Machine Learning

Dimensionality Reduction using SVD

Satishkumar L. Varma

Department of Information Technology
SVKM's Dwarkadas J. Sanghvi College of Engineering, Vile Parle, Mumbai.
ORCID | Scopus | Google Scholar | Google Site | Website

Outline

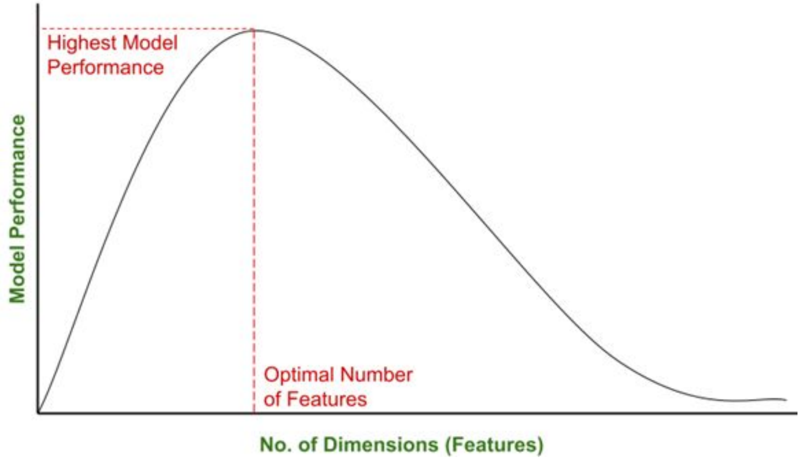
- Dimensionality Reduction
  - Dimensionality Reduction Techniques
    - Principal Components Analysis (Eigenvalues, Eigen vectors, Orthogonality)
    - Independent Component Analysis (ICA)
    - Single Value Decomposition (SVD)

Singular Value Decomposition

- Data Science and ML helps to solve several complex regression and classification problems.
- However, the performance of all these models depends on the input dataset.
- So, It is important to provide optimal dataset to ML models.
- Large dataset leads to increased computational demands
- Large dataset leads to overfitting.
- If we provide large dataset (with a large no. of features/columns) to ML models
  - it gives rise to the problem of overfitting,
  - wherein the model starts getting influenced by outlier values and noise.
  - This is called the Curse of Dimensionality.

Dimensionality Reduction

- Dimensionality Reduction is a statistical/ML-based technique
- It helps to reduce the number of features in our dataset.
- It helps to obtain a dataset with an optimal number of dimensions.
- It is useful in Feature Extraction
  - By reduce the number of dimensions by mapping a higher dimensional feature space to a lower-dimensional feature space.
- Effect of the change in model performance with the increase in the number of dimensions of the dataset.
- The model performance is best only at an option dimension, beyond which it starts decreasing.
- Technique of Dimensionality Reduction (Feature Extraction)
  - Principal Component Analysis (PCA)
  - Independent Component Analysis (ICA)
  - Singular Value Decomposition (SVD)
  - Linear Discriminant Analysis (LDA)



Singular Value Decomposition

- Singular Value Decomposition (SVD)
- It is a method used in linear algebra to decompose a matrix into three simpler matrices,
  - making it easier to analyze and manipulate.
- SVD is one of the important concepts in linear algebra.
- To understand the meaning of SVD, one must be aware of the related concepts such as matrix, types of matrices, transformations of a matrix, etc.
- SVD concept is connected to various concepts of linear algebra,

Singular Value Decomposition

- Example: Understanding SVD
- Imagine you have a table of data, like a set of ratings where rows are people, and columns are products.
- The numbers in the table show how much each person likes each product.
- SVD helps you split that table into three parts:
  - U: This part tells you about the people people's preferences.
  - Σ: This part shows how important each book is (how much each rating matters).
  - V^T: This part tells you about the books (how similar books are to each other)
- So, mathematically, the SVD of a matrix A (of size m×n) is represented as: $A = U \Sigma V^T$
  - U: An m×m orthogonal matrix whose columns are the left singular vectors of A.
  - Σ: A diagonal m×n matrix containing the singular values of A in descending order.
  - V^T: The transpose of an n×n orthogonal matrix, where the columns are the right singular vectors of A.

