

Bayesian alignment using hierarchical models, with applications in protein bioinformatics

BY PETER J. GREEN

School of Mathematics, University of Bristol, Bristol BS8 1TW, U.K.

p.j.green@bristol.ac.uk

AND KANTI V. MARDIA

Department of Statistics, School of Mathematics, University of Leeds, Leeds LS2 9JT, U.K.

k.v.mardia@leeds.ac.uk

SUMMARY

An important problem in shape analysis is to match configurations of points in space after filtering out some geometrical transformation. In this paper we introduce hierarchical models for such tasks, in which the points in the configurations are either unlabelled or have at most a partial labelling constraining the matching, and in which some points may only appear in one of the configurations. We derive procedures for simultaneous inference about the matching and the transformation, using a Bayesian approach. Our hierarchical model is based on a Poisson process for hidden true point locations; this leads to considerable mathematical simplification and efficiency of implementation of EM and Markov chain Monte Carlo algorithms. We find a novel use for classical distributions from directional statistics in a conditionally conjugate specification for the case where the geometrical transformation includes an unknown rotation. Throughout, we focus on the case of affine or rigid motion transformations. Under a broad parametric family of loss functions, an optimal Bayesian point estimate of the matching matrix can be constructed that depends only on a single parameter of the family. Our methods are illustrated by two applications from bioinformatics. The first problem is of matching protein gels in two dimensions, and the second consists of aligning active sites of proteins in three dimensions. In the latter case, we also use information related to the grouping of the amino acids, as an example of a more general capability of our methodology to include partial labelling information. We discuss some open problems and suggest directions for future work.

Some key words: Bioinformatics; Markov chain Monte Carlo; Matching; Poisson process; Protein gel; Protein structure; Shape analysis; Von Mises-Fisher distribution.

1. INTRODUCTION

Various new challenging problems in shape matching have been appearing from different scientific areas including bioinformatics and image analysis. In a class of problems in shape analysis, one assumes that the points in two or more configurations are labelled and these configurations are to be matched after filtering out some transformation. Usually the transformation is a rigid transformation or similarity transformation. Several new problems are appearing in which either the points of configuration are not labelled or

the labelling is ambiguous, and in which some points do not appear in each of the configurations. An example of ambiguous labelling arises in understanding the secondary structure of proteins, where we are given not only the three-dimensional molecular configuration but also the types of amino acid at each point. A generic problem is to match two such configurations, where the matching has to be invariant under some transformation group. Descriptions of such problems can be found in the review article by Mardia et al. (2003).

We now describe two datasets related to protein structure. One is of two-dimensional gel data, where each point is a protein itself and the transformation group is affine. In this case we have a partial matching, identified already by experts, that we can use to assess our procedures. In the second example we have a three-dimensional configuration of two active sites of two proteins which also has additional chemical information. Here the underlying transformation to be filtered out is rigid motion. In this problem, one of the main aims is to take a query active site and find matches to a given database, in some ranking order. The matches will give some idea of the functions of the unknown proteins, leading to the design of new enzymes, for example.

There are other related examples from image analysis such as the matching of buildings when one has multiple two-dimensional views of three-dimensional objects; see for example Cross & Hancock (1998). The problem here requires one to filter out the projective transformations before matching. Other examples involve matching outlines or surfaces; see for example Chui & Rangarajan (2000) and an unpublished 2002 Ph.D. thesis from the Technical University of Denmark by L. Pedersen. Here no labelling of points is involved, and we are dealing with a continuous contour or surface rather than a finite number of points. Such problems are not addressed in this paper.

The principal innovations in our approach are the fully model-based approach to alignment, the fact that the model formulation allows us to integrate out the hidden point locations, the prior specification for the rotation matrix, and the Markov chain Monte Carlo algorithm.

2. HIERARCHICAL MODELLING OF ALIGNMENT AND MATCHING PROBLEMS

2.1. *Point process model, with geometrical transformation and random thinning*

Suppose we are given two point configurations

$$\{x_j, j = 1, 2, \dots, m\}, \quad \{y_k, k = 1, 2, \dots, n\},$$

in d -dimensional space \mathcal{R}^d . The points are labelled for identification, but arbitrarily.

Both point sets are regarded as noisy observations on subsets of a set of true locations $\{\mu_i\}$, where we do not know the mappings from j and k to i . There may be a geometrical transformation between the x -space and the y -space, which may also be unknown. The objective is to make model-based inference about these mappings, and in particular make probability statements about matching: which pairs (j, k) correspond to the same true location?

The geometrical transformation between the x -space and the y -space will be denoted by \mathcal{A} ; thus y in y -space corresponds to $x = \mathcal{A}y$ in x -space. The notation does not imply that the transformation \mathcal{A} is necessarily linear. It may be a rotation or more general linear transformation, a translation, both of these, or some nonrigid motion. We regard the true locations $\{\mu_i\}$ as being in x -space.

The mappings between the indexing of the $\{\mu_i\}$ and that of the data $\{x_j\}$ and $\{y_k\}$ are captured by indexing arrays $\{\xi_j\}$ and $\{\eta_k\}$; to be specific we assume that

$$x_j = \mu_{\xi_j} + \varepsilon_{1j}, \quad (1)$$

for $j = 1, 2, \dots, m$, where $\{\varepsilon_{1j}\}$ have probability density f_1 , and

$$\mathcal{A}y_k = \mu_{\eta_k} + \varepsilon_{2k}, \quad (2)$$

for $k = 1, 2, \dots, n$, where $\{\varepsilon_{2k}\}$ have density f_2 . Multiple matches are excluded, and thus each hidden point μ_i is observed at most once in each of the x and y configurations; equivalently, the ξ_j are distinct, as are the η_k . All $\{\varepsilon_{1j}\}$ and $\{\varepsilon_{2k}\}$ are independent of each other, and independent of the $\{\mu_i\}$.

2.2. Formulation of Poisson process prior

Suppose that the set of true locations $\{\mu_i\}$ forms a homogeneous Poisson process with rate λ over a region $V \subset \mathcal{R}^d$ of volume v , and that N points are realised in this region. Some of these give rise to both x and y points, some to points of one kind and not the other, and some are not observed at all. We suppose that these four possibilities occur independently for each realised point, with probabilities parameterised so that with probabilities $(1 - p_x - p_y - \rho p_x p_y, p_x, p_y, \rho p_x p_y)$ we observe neither, x alone, y alone, or both x and y , respectively. The parameter ρ is a certain measure of the tendency a priori for points to be matched: the random thinnings leading to the observed x and y configurations can be dependent, but remain independent from point to point.

Given N , m and n , there are L matched pairs of points in our sample if and only if the numbers of these four kinds of occurrence among the N points are

$$(N - m - n + L, m - L, n - L, L).$$

Under the above assumptions these four counts will be independent Poisson-distributed variables, with means $(\lambda v(1 - p_x - p_y - \rho p_x p_y), \lambda v p_x, \lambda v p_y, \lambda v \rho p_x p_y)$. The prior probability distribution of L conditional on m and n is therefore proportional to

$$\frac{e^{-\lambda v p_x} (\lambda v p_x)^{m-L}}{(m-L)!} \times \frac{e^{-\lambda v p_y} (\lambda v p_y)^{n-L}}{(n-L)!} \times \frac{e^{-\lambda v \rho p_x p_y} (\lambda v \rho p_x p_y)^L}{L!},$$

so that

$$p(L) \propto \frac{(\rho/\lambda v)^L}{(m-L)!(n-L)!L!}, \quad (3)$$

for $L = 0, 1, \dots, \min\{m, n\}$. The normalising constant here is the reciprocal of $H\{m, n, \rho/(\lambda v)\}$, where H can be written in terms of the confluent hypergeometric function:

$$H(m, n, d) = \frac{d^m}{m!(n-m)!} {}_1F_1(-m, n-m+1, -1/d),$$

assuming without loss of generality that $n > m$; see Abramowitz & Stegun (1970, p. 504). Here and later, we use the generic $p(\cdot)$ notation for densities and conditional densities in our hierarchical model.

