Solution HW ANTIPODAL

Given

$$|p\rangle = \cos\theta/2\,|0\rangle + e^{i\phi}\sin\theta/2\,|1\rangle$$

$|p'\rangle$ is at $\theta' =$

$0 \to \pi$

$\pi \to 0$

$\pi/2 \to \pi/2$

$\theta \quad \pi - \theta$

$\phi' =$

$0 \to \pi$

$\pi \to 0 \quad (=2\pi)$

$\pi/2 \to 3\pi/2$

$$\phi' = \phi + \pi$$

$$|p'\rangle = \cos\left(\frac{\pi-\theta}{2}\right)|0\rangle + \underbrace{e^{i(\phi+\pi)}}_{e^{i\phi}e^{i\pi}} \sin\left(\frac{\pi-\theta}{2}\right)|1\rangle$$

$$e^{i\pi} = -1$$

$$= \cos\left(\frac{\pi-\theta}{2}\right)|0\rangle - e^{i\phi}\sin\left(\frac{\pi-\theta}{2}\right)|1\rangle$$

$$\langle p'| = \cos\left(\frac{\pi-\theta}{2}\right)|0\rangle - e^{-i\phi}\sin\left(\frac{\pi-\theta}{2}\right)|1\rangle$$

$$\langle p'|p\rangle = \cos\frac{\theta}{2}\cos\left(\frac{\pi-\theta}{2}\right) - e^{(i\phi-i\phi)}\sin\frac{\theta}{2}\sin\left(\frac{\pi-\theta}{2}\right)$$

$$= \cos\frac{\theta}{2}\cos\left(\frac{\pi-\theta}{2}\right) - \sin\frac{\theta}{2}\sin\left(\frac{\pi-\theta}{2}\right)$$

$$\left[\cos(a+b) = \cos a\cos b - \sin a\sin b\right]$$

$$a = \frac{\theta}{2} \quad b = \frac{\pi}{2} - \frac{\theta}{2}$$

$$= \cos \frac{\theta}{2} + \pi/2 - \theta/2 = \cos\pi/2 = 0 \quad \square$$

New 5C

Beyond 1 qubit

Composite Systems

Need to reason about qubits interacting with each other

When 2 physical systems are treated as one combined system the state space is the tensor product space $H_a \otimes H_b$ of the state spaces H_a + H_b

If H_a is in state $|\psi_a\rangle$
H_b $|\psi_b\rangle$

then the state of the combined system is

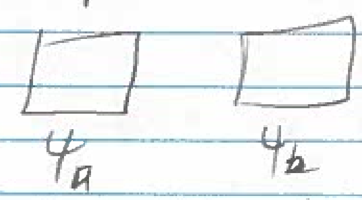
$$|\psi_a\rangle \otimes |\psi_b\rangle$$

written $|\psi_a\rangle|\psi_b\rangle$ or $|\psi_a\psi_b\rangle$

Lets be clear

$$H_1 \quad \alpha_{a0}|0\rangle + \alpha_{a1}|1\rangle = |\psi_a\rangle$$
$$H_2 \quad \alpha_{b0}|0\rangle + \alpha_{b1}|1\rangle = |\psi_b\rangle$$

We do not keep track separately of H_1 + H_2



No!

We use the tensor product to reason about the combined system

We must keep consistent about the prob amplitude associated with a given outcome

measure $|\psi_a\rangle$ get $|0\rangle$ or $|1\rangle$
$|\psi_b\rangle$ get $|0\rangle$ or $|1\rangle$

Tensor product

$$\begin{pmatrix} \alpha_{a0} \\ \alpha_{a1} \end{pmatrix} \begin{pmatrix} \alpha_{b0} \\ \alpha_{b1} \end{pmatrix} \text{ IS NOT } \begin{pmatrix} \alpha_{a0} \\ \alpha_{a1} \\ \alpha_{b0} \\ \alpha_{b1} \end{pmatrix} \begin{matrix} \}\, q_a \\ \\ \}\, q_b \end{matrix}$$