Rating	Book 1	Book 2
Satish	5	3
Harsh	4	2
Saransh	2	5

Singular Value Decomposition

- Steps to Perform Singular Value Decomposition
  - Step 1: Compute AA^T
  - Step 2: Find the Eigenvalues of AA^T
  - Step 3: Find the Right Singular Vectors (Eigenvectors of A^TA)
  - Step 4: Compute the Left Singular Vectors (Matrix U)
  - Step 5: Final SVD Equation

Singular Value Decomposition

- **Example 1:**
- Step 1: Compute AA^T
- Step 2: Find the Eigenvalues of AA^T
- Thus, the eigenvalues are $\lambda_1=25$ & $\lambda_2=9$
- These eigenvalues correspond to the singular values $\sigma_1=5$ and $\sigma_2=3$
- Since the singular values are the square roots of the eigenvalues.

Step 1: Compute AA^T

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \quad A^T = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \quad AA^T = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$$

Step 2: Find the Eigenvalues of AA^T

$$\det(AA^T - \lambda I) = 0 \quad \det \begin{bmatrix} 17-\lambda & 8 \\ 8 & 17-\lambda \end{bmatrix} = 0 \quad (\lambda-25)(\lambda-9) = 0$$

Step 3: Find the Right Singular Vectors (Eigenvectors of A^TA for $\lambda=25$ and $\lambda=9$).

For $\lambda = 25$: Solve $(A^TA - 25I)v = 0$:

$$A^TA - 25I = \begin{bmatrix} -12 & 12 & 2 \\ 12 & -12 & -2 \\ 2 & -2 & -17 \end{bmatrix}$$ Row-reduce this matrix to: $$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The eigenvector corresponding to $\lambda = 25$ is: $v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$

For $\lambda = 9$: Solve $(A^TA - 9I)v = 0$: The eigenvector corresponding to $\lambda = 9$ is: $v_2 = \begin{bmatrix} \frac{1}{\sqrt{18}} \\ \frac{-1}{\sqrt{18}} \\ \frac{4}{\sqrt{18}} \end{bmatrix}$

For the third eigenvector v_3:

Since v_3 must be perpendicular to v_1 and v_2, solve $v_1^T v_3 = 0$ and $v_2^T v_3 = 0$, : $v_3 = \begin{bmatrix} \frac{2}{3} \\ \frac{-2}{3} \\ \frac{-1}{3} \end{bmatrix}$

Singular Value Decomposition

- Example 1:
- Steps to Perform Singular Value Decomposition
  - Step 4: Compute the Left Singular Vectors (Matrix U)
  - Step 5: Final SVD Equation

Step 4: Compute the Left Singular Vectors (Matrix U)

To compute the left singular vectors U, use the formula $u_i = \frac{1}{\sigma_i} A v_i$: $U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$

Step 5: Final SVD Equation

Finally, the Singular Value Decomposition of matrix A is: $A = U\Sigma V^T$

Where: $U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$ $\Sigma = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$ $V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{18}} & \frac{-1}{\sqrt{18}} & \frac{4}{\sqrt{18}} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix}$

Thus, the SVD of matrix A is: $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{18}} & \frac{-1}{\sqrt{18}} & \frac{4}{\sqrt{18}} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix}$

Singular Value Decomposition

- Applications of Singular Value Decomposition (SVD)
- **Rank, Range, and Null Space**
  - The rank, range, and null space of a matrix M can be derived from its SVD.
  - Rank: The rank of matrix M is the number of non-zero singular values in Σ.
  - Range: The range of matrix M is the span of the left singular vectors in matrix U corresponding to the non-zero singular values.
  - Null Space: The null space of matrix M is the span of the right singular vectors in matrix V corresponding to the zero singular values.
- **Digital Signal Processing**
  - SVD can be used to analyze signals and filter noise.
- **Digital Image Processing**
  - SVD is used for image compression and denoising.
  - It helps in reducing the dimensionality of image data by preserving the most significant singular values and discarding the rest.