The matching of the configurations is represented by the ‘matching matrix’ M , where M_{jk} indicates whether or not x_j and y_k are derived from the same μ_i point; that is,

$$M_{jk} = \begin{cases} 1, & \text{if } \xi_j = \eta_k, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\sum_{j,k} M_{jk} = L$, and that, since multiple matches are ruled out, there is at most one 1 in each row and in each column of M : $\sum_j M_{jk} \leq 1$ for all k , $\sum_k M_{jk} \leq 1$ for all j . We assume for the moment that, conditional on L , M is a priori uniform: there are

$$L! \binom{m}{L} \binom{n}{L}$$

different M matrices consistent with a given value of L , and these are taken as equally likely. Thus

$$p(M) = p(L)p(M|L) \propto \frac{(\rho/\lambda v)^L}{(m-L)!(n-L)!L!} \left\{ L! \binom{m}{L} \binom{n}{L} \right\}^{-1} \propto (\rho/\lambda v)^L,$$

where here and later ‘ \propto ’ means proportional to, as functions of the variable(s) to the left of the conditioning sign, in this case, M . Thus,

$$p(M) = \frac{(\rho/\lambda v)^L}{\sum_{\ell=0}^{\min\{m,n\}} \ell! \binom{m}{\ell} \binom{n}{\ell} (\rho/\lambda v)^\ell}. \quad (4)$$

Note that, because of the choice of parameterisation for the probabilities that hidden points are observed, this expression does not involve p_x and p_y .

2.3. Likelihood of data

We now have to specify the likelihood of the observed configurations of points, given M . For simplicity, we will henceforth assume that \mathcal{A} is an affine transformation: $\mathcal{A}y = Ay + \tau$. From (1) and (2), the densities of x_j and y_k , conditional on A , τ , $\{\mu_i\}$, $\{\xi_j\}$ and $\{\eta_k\}$ are $f_1(x_j - \mu_{\xi_j})$ and $|A|f_2(Ay_k + \tau - \mu_{\eta_k})$, respectively, $|A|$ denoting the absolute value of the determinant of A .

The locations $\{\mu_i\}$ of the $m - L$ points that generate an x observation but not a y observation are independently and uniformly distributed over the region V , so that the likelihood contribution of these $m - L$ observations, namely $\{x_j: M_{jk} = 0 \text{ for all } k\}$, is

$$\prod_{j: M_{jk}=0 \text{ for all } k} v^{-1} \int_V f_1(x_j - \mu) d\mu.$$

Similarly, the contributions from the unmatched y observations, and from the matched pairs, are

$$\prod_{k: M_{jk}=0 \text{ for all } j} v^{-1} \int_V |A|f_2(Ay_k + \tau - \mu) d\mu,$$

$$\prod_{j,k: M_{jk}=1} v^{-1} \int_V f_1(x_j - \mu) |A|f_2(Ay_k + \tau - \mu) d\mu,$$

respectively. These integrals all exhibit ‘edge effects’ from the boundary of the region V , which can be neglected if V is large relative to the supports of f_1 and f_2 . In this case these three expressions approximate to

$$v^{-(m-L)}, \quad (|A|/v)^{n-L}, \quad (|A|/v)^L \prod_{j,k:M_{jk}=1} \int_{\mathcal{R}^d} f_1(x_j - \mu) f_2(Ay_k + \tau - \mu) d\mu,$$

respectively. The last expression can be written

$$(|A|/v)^L \prod_{j,k:M_{jk}=1} g(x_j - Ay_k - \tau),$$

where $g(z) = \int f_1(z + u) f_2(u) du$, the density of $\varepsilon_{1j} - \varepsilon_{2k}$.

Combining these terms, we obtain the complete likelihood as

$$p(x, y | M, \mathcal{A}) = v^{-(m+n)} |A|^n \prod_{j,k:M_{jk}=1} g(x_j - Ay_k - \tau). \quad (5)$$

Multiplying (4) and (5), we then have

$$p(M, x, y | \mathcal{A}) \propto |A|^n \prod_{j,k:M_{jk}=1} \{(\rho/\lambda) g(x_j - Ay_k - \tau)\}.$$

Note that the constant of proportionality involves m , n , λ , ρ and v , but not A , τ , any parameters in f_1 or f_2 , or M of course.

If we further specialise by making assumptions of spherical normality for f_1 and f_2 , so that

$$x_j \sim N_d(\mu_{\xi_j}, \sigma_x^2 I), \quad Ay_k + \tau \sim N_d(\mu_{\eta_k}, \sigma_y^2 I),$$

with $\sigma_x = \sigma_y = \sigma$, say, then

$$g(z) = \frac{1}{(\sigma\sqrt{2})^d} \phi\{z/(\sigma\sqrt{2})\},$$

where ϕ is the standard normal density in \mathcal{R}^d , and our final joint model is

$$p(M, A, \tau, \sigma, x, y) \propto |A|^n p(A) p(\tau) p(\sigma) \prod_{j,k:M_{jk}=1} \frac{[\rho \phi\{(x_j - Ay_k - \tau)/(\sigma\sqrt{2})\}]}{\lambda(\sigma\sqrt{2})^d}. \quad (6)$$

Note that not only p_x and p_y but also v does not appear in this expression, principally from our choice of parameterisation, and that only the ratio ρ/λ is identifiable. The directed acyclic graph representing this joint probability model, including the variables μ , ξ and η that we have integrated out, is displayed in Fig. 1.

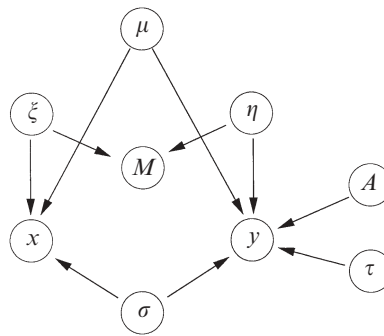


Fig. 1. Directed acyclic graph representing our model, showing all data and parameters treated as variables.

It is interesting to note that a likelihood algebraically similar to (5) also arises under a conceptually different model, in which unmatched points are drawn independently uniformly in a region, while, among matched pairs, the transformed y points arise through independent identically distributed perturbations to the corresponding x points; see also Kent et al. (2004), for the case of Gaussian noise and rigid motion.

3. PRIOR DISTRIBUTIONS AND COMPUTATIONAL IMPLEMENTATION

3.1. Preamble

We will henceforth treat ρ and λ as fixed, and consider inference for the remaining unknowns M , τ , σ^2 and sometimes A , given the data $\{x_j\}$ and $\{y_k\}$. Markov chain Monte Carlo methods must be used for the computation; several introductions and overviews of these methods are available; see for example Green (2001). In § 3.7, we discuss the relevance and applicability of an EM algorithm for making inference with an approximation of our model.

We suppose that prior information about τ , σ^2 and A will be at best weak, and so we concentrate on generic prior formulations that facilitate the posterior analysis. Prior assumptions are therefore discussed in parallel with Markov chain Monte Carlo implementation. Note that our formulation has some affinity with mixture models, the matching matrix M playing a similar role to the allocation variables often used in computing with mixtures; see for example Richardson & Green (1997). As in that paper, the fully Bayesian analysis here aims at simultaneous joint inference about both the discrete and continuously varying unknowns, in contrast to frequentist approaches.

