Universidad Autónoma de la Ciudad de México

Notas de Probabilidad

Apuntes y ejercicios seleccionados

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Índice

1. Fundamentos

Definición 1. *Parámetro es una Característica de la población medias y varianzas en la distribución normal o binomial. Si se conocen sus valores o se puede aproximar con suficiente precisión, se puede responder cualquier pregunta sobre la población.*

Definición 2. *Un Estadístico es Cualquier función de la muestra, por ejemplo la media o varianza muestral.*

Definición 3. *Los Estimadores son estadísticos independientes de los parámetros de la población y se utilizan para aproximar. Si θ es el parámetro de interés, su estimador es denotado por $\hat{\theta}$. Por ejemplo en la distribución normal para μ s tiene que $\bar{X} = \hat{\mu}$ y para σ^2, $S^2 = \hat{\sigma}^2$. en este sentido $\bar{X}$, S^2 son estimadores puntuales de μ y σ^2 para la distribución normal.*

Definición 4. *Muestreo: Se considerará el muestreo aleatorio simple para determinar el tamaño de la muestra, n.*

Los estimadores se obtienen a partir de una Muestra Aleatoria Simple (MAS), X_1, X_2, ..., X_n. de la variable aleatoria X. Para cada muestreo los resultados serán diferentes, es decir, por ser realizados a partir de una muestra aleatoria, los estimadores son aleatorios y por tanto tienen una distribución, a la que se le denomina distribución de muestreo.

Ejemplo 1. ***Distribución Binomial***: *Supóngase que $X \sim Bin(n,p)$, con $p = P\{X = 1\}$ y $n =$ tamaño de la muestra. Para determinar p se selecciona una MAS $X_1, X_2, \ldots, X_n$ de v.a. $Bin(1,p) = Bern(p)$. La proporción muestral está dada por:*

$$\hat{p} = \sum_{i=1}^{n} \frac{X_i}{n} \tag{1}$$

la proporción muestral $\hat{p}$ es una v.a. para n suficientemente grande

$$\hat{p} \sim N\left(p, \frac{p(1-p)}{n}\right) \text{ (Teorema Central del Límite)} \tag{2}$$

El error típico definido por

$$ET(\hat{p}) = \sqrt{\frac{p(1-p)}{n}}, \tag{3}$$

p desconocido y por tanto $ET(\hat{p})$ también, para encontrar una aproximación se puede sustituir el valor de p por el de $\hat{p}$. Por ejemplo, si $X \sim Bin(15,p)$ y se quiere estimar p, se toman 500 muestras de tamaño 100 ($X_1^i, X_2^i, \ldots, X_{100}^i$, $i = 1, 2, \ldots, 500$) y se calcula la proporción muestral en cada una de ellas obteniendo 500 valores para $\hat{p}$ ($\hat{\mu} = 0{,}7$ y $\hat{\sigma}^2 = 0{,}7 \times 0{,}3/100$)

Ejemplo 2. *2) Dist. Normal* $N(\mu, \sigma^2)$
Def.: $X \sim N(\mu, \sigma^2)$. Sup. $X_1, X_2, \ldots, X_n$ MAS, $X \sim N(\mu, \sigma^2)$.

$$\hat{\mu} = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \quad T.O. \tag{4}$$

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right) \quad \Rightarrow \quad Z_1 = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1) \tag{5}$$

Esto es válido si la varianza pob. σ^2 *es conocida.*

$$E(\bar{X}) = \mu \qquad ET(\bar{X}) = \frac{\sigma}{\sqrt{n}}, \quad \textit{ejemplo } n = 20, 100 \ \textit{y} \ 500 \tag{6}$$

$$\Rightarrow \quad \bar{X} \sim N(\mu = 5, \sigma^2 = 4/n) \tag{7}$$

Si σ^2 *es desconocida,*

$$\hat{\sigma}^2 \leftarrow S^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 \tag{8}$$

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 \tag{9}$$

En (1) $\sigma^2 \to s^2$*:*

$$\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim \begin{cases} t_{n-1}, & n \leq 30 \\ N(0,1), & n > 30 \end{cases} \tag{10}$$

y

$$ET(\bar{X}) = \frac{S}{\sqrt{n-1}} \simeq \frac{S}{\sqrt{n}} \quad \textit{(error residual)} \tag{11}$$

1) σ^2 *conocida:*

$$\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1) \tag{12}$$

2) σ^2 *desc.,* $n > 30$*:*

$$\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim N(0,1) \tag{13}$$

3) σ^2 desc., $n \leq 30$:

$$\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{n-1} \tag{14}$$

3) Est. de varianza σ^2:
Si $X_1, X_2, \ldots, X_n$ MAS de v.a. $N(\mu, \sigma^2)$, entonces

$$\frac{nS^2}{\sigma^2} \sim \chi^2_n \qquad o \qquad \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1} \tag{15}$$

Para n pequeña.
Si n es suf. grande se puede aprox χ^2_n por $N(n, 2n)$.

Un estimador de un parámetro pob es función de la muestra.
Es preciso dar una definición de las prop. para que el estimador sea bueno.
Sea $X_1, X_2, \ldots, X_n$ una MAS de tamaño n, se dice que $\hat{\theta}$ es un estimador de θ si es estadístico calculado p/ cualquier dicho parámetro desconocido es ese.

$$\hat{\theta} = h(X_1, X_2, \ldots, X_n) \tag{16}$$

3) Sea X v.a. con dist. $F_X(x)$, $F_X(x; \theta)$, θ parámetro desconocido.
Sea $X_1, X_2, \ldots, X_n$ MAS de tamaño n. A $\hat{\theta} = h(X_1, X_2, \ldots, X_n)$ se le llama estimador puntual de θ, $\hat{\theta}$ es una v.a. por ser función de v.a. y por tanto tiene dist.
Al seleccionar una muestra, $\hat{\theta}$ toma un valor llamado estimación puntual.
Se dice que $\hat{\theta}$ es un estimador insesgado de θ si

$$E(\hat{\theta}) = \theta \tag{17}$$

Si $\hat{\theta}$ no es insesgado $\Rightarrow$

$$E(\hat{\theta}) - \theta \tag{18}$$

es el sesgo del estimador.
De los posibles estimadores, el de menor varianza es considerado mejor estimador (MVUE). Si es insesgado $\Rightarrow$ (UMVUE) es mejor estimador de θ.
Para $X_1, X_2, \ldots, X_n$ MAS de $N(\mu, \sigma^2)$, μ es UMVUE.
Para $X_1, X_2, \ldots, X_n$ MAS de $N(\mu, \sigma^2)$,

$$\hat{\sigma}_{\hat{\theta}} = \sqrt{V(\hat{\theta})} \tag{19}$$

El error estándar.
Para MAS $N(\mu, \sigma^2)$,

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right) \quad \text{con error estándar} \tag{20}$$

$$\hat{\sigma}_{\bar{X}} = \frac{S}{\sqrt{n}} \tag{21}$$

Def. 1. Dado un espacio de probabilidad

$$(\Omega, \mathcal{A}, P), \tag{22}$$

una variable aleatoria.

Una v.a. X es una función

$$X : (\Omega, \mathcal{A}) \to \mathbb{R} \quad \text{t.q.} \quad A_x = \{\omega : X(\omega) \leq x\} \in \mathcal{A}, \ \forall x \in \mathbb{R}. \tag{23}$$

Def. 2. Sea Ω espacio muestral, y v.a. La función de prob inducida por Y es

$$P_Y(y = y_i) = P[Y = y_i] = P_\Omega\{\omega \in \Omega : Y(\omega) = y_i\}. \tag{24}$$

El espacio muestral para Y es

$$S_Y = \{y_i \in \mathbb{R} \mid \exists s \in \Omega, \ Y(s) = y_i\}. \tag{25}$$

Def. La población es el conjunto (grupo) de elementos de los cuales se desea tener información. La muestra es una parte de la población bajo estudio.

Def. La función distribución acumulada (CDF) de la v.a. X es

$$F_X(x) = P[X \leq x], \quad \forall x \in \mathbb{R}. \tag{26}$$

$$F_X(x) = P[X \leq x] = P\{\omega : X(\omega) \leq x\}, \quad \forall x \in \mathbb{R}. \tag{27}$$

Prop. de la FDA (CDF) $F_X(\cdot)$:

a)

$$F_X(-\infty) = \lim_{x \to -\infty} F_X(x) = 0, \qquad F_X(\infty) = \lim_{x \to \infty} F_X(x) = 1. \tag{28}$$

b) $F_X(\cdot)$ es monótona no decreciente, i.e.

$$F_X(a) \leq F_X(b), \quad \text{para } a < b. \tag{29}$$

c) $F_X(\cdot)$ es continua por la derecha:

$$\lim_{h \to 0^+} F_X(x + h) = F_X(x). \tag{30}$$

Def. Sea X v.a.

a) X es una v.a. discreta si sólo puede tomar una cantidad finita o numerable de valores distintos, i.e. su rango es discreto.

Def. de prob.: Sea $\mathcal{A}$ σ-álgebra en Ω, $P : \mathcal{A} \to [0, 1]$ es una función de conjuntos t.q.

1) $P(A) \geq 0 \quad \forall A \in \mathcal{A}$

2) $P(\Omega) = 1$

3) Si $\{A_\alpha\}_{\alpha=1}^{\infty}$ son eventos mutuamente excluyentes,

$$A_\alpha \cap A_{\alpha'} = \quad \text{para } \alpha \neq \alpha', \tag{31}$$

entonces

$$P\left(\bigcup_{\alpha=1}^{\infty} A_\alpha\right) = \sum_{\alpha=1}^{\infty} P(A_\alpha). \tag{32}$$

Def. Si X es una v.a. discreta con valores $x_1, x_2, \ldots, x_n, \ldots$, la función $f_X(\cdot)$ definida por

$$f_X(x) = \begin{cases} P[X = x_j], & x = x_j, \quad j = 1, 2, \ldots \\ 0, & x \neq x_j \end{cases} \tag{33}$$

es llamada función de densidad discreta de X.

Los valores x_j son llamados puntos de masa de X y $f_X(x_j)$ son las masas asociadas a los puntos de masa x_j.

Teo. Si X es una v.a. discreta, $F_X(\cdot)$ puede obtenerse de $f_X(\cdot)$ y viceversa.

Nota: La CDF $F_X(\cdot)$ es una función escalón.

Def.

Def. Una v.a. X es continua si existe una función $f_X(\cdot)$ t.q.

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\, du, \quad \forall x \in \mathbb{R}. \tag{34}$$

Def. Si X es una v.a. continua, la función

$$f_X(x) = \frac{d}{dx} F_X(x) \tag{35}$$

es llamada función de densidad de probabilidad de X.

Teo. Si X tiene densidad $f_X(\cdot)$, entonces

$$f_X(x) = \frac{d}{dx} F_X(x) = F'_X(x). \tag{36}$$

Def.