We expect the new system to have basis

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

each corresponds to a measurement of the system's 2 qubits

both 0
a 0 b 1
a 1 b 0
both 1

so

$$\begin{pmatrix} \alpha_{a0} \\ \alpha_{a1} \end{pmatrix} \otimes \begin{pmatrix} \alpha_{b0} \\ \alpha_{b1} \end{pmatrix} = \begin{pmatrix} \alpha_{a0}\,\alpha_{b0} \\ \alpha_{a0}\,\alpha_{b1} \\ \alpha_{a1}\,\alpha_{b0} \\ \alpha_{a1}\,\alpha_{b1} \end{pmatrix} \begin{matrix} \to |00\rangle \\ \to |01\rangle \\ \to |10\rangle \\ \to |11\rangle \end{matrix}$$

this matches what we know about
prob amplitudes and probability

$$\left.\begin{matrix}\alpha_{a0}\\ \alpha_{b0}\end{matrix}\right\}\text{ amplitudes} \to \begin{matrix}|\alpha_{a0}|^2\\ |\alpha_{b0}|^2\end{matrix}\text{ prob}$$

$$\alpha_{a0}\alpha_{b0} \to |\alpha_{a0}\alpha_{b0}|^2$$

$$\left.\begin{matrix}\alpha_{a0} = r_a e^{i\theta_a}\\ \alpha_{b0} = r_b e^{i\theta_b}\end{matrix}\right\}\text{ so } \alpha_{a0}\alpha_{b0} = r_a r_b e^{i(\theta_a+\theta_b)}$$

$$|\alpha_{a0}|^2 = r_a^2$$
$$|\alpha_{b0}|^2 = r_b^2$$

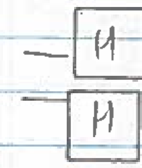
$$|\alpha_{a0}\alpha_{b0}|^2 = r_a^2 r_b^2$$

So prob $|00\rangle$ is
prob ψ_0 $|0\rangle$
$\times$ prob ψ_1 $|0\rangle$

We can also tensor matrices

$$H = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$H \otimes H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$



$$= \frac{1}{2}\begin{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & \begin{matrix} -1 & -1 \\ -1 & 1 \end{matrix} \end{pmatrix}$$

$|00\rangle$

$$H \otimes H \left(|00\rangle\right) = H \otimes H \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$$

prob each $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$ each equally likely

Factor a tensored vector

If $\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$

$$\left.\begin{aligned} ac &= v_1 \\ ad &= v_2 \\ bc &= v_3 \\ bd &= v_4 \end{aligned}\right\}$$ then we can factor

Can we always? Consider EPR $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$ac = 1 \rightarrow a \neq 0$

$ad = 0$?

$bc = 0$

$bd = 1 \rightarrow d \neq 0$

$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \leftarrow$ Not a pure state

But a, b, c, d can be complex — does that help?

Not always possible

If $w = r + si$ · $z = t + ui = 0$ one must be 0

$(r + si)(t + ui)$

NB better proof → next page

$rt - su + (ru + st)i$

$rt = su \qquad ru = -st$

$ru = \frac{su^2}{t} \qquad \frac{su^2}{t} = -st$

$u^2 = -t^2 \rightarrow u = t = 0$

§2.7

Better proof [math stackexchange]

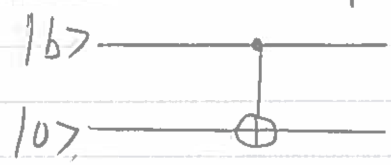
suppose $wz = 0$. but $w \neq 0$ & $z \neq 0$

then $w = 0/z = 0$, contradiction

$\square$

No cloning theorem

Consider a possible cloning circuit



$|b\rangle = |0\rangle$ or $|1\rangle$

$$\overset{\text{CNOT}}{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |00\rangle$$

$$[\text{CNOT}] \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = |11\rangle$$

Looks like CNOT can <u>clone</u> an input!