Our model has another similarity with a mixture formulation, in that, as M varies, the number of hidden points needed to generate all the observed data also varies, and thus there seems to be a ‘variable-dimension’ aspect to the model. However, here our approach of integrating out the hidden point locations eliminates the variable-dimension parameter, so that reversible jump Markov chain Monte Carlo is not needed.

3.2. Priors and Markov chain Monte Carlo updating for a rotation matrix

We are interested in alignment and matching problems in which A is either given, and treated as fixed, or one of the objects of inference. In the latter case, we consider in this paper only the case of rotation matrices in two and three dimensions. We therefore focus on the full conditional distribution of A , which from (6) is

$$p(A|M, \tau, \sigma, x, y) \propto |A|^n p(A) \prod_{j,k:M_{jk}=1} \phi\{(x_j - Ay_k - \tau)/(\sigma\sqrt{2})\}.$$

Viewing this as a density for A , we are still free to choose the dominating measure for $p(A)$, which is arbitrary: this full conditional density is then with respect to the same measure.

Let us restrict attention to rotations, defined by orthogonal matrices A with positive determinant, so that $|A| = 1$. Expanding the expression above, we then find

$$\begin{aligned} p(A|M, \tau, \sigma, x, y) &\propto p(A) \exp \left[\sum_{j,k:M_{jk}=1} -0.5 \{ \|x_j - Ay_k - \tau\| / (\sigma\sqrt{2}) \}^2 \right] \\ &\propto p(A) \exp \left[\text{tr} \left\{ 1/(2\sigma^2) \sum_{j,k:M_{jk}=1} y_k (x_j - \tau)^T A \right\} \right]. \end{aligned}$$

Note a remarkable opportunity for conditional conjugacy: if $p(A)$ has the form $p(A) \propto \exp\{\text{tr}(F_0^T A)\}$ for some matrix F_0 , then the posterior has the same form with F_0 replaced by

$$F = F_0 + \left(\frac{1}{2\sigma^2}\right) \sum_{j,k: M_{jk}=1} (x_j - \tau)y_k^T.$$

This form of $p(A)$ is known as the matrix Fisher distribution (Downs, 1972; Mardia & Jupp, 2000, p. 289). To the best of our knowledge, this unique role of the matrix Fisher distribution, or in the two-dimensional case the von Mises distribution, as the prior distribution for a rotation conjugate to spherical Gaussian error distributions has not previously been noted, although Mardia & El-Atoum (1976) have identified the von Mises–Fisher distribution as the conjugate prior for the mean direction. This may have relevance in models for other situations, including the simpler case where there is no uncertainty in the matching. The conjugacy is presumably related to the interpretation of the matrix Fisher distribution as a conditional multivariate Gaussian (Mardia & Jupp, 2000, p. 289).

We now consider the two-dimensional case, $d = 2$. An arbitrary rotation matrix A can be written

$$A = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

and the natural dominating measure for θ is Lebesgue on $(0, 2\pi)$. Then a uniformly distributed choice of A corresponds to $p(A) \propto 1$. More generally, the von Mises density for θ ,

$$p(\theta) \propto \exp\{\kappa \cos(\theta - \nu)\} = \exp(\kappa \cos \nu \cos \theta + \kappa \sin \nu \sin \theta),$$

can indeed be expressed as $p(A) \propto \exp\{\text{tr}(F_0^T A)\}$, where a nonunique choice for F_0 is

$$F_0 = \kappa/2 \begin{pmatrix} \cos \nu & -\sin \nu \\ \sin \nu & \cos \nu \end{pmatrix}.$$

Thus, the full conditional distribution for θ is of the same von Mises form, with $\kappa \cos \nu$ updated to $(\kappa \cos \nu + S_{11} + S_{22})$ and $\kappa \sin \nu$ to $(\kappa \sin \nu - S_{12} + S_{21})$, where S is the 2×2 matrix $1/(2\sigma^2) \sum_{j,k: M_{jk}=1} (x_j - \tau)y_k^T$.

It is therefore trivial to implement a Gibbs sampler move to allow inference about A , assuming a von Mises prior distribution on the rotation angle θ , including the uniform case, $\kappa = 0$. We can use the Best/Fisher algorithm, an efficient rejection method (Mardia & Jupp, 2000, p. 43), to sample from the full conditional for θ .

In the three-dimensional case, we can represent A as the product of elementary rotations

$$A = A_{12}(\theta_{12})A_{13}(\theta_{13})A_{23}(\theta_{23}), \quad (7)$$

as in Raffenetti & Ruedenberg (1970) and Khatri & Mardia (1977). Here, for $i < j$, $A_{ij}(\theta_{ij})$ is the matrix with $m_{ii} = m_{jj} = \cos \theta_{ij}$, $-m_{ij} = m_{ji} = \sin \theta_{ij}$, $m_{rr} = 1$ for $r \neq i, j$ and other entries 0. We can then update each of the generalised Euler angles θ_{ij} in turn, conditioning on the other two angles and the other variables (M, τ, σ, x, y) entering the expression for F .

The joint full conditional density of the Euler angles is proportional to

$$\exp\{\text{tr}(F^T A)\} \cos \theta_{13},$$

for $\theta_{12}, \theta_{23} \in (-\pi, \pi)$ and $\theta_{13} \in (-\pi/2, \pi/2)$. The cosine term arises since the natural dominating measure, corresponding to uniform distribution of rotation, has volume element $\cos \theta_{13} d\theta_{12} d\theta_{13} d\theta_{23}$ in these coordinates.

Substituting the representation (7) and simplifying, we find that the trace can be written variously as

$$\begin{aligned}\text{tr}(F^T A) &= a_{12} \cos \theta_{12} + b_{12} \sin \theta_{12} + c_{12} = a_{13} \cos \theta_{13} + b_{13} \sin \theta_{13} + c_{13} \\ &= a_{23} \cos \theta_{23} + b_{23} \sin \theta_{23} + c_{23},\end{aligned}$$

where

$$\begin{aligned}a_{12} &= (F_{22} - \sin \theta_{13} F_{13}) \cos \theta_{23} + (-F_{23} - \sin \theta_{13} F_{12}) \sin \theta_{23} + \cos \theta_{13} F_{11}, \\ b_{12} &= (-\sin \theta_{13} F_{23} - F_{12}) \cos \theta_{23} + (F_{13} - \sin \theta_{13} F_{22}) \sin \theta_{23} + \cos \theta_{13} F_{21}, \\ a_{13} &= \sin \theta_{12} F_{21} + \cos \theta_{12} F_{11} + \sin \theta_{23} F_{32} + \cos \theta_{23} F_{33}, \\ b_{13} &= (-\sin \theta_{23} F_{12} - \cos \theta_{23} F_{13}) \cos \theta_{12} + (-\sin \theta_{23} F_{22} - \cos \theta_{23} F_{23}) \sin \theta_{12} + F_{31}, \\ a_{23} &= (F_{22} - \sin \theta_{13} F_{13}) \cos \theta_{12} + (-\sin \theta_{13} F_{23} - F_{12}) \sin \theta_{12} + \cos \theta_{13} F_{33}, \\ b_{23} &= (-F_{23} - \sin \theta_{13} F_{12}) \cos \theta_{12} + (F_{13} - \sin \theta_{13} F_{22}) \sin \theta_{12} + \cos \theta_{13} F_{32},\end{aligned}$$

and the c_{ij} can be ignored, combined into the normalising constants. Thus, the full conditionals for θ_{12} and θ_{23} are von Mises distributions, and so these two variables can be updated by Gibbs sampling. That of θ_{13} is proportional to

$$\exp(a_{13} \cos \theta_{13} + b_{13} \sin \theta_{13}) \cos \theta_{13}$$

and we use a random walk Metropolis update for this variable, with a perturbation uniformly distributed on $[-0.1, 0.1]$. The latter distribution has been studied in Mardia & Gadsden (1977) but with no discussion on how to simulate from it.

