The Cosmic Speed Limit

Preamble

This discussion assumes that you are already familiar with the Special Relativity concepts of time dilation and length contraction:

$$t = \frac{t_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$l = l_0\sqrt{1 - \frac{v^2}{c^2}}$$

Einstein reached these conclusions in an attempt to reconcile Maxwell's equations, which predict a constant velocity for electromagnetic waves in a vacuum:

$$c = \sqrt{\frac{1}{\mu_0 \varepsilon_0}} = 3 \times 10^8 \text{ m s}^{-1}$$

with Galileo's Principle of Relativity, which in a nutshell, states that all motion is relative. More formally, the Principle of Relativity states there are no special or absolute frames of reference. Fundamental to the Principle of Relativity is Newton's conclusion that in all inertial frames of reference (ie constant velocity) it is impossible to do an experiment to determine if you are moving. This may seem absurd and you could argue that all you have to do is look out the window and you can tell whether you are moving but, if you think more deeply about it, there is actually no way to prove that you are moving relative to the landscape or whether the landscape is moving relative to you.

Relativistic Momentum

One of the consequences of the Principle of Relativity is that your location should make no difference to the outcome of your experiment. Let's say you are measuring the results of collisions between balls. A lab in Melbourne should measure the same results as a lab in Sydney. Nor should your direction matter, if you turn your bench through 90 degrees you should get the same result. In fact, if you were to put your lab on a train with perfectly smooth, straight tracks and wheels, and travel at a constant velocity, then you should also get the same results no matter what velocity you are travelling at. This idea is called 'translational symmetry', and we can prove, using a wonderful piece of mathematics called Noether's Theorem which is sadly beyond the scope of High School Physics, that wherever there is translational symmetry, momentum must be conserved.

Now if you sit down and do the maths for momentum at relativistic velocities you will find that the equation $p = mv$ is not sufficient to conserve momentum. In order to conserve momentum at relativistic velocities we need to generalise the momentum to:

$$p = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Note that for everyday velocities, where v is much smaller than c, the denominator approaches 1 and we recover the Newtonian equation $p = mv$.

Total Energy

We can use the equation for momentum to derive an expression for the total energy of a particle by considering what happens if we apply a force to accelerate a particle from rest over a given distance. Unfortunately this analysis requires calculus and so it is beyond the scope of non-calculus courses, but for those who are interested and have a good knowledge of differentiation rules and standard integrals, I have included an appendix at the end of this discussion. However, the end result of such an analysis is Einstein's general equation for total energy of a particle:

$$E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Which, with a bit of substitution and rearranging can be written as:

$$E^2 = (mc^2)^2 + (pc)^2$$

This, not the equation you read on t-shirts and coffee cups, is the full version of Einstein's energy equation, and hidden within this equation are two amazing insights that changed our understanding of physics. The first is that all objects have a rest energy that is related to their mass. The second is that even particles without mass can have energy related to their momentum.

Rest Energy

At rest a particle has no momentum. If we substitute p = 0 into the equation above, we can simplify it to the t-shirt form of Einstein's equation that most people are familiar with:

$$E = mc^2$$

This is more correctly written as:

$$E_0 = mc^2$$

where E_0 indicates that we are talking about rest energy[1]. This equation implies that it is possible to convert mass into an enormous amount of energy (c^2 is a staggeringly big number), a conclusion which led to us understanding what fuels the stars and also led to nuclear power and atomic bombs. It also leads to the amazing insight that what we call mass is actually a measure of the rest energy of a particle:

$$m = \frac{E_0}{c^2}$$

Massless particles

For particles without mass, the general equation for energy simplifies to:

$$E = pc$$

So that massless particles have energy related to their momentum, but more astounding stuff on massless particles later.

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[1] Einstein consistently related the "rest energy" of a system to its invariant inertial mass, and rejected the idea of mass dilation. Hecht, E. (2009) 'Einstein on mass and energy', *American Journal of Physics*, 77(9), pp. 799–806. doi: 10.1119/1.3160671.