Def. Sea X v.a., la media de X, μ_X, $E[X]$ se define por

$$E[X] = \sum_{j=1}^{\infty} x_j f_X(x_j) \qquad \text{VAD} \tag{37}$$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx \qquad \text{VAC} \tag{38}$$

Si g es una función de $\bar{X}$, entonces

$$E[g(X)] = \sum_{x: g(x) \geq 0} g(x)\, f_X(x) \tag{39}$$

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx \tag{40}$$

siempre que la integral exista.
Def. Si $E[X^2]$ existe, entonces la varianza de la v.a. X es

$$\mathrm{Var}(X) = V[X] = E\big[X - (E[X])^2\big] \tag{41}$$

y la desviación estándar

$$SD(X) = \sqrt{V[X]}. \tag{42}$$

Si $E[X^2]$ no existe, la varianza no existe.
Teo.

$$V[Y] = E[Y^2] - (E[Y])^2. \tag{43}$$

Def. Sea X v.a., $\mu_X = E[X]$, la varianza σ_X^2 se define por

$$\mathrm{Var}[X] = \sum_{j=1}^{\infty} (x_j - \mu_X)^2\, f_X(x_j) \tag{44}$$

$$\mathrm{Var}[X] = \int_{-\infty}^{\infty} (x - \mu_X)^2\, f_X(x)\, dx. \tag{45}$$

Teo. Sea X v.a. entonces
i) $E[c] = c$, c constante.
ii) $E[cg(X)] = c\, E[g(X)]$.
iii) $E[c_1 g_1(X) + c_2 g_2(X)] = c_1 E[g_1(X)] + c_2 E[g_2(X)]$.
iv) $E[g_1(X)] \leq E[g_2(X)]$ si $g_1(x) \leq g_2(x)$ $\forall x$.
Teo. Sea X v.a. con $g(\cdot) \geq 0$, $g : \mathbb{R} \to \mathbb{R}$,

$$P[g(X) \geq k] \leq \frac{E[g(X)]}{k}, \qquad \forall k > 0. \tag{46}$$

(Chebyshov)
Corolario: (Des. Chebyshev) Si X v.a. con varianza finita,

$$P\left(|X - \mu_X| \geq r\sigma_X\right) = P\left((X - \mu_X)^2 \geq r^2 \sigma_X^2\right) \leq \frac{1}{r^2}, \quad r > 0. \tag{47}$$

Obs.

$$P\left(|X - \mu_X| < r\sigma_X\right) \geq 1 - \frac{1}{r^2}. \tag{48}$$

Teo.

$$V[aY+b]=a^2V[Y]. \tag{49}$$

Def. La FGM de la v.a. X es

$$M(t)=E[e^{tY}] \tag{50}$$

y por Taylor alrededor,

$$\mu_r=E[X^r]=\sum\int e^{ty}f(y)=\int_{-\infty}^{\infty}e^{ty}f_Y(y)\,dy. \tag{51}$$

La función característica (CF) de $\bar{X}$

$$C(t)=E\left[e^{itX}\right], \qquad i=\sqrt{-1}. \tag{52}$$

Prop. Sea X v.a. con MGF $M_X(t)$ que existe para $|t|<b$, $b>0$. Entonces

$$C(t)=M_X(it). \tag{53}$$

Def. Sean X,Y v.a., X,Y son v.a.i.d. $X\sim Y$ o $Y\sim F_X$ si $F_Y(y)=F_X(y)$, $\forall y\in\mathbb{R}$.

Prop. Sean X,Y v.a., X y Y son idénticamente distribuidas si se cumple cualquiera de las sig. condiciones:

a) $F_X(y)=F_Y(y) \quad \forall y$

b) $f_X(y)=f_Y(y) \quad \forall y$

c) $C_X(t)=C_Y(t) \quad \forall t$

d) $M_X(t)=M_Y(t) \quad \forall t$ en una vecindad de cero.

Def. El k-ésimo momento de X es $E[X^k]$ y su k-ésimo momento central es

$$E\left[(X-E[X])^k\right]. \tag{54}$$

Teo. Sup. MGF $m(t)$ existe para $|t|<b$, b constante positiva. Sup. k-ésima derivada $m^{(k)}(t)$ existe para $|t|<b$. Entonces

$$E[X^k]=m^{(k)}(0). \tag{55}$$

Def. El q-ésimo cuantil de una v.a. $\bar{X}$, q_α, es el mínimo valor ξ t.q.

$$F_X(\xi)\geq q. \tag{56}$$

Def. La mediana de una v.a. $\bar{X}$, $\mathrm{Med}_X=\xi_{0,5}$.

Nota. El 3er momento sobre la media es llamada medida de simetría o sesgo.

$$\gamma = \frac{\mu_3}{\sigma^3} \tag{57}$$

es el coeficiente de sesgo.

$$\frac{\mu_4}{\sigma^4} - 3 \tag{58}$$

coeficiente de exceso de kurtosis.

Def. Sea $f(y) = f_Y(y \mid \theta)$ la PDF de Y v.a.

$$\mathcal{Y} = \{y \mid f(y) > 0\} \tag{59}$$

es el soporte de valores de Y.

Sea Θ el conjunto de parámetros θ de interés.

$\mathcal{H}$ es el espacio de parámetros de Y.

Def. La función indicadora

$$\mathbf{1}_A(x) = \mathbf{1}(x \in A) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases} \tag{60}$$

Nota: Algunas PMF / PDF / CDF están restringidas solamente en el soporte. Apéndice 2.

Def. Sea $f_Y(y)$ PDF de Y v.a. Sup.

$$f_Y(y \mid \theta) = c(\theta)\, k(y \mid \theta), \tag{61}$$

entonces $k(y \mid \theta)$ es el kernel de f_Y y $c(\theta) > 0$ es el término constante que hace que f_Y sume / integre 1. Así,

$$\int_{-\infty}^{\infty} k(y \mid \theta)\, dy = \frac{1}{c(\theta)}. \tag{62}$$

$$\sum_{y \in \mathcal{Y}} k(y \mid \theta) = \frac{1}{c(\theta)}. \tag{63}$$

Sup. Y es v.a. con $f_Y(\cdot)$ PDF,

$$E[g(Y)] = \int_{-\infty}^{\infty} g(y)\, f(y \mid \theta)\, dy = \int_{y \in \text{soporte}} g(y)\, f(y \mid \theta)\, dy. \tag{64}$$

Sup. que desp. de operaciones,

$$E[g(Y)] = a\, c(\theta) \int_{-\infty}^{\infty} k(y \mid \theta)\, dy, \quad a \text{ cte.} \tag{65}$$

De manera similar si f_Y es PMF,

$$E[g(Y)] = \sum_{y \in \mathcal{Y}} g(y)\, f(y \mid \theta), \tag{66}$$

y soporte de Y.

Sup. que desp. operaciones algebraicas,

$$E[g(Y)] = a\, c(\theta) \sum_{y \in \mathcal{Y}} k(y \mid \tau), \quad a \text{ cte.} \tag{67}$$

$$\Rightarrow \quad E[g(Y)] = a\, c(\theta) \sum_{y \in \mathcal{Y}} \frac{c(\tau)}{c(\tau)}\, k(y \mid \tau) \tag{68}$$

$$= a\, \frac{c(\theta)}{c(\tau)} \sum_{y \in \mathcal{Y}} c(\tau)\, k(y \mid \tau) = a\, \frac{c(\theta)}{c(\tau)}. \tag{69}$$

Ej. La función $\zeta(v)$ tiene PMF

$$f(y) = P[Y = y] = \frac{1}{y^v\, \zeta(v)}, \qquad v > 1,\ y = 1, 2, 3, \ldots \tag{70}$$

$$\zeta(v) = \sum_{y=1}^{\infty} \frac{1}{y^v}. \tag{71}$$

Luego, para $v > 1$,

$$E[Y] = \sum_{y=1}^{\infty} y\, \frac{1}{\zeta(v)}\, \frac{1}{y^v} = \frac{1}{\zeta(v)} \sum_{y=1}^{\infty} y^{1-v} \tag{72}$$

$$= \frac{\zeta(v-1)}{\zeta(v)} \sum_{y=1}^{\infty} \frac{1}{y^{v-1}} = \frac{\zeta(v-1)}{\zeta(v)}. \tag{73}$$

Mezclas de distribuciones

Def. La dist. de una v.a. $\bar{X}$ es una mezcla si la CDF de $\bar{X}$ es de la forma

$$F_{\bar{X}}(x) = \sum_{i=1}^{k} \alpha_i\, F_{X_i}(x), \qquad 0 < \alpha_i < 1, \quad \sum_{i=1}^{k} \alpha_i = 1, \quad k \geq 2, \tag{74}$$

y $F_{X_i}(x)$ es la CDF de una v.a. X_i, $i = 1, 2, \ldots, k$.

Def. Sea $\bar{X}$ v.a. con CDF $F_{\bar{X}}(x)$. Sea h función t.q. $E[h(\bar{X})]$ existe. Entonces

$$E[h(\bar{X})] = \int_{-\infty}^{\infty} h(x)\, dF_{\bar{X}}(x). \tag{75}$$

Prop.
a) Si $\bar{X}$ es una v.a. discreta con PMF $f_{\bar{X}}(x)$, tiene soporte $\mathcal{Y}$, entonces

$$E[h(\bar{X})] = \int_{-\infty}^{\infty} h(x)\, dF_{\bar{X}}(x) = \sum_{y\in\mathcal{Y}} h(y)\, f_{\bar{X}}(y). \tag{76}$$

b) Si $\bar{X}$ es una v.a. continua con CDF $F_{\bar{X}}(x)$ y PDF $f_{\bar{X}}(x)$, entonces

$$E[h(\bar{X})] = \int_{-\infty}^{\infty} h(x)\, dF_{\bar{X}}(x) = \int_{-\infty}^{\infty} h(x)\, f_{\bar{X}}(x)\, dx. \tag{77}$$

c) Si $\bar{X}$ es una v.a. con dist. mezcla y

$$F_{\bar{X}}(x) = \sum_{i=1}^{k} \alpha_i\, F_{X_i}(x), \tag{78}$$

entonces

$$E[h(\bar{X})] = \int_{-\infty}^{\infty} h(x)\, dF_{\bar{X}}(x) = \sum_{i=1}^{k} \alpha_i\, E_{X_i}[h(X_i)]. \tag{79}$$

Ej. Sup. CDF de X es

$$F_X(x) = (1-\varepsilon)\,\Phi(x) + \varepsilon\,\Phi(x \mid k), \tag{80}$$

$0 < \varepsilon < 1$ y $\Phi(x)$ es la CDF de $W_1 \sim N(0,1)$, entonces

$$\Phi(x \mid k) \tag{81}$$

es la CDF de $W_2 \sim N(0,k^2)$.
$E[Y] =?$ Usar $h(y) = y$. Entonces

$$E[Y] = (1-\varepsilon)E[W_1] + \varepsilon E[W_2] = (1-\varepsilon)0 + \varepsilon(0) = 0. \tag{82}$$

Para encontrar $E[Y^2]$ utilizar $h(y) = y^2$.

$$E[Y^2] = (1-\varepsilon)E[W_1^2] + \varepsilon E[W_2^2] = (1-\varepsilon)1 + \varepsilon k^2 = 1 - \varepsilon + \varepsilon k^2. \tag{83}$$

Entonces

$$V[Y] = E[Y^2] - E^2[Y] = 1 - \varepsilon + \varepsilon k^2. \tag{84}$$

Distribución discreta.
1)

$$f(x) = f(x; N) = \begin{cases} \frac{1}{N}, & x = 1, 2, \ldots, N, \\ 0, & \text{otro caso.} \end{cases} \tag{85}$$

X v.a. uniforme.

$$E[X] = \frac{N+1}{2}, \qquad V[X] = \frac{N^2 - 1}{12}. \tag{86}$$

$$M_X(t) = E[e^{tX}] = \sum_{x=1}^{N} e^{tx} \, \frac{1}{N}. \tag{87}$$

2) Si X v.a. con

$$f_X(x) = f_X(x; p) = \begin{cases} p\, x^{p-1}, & 0 < x < 1, \\ 0, & \text{otro caso,} \end{cases} \qquad 0 \leq p \leq 1, \tag{88}$$

$$F_X(x) = \int_0^x p\, t^{p-1} \, dt = x^p, \qquad 0 < x < 1. \tag{89}$$

Sea $X \sim \text{Beta}(p)$.

$$E[X] = p, \qquad V[X] = \frac{p(1-p)}{(p+1)}. \tag{90}$$

3) Sea X v.a. con

$$f_X(x) = f_X(x; n, p) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \ldots, n. \tag{91}$$

$X \sim \text{Bin}(n, p)$.

$$E[X] = np, \qquad V[X] = np(1-p). \tag{92}$$

4) Sea X v.a. con

$$f_X(x) = f_X(x;\lambda) = \frac{(\lambda x)^k e^{-\lambda x}}{k!}, \quad x = 0, 1, 2, \dots \tag{93}$$

$X \sim \text{Poisson}(\lambda)$.

$$E[X] = \lambda, \qquad V[X] = \lambda. \tag{94}$$

5) Sea X v.a. con

$$f_X(x;\mu,n) = \frac{\left(\frac{n}{\mu}\right)^n x^{n-1} e^{-nx/\mu}}{(n-1)!}, \quad x > 0. \tag{95}$$

$X \sim \text{Gamma}(n,\mu)$.

$$E[X] = \mu, \qquad V[X] = \frac{\mu^2}{n}. \tag{96}$$

6) Sea X v.a. con

$$f_X(x;\alpha,\beta) = \frac{1}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-(x/\beta)^\alpha}, \quad x > 0. \tag{97}$$

$X \sim \text{Weibull}(\alpha,\beta)$.

$$E[X] = \beta\,\Gamma\left(1 + \frac{1}{\alpha}\right), \qquad V[X] = \beta^2\left[\Gamma\left(1 + \frac{2}{\alpha}\right) - \Gamma^2\left(1 + \frac{1}{\alpha}\right)\right]. \tag{98}$$