Sadly, no. <u>We should have:</u>

$$|\psi\rangle \oplus |\psi\rangle \text{ is } \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \otimes \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$= \begin{pmatrix} \alpha^2 \\ \alpha\beta \\ \alpha\beta \\ \beta^2 \end{pmatrix}$$

But CNOT gives

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \underbrace{\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\begin{pmatrix} \alpha \\ 0 \\ \beta \\ 0 \end{pmatrix}}$$

$$= \begin{pmatrix} \alpha \\ 0 \\ 0 \\ \beta \end{pmatrix} = \alpha |00\rangle + \beta |11\rangle$$

$$\begin{pmatrix} \alpha \\ 0 \\ 0 \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha^2 \\ \alpha\beta \\ \alpha\beta \\ \beta^2 \end{pmatrix}$$ only when $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

the classical states

Maybe some other circuit could create from

$$\begin{pmatrix} \alpha \\ \beta \\ \vdots \\ \vdots \end{pmatrix} \quad \begin{pmatrix} \alpha^2 \\ \alpha\beta \\ \alpha\beta \\ \beta^2 \\ \vdots \end{pmatrix}$$

ancilla

garbage of $n-4$

thm $\forall n \geq 2$

input $|\psi\rangle \otimes \underbrace{|0^{n-1}\rangle}$ state, $n-1$ qubits 2^{n-1} col vector entries!

output $|\psi\rangle \otimes |\psi\rangle \otimes \underbrace{f(|\psi\rangle)}_{n-2 \text{ qubits}}$

Suppose $\exists$ C that could compute the above with U as its unitary matrix

U must do the following

$$U(|0\rangle \otimes |0^{n-1}\rangle) = |00\rangle \otimes f(|0\rangle)$$

$$U(|1\rangle \otimes |0^{n-1}\rangle) = |11\rangle \otimes f(|1\rangle)$$

the matrix copies $|0\rangle$ and $|1\rangle$ faithfully, puts out $n-2$ garbage qubits

Let's try to copy $|+\rangle$

$$|+\rangle = 1/\sqrt{2}\,|0\rangle + 1/\sqrt{2}\,|1\rangle$$

$$= 1/\sqrt{2}\,\left(|0\rangle + |1\rangle\right)$$

Linearity of U

$$U\left(|+\rangle \otimes |0^{n-1}\rangle\right) = 1/\sqrt{2}\, U\left([|0\rangle + |1\rangle] \otimes |0^{n-1}\rangle\right)$$

$$1/\sqrt{2}\, U\left(|0\rangle|0^{n-1}\rangle\right) + 1/\sqrt{2}\, U\left(|1\rangle|0^{n-1}\rangle\right)$$

$$= 1/\sqrt{2}\left[\underbrace{|00\rangle \otimes f(|0\rangle)}_{\text{from before}}\right] + 1/\sqrt{2}\left[\underbrace{|11\rangle \times f(|1\rangle)}_{\text{from before}}\right]$$

If we measure now the first 2 qubits we see

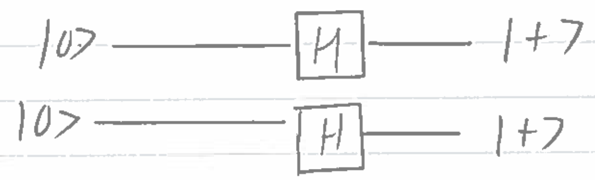
$|00\rangle$ prob $1/2$
$|11\rangle$ prob $1/2$

which is <u>NOT</u> a clone of $\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

$$= \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$$

So $|+\rangle$ cannot be cloned $\rightarrow$ general cloning impossible

You can create any state you want
(subject to rules governing states)
including 2 copies of $|+\rangle$



Circuit
$H \otimes H$

But we cannot clone some state
$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

Measuring it doesn't help - does not
reveal α or β

EPR pairs - entangled states

Consider Circuit HC

?