Cosmic speed Limit

If we look again at the equations for relativistic momentum and total energy of particles:

$$p = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$E^2 = (mc^2)^2 + (pc)^2$$

we come to another very important and mind-blowing conclusion. As the velocity of a particle approaches c, the denominator of the momentum equation approaches 0 and so the momentum approaches infinity. As the momentum of the particle approaches infinity, the energy also approaches infinity. So to accelerate a particle with mass to c would require an infinite amount of energy, which is something the Universe simply doesn't have. We must therefore conclude that no particle with mass can have a velocity equal to or greater than c. In other words c is a cosmic speed limit!

Massless particles must travel at c

Looking at the equation for momentum we might reasonably conclude that for an object with zero mass the numerator is 0 so the momentum must be zero. However, in the special case where v = c, the denominator is also 0 so the momentum formula gives 0 divided by 0 which is undefined.

By undefined, we mean that mathematically there is not enough information to solve the equation so we are not able to say the result is zero or any other value. In this case, p = 0/0 which can be rewritten as 0p = 0 and because all values of p satisfy this equation we can't use it to determine the actual value of p. In summary, in the special case of m = 0 and v = c, we can't work out what p is but we can't justifiably conclude it is 0.

We can apply the similar arguments to the equation for total energy:

$$E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

When m = 0 and v = c the solution to this equation is also undefined, E = 0/0. Again, this relationship doesn't allow us to work out what the energy of a massless particle is but it also doesn't allow us to conclude the energy is 0.

If we can't say anything defined about the momentum or energy of a massless particle then none of this seems very useful. Fortunately, mathematics allows us a little trick to get around this impasse. We can look at their ratios. If we divide E by p we get:

$$\frac{E}{p} = \frac{c^2}{v}$$

(go ahead, do the maths) and for the special case where v = c:

$$E = pc$$

Let's reconsider what all of this means. Any 'particle' with zero mass travelling at less than the speed of light will have zero momentum and zero energy and therefore will be unable to interact with anything. Such a particle may as well not exist and we can ignore it. However, in the special case of a massless particle with velocity c we are able to assign meaningful values of energy and momentum, so we can't ignore their potential to interact with the universe. In other words Special Relativity allows for the existence of massless particles but only if they travel at c.

Speed of light

The exchange particle for light and all other electromagnetic radiation is the photon. It is, as far as we can tell, massless and therefore must travel at c. The first reasonably accurate measurement of the speed of light was made by Hippolyte Fizeau in 1849, well before anybody had heard of exchange particles, Special Relativity, or other massless particles such as gluons. Because the cosmic speed limit was first glimpsed through the study of light, the association of c with the Speed of Light is historically understandable and we are probably stuck with it, however c is so much more that just the speed of light, it is a cosmic speed limit for particles with mass, and a mandatory speed for massless particles.

Appendix: Derivation of the energy equations involving some pretty ugly maths – not for the faint hearted

We start with relativistic momentum

$$p = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}}$$

And the recognition that force results in a change of momentum over time:

$$F = \frac{dp}{dt} = \frac{d}{dt}\frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Now we have to set up some relationships that will allow us to apply chain rules and other mathemagic to this problem. Pay attention and mind the step. Let:

$$i = mv$$

$$k = 1 - \frac{v^2}{c^2}$$

$$j = \sqrt{1 - \frac{v^2}{c^2}} = \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} = k^{\frac{1}{2}}$$

So

$$F = \frac{i}{j}\frac{d}{dt} = \frac{\frac{di}{dt}j - \frac{dj}{dt}i}{j^2}$$

So now let’s work out some of these derivatives:

$$\frac{dv}{dt} = a$$

$$\frac{di}{dv} = m$$

$$\frac{di}{dt} = \frac{di}{dv} \times \frac{dv}{dt} = ma$$

$$\frac{dk}{dv} = \frac{-2v}{c^2}$$

$$\frac{dj}{dk} = \frac{1}{2}k^{-\frac{1}{2}} = \frac{1}{2}\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$

$$\frac{dj}{dt} = \frac{dj}{dk} \times \frac{dk}{dv} \times \frac{dv}{dt} = \frac{1}{2}\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \times \frac{-2v}{c^2} \times a$$

