

The Bootstrap and its Application in Signal Processing

An Attractive Tool for Assessing the Accuracy
of Estimators and Testing Hypothesis for Parameters
in Small Data-Sample Situations

The bootstrap is a powerful technique for assessing the accuracy of a parameter estimator in situations where conventional techniques are not valid. In this article we highlight the motivations for using the bootstrap in typical signal-processing applications and we give several practical examples. Bootstrap methods for testing statistical hypotheses are described and we provide an analysis of the accuracy of bootstrap tests. We also discuss how the bootstrap can be used to estimate a variance-stabilizing transformation to define a pivotal statistic, and we demonstrate the use of the bootstrap for constructing confidence intervals for flight parameters in a passive acoustic emission problem.

Development of the Bootstrap

In many signal-processing applications one is interested in forming estimates of a certain number of unknown parameters of a random process, using a set of sample values. Further, one is interested in finding the sampling distribution of the estimators, so that the respective means, variances, and cumulants can be calculated, or in making some kind of probability statements with respect to the unknown true values of the parameters. For example, one could be interested in assigning two limits to a certain parameter, and in asserting that, with some specified probability, the true value of the parameter will be situated between these limits, which constitute the so-called confidence interval [13].

Most techniques for computing variances of parameter estimators or for setting confidence intervals for the



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true parameters assume that the size of the available set of sample values is sufficiently large, so that “asymptotic” results can be applied. However, in most signal-processing problems this assumption cannot be made either because of time constraints or because the process is nonstationary and only small portions of stationary data are considered. Thus, often in practice, large sample methods are inapplicable.

The bootstrap was introduced by Efron [15-18] as an approach to calculate confidence intervals for parameters in circumstances where standard methods cannot be applied (see also [20]). An example of this would be one in which few data are available, so that approximate large sample methods are inapplicable. The bootstrap has subsequently been used to solve many other problems that would be too complicated for traditional statistical analysis [21, 32].

In simple words, the bootstrap does with a computer what the experimenter would do in practice, if it were possible: he or she would repeat the experiment. With the bootstrap, the observations are randomly reassigned, and the estimates recomputed. These assignments and recomputations are done thousands of times and treated as repeated experiments.

The bootstrap is an extremely attractive tool in that it requires very little in the way of modeling, assumptions, or analysis, and it can be applied in an automatic way. The bootstrap is essentially a computer-based method that substitutes considerable amounts of computation in place of theoretical analysis. In an era of exponentially declining computational costs, such computer-intensive methods are becoming increasingly attractive.

To illustrate the importance of the bootstrap in a signal-processing context, consider the problem of estimating the spectral density of a stationary random signal. The two main questions asked are: (1) what estimator should be used?, and (2), having decided to use a particular estimator, how accurate is it? The bootstrap is a methodology for answering the second question with very little assumption; for example, it does not assume that a large number of observations of the signal is available so as to use large sample results. Recent research has also been devoted to question (1), i.e., the choice of an estimator among a family of estimators using the bootstrap [43].

Applications of bootstrap methods to real-life problems have been reported in radar signal processing [48, 49], sonar signal processing [6, 41, 45, 59], geophysics [25-27, 63], biomedical engineering [38, 2], control [14], atmospheric environmental research [36], and vibration analysis [71]. In all these fields, bootstrap methods have been used to approximate the distribution of an estimator or some of its characteristics.

Nagaoka and Amai discuss in [48, 49] an application in which the distribution of the estimated “close approach probability” is derived to be used as an index of collision risk in air traffic control. In [6], Böhme and Maiwald apply bootstrap procedures to signal detection and location

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using sensor arrays in passive sonar. In [45] the authors also analyze seismic data using the bootstrap. Krolik et al. [41] use bootstrap methods for evaluating the performance of source localization methods on real sensor array data without precise a priori knowledge of true source positions and the underlying data distribution. In [25-27] Fisher and Hall apply the bootstrap to the problem of deciding whether or not paleomagnetic specimens sampled from a folded rock surface were magnetized before or after folding occurred. They conclude that the bootstrap method provides the only feasible approach in this common paleomagnetic problem. Another application in paleomagnetism has been reported in [63]. In [38], Haynor and Woods use the bootstrap for estimating the regional variance in emission tomography images. Banga and Ghorbel [2] introduce a bootstrap sampling scheme to remove the dependence effect of pixels in retina images. Another application can be found in [36], where Hanna uses the related jackknife procedure and the bootstrap for estimating the confidence limits for air-quality models.

Dejian and Guanrong [14] apply bootstrap techniques for estimating the distribution of the Lyapunov exponent of an unknown dynamic system using its time-series data. Zoubir and Böhme [71] apply bootstrap techniques to construct multiple hypotheses tests for finding optimal sensor locations for knock detection in spark ignition engines. Bootstrap techniques have been also applied to nonstationary data in [73] where Zoubir et al. use the bootstrap to determine confidence bounds for the instantaneous frequency. More signal-processing applications of the bootstrap can be found in [60].

Recently, bootstrap techniques were also applied in the area of artificial neural networks. Tibshirani [66] discusses a number of methods for estimating the standard error of predicted values from a multilayered perceptron. He found that the bootstrap methods perform best, partly because they capture variability due to the choice of starting weights. Bhide et al. [3] demonstrate the use of bootstrap methods also in the context of an artificial neural network to estimate a distillation process bottoms' composition.

This list is by no means complete and does not include applications of the jackknife, such as the work by Thomson and Chave [64], where the authors approximate confidence intervals for spectra, coherences, and transfer functions for diverse geophysical data.

Theoretical and practical work have shown that bootstrap methods are potentially superior to large-sample techniques. A danger, however, exists in that the practi-

titioner may well be attracted to applying bootstrap techniques in some circumstances where standard approaches that invoke strong assumptions are judged inappropriate—in such circumstances the bootstrap may also fail [67]. Special care is therefore required when applying the bootstrap in real-life situations [52]. This article provides the fundamental concepts and methods needed by the signal-processing practitioner to decide when and how to apply the bootstrap successfully. The theoretical basis of the bootstrap, its assumptions, and pitfalls are provided in Politis' companion article that appears in this issue [52]. This article considers the independent data bootstrap only and we shall modify the bootstrap procedure to cater to dependent data models as is often done in practice [29, 72, 74]. The dependent data bootstrap is omitted here but details and some applications can be found in [7, 11, 12, 42, 50, 52, 53, 54, 68]. The reader interested in only real-life application is advised to first read through the Principle and Variance Estimation subsections of the Hypothesis Testing section, Example 3, and the Variance Stabilization section. Those specialists who are interested in the theoretical aspects of the bootstrap are encouraged to read the examples that are a natural follow up of Politis' article.

Bootstrap Methods

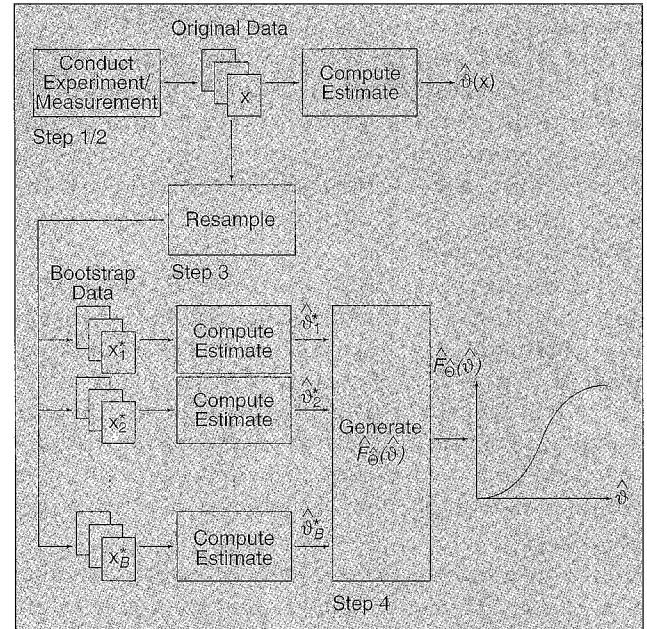
Basic Principle

Let $X = \{X_1, X_2, \dots, X_n\}$ be a sample, i.e., a collection of n numbers drawn at random from a completely unspecified distribution, F . By “at random” it is meant that the X_i 's are *independent* and *identically distributed* random variables, each having distribution F . Let ϑ denote an unknown characteristic of F , such as its mean or variance. The problem considered in this article is to find the distribution of $\hat{\vartheta}$, an estimator of ϑ , derived from the sample X . This is of great practical importance as we need to infer ϑ based on $\hat{\vartheta}$. For example, in a spectral estimation problem, we could be interested in testing whether the spectral density at a given frequency is zero or whether it exceeds a certain bound from the estimate constructed from the observations of the stationary process.

A way to obtain the distribution of $\hat{\vartheta}$ or its characteristics is to repeat the experiment a sufficient number of times and approximate the distribution of $\hat{\vartheta}$ by the so obtained empirical distribution. In many practical situations, this method is inapplicable for cost reasons or because the experimental conditions are not reproducible.

The bootstrap paradigm suggests that we resample from a distribution chosen to be close to F in some sense, for example, the sample (or empirical) distribution, \hat{F} , that approaches F as $n \rightarrow \infty$. The bootstrap principle is illustrated in Table 1 and in Fig. 1.

Note that the choice of \hat{F} is not unique; any distribution that approaches F as $n \rightarrow \infty$ can be used. This is of special interest if one has partial information on F . For example, if F is known to be the normal distribution with

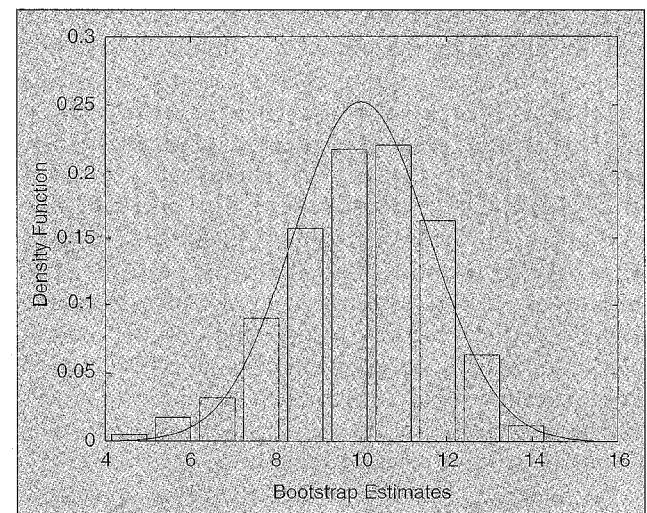


▲ 1. The bootstrap principle for estimating a distribution function.

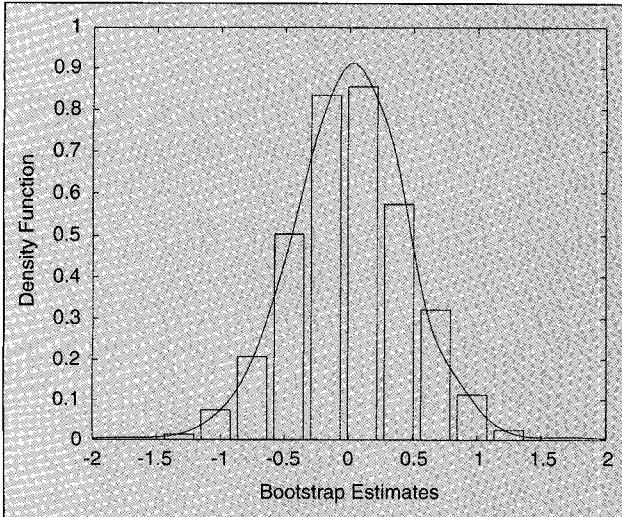
unknown mean μ and variance σ^2 , then we would draw a resample of size n from the normal distribution with mean $\hat{\mu}$ and variance $\hat{\sigma}^2$ where $\hat{\mu}$ and $\hat{\sigma}^2$ are estimated from x . With this method, known as the *parametric bootstrap* [21, 32], one hopes to improve upon the nonparametric bootstrap in which the resample, $x^* = \{X_1^*, X_2^*, \dots, X_n^*\}$, is an unordered collection of n items drawn randomly from x with replacement, so that each X_i^* has probability n^{-1} of being equal to any one of the X_j 's,

$$P(X_i^* = X_j | x) = n^{-1}, \quad 1 \leq i, j \leq n.$$

That is, the X_i^* 's are independent and identically distributed, conditional on x , with this distribution [32]. This



▲ 2. Histogram of $\hat{\mu}_1^*, \hat{\mu}_2^*, \dots, \hat{\mu}_{1000}^*$ based on the random sample $x = \{-2.41, 4.86, 6.06, 9.11, 10.20, 12.81, 13.17, 14.10, 15.77, 15.79\}$. The solid line indicates the probability density function of a Gaussian variable with mean 10 and variance 2.5.



▲ 3. Histogram of 1000 bootstrap estimates of the mean of the t -distribution with four degrees of freedom. The solid line indicates the kernel probability density function obtained from 1000 Monte Carlo simulations.

means that X is likely to contain repeats. For example, when $n = 4$ the collection $x^* = \{1.5, 1.7, 1.7, 1.8\}$ should not be mistaken for the set $\{1.5, 1.7, 1.8\}$ and, of course, x^* is the same as $\{1.5, 1.7, 1.8, 1.7\}, \{1.7, 1.5, 1.8, 1.7\}$, etc. [21, 32]. In all the applications presented here, we shall use the nonparametric bootstrap.