Distribuciones continuas
6) Binomial negativa

$$P(X = x \mid r, p) = \binom{x-1}{r-1} p^r (1-p)^{x-r}, \qquad x = r, r+1, \dots \tag{99}$$

$$E[X] = \frac{r}{p}, \qquad \text{Var}[X] = \frac{r(1-p)}{p^2}, \tag{100}$$

$$M_X(t) = \left(\frac{p}{1-(1-p)e^t}\right)^r. \tag{101}$$

7) Distribución geométrica

$$P(X = x \mid p) = p(1-p)^{x-1}, \qquad x = 1, 2, \ldots \tag{102}$$

$$E[X] = \frac{1}{p}, \qquad \mathrm{Var}[X] = \frac{1-p}{p^2}, \tag{103}$$

$$M_X(t) = \frac{pe^t}{1-(1-p)e^t}. \tag{104}$$

8) Distribución uniforme

$$f(x \mid a, b) = \begin{cases} \frac{1}{b-a}, & a < x < b, \\ 0, & \text{otro caso.} \end{cases} \tag{105}$$

$$E[X] = \frac{a+b}{2}, \qquad \mathrm{Var}[X] = \frac{(b-a)^2}{12}, \tag{106}$$

$$M_X(t) = \frac{e^{bt} - e^{at}}{(b-a)t}. \tag{107}$$

9) Distribución gamma

$$f(x \mid \alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \qquad x > 0. \tag{108}$$

$$E[X] = \alpha\beta, \qquad \mathrm{Var}[X] = \alpha\beta^2, \tag{109}$$

$$M_X(t) = (1-\beta t)^{-\alpha}, \qquad t < \frac{1}{\beta}. \tag{110}$$

10) Distribución exponencial $\quad (\alpha = 1)$

$$f(x \mid \beta) = \frac{1}{\beta} e^{-x/\beta}, \qquad x > 0. \tag{111}$$

$$E[X] = \beta, \qquad \mathrm{Var}[X] = \beta^2, \tag{112}$$

$$M_X(t) = \frac{1}{1-\beta t}, \qquad t < \frac{1}{\beta}. \tag{113}$$

5) Distribución Weibull
Sea $Y = X^{1/r}$.

$$f_Y(y \mid \beta) = \frac{r}{\beta^r}\, y^{r-1} e^{-(y/\beta)^r}, \qquad y > 0,\ r > 0,\ \beta > 0 \tag{114}$$

$$E[Y] = \beta\,\Gamma\left(1+\frac{1}{r}\right), \qquad \mathrm{Var}[Y] = \beta^2\left[\Gamma\left(1+\frac{2}{r}\right) - \Gamma^2\left(1+\frac{1}{r}\right)\right] \tag{115}$$

6) Distribución normal general

$$X \sim N(\mu, \sigma^2) \tag{116}$$

$$f(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \tag{117}$$

$$E[X] = \mu, \qquad \mathrm{Var}[X] = \sigma^2, \qquad M_X(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right) \tag{118}$$

7) Distribución Beta

$$f(x \mid \alpha, \beta) = \frac{1}{B(\alpha,\beta)}\, x^{\alpha-1}(1-x)^{\beta-1}, \qquad 0 < x < 1,\ \alpha > 0,\ \beta > 0 \tag{119}$$

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{120}$$

$$E[X] = \frac{\alpha}{\alpha+\beta}, \qquad \mathrm{Var}[X] = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} \tag{121}$$

8) Distribución Gamma

$$f(x \mid \alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \qquad x > 0 \tag{122}$$

$$E[X] = \alpha\beta, \qquad \mathrm{Var}[X] = \alpha\beta^2 \tag{123}$$

$$M_X(t) = (1-\beta t)^{-\alpha}, \qquad t < \frac{1}{\beta} \tag{124}$$

9) Dist. Lognormal
Sea X v.a. y sea $Y = \log X$ v.a.

$$\log X \sim N(\mu, \sigma^2) \tag{125}$$

$$f(x \mid \mu, \sigma^2) = \frac{1}{x\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\log x - \mu)^2}{2\sigma^2}\right), \qquad x > 0 \tag{126}$$

$$E[X] = e^{\mu + \frac{\sigma^2}{2}}, \qquad \mathrm{Var}[X] = \left(e^{\sigma^2} - 1\right)e^{2\mu + \sigma^2} \tag{127}$$

10) Dist. Doble exponencial

$$f(x \mid \mu, b) = \frac{1}{2b} e^{-|x-\mu|/b}, \qquad x \in \mathbb{R},\ b > 0 \tag{128}$$

$$E[X] = \mu, \qquad \mathrm{Var}[X] = 2b^2 \tag{129}$$

11) Distribución logística

$$f(x \mid \alpha, \beta) = \frac{e^{-(x-\alpha)/\beta}}{\beta\left(1 + e^{-(x-\alpha)/\beta}\right)^2}, \qquad x \in \mathbb{R},\ \beta > 0 \tag{130}$$

$$E[X] = \alpha, \qquad \mathrm{Var}[X] = \frac{\pi^2}{3}\beta^2 \tag{131}$$

12) Dist. Pareto

$$f(x \mid x_0, \theta) = \frac{\theta x_0^\theta}{x^{\theta+1}}, \qquad x \geq x_0,\ \theta > 0 \tag{132}$$

$$E[X] = \frac{\theta x_0}{\theta - 1}, \qquad \theta > 1 \tag{133}$$

$$\mathrm{Var}[X] = \frac{\theta x_0^2}{(\theta-1)^2(\theta-2)}, \qquad \theta > 2 \tag{134}$$

13) Dist. Gumbel

$$f_X(x;\alpha,\beta) = \frac{1}{\beta}\exp\left(-\frac{x-\alpha}{\beta} - e^{-(x-\alpha)/\beta}\right), \qquad \beta > 0 \tag{135}$$

Distribución Normalizada
Sea

$$I = \int_{-\infty}^{\infty} e^{-z^2/2}\,dz \tag{136}$$

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(z_1^2+z_2^2)/2}\,dz_1\,dz_2 \tag{137}$$

Cambio a coordenadas polares:

$$z_1 = r\cos\theta, \qquad z_2 = r\sin\theta, \qquad dz_1\,dz_2 = r\,dr\,d\theta \tag{138}$$

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2/2} r\,dr\,d\theta \tag{139}$$

$$= \int_0^{2\pi} d\theta \int_0^{\infty} re^{-r^2/2}\,dr \tag{140}$$

Sea

$$u = \frac{r^2}{2} \quad \Rightarrow \quad du = r\,dr \tag{141}$$

$$\int_0^{\infty} re^{-r^2/2}\,dr = \int_0^{\infty} e^{-u}\,du = 1 \tag{142}$$

$$I^2 = 2\pi \quad \Rightarrow \quad I = \sqrt{2\pi} \tag{143}$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\,dz = 1 \tag{144}$$

Sea X v.a. con f.d.p. $f(x)$.
I)

$$f(x) = xe^{-x^2/2}, \qquad x > 0 \tag{145}$$

II)

$$f(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}, \qquad x \in \mathbb{R} \tag{146}$$

III)

$$f(x) = \frac{4}{\pi} x^2 e^{-x^2}, \qquad x > 0 \tag{147}$$

IV)

$$f(x) = \frac{4}{\sqrt{\pi}} x^2 e^{-x^2}, \qquad x > 0 \tag{148}$$

V)

$$f(x) = 2x e^{-x^2}, \qquad x > 0 \tag{149}$$

VI)

$$f(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}, \qquad x > 0 \tag{150}$$

VII)

$$f(x) = \frac{4}{\sqrt{\pi}} x e^{-x^2}, \qquad x > 0 \tag{151}$$

VIII)

$$f(x) = 2x^3 e^{-x^2}, \qquad x > 0 \tag{152}$$

IX)

$$f(x) = c e^{-x^2}, \qquad x \in \mathbb{R} \tag{153}$$

X)

$$f(x) = k x^2 e^{-x^2}, \qquad x > 0 \tag{154}$$

—
Sea

$$I = \int_0^\infty e^{-z^2}\, dz \tag{155}$$

$$I^2 = \int_0^\infty \int_0^\infty e^{-(z_1^2 + z_2^2)}\, dz_1\, dz_2 \tag{156}$$

Cambio a coordenadas polares:

$$z_1 = r\cos\theta, \qquad z_2 = r\sin\theta, \qquad dz_1 dz_2 = r\,dr\,d\theta \tag{157}$$

$$I^2 = \int_0^{\pi/2}\int_0^{\infty} e^{-r^2} r\,dr\,d\theta \tag{158}$$

$$= \int_0^{\pi/2} d\theta \int_0^{\infty} re^{-r^2}\,dr \tag{159}$$

Sea

$$u = r^2 \quad \Rightarrow \quad du = 2r\,dr \tag{160}$$

$$\int_0^{\infty} re^{-r^2}\,dr = \frac{1}{2}\int_0^{\infty} e^{-u}\,du = \frac{1}{2} \tag{161}$$

$$I^2 = \frac{\pi}{4} \quad \Rightarrow \quad I = \frac{\sqrt{\pi}}{2} \tag{162}$$

$(X)\,dx \sim X$ i.s.

$$\frac{k}{T} = \theta = \left(\frac{2k}{T}, \frac{k}{T}\right) = \theta \leq \frac{2k}{T} = 0 \qquad k = n \tag{163}$$

$$\frac{2k}{T} = 0 \leq \frac{2k}{T} = \frac{2k}{2} = \left(\frac{y}{T}\right) - \frac{2k}{2} = E[Y] \tag{164}$$

$$\frac{2k}{2} = \frac{k}{T} = M_X(1) \tag{165}$$

$$M_X(t) = \frac{2(x-t)}{2x(x-t)} = \frac{(x-t)}{2x(x-t)} \tag{166}$$

$$\frac{2(x-t)}{x} \tag{167}$$

$$M_X(t) = \frac{x-t}{x} \tag{168}$$

$$M_X(t) = \frac{2(x-t)}{x} \tag{169}$$

$$M_X(t) = \frac{x-t}{x} \tag{170}$$

$$M_X(t) = \frac{x-t}{x} \tag{171}$$

$$M'_X(t) = -\frac{1}{x} \tag{172}$$

$$M''_X(t) = \frac{2(x-t)}{x^2} \tag{173}$$

$$M'_X(1) = -\frac{1}{x} \tag{174}$$

$$\int_0^\infty e^{-(x-t)x}\,dx \tag{175}$$

$$\int_0^\infty xe^{-x^2}\,dx \tag{176}$$

$$\int_0^\infty x^2e^{-x^2}\,dx \tag{177}$$

$$E[X] = \int_0^\infty xe^{-x^2}\,dx \tag{178}$$

$$E[X^2] = \int_0^\infty x^2e^{-x^2}\,dx \tag{179}$$

$$K(X|\theta) \sim \exp(\theta) \tag{180}$$

$$K(X|\theta) = \lambda e^{-\lambda x} \tag{181}$$

$$f_X(x) = \lambda e^{-\lambda x}, \quad x > 0 \tag{182}$$

$$\int_0^\infty \lambda e^{-\lambda x}\, dx \tag{183}$$

$$E[X] = \int_0^\infty x\lambda e^{-\lambda x}\, dx \tag{184}$$

$$= \frac{1}{\lambda} \tag{185}$$

$$E[X^2] = \int_0^\infty x^2\lambda e^{-\lambda x}\, dx \tag{186}$$

$$= \frac{2}{\lambda^2} \tag{187}$$

$$V[X] = \frac{1}{\lambda^2} \tag{188}$$

$$f(x) = ke^{-2x}, \quad x > 0 \tag{189}$$

$$\int_0^\infty ke^{-2x}\, dx \tag{190}$$

$$k = \frac{1}{2} \tag{191}$$

$$E[X] = \int_0^\infty x\frac{1}{2}e^{-2x}\, dx \tag{192}$$

$$= \frac{1}{4} \tag{193}$$

$$\theta = \frac{1}{\lambda} \tag{194}$$

$$\theta = \frac{X}{2} \tag{195}$$

$$\theta = \frac{X}{2} \tag{196}$$

$$K(\theta|X) \sim \exp(k) \tag{197}$$

(24)

$$\theta = (\mu, \sigma^2) \tag{198}$$

$$\lambda = \frac{1}{2} \tag{199}$$

$$X \sim \text{Exp}\left(\frac{1}{2}\right) \tag{200}$$

$$f_X(x;\theta) \tag{201}$$

$$\Phi = \begin{pmatrix} \mu = \frac{1}{\lambda} \\ \sigma^2 = \frac{1}{\lambda^2} \end{pmatrix} \tag{202}$$