3.3. Priors and updating for other parameters

We make the standard normal/inverse gamma assumptions:

$$\tau \sim N_d(\mu_\tau, \sigma_\tau^2 I), \quad \sigma^{-2} \sim \Gamma(\alpha, \beta).$$

Under the assumptions of (6), there is conjugacy for τ and σ , and we have explicit full conditionals:

$$\begin{aligned}\tau | M, A, \sigma, x, y &\sim N_d \left(\frac{\mu_\tau / \sigma_\tau^2 + \sum_{j,k: M_{jk}=1} (x_j - Ay_k) / (2\sigma^2)}{1/\sigma_\tau^2 + L/(2\sigma^2)}, \frac{1}{1/\sigma_\tau^2 + L/(2\sigma^2)} I \right) \\ \sigma^{-2} | M, A, \tau, x, y &\sim \Gamma \left(\alpha + (d/2)L, \beta + \frac{1}{4} \sum_{j,k: M_{jk}=1} \|x_j - Ay_k - \tau\|^2 \right).\end{aligned}$$

and so it is trivial to implement Gibbs sampler updates for these parameters.

3.4. Updating M

The matching matrix M is updated in detailed balance using Metropolis–Hastings moves that only propose changes to a few entries: the number of matches $L = \sum_{j,k} M_{jk}$

can only increase or decrease by 1 at a time, or stay the same. The possible changes are as follows:

- (a) adding a match, which changes one entry M_{jk} from 0 to 1;
- (b) deleting a match, which changes one entry M_{jk} from 1 to 0;
- (c) switching a match, which simultaneously changes one entry from 0 to 1 and another in the same row or column from 1 to 0.

The proposal proceeds as follows. First a uniform random choice is made from all the $m + n$ data points $x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n$. Suppose without loss of generality, by the symmetry of the set-up, that an x is chosen, x_j say. There are two possibilities: either x_j is currently matched, in that there is some k such that $M_{jk} = 1$, or not, in that there is no such k .

If x_j is matched to y_k , with probability p^* we propose deleting the match, and with probability $1 - p^*$ we propose switching it from y_k to $y_{k'}$, where k' is drawn uniformly at random from the currently unmatched y points. On the other hand, if x_j is not currently matched, we propose adding a match between x_j and a y_k , where again k is drawn uniformly at random from the currently unmatched y points.

The acceptance probabilities for these three possibilities are easily derived from the expression (6) for the joint distribution, since in each case the proposed new matching matrix M' is only slightly perturbed from M , so that the ratio $p(M', \tau, \sigma | x, y) / p(M, \tau, \sigma | x, y)$ has only a few factors. Taking into account also the proposal probabilities, whose ratio is $(1/n_u) \div p^*$, where $n_u = \#\{k \in 1, 2, \dots, n : M_{jk} = 0 \text{ for all } j\}$ is the number of unmatched y points in M , we find that the acceptance probability for adding a match (j, k) is

$$\min \left[1, \frac{\rho \phi \{(x_j - Ay_k - \tau) / (\sigma \sqrt{2})\} p^* n_u}{\lambda (\sigma \sqrt{2})^d} \right]. \quad (8)$$

Similarly, the acceptance probability for switching the match of x_j from y_k to $y_{k'}$ is

$$\min \left[1, \frac{\phi \{(x_j - Ay_{k'} - \tau) / (\sigma \sqrt{2})\}}{\phi \{(x_j - Ay_k - \tau) / (\sigma \sqrt{2})\}} \right] \quad (9)$$

and for deleting the match (j, k) it is

$$\min \left[1, \frac{\lambda (\sigma \sqrt{2})^d}{\rho \phi \{(x_j - Ay_k - \tau) / (\sigma \sqrt{2})\} p^* n'_u} \right],$$

where $n'_u = \#\{k \in 1, 2, \dots, n : M'_{jk} = 0 \text{ for all } j\} = n_u + 1$. Along with just one of each of the other updates, we typically make several moves updating M per sweep, since the changes effected are so modest.

3.5. Loss functions

The output from the Markov chain Monte Carlo sampler derived above, once equilibrated, is a sample from the posterior distribution determined by (6). As always with sample-based computation, this provides an extremely flexible basis for reporting aspects of the full joint posterior that are of interest.

The matching matrix M will often be of particular inferential interest, and for some purposes a point estimate is desirable; in this section we discuss how to obtain a Bayesian point estimate of the matching matrix M .

The most easily understood estimator of M would be its posterior mode, the maximum a posteriori estimator. However, there are difficulties here. First, the notion is itself

ambiguous, in that the unknown ‘parameter’ in our model consists of the matching matrix M and some real parameters. ‘Maximum a posteriori’ might refer to the M component of the overall maximum, or the mode of the marginal posterior for M alone. Secondly, the posterior is multi-modal, and different modes may have different ‘widths’ appropriately measured. There is therefore no intrinsic attraction to the maximum a posteriori estimator. We should return to basic principles.

By standard theory, this requires specification of a loss function, $L(M, \hat{M})$, giving the cost incurred in declaring the matching matrix to be \hat{M} when it is in fact M . The optimal estimator given data (x, y) is the matching matrix \hat{M} that minimises the posterior expected loss

$$E\{L(M, \hat{M})|x, y\},$$

the expectation over M being taken with respect to the posterior determined by (6). In this language, the maximum a posteriori estimator is optimal for the ‘zero-one’ loss function under which a fixed total cost is paid if there is a single error in any value M_{jk} ; this is logically unappealing, and a further argument against using the maximum a posteriori estimation.

We consider instead loss functions $L(M, \hat{M})$ that penalise different kinds of error and do so cumulatively. The simplest of these are additive over pairs (j, k) . Suppose that the loss when $M_{jk} = a$ and $\hat{M}_{jk} = b$, for $a, b = 0, 1$, is ℓ_{ab} ; we set $\ell_{00} = \ell_{11} = 0$. For example, ℓ_{01} is the loss associated with declaring a match between x_j and y_k when there is really none, that is, a ‘false positive’. Then it is readily shown that

$$E\{L(M, \hat{M})|x, y\} = \text{const} - (\ell_{10} + \ell_{01}) \sum_{j,k: \hat{M}_{jk}=1} (p_{jk} - K),$$

where

$$K = \ell_{01}/(\ell_{10} + \ell_{01}).$$

and $p_{jk} = \text{pr}(M_{jk} = 1|x, y)$ is the posterior probability that (j, k) is a match, which is estimated from a Markov chain Monte Carlo run by the empirical relative frequency of this match. Thus, provided that $\ell_{10} + \ell_{01} > 0$ and $\ell_{01} > 0$, as is natural, the optimal estimate is that maximising the sum of marginal posterior probabilities of the declared matches $\sum_{j,k: \hat{M}_{jk}=1} p_{jk}$, penalised by a multiple K times the number of matches. The optimal match therefore depends on the loss function parameters only through the cost ratio K . If false positive and false negative matches are equally undesirable, one can simply choose $K = 0.5$.

Computation of the optimal match \hat{M} would be trivial but for the constraint that there can be at most one positive entry in each row and column of the array. This weighted bipartite matching problem is equivalent to a mathematical programming assignment problem, and can be solved by special-purpose or general linear programming methods; see Burkard & Cela (1999). For problems of modest size, the optimal match can be found by informal heuristic methods. These may not even be necessary, especially if K is not too small. In particular, it is immediate that, if the set of all (j, k) pairs for which $p_{jk} > K$ includes no duplicated j or k value, the optimal \hat{M} consists of precisely these pairs.