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$H\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$ - superpos of $|0\rangle$ & $|1\rangle$ equally likely

$$\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \otimes \overset{|0\rangle}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} = \begin{array}{cl} \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{pmatrix} & \begin{array}{l} -|00\rangle \\ -|01\rangle \\ -|10\rangle \\ -|11\rangle \end{array} \end{array}$$

$$\overset{\text{CNOT}}{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}} \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$$

$$= 1/\sqrt{2}\,|00\rangle + 1/\sqrt{2}\,|11\rangle$$

From two $|0\rangle$ inputs we obtain
2 qubits that are both $|0\rangle$
or both $|1\rangle$
with equal probability

→ measuring one implies what
you see by measuring the other
even if they are really far apart

EPR pair Einstein Podolsky Rosen
1935!

All other classical inputs to HC
$|01\rangle$ $|10\rangle$ $|11\rangle$ to
also produce Bell states

* What state results if

$|1\rangle$ are the inputs to HC
$|1\rangle$

Answer $\frac{1}{\sqrt{2}}\left[|01\rangle - |10\rangle\right]$

EPR Paradox ?

Perhaps we would like to believe

1) Reality principle physical quantities have physical reality independently of measurement

My pencil weighs 7 grams whether I measure it or not

2) Locality principle the result of a measurement on one system cannot instantaneously influence the result of measuring another system

5 light years

measure A

measure B

But

take $\frac{1}{\sqrt{2}}\left(\underbrace{|0}_{\text{Alice}}\overbrace{1\rangle + |1}^{\text{Bob}}0\rangle\right)$

Alice gets first qubit
Bob gets second qubit

They separate by a long distance

If Alice measures 1 in std basis
Bob will see 0 in std basis

Maybe the qubits agree they'll behave that way, somehow

Alice's qubit is left in the $|1\rangle$ state
Bob's in $|0\rangle$

But suppose Alice decides at the last moment to measure her bit in $|+\rangle$ $|-\rangle$ basis instead of $|0\rangle$ $|1\rangle$

* We can show

$$\frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right)$$

Recall $|+\rangle \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ $|-\rangle \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$

If Alice measures $|+\rangle$ on her qubit
Bob measures $|-\rangle$ in Hadamard basis

Bob is left with state $|0\rangle$ in one case
$|-\rangle$ in other

but these are very different states!

Bob's state depends on the measurement done by Alice!

* What would Bob see if
  Alice measures in $|+\rangle\ |-\rangle$
  Bob measures in $|0\rangle\ |1\rangle$?

Violates locality or reality. How can Alice's measurement change the state of Bob's system instantaneously?
PARADOX! - Is Q theory incomplete?
"God doesn't throw dice" Well yes, & Heisenberg says it must be so
Some ideas

1) Maybe the particles conspire before separating and create a list of scenarios & how they will act in each
But, infinite # of scenarios

2) Maybe there is some other hidden local variable they use
3) Maybe a global presence affects them

Quantum teleportation

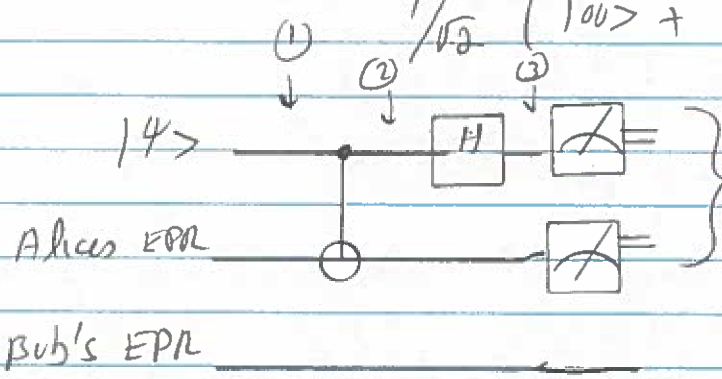
Alice has a qubit $|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$
she wants to send to Bob

She can't determine α_0 & α_1
& send those -
She'd have to measure, collapsing $|\psi\rangle$
& even then one measurement
doesn't reveal α_0 or α_1