Examples

Example 1. Confidence interval for the mean

Let X_1, \dots, X_n be n independent and identically distributed random variables from some unknown distribution, and suppose we wish to find an estimator and a $(1 - \alpha)100\%$ interval for the mean μ . Usually, we estimate μ by the sample mean

$$\hat{\mu} = \frac{X_1 + \dots + X_n}{n}.$$

A confidence interval for μ can be found by determining the distribution of $\hat{\mu}$ (over repeated samples of size n from the underlying distribution), and finding values $\hat{\mu}_L, \hat{\mu}_U$ such that

$$P(\hat{\mu}_L \leq \mu \leq \hat{\mu}_U) = 1 - \alpha.$$

However, the distribution of $\hat{\mu}$ depends on the distribution of the X_i 's, which is unknown. In the case where n is large the distribution of $\hat{\mu}$ could be approximated by the normal distribution as per the central limit theorem [39, 46], but such an approximation is not valid in applications where n is small.

The bootstrap paradigm suggests that we assume that the sample $X = \{X_1, \dots, X_n\}$ itself constitutes the underlying distribution; then by resampling from X many times and computing $\hat{\mu}$ for each of these resamples, we get a bootstrap distribution for $\hat{\mu}$ that approximates the actual distribution of $\hat{\mu}$, and from which a confidence interval for μ is derived. This procedure is described in Table 2, where a sample of size 10 is taken from the normal distribution with mean $\mu = 10$ and variance $\sigma^2 = 25$. The same data and algorithm of Table 2 were used with other α values. We found the 99% confidence interval to be $(4.72, 14.07)$ and with only $N = 100$ resamples, we found $(7.33, 12.93)$ to be the 90% confidence interval.

We also ran the algorithm of Table 2 using a random sample of size 10 from the t -distribution with four degrees of freedom. The histogram of the bootstrap estimates so obtained is shown in Fig. 3. As the theoretical fit in this case is not available, we compare the result with the smoothed empirical density function (kernel) based on 1000 Monte Carlo replications. In this example we used the Gaussian kernel with the optimum width $b = 1.06N^{(-1/5)}\hat{\sigma}_0 = 0.12$, where $\hat{\sigma}_0$ is the standard deviation of the estimates of μ , obtained through Monte Carlo replications [62]. The 95% confidence interval for μ was found to be $(-0.896, 0.902)$ and $(-0.886, 0.887)$ based on the bootstrap and Monte Carlo estimates, respectively.

In practice, the procedure described in Table 2 can be substantially improved because the interval calculated is, in fact, an interval with coverage less than the nominal value [31]. Later in this article we shall discuss another way for constructing confidence intervals for the mean that would lead to a more accurate result. The example here suffices to show the single steps of the bootstrap procedure. The computational expense to calculate the confidence interval for μ is approximately n times greater than the one needed to compute $\hat{\mu}$.

A reliable pseudo-random number generator is essential for valid application of the bootstrap method. In all our applications, we used a pseudo-random number gen-

Table 1. The bootstrap principle.

1. Conduct the experiment to obtain the random sample $X = \{X_1, X_2, \dots, X_n\}$ and calculate the estimate $\hat{\theta}$ from the sample X .
2. Construct the empirical distribution, \hat{F} , which puts equal mass, $1/n$, at each observation^a, $X_1 = x_1, X_2 = x_2, \dots, X_n = x_n$.
3. From the selected \hat{F} , draw a sample, $x^* = \{X_1^*, X_2^*, \dots, X_n^*\}$, called the bootstrap (re)sample.
4. Approximate the distribution of $\hat{\theta}$ by the distribution of $\hat{\theta}^*$ derived from x^* .

^aIn other words, the empirical distribution \hat{F} is that probability measure that assigns to a set A in the sample space of X a measure equal to the proportion of sample values that lie in A .

erator proposed by Park and Miller [51], which is a built-in function in MATLAB®. A detailed treatment of pseudo-random number generators can be found in [8, 40].

Example 2. Variance estimation

This example illustrates an application of the bootstrap for estimating the variance of the parameter of a first-order autoregressive (AR) time series. One may choose to use dependent-data bootstrap techniques to solve this problem. As we mentioned earlier, we shall adapt the independent data bootstrap for this dependent-data model. We generate T observations, $x_t, t = 0, \dots, T - 1$, from the first-order AR model

$$X_t + \alpha X_{t-1} = Z_t \quad (1)$$

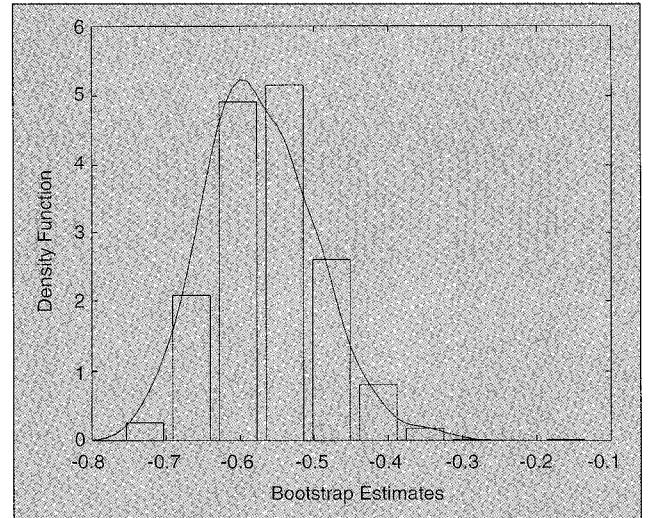
where Z_t is stationary white Gaussian noise with $EZ_t = 0$ and auto-covariance function $c_{ZZ}(u) = \sigma_Z^2 \delta(u)$, where $\delta(u)$ is Kronecker's delta function, which is zero unless $u = 0$ when $\delta(0) = 1$, and α is a real number, satisfying $|\alpha| < 1$. After detrending the data (replacing x_t by $x_t - \frac{1}{T} \sum_{t=0}^{T-1} x_t$), we fit the first-order AR model to the observation x_t . With the empirical auto-covariance function of x_t ,

$$\hat{c}_{xx}(u) = \begin{cases} \frac{1}{T} \sum_{t=0}^{T-|u|-1} x_t x_{t+|u|}, & 0 \leq |u| \leq T-1, \\ 0 & \text{otherwise,} \end{cases} \quad (2)$$

we calculate the maximum-likelihood estimate of α ,

$$\hat{\alpha} = -\frac{\hat{c}_{xx}(1)}{\hat{c}_{xx}(0)} \quad (3)$$

which has approximate variance [56]



▲ 4. Histogram of $\hat{\alpha}_1^*, \hat{\alpha}_2^*, \dots, \hat{\alpha}_{1000}^*$ for an AR(1) process with parameter $\alpha = -0.6$. The solid line indicates the kernel probability density function obtained from 1000 Monte Carlo simulations.

$$\hat{\sigma}_{\hat{\alpha}}^2 = \frac{1 - \alpha^2}{T} \quad (4)$$

It is necessary to assume normality of Z_t for obtaining Eq. (4) [56]. We note, however, that under some regularity conditions an asymptotic formula for $\hat{\sigma}_{\hat{\alpha}}^2$ can be found in the non-Gaussian case and is a function of α and the variance and kurtosis of Z_t [30, 55]. In Table 3 we show how we can approximate $\hat{\sigma}_{\hat{\alpha}}^2$ without knowledge of the distribution of Z_t .

In an experiment we chose $\alpha = -0.6$, $T = 128$ and, for comparative purposes, Z_t to be Gaussian. The maximum likelihood estimate, derived from Eq. (3), was found to be $\hat{\alpha} = -0.6351$, and the standard deviation $\hat{\sigma}_{\hat{\alpha}} = 0.0707$, when applying Eq. (4). Using the procedure described in

Table 2. The bootstrap principle for calculating a confidence interval for the mean.

- Step 0.** *Experiment.* Conduct the experiment. Suppose our sample is $X = \{-2.41, 4.86, 6.06, 9.11, 10.20, 12.81, 13.17, 14.10, 15.77, 15.79\}$ of size 10, with $\bar{X} = 9.946$ being the mean of all values in X .
- Step 1.** *Resampling.* Using a pseudo-random number generator, draw a random sample of 10 values, with replacement, from X . Thus, one might obtain the *bootstrap resample* $X^* = \{9.11, 9.11, 6.06, 13.17, 10.20, -2.41, 4.86, 12.81, -2.41, 4.86\}$. Note that some of the original sample values appear more than once, and others not at all.
- Step 2.** *Calculation of the bootstrap estimate.* Calculate the mean of all values in X^* . The mean of all 10 values in X^* is $\hat{\mu}_1^* = 6.54$.
- Step 3.** *Repetition.* Repeat Steps 1 and 2 a large number of times to obtain a total of n bootstrap estimates $\hat{\mu}_1^*, \dots, \hat{\mu}_N^*$. For example, let $N = 1000$.
- Step 4.** *Approximation of the distribution of $\hat{\mu}$.* Sort the bootstrap estimates into increasing order to obtain $\hat{\mu}_{(1)}^* \leq \hat{\mu}_{(2)}^* \leq \dots \leq \hat{\mu}_{(1000)}^*$, where $\hat{\mu}_{(k)}^*$ is the k th smallest of $\hat{\mu}_1^*, \dots, \hat{\mu}_N^*$. For example, we might get 3.48, 3.39, 4.46, ..., 8.86, 8.88, 8.89, ..., 10.07, 10.08, ..., 14.46, 14.53, 14.66. A histogram of the obtained bootstrap estimates $\hat{\mu}_1^*, \dots, \hat{\mu}_N^*$ is given in Fig. 2 along with the density function of the normal distribution with mean $\mu = 10$ and variance $\sigma^2 / n = 2.5$.
- Step 5.** *Confidence interval.* The desired $(1 - \alpha)100\%$ bootstrap confidence interval is $(\hat{\mu}_{(q_1)}, \hat{\mu}_{(q_2)})$, where $q_1 = \lfloor N\alpha / 2 \rfloor$ is the integer part of $N\alpha / 2$ and $q_2 = N - q_1 + 1$. For $\alpha = 0.05$ and $N = 1000$, we get $q_1 = 25$ and $q_2 = 976$, and the 95% confidence interval is found to be (6.27, 13.19).

There is a well established duality between confidence intervals and hypothesis testing.

Table 3, we obtained the histogram of $N = 1000$ bootstrap estimates of $\hat{\alpha}, \hat{\alpha}_1^*, \hat{\alpha}_2^*, \dots, \hat{\alpha}_{1000}^*$, shown in Fig. 4. We then found an estimate of the standard deviation of $\hat{\alpha}$, $\hat{\sigma}_{\hat{\alpha}}^* = 0.0712$, to be close to the theoretical value, $\hat{\sigma}_{\hat{\alpha}}$. For comparison purposes we also show in Fig. 4 the kernel density estimator of $\hat{\alpha}$ based on 1000 Monte Carlo simulations (solid line). The estimate of the standard deviation of $\hat{\alpha}$, $\hat{\sigma}_{\hat{\alpha}}$, in this case is found to be 0.0694.

Notice that in the bootstrap procedure neither the assumption of Gaussian distribution for the noise process Z_t , nor knowledge of any characteristic of the non-Gaussian distribution is necessary. The only assumption we made is that the variables Z_0, Z_1, \dots, Z_{T-1} are independently and identically distributed.

Note also that we could have obtained $\hat{\sigma}_{\hat{\alpha}}^*$ using the parametric bootstrap. Herein, we could also sample \hat{z}_t^* from a fitted normal distribution, i.e., a normal distribution with mean zero and variance $\hat{\sigma}_Z^2 = (1 - \hat{\alpha}^2) \hat{\sigma}_{xx}^2(0)$, instead of resampling from the residuals $\hat{z}_1, \hat{z}_2, \dots, \hat{z}_{T-1}$ and performed the computation of $\hat{\alpha}^*$ from the so obtained resamples. We would have taken a similar approach to estimate the variances and the covariances of the parameter estimates of an AR(p) process, where p is the order of the autoregressive time series. For more details on regression analysis using the bootstrap see [52].

Hypothesis Testing Using the Bootstrap Principle

In this section, we shall discuss in detail the use of bootstrap techniques for hypotheses testing, a key element in many signal-processing applications such as radar and sonar.

Consider a situation in which a random sample, $x = \{X_1, \dots, X_n\}$, is observed from its unspecified probability distribution, F . Let ϑ denote an unknown characteristic of F . We consider the problem of testing, for example, the hypothesis $H: \vartheta \leq \vartheta_0$ against the alternative $K: \vartheta > \vartheta_0$, where ϑ_0 is a given bound. Let $\hat{\vartheta}$ be an estimator of ϑ and $\hat{\sigma}^2$ an estimator of the variance σ^2 of $\hat{\vartheta}$.

For testing H against K , we define the statistic

$$\hat{T} = \frac{\hat{\vartheta} - \vartheta_0}{\hat{\sigma}}, \quad (5)$$

The inclusion of the scale factor, $\hat{\sigma}$, to be defined later, ensures that \hat{T} is asymptotically pivotal as $n \rightarrow \infty$, i.e., the asymptotic distribution of \hat{T} does not depend on any unknown parameter [34, 35]. This means that we only need to deal with the appropriate standard distribution rather than a whole family of distributions.