We could also consider loss functions that penalise mismatches differently from the sum of the losses of the individual errors. For example, declaring (j, k) to be a match when it should be (j, k') might deserve a relative loss greater or less than $(\ell_{10} + \ell_{01})$, depending on context. Such loss functions could be handled in a broadly similar way, but this is left for future work.

3.6. Using partial labelling information

When the points in each configuration are ‘coloured’, with the interpretation that like-coloured points are more likely to be matched than unlike-coloured ones, it is appropriate to use a modified likelihood that allows us to exploit such information. Let the colours for the x and y points be $\{r_j^x, j = 1, 2, \dots, m\}$ and $\{r_k^y, k = 1, 2, \dots, n\}$ respectively. The hidden-point model is augmented to generate the point colours, as follows. Independently for each hidden point, with probability $(1 - p_x - p_y - \rho p_x p_y)$ we observe neither x nor y point, as before. With probabilities $p_x \pi_r^x$ and $p_y \pi_r^y$, respectively, we observe only an x or y point, with colour r from an appropriate finite set. With probability

$$\rho p_x p_y \pi_r^x \pi_s^y \exp(\gamma I[r = s] + \delta I[r \neq s]),$$

where $I[.]$ is an indicator function, we observe an x point coloured r and a y point coloured s . Our original likelihood is equivalent to the case $\gamma = \delta = 0$, where colours are independent and so carry no information about matching. If γ and δ increase, then matches are more probable, a posteriori, and, if $\gamma > \delta$, matches between like-coloured points are more likely than those between unlike-coloured ones. The case $\delta \rightarrow -\infty$ allows the prohibition of matches between unlike-coloured points, a feature that might be adapted to other contexts such as the matching of shapes with given landmarks.

In implementation of this modified likelihood, the Markov chain Monte Carlo acceptance ratios in § 3.4 have to be modified accordingly. For example, if $r_j^x = r_k^y$ and $r_j^x \neq r_{k'}^y$, then (8) has to be multiplied by $\exp(-\gamma)$ and (9) by $\exp(\delta - \gamma)$.

Other, more complicated, colouring distributions in which the log probability can be expressed linearly in entries of M can be handled similarly.

3.7. Alternative approach using the EM algorithm

The interplay between matching, i.e. allocation, and parameter uncertainty has something in common with mixture estimation. This might suggest considering maximisation of the posterior by using the EM algorithm, which could of course in principle be applied either to maximum likelihood estimation based on (5) or to maximum a posteriori estimation based on (6). For the EM formulation, the ‘missing data’ are the matches.

In an exponential family, the EM algorithm alternates between finding expectations of missing values given data, at current parameter values, and maximising the log-posterior, with missing values replaced by these expectations.

The ‘expectations of missing values’ are just probabilities of matching. These are only tractable if we drop the assumption that a point can only be matched with at most one other point; that is, that $\sum_j M_{jk} \leq 1$ for all k , $\sum_k M_{jk} \leq 1$ for all j . With this approximation, the E-step is trivial: the expectation of $I[M_{jk} = 1]$ is $p_{jk} = w_{jk}/(1 + w_{jk})$, where w_{jk} is the (j, k) factor in the joint model,

$$w_{jk} = \{(\rho/\lambda) g_\sigma(x_j - Ay_k - \tau)\}.$$

The M-step then requires one to maximise, for given p_{jk} ,

$$\log \{|A|^n p(A) p(\tau) p(\sigma)\} + \sum_{j,k} p_{jk} \log \{w_{jk}(A, \tau, \sigma)\}$$

over A, τ, σ ; note that here w_{jk} is a function of all three. Although for some individual parameters this seems to be explicit, in the general case we need numerical optimisation.

In summary, EM allows us to study only certain aspects of an approximate version of our model, and is not trivial numerically. Consequently, we do not pursue this approach. Obtaining the complete posterior by Markov chain Monte Carlo sampling gives much greater freedom in inference.

4. APPLICATIONS

4.1. Matching protein gels

The objective in this example is to match two electrophoretic gels automatically, given the locations of the centres of 35 proteins on each of the two gels. The data are presented in the supplementary information on the web, at <http://www.stats.bris.ac.uk/~mapjg/Align>. The correspondence between pairs of proteins, one protein from each gel, is unknown, so our aim is to match the two gels based on these sets of unlabelled points. We suppose that it is known that the transformation between the gels is affine. In this case, experts have already identified 10 points; see Horgan et al. (1992). Based on these 10 matches, the linear part of the transformation is estimated a priori to be

$$A = \begin{pmatrix} 0.9731 & 0.0394 \\ -0.0231 & 0.9040 \end{pmatrix}. \quad (10)$$

(Dryden & Mardia, 1998, pp. 20–1, 292–6).

Here, we have only to make inference about the translation τ and the unknown matching between certain of the proteins. The model (6) will therefore be taken to apply, with $d = 2$ and with A held fixed at (10). The Markov chain Monte Carlo sampler described in §3 was run for 100 000 sweeps, after a burn-in period of 20 000 sweeps, considered on the basis of an informal visual assessment of time series traces to be adequate for convergence. Prior and hyperprior settings were $\alpha = 1$, $\beta = 16$, $\mu_\tau = (0, 0)^T$, $\sigma_\tau = 20.0$ and $\lambda/\rho = 0.0001$. The sampler parameter p^* was set to 0.5. Such a run took about 2 seconds on a 800 MHz PC. Acceptance rates for the moves updating M were between 0.6% and 2.1%.

The posterior expectation and variance matrix of τ were estimated to be $(-35.950, 66.685)^T$, to be compared with $(-36.08, 66.64)^T$ obtained by Dryden & Mardia (1998) and

$$\begin{pmatrix} 0.5776 & -0.0227 \\ -0.0227 & 0.6345 \end{pmatrix}.$$

The posterior mean and variance of σ are 2.050 and 0.1192.

The 20 most probable matches between x and y points are listed in Table 1. Note that there is no duplication in their indices until the 19th match, with $j = 26$ also appearing in the 16th match; recall that there is a simple rule for identifying the optimal \hat{M} if there is no duplication among the matches with p_{jk} above the threshold K . We can conclude that, for all values of $K = \ell_{01}/(\ell_{10} + \ell_{01})$ from 1 down to 0.1112, the optimal Bayesian matching is given by an appropriate subset of Table 1, reading down from the top. For example if this cost ratio is 0.8 we take the first 15 rows of the table, while if the ratio is 0.6 or 0.4 we include the 16th and 17th rows as well. The 17 most probable matches are displayed graphically in Fig. 2.

Note that all of the expert-identified matches, points 1 to 10 in each set, are declared to be matches with high probability in the Bayesian analysis. We also repeated the analysis

Table 1. The 20 marginally most probable matches in the analysis of the gel data

Rank	j	k	p_{jk}	Rank	j	k	p_{jk}
1	15	21	1	11	24	23	0.9855
2	19	19	1	12	7	7	0.9824
3	8	8	1	13	32	31	0.9776
4	3	3	1	14	1	1	0.9763
5	2	2	1	15	9	9	0.9677
6	31	30	0.9989	16	26	32	0.7910
7	6	6	0.9987	17	12	13	0.7552
8	4	4	0.9966	18	21	33	0.3998
9	5	5	0.9946	19	26	27	0.1931
10	10	10	0.9927	20	35	35	0.0025

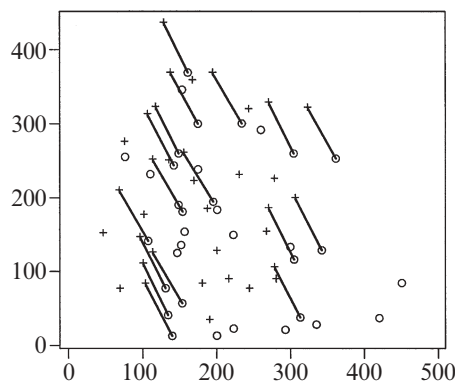


Fig. 2. The 17 most probable matches in the gel data, the optimal match for any $K \in (0.3998, 0.7552)$; crosses signify x points, circles the y points, linearly transformed by premultiplication by the fixed affine transformation A given in (10). The solid line for each of the 17 matches joins the matched points, and represents the inferred translation τ plus noise.

with these 10 pairs held fixed. The next 9 most probable matches, together with these 10, are identical to those in the first 19 lines of Table 1, and the posterior probabilities differ by less than 0.037 in all 19 cases.