If she knew $\alpha_1 = 1/\sqrt{2}$ that takes
unbounded precision - she can't transmit
$1/\sqrt{2}$ over a channel using bits.
There are uncountably many irrationals
between 0 & 1 so we can't hope
to name them all

<u>BUT</u>
Suppose Alice & Bob share
an EPR pair

$$\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$



Alice measures

① ② ③ are places of
analysis on following
pages

① At the start we have $|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$

$$|\psi\rangle \otimes \boxed{1/\sqrt{2}\left(|00\rangle + |11\rangle\right)}$$ — Alice + Bob's EPR pair

$$= \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \otimes 1/\sqrt{2}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 1/\sqrt{2}\begin{pmatrix} \alpha_0 \\ 0 \\ 0 \\ \alpha_0 \\ \alpha_1 \\ 0 \\ 0 \\ \alpha_1 \end{pmatrix}$$

$$= 1/\sqrt{2}\left(\alpha_0|000\rangle + \alpha_0|011\rangle + \alpha_1|100\rangle + \alpha_1|111\rangle\right)$$

START analysis of ②

CNOT $\otimes$ I

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \end{pmatrix} 1/\sqrt{2}\begin{pmatrix} \alpha_0 \\ 0 \\ 0 \\ \alpha_0 \\ \alpha_1 \\ 0 \\ 0 \\ \alpha_1 \end{pmatrix} = 1/\sqrt{2}\begin{pmatrix} \alpha_0 \\ 0 \\ 0 \\ \alpha_0 \\ 0 \\ \alpha_1 \\ \alpha_1 \\ 0 \end{pmatrix}$$ ②

$H \otimes I \otimes I$

$$\tfrac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes I$$

$$= \tfrac{1}{\sqrt{2}}\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&-1&0\\0&1&0&-1\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$= \tfrac{1}{\sqrt{2}}\begin{pmatrix}1&0&0&0&1&0&0&0\\0&1&0&0&0&1&0&0\\0&0&1&0&0&0&1&0\\0&0&0&1&0&0&0&1\\1&0&0&0&-1&0&0&0\\0&1&0&0&0&-1&0&0\\0&0&1&0&0&0&-1&0\\0&0&0&1&0&0&0&-1\end{pmatrix}$$

$$(H\otimes I\otimes I)\ \tfrac{1}{\sqrt{2}}\begin{pmatrix}\alpha_0\\0\\0\\\alpha_0\\0\\\alpha_1\\\alpha_1\\0\end{pmatrix} = \tfrac{1}{2}\begin{pmatrix}\alpha_0\\\alpha_1\\\alpha_1\\\alpha_0\\\alpha_0\\-\alpha_1\\-\alpha_1\\\alpha_0\end{pmatrix}$$ ③

$$= \tfrac{1}{2}\Big[\alpha_0|000\rangle + \alpha_1|001\rangle + \alpha_1|010\rangle + \alpha_0|011\rangle + \alpha_0|100\rangle - \alpha_1|101\rangle - \alpha_1|110\rangle + \alpha_0|111\rangle\Big]$$

Orig $|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$ / $|\alpha_0|^2 + |\alpha_1|^2 = 1$

Let's look at Alice's measurements (partial)

Suppose she sees $|00\rangle$ [Bob hasn't measured]
What are the chances?

$|000\rangle$ or $|001\rangle$

$$\left(\frac{|\alpha_0|}{2}\right)^2 + \left(\frac{|\alpha_1|}{2}\right)^2$$

$$= \frac{|\alpha_0|^2 + |\alpha_1|^2}{4} = 1/4$$

Collapsed state after partial measurement

$$\underbrace{|00\rangle}_{\text{prob } 1} \otimes \boxed{\quad ? \quad}$$

We must normalize $\boxed{\quad}$ so that its total probability is $\underline{1}$