If the distribution function G , say, of \hat{T} under H , were known, an exact α -level test would suggest to reject H if $\hat{T} \geq t_\alpha$, where t_α is determined by $G(t_\alpha) = 1 - \alpha$ [44]. For example, if F has mean μ and unknown variance,

$$\hat{T} = \frac{\frac{1}{n} \sum_{i=1}^n X_i - \mu_0}{\sqrt{\frac{1}{n(n-1)} \sum_{i=1}^n \left(X_i - \frac{1}{n} \sum_{i=1}^n X_i \right)^2}}$$

is used to test $\mu \leq \mu_0$ against $\mu > \mu_0$, given the random sample $x = \{X_1, X_2, \dots, X_n\}$. For large n , \hat{T} is asymptotically t -distributed with $n - 1$ degrees of freedom. (If Z and X^2 are independent random variables having a standard normal distribution and a chi-square distribution of v degrees of freedom, respectively, then $Z / \sqrt{X^2 / v}$ has a t -distribution of v degrees of freedom.) [39, 44, 46].

Two common problems can be encountered when solving a test problem. The first one is when the size of the random sample is small and asymptotic methods do not apply, such as in Example 1 discussed earlier. The second

Table 3. The bootstrap principle for estimating the variance of the parameter estimate of an AR(1) process.

- Step 0.** *Experiment.* Conduct the experiment and generate T observations $x_t, t = 0, \dots, T - 1$, from a first-order AR process, X_t .
- Step 1.** *Calculation of the residuals.* Having estimated the parameter α from Eq. (3), define the residuals, $\hat{z}_t = x_t + \hat{\alpha} \cdot x_{t-1}$, for $t = 1, 2, \dots, T - 1$.
- Step 2.** *Resampling.* Create a bootstrap sample, $x_0^*, x_1^*, \dots, x_{T-1}^*$, by sampling $\hat{z}_1^*, \hat{z}_2^*, \dots, \hat{z}_{T-1}^*$, with replacement, from the residuals $\hat{z}_1, \hat{z}_2, \dots, \hat{z}_{T-1}$, then letting $x_0^* = x_0$ and $x_t^* = -\hat{\alpha}x_{t-1}^* + \hat{z}_t^*$, $t = 1, 2, \dots, T - 1$.
- Step 3.** *Calculation of the bootstrap estimate.* After centering the time series $x_0^*, x_1^*, \dots, x_{T-1}^*$, obtain $\hat{\alpha}^*$ from Eqs. (2) and (3) but based on $x_0^*, x_1^*, \dots, x_{T-1}^*$ rather than x_0, x_1, \dots, x_{T-1} .
- Step 4.** *Repetition.* Repeat steps 2-3 a large number of times, say $N = 1000$, to obtain $\hat{\alpha}_1^*, \hat{\alpha}_2^*, \dots, \hat{\alpha}_N^*$.
- Step 5.** *Variance estimation.* From $\hat{\alpha}_1^*, \hat{\alpha}_2^*, \dots, \hat{\alpha}_N^*$ approximate the variance of $\hat{\alpha}$ by

$$\hat{\sigma}_{\hat{\alpha}}^{*2} = (N - 1)^{-1} \sum_{i=1}^N (\hat{\alpha}_i^* - N^{-1} \sum_{i=1}^N \hat{\alpha}_i^*)^2$$
an estimate of the variance of $\hat{\alpha}^*$.

Table 4. The bootstrap principle for testing the hypothesis $H: \vartheta \leq \vartheta_0$ against $K: \vartheta > \vartheta_0$.

Step 0. *Experiment.* Conduct the experiment and collect the random data into the sample $X = \{X_1, \dots, X_n\}$.

Step 1. *Resampling.* Using a pseudo-random number generator, draw a random sample, X^* , of the same size as X , with replacement, from X .

Step 2. *Calculation of the bootstrap statistic.* From X^* , calculate

$$\hat{T}^* = \frac{\hat{\vartheta}^* - \hat{\vartheta}}{\hat{\sigma}^*}, \quad (6)$$

where $\hat{\vartheta}$ replaces^a ϑ_0 , and $\hat{\vartheta}^*$ and $\hat{\sigma}^*$ are versions of $\hat{\vartheta}$ and $\hat{\sigma}$ computed in the same manner as $\hat{\vartheta}$ and $\hat{\sigma}$, respectively, but with the resample X^* replacing the sample X .

Step 3. *Repetition.* Repeat steps 1 and 2 many times to obtain a total of n bootstrap statistics, $\hat{T}_1^*, \hat{T}_2^*, \dots, \hat{T}_N^*$.

Step 4. *Ranking.* Rank the collection $\hat{T}_1^*, \hat{T}_2^*, \dots, \hat{T}_N^*$ into increasing order to obtain $\hat{T}_{(1)}^* \leq \hat{T}_{(2)}^* \leq \dots \leq \hat{T}_{(N)}^*$.

Step 5. *Test.* A bootstrap test has then the following form: reject H if $\hat{T} = \hat{T}_{(M)}$, where the choice of M determines the level of significance of the test and is given by $\alpha = (N+1-M)(N+1)^{-1}$, where α is the nominal level of significance [34].

^aNote that the constant ϑ_0 has been replaced in Eq. (6) by the estimate of ϑ , $\hat{\vartheta}$, derived from X . This is crucial if the test is to have good power properties. It is also important in the context of the accuracy of the level of the test [32, 34, 35].

possible problem is that the distribution of the statistic used cannot be determined analytically. One can overcome both problems when bootstrap techniques are used.

The bootstrap approach for testing $H: \vartheta \leq \vartheta_0$ against $K: \vartheta > \vartheta_0$, given $\hat{\vartheta}$ and $\hat{\sigma}$ (see the Variance Estimation section), derived from X , is illustrated in Table 4. In the approach, we retain the asymptotically pivotal nature of the test statistic because the bootstrap approximation of the distribution of \hat{T} is better than the approximation of the distribution of $\hat{\vartheta}$ [31].

Note that in the case where one is interested in the hypothesis $H: \vartheta = \vartheta_0$ against the two-sided alternative $K: \vartheta \neq \vartheta_0$, the procedure shown in Table 4 is still valid, except that $\vartheta - \vartheta_0$ is replaced by $|\hat{\vartheta} - \vartheta_0|$ in Eq. (5) so that \hat{T} is given by $\hat{T} = |\hat{\vartheta} - \vartheta_0| / \hat{\sigma}$ and correspondingly $\hat{T}^* = |\hat{\vartheta}^* - \vartheta_0| / \hat{\sigma}^*$ [34].

Variance Estimation

The tests described in Table 4 require the estimation of $\hat{\sigma}$ and its bootstrap counterpart $\hat{\sigma}^*$. In this section, we discuss how one can use the bootstrap to achieve this.

Suppose that X is a real-valued random variable with unknown probability distribution F with mean μ_x and variance σ_x^2 . Let $X = \{X_1, X_2, \dots, X_n\}$ be a random sample of size n from F . We wish to estimate μ_x and assign to it a measure of accuracy.

The sample mean $\hat{\mu}_x = n^{-1} \sum_{i=1}^n X_i$ is a natural estimate for μ_x , which has expectation μ_x and variance σ_x^2/n . The standard deviation of the sample mean, $\hat{\mu}_x$, is the square root of its variance and is the most common way of indicating statistical accuracy. The standard deviation of $\hat{\mu}_x$ amounts therefore to σ_x / \sqrt{n} . The mean value and standard deviation of $\hat{\mu}_x$ are exact but the usual as-

sumption of normality of $\hat{\mu}_x$ is an approximation only and is valid under general conditions on F as n grows.

In this example, we could use

$$\hat{\sigma}_x = \sqrt{\frac{1}{n-1} \sum_{i=1}^n (X_i - \hat{\mu}_x)^2} \quad (7)$$

to estimate $\sigma_x = \sqrt{E(X - \mu_x)^2}$. This gives the estimate of the standard deviation of $\hat{\mu}_x$,

$$\hat{\sigma} = \hat{\sigma}_x / \sqrt{n} = \sqrt{\frac{1}{n(n-1)} \sum_{i=1}^n (X_i - \hat{\mu}_x)^2}. \quad (8)$$

In this particular example where $\hat{\vartheta} = \hat{\mu}_x$, the estimate of the standard deviation is just the usual estimate of the standard deviation of F . However, for any estimate, $\hat{\vartheta}$, other than the mean, there is no such neat formula that enables us to compute the numerical value of the ideal estimate exactly.

The bootstrap can be used for estimating the standard deviation of $\hat{\vartheta}$, it does not require any theoretical calculation, and it is available no matter how mathematically complicated the estimate $\hat{\vartheta}$ may be. The procedure to estimate $\hat{\sigma}$, the standard deviation of $\hat{\vartheta}$, is given in Table 5 [21, 32, 52].

In the case of estimating $\hat{\sigma}^*$, we would proceed similarly as in Table 5, except that the procedure involves two nested levels of resampling. Herein, for each resample X_b^* , $b = 1, \dots, B_1$ we draw resamples X_b^{**} , $b = 1, \dots, B_2$, evaluate $\hat{\vartheta}_b^{**}$ from each resample to obtain B_2 replications, and calculate Eq. (9), replacing $\hat{\vartheta}_b^*$, B by $\hat{\vartheta}_b^{**}$, and B_2 , respectively.

The jackknife [47, 52] is another technique for estimating the standard deviation. As an alternative to the bootstrap, the jackknife method can be thought of as drawing n samples of size $n-1$ each *without* replacement from the original sample of size n [47, 52].

Suppose we are given the sample $X = \{X_1, \dots, X_n\}$ and an estimate, $\hat{\vartheta}$, from X . The jackknife method is based on the sample *delete-one observation at a time*,

$$X^{(i)} = \{X_1, X_2, \dots, X_{i-1}, X_{i+1}, \dots, X_n\}$$

for $i = 1, 2, \dots, n$, called the jackknife sample [21]. The i th jackknife sample consists of the data set with the i th observation removed. For each i th jackknife sample, we calculate the i th jackknife estimate, $\hat{\vartheta}^{(i)}$ of ϑ , $i = 1, \dots, n$. The jackknife estimate of standard deviation of $\hat{\vartheta}$ is defined by

$$\hat{\sigma} = \sqrt{\frac{n-1}{n} \sum_{i=1}^n \left(\hat{\vartheta}^{(i)} - n^{-1} \sum_{i=1}^n \hat{\vartheta}^{(i)} \right)^2}. \quad (10)$$

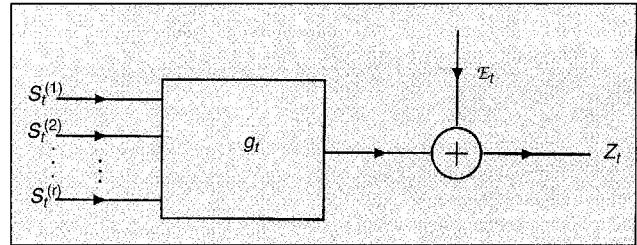
The jackknife is computationally less expensive if n is less than the number of replicates used by the bootstrap for standard deviation estimation because it requires computation of $\hat{\vartheta}$ only for the n jackknife data sets. For example, if $B = 25$ resamples are necessary for standard deviation estimation with the bootstrap, and the sample size is $n = 10$, then clearly the jackknife would be computationally less expensive than the bootstrap.

Examples

This subsection illustrates some applications of the bootstrap to testing parameters in some statistical models.

Example 3. Regression analysis

In many signal-processing applications, we face the situation illustrated by Fig. 5, where an r vector-valued stationary process, $S_t = (S_{1,t}, \dots, S_{r,t})'$, is transmitted through a linear time-invariant system having an r vector-valued impulse response, $g_t = (g_{1,t}, \dots, g_{r,t})'$, where $'$ denotes transpose. Assuming the linear system to be stable, the filtered signal is then buried in a stationary zero-mean noise process, ε_t , and received as a stationary process Z_t , where ε_t and S_t are assumed to be independent for any $t = 0, \pm 1, \pm 2, \dots$. For the model of Fig. 5, we can write



▲ 5. Multiple input/single output system.

$$Z_t = \sum_{u=-\infty}^{\infty} g'_u S_{t-u} + \varepsilon_t. \quad (11)$$

We denote the linear system's unknown transfer function by $\mathbf{G}(\omega) = (G_1(\omega), \dots, G_r(\omega))' = \sum_{t=-\infty}^{\infty} g_t e^{-j\omega t}$.

The problem considered here is to answer the question of which element, $G_l(\omega)$, $1 \leq l \leq r$, is zero at a certain frequency, ω . This would imply that Z_t does not contain any information at ω , contributed by the l th signal component of S_t , $S_{l,t}$, $1 \leq l \leq r$. This situation occurs in many applications where one is interested in approximating a vector-valued time series by a version of itself plus noise, but restraining the new series to be of reduced dimension (in this case a scalar). Then, the problem is to detect channels (frequency responses) that do have bandstop behavior at certain frequencies.

