cdots)}$$

$$= e^{2\pi i (0.x_{k+1} x_{k+2} \cdots)}$$

Example $n = 1$ single qubit

$w = 0.x_1$, $|\psi\rangle$ is

$$1/\sqrt{2} \sum_{y=0}^{1} e^{2\pi i (0.x_1) y} |y\rangle$$

$$= 1/\sqrt{2} \sum_{y=0}^{1} e^{2\pi i x_1 y/2} |y\rangle = 1/\sqrt{2} \sum_{y=0}^{1} e^{\pi i x_1 y} |y\rangle$$

(can mpy exponent by 2^k)

$$= \frac{1}{\sqrt{2}} \sum_{y=0}^{1} \left(e^{i\pi}\right)^{x_1 y} |y\rangle$$

$$= \frac{1}{\sqrt{2}} \sum_{y=0}^{1} (-1)^{x_1 y} |y\rangle$$

$$= \frac{1}{\sqrt{2}} \left(|0\rangle + (-1)^{x_1} |1\rangle \right)$$

$x_1 = 0 \quad |\psi\rangle = |+\rangle$
$x_1 = 1 \quad |\psi\rangle = |-\rangle$

So $H(|\psi\rangle) = |x_1\rangle$

This gives the value of x_1 & therefore $w = .x_1$

Thm

$$\frac{1}{\sqrt{2^n}} \sum_{y=0}^{2^n-1} e^{2\pi i w y} |y\rangle =$$

$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i (2^{n-1} w)} |1\rangle\right) \otimes$$

$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i (2^{n-2} w)} |1\rangle\right) \otimes \dots \otimes$$

$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i (w)} |1\rangle\right)$$

Proof by induction

P(n) as stated above

P(1) :

$$\frac{1}{\sqrt{2}} \sum_{y=0}^{1} e^{2\pi i w y} |y\rangle$$

$$= \frac{|0\rangle + e^{2\pi i (0.x_1)} |1\rangle}{\sqrt{2}} \checkmark$$

$P(n) \rightarrow P(n+1)$

$$\frac{1}{\sqrt{2^n}} \sum_{y=0}^{2^n-1} e^{2\pi i w y} |y\rangle + \sum_{y=2^n}^{2^{n+1}-1} e^{2\pi i w y} |y\rangle$$

$=$

2-qubit example

$$|\psi\rangle = \frac{1}{\sqrt{2^2}} \sum_{y=0}^{2^2-1} e^{2\pi i w y} |y\rangle$$

$w = 0.x_1 x_2$ 00, 01, 10, 11 possible

0, 1/4, 1/2, 3/4

These are integer multiples of $1/2^2$

We'll look at the error we get when w is not like that later

So $$|\psi\rangle = \frac{1}{\sqrt{2^2}} \sum_{y=0}^{2^2-1} e^{2\pi i (0.x_1 x_2) y} |y\rangle$$

$$= \frac{|0\rangle + e^{2\pi i (0.x_2)} |1\rangle}{\sqrt{2}}$$

$$\circledast$$

$$\frac{|0\rangle + e^{2\pi i (0.x_1 x_2)} |1\rangle}{\sqrt{2}}$$

x_2 is either 0 or 1
Let's look at the first qubit

$$\frac{|0\rangle + e^{2\pi i (0.x_2)} |1\rangle}{\sqrt{2}}$$

if $x_2 = 0$, it's $\frac{|0\rangle + |1\rangle}{\sqrt{2}}$

if $x_2 = 1$, $2\pi i (1/2) = \pi i$, $e^{i\pi} = -1$

it's $\frac{|0\rangle - |1\rangle}{\sqrt{2}}$

Applying $H(\quad)$ measures $|x_2\rangle$ in std basis ± 1

So we now know x_2. How about x_1

Look at $.x_1 x_2$ if $x_2 = 0$
we can measure x_1 as we did for the first qubit

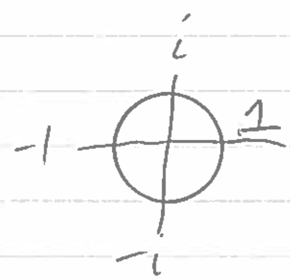
If $x_2 = 1$ then we have
either $.x_1 x_2 =$ $.01 = 1/4$
$.11 = 3/4$

H( ) wont work because the state of qubit 2 is not $|+\rangle$ or $|-\rangle$ so H( ) and measuring will not provide a certain result