4.2. Aligning proteins in three dimensions

We now apply the matching method to a problem in three-dimensional structural biology, previously considered by Gold et al. (2002). The problem consists of finding the matches for two Active sites 1 and 2 corresponding to two Proteins, A and B respectively. The corresponding coordinates x and y of these sites are presented in the supplementary information on the website mentioned in § 4.1; these coordinates are the centres of gravity of the amino acids of the two sites. Here $m = 40$ and $n = 63$. The biological details of the two proteins are as follows. Protein 1 is the human protein 17-beta hydroxysteroid dehydrogenase and is involved in the synthesis of oestrogens. This protein binds the ligands, which are molecules comparatively smaller than proteins, oestradiol and NADP. Protein 2 is the mouse protein carbonyl reductase and is involved in metabolism of

carbonyl compounds. This protein binds the ligands 2-Propanol and NADP. The common element between these two sets of ligands is NADP. Since we have the NADP molecule as the common ligand for the two active sites, the matched points can be regarded as representing the ‘binding’ sites due to NADP, so this common matched configuration could be used as a template in discovering such sites in other proteins.

From chemical properties of the sites, the relevant matching should be invariant under rigid transformation. There is information about the identities of the amino acids in the two configurations: we defer use of this to § 4.4.

The sampler described in § 3 was for 1000 000 sweeps, after a burn-in period of 200 000 sweeps, considered on the basis of an informal visual assessment of time series traces to be adequate for convergence. Prior and hyperprior settings were $\alpha = 1$, $\beta = 36$, $\mu_\tau = (0, 0, 0)^T$, $\sigma_\tau = 50.0$, $\lambda/\rho = 0.003$ and the matrix F_0 defining the prior for A set to the zero matrix. The sampler parameter p^* was set to 0.5, and we made 10 updates to M in each sweep. Such a run took about 71 seconds on a 800 MHz PC. Acceptance rates for the moves updating M were between 0.41% and 5.6%.

The posterior expectation and variance matrix of τ were estimated to be $(31.60, 8.89, 17.44)^T$ and

$$\begin{pmatrix} 0.227 & 0.120 & -0.044 \\ 0.120 & 0.307 & 0.176 \\ -0.044 & 0.176 & 0.428 \end{pmatrix}.$$

The posterior mean and variance of σ are 1.051 and 0.00996. In representing the centre of the posterior distribution for the rotation matrix A , we need to use a definition of mean appropriate to the geometry. We form the mean elementwise from a thinned sample of 2000 values of A from the post-burn-in Markov chain Monte Carlo run. This mean matrix \bar{A} is of course not a rotation matrix, but postmultiplication by the positive definite symmetric square root of $\bar{A}^T \bar{A}$ yields a rotation matrix that is known as its polar part (Mardia & Jupp, pp. 286, 290). This is an appropriate measure of location of the posterior, and takes the value

$$\hat{A} = \begin{pmatrix} 0.4339 & -0.8444 & 0.3140 \\ -0.7118 & -0.5350 & -0.4550 \\ 0.5522 & -0.0261 & -0.8333 \end{pmatrix} \quad (11)$$

in this case.

The 40 most probable matches between x and y points are listed in the supplementary information on the web; there is no duplication in their indices until the 39th match, with $k = 12$ also appearing in the 38th match. We can conclude that for all values of K greater than 0.2895, which is the marginal posterior probability associated with the 39th match, the optimal Bayesian matching is given by an appropriate leading subset of the matches. For example, if this cost ratio is 0.5 we take the first 36 matches; these are displayed graphically in Fig. 3. In this three-dimensional example, the axes signify the first two principle coordinates of the combined cloud of data.

As would be anticipated, simultaneous inference for the rotation A and the matching matrix M , as well as for τ and σ , is a considerably greater challenge for Markov chain Monte Carlo than is the problem of § 4.1, where the rotation matrix is held fixed. It is clear that there is a possibility of severe multi-modality in the posterior, with the conditional

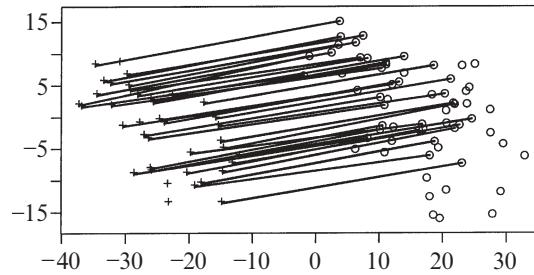


Fig. 3. The optimal alignment, 36 matches, when $K = 0.5$ for the protein alignment analysis data, without using colouring information; crosses signify x points, circles the y points, rotated according to the inferred \hat{A} matrix given by (11). The entire joint configuration has been rotated to its first two principal axes; axes signify the first two principle coordinates of the combined cloud of data. Solid lines represent the 36 marginally most probable matches, and indicate the inferred translation τ plus noise.

posterior for M and τ given A depending strongly on A . This challenge is quantified empirically by a heavy-tailed distribution of times to convergence, and by ‘meta-stability’ in the time series plots of various monitoring statistics against simulation time. We found the log-posterior to be a useful summary statistic for quality of fit, and pilot runs provided experience for choosing a threshold value, exceedance of which we hypothesised diagnosed convergence to the main mode of the posterior.

To investigate multimodality and convergence time, we conducted a study in which the Markov chain Monte Carlo run described was repeated, with the same parameters, from 100 different initial configurations, obtained by independent random rotations as initial settings for A . After short runs of 50 000 sweeps, we tested whether or not the threshold log-posterior value had been exceeded, and if not the run was abandoned. Eighty-three out of the 100 runs passed this test, and these were allowed to run on for a further 450 000 sweeps. Every one of these 83 long runs provided exactly the same set of 36 most probable matches. We therefore felt justified in concluding that they had not been trapped in a subsidiary mode of the posterior, and that it was safe to draw inference from the results. This conclusion is specific to the dataset and parameter settings used, and it would be straightforward to contrive artificial data where multiple modes were more equal in probability content. In such cases more sophisticated Markov chain Monte Carlo samples would be needed.

4.3. Prior settings and sensitivity

Our analysis depends of course on the settings of the hyperparameters λ/ρ , see § 2.2, F_0 , see § 3.2, and μ_τ , σ_τ , α , β , see § 3.3. These allow the provision of real prior information from the experimental context, if it is available.

For a default analysis in the absence of such information, we would set F_0 to the zero matrix, corresponding to a uniform prior on A , μ_τ to be the zero vector, and σ_τ of the order of twice the distance between the centres of gravity of the two configurations. We fix $\alpha = 1$, giving an exponential prior distribution for σ^{-2} . Here we briefly discuss settings of, and sensitivity to, the remaining two parameters, the scalars λ/ρ and β .

