In HW you show that every pair of antipodal points on the Bloch Sphere are orthogonal

Each such pair forms an orthonormal basis for any quantum state

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

are the standard pair

$$|+\rangle = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \qquad |-\rangle = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

also work etc

Any state $|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

can be written uniquely in $|0\rangle$ $|1\rangle$

as $\alpha|0\rangle + \beta|1\rangle$

but also uniquely in $|+\rangle$ $|-\rangle$

as $$\alpha' \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} + \beta' \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

by solving (change of basis)
$$\alpha' + \beta' = \sqrt{2}\,\alpha$$
$$\alpha' - \beta' = \sqrt{2}\,\beta$$

For example

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} \alpha' + \beta' &= \sqrt{2} \\ \alpha' - \beta' &= 0 \end{aligned}\right\} \quad 2\alpha' = \sqrt{2}$$

$$\alpha' = 1/\sqrt{2} \qquad \beta' = 1/\sqrt{2}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1/\sqrt{2} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} + 1/\sqrt{2} \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

$$|0\rangle = 1/\sqrt{2}\,|+\rangle + 1/\sqrt{2}\,|-\rangle$$

Generally if $\{|v_j\rangle\}$ is a set of such basis vectors

We say that set is complete

We can uniquely express vector $|x\rangle = \sum_j \underbrace{x_j}_{\text{complex value}} \underbrace{|v_j\rangle}_{\text{basis vector}}$

where x_j is computed as

$$x_j = \underbrace{\langle v_j | x \rangle}_{\text{inner product}}$$

How much is x like v_j?

Recall

$\langle v_j|$ is bra of $|v_j\rangle$

In the standard basis

$$v_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$ these simply select the right element of x

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha$$

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \beta$$

& so it's no surprise

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Otherwise consider

$$\langle + | 0 \rangle = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = (\alpha + \beta)/\sqrt{2}$$

$$\langle - | 0 \rangle = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad (\alpha - \beta)/\sqrt{2}$$

$$\text{So } \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{(\alpha + \beta)}{\sqrt{2}} |+\rangle + \frac{(\alpha - \beta)}{\sqrt{2}} |-\rangle$$

Consider

$|0\rangle\langle 0|$ the outer product

$$\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$$

$|1\rangle\langle 1|$

$$\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$$

these sum to $I=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$

$|+\rangle\langle +|$

$$\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}\begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\\frac{1}{2} & \frac{1}{2}\end{pmatrix}$$

$|-\rangle\langle -|$

$$\begin{pmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{pmatrix}\begin{pmatrix}1/\sqrt{2} & -1/\sqrt{2}\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & -1/2\\-1/2 & 1/2\end{pmatrix}$$

these sum to $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$

More generally

$$\sum_j |v_j\rangle\langle v_j| = I$$

Properties of Unitary matrices

The following conditions are equivalent

1) U is unitary
2) U^* is unitary
3) $U^*U = UU^* = I$
4) The columns of U form an orthonormal basis of C^n
5) So do the rows

orthonormal: Each vector is a unit vector $|v_j| = 1$
and each is orthogonal to all others

[THX Arthur Rattew]

Universal 3-param quantum gate theorem

Up to a global phase any 2×2 unitary matrix can be expressed as

$$U3(\theta, \phi, \lambda) =$$

$$\begin{pmatrix} \cos \theta/2 & -e^{i\lambda} \sin(\theta/2) \\ e^{i\phi} \sin(\theta/2) & e^{i(\phi+\lambda)} \cos(\theta/2) \end{pmatrix}$$

Ex $\quad 1/\sqrt{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$\theta = \pi/2 \qquad \lambda = \pi$

$\phi = 0$

yields $\begin{pmatrix} \cos \pi/4 & -e^{-i\pi} \sin \pi/4 \\ e^{i0} \sin \pi/4 & e^{i\pi} \cos \pi/4 \end{pmatrix}$