A specific example is a situation where one is interested in finding suitable vibration sensor positions to detect tool wear or breaks in a milling process. One would distribute sensors on the spindle fixture and one sensor as a reference on the work piece, which would not be accessible in a serial production. Based on observations of the vibration sensors, one would decide the suitability of a sensor position on the spindle fixture based on the structural behavior of the fixture at some given frequencies, which would have been assumed to be linear and time invariant. This problem is currently resolved using heuristic arguments. A similar problem is discussed in [71].

Another application in vertical seismic profiling requires a detailed knowledge of the filter constituted by the various layers constituting the vertical earth profile at a particular point in space. In such an application, waves are

Table 5. The bootstrap principle for estimating the standard deviation of a parameter estimator.

- Step 0.** *Experiment.* Conduct the experiment and collect the random data into the sample $X = \{X_1, \dots, X_n\}$.
- Step 1.** *Resampling.* Draw a random sample of size n , with replacement, from X .
- Step 2.** *Calculation of the bootstrap estimate.* Evaluate the bootstrap estimate $\hat{\vartheta}^*$ from X^* calculated in the same manner as $\hat{\vartheta}$ but with the resample X^* replacing X .
- Step 3.** *Repetition.* Repeat steps 1 and 2 many times to obtain a total of B bootstrap estimates, $\hat{\vartheta}_1^*, \dots, \hat{\vartheta}_B^*$. Typical values for B are between 25 and 200.
- Step 4.** *Standard deviation estimation of $\hat{\vartheta}$.* Estimate the standard deviation, $\hat{\sigma}$, of $\hat{\vartheta}$ by the sample standard deviation of the B bootstrap estimates,

$$\hat{\sigma} = \sqrt{\frac{1}{B-1} \sum_{b=1}^B \left(\hat{\vartheta}_b^* - B^{-1} \sum_{b=1}^B \hat{\vartheta}_b^* \right)^2}. \quad (9)$$

Table 6. The bootstrap principle for the regression analysis example.

- Step 0.** *Experiment.* Conduct the experiment and calculate the frequency data, $\mathbf{d}_s(\omega, 1), \dots, \mathbf{d}_s(\omega, n)$, and $\mathbf{d}_z(\omega, 1), \dots, \mathbf{d}_z(\omega, n)$.
- Step 1.** *Resampling.* Conduct two totally independent resampling operations in which a random sample $\{\mathbf{d}_s^*(\omega, 1), \dots, \mathbf{d}_s^*(\omega, n)\}$, is drawn, with replacement, from $\{\mathbf{d}_s(\omega, 1), \dots, \mathbf{d}_s(\omega, n)\}$, where $\mathbf{d}_s(\omega, i) = (d_{s_1}(\omega, i), \dots, d_{s_r}(\omega, i))$, $i = 1, \dots, n$, and a resample, $\{\mathbf{d}_{\hat{z}}^*(\omega, 1), \dots, \mathbf{d}_{\hat{z}}^*(\omega, n)\}$, is drawn, with replacement, from the random sample $\mathbf{d}_{\hat{z}}^*(\omega, 1), \dots, \mathbf{d}_{\hat{z}}^*(\omega, n)$, collected into the vector $\mathbf{d}_{\hat{z}}(\omega) = (\mathbf{d}_{\hat{z}}(\omega, 1), \dots, \mathbf{d}_{\hat{z}}(\omega, n))'$, so that
- $$\mathbf{d}_{\hat{z}}(\omega) = \mathbf{d}_z(\omega) - \mathbf{d}_s(\omega) \hat{\mathbf{G}}(\omega)$$
- are the residuals of the regression model (Eq. (12)).
- Step 2.** *Generation of bootstrap data.* Center the frequency data resamples and compute
- $$\mathbf{d}_z^*(\omega) = \mathbf{d}_s^*(\omega) \hat{\mathbf{G}}(\omega) + \mathbf{d}_{\hat{z}}(\omega).$$
- The joint distribution of $\{(\mathbf{d}_s^*(\omega, i), \mathbf{d}_z^*(\omega, i)), 1 \leq i \leq n\}$, conditional on $\mathcal{X}(\omega) = \{(\mathbf{d}_s(\omega, 1), \mathbf{d}_z(\omega, 1)), \dots, (\mathbf{d}_s(\omega, n), \mathbf{d}_z(\omega, n))\}$, is the bootstrap estimate of the unconditional joint distribution of $\mathcal{X}(\omega)$.
- Step 3.** *Calculation of bootstrap estimates.* With the new $\mathbf{d}_z^*(\omega)$ and $\mathbf{d}_s^*(\omega)$, calculate $\hat{\mathbf{G}}^*(\omega)$, using (14) but with the resamples $\mathbf{d}_z^*(\omega)$ and $\mathbf{d}_s^*(\omega)$ replacing $\mathbf{d}_z(\omega)$ and $\mathbf{d}_s(\omega)$, respectively.
- Step 4.** *Calculation of the bootstrap statistic.* Calculate the statistic given in Eq. (15), replacing $\mathbf{d}_s(\omega)$, $\mathbf{d}_{s^{(l)}}(\omega)$, $\mathbf{d}_z(\omega)$, $\hat{\mathbf{G}}(\omega)$, and $\hat{\mathbf{G}}^{(l)}(\omega)$ by their bootstrap counterparts to yield $\hat{\vartheta}^*(\omega)$.
- Step 5.** *Repetition.* Repeat steps 1-4 a large number of times, say N , to obtain $\hat{\vartheta}_1^*(\omega), \dots, \hat{\vartheta}_N^*(\omega)$
- Step 6.** *Distribution estimation.* Approximate the distribution of $\hat{\vartheta}(\omega)$, given in Eq. (15), by the distribution of $\hat{\vartheta}^*(\omega)$ obtained.

emitted in the ground that propagate through and reflect on the dioptrre, which separate the various layers characterized by different acoustic impedances. The transmission function for the range of relevant frequencies is of crucial importance, as the filter characteristics vary from location to location. Knowledge of the frequency transfer function of the various earth-layers filter contributes to a proper modeling of the earth surface, and then to a decision as to the likelihood of the presence of gas and/or oil in a particular place.

To test whether $G_l(\omega)$, $1 \leq l \leq r$ is zero, we would let $\mathbf{G}(\omega) = (\mathbf{G}^{(l)}(\omega)', G_l(\omega))'$, where $G_l(\omega)$ is an arbitrary frequency response that represents the transfer function of the filter that transforms the signal $S_{t,i}$ by time-invariant and linear operations, and $\mathbf{G}^{(l)}(\omega) = (G_1(\omega), G_2(\omega), \dots, G_{l-1}(\omega), G_{l+1}(\omega), \dots, G_r(\omega))'$ is the vector of transfer func-

tions obtained from $\mathbf{G}(\omega)$ by deleting the component $G_l(\omega)$. Then, we test the hypothesis $H: G_l(\omega) = 0$ ($\mathbf{G}^{(l)}(\omega)$ unspecified) against the two-sided alternative.

Let \mathbf{S}_t and Z_t be given for n independent observations of length T each. By taking the finite Fourier transform of both sides of Eq. (11), we obtain (omitting the error term $\sigma_{a.s.}(1)$, which is an error term that tends to zero almost surely as $T \rightarrow \infty$ [9]) the complex regression

$$\mathbf{d}_z(\omega) = \mathbf{d}_s(\omega) \mathbf{G}(\omega) + \mathbf{d}_{\hat{z}}(\omega) \quad (12)$$

where $\mathbf{d}_s(\omega) = (\mathbf{d}_{s_1}(\omega), \dots, \mathbf{d}_{s_r}(\omega))$, $\mathbf{d}_{s_l}(\omega) = (d_{s_1}(\omega, 1), \dots, d_{s_l}(\omega, n))'$, $l = 1, \dots, r$, $\mathbf{d}_{\hat{z}}(\omega) = (d_z(\omega, 1), \dots, d_z(\omega, n))'$, and

$$d_z(\omega, i) = \sum_{t=0}^{T-1} w(t/T) \cdot Z_{t,i} e^{-j\omega t}, \quad i = 1, \dots, n, \quad (13)$$

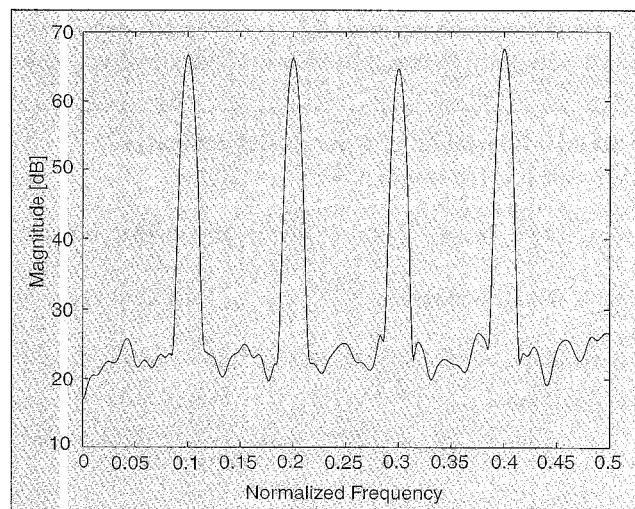
is the normalized finite Fourier transform of the i th data block $Z_{t,i}$, $i = 1, \dots, n$, of Z_t and $w(u)$, $u \in \mathbb{R}$, is a smooth window that vanishes outside the interval $[0, 1]$.

Based on the observations of \mathbf{S}_t and Z_t , for $t = 0, 1, \dots, T-1$, and $i = 1, \dots, n$, we first calculate the least-squares estimate of $\mathbf{G}(\omega)$,

$$\begin{aligned} \hat{\mathbf{G}}(\omega) &= (\mathbf{d}_s(\omega)^H \mathbf{d}_s(\omega))^{-1} (\mathbf{d}_s(\omega)^H \mathbf{d}_z(\omega)) \\ &= \hat{\mathbf{C}}_{ss}(\omega)^{-1} \hat{\mathbf{C}}_{zs}(\omega)', \end{aligned} \quad (14)$$

where $\hat{\mathbf{C}}_{ss}(\omega)$ and $\hat{\mathbf{C}}_{zs}(\omega)$ are spectral densities obtained by averaging the corresponding periodograms over n independent data records, and $\hat{\mathbf{C}}^H$ denotes Hermitian operation.

Conventional techniques assume the number of observations, T , to be large so that the finite Fourier transform



▲ 6. Spectrum of $S_t^{(l)}$, obtained by averaging 20 periodograms.

The results show that the bootstrap method is reasonable for estimating the variance-stabilizing transformation.

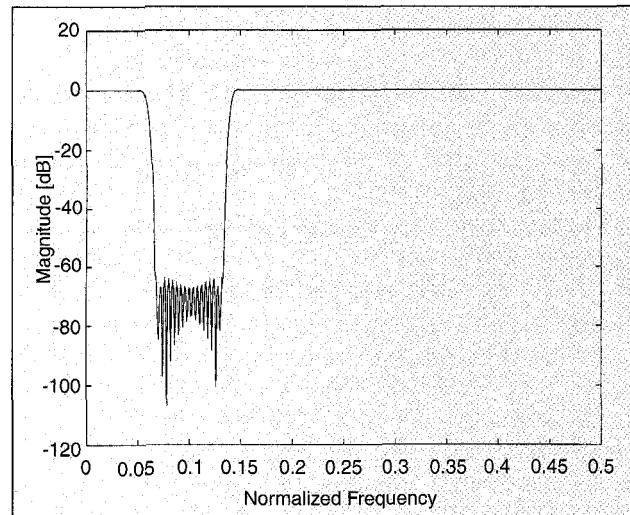
$\mathbf{d}_z(\omega)$ becomes complex Gaussian [9]. Under this condition and H , the statistic

$$\hat{\vartheta}(\omega) = (n - r) \frac{\|\mathbf{d}_z(\omega) - \mathbf{d}_{s^{(t)}}(\omega)\hat{\mathbf{G}}^{(t)}(\omega)\|^2 - \|\mathbf{d}_z(\omega) - \mathbf{d}_s(\omega)\hat{\mathbf{G}}(\omega)\|^2}{\|\mathbf{d}_z(\omega) - \mathbf{d}_s(\omega)\hat{\mathbf{G}}(\omega)\|^2} \quad (15)$$

is assumed to be F -distributed with 2 and $2(n - r)$ degrees of freedom (if X_1^2 and X_2^2 are independent random variables having chi-square distributions of v_1 and v_2 degrees of freedom, respectively, then $(X_1^2/v_1)/(X_2^2/v_2)$ has an F -distribution with v_1 and v_2 degrees of freedom), where $\mathbf{d}_{s^{(t)}}(\omega) = (\mathbf{d}_{s_1}(\omega), \dots, \mathbf{d}_{s_{l-1}}(\omega), \mathbf{d}_{s_{l+1}}(\omega), \dots, \mathbf{d}_{s_r}(\omega))'$ is obtained from $\mathbf{d}_s(\omega) = (\mathbf{d}_{s_1}(\omega), \mathbf{d}_{s_r}(\omega))$ by deleting the l th vector $\mathbf{d}_{s_l}(\omega)$, and $\hat{\mathbf{G}}^{(t)}(\omega) = (\mathbf{d}_{s^{(t)}}(\omega)^H \mathbf{d}_{s^{(t)}}(\omega))^{-1} (\mathbf{d}_{s^{(t)}}(\omega)^H \mathbf{d}_z(\omega))$ [61].