We have

$$\frac{|0\rangle + e^{\frac{\pi}{2}i}|1\rangle}{\sqrt{2}} \quad \text{or} \quad \frac{|0\rangle + e^{\frac{3\pi}{2}i}|1\rangle}{\sqrt{2}}$$

$$= \frac{|0\rangle + i|1\rangle}{\sqrt{2}} \quad \text{or} \quad \frac{|0\rangle - i|1\rangle}{\sqrt{2}}$$



We need a transform that leaves the $|0\rangle$ alone but rotates the $|1\rangle$

$$\begin{pmatrix} 1 & 0 \\ 0 & ? \end{pmatrix} \begin{pmatrix} 1 \\ \pm i \end{pmatrix} = \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$$

i, or $-i$

Say we want $i \to \underline{1}$
$-i \to \underline{-1}$

then we pick

$$U = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$$

Like the beam splitter level 4 quantum games

Rotates clockwise by $\pi/2$ radians

If we apply that gate & then Hadamard, then measure 2nd qubit we now know X_1

Is U a real thing? Sure, recall

$$U_3(\theta, \phi, \lambda) = \begin{pmatrix} \cos\theta/2 & -e^{i\lambda}\sin\theta/2 \\ e^{i\phi}\sin\theta/2 & e^{i(\lambda+\phi)}\cos\frac{\theta}{2} \end{pmatrix}$$

$\theta = 0$
$\phi = 0$
$\lambda = 3\pi/2$

$$\begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$$

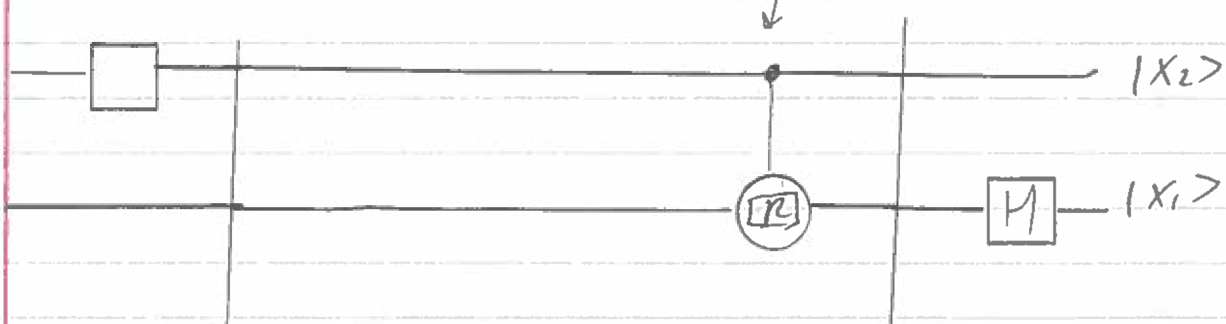
Putting this together.

We need to use x_2 to control whether we apply the $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ rotation

$$1/\sqrt{2}\left(|0\rangle + e^{2\pi i (0.x_2)}|1\rangle\right.$$

$$1/\sqrt{2}\left(|0\rangle + e^{2\pi i (0.x_1 x_2)}|1\rangle\right.$$

Controlled R



$$|x_2\rangle\ 1/\sqrt{2}\left(|0\rangle + e^{2\pi i (0.x_1 x_2)}|1\rangle\right)$$

$$|x_2\rangle\ 1/\sqrt{2}\left(|0\rangle + e^{2\pi i (0.x_1)}|1\rangle\right)$$

$$R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

3 qubits

$|0.x_3\rangle$ ————————

$|0.x_2 x_3\rangle$ ————————

$|0.x_1 x_2 x_3\rangle$ ————————

each $\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i(\;\;)}|1\rangle\right)$

Define $$R_K = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^K} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i(0.0..0\underline{1})} \end{pmatrix}$$

$1/2^K$ position

& its inverse R_K^{-1}

$$\begin{pmatrix} 1 & 0 \\ 0 & ? \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^K} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$? = e^{-2\pi i/2^K}$$