Proof of the Theorem

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

so $$U^* = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$$

Columns of U are orthonormal

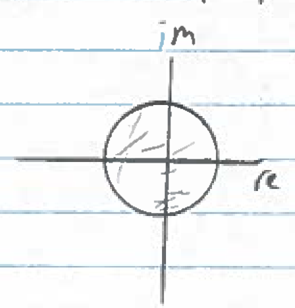
Normality

$$a^*a + c^*c = |a|^2 + |c|^2 = 1$$
$$b^*b + d^*d = |b|^2 + |d|^2 = 1$$

Columns of U^* are orthonormal (rows of U)

$$|a|^2 + |b|^2 = 1$$
$$|c|^2 + |d|^2 = 1$$

Consider a – some complex number such that $|a| \le 1$



a is a point on or in unit circle

It has some magnitude $r_a \le 1$

$e^{i\phi_a}$ point on circle

$\times \cos(\theta/2)$ to scale it

[$\theta/2$ helps later]

$$\boxed{a = e^{i\phi_a} \cos \theta/2}$$

then

$$\left| e^{i\phi_a} \cos(\theta/2) \right|^2 + |b|^2 = 1$$

$$|b|^2 = \sin^2 \theta/2$$

So

$$b = e^{i\phi_b} \sin \theta/2$$

$$c = e^{i\phi_c} \sin \theta/2$$

$$d = e^{i\phi_d} \cos \theta/2$$

End of Normality   5 parameters
$\phi_a \; \phi_b \; \phi_c \; \phi_d \; \theta$

Orthogonality of columns in U

$$(a^* \;\; c^*) \begin{pmatrix} b \\ d \end{pmatrix} = 0$$

$$a^* b + c^* d = 0 \qquad [1]$$

Recall conjugate of e^{ix} is e^{-ix}

Plug a, b, c, d into [1]

$$e^{-i\phi_a} \cos\theta/2 \; e^{i\phi_b} \sin\theta/2$$
$$+ \; e^{-i\phi_c} \cos\theta/2 \; e^{i\phi_d} \sin\theta/2 = 0$$

$$\cos\theta/2 \; \sin\theta/2 \left[e^{i(\phi_b - \phi_a)} + e^{i(\phi_d - \phi_c)} \right] = 0$$

θ needs to be arbitrary so we then require

$$e^{i(\phi_b - \phi_a)} = -e^{i(\phi_d - \phi_c)}$$
$$= e^{i\pi} e^{i(\phi_d - \phi_c)}$$
$$= e^{i(\phi_d - \phi_c + \pi)}$$

$$\phi_d = \phi_b + \phi_c - \phi_a - \pi$$
$$= \phi_b + \phi_c - \phi_a - \pi + 2\pi$$
$$= (\phi_b + \pi) + \phi_c - \phi_a$$

Produces

$$\begin{pmatrix} e^{i\phi_a}\cos\theta/2 & e^{i\phi_b}\sin\theta/2 \\ e^{i\phi_c}\sin\theta/2 & e^{i(\phi_b+\pi+\phi_c-\phi_a)}\cos\theta/2 \end{pmatrix}$$

Factor out $e^{i\phi_a}$ as global phase

$$= e^{i\phi_a}\begin{pmatrix} \cos\theta/2 & e^{i(\phi_b-\phi_a)}\sin\theta/2 \\ e^{i(\phi_c-\phi_a)}\sin\theta/2 & e^{i(\phi_b+\pi+\phi_c-2\phi_a)}\cos\theta/2 \end{pmatrix}$$

let $\phi = \phi_c - \phi_a$

$\lambda = \phi_b - \phi_a - \pi$

$a' = \cos\theta/2$

$b' = e^{i\pi} e^{i\lambda}\sin\theta/2$

$= -e^{i\lambda}\sin\theta/2$

$c' = e^{i\phi}\sin\theta/2$

$$d' = e^{i(\phi_b + \pi + \phi_c - 2\phi_a)}$$

$$= e^{i(\lambda + \phi + 2\pi)} \cos \theta/2$$

$$= e^{i(\lambda + \phi)} \cos \theta/2$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = e^{i\phi_a} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

$$= e^{i\phi_a} \begin{pmatrix} \cos \theta/2 & -e^{\lambda} \sin \theta/2 \\ e^{i\phi} \sin \theta/2 & e^{i(\lambda+\phi)} \cos \theta/2 \end{pmatrix}$$

$\square$