SVD: Example 1

For symmetric matrices, eigenvectors for distinct eigenvalues are orthogonal

All eigenvalues of a real symmetric matrix are real

All eigenvalues of a positive semi-definite matrix are non-negative

A positive semi-definite matrix is a symmetric matrix with non-negative eigenvalues.

$$S = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Calculate Eigenvalues

$$S - \lambda * I = 0$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = 0$$

$$(2-\lambda)*(2-\lambda)-1 \quad 0$$

λ = 1 $\qquad$ λ = 0

λ = 3 $\qquad$ λ = 0

Plug in these eigenvalues values to find eigenvectors

$$(S - \lambda * I) V = 0$$

Eigenvector for λ = 1

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} v1 \\ v2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} * \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Eigenvector for λ = 3

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} v1 \\ v2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} * \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

OR

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} * \begin{bmatrix} -1 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The eigenvectors are orthogonal (and real):

Columns of U are eigenvectors of S; (Recall UU–1 =1)

$$U = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \qquad U^{-1} = \begin{bmatrix} 0.5 & -0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

Diagonal elements of Σ are eigenvalues of S

$$\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

Reconstructed of S from U, U^{-1} and Σ

$$S = U\Sigma U^{-1}$$

$$S = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} * \begin{bmatrix} 0.5 & -0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & 3 \\ -1 & 3 \end{bmatrix} * \begin{bmatrix} 0.5 & -0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

Reconstructed S

$$S = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$