Sensitivity to λ/ρ is pronounced, as might be anticipated. This parameter ratio has a very direct role in determining whether or not an (x_j, y_k) pair are noisy observations of the same hidden μ_i , after transformation, since it controls the density of hidden points. In practice, we should not expect to be able to draw inference about matching without real prior knowledge about this ratio or an equivalent measure of the prior tendency of points to be matched.

The prior for the number of matches L is parameterised by λ/ρ ; see (3). This distribution is nonstandard, but very well behaved. It is clear from inspection that setting λ/ρ equal to $(m - \bar{L})(n - \bar{L})/(\bar{L}v)$ yields a mode of L that is within 1 of \bar{L}_1 , and numerical calculation in the context of the example in § 4.2 verifies that, for all possible ‘prior guesses’ \bar{L} for L , the prior expectation and median are also both equal to \bar{L} to the nearest integer. Thus, prior information about L is directly informative about the parameter ratio λ/ρ . As long as v is known, or at least a representative value is provided, and the analyst is able to make a prior guess \bar{L} at the number of matches, this suggests a reasonable way of specifying λ/ρ . The posterior distribution for L tracks the prior rather closely, confirming that the raw data carry little information about the number of matches.

The hyperparameter β is an inverse scale parameter for the precision of the noise terms ε ; thus, as β increases, we expect that $\sigma^2 = \text{var}(\varepsilon)$ increases too. The runs we have presented used $\beta = 36$; reducing this by a factor of 2 makes minimal difference to the posterior inference for either σ^2 or M . However, increasing β by a factor of 2 leads to a three-fold increase in σ and a sharp reduction in the number of matches, the posterior expectation of L going down from around 34 to 26. The latter observation is perhaps counter-intuitive, until one realises that, when σ is larger, it becomes relatively less likely that points that are nearly coincident, after transformation, are in fact matched.

Finally, it would be desirable to assess the sensitivity to the Poisson assumption for the hidden point model, but it would be extremely onerous to do this directly, since alternatives would require a substantially modified formulation and implementation. There is scientific reason to doubt the Poisson assumption; for example, the minimum spacing between the centres of gravity of the amino acids in proteins is approximately 3.8 Angstroms. However, experiments reported in unpublished work by K. V. Mardia, V. Nyirongo and D. R. Westhead do at least suggest strongly that the ability of our method to detect matches is little affected by real hard-core effects.

4.4. *Using information about types of amino acid*

The protein alignment data include identifiers of the type of amino acid at each point; see the supplementary information. There are 20 different types, which can be categorised into four groups, namely hydrophobic, charged, polar and glycine; we use the group identifiers as colours in defining a modified likelihood as in § 3.6. The parameter values taken were $\gamma = 1.0$ and $\delta = -0.5$, providing a strong preference for like-coloured matching since $\exp(\gamma - \delta) \simeq 4.48$. The analysis was repeated with this modified model, leaving all other details unchanged.

The 40 most marginally probable matches are listed in the supplementary information, along with displays of the optimal alignment. The 36 most probable matches, which together form the optimal matching when $K = 0.4$, are identical to those found in § 4.2; however, there are modest variations in the posterior probabilities attached to individual matches.

The posterior expectation and variance of τ were now estimated to be $(31.90, 9.11, 17.65)^T$, which are slightly shifted from that obtained in the analysis of § 4.3, and

$$\begin{pmatrix} 0.764 & 0.325 & -0.185 \\ 0.325 & 0.961 & 0.417 \\ -0.185 & 0.417 & 1.213 \end{pmatrix}.$$

The posterior mean and variance of σ are 1.222 and 0.0869. The increased estimate of σ is perhaps anticipated. The centre of the posterior distribution of A is in this case

$$\hat{A} = \begin{pmatrix} 0.4220 & -0.8530 & 0.3071 \\ -0.7298 & -0.5206 & -0.4431 \\ 0.5378 & -0.0371 & -0.8422 \end{pmatrix}. \quad (12)$$

In the approach to the analysis of these data taken by Gold et al. (2002), the matching between the configurations was performed in two stages, and was not driven by an explicit probability model. First, interpoint distances $d(\cdot, \cdot)$ were calculated within each configuration. These distances are invariant under the rigid body motions considered here. A maximal set of pairs of indices $\{(j_1, k_1), (j_2, k_2), \dots\}$, with no ties among the j 's or k 's, is found such that $|d(x_{j_r}, x_{j_s}) - d(y_{k_r}, y_{k_s})|$ is less than some threshold, for all $s \neq r$. This is done using graph theoretical algorithms of Bron & Kerbosch (1973) and Carraghan & Pardos (1990), applied to a product graph whose vertices are labelled with (j, k) pairs. This first stage of the matching algorithm was formulated by Kuhl et al. (1984).

In the second stage, the matches are scored using the amino acid information, assigning a score of 1 for identity of the amino acids and 0.5 when the amino acids are different but fall in the same group. The initial list of matches from stage one is then permuted so as to maximise the total score.

Once the matches are found the rigid body transformation is estimated by Procrustes analysis; see for example Dryden & Mardia (1998, pp. 176–8).

It is interesting to compare the rotation matrix resulting from this method, namely

$$A = \begin{pmatrix} 0.441 & -0.841 & 0.312 \\ -0.678 & -0.541 & -0.498 \\ 0.588 & 0.008 & -0.809 \end{pmatrix},$$

with that obtained by our method. The trace of the orthogonal matrix taking A to \hat{A} is approximately $1 + 2 \cos 0.079$, so the two differ by a rotation of only 0.079 radians. Figure 4 provides a comparison between the matchings achieved by the two approaches. Of the 27 matches identified by Gold et al., 14 are among the most probable 20 that we find, and all 27 are among the first 35.

In addition to the amino acid information used here for the two active sites, there is also sequence ordering information attached to the two configurations, which we have not used. Matching methods using only three-dimensional information are called 'sequence order-independent structural' methods in the bioinformatics literature; see for example Fischer et al. (1994) and Eidhammer et al. (2004, pp. 333–4). For our case, Fig. 4 shows the sequence number of the point in the x configuration compared to that of the matched point in the y configuration; that is, for example, along the axis for x , the first segment

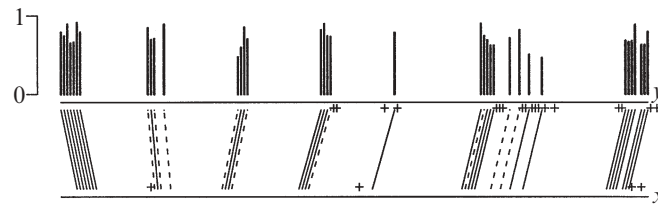


Fig. 4. The optimal matching, 36 matches, when $K = 0.4$, in the protein alignment analysis data, using colouring information, with $\gamma = 1.0$ and $\delta = -0.5$; matches are signified by line segments joining the sequence number of the point in the x configuration to that of the matched point in the y configuration. The solid lines indicate the 27 matches identified by Gold et al. (2002); our method discovers all of these, together with the 9 further matches shown with broken lines. The heights of the vertical bars indicate the marginal probabilities of each match. The crosses denote points that are present in either configuration but are not matched.

contains six contiguous sequence points which match the six contiguous points along the axis for y . Note that there is a gap in sequence numbering for the next match. The figure also shows the posterior probabilities of these matches together with comparison with matches from the Gold et al. (2002) method. It is interesting to note that the sequence ordering has been perfectly preserved in matching though we did not constrain our method to preserve the ordering. This point is a part of a bigger biological question having some evolutionary implications; for details see for example Fischer et al. (1994) and the various other papers cited in this reference.