$$f(x) = \frac{1}{2}e^{-x/2}, \quad x > 0 \tag{203}$$

$$\int_0^\infty f(x)\,dx = \int_0^\infty \frac{1}{2}e^{-x/2}\,dx \tag{204}$$

$$u = \frac{x}{2} \quad \Rightarrow \quad du = \frac{1}{2}dx \tag{205}$$

$$\int_0^\infty \frac{1}{2}e^{-x/2}\,dx = \int_0^\infty e^{-u}\,du = 1 \tag{206}$$

$$E[X] = \int_0^\infty xf(x)\,dx = \int_0^\infty x\frac{1}{2}e^{-x/2}\,dx \tag{207}$$

$$u = \frac{x}{2} \quad \Rightarrow \quad x = 2u \tag{208}$$

$$E[X] = \int_0^\infty 2ue^{-u}\,du \tag{209}$$

$$= 2\int_0^\infty ue^{-u}\,du \tag{210}$$

$$= 2 \tag{211}$$

$$E[X^2] = \int_0^\infty x^2f(x)\,dx = \int_0^\infty x^2\frac{1}{2}e^{-x/2}\,dx \tag{212}$$

$$= \int_0^\infty 4u^2e^{-u}\,du \tag{213}$$

$$= 4\int_0^\infty u^2e^{-u}\,du \tag{214}$$

$$= 8 \tag{215}$$

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = 8 - 4 = 4 \tag{216}$$

$$\mu = 2, \qquad \sigma^2 = 4 \tag{217}$$

$$M_X(t) = \int_0^\infty e^{tx} \frac{1}{2} e^{-x/2}\, dx \tag{218}$$

$$= \int_0^\infty \frac{1}{2} e^{x(t-1/2)}\, dx \tag{219}$$

$$= \int_0^\infty \frac{1}{2} e^{-x(1/2-t)}\, dx \tag{220}$$

$$= \frac{1}{2} \int_0^\infty e^{-x(1/2-t)}\, dx \tag{221}$$

$$= \frac{1}{2} \cdot \frac{1}{(1/2-t)} \tag{222}$$

$$M_X(t) = \frac{1}{1-2t} \tag{223}$$

$$t < \frac{1}{2} \tag{224}$$

$$f_X(x) = x^{n-1} e^{-x/\lambda} \tag{225}$$

$$f_X(x) = x^{n-1} e^{-x/\lambda} \tag{226}$$

$$X \sim \text{Gamma}(n, \lambda) \tag{227}$$

$$\mathbb{E}[X] = n\lambda, \qquad \text{Var}(X) = n\lambda^2 \tag{228}$$

$$M_X(t) = \frac{1}{(1-\lambda t)^n} \tag{229}$$

Supongamos ahora

$$f_X(x) = c\,xe^{-x^2}, \qquad x \geq 0 \tag{230}$$

Averiguar c.

$$\int_0^\infty c\,xe^{-x^2}\,dx = 1 \tag{231}$$

Sea $u = x^2$, entonces $du = 2x\,dx$.

$$\int_0^\infty xe^{-x^2}\,dx = \frac{1}{2}\int_0^\infty e^{-u}\,du = \frac{1}{2} \tag{232}$$

Por lo tanto,

$$c = 2 \tag{233}$$

Luego,

$$f_X(x) = 2xe^{-x^2}, \qquad x \geq 0 \tag{234}$$

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2} \tag{235}$$

$$\int_{-\infty}^\infty e^{-x^2}\,dx = \sqrt{\pi} \tag{236}$$

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \frac{\pi}{4} \tag{237}$$

Pasando a coordenadas polares:

$$x = r\cos\theta, \qquad y = r\sin\theta \tag{238}$$

$$\int_0^{2\pi} \int_0^\infty e^{-r^2} r\,dr\,d\theta \tag{239}$$

$$= 2\pi \int_0^\infty re^{-r^2}\,dr \tag{240}$$

$$= 2\pi \cdot \frac{1}{2} = \pi \tag{241}$$

Def. Sea X una v.a. con función de distribución $F_X(x)$. Se define la función de probabilidad acumulada para X como

$$F_X(x) = \mathbb{P}(X \leq x). \tag{242}$$

Definición:

$$F_X(a, b) = \mathbb{P}(a < X \leq b). \tag{243}$$

Las funciones acumuladas $F_X(x)$ satisfacen las siguientes propiedades:

$$F_X(b) = \mathbb{P}(X \leq b), \qquad F_X(a) = \mathbb{P}(X \leq a). \tag{244}$$

Entonces

$$F_X(a, b) = F_X(b) - F_X(a). \tag{245}$$

Caso discreto
Def. Sean (X_1, X_2) v.a. discretas. Se define la función de distribución conjunta como

$$F_{X_1,X_2}(x, y) = \mathbb{P}(X_1 \leq x,\ X_2 \leq y). \tag{246}$$

Def. La función de densidad de probabilidad conjunta de X_1, X_2 se denota por $f_{X_1,X_2}(x, y)$.
La función de densidad marginal de X_1 es

$$f_{X_1}(x) = \sum_y f_{X_1,X_2}(x, y). \tag{247}$$

La función de densidad marginal de X_2 es

$$f_{X_2}(y) = \sum_x f_{X_1,X_2}(x, y). \tag{248}$$

Def. La función de densidad condicional de X_1 dado $X_2 = y$ es

$$f_{X_1|X_2}(x|y) = \frac{f_{X_1,X_2}(x, y)}{f_{X_2}(y)}, \qquad f_{X_2}(y) > 0. \tag{249}$$

Análogamente,

$$f_{X_2|X_1}(y|x) = \frac{f_{X_1,X_2}(x, y)}{f_{X_1}(x)}, \qquad f_{X_1}(x) > 0. \tag{250}$$

Def. Decimos que X, Y son continuas si existen funciones $f_{X,Y}(x, y)$ tales que

$$\forall A \subset \mathbb{R}^2, \qquad \mathbb{P}\{(X, Y) \in A\} = \iint_A f_{X,Y}(x, y)\, dx\, dy. \tag{251}$$

$f_{X,Y}(x, y)$ es función de densidad de probabilidad conjunta de X y Y.
La densidad de X se obtiene observando

$$\mathbb{P}\{X \in \Delta\} = \int_{-\infty}^{\infty} \left(\int_\Delta f_{X,Y}(x, y)\, dx \right) dy = \int_\Delta f_X(x)\, dx. \tag{252}$$

De donde

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy. \tag{253}$$

Análogamente,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx. \tag{254}$$

Def. La densidad condicional de X dado $Y = y$ es

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}, \qquad f_Y(y) > 0. \tag{255}$$

La densidad condicional de Y dado $X = x$ es

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}, \qquad f_X(x) > 0. \tag{256}$$

Nota: La variable $X_1, \ldots, X_k$ se dice k-dimensional si

$$(x_1, \ldots, x_k) \in \mathbb{R}^k. \tag{257}$$

Si existe una función de densidad conjunta $f_{X_1,\ldots,X_k}(x_1, \ldots, x_k)$, entonces

$$\mathbb{P}\{(X_1, \ldots, X_k) \in A\} = \int \cdots \int_A f_{X_1,\ldots,X_k}(x_1, \ldots, x_k)\,dx_1 \cdots dx_k. \tag{258}$$

Def. La variable aleatoria $(X_1, \ldots, X_k)$ es absolutamente continua si existe $f_{X_1,\ldots,X_k}(x_1, \ldots, x_k)$ tal que

$$\mathbb{P}\{(X_1, \ldots, X_k) \in A\} = \int \cdots \int_A f_{X_1,\ldots,X_k}(x_1, \ldots, x_k)\,dx_1 \cdots dx_k, \quad \forall A. \tag{259}$$

Nota: Sean (X_1, X_2) v.a. continuas en $\mathbb{R}^2$, sea $f_{X_1,X_2}(x_1, x_2)$ su densidad conjunta.

$$\mathbb{P}\{(X_1, X_2) \in A\} = \iint_A f_{X_1,X_2}(x_1, x_2)\,dx_1\,dx_2. \tag{260}$$

Sea

$$R = \{(x_1, x_2) : a_1 \leq x_1 \leq b_1,\ a_2 \leq x_2 \leq b_2\}. \tag{261}$$

Entonces

$$\mathbb{P}\{a_1 \leq X_1 \leq b_1,\ a_2 \leq X_2 \leq b_2\} = \int_{a_2}^{b_2} \int_{a_1}^{b_1} f_{X_1,X_2}(x_1, x_2)\,dx_1\,dx_2. \tag{262}$$

Ej. Considere

$$f(x,y) = k(x+y), \qquad 0 < x < 1,\ 0 < y < 1, \tag{263}$$

y $f(x, y) = 0$ en otro caso.

Primero,

$$1 = \int_0^1 \int_0^1 k(x + y)\, dx\, dy. \tag{264}$$

Luego,

$$\int_0^1 \int_0^1 (x + y)\, dx\, dy = \int_0^1 \left(\int_0^1 x\, dx + \int_0^1 y\, dx \right) dy. \tag{265}$$

Se obtiene

$$k = \frac{1}{\frac{1}{2}} = 2. \tag{266}$$

Por lo tanto,

$$f(x, y) = 2(x + y). \tag{267}$$

Calcular:

$$\mathbb{P}\{X < \tfrac{1}{2}\} = \int_0^{1/2} \int_0^1 2(x + y)\, dy\, dx. \tag{268}$$

Además,

$$\int_0^{1/2} \int_0^1 2(x + y)\, dy\, dx = \frac{3}{8}. \tag{269}$$

$$\text{Vale } y = x + y. \tag{270}$$

Teo: Sean X, Y v.a.c.c. y $f_{X,Y}(x, y)$ su densidad conjunta. Se procede a obtener $f_X(x)$ y $f_Y(y)$.

Def:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy, \qquad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx. \tag{271}$$

Def: Sean X, Y v.a.c.c. con densidad conjunta $f_{X,Y}(x, y)$. Entonces, para todo conjunto medible $A \subset \mathbb{R}$,

$$\mathbb{P}\{X \in A\} = \int_A \left(\int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy \right) dx. \tag{272}$$

Análogamente,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx. \tag{273}$$

Teo: Sean $X_1, X_2, \ldots, X_n$ v.a. independientes. Entonces la densidad conjunta es

$$f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = \prod_{i=1}^{n} f_{X_i}(x_i), \qquad x_i \geq 0. \tag{274}$$

Ej: Sea

$$f_{X,Y}(x, y) = 2(x + y), \qquad 0 < x < 1,\ 0 < y < 1. \tag{275}$$

Entonces

$$f_X(x) = \int_0^1 2(x + y)\, dy, \qquad f_Y(y) = \int_0^1 2(x + y)\, dx. \tag{276}$$

Además,

$$\int_0^1 2(x + y)\, dy = 2x + 1. \tag{277}$$

$$\int_0^1 2(x + y)\, dx = 2y + 1. \tag{278}$$

Obs:

$$\int_0^1 f_X(x)\, dx = 1, \qquad \int_0^1 f_Y(y)\, dy = 1. \tag{279}$$

$$\int_0^\infty \left(\int_0^\infty (x + y)\, dy \right) dx = \int_0^\infty \left[x + \frac{y^2}{2} \right]_0^\infty dx \tag{280}$$

$$f_X(x) = \int_0^\infty (x + y)\, dy \tag{281}$$

$$= \int_0^\infty x\, dy + \int_0^\infty y\, dy \tag{282}$$

$$= x \int_0^\infty dy + \int_0^\infty y\, dy \tag{283}$$

$$f_Y(y) = \int_0^\infty (x + y)\, dx \tag{284}$$

$$= \int_0^\infty x\, dx + y \int_0^\infty dx \tag{285}$$

$$\Rightarrow f_{X,Y}(x,y) = f_X(x)f_Y(y) \tag{286}$$

Ej: Una fábrica produce cuerdas que se venden y que se miden en metros. Sean X y Y las longitudes respectivas.