From ☆ if Alice sees $|00\rangle$ on her bits the collapsed states are

$$\alpha_0|000\rangle + \alpha_1|001\rangle$$

$$= |00\rangle \otimes (\alpha_0|0\rangle + \alpha_1|1\rangle)$$

since $|\alpha_0|^2 + |\alpha_1|^2 = \underline{1}$

phone call

So, Alice tells Bob (classical channel) that she saw $|00\rangle$

Bob now has $\alpha_0|0\rangle + \alpha_1|1\rangle$ on his qubit

All possibilities from ☆

Alice sees			
0	$\|00\rangle$	$\frac{\|\alpha_0\|^2 + \|\alpha_1\|^2}{4} = 1/4$	$\|00\rangle \otimes (\alpha_0\|0\rangle + \alpha_1\|1\rangle)$
1	$\|01\rangle$	$\frac{\|\alpha_1\|^2 + \|\alpha_0\|^2}{4} = 1/4$	$\|01\rangle \otimes (\alpha_1\|0\rangle + \alpha_0\|1\rangle)$
2	$\|10\rangle$	$\frac{\|\alpha_0\|^2 + \|-\alpha_1\|^2}{4} = 1/4$	$\|10\rangle \otimes (\alpha_0\|0\rangle - \alpha_1\|1\rangle)$
3	$\|11\rangle$	$\frac{\|-\alpha_1\|^2 + \|\alpha_0\|^2}{4} = 1/4$	$\|11\rangle \otimes (-\alpha_1\|0\rangle + \alpha_0\|1\rangle)$

Case 0, Bob receives $\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = |\psi\rangle$

Case 1

Bob uses

$$\underset{\substack{\uparrow\\ X \text{ gate}\\ \text{"Not"}}}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \begin{pmatrix} \alpha_1 \\ \alpha_0 \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}$$

Case 2

Bob uses ?

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{Z \text{ gate}} \begin{pmatrix} \alpha_0 \\ -\alpha_1 \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}$$

Case 3

Bob uses ? $X \times Z$

$e^{i\pi}$ global phase doesn't matter

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -\alpha_1 \\ \alpha_0 \end{pmatrix} = -1 \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}$$

$$X \times Z \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Bob now has $|\psi\rangle$!

1) Why doesn't this break the speed of light barrier?

Why doesn't a separated EPR pair break the speed of light barrier

2) Why doesn't this break the NO CLONING theorem ?

For (1) - if Bob measures his EPR qubit pair before Alice does anything

He sees $|0\rangle$ or $|1\rangle$ equally likely

* and what does this do to Alice?

If Bob measures his qubit after Alice measures but before the phone call, from ☆, sees $|0\rangle$ w/ prob

$$\frac{1}{4}|\alpha_0|^2 + \frac{1}{4}|\alpha_1|^2 + \frac{1}{4}|\alpha_0|^2 + \frac{1}{4}|-\alpha_1|^2$$

$$= \frac{1}{2}\left(|\alpha_0|^2 + |\alpha_1|^2\right) = 1/2$$

Not w/ prob α_0 ! □

Teleportation has been demonstrated in practice

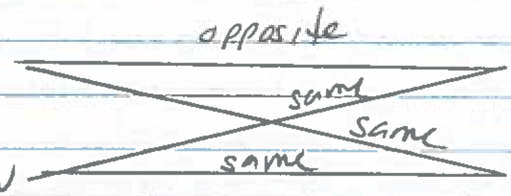
CHSH game
Clauser, Horn, Shimony, Holt

Game

Ref calls Red, Yellow — Alice says a

Ref calls Green, Orange — Bob says b

x				y
0	Red		Green	0
1	Yellow		Orange	1



$a, b \in$ {Solid, Dotted} or $\{0, 1\}$

Win iff $a \oplus b = x \wedge y$

	x	y	$x \wedge y$
same	0	0	0
"	0	1	0
"	1	0	0
diff	1	1	1

Alice + Bob initially meet, discuss strategy, then separate & can't communicate (classically) after then.