The hypothesis H is rejected at a level, α , if the statistic in Eq. (15) exceeds the $(1 - \alpha)\%$ quantile of the F -distribution. However, the use of the F -distribution in the case where $\mathbf{d}_z(\omega)$ is non-Gaussian is not valid. To find the distribution of the statistic in Eq. (15) in the more general case, we could use a procedure based on the bootstrap described in Table 6 (for a more detailed discussion on the bootstrap for regression analysis, see [52]).

Note that several regression models are available and, depending upon which model the analysis is based, we have a different resampling procedure [32]. In the procedure described in Table 6, we assumed the pairs $(\mathbf{d}_s(\omega, i), \mathbf{d}_z(\omega, i))$ to be independent and identically distributed, with $\mathbf{d}_s(\omega, i)$ and $\mathbf{d}_z(\omega, i)$ independent.



▲ 7. Frequency response of the first channel, $G_1(\omega)$, obtained using an FIR filter with 256 coefficients.

An alternative bootstrap approach to the one described in Table 6 is based on the fact that Eq. (15) can be written as

$$\hat{\vartheta}(\omega) = (n - r) \frac{|\hat{R}_{zs}(\omega)|^2 - |\hat{R}_{zs^{(t)}}(\omega)|^2}{1 - |\hat{R}_{zs}(\omega)|^2}, \quad (16)$$

where

$$|\hat{R}_{zs}(\omega)|^2 = \frac{\hat{\mathbf{C}}_{zs}(\omega)\hat{\mathbf{C}}_{ss}(\omega)^{-1}\hat{\mathbf{C}}_{sz}(\omega)}{\hat{\mathbf{C}}_{zz}(\omega)} \quad (17)$$

and

$$|\hat{R}_{zs^{(t)}}(\omega)|^2 = \frac{\hat{\mathbf{C}}_{zs^{(t)}}(\omega)\hat{\mathbf{C}}_{s^{(t)}s^{(t)}}(\omega)^{-1}\hat{\mathbf{C}}_{s^{(t)}z}(\omega)}{\hat{\mathbf{C}}_{zz}(\omega)} \quad (18)$$

are respectively the sample multiple coherence of Z_t with S_t and Z_t with $S_t^{(t)} = (S_{1,t}, S_{2,t}, \dots, S_{l-1,t}, S_{l+1,t}, \dots, S_{r,t})'$ at frequency ω . Herein, the spectral densities, $\hat{\mathbf{C}}_{zs}(\omega)$, $\hat{\mathbf{C}}_{zs^{(t)}}(\omega)$, $\hat{\mathbf{C}}_{ss}(\omega)$, $\hat{\mathbf{C}}_{s^{(t)}s^{(t)}}(\omega)$, and $\hat{\mathbf{C}}_{zz}(\omega)$, in Eq. (17) and Eq. (18) are obtained by averaging periodograms of n independent data records.

Table 7. Alternative bootstrap approach for the regression analysis example.

- Step 0. *Experiment.* Conduct the experiment and calculate the frequency data, $\mathbf{d}_s(\omega, 1), \dots, \mathbf{d}_s(\omega, n)$ and $\mathbf{d}_z(\omega, 1), \dots, \mathbf{d}_z(\omega, n)$.
- Step 1. *Resampling.* Using a pseudo-random number generator, draw a random sample, $X^*(\omega)$ (of the same size), with replacement, from $X(\omega) = \{(\mathbf{d}_s(\omega, 1), \mathbf{d}_z(\omega, 1)), \dots, (\mathbf{d}_s(\omega, n), \mathbf{d}_z(\omega, n))\}$.
- Step 2. *Calculation of the bootstrap statistic.* From $X^*(\omega)$, calculate $\hat{\vartheta}^*(\omega)$, the bootstrap analogue of $\hat{\vartheta}(\omega)$ given by Eq. (16).
- Step 3. *Repetition.* Repeat steps 1 and 2 many times to obtain a total of N bootstrap statistics, $\hat{\vartheta}_1^*(\omega), \dots, \hat{\vartheta}_N^*(\omega)$.
- Step 4. *Distribution estimation.* Approximate the distribution of $\hat{\vartheta}(\omega)$, given in Eq. (16), by the so obtained bootstrap distribution.

Alternatively to Table 6, we could proceed as described in Table 7 to estimate the distribution of $\hat{\vartheta}(\omega)$. The main difference with this approach compared to the previous one is that the resampling procedure does not take into consideration the assumed complex regression model (Eq. (12)).

The procedure described in Table 6 reflects the complex regression model (Eq. (12)) while the one described in Table 7 does not take this model into account. It is worthwhile emphasizing that in practice bootstrap resampling should be performed to reflect the model characteristics. If we assume that the data is generated from the model in Eq. (12), we should use the method given in Table 6 to estimate the distribution of the test statistic. *Resampling* from $X(\omega)$ will not necessarily generate data satisfying the *assumed* model. In our application, the regression model (Eq. (12)) is an approximation only and its validity is questionable in the case where the number of observations is small. Notice that Eq. (16) is a measure (see [71]) of the extent to which the signal $S_{l,t}$ contributes in Z_t and can be derived heuristically, without use of Eq. (15), which is based on regression (Eq. (12)).

Comparative studies of the two indicated resampling methods will show that bootstrapping the coherences as discussed in Table 7 gives similar test results as the approach in Table 6.

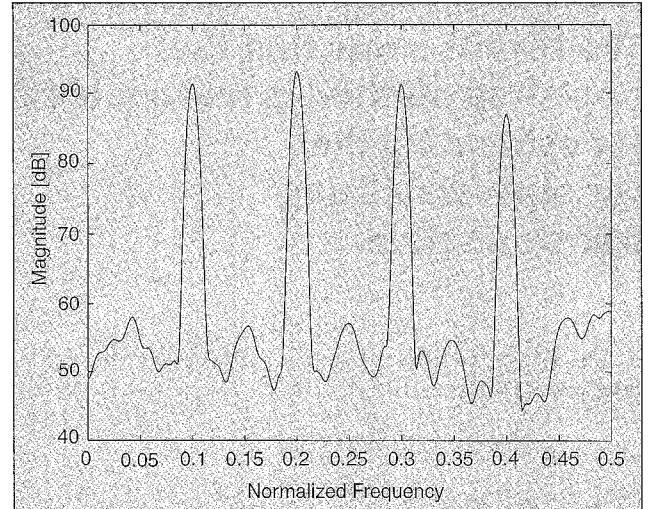
Simulation Results. We have simulated $n = 20$ independent records of a vector-valued signal, S_t , with $r = 5$. The model used to generate a component, $S_{l,t}$, $l = 1, \dots, 5$, is as follows:

$$S_{l,t} = \sum_{k=1}^K A_{k,l} \cos(\omega_k t + \Phi_{k,l}) + U_{l,t} \quad l = 1, \dots, 5. \quad (19)$$

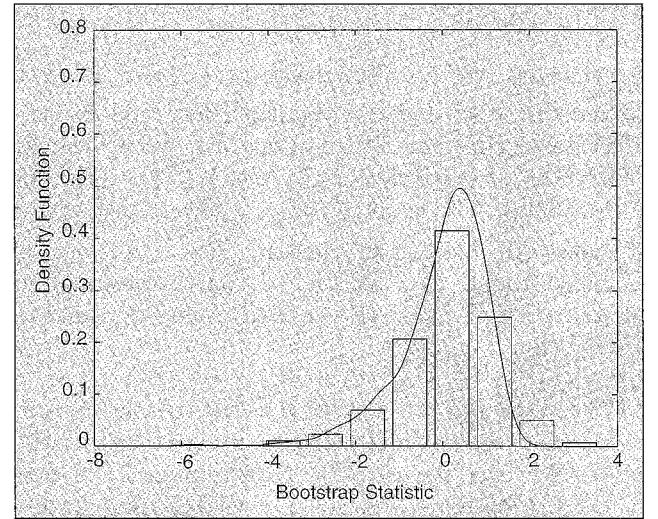
Herein, $A_{k,l}$ and $\Phi_{k,l}$ are mutually independent random amplitudes and phases, respectively; ω_k are arbitrary resonance frequencies for $k = 1, \dots, K$; and $U_{l,t}$ is a white noise process, $l = 1, \dots, r$. We have fixed $K = 4$ and generated records of length $T = 128$ each, using a uniform distribution for both the phase and the amplitude on the interval $[0, 2\pi)$ and $[0, 1)$, respectively. We have chosen resonance frequencies at $f_1 = 0.1$, $f_2 = 0.2$, $f_3 = 0.3$, and $f_4 = 0.4$, all normalized, where $f_k = \omega_k / 2\pi$, $k = 1, \dots, 4$. We have then added uniformly distributed noise, U_t , to the generated signal. A typical spectrum of $S_{l,t}$, $l = 1, \dots, r$ obtained by averaging 20 periodograms is depicted in Fig. 6.

We then generated bandstop filters (FIR filters with 256 coefficients) with bands centered about the four resonance frequencies, f_1 , f_2 , f_3 , and f_4 . As an example, the frequency response of the first filter, $G_1(\omega)$, is given in Fig. 7.

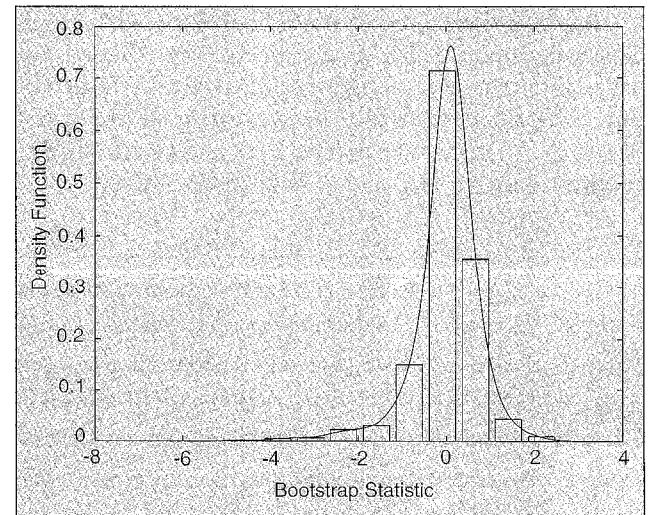
We filtered S_t and added independent uniformly distributed noise, E_t , to the filtered signal to generate Z_t . The SNR of the output signal was 5 dB with respect to the component $S_{l,t}$ with highest power. A plot of a spectral estimate of Z_t , obtained by averaging 20 periodograms, is given in Fig. 8.



▲ 8. Spectrum of Z_t , obtained by averaging 20 periodograms.



▲ 9. Histogram of 1000 bootstrap values of the statistic $(\hat{\vartheta}^*(\omega) - \hat{\vartheta}(\omega)) / \hat{\sigma}^*(\omega)$, where $\hat{\vartheta}(\omega)$ is given in Eq. (15), at a frequency bin where the hypothesis $H: G_2(\omega) = 0$ was retained.



▲ 10. Histogram of 1000 bootstrap values of the statistic $(\hat{\vartheta}^*(\omega) - \hat{\vartheta}(\omega)) / \hat{\sigma}^*(\omega)$, where $\hat{\vartheta}(\omega)$ is given in Eq. (16), at a frequency bin where the hypothesis $H: G_2(\omega) = 0$ was retained.

Table 8. The percentile-t bootstrap principle for calculating a confidence interval for the mean.

- Step 0.** *Experiment.* Conduct the experiment and collect the random data into sample $X = \{X_1, X_2, \dots, X_n\}$.
- Step 1.** *Parameter estimation.* Based on X , calculate $\hat{\mu}_X$ and $\hat{\sigma}_X$.
- Step 2.** *Resampling.* Draw a random sample, X^* of n values, with replacement, from X .
- Step 3.** *Calculation of the pivotal statistic.* Calculate the mean of all values in X^* and using a nested bootstrap calculate $\hat{\sigma}^*$. Then, form
- $$\hat{\mu}_Y^* = \frac{\hat{\mu}_X^* - \hat{\mu}_X}{\hat{\sigma}^*} \quad (21)$$
- whose distribution approximates the distribution of Eq. (20).
- Step 4.** *Repetition.* Repeat steps 2 and 3 many times to obtain a total of N bootstrap estimates, $\hat{\mu}_{Y,1}^*, \hat{\mu}_{Y,2}^*, \dots, \hat{\mu}_{Y,N}^*$.
- Step 5.** *Ranking.* Sort the bootstrap estimates into increasing order to obtain $\hat{\mu}_{Y,(1)}^* \leq \hat{\mu}_{Y,(2)}^* \leq \dots \leq \hat{\mu}_{Y,(1000)}^*$, where $\hat{\mu}_{Y,(k)}$ is the k th smallest of $\hat{\mu}_{Y,1}^*, \dots, \hat{\mu}_{Y,N}^*$.
- Step 6.** *Confidence interval.* If $(\hat{\mu}_{Y,(q_1)}, \hat{\mu}_{Y,(q_2)})$ is an interval containing $(1-\alpha)N$ of the means $\hat{\mu}_Y^*$, where $q_1 = \lfloor N\alpha/2 \rfloor$ and $q_2 = N - q_1 + 1$, then
- $$(\hat{\mu}_X - \hat{\sigma}\hat{\mu}_{Y,(q_2)}^*, \hat{\mu}_X - \hat{\sigma}\hat{\mu}_{Y,(q_1)}^*) \quad (22)$$
- is a $(1-\alpha)100\%$ confidence interval for μ_X .