Consider

$$e^{2\pi i(0.x_1 x_2 \ldots x_k)} \quad \text{when } x_k = 1$$

$$e^{2\pi i(0.x_1 x_2 \cdots x_k)} \cdot e^{-2\pi i/2^k}$$

$$= e^{2\pi i(0.x_1 x_2 \cdots x_{k-1} 1 \; - \; 0.00..01)}$$

k^{th} position

$$= e^{2\pi i(0.x_1 x_2 \cdots x_{k-1})}$$

If rotate away $x_2 - x_k$

we get $e^{2\pi i(0.x_1)}$

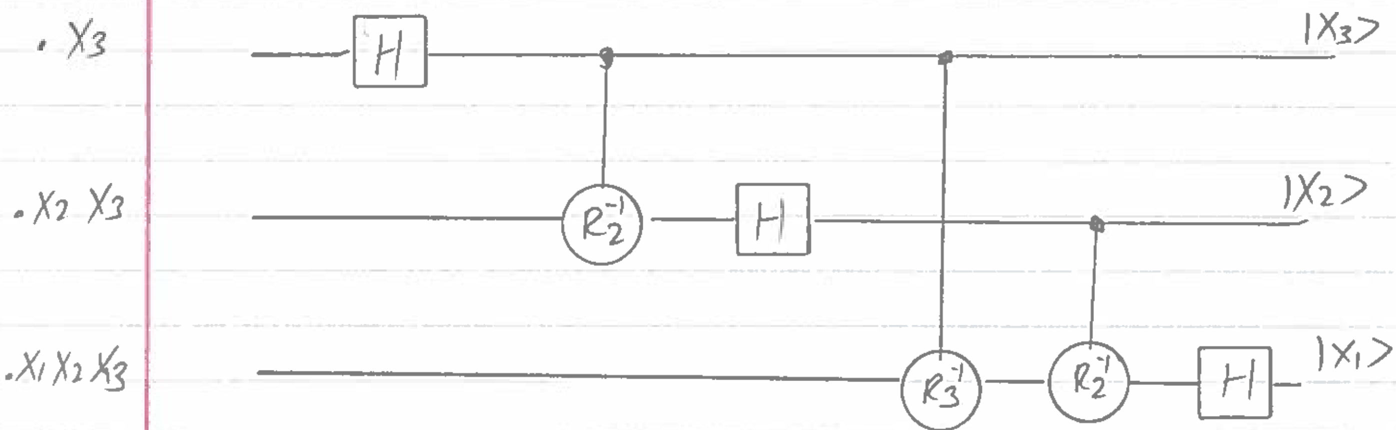
and $H\left(e^{2\pi i(0.x_1)}\right) \rightarrow |0\rangle \quad x_1 = 0$
$|1\rangle \quad x_1 = 1$

3 bits

$.x_3$ — H, measure

$.x_2 x_3$ — Pass R_2^{-1}, H, measure

$.x_1 x_2 x_3$ — Pass R_3^{-1},
Pass R_2^{-1}, H, measure



We get a precise answer if

$$w = x/2^3 \qquad 0/8 \;\; 1/8 \; \cdots \; 7/8$$

If w is an arbitrary value $0 \le w < 1$
We get 3 bits of resolution on w

For n qubits

$1 + 2 + \dots + n$ R^{-1} gates $= \theta(n^2)$
and n H gates

Delay of $\theta(n)$ gates

this computes

$$1/\sqrt{2^n} \sum_{y=0}^{2^n-1} e^{\frac{2\pi i x}{2^n} y} \; |y\rangle$$

$\rightarrow |x\rangle$

Well, $|x^R\rangle$ reversed, we can swap to get $|x\rangle$

Arthur covered VQE in 4 lectures

Quantum Fourier

The phase estimation algorithm takes

$$|\psi\rangle = 1/\sqrt{2^n} \sum_{y=0}^{2^n-1} e^{2\pi i \frac{x}{2^n} y} |y\rangle$$

and computes $|x\rangle$

Going in the other direction is the

Quantum Fourier Transform QFT

$$|x\rangle \rightarrow 1/\sqrt{2^n} \sum_{y=0}^{2^n-1} e^{2\pi i \, x/2^n \, y} |y\rangle$$

BASIS state for now

$$= 1/\sqrt{2^n} \sum_{y=0}^{2^n-1} e^{2\pi i \, y/2^n \, x} |y\rangle$$

$$= 1/\sqrt{2^n} \Big[e^{2\pi i \, 0/2^n \, x} |0\rangle$$

$$+ e^{2\pi i \, 1/2^n \, x} |1\rangle$$

$$+ e^{2\pi i \, 2/2^n \, x} |2\rangle$$

$$\vdots$$

$$+ e^{2\pi i \frac{2^n-1}{2^n} x} |2^n-1\rangle \Big]$$