Refs call 0, 1 to each
2 refs, 1 ref doesn't matter
but each of the 4 calls is equally likely

Lemma

No classical strategy allows a win with probability greater than 3/4

<u>Proof</u>

Alice (+ Bob) has 4 possible strategies

$a = 0$ always respond this
$a = 1$
$a = x$ Respond x
$a = \bar{x}$ Respond $\bar{x}$

Same with Bob – 4 strategies, so:

	$a=0$	$a=1$	$a=x$	$a=\bar{x}$
$b=0$	3/4 same	1/4	3/4 Ⓐ so	1/4
$b=1$	1/4 diff	3/4	1/4 Ⓑ so	3/4
$b=y$	3/4 Ⓐ'	1/4	1/4	3/4
$b=\bar{y}$	so 1/4	3/4	3/4	1/4

Ⓐ 0, X

x	y	$x \wedge y$	a=x	b	WIN
0	0	0	0	0	✓
0	1	0	0	0	✓
1	0	0	1	0	X
1	1	1	1	0	✓

Ⓑ

x	y	$x \wedge y$	a=x	b	WIN
0	0	0	0	1	X
0	1	0	0	1	X
1	0	0	1	1	✓
1	1	1	1	1	X

For any deterministic strategy, the best they do is

$$3/4 \; !$$

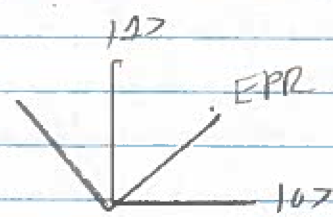
If they randomize their strategies that's just a distr over the deterministic ones, & so no better than 3/4

Even if they shared a random bit they can't do better than 3/4 ! } Hidden var theory

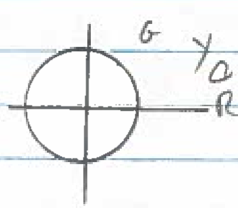
Suppose when they first meet they create an EPR pair and each keeps one of the bits

$$|\psi\rangle = \tfrac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

Idea: they agree to measure in a basis related to what the referee says



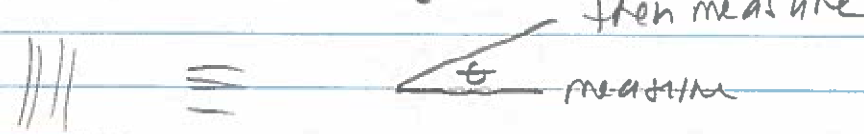
Arrange bases to be near each other when Alice & Bob must agree on their outcomes



R – O
O – Y
Y – G

} each $\pi/8$ radians ($22.5°$)

Recall from polarized light



$\cos^2\theta$ is the fraction of light that goes through

$\theta = 0$ 1
$\theta = \pi/2$ 0

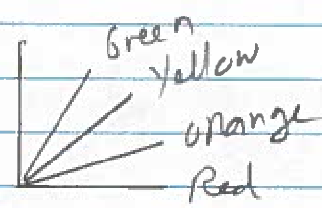
Generally $\cos^2\theta$ is the chance of agreement

Suppose Alice measures first, then Bob

Alice: Red or Yellow

Situations when they should agree

ALICE	BOB
Red	Orange
Yellow	Either Orange or Green



θ between each pair is $\pi/8$

$$\cos^2 \pi/8 \cong .853$$

So

Alice measures her EPR bit
in basis Red or Yellow dep on
referee call

Bob does the same

Each reports his / her measurement

* The measurement order doesn't matter

1st to measure collapses system in
that basis

2nd is likely to see the same result
w/ prob $\cos^2 \theta$

$\theta = \pi/8$ for the "should agree" choices

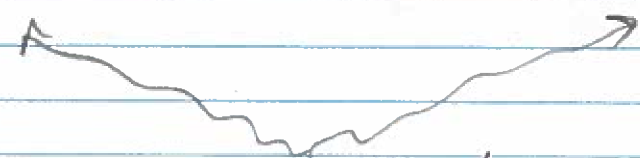
$\theta = \pi/2 - \pi/8 = 3\pi/8$ for should disagree