$$f_{X,Y}(x,y) = \frac{2}{3}(2x+3y), \quad 0 < x < 2,\ 0 < y < 1 \tag{287}$$

a)

$$f_X(x) = \int_0^1 \frac{2}{3}(2x+3y)\,dy \tag{288}$$

$$= \frac{2}{3}\left[2xy + \frac{3}{2}y^2\right]_0^1 \tag{289}$$

$$= \frac{2}{3}\left(2x + \frac{3}{2}\right) \tag{290}$$

$$f_Y(y) = \int_0^2 \frac{2}{3}(2x+3y)\,dx \tag{291}$$

$$= \frac{2}{3}\left[x^2 + 3xy\right]_0^2 \tag{292}$$

$$= \frac{2}{3}(4+6y) \tag{293}$$

Sol:
a)

$$\mathbb{P}(X+Y \leq 1) \tag{294}$$

Ej:
Se seleccionan al azar 2 refrescos. Probabilidad de que ambos sean rojos.
Sean $X = \#$ de refrescos verdes, $Y = \#$ de refrescos rojos.
a)

$$\mathbb{P}\big((X,Y) \in \Delta\big) =? \tag{295}$$

$$\Delta = \{(x, y) : x + y \leq 1\} \tag{296}$$

Sol:

$$\mathbb{R}^2 = \{(0,0), (0,1), (0,2), (1,0), (1,1), (2,0)\} \tag{297}$$

$$C_2^8 = \frac{8!}{2!6!} \tag{298}$$

$$f(0,1) = \frac{(2)(1)(3)}{(3)(7)} \tag{299}$$

$$= \frac{6}{21} \tag{300}$$

$$f(0,0) = \frac{(3)(2)}{(5)(7)} \tag{301}$$

$$= \frac{6}{35} \tag{302}$$

$$\frac{(3)(2)}{(5)(7)} = \frac{6}{35} \tag{303}$$

$$f_{X,Y}(x, y) \tag{304}$$

$$\begin{array}{c|ccc|c} X \backslash Y & 0 & 1 & 2 & I \\ \hline 0 & \frac{7}{28} & \frac{3}{14} & \frac{7}{28} & \frac{1}{2} \\ 1 & \frac{4}{28} & \frac{3}{14} & 0 & \frac{15}{28} \\ 2 & \frac{3}{28} & 0 & 0 & \frac{3}{28} \\ \hline & \frac{5}{14} & \frac{3}{7} & \frac{1}{28} & \end{array} \tag{305}$$

$$f_X(x) \tag{306}$$

$$\sum_{x,y} f_{X,Y}(x,y) \tag{307}$$

$$\sum_{x} f_X(x) \tag{308}$$

$$f_{X,Y}(x,y) \tag{309}$$

Ej: Definamos

$$f_{X,Y}(x,y) = \frac{2}{5}(2x+3y), \qquad \mathbf{1}_{\{0\leq x\leq 1\}}\mathbf{1}_{\{0\leq y\leq 1\}} \tag{310}$$

$$f_X(x) = \int_0^{\infty} f_{X,Y}(x,y)\,dy \tag{311}$$

$$f_X(x) = \int_0^{1} \frac{2}{5}(2x+3y)\,dy \tag{312}$$

$$f_Y(y) = \int_0^{\infty} f_{X,Y}(x,y)\,dx \tag{313}$$

$$f_Y(y) = \int_0^{1} \frac{2}{5}(2x+3y)\,dx \tag{314}$$

Def:
Sea X, Y v.a. con Δ dist. continua.
Dado que $X = x$ se tiene

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}, \qquad f_X(x) > 0 \tag{315}$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}, \qquad f_Y(y) > 0 \tag{316}$$

Notas:

$$\mathbb{P}(a < X < b \mid Y = y) = \int_a^b f_{X|Y}(x|y)\,dx \tag{317}$$

Ej: 3.18 (libro)

$$f_{X,Y}(x,y) = \frac{3}{7} \tag{318}$$

$$f_{X,Y}(x,1) = \frac{f_{X,Y}(x,1)}{f_Y(1)} \tag{319}$$

$$f_{X|Y}(x|1) = \frac{f_{X,Y}(x,1)}{f_Y(1)} = \frac{3}{7} \tag{320}$$

$$f_{X|Y}(x|1) = f_{X,Y}(x,1) + f_{X,Y}(2,1) \tag{321}$$

$$= \frac{3}{7} \tag{322}$$

Nota:

$$\begin{array}{c|ccc} x & 0 & 1 & 2 \\ \hline f_X(x) & \frac{1}{2} & \frac{1}{2} & 0 \end{array} \tag{323}$$

$$f_{X,Y}(x,y) = 10xy^2\,\mathbf{1}_{\{0\le x\le 1\}}\mathbf{1}_{\{0\le y\le 1\}} \tag{324}$$

$$f_X(x) = ? \tag{325}$$

$$f_Y(y) = ? \tag{326}$$

Ej: Sea X, Y v.a.

$$f_X(x) = \int_0^\infty f_{X,Y}(x,y)\,dy \tag{327}$$

$$f_X(x) = \int_0^1 10xy^2\,dy \tag{328}$$

$$f_Y(y) = \int_0^\infty f_{X,Y}(x,y)\,dx \tag{329}$$

$$f_Y(y) = \int_0^1 10xy^2\,dx \tag{330}$$

$$\Rightarrow f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \tag{331}$$

$$\mathbb{P}\left(\frac{1}{2} < X < 1\right) = \int_{1/2}^1 f_{X|Y}(x|y)\,dx \tag{332}$$

Observa:

$$f_{X|Y}(x|0) \tag{333}$$

$$f_{X|Y}(x|1) \tag{334}$$

$$f_{X|Y}(x|2) \tag{335}$$

$$f_Y(y) = \begin{cases} \frac{15}{28}, & y = 0 \\ \frac{3}{7}, & y = 1 \\ \frac{3}{28}, & y = 2 \end{cases} \tag{336}$$

$$f_X(0) = \frac{15}{28} \tag{337}$$

$$f_X(1) = \frac{3}{7} \tag{338}$$

$$f_X(2) = \frac{3}{28} \tag{339}$$

$$f_{X|Y}(y) = f_X(x|y) \tag{340}$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \tag{341}$$

$$f_X(x) = \int f_{X,Y}(x,y)\,dy \qquad f_Y(y) = \int f_{X,Y}(x,y)\,dx \tag{342}$$

$$\mathbb{P}\left\{\frac{1}{2} < X < 1 \mid Y = \frac{1}{2}\right\} = \int_{1/2}^{1} f_{X|Y}(x|y)\,dx \tag{343}$$

Ejercicios

$$3{,}40,\ 3{,}41,\ 3{,}42,\ 3{,}43,\ 3{,}44 \tag{344}$$

$$3{,}45,\ 3{,}49,\ 3{,}50,\ 3{,}53,\ 3{,}63 \tag{345}$$

$$3{,}68 \tag{346}$$

Marque verdaderas

$$f_{X,Y}(x,y) = \frac{1}{500} \quad 0 \le x \le y \le 200 \tag{347}$$

Ejercicio

$$\mathbb{P}\{1 \le X \le 2\} \tag{348}$$

$$f_X(x) = \int_0^{\infty} f_{X,Y}(x,y)\,dy \tag{349}$$

$$f_Y(y) = \int_0^{\infty} f_{X,Y}(x,y)\,dx \tag{350}$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \tag{351}$$

Ej:

$$f_X(x) = \int_0^{\infty} f_{X,Y}(x,y)\,dy \tag{352}$$

$$f_Y(y) = \int_0^{\infty} f_{X,Y}(x,y)\,dx \tag{353}$$

$$f_{X|Y}(x|y) \tag{354}$$

Continúa Nota 1

$$\begin{array}{c|ccc} & f_{Y|X}(y|x) & f_X(x) & f_{X,Y}(x,y) \\ \hline x=0 & \frac{3}{10} & \frac{3}{14} & \frac{3}{28} \\ x=1 & \frac{3}{5} & \frac{15}{28} & \frac{15}{28} \\ x=2 & \frac{1}{10} & 0 & 0 \end{array} \tag{355}$$

$$f_X(0) = \frac{3}{14} \tag{356}$$

$$f_X(1) = \frac{15}{28} \tag{357}$$

$$f_X(2) = \frac{3}{28} \tag{358}$$

$$f_{X|Y}(x|y) = \frac{2x}{3}, \qquad x \in (0,2) \tag{359}$$

$$f_X(x) = \int_0^2 f_{X|Y}(x|y)\,dy, \qquad f_Y(y) = \int_0^2 f_{X|Y}(x|y)\,dx \tag{360}$$

$$f_{X|Y}(x|y) \tag{361}$$

$$f_{Y|X}(y|x) \tag{362}$$

Teorema

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}, \qquad f_Y(y) \neq 0 \tag{363}$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}, \qquad f_X(x) \neq 0 \tag{364}$$

Ejemplos, Muestreos y Ejercicios

$$\text{Ejs: } 4{,}4,\ 4{,}6,\ 4{,}8,\ 4{,}9,\ 4{,}10 \tag{365}$$

Proposición

$$f_{Y|X}(y|x) \text{ es una función de densidad condicional} \tag{366}$$

$$\text{cuando } x \text{ es un número fijo} \tag{367}$$

$$\text{y para todo } x \text{ cumple con las propiedades de una densidad.} \tag{368}$$

Proposición

$$f_{Y|X}(\cdot|x) \text{ es positiva (no negativa)} \tag{369}$$

$$\int_{-\infty}^{\infty} f_{Y|X}(y|x)\, dy = 1 \tag{370}$$

$$\int_{-\infty}^{\infty} \frac{f_{X,Y}(x,y)}{f_X(x)}\, dy = 1 \tag{371}$$

$$\frac{1}{f_X(x)} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy = 1 \tag{372}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy \tag{373}$$

Nota: En $f_{Y|X}(\cdot|x)$, x es fija y permite considerarla como una distribución acumulada.
Def: Función de dist. acumulada condicional
Si X, Y son V.A.C.C., la dist. acumulada condicional de Y dado $X = x$ se define por

$$F_{Y|X}(y|x) = \int_{-\infty}^{y} f_{Y|X}(t|x)\, dt, \qquad x \in \mathbb{R},\ f_X(x) > 0. \tag{374}$$

Sup:

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}. \tag{375}$$

Ej:

$$f_{Y|X}(y|x) = \int_0^y \frac{2x+t}{x+\frac{1}{2}}\,dt, \qquad 0 \leq y \leq 1. \tag{376}$$

$$f_{Y|X}(y|x) = \frac{1}{x+\frac{1}{2}}\int_0^y (2x+t)\,dt = \frac{1}{x+\frac{1}{2}}\left(2xy+\frac{y^2}{2}\right). \tag{377}$$

Recordando:

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}. \tag{378}$$

$$f_{X,Y}(x,y) = f_{Y|X}(y|x)f_X(x). \tag{379}$$

Si X y Y son independientes, entonces

$$f_{X,Y}(x,y) = f_X(x)f_Y(y). \tag{380}$$

$$\Rightarrow f_{Y|X}(y|x) = f_Y(y). \tag{381}$$

$$f_{Y|X}(y|x)\,dy = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy. \tag{382}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx. \tag{383}$$

$$f_{X,Y}(x,y) = f_X(x)\,f_Y(y) \tag{384}$$

Def: Sean X, Y V.A.D.C. o V.A.C.C. con densidades $f_{X,Y}(x,y)$ y marginales $f_X(x)$, $f_Y(y)$.

X y Y son estocásticamente independientes si y solo si

$$f_{X,Y}(x,y) = f_X(x)\,f_Y(y). \tag{385}$$

Ej: Retomar Ej. 3 p. 20 y 3 p. 24 y verificar si X, Y son independientes.

Def: Sean $(X_1, X_2, \dots, X_k)$ V.A. discretas o continuas. $X_1, X_2, \dots, X_k$ son estocásticamente independientes si

$$f_{X_1,X_2,\dots,X_k}(x_1, x_2, \dots, x_k) = f_{X_1}(x_1)\, f_{X_2}(x_2) \cdots f_{X_k}(x_k). \tag{386}$$

Def: Sean $(X_1, X_2, \dots, X_k)$ V.A. discretas o continuas. $X_1, X_2, \dots, X_k$ son estocásticamente independientes si

$$f_{X_i|X_j}(x_i|x_j) = f_{X_i}(x_i), \qquad i \neq j. \tag{387}$$

Ej: Inferir

$$f_{X,Y}(x, y) = e^{-(2x)}\, e^{-(y)}, \qquad x > 0,\ y > 0. \tag{388}$$

Luego:

$$f_{X,Y}(x, y) = f_X(x)\, f_Y(y) \qquad \text{(VERIFICAR)} \tag{389}$$

$$f_{X,Y}(x, y) = f_X(x) f_Y(y) \tag{390}$$

Def: Sean X, Y VADC o VACC con $f_{X,Y}(x, y)$ y densidades marginales $f_X(x)$ y $f_Y(y)$. X, Y son estocásticamente independientes si y solo si

$$f_{X,Y}(x, y) = f_X(x) f_Y(y). \tag{391}$$

Ej: Probar si 3 R.V. X_1, X_2, X_3 son independientes. Verificar si

$$f_{X_1,X_2,X_3}(x_1, x_2, x_3) = f_{X_1}(x_1)\, f_{X_2}(x_2)\, f_{X_3}(x_3). \tag{392}$$

Def: Sean $(X_1, X_2, \dots, X_k)$ un vector aleatorio. Son estocásticamente independientes si

$$f_{X_1,\dots,X_k}(x_1, \dots, x_k) = \prod_{i=1}^{k} f_{X_i}(x_i). \tag{393}$$

Def: Sean $(X_1, X_2, \dots, X_k)$ V.A. discretas. Son estocásticamente independientes si

$$p_{X_1,\dots,X_k}(x_1, \dots, x_k) = \prod_{i=1}^{k} p_{X_i}(x_i). \tag{394}$$

Ej: Sea

$$f_{X,Y}(x, y) = e^{-(x+y)}. \tag{395}$$

Teo: Sean $X_1, X_2, \dots, X_k$ V.A. independientes y $g_1(\cdot), g_2(\cdot), \dots, g_k(\cdot)$ sean funciones tales que $Y_i = g_i(X_i)$, $i = 1, 2, \dots, k$. Entonces $Y_1, \dots, Y_k$ son independientes.