Singular Value Decomposition

- Key points:
- SVD decomposition function simplifies complex data by breaking it into three smaller parts.
- SVD helps uncover hidden patterns and relationships
- It help to analyze and work with large datasets.
- SVD is useful in tasks like
  - recommendations,
  - data compression, and
  - finding important features,
  - making data simpler

Singular Value Decomposition

- Example 2
- The Singular Value Decomposition of a matrix is a factorization of the matrix into three matrices.
- The SVD of matrix A can be expressed in terms of the factorization of A into the product of three matrices as
  - $A = UDV^T$
- The columns of U and V are orthonormal, and the matrix D is diagonal with real positive entries.

Singular Value Decomposition

- Example 2: Singular Value Decomposition 2×2 Matrix Example

$$A = \begin{bmatrix} -4 & -7 \\ 1 & 4 \end{bmatrix} \quad A^T = \begin{bmatrix} -4 & 1 \\ -7 & 4 \end{bmatrix} \quad AA^T = \begin{bmatrix} -4 & -7 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -4 & 1 \\ -7 & 4 \end{bmatrix} = \begin{bmatrix} 65 & -32 \\ -32 & 17 \end{bmatrix}$$

Find the Eigenvalues of AA^T

$$|A \cdot A' - \lambda I| = 0 \qquad \begin{vmatrix} (65 - \lambda) & -32 \\ -32 & (17 - \lambda) \end{vmatrix} = 0$$

$$(65 - \lambda) \times (17 - \lambda) - (-32) \times (-32) = 0$$

$$\left(1105 - 82\lambda + \lambda^2\right) - 1024 = 0 \qquad \left(\lambda^2 - 82\lambda + 81\right) = 0$$

$$(\lambda - 1)(\lambda - 81) = 0 \qquad (\lambda - 1) = 0 \text{ or } (\lambda - 81) = 0$$

The eigenvalues of the matrix $A \cdot A'$ are given by $\lambda = 1, 81$.

Eigenvectors for $\lambda = 81$: $v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ Eigenvectors for $\lambda = 1$: $v_2 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$

Similarly, find the eigenvectors for A'.A as:

Eigenvectors for $\lambda = 81$: $v_1 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$ Eigenvectors for $\lambda = 1$: $v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Singular Value Decomposition

- Normalize Eigenvectors

Normalize Eigenvectors:

Eigenvectors for λ = 1 : $(0.5, 1)$, Length L = $\sqrt{0.5^2 + 1^2} = 1.118$

So, normalizing gives $v_1 = \left(\frac{0.5}{1.118}, \frac{1}{1.118}\right) = (0.447, 0.894)$

Eigenvectors for λ = 81 : $(-2, 1)$, Length L = $\sqrt{(-2)^2 + 1^2} = 2.236$

So, normalizing gives $v_2 = \left(\frac{-2}{2.236}, \frac{1}{2.236}\right) = (-0.894, 0.447)$

Singular Value Decomposition

- Using these values, we can write the solution as:

$$\Sigma = \begin{bmatrix} \sqrt{81} & 0 \\ 0 & \sqrt{1} \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}$$

$$U = [u_1, u_2] = \begin{bmatrix} -0.894 & 0.447 \\ 0.447 & 0.894 \end{bmatrix}$$

V is found using formula $v_i = \frac{1}{\sigma_i} A^T \cdot u_i$

$$V = \begin{bmatrix} 0.447 & -0.894 \\ 0.894 & 0.447 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sqrt{81} & 0 \\ 0 & \sqrt{1} \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}$$

$$V = [v_1, v_2] = \begin{bmatrix} 0.447 & -0.894 \\ 0.894 & 0.447 \end{bmatrix}$$

U is found using formula $u_i = \frac{1}{\sigma_i} A \cdot v_i$

$$U = \begin{bmatrix} -0.894 & 0.447 \\ 0.447 & 0.894 \end{bmatrix}$$

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Thank You.