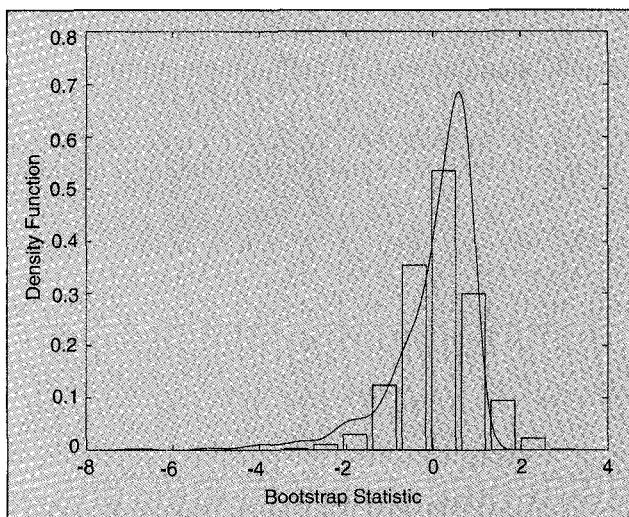
We selected arbitrarily one transfer function, $G_l(\omega)$, $l = 1, \dots, r$, and tested H . Using the procedure of Table 4 with 1000 and 30 bootstrap resamples for a quantile and variance estimate (see Table 5), respectively, we could reject H at a level of significance of 5% if ω does not fall in the stopband; otherwise, we could retain H . In the simulations we have performed, with both methods described in Table 6 and Table 7, the level of significance obtained was never above the nominal value.

Figures 9 and 10 show the bootstrap distribution of the statistic $(\hat{\vartheta}^*(\omega) - \hat{\vartheta}(\omega))\hat{\sigma}^*(\omega)$, using the procedures described in Table 6 and Table 7, where $\hat{\vartheta}(\omega)$ is given by Eqs. (15) and (16), respectively. In this case, we tested $H_2: G_2(\omega) = 0$, against the two-sided alternative, where ω was taken to be the center frequency of the bandstop. With both methods, the hypothesis was retained. Another example is shown in Figs. 11 and 12, where we tested $H_4: G_4(\omega) = 0$ and $H_3: G_3(\omega) = 0$

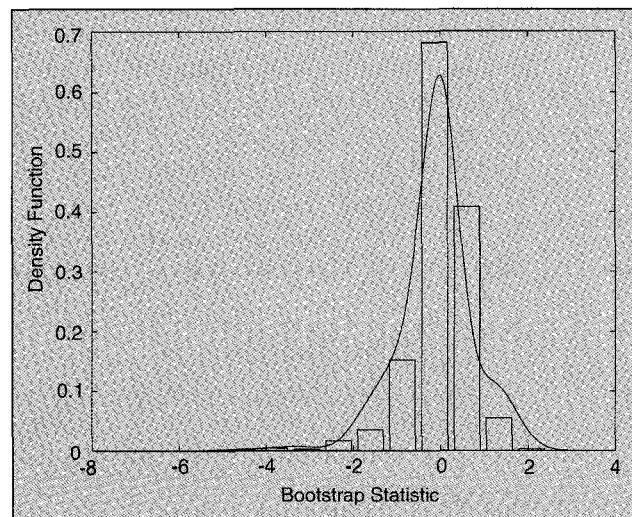
$H_3: G_3(\omega) = 0$ against two-sided alternatives at a frequency outside and inside the bandstop, with the procedures of Table 6 and Table 7, respectively. Here, again, both procedures (see Table 6 and Table 7) rejected H_4 and accepted H_3 , respectively.

These examples have shown that it is possible to perform tests in complicated situations but with minimal assumptions. In this situation, classical (F -) tests are only valid if the number of observations, T , is assumed to be large and asymptotic results for the finite Fourier transform [10] of stationary data hold.

Note that in practice one is interested in performing the above-discussed tests not only at one frequency but at a set of multiple frequencies simultaneously. A discussion of bootstrap procedures to perform multiple tests can be found in [70]. A practical application of the method described in Table 7 was discussed in [71].



▲ 11. Histogram of 1000 bootstrap values of the statistic $(\hat{\vartheta}^*(\omega) - \hat{\vartheta}(\omega))\hat{\sigma}^*(\omega)$ where $\hat{\vartheta}(\omega)$ is given in Eq. (15), at a frequency bin where the hypothesis $H: G_4(\omega) = 0$ was rejected.



▲ 12. Histogram of 1000 bootstrap values of the statistic $(\hat{\vartheta}^*(\omega) - \hat{\vartheta}(\omega))\hat{\sigma}^*(\omega)$ where $\hat{\vartheta}(\omega)$ is given in Eq. (16), at a frequency bin where the hypothesis $H: G_4(\omega) = 0$ was retained.

Table 9. Procedure of the transformed percentile-t method.

1. Transform the parameter estimate to a statistic for which a reasonably stable variance estimate is available.
2. Use the percentile-t method to construct a $(1 - \alpha)100\%$ confidence interval for the transformed parameter.
3. Invert this interval into a $(1 - \alpha)100\%$ confidence interval for the parameter of interest.

Example 4. Confidence interval for the mean

There is a well established duality between confidence intervals and hypothesis testing. If I is a confidence interval for an unknown parameter, ϑ , with coverage probability α , then a $100(1 - \alpha)\%$ level test of, for example, the hypothesis $H: \vartheta = \vartheta_0$ against $K: \vartheta \neq \vartheta_0$ is to reject H if $\vartheta_0 \notin I$. Thus, the issue of pivoting in the context of hypothesis testing also arises in the construction of confidence intervals using bootstrap methods.

In this example, we consider the construction of a confidence interval for the mean as in Example 1 presented earlier. Let $X = \{X_1, \dots, X_n\}$ be a random sample from some unknown distribution with mean μ_X and variance σ_X^2 . We wish to find an estimator of μ_X with a $(1 - \alpha)100\%$ confidence interval. Let $\hat{\mu}_X$ and $\hat{\sigma}_X^2$ be the sample mean and the sample variance of X , respectively. Alternatively to Example 1, we will base our method for finding a confidence interval for μ_X on the statistic

$$\hat{\mu}_Y = \frac{\hat{\mu}_X - \mu_X}{\hat{\sigma}} \quad (20)$$

which asymptotically has a distribution that is free of unknown parameters. In Eq. (20) $\hat{\sigma}$ can be obtained using either Eqs. (9) or (10). A bootstrap procedure for calculating a confidence interval is described in Table 8.

Such an interval is known as a *percentile-t confidence interval* [19, 31, 52]. For the same random sample, X , as in Example 1, we obtained the confidence interval (3.54, 13.94) for the mean. The percentile-t method for

constructing a confidence interval for the mean as discussed above improves upon the one discussed in Table 2. This interval is larger than the one obtained using the procedure of Table 2 and enforces the statement given in Example 1 that the interval obtained there has coverage less than the nominal 95%. It also yields better results than an interval derived using the assumption that \hat{T} is $\mathcal{N}(0,1)$ distributed or the (better) approximation that \hat{T} is t -distributed with $n - 1$ degrees of freedom because the interval obtained with the percentile-t method accounts for skewness in the underlying population or other errors [21, 31, 32, 52].

The percentile-t method is particularly applicable to location statistics, such as the sample mean, sample median, etc. [21]. However, for more general statistics, the percentile-t method may not be accurate. In the next section, we discuss a method to improve the percentile-t method in the context of hypothesis testing.

Variance Stabilization

Bootstrap tests have generally excellent power properties even for relatively low fixed values of N . This, as well as the accuracy level claimed, holds whenever the test statistic is asymptotically pivotal [34]. To ensure pivoting, usually the statistic is “studentized.” However, in many situations, standard estimates of variance, such as the jackknife estimate, to studentise, result in confidence intervals with erratically varying lengths and end points. Practical experience showed that pivoting often does not hold unless an appropriate variance-stabilizing transfor-

Table 10. Procedure for the estimation of a variance-stabilizing transformation and a bootstrap test.**Step 1. Estimation of the variance-stabilizing transformation.**

- (a) Generate B_1 bootstrap samples, X_i^* , from X and for each calculate the value of the statistic $\hat{\vartheta}_i^*$, $i = 1, \dots, B_1$.
- (b) Generate B_2 bootstrap samples from X_i^* , $i = 1, \dots, B_1$ and calculate $\hat{\sigma}_i^{*2}$, a bootstrap estimate for the variance of $\hat{\vartheta}_i^*$, for example, using Eq. (9) with $B = B_2$.
- (c) Estimate the variance function $\zeta(\vartheta)$ by smoothing the values of $\hat{\sigma}_i^{*2}$ against $\hat{\vartheta}_i^*$.
- (d) Estimate the variance-stabilizing transformation^a, $b(\hat{\vartheta})$, from

$$b(\hat{\vartheta}) = \int_{\hat{\vartheta}_l}^{\hat{\vartheta}} \{\zeta(s)\}^{-1/2} ds.$$

using some sort of numerical integration, where $\hat{\vartheta}_l$ is the lowest permissible value of $\hat{\vartheta}$.

Step 2. Bootstrap test for ϑ (or $b(\vartheta)$). Generate B_3 bootstrap samples and compute $\hat{\vartheta}_i^*$ and thus $b(\hat{\vartheta}_i^*)$ for each sample. Approximate \hat{t}_α given $P(b(\hat{\vartheta}) - b(\vartheta) \geq \hat{t}_\alpha) = \alpha$ by the α th critical point of $b(\hat{\vartheta}^*) - b(\hat{\vartheta})$.

^aIf T is a statistic that in large samples tends to a parent parameter ϑ , and the variance of T is some function of ϑ , say $\zeta(\vartheta)/n + o(n^{-1})$, then to order n the variate $Z = \int^T \{\zeta(\vartheta)^{-1/2} d\vartheta\}$ has variance $1/n + o(n^{-1})$.

Table 11. The bootstrap procedure to find a confidence interval for ϑ based on the unwrapped phase estimator.

- Step 0.** Collect and sample the data to obtain Z_t , $t = -\frac{T}{2}, \dots, \frac{T}{2} - 1$.
- Step 1.** Unwrap the phase of the T point signal Z_t to provide a non-decreasing function $\hat{\phi}_t$, which approximates the true phase ϕ_t .
- Step 2.** Obtain $\hat{\vartheta}$, an initial estimate of the aircraft parameters by fitting the nonlinear observer phase model (Eq.(25)) $\phi_{t;\hat{\vartheta}}$ to $\hat{\phi}_t$ in a least-squares sense.
- Step 3.** Compute the residuals
- $$\hat{\varepsilon}_t = \phi_{t;\hat{\vartheta}} - \hat{\phi}_t, \quad t = -T/2, \dots, T/2 - 1$$
- Step 4.** Compute $\hat{\sigma}_i$, a bootstrap estimate of the standard deviation of $\hat{\vartheta}_i$, $i = 1, \dots, 4$.
- Step 5.** Draw a random sample $X^* = \{\hat{\varepsilon}_{-T/2}, \dots, \hat{\varepsilon}_{T/2-1}\}$, with replacement, from $X = \{\hat{\varepsilon}_{-T/2}, \dots, \hat{\varepsilon}_{T/2-1}\}$ and construct
- $$\hat{\phi}_t^* = \phi_{t;\hat{\vartheta}} + \hat{\varepsilon}_t^*.$$
- Step 6.** Obtain and record the bootstrap estimates of the aircraft parameters, $\hat{\vartheta}^*$, by fitting the observer phase model to $\hat{\phi}_t^*$ in a least-squares sense.
- Step 7.** Estimate the standard deviation of $\hat{\vartheta}_i^*$ using nested bootstrap step and compute and record the bootstrap statistics
- $$\hat{T}_i^* = \frac{\hat{\vartheta}_i^* - \hat{\vartheta}_i}{\hat{\sigma}_i^*}, \quad i = 1, \dots, 4.$$
- Step 8.** Repeat steps 5 through 7 a large number of times, N .
- Step 9.** For each parameter, order the bootstrap estimates as $\hat{T}_{i,(1)}^* \leq \hat{T}_{i,(2)}^* \leq \dots \leq \hat{T}_{i,(N)}^*$ and compute the $(1-\alpha)100\%$ confidence interval as
- $$(\hat{\vartheta}_i - \hat{T}_{i(U)}^* \hat{\sigma}_i, \hat{\vartheta}_i - \hat{T}_{i(L)}^* \hat{\sigma}_i),$$
- where $U = N - \lfloor N\alpha/2 \rfloor + 1$ and $L = \lfloor N\alpha/2 \rfloor$.

mation for the parameter estimate of interest is applied first. Hall et al. [33] have discussed the construction of a confidence interval for the correlation coefficient when using a variance-stabilizing transformation. The idea suggested in [33] is illustrated in Table 9 (see also [21, 52]).