Phew…now let's substitute back into the equation for F:

$$F = \frac{ma\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} - mv\frac{1}{2}\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \times \frac{-2v}{c^2} \times a}{1 - \frac{v^2}{c^2}}$$

Yuk…that is ugly. Let's multiply all the terms by $\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}$

$$F = \frac{ma\left(1 - \frac{v^2}{c^2}\right) + ma\frac{v^2}{c^2}}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}} = \frac{ma}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}}$$

Now if we apply a force to an object over a distance we do work. So:

$$W = \int F\, dx = \int \frac{ma}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}} dx$$

And

$$a\, dx = \frac{dv}{dt} dx = \frac{dx}{dt} dv = v\, dv$$

So

$$W = \int \frac{mv}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}} dv$$

Conservation of energy allows us to consider kinetic energy, K, as the work necessary to accelerate an object from rest to a velocity v.

$$K = W_{0\to v} = \int_0^v \frac{mv}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}dv$$

Which we can rewrite as:

$$K = m\int_0^v v\left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}} dv$$

Now we have to resort to standard integrals, particularly this one:

$$\int x(ax^2+b)^n\,dx = \frac{1}{2a}\frac{(ax^2+b)^{n+1}}{n+1}$$

So our equation for K becomes:

$$K = \; m\left[\frac{1}{2}\cdot\frac{-c^2}{1}\cdot\frac{\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}}{-\frac{1}{2}}\right]_0^v$$

$$K = m\left[c^2\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}\right]_0^v$$

$$K = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} - mc^2$$

Now here comes the tricky bit. We re-arrange this equation to :

$$\frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} = K + mc^2$$

And this can be interpreted as indicating that a particle has some total energy (the left hand side) that is a combination of kinetic energy plus some other energy that is independent of motion (the right hand side).

The total energy is therefore:

$$E = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

Really we should stop there, but if you have made it this far you are a mathemasochist so let's keep going and derive the other version of the energy equation.

We can square both sides of the energy equation then rearrange it to:

$$\frac{E^2}{m^2c^4} = \frac{1}{1 - \frac{v^2}{c^2}}$$

And we can do a similar job on the momentum equation to get:

$$\frac{p^2}{m^2c^2} = \frac{\frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}}$$

Now if we subtract these equations from each other we get:

$$\frac{E^2}{m^2c^4} - \frac{p^2}{m^2c^2} = \frac{1}{1 - \frac{v^2}{c^2}} - \frac{\frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}}$$

$$\frac{E^2}{m^2c^4} - \frac{p^2}{m^2c^2} = \frac{1 - \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}}$$

$$\frac{E^2}{m^2c^4} - \frac{p^2}{m^2c^2} = 1$$

$$\frac{E^2}{m^2c^4} = 1 + \frac{p^2}{m^2c^2}$$

$$E^2 = m^2c^4 + \frac{p^2m^2c^4}{m^2c^2}$$

$$E^2 = m^2c^4 + p^2c^2$$

$$E^2 = (mc^2)^2 + (pc)^2$$

Addendum: How can massless particles have momentum?

here is a quick explanation. We normally associate momentum with mass: $p = mv$, but a quick thought experiment (actually you can do the physical experiment) demonstrates that waves have momentum too. Imagine sending a wave down a slinky which is touching a ball. As the wave passes the ball it will transfer some energy to the ball which will start moving. The ball takes some of the wave's energy as kinetic energy. But now the ball also has momentum, which must have come from somewhere because momentum is conserved. Yep, you guessed it it came from the wave - so waves transfer momentum even though waves don't transfer mass. If you have a spare couple of days, you can actually use the principles of conservation of momentum and conservation of energy to show that the momentum of a wave is its energy divided by its velocity: $p = E/v$. Of course this can be rearranged to give $E = pv$ - look familiar? Now as you know, when we get down to the level of fundamental 'particles' the distinction between their particle-like properties and their wave-like properties becomes blurred. You can think of momentum as one of the wave-like properties of a massless 'particle'. Does that help?