5. DISCUSSION

The main conclusion of this paper is that a probability-model-based approach is successful in allowing simultaneous inference about partial matching between two point configurations, and a geometrical transformation between the coordinate systems in which the configurations are measured. This seems an advance over previous more ad hoc methods.

We have only used the translation and rigid motion groups in illustrating our methodology. However, the formulation allows inference about various other group transformations such as affine transformation, and so on. The fairly straightforward Markov chain Monte Carlo implementation presented here has proved adequate for the models and datasets considered, although allowing rotations did increase considerably the needed run lengths. We anticipate that, at least for models allowing rotations, dealing with larger datasets will be much more challenging, since small rotational perturbations generate large displacements at sites far from the axis of rotation; moves that simultaneously perturb allocations and geometrical and error distribution parameters will be necessary for good performance. We also anticipate more severe difficulties from multimodality than were exposed in § 4.2.

An important task left for future work is a formulation that allows smooth nonparametric transformations between coordinate systems, setting warping into a model-based framework; this would be important in dealing more comprehensively with gel-matching problems.

We have only used pairwise comparisons but there is scope for taking multiple combinations such as triads. The transformations considered above are parametric but some nonparametric alternatives such as nonlinear deformations may be useful in some cases, for example to deal with dynamic aspects of the atoms in a protein. We have considered only two configurations, but a natural extension would be to take three or more point configurations simultaneously, and make joint inference about patterns of matching between the configurations and the various geometrical transformations involved. More straightforward extensions would be to allow for non-Gaussian noise, other types of prior and so on.

Kent et al. (2004) have treated the unlabelled case by using a different model. While matching two configurations, one of them is taken as the population and the second as a random sample from this population after an unknown transformation. This approach is different from the symmetrical model for the two configurations proposed here. Furthermore, the emphasis in Kent et al. (2004) is on maximum likelihood inference using the EM algorithm.

Recent independent work by Dryden et al. (2006) addresses a similar problem of matching unlabelled point sets. Their approach has some substantial differences. For example, there is asymmetry in comparing two configurations, one being treated as a perturbation of the other. The geometrical transformation parameters are given uniform priors and maximised out, using standard ideas from shape analysis, rather than integrated out as in our fully Bayesian approach. The loss function basis for estimating matches does not appear, and the treatment of partial labelling is different.

There is other statistical work on alignment and matching in proteins by Wu et al. (1998) and in unpublished work by S. C. Schmidler, which in contrast does use sequence information. Further work is needed to clarify the relationships between all these methods and their comparative performance.

Finally, in the context of using methods such as ours in database search, often the reason for assessing protein alignment, there are issues related to multiple comparisons. These are not discussed here, but the answers will depend on the size of the database as well as on the number of points in the query site.

ACKNOWLEDGEMENT

We are grateful to Nicola Gold and Dave Westhead for their many helpful discussions, and in particular for the data used in § 4.3, and to Vysaul Nyirongo and Charles Taylor for various helpful comments.

REFERENCES

- ABRAMOWITZ, M. & STEGUN, I. A. (1970). *Handbook of Mathematical Functions*. New York: Dover.
- BRON, C. & KERBOSCH, J. (1973). Algorithm 457; finding all cliques of an undirected graph. *Commun. ACM* **16**, 575–7.
- BURKARD, R. E. & CELA, E. (1999). Linear assignment problems and extensions. In *Handbook of Combinatorial Optimization*, Vol. 4, Ed. P. Pardalos and D.-Z. Du, pp. 75–149. Boston: Kluwer Academic Press.
- CARRAGHAN, R. & PARDALOS, P. M. (1990). Exact algorithm for the minimal clique problem. *Oper. Res. Lett.* **9**, 375.
- CHUI, H. & RANGARAJAN, A. (2000). A new algorithm for non-rigid point matching. *IEEE Conf. Comp. Vision Pat. Recog.* **2**, 44–51.
- CROSS, A. D. J. & HANCOCK, E. R. (1998). Graph matching with dual-step EM algorithm. *IEEE Trans. Pat. Anal. Mach. Intel.* **20**, 1236–53.

- DOWNES, T. D. (1972). Orientation statistics. *Biometrika* **59**, 665–76.
- DRYDEN, I. L. & MARDIA, K. V. (1998). *Statistical Shape Analysis*, Chichester: Wiley.
- DRYDEN, I. L., HIRST, J. D. & MELVILLE, J. L. (2006). Statistical analysis of unlabelled point sets: comparing molecules in chemoinformatics. *Biometrics*. To appear.
- EIDHAMMER, T., JONASSEN, T. & TAYLOR, W. R. (2004). *Protein Bioinformatics*. Chichester: Wiley.
- FISCHER, D., WOLFSON, H. J., LIN, S. L. & NUSSINOV, R. (1994). Three dimensional, sequence order-independent structural comparison of a serine protease against the crystallographic data base reveals active site similarities: Potential implications to evolution and to protein folding. *Protein Sci.* **3**, 769–78.
- GOLD, N. D., PICKERING, S. J. & WESTHEAD, D. R. (2002). Protein functional site matching using graph theory techniques. In *Proceedings of the International Conference on Bioinformatics*, Bangkok, Thailand, p. 79.
- GREEN, P. J. (2001). A primer on Markov chain Monte Carlo. In *Complex Stochastic Systems*, Ed. O. E. Barndorff-Nielsen, D. R. Cox and C. Klüppelberg, pp. 1–62. London: Chapman and Hall.
- HORGAN, G. W., CREASEY, A. & FENTON, B. (1992). Superimposing two dimensional gels to study genetic variation in malaria parasites. *Electrophoresis* **13**, 871–5.
- KENT, J. T., MARDIA, K. V. & TAYLOR, C. C. (2004). Matching problems for unlabelled configurations. In *Bioinformatics, Images and Wavelets*, Proceedings of LASR 2004, Ed. R. G. Aykroyd, S. Barber and K. V. Mardia, pp. 33–6. Leeds: Leeds University Press.
- KHATRI, C. G. & MARDIA, K. V. (1977). The von Mises-Fisher distribution in orientation statistics. *J. R. Statist. Soc. B* **39**, 95–106.
- KUHL, F. S., CRIPPEN, G. M. & FRIESEN, D. K. (1984). A combinatorial algorithm for calculating ligand binding. *J. Comp. Chem.* **5**, 24–34.
- MARDIA, K. V. & EL-ATOUM, S. A. M. (1976). Bayesian inference for the von Mises-Fisher distribution. *Biometrika* **63**, 203–5.
- MARDIA, K. V. & GADSDEN, R. J. (1977). A circle of best fit for spherical data and areas of vulcanism. *Appl. Statist.* **26**, 238–45.
- MARDIA, K. V. & JUPP, P. E. (2000). *Directional Statistics*, Chichester: Wiley.
- MARDIA, K. V., TAYLOR, C. C. & WESTHEAD, D. R. (2003). Structural bioinformatics revisited. In *Stochastic Geometry, Biological Structures and Images*, Proceedings of LASR 2003, Ed. R. G. Aykroyd, S. Barber and K. V. Mardia, pp. 11–8. Leeds: Leeds University Press.
- RAFFENETTI, R. C. & RUEDENBERG, K. (1970). Parameterization of an orthogonal matrix in terms of generalized Eulerian angles. *Int. J. Quant. Chem.* **IIIS**, 625–34.
- RICHARDSON, S. & GREEN, P. J. (1997). On Bayesian analysis of mixtures with an unknown number of components (with Discussion). *J. R. Statist. Soc. B* **59**, 731–92.
- WU, T. D., SCHMIDLER, S. C., HASTIE, T. & BRUTLAG, G. (1998). Regression analysis of multiple protein structures. *J. Comp. Biol.* **5**, 585–95.

[Received August 2004. Revised December 2005]