The problem with this procedure is that in many applications a variance-stabilizing transformation is not known. Tibshirani has introduced an “automatic” method of obtaining a variance-stabilizing transformation for use in constructing percentile- t confidence intervals [65]. The method estimates a variance-stabilizing transformation using an initial double bootstrap step, computes a percentile- t interval for the transformed statistic, and then transforms this interval back to the original scale. Tibshirani’s idea can be used to conduct hypothesis tests with good power properties as illustrated in Table 10. Advantages of this are that it is “automatic”—one does not need to know the variance-stabilizing transformation in advance and that it is invariant under monotonic transformations in some settings.

The total number of bootstrap samples required by this procedure is $B_1 B_2 + B_3$. In the example given by Tibshirani [65] $B_1 = 100$, $B_2 = 25$ and $B_3 = 1000$ were used so that the total number of bootstrap samples is 3500. $B_2 = 25$ and $B_3 = 1000$ were suggested by Efron [21] to estimate a quantile and a variance, respectively. Tibshirani [65] believes that $B_1 = 100$ is sufficient for the estimator of the variance-stabilizing transformation because

of the smoothing that is inherent in the computation explicitly in step 1(c) and implicitly in step 1(d) as per Table 10. Note that step 1(b) in Table 10 can be replaced by the jackknife method. A jackknife estimate might help to reduce the number of computations, especially when the sample size is small.

Example 5. Confidence interval for the correlation coefficient
Let $\vartheta = \rho$ be the correlation coefficient of two populations, assumed to be unknown, and let $\hat{\rho}$ and $\hat{\sigma}^2$ be estimates of ρ and the variance of $\hat{\rho}$, respectively, based on the samples $X = \{X_1, \dots, X_n\}$ and $Y = \{Y_1, \dots, Y_n\}$. Let X^* and Y^* be resamples, drawn with replacement from X and Y , respectively, and let $\hat{\rho}^*$ and $\hat{\sigma}^{*2}$ be versions of $\hat{\rho}$ and $\hat{\sigma}^2$ computed using X^* and Y^* rather than X and Y . By repeated resampling from X and Y we compute \hat{s}_α and \hat{t}_α , such that with $0 < \alpha < 1$

$$\begin{aligned} P((\hat{\rho}^* - \hat{\rho}) / \hat{\sigma}^* \leq \hat{s}_\alpha | X, Y) &= \frac{\alpha}{2} \\ &= P((\hat{\rho}^* - \hat{\rho}) / \hat{\sigma}^* \geq \hat{t}_\alpha | X, Y) \end{aligned}$$

The percentile- t confidence interval for ρ is given by $I(X, Y) = (\hat{\rho} - \hat{\sigma} \hat{t}_\alpha, \hat{\rho} + \hat{\sigma} \hat{t}_\alpha)$. A transformed percentile- t confidence interval is obtained as shown below.

In the case of the correlation coefficient, there exists a transformation called Fisher’s z -transform [28] that is

stabilizing and normalizing [1]. The transformation maps the parameter estimate \hat{p} into $\check{p} = \frac{1}{2} \log \frac{1+\hat{p}}{1-\hat{p}}$. The transformed percentile- t method consists of first finding a confidence interval for $\xi = \tanh^{-1} \hat{p} = \frac{1}{2} \log \frac{1+\hat{p}}{1-\hat{p}}$ and then of transforming the endpoints back with the inverse transformation $p = \tanh \xi = (e^{2\xi} - 1) / (e^{2\xi} + 1)$ to obtain a confidence interval for p .

Under the assumption that X and Y are bivariate normal, the estimator \check{p} is normally distributed with mean ξ and variance $1 / (n - 3)$ [1]. Thus,

$$P(-1.96 \leq \sqrt{n-3}(\check{p} - \xi) \leq 1.96) = 0.95,$$

and therefore the 95% confidence interval for ξ is given by

$$\left(\frac{-1.96}{\sqrt{n-3}} + \check{p}, \frac{1.96}{\sqrt{n-3}} + \check{p} \right).$$

A 95% confidence interval for p is then obtained from

$$\left(\tanh\left(\frac{-1.96}{\sqrt{n-3}} + \check{p} \right), \tanh\left(\frac{1.96}{\sqrt{n-3}} + \check{p} \right) \right), \quad (23)$$

if we were to assume the Gaussian distribution for \check{p} .

Simulation Results. Suppose that $X = Z_1 + W$ and $Y = Z_2 + W$, where Z_1, Z_2 , and W are pairwise independent and identically distributed. In this case, the correlation coefficient of X and Y is $p = 0.5$. We drew $n = 15$ realizations $z_{1,i}, z_{2,i}$, and $w_i, i = 1, \dots, 15$, from the normal distribution and calculated $x_i, y_i, i = 1, \dots, 15$. We found $\hat{p} = 0.36$ and the 95% confidence interval $(-0.18, 0.74)$ for p , using Eq. (23). On the other hand, we used the bootstrap percentile- t method similar to the one described in Table 8 for the mean and found with $N = 1000$ the 95% confidence interval $(-0.05, 1.44)$. We also used Fisher's z -transform and calculated, based on the bootstrap (without assuming bivariate normal distribution of (X, Y)), a

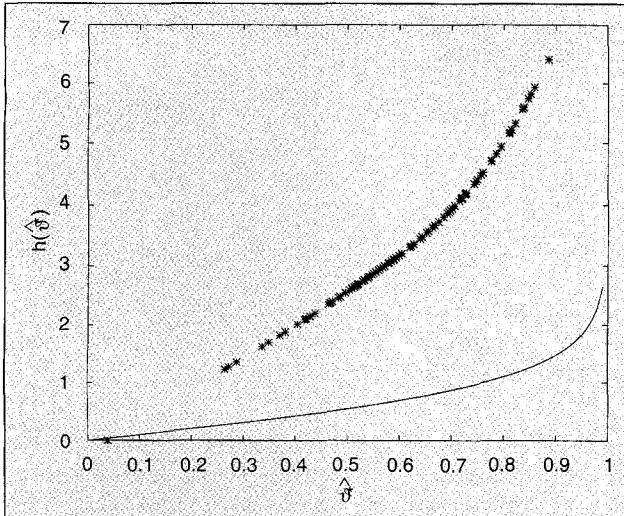
Bootstrap methods are extremely valuable in situations where data sizes are too small to invoke asymptotic results.

confidence interval for the transformed parameter $\xi = \tanh^{-1} p$. Its endpoints were then transformed back with the inverse transformation \tanh to yield the confidence interval $(-0.28, 0.93)$ for p . In both bootstrap approaches we have used a jackknife variance estimate. Clearly, the interval found using the percentile- t method is over-covering (Note that p is bounded within the interval $[-1, 1]$), being larger than the two other ones. Therefore, finding a confidence interval for the transformed parameter and then transforming the endpoints back with the inverse transformation yields a better interval than the one obtained using the percentile- t method. We never observed in the simulations we ran that the transformed percentile- t confidence interval obtained contained values outside the interval $[-1, 1]$.

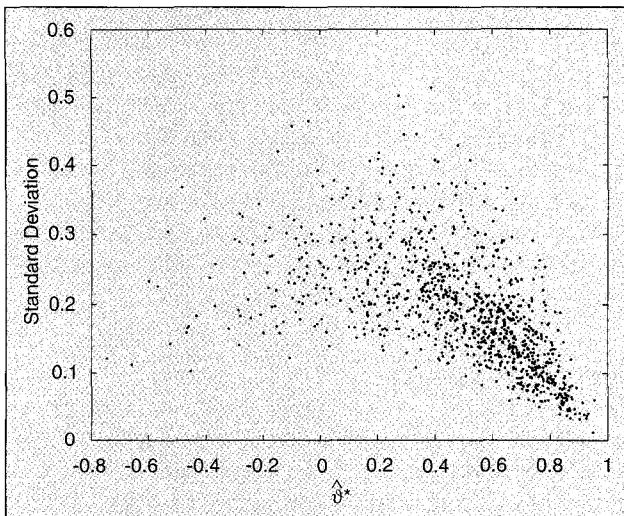
Furthermore, we considered a bootstrap-based variance-stabilizing transformation as an alternative to Fisher's z -transform, as illustrated in Table 10. The smoothing in stage 1(c) was performed by using a fixed-span 50% "running lines" smoother described by Hastie and Tibshirani [37]. This smoother fits a least-squares regression line in symmetric windows centered at each $\hat{\theta}_i^*$, $i = 1, \dots, B_1$. It is simple to implement and generally performs well. The integration in step 1(d) was approximated by a trapezoid rule. We used $B_1 = 100$, $B_3 = 1000$, and a bootstrap variance estimate with $B_2 = 25$. The so obtained variance-stabilizing transformation is depicted in Fig. 13 along with Fisher's z -tran-

Table 12. Results for parameter estimates based on the unwrapped phase. The 95% confidence bounds based on the bootstrap for each of the parameters are compared with the 95% confidence bounds determined by Monte Carlo simulation. These results are shown for 30 dB, 20 dB and 15 dB SNR.

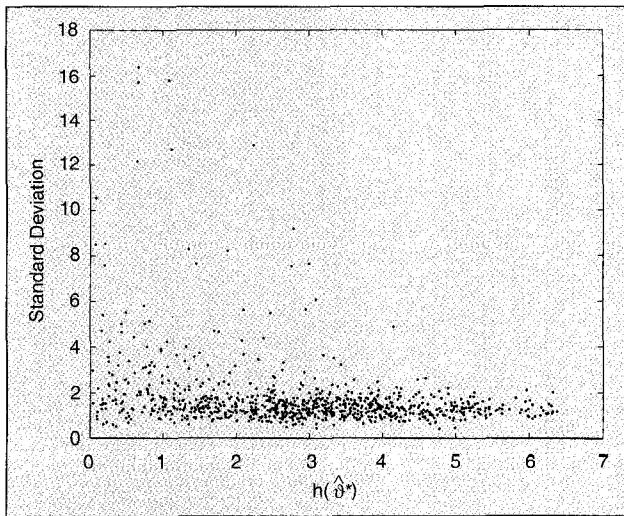
		Height (m)		Velocity (ms^{-1})		Time Ref. (s)		Source Freq.	
	Actual	304.8		102.89		0		1.000	
SNR		BS	MC	BS	MC	BS	MC	BS	MC
30 dB	Upper Bound	309.1	308.0	103.09	103.04	0.004	0.005	1.0001	1.0001
	Lower Bound	302.2	301.9	102.76	102.74	-0.006	-0.005	0.9999	0.9999
	Interval length	6.9	6.1	0.33	0.30	0.010	0.010	0.0002	0.0002
20 dB	Upper Bound	305.6	314.5	103.06	103.34	0.015	0.016	1.0002	1.0004
	Lower Bound	287.4	295.2	102.20	102.44	-0.017	-0.016	0.9994	0.9996
	Interval length	18.2	19.4	0.86	0.92	0.032	0.032	0.0008	0.0008
15 dB	Upper Bound	328.2	320.7	103.52	103.65	0.021	0.027	1.0003	1.0007
	Lower Bound	290.7	288.6	102.30	102.12	-0.032	-0.027	0.9992	0.9993
	Interval length	37.5	32.1	1.21	1.52	0.053	0.054	0.0011	0.0014



▲ 13. Variance-stabilizing transformation for the correlation coefficient estimated using $B_1 = 100$ and $B_2 = 25$. The solid line is a plot of Fisher's z -transform.



▲ 14. Bootstrap estimates of the standard deviation of $B_3 = 1000$ (bootstrap) estimates of the correlation coefficient before applying the variance-stabilizing transformation.



▲ 15. Bootstrap estimates of the standard deviation of $B_3 = 1000$ (bootstrap) estimates of the correlation coefficient after applying the variance-stabilizing transformation of Fig. 13.

sform applied to the same bootstrap values, $\hat{\vartheta}_i^*, i = 1, \dots, 1000$.

To demonstrate the effect of the variance-stabilizing transformation, we estimated the standard deviation of 1000 bootstrap estimates of $\vartheta = \rho$ using the bootstrap (see Table 5 with $B = 25$) resulting in the graph of Fig. 14. The graph shows the dependence of the standard deviation with respect to $\hat{\vartheta}^* = \hat{\rho}^*$. After taking 1000 new bootstrap estimates $\hat{\rho}_i^*, i = 1, \dots, 1000$, and applying the transformation of Fig. 13, we obtained the more stable standard deviations (estimated using Table 5) of Fig. 15. For comparison, we have also reproduced in Fig. 16 the standard deviation of new 1000 bootstrap estimates, $\hat{\rho}_i^*, i = 1, \dots, 1000$, after applying Fisher's z -transform depicted in Fig. 13 (solid line). The results show that the bootstrap method is reasonable for estimating the variance-stabilizing transformation, which will lead to more accurate confidence intervals in situations where a variance-transformation situation is not known.

To construct a confidence interval for ϑ with the method of Table 10 we need first to find an interval for $h(\vartheta)$ and then back-transform the interval for $h(\vartheta)$, to give $(h^{-1}(h(\hat{\vartheta}) - \hat{t}_{1-\alpha}), h^{-1}(h(\hat{\vartheta}) - \hat{t}_\alpha))$, where \hat{t}_α is the α th critical point of the bootstrap distribution of $h(\hat{\vartheta}^*) - h(\hat{\vartheta})$. For the same data, $z_{1,i}, z_{2,i}$, and w_i , $i = 1, \dots, 15$, we obtained the confidence interval $(0.06, 0.97)$. This interval is much tighter than the one obtained using the transformed percentile- t method based on Fisher's z -transform.