Esperanza

Def: Sean X, Y V.A. con densidad conjunta $f_{X,Y}(x, y)$. Se define

$$\mathbb{E}[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{X,Y}(x,y)\, dx\, dy. \tag{396}$$

Teo: Si $g(X,Y) = X$, entonces

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx. \tag{397}$$

Ej: Sea

$$f_{X,Y}(x,y) = e^{-(x+y)}, \quad x > 0,\ y > 0. \tag{398}$$

Calcular

$$\mathbb{E}[X]. \tag{399}$$

$$\mathbb{E}[X] = \int_0^{\infty}\int_0^{\infty} x\, e^{-(x+y)}\, dx\, dy = \int_0^{\infty} xe^{-x}\left(\int_0^{\infty} e^{-y}\, dy\right) dx = \int_0^{\infty} xe^{-x}\, dx. \tag{400}$$

$$\mathbb{E}[X] = 1. \tag{401}$$

Ej: Sea

$$g(X,Y) = X + Y. \tag{402}$$

$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]. \tag{403}$$

$$\mathbb{E}[X+Y] = \int_0^{\infty}\int_0^{\infty} (x+y)\, f_{X,Y}(x,y)\, dx\, dy = \int_0^{\infty}\int_0^{\infty} (x+y)e^{-(x+y)}\, dx\, dy = 2. \tag{404}$$

Prop:

$$\mathbb{E}[aX + bY] = a\,\mathbb{E}[X] + b\,\mathbb{E}[Y]. \tag{405}$$

Teo: Sean X, Y V.A. independientes. Sea $g(X,Y) = g(X)h(Y)$. Entonces

$$\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\,\mathbb{E}[h(Y)]. \tag{406}$$

Dem:

$$\mathbb{E}[g(X)h(Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)\, f_{X,Y}(x,y)\, dx\, dy \tag{407}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x)h(y)\, f_X(x)f_Y(y)\, dx\, dy \tag{408}$$

$$= \left(\int_{-\infty}^{\infty} g(x)f_X(x)\, dx \right) \left(\int_{-\infty}^{\infty} h(y)f_Y(y)\, dy \right) \tag{409}$$

$$= \mathbb{E}[g(X)]\, \mathbb{E}[h(Y)]. \tag{410}$$

Sup:

$$f_{X,Y}(x,y) = f_X(x)f_Y(y). \tag{411}$$

Ej:

$$\mathbb{E}[X+Y] = \tag{412}$$

$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]. \tag{413}$$

Def: (Covarianza)
Sean X, Y V.A. con varianzas finitas.

$$\text{Cov}(X,Y) = \mathbb{E}\big[(X-\mu_X)(Y-\mu_Y)\big], \tag{414}$$

donde $\mu_X = \mathbb{E}[X]$ y $\mu_Y = \mathbb{E}[Y]$.
Def: (Correlación)
El coeficiente de correlación de X y Y se define por

$$\rho_{X,Y} = \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y}, \tag{415}$$

donde $\sigma_X > 0$ y $\sigma_Y > 0$.
Nota:

$$\text{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]. \tag{416}$$

$$\mathbb{E}\big[(X-\mu_X)(Y-\mu_Y)\big] = \mathbb{E}[XY] - \mu_Y\mathbb{E}[X] - \mu_X\mathbb{E}[Y] + \mu_X\mu_Y \tag{417}$$

$$= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]. \tag{418}$$

Nota: Si X, Y son independientes, entonces

$$\text{Cov}(X,Y) = 0. \tag{419}$$

Ej: Funcional para X, Y.

$$f_{X,Y}(x,y) = (x+y)\,\mathbf{1}_{(0,1)}(x)\,\mathbf{1}_{(0,1)}(y). \tag{420}$$

Def: Sean $X_1, X_2, \ldots, X_k$ funciones aleatorias con

$$\mathbb{E}[X_i] = \mu_i, \qquad i = 1, \ldots, k. \tag{421}$$

Los momentos centrales están definidos por

$$\mathbb{E}\big[(X_1 - \mu_1)\cdots(X_k - \mu_k)\big]. \tag{422}$$

Def: (Función generadora de momentos)

$$M_{X_1,\ldots,X_k}(t_1,\ldots,t_k) = \mathbb{E}\left[\exp\left(\sum_{i=1}^{k} t_i X_i\right)\right]. \tag{423}$$

Nota: Las funciones generadoras satisfacen

$$\mathbb{E}\big[g(X,Y)\big] = \iint g(x,y)\, f_{X,Y}(x,y)\, dx\, dy. \tag{424}$$

Ej:
Sea $X = \mathbf{1}_A$, con $A \in \mathcal{F}$.

$$\mathbb{E}(X) = \mathbb{E}[\mathbf{1}_A] = P(A). \tag{425}$$

$$X^2 = X \;\Rightarrow\; \mathbb{E}(X^2) = \mathbb{E}(X). \tag{426}$$

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \big(\mathbb{E}(X)\big)^2 = P(A) - P(A)^2 = P(A)\big(1 - P(A)\big). \tag{427}$$

Prop: Sean $X_1, \ldots, X_n$ V.A. independientes.

a) $\mathrm{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$.

b) $\sigma_X^2 = \mathrm{Var}(X)$.

c) $\sigma_{X+Y}^2 = \sigma_X^2 + \sigma_Y^2$.

Prop:

$$\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{Var}(X_i). \tag{428}$$

$$\mathrm{Cov}(X_i, X_j) = 0, \qquad i \neq j. \tag{429}$$

Corolario:
Sean $X_1, X_2, \ldots, X_n$ V.A. independientes. Entonces

$$\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{Var}(X_i). \tag{430}$$

Def: Sean $X_1, X_2, \ldots, X_n$ V.A. independientes i.i.d. Entonces

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \tag{431}$$

es una V.A. y se llama *media muestral.*

Prop: Si $X_1, X_2, \ldots, X_n$ son V.A. i.i.d. con

$$\mathbb{E}(X) = \mu, \qquad \mathrm{Var}(X) = \sigma^2, \tag{432}$$

entonces:

a) $\mathbb{E}(\bar{X}) = \mu$.

b) $\mathrm{Var}(\bar{X}) = \dfrac{\sigma^2}{n}$.

c) $\mathrm{Cov}(X_i, \bar{X}) = \dfrac{\sigma^2}{n}$.

Def:

b)

$$\mathrm{Var}(\bar{X}) = \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \frac{\sigma^2}{n}. \tag{433}$$

c)

$$\mathrm{Cov}(X_i, \bar{X}) = \mathrm{Cov}\left(X_i, \frac{1}{n}\sum_{j=1}^{n} X_j\right) = \frac{1}{n}\mathrm{Cov}(X_i, X_i) = \frac{\sigma^2}{n}. \tag{434}$$

Corolario:
Sean $X_1, X_2, \ldots, X_n$ V.A. independientes. Entonces

$$\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{Var}(X_i). \tag{435}$$

Def: Sean $X_1, X_2, \ldots, X_n$ V.A. i.i.d. Entonces

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \tag{436}$$

es una V.A. y se llama *media muestral.*

Prop: Si $X_1, X_2, \ldots, X_n$ son V.A. i.i.d. con

$$\mu = \mathbb{E}(X), \qquad \sigma^2 = \mathrm{Var}(X), \tag{437}$$

entonces:

a) $\mathbb{E}(\bar{X}) = \mu$.

b) $\mathrm{Var}(\bar{X}) = \dfrac{\sigma^2}{n}$.

c) $\mathrm{Cov}(X_i, \bar{X}) = \dfrac{\sigma^2}{n}$.

Def:

b)

$$\mathrm{Var}(\bar{X}) = \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \frac{\sigma^2}{n}. \tag{438}$$

c)

$$\mathrm{Cov}(X_i, \bar{X}) = \mathrm{Cov}\left(X_i, \frac{1}{n}\sum_{j=1}^{n} X_j\right) = \frac{1}{n}\mathrm{Cov}(X_i, X_i) = \frac{\sigma^2}{n}. \tag{439}$$

$$\iint h(y)\, f_Y(y)\, f_X(x)\, dx\, dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(y)\, f_Y(y)\, f_X(x)\, dx\, dy \tag{440}$$

$$= \int_{-\infty}^{\infty} h(y)\, f_Y(y) \left(\int_{-\infty}^{\infty} f_X(x)\, dx\right) dy \tag{441}$$

$$= \int_{-\infty}^{\infty} h(y)\, f_Y(y)\, dy \tag{442}$$

Sea Z = X + Y.

$$X, Y \text{ V.A. indep.} \quad \Rightarrow \quad f_{X,Y}(x,y) = f_X(x) f_Y(y) \tag{443}$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z-x)\, dx = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx \tag{444}$$

Propiedades

$$\text{Traslación} \quad u \text{ unid.} \tag{445}$$

$$\text{Proporcionalidad} \quad u/n \text{ unid.} \tag{446}$$

$$\text{Prácticas} \quad 2q/n \text{ unid.} \tag{447}$$

$$\frac{1}{n}\,\mathrm{Cov}\left(\sum_{i=1}^{n} X_i, X_j\right) = \frac{1}{n}\,\mathrm{Cov}(X_j, X_j) = \frac{\sigma^2}{n} \tag{448}$$

$$\frac{1}{n}\,\mathrm{Cov}\left(\sum_{i=1}^{n} X_i, X_i\right) = \frac{1}{n}\,\sigma^2 \tag{449}$$

$$\frac{1}{n}\,\mathrm{Cov}\left(\sum_{i=1}^{n} X_i, X_i\right) - \frac{\sigma^2}{n} = 0 \tag{450}$$

$$\text{para todo valor válido.} \tag{451}$$

$$\int_{-\infty}^{\infty} f_X(a-y)\, f_Y(y)\, dy \tag{452}$$

n.e. la f.d.p. de la convolución

$$X + Y \tag{453}$$

Definimos

$$f_{X+Y}(a) = \int_{-\infty}^{\infty} f_X(a-y)\, f_Y(y)\, dy \tag{454}$$

$$= \int_{-\infty}^{\infty} f_X(a-y)\, f_Y(y)\, dy \tag{455}$$

Ej:

$$X, Y \sim U(0,1) \tag{456}$$

$$f_{X+Y}(a) = \int_{-\infty}^{\infty} f_X(a-y)\, f_Y(y)\, dy \tag{457}$$

$$\text{si } 0 < a < 1 \tag{458}$$

$$f_{X+Y}(a) = \int_0^a 1\,dy \tag{459}$$

$$= a \tag{460}$$

$$\text{si } 1 < a < 2 \tag{461}$$

$$f_{X+Y}(a) = \int_{a-1}^1 1\,dy \tag{462}$$

$$= 2 - a \tag{463}$$

$$f_{X+Y}(a) = \begin{cases} a, & 0 < a < 1, \\ begineqnarray4pt]2 - a, & 1 < a < 2. \end{cases} \tag{464}$$

$$a \in (0,1) \qquad a \in (1,2) \tag{465}$$

$$\int_{-\infty}^{\infty} f_X(a-y)\, f_Y(y)\, dy \tag{466}$$

n.e. la f.d.p. de la convolución

$$X + Y \tag{467}$$

Definimos

$$f_{X+Y}(a) = \int_{-\infty}^{\infty} f_X(a-y)\, f_Y(y)\, dy \tag{468}$$

$$= \int_{-\infty}^{\infty} f_X(a-y)\, f_Y(y)\, dy \tag{469}$$

Ej:

$$X, Y \sim U(0,1) \tag{470}$$

$$f_{X+Y}(a) = \int_{-\infty}^{\infty} f_X(a-y)\, f_Y(y)\, dy \tag{471}$$

$$\text{si } 0 < a < 1 \tag{472}$$

$$f_{X+Y}(a) = \int_0^a 1\, dy \tag{473}$$

$$= a \tag{474}$$

$$\text{si } 1 < a < 2 \tag{475}$$

$$f_{X+Y}(a) = \int_{a-1}^{1} 1\, dy \tag{476}$$

$$= 2 - a \tag{477}$$

$$f_{X+Y}(a) = \begin{cases} a, & 0 < a < 1, \\ begineqnarray4pt]2 - a, & 1 < a < 2. \end{cases} \tag{478}$$

$$a \in (0,1) \qquad a \in (1,2) \tag{479}$$

Ej 2: X,Y v.a. ind

$$f_{X+Y}(n) \tag{480}$$

$$0 \le k \le n \tag{481}$$

$$\text{Caso } X+Y=n \iff X=k,\ Y=n-k \tag{482}$$

$$P(X+Y=n)=\sum_{k=0}^{n} P(X=k,\ Y=n-k) \tag{483}$$

$$=\sum_{k=0}^{n} P(X=k)\,P(Y=n-k) \tag{484}$$

$$=\sum_{k=0}^{n} \frac{\lambda_1^k}{k!}e^{-\lambda_1}\frac{\lambda_2^{n-k}}{(n-k)!}e^{-\lambda_2} \tag{485}$$

$$=e^{-(\lambda_1+\lambda_2)}\sum_{k=0}^{n}\frac{\lambda_1^k\lambda_2^{n-k}}{k!(n-k)!} \tag{486}$$

$$=e^{-(\lambda_1+\lambda_2)}\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}\lambda_1^k\lambda_2^{n-k} \tag{487}$$

$$=e^{-(\lambda_1+\lambda_2)}\frac{(\lambda_1+\lambda_2)^n}{n!} \tag{488}$$

$$X+Y\sim P(\lambda_1+\lambda_2) \tag{489}$$

$$\text{V.A.C.C.} \tag{490}$$

X_1, X_2 V.A.C.C. con densidad*to* $f_{X_1,X_2}(x_1,x_2)$(491)
Sup. $\mathrm{Y}_1=g_1(x_1,x_2),\quad Y_2=g_2(x_1,x_2)$ T–Q

$$y_1=y_1(x_1,x_2),\quad y_2=y_2(x_1,x_2) \tag{492}$$

A) se pueden resolver las ecuaciones en forma única

$$\text{para } x_1 \text{ y } x_2 \tag{493}$$

$$x_1=h_1(y_1,y_2),\quad x_2=h_2(y_1,y_2) \tag{494}$$

B) las funciones y_1, y_2 tienen derivadas*to* parciales $\forall (x_1, x_2)$(495)

$$J(x_1, x_2) = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ begineqnarray1ex]\frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{vmatrix} \neq 0 \tag{496}$$

Entonces Y_1 y Y_2 son V.A.C.C.

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(x_1, x_2)\,|J(x_1, x_2)| \tag{497}$$

$$x_1 = h_1(y_1, y_2), \quad x_2 = h_2(y_1, y_2) \tag{498}$$

Ej:
Sean X i.i.d. V.A. con

$$X \sim \text{Ga}(\alpha, \lambda), \qquad Y \sim \text{Ga}(\beta, \lambda), \qquad U = X + Y, \qquad V = \frac{X}{X + Y}. \tag{499}$$

$$f_{X,Y}(x, y) = \frac{\lambda^\alpha e^{-\lambda x} x^{\alpha-1}}{\Gamma(\alpha)} \frac{\lambda^\beta e^{-\lambda y} y^{\beta-1}}{\Gamma(\beta)} \tag{500}$$

Sea

$$y_1 = g_1(x, y) = x + y, \qquad y_2 = g_2(x, y) = \frac{x}{x + y}. \tag{501}$$

Entonces

$$\frac{\partial y_1}{\partial x} = 1, \qquad \frac{\partial y_1}{\partial y} = 1, \tag{502}$$

$$\frac{\partial y_2}{\partial x} = \frac{y}{(x + y)^2}, \qquad \frac{\partial y_2}{\partial y} = -\frac{x}{(x + y)^2}. \tag{503}$$

$$U = x + y, \qquad V = \frac{x}{x + y} \Rightarrow x = uv, \qquad y = u(1 - v). \tag{504}$$

$$x = uv \tag{505}$$

$$v = \frac{uv}{uv + u(1 - v)} = uv \tag{506}$$

$$uv = uv \ \Rightarrow \ v = uv + u(1-v) \tag{507}$$

$$uv = u \tag{508}$$

$$v = uv \tag{509}$$

$$\Rightarrow \ v = uv + u(1-v) \tag{510}$$

$$\Rightarrow \ y = u(1-v). \tag{511}$$

$$f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(x_1,x_2)\ |J(x_1,x_2)|^{-1} \tag{512}$$

Donde

$$|J(x_1,x_2)|^{-1} = \begin{vmatrix} \dfrac{\frac{1}{x}}{(x+y)^2} & \dfrac{\frac{1}{y}}{(x+y)^2} \end{vmatrix}^{-1} = \frac{xy}{(x+y)^2} = \frac{1}{x+y} \tag{513}$$

$$= \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)} e^{-\lambda(x+y)} x^{\alpha-1} y^{\beta-1}(x+y) \tag{514}$$

$$= \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)} e^{-\lambda u}(uv)^{\alpha-1}\big(u(1-v)\big)^{\beta-1} u \tag{515}$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\big(\lambda e^{-\lambda u}\big)\big(u^{\alpha-1}u^{\beta}\big)\big(v^{\alpha-1}(1-v)^{\beta-1}\big) \tag{516}$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\big(\lambda e^{-\lambda u}\big)(\lambda u)^{\alpha-1}(\lambda u)^{\beta} v^{\alpha-1}(1-v)^{\beta-1} \tag{517}$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\lambda e^{-\lambda u}(\lambda u)^{\alpha+\beta-1} v^{\alpha-1}(1-v)^{\beta-1} \tag{518}$$

$$= \frac{\lambda e^{-\lambda u}(\lambda u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} v^{\alpha-1}(1-v)^{\beta-1} \tag{519}$$

$$U \sim \mathrm{Ga}(\alpha+\beta, \lambda), \qquad V \sim \mathrm{Be}(\alpha, \beta). \tag{520}$$

$$\text{Sean } X, Y \text{ v.a. c.c. con densidad} \tag{521}$$

$$Z = XY \tag{522}$$

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{|y|} f_{X,Y}\left(\frac{z}{y},\, y\right)\, dy \tag{523}$$

$$\text{Sean } X, Y \text{ v.a. c.c. con densidad} \tag{524}$$

$$Z = X + Y, \qquad V = X - Y \tag{525}$$

$$x = \frac{z+v}{2}, \qquad y = \frac{z-v}{2} \tag{526}$$

$$f_{Z,V}(z,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy \tag{527}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}\left(\frac{z+v}{2},\, \frac{z-v}{2}\right) |J|\, dz\, dv \tag{528}$$

$$f_X(w) = \int_{-\infty}^{\infty} f_{X,Y}(x,\, x-w)\, dx \tag{529}$$

$$= \int_{-\infty}^{\infty} f_{X,Y}(x,\, x-v)\, dx \tag{530}$$

$$\text{Sean } X, Y \text{ variables aleatorias i.i.d.} \tag{531}$$

$$\mathrm{Var}(X - Y) \tag{532}$$

$$\text{Entonces:} \tag{533}$$

$$\mathbb{E}(X - Y) = \mathbb{E}(X) - \mathbb{E}(Y) \tag{534}$$

$$\text{y} \tag{535}$$

$$\mathrm{Var}(X - Y) = \mathbb{E}\big[(X - Y)^2\big] - [\mathbb{E}(X - Y)]^2 \tag{536}$$

$$= \mathbb{E}(X^2 - 2XY + Y^2) - [\mathbb{E}(X - Y)]^2 \tag{537}$$

$$= \mathbb{E}(X^2) - 2\mathbb{E}(XY) + \mathbb{E}(Y^2) - [\mathbb{E}(X - Y)]^2 \tag{538}$$

—

$$\text{Def:} \tag{539}$$

$$\mathbb{E}(X^2) = \mathrm{Var}(X) + [\mathbb{E}(X)]^2 \tag{540}$$

$$\mathbb{E}(Y^2) = \mathrm{Var}(Y) + [\mathbb{E}(Y)]^2 \tag{541}$$

$$\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y) \quad \text{(i.i.d.)} \tag{542}$$

—

$$\text{Así:} \tag{543}$$

$$\text{Var}(X-Y) = \text{Var}(X) + \text{Var}(Y) + [\mathbb{E}(X)]^2 + [\mathbb{E}(Y)]^2 - 2\mathbb{E}(X)\mathbb{E}(Y) - (\mathbb{E}(X) - \mathbb{E}(Y))^2 \tag{544}$$

$$= \text{Var}(X) + \text{Var}(Y) \tag{545}$$

—

$$\boxed{\text{Var}(X-Y) = \text{Var}(X) + \text{Var}(Y)} \tag{546}$$

$$\Rightarrow \ \mathbb{E}[XY] = \mu_X\mu_Y + \mu_Y\mathbb{E}[X-\mu_X] + \mu_X\mathbb{E}[Y-\mu_Y] + \mathbb{E}[(X-\mu_X)(Y-\mu_Y)] \tag{547}$$

$$\mathbb{E}[(X-\mu_X)(Y-\mu_Y)] = \mu_X\mu_Y + \text{Cov}(X,Y) \tag{548}$$

$$\mathbb{E}\left[(XY)^2\right] = \sim\sim\sim \tag{549}$$

—

$$\text{Cor: Si } X, Y \text{ son indep.} \tag{550}$$

$$\mathbb{E}[XY] = \mu_X\mu_Y \tag{551}$$

$$\text{y} \tag{552}$$

$$\text{Var}(XY) = \mu_Y^2 \, \text{Var}(X) + \mu_X^2 \, \text{Var}(Y) + \text{Var}(X)\text{Var}(Y) \tag{553}$$

—

$$\text{Tarea: } \mathbb{E}\left[\frac{X}{Y}\right] \tag{554}$$

—

$$\text{Ej: Sea } X \sim \mathcal{N}(0,1). \quad \text{Sea } Y = X^2 = g(X) \tag{555}$$

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(X^2 \leq y) = \mathbb{P}(-\sqrt{y} \leq X \leq \sqrt{y}) \tag{556}$$

$$= \Phi(\sqrt{y}) - \Phi(-\sqrt{y}) = 2\int_0^{\sqrt{y}} \phi(u)\, du \tag{557}$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\sqrt{y}} e^{-u^2/2}\, du \tag{558}$$

—

$$u = \sqrt{z} \quad \iff \quad u^2 = z \quad \Rightarrow \quad 2u\, du = dz \tag{559}$$

$$du = \frac{dz}{2\sqrt{z}} \tag{560}$$

$$0 \leq u^2 \leq y \quad \Rightarrow \quad 0 \leq z \leq y \tag{561}$$

$$\frac{2}{\sqrt{2\pi}} \int_0^y \frac{e^{-z/2}}{2\sqrt{z}}\, dz = \frac{1}{\sqrt{\pi}} \int_0^y \frac{e^{-z/2}}{\sqrt{2z}}\, dz \tag{562}$$

—

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi} \tag{563}$$

$$X \sim \Gamma(\alpha, \beta) \quad \Rightarrow \quad f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} \tag{564}$$

$$\alpha = \tfrac{1}{2}, \quad \beta = 2 \tag{565}$$

$$\Rightarrow \int_0^y \frac{e^{-z/2}}{\Gamma(\frac{1}{2})\sqrt{2z}}\, dz \tag{566}$$

$$\text{Sea } X \sim \mathcal{N}(0,1), \quad Y = X^2 \tag{567}$$

—

$$\mathbb{E}(Y) = \mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 e^{-x^2/2}\, dx \tag{568}$$

$$= \int_{-\infty}^{\infty} e^{-x^2/2} x^2\, dx \tag{569}$$

$$= \left[-xe^{-x^2/2}\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-x^2/2}\, dx \tag{570}$$

$$= 0 + \sqrt{2\pi} \tag{571}$$

$$\Rightarrow \mathbb{E}(Y) = 1 \tag{572}$$

—

$$\text{Por } X \sim \mathcal{N}(0,1) \quad \Rightarrow \quad Y = X^2 \sim \chi^2_{(1)} \tag{573}$$