$\cos^2(3\pi/8) = 1 - .85 = .15$

They get the winning answer 85%
of the time — strictly better than
can be done classically

Bell's inequalities
Bounds on classical strategies

Alice Bob



Shared random bits
Hidden or local vars

Still they couldn't beat $3/4$ 75%

But sharing an EPR pair → 83% wins

→ There is something in QC that is better than (beyond) hidden variables

Math version CHSH / Bell

Alice has particle a Bob particle b

causally disconnected

randomly chose: can measure $Q = \pm 1$, $R = \pm 1$

can measure $S = \pm 1$, $T = \pm 1$

Look at $QS + RS + RT - QT$ ☆

$$= (Q+R)S + (R-Q)T$$

Q	R	$Q+R$	$R-Q$
-1	-1	-2	0
-1	1	0	2
1	-1	0	-2
1	1	2	0

either $Q+R=0$ or $R-Q=0$

When $Q+R=0$ $\quad R-Q=\pm 2$
When $R-Q=0$ $\quad Q+R=\pm 2$

S, T are ± 1, we have $\star = (\pm 2)(\pm 1)$

so $\star$ is ± 2 always

What the expected value

$$E(\star) = \sum_{q,r,s,t} p(q,r,s,t)(qs+rs+rt-qt)$$

$$\leq \sum_{q,r,s,t} p(q,r,s,t)\cdot 2$$

$$\leq 2 \cdot 1$$

$$\leq 2$$

Also $E(\star) = E(QS)+E(RS)+E(RT)-E(QT)$

So

$$E(QS)+E(RS)+E(RT)-E(QT) \leq 2$$

Bell's inequality / CHSH inequality

Suppose instead Alice + Bob receive a qubit each of

$$|\psi\rangle = 1/\sqrt{2}\,(|00\rangle + |11\rangle) \quad \text{EPR}$$

Alice measures using

$$R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z$$

OR

$$Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X$$

Bob measures using

$$S = 1/\sqrt{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = H$$

OR

$$T = 1/\sqrt{2}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

If a state $|\psi\rangle$ is measured using A the expected outcome is $\langle\psi|A|\psi\rangle$

Quantum $E(\cancel{A})$ becomes

$$1/4\,\langle\psi|\,Q\otimes S + R\otimes S + R\otimes T - Q\otimes T\,|\psi\rangle$$

Look at

$$\langle \psi | R \otimes S | \psi \rangle$$

$$R \otimes S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes 1/\sqrt{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= 1/\sqrt{2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

$$\langle \psi | = 1/\sqrt{2} \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$$ EPR

$$\langle \psi | R \otimes S = 1/2 \begin{pmatrix} 1 & 1 & -1 & 1 \end{pmatrix}$$

$$\langle \psi | R \otimes S | \psi \rangle = 1/2 \begin{pmatrix} 1 & 1 & -1 & 1 \end{pmatrix} 1/\sqrt{2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= 1/2 \, 1/\sqrt{2} \, (2) = 1/\sqrt{2}$$

Others work out similarly

$$\langle \psi | Q \otimes S | \psi \rangle = 1/\sqrt{2}$$

$$\langle \psi | R \otimes S | \psi \rangle = 1/\sqrt{2}$$

$$\langle \psi | R \otimes T | \psi \rangle = 1/\sqrt{2}$$

$$\langle \psi | Q \otimes T | \psi \rangle = -1/\sqrt{2}$$

Quantum E(☆)

$$\left[1/\sqrt{2} + 1/\sqrt{2} + 1/\sqrt{2} - \left(-1/\sqrt{2} \right) \right]$$

$$= 4/\sqrt{2} = 2\sqrt{2}$$

Violates Bell's inequality

Experiments have verified this!
Repeatedly randomly pick

Q or R — Alice

S or T — Bob

to empirically evaluate Quantum E(☆)
it does tend to $2\sqrt{2}$

Something about Bell must be wrong

Local realism assumption

Must not hold

Either or both assumptions must go

Entanglement brings possibilities beyond classical notions of computing