The importance of variance stabilization for hypothesis testing and confidence-interval construction is elaborated in [33, 65] where Monte Carlo simulation results and extensive analysis can be found.

Determination of Confidence Intervals for Passive Acoustic Aircraft Parameters

The application reported in this section illustrates the concepts presented earlier for confidence interval estimation using the percentile- t method.

Introduction

We consider the problem of estimating an aircraft's constant height, velocity, range, and acoustic frequency based on a single acoustic recording of the aircraft passing overhead. Information about these physical parameters is contained in the phase- and time-varying Doppler frequency shift, or instantaneous frequency (IF), of the observed acoustic signal. An estimation scheme has been previously demonstrated [22-24, 57, 58] in this application using a model for the IF.

To establish some statistical confidence of the parameters based on these estimates, we could use the theoretical distribution of the parameter estimates. However, the theoretical derivation of the parameter distribution is often mathematically intractable particularly when the dis-

Table 13. Estimated 95% confidence bounds for real passive acoustic data. The parameter estimates were based on the unwrapped phase of the signal.

	Height (m)	Velocity (m/s)	Time Ref. (s)	Source freq. (Hz)
Nominal Value	422	72	0	68.2
Upper Bound	460.9	74.53	0.08	68.34
Lower Bound	378.8	70.46	-0.06	68.11
Interval length	82.1	4.07	0.14	0.23

tribution of the noise is unknown. Alternatively, if multiple realizations of the acoustic signal were available it would be a straightforward task to empirically determine the distribution of the parameter estimates. In practice this is not possible since only a single realization of the acoustic signal is available. In this section we use bootstrap techniques to provide a practical means of determining the confidence bounds for the aircraft parameter estimates without assuming any distribution for the noise [59].

A simple model for the aircraft acoustic signal, as heard by a stationary observer, is expressed as

$$Z(t) = Ae^{j\phi(t)} + U(t), \quad t \in \mathbb{R} \quad (24)$$

where $U(t)$ is a continuous time, zero-mean complex white noise process with variance σ^2 , A is a constant assumed herein, without loss of generality, to be unity, and $\phi(t)$ is given by

$$\begin{aligned} \phi(t) = & 2\pi \frac{f_a c^2}{c^2 - v^2} \left(t - \sqrt{\frac{h^2 c + v^2 t^2 c + 2v^2 tb}{c^3}} \right) \\ & + \phi_0, \quad t \in \mathbb{R} \end{aligned} \quad (25)$$

where t_0 is the time when the aircraft is directly overhead, f_a is the source acoustic frequency, c is the speed of sound in the medium, v is the constant velocity of the aircraft, h is the constant altitude of the aircraft, and ϕ_0 is an initial phase constant.

From Eq. (25) the IF, relative to the stationary observer, can be expressed as

$$\begin{aligned} f(t) = & \frac{1}{2\pi} \frac{d\phi(t)}{dt} = \frac{f_a c^2}{c^2 - v^2} \\ & \left(1 - \frac{v^2 (t + h/c)}{\sqrt{h^2 (c^2 - v^2) + v^2 c^2 (t + h/c)^2}} \right), \quad t \in \mathbb{R} \end{aligned} \quad (26)$$

For a given $f(t)$ or $\phi(t)$, and c , the aircraft parameters collected in the vector $\boldsymbol{\theta} = (f_a, h, v, t_0)'$ can be uniquely determined from the phase model (Eq. (25)) or observer IF model (Eq. (26)).

To illustrate the use of the bootstrap, we consider an estimate of the unwrapped phase of the observed signal,

as modeled by Eq. (25). The phase estimate is then fitted to the model in Eq. (25) in a least-squares sense to provide an estimate of the aircraft parameters. Bootstrap techniques are then applied to provide confidence bounds for the parameters based on this estimate.

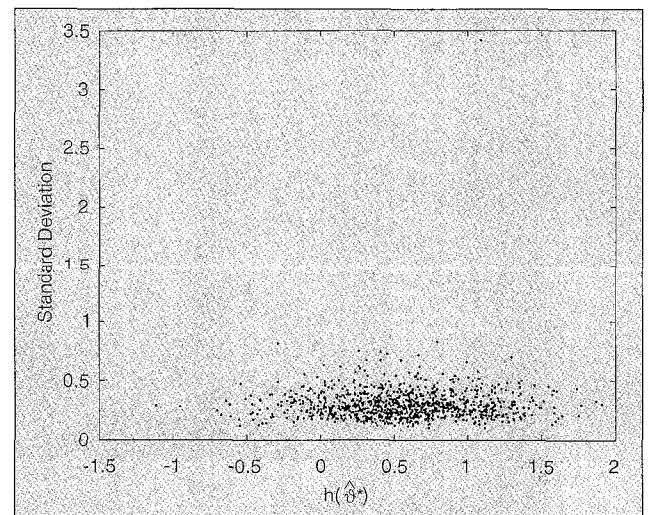
Alternatively, one may choose to use the observer IF, as modeled by Eq. (26), using central finite difference (CFD) methods [4, 5]. The CFD estimate is then fitted to the model in Eq. (26) in a least-squares sense to provide an estimate of the aircraft parameters. Bootstrap confidence bound estimates are then obtained. However, special care must be taken with the CFD approach due to the correlation of the residuals (see, for example, [73]).

We first validate the method presented in Table 11 using synthetic data and then we demonstrate the application of the bootstrap to real passive acoustic data.

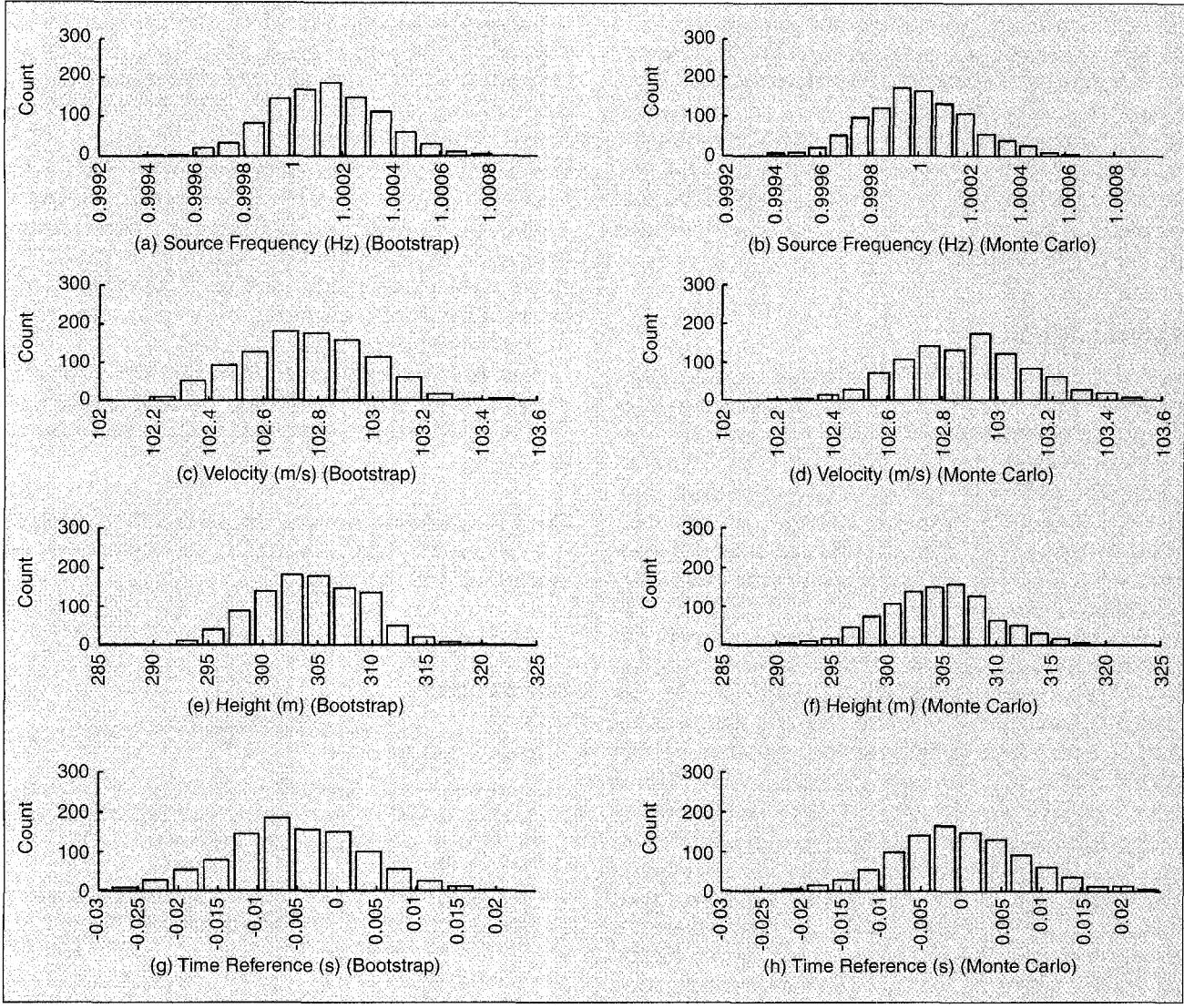
Simulation Results

We consider only discrete-time signals as appropriate sampled versions of the continuous-time signals and denote by ϕ_t and Z_t , the phase and the observed signal as functions of the discrete time parameter $t \in \mathbb{Z}$.

We estimate the aircraft parameters using a $T = 320$ -point test signal Z_t described by Eq. (24) at



▲ 16. Bootstrap estimates of the standard deviation of $B_3 = 1000$ (bootstrap) estimates of the correlation coefficient after applying Fisher's variance-stabilizing transformation \tanh^{-1} .



▲ 17. The histograms for the unwrapped phase-based bootstrap estimates (left column) are compared with the histograms of the Monte Carlo estimates for each of the parameters (right column) at 20 dB SNR: (a, b) source frequency, (c, d) velocity, (e, f) height, and (g, h) time reference.

three levels of SNR (15 dB, 20 dB, and 30 dB). The parameter estimates are derived from the phase of the observed signal, Z_t . The steps in this procedure are given in Table 11. Zero-mean white Gaussian noise was used in the simulation. The parameters of the test signal are; sampling frequency $f_s = 8$ Hz, $h = 304.8$ m, $v = 102.89$ ms $^{-1}$, $f_a = 1$ Hz, and $t_0 = 0$. The number of resamples was $N = 1000$.

The bootstrap results are then compared with those computed by Monte Carlo simulation where the parameter estimates are computed for $N = 1000$ independent realizations of Z_t having the same parameter values and SNR levels as for the bootstrap experiments. The confidence bounds for each of the parameters are then computed as for the bootstrap procedure. We assess the performance of the bootstrap experiments by comparing them with the corresponding Monte Carlo results.

The results of this experiment are summarized in Table 12. The bootstrap-based bounds, as shown in the Table

12, agree closely with the Monte Carlo results at each of the three SNR levels considered. In Figures 17a-h bootstrap based histograms for each of the parameter estimates (left column) are compared with the histograms of their Monte Carlo counterparts (right column) at 20 dB SNR.

Results with Real Passive Acoustic Data

The aircraft parameter-estimation technique, using the unwrapped phase of the signal, and the bootstrap technique are now applied to real passive acoustic data. The physical parameters of the aircraft in this single acoustic recording are nominally: $h = 422$ m, $v = 72$ ms $^{-1}$, $t_0 \approx 0$, $f_a = 682$ Hz. The 95% confidence bounds for the parameters, with $N = 1000$ bootstrap resamples, are shown in Table 13.

The results confirm that bootstrap techniques can be used in the aircraft passive acoustic parameter estimation problem to provide confidence bounds for the parame-

ters, without assuming any statistical distribution for the parameter estimates and in the absence of multiple acoustic realizations. The bootstrap-derived bounds agree closely with those obtained by Monte Carlo simulation. We performed similar experiments with CFD estimates of Eq. (26) and observed that the confidence bounds presented here are much tighter than those of the CFD-based estimates. Thus, the unwrapped phase-based method provided superior performance.

Conclusions

In this article we attempted to introduce the reader to the powerful bootstrap, which is to date unavailable to most engineers. The bootstrap is an extremely attractive tool because it requires very little in the way of modeling, assumptions, or analysis, and it can be applied in an automatic way. Further, bootstrap methods are extremely valuable in situations where data sizes are too small to invoke asymptotic results, which is often the case in signal-processing applications.

We described the basic concept of the bootstrap, discussed its application to testing statistical hypotheses, and looked at accuracy related issues. In particular, we discussed the principle of pivoting, which is standardizing for scale so that large-sample distributions of test statistics do not depend on unknown parameters. We found that bootstrap tests based on pivots have simultaneously greater accuracy of level of significance and greater accuracy in terms of position of critical point than tests that do not employ pivots. We considered a bootstrap-based method for finding variance-stabilizing transformations to ensure pivoting. Several examples and one real-life application to passive acoustic aircraft parameter estimation were given to illustrate the use of bootstrap techniques.

The bootstrap substitutes considerable amounts of computation in place of theoretical analysis. In an era of exponentially declining computational costs, such computer-intensive methods are becoming increasingly attractive. We recommend that the reader implements the algorithms presented in this article to his or her particular application to discover the power of the bootstrap for signal-processing applications. Matlab® codes that can assist the reader are available upon request.

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