—

$$Y_1 = g_1(X_1, X_2) \quad , \quad Y_2 = X_1 + X_2 \tag{574}$$

$$\text{(Transformación uno–a–uno)} \tag{575}$$

—

$$\mathbb{E}\left[e^{t_1 Y_1 + t_2 Y_2}\right] = \mathbb{E}\left[e^{t_1(X_1^2 + X_2^2) + t_2(X_1 + X_2)}\right] \tag{576}$$

$$= \mathbb{E}\left[e^{t_1 X_1^2 + t_2 X_1}\right] \mathbb{E}\left[e^{t_1 X_2^2 + t_2 X_2}\right] \tag{577}$$

—

$$M_{Y_1,Y_2}(t_1,t_2) = \mathbb{E}\left[e^{t_1(X_1^2+X_2^2)+t_2(X_1+X_2)}\right] \tag{578}$$

$$= \mathbb{E}\left[e^{t_1X_1^2+t_2X_1}\right]\mathbb{E}\left[e^{t_1X_2^2+t_2X_2}\right] \tag{579}$$

—

$$M(t_1,t_2) = \exp\left(\frac{t_2^2}{2(1-2t_1)}\right)\cdot\frac{1}{\sqrt{1-2t_1}} \tag{580}$$

—

$$= \exp\left(\frac{t_2^2}{1-2t_1}\right)\cdot\frac{1}{1-2t_1} \tag{581}$$

—

$$M(t_1,t_2) = \frac{1}{1-2t_1}\exp\left(\frac{t_2^2}{1-2t_1}\right) \tag{582}$$

—

$$M(t) = \mathbb{E}(e^{tY}) = e^{2t^2/2} = e^{t^2} \tag{583}$$

$$\Rightarrow \quad Y \sim \mathcal{N}(0,2) \tag{584}$$

$$\text{Sea } X_1, X_2 \sim \mathcal{N}(0,1) \quad \text{independientes} \tag{585}$$

$$Z = \frac{X_2 - X_1}{\sqrt{2}} \tag{586}$$

—

$$\text{Sea } Y = X_2 - X_1 \tag{587}$$

$$\Rightarrow \quad Y \sim \mathcal{N}(0,2) \tag{588}$$

—

$$\text{Por binomiales:} \tag{589}$$

$$M_Y(t) = M_{X_2 - X_1}(t) = M_{X_2}(t)\, M_{-X_1}(t) \tag{590}$$

$$= \left(e^{t^2/2}\right)\left(e^{t^2/2}\right) = e^{t^2} \tag{591}$$

—

$$\Rightarrow \quad Y \sim \mathcal{N}(0,2) \tag{592}$$

—

$$\text{Sea } Z = \frac{X_2 - X_1}{\sqrt{2}} \tag{593}$$

$$M_Z(t) = \mathbb{E}\big(e^{tZ}\big) = \mathbb{E}\Big(e^{t(X_2 - X_1)/\sqrt{2}}\Big) \tag{594}$$

$$= M_Y\left(\frac{t}{\sqrt{2}}\right) = e^{(t/\sqrt{2})^2} = e^{t^2/2} \tag{595}$$

—

$$\Rightarrow \quad Z \sim \mathcal{N}(0,1) \tag{596}$$

—

$$\text{Suma de normales} \tag{597}$$

$$M_{X+Y}(t) = M_X(t)\, M_Y(t) = e^{\mu_X t + \sigma_X^2 t^2/2}\, e^{\mu_Y t + \sigma_Y^2 t^2/2} \tag{598}$$

$$= e^{(\mu_X + \mu_Y)t + (\sigma_X^2 + \sigma_Y^2)t^2/2} \tag{599}$$

—

$$\Rightarrow \quad X + Y \sim \mathcal{N}(\mu_X + \mu_Y,\ \sigma_X^2 + \sigma_Y^2) \tag{600}$$

Resultado:
Sean $Z_1, Z_2, \ldots, Z_n \sim \mathcal{N}(0,1)$ independientes.
Sean $a_{ij} \in \mathbb{R}$,

$$i = 1, 2, \ldots, m, \qquad j = 1, 2, \ldots, n \tag{601}$$

Sea $\mu_i \in \mathbb{R}$, $i = 1, 2, \ldots, m$.
—
Definición:

$$X_1 = \sum_{j=1}^{n} a_{1j} Z_j + \mu_1 \tag{602}$$

$$X_2 = \sum_{j=1}^{n} a_{2j} Z_j + \mu_2 \tag{603}$$

$$\vdots \tag{604}$$

$$X_m = \sum_{j=1}^{n} a_{mj} Z_j + \mu_m \tag{605}$$

En general,

$$X_k = \sum_{j=1}^{n} a_{kj} Z_j + \mu_k, \qquad k = 1, 2, \ldots, m \tag{606}$$

—
Entonces:

$$X_k \sim \mathcal{N}\left(\mu_k,\ \sum_{j=1}^{n} a_{kj}^2\right) \tag{607}$$

—
Luego:

$$M_{X_1+X_2+\cdots+X_m}(t_1, t_2, \ldots, t_m) = \mathbb{E}\left[e^{\,t_1X_1+t_2X_2+\cdots+t_mX_m}\right] \tag{608}$$

—

$$M_{\sum_{j=1}^{m} X_j}(t_1, t_2, \ldots, t_m) = \mathbb{E}\left[e^{\sum_{j=1}^{m} t_j X_j}\right] \tag{609}$$

—

A saber:

$$\mathbb{E}\left[\sum_{j=1}^{m} t_j X_j\right] = \sum_{j=1}^{m} \mathbb{E}[t_j X_j] = \sum_{j=1}^{m} t_j \mathbb{E}[X_j] \tag{610}$$

$$= \sum_{j=1}^{m} t_j \mu_j \tag{611}$$

—

$$\operatorname{Var}\left(\sum_{j=1}^{m} t_j X_j\right) = \operatorname{Cov}\left(\sum_{j=1}^{m} t_j X_j,\ \sum_{j=1}^{m} t_j X_j\right) \tag{612}$$

$$= \sum_{j=1}^{m} \sum_{i=1}^{m} \operatorname{Cov}(X_i, X_j)\, t_i t_j \tag{613}$$

Ejemplo:
Sean X_1, X_2 v.a. independientes con distribución $\mathcal{N}(0,1)$.
Sea

$$Y = \frac{(X_2 - X_1)^2}{2} \tag{614}$$

y encontrar la distribución de Y.

—

Función generadora de momentos de Y:

$$M_Y(t) = \mathbb{E}\left[e^{tY}\right] = \mathbb{E}\left[e^{\frac{(X_2 - X_1)^2}{2} t}\right] \tag{615}$$

$$= \mathbb{E}\left[\exp\left(\frac{X_2^2 - 2X_1X_2 + X_1^2}{2}\, t\right)\right] \tag{616}$$

—

Como $X_1, X_2 \sim \mathcal{N}(0,1)$ independientes,

$$f_{X_1,X_2}(x_1, x_2) = \frac{1}{2\pi} \exp\left(-\frac{x_1^2 + x_2^2}{2}\right) \tag{617}$$

—

$$M_Y(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi} \exp\left(\frac{(x_2 - x_1)^2}{2} t\right) \exp\left(-\frac{x_1^2 + x_2^2}{2}\right) dx_1 dx_2 \tag{618}$$

—

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi} \exp\left(\frac{x_2^2(t-1) + x_1^2(t-1) - 2x_1x_2t}{2}\right) dx_1 dx_2 \tag{619}$$

—

Reordenando términos:

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi} \exp\left(\frac{x_2^2(1-t)}{2}\right) \left[\int_{-\infty}^{\infty} \exp\left(-\frac{(1-t)}{2}\left(x_1^2 + \frac{2x_1x_2t}{1-t}\right)\right) dx_1\right] dx_2 \tag{620}$$

Ejemplo (continuación):
Sean X_1, X_2 v.a. independientes con distribución $\mathcal{N}(0,1)$. Sea

$$Y = \frac{(X_2 - X_1)^2}{2} \tag{621}$$

y encontrar la distribución de Y.

—

Función generadora de momentos:

$$M_Y(t) = \mathbb{E}\left[e^{Yt}\right] = \mathbb{E}\left[e^{\frac{(X_2 - X_1)^2}{2} t}\right] \tag{622}$$

$$= \mathbb{E}\left[\exp\left(\frac{X_2^2 - 2X_1X_2 + X_1^2}{2} t\right)\right] \tag{623}$$

—

Como $X_1, X_2 \sim \mathcal{N}(0,1)$ independientes,

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi} \exp\left(\frac{(x_2 - x_1)^2}{2} t\right) \exp\left(-\frac{x_1^2 + x_2^2}{2}\right) dx_1 dx_2 \tag{624}$$

—

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi} \exp\left\{\frac{x_2^2(t-1) + x_1^2(t-1) + 2x_1x_2t}{2}\right\} dx_1 dx_2 \tag{625}$$

—

Reescribiendo:

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi} \exp\left(\frac{x_2^2(1-t)}{2}\right) \left[\int_{-\infty}^{\infty} \exp\left(-\frac{(1-t)}{2}\left(x_1^2 + \frac{2x_1x_2t}{1-t}\right)\right) dx_1\right] dx_2 \tag{626}$$

—
Factorizando y completando cuadrados en la integral interna:

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi} \exp\left(\frac{x_2^2(1-t)}{2}\right) \left[\int_{-\infty}^{\infty} \exp\left(-\frac{(1-t)}{2}\left(x_1 + \frac{x_2t}{1-t}\right)^2\right) dx_1\right] dx_2 \tag{627}$$

—
Evaluando la integral gaussiana:

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{1-t}} \exp\left(-\frac{x_2^2}{2}\frac{1-t-t^2}{1-t}\right) dx_2 \tag{628}$$

—

$$= \frac{1}{\sqrt{1-t}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x_2^2}{2(1-2t)}\right) dx_2 \tag{629}$$

—
Finalmente:

$$M_Y(t) = (1-2t)^{-1/2}, \qquad t < \frac{1}{2} \tag{630}$$

Teorema. Sean $X_1, X_2, \dots, X_n$ v.a. independientes t.q. su FGM existe $\forall |t| < h$, p.a. $h > 0$. Sea

$$Y = \sum_{i=1}^{n} X_i. \tag{631}$$

Entonces,

$$M_Y(t) = \mathbb{E}\left[e^{\sum_{i=1}^{n} X_i t}\right] = \prod_{i=1}^{n} M_{X_i}(t), \qquad |t| < h. \tag{632}$$

Demostración.

$$M_Y(t) = \mathbb{E}\left[\exp\left(\sum_{i=1}^{n} X_i t\right)\right] = \mathbb{E}\left[\prod_{i=1}^{n} e^{X_i t}\right] \tag{633}$$

$$= \prod_{i=1}^{n} \mathbb{E}\left[e^{X_i t}\right] = \prod_{i=1}^{n} M_{X_i}(t). \tag{634}$$

—

Ejemplo. Sean $X_1, X_2, \ldots, X_n$ v.a. independientes con distribución Ber(p).

$$M_X(t) = pe^t + q, \tag{635}$$

entonces,

$$M_{\sum X_i}(t) = \prod_{i=1}^{n} M_{X_i}(t) = (pe^t + q)^n. \tag{636}$$

Se tiene que

$$\sum_{i=1}^{n} X_i \sim \text{Binomial}(n, p). \tag{637}$$

—

Ejemplo. Sean $X_1, X_2, \ldots, X_n$ v.a. independientes con

$$X_i \sim \text{Po}(\lambda_i). \tag{638}$$

$$M_{X_i}(t) = \mathbb{E}\big[e^{X_i t}\big] = \exp\big(\lambda_i(e^t - 1)\big). \tag{639}$$

Entonces,

$$M_{\sum X_i}(t) = \prod_{i=1}^{n} M_{X_i}(t) = \prod_{i=1}^{n} \exp\big(\lambda_i(e^t - 1)\big) \tag{640}$$

$$= \exp\left(\sum_{i=1}^{n} \lambda_i(e^t - 1)\right), \quad \text{i.e. } \sum_{i=1}^{n} X_i \sim \text{Po}\left(\sum_{i=1}^{n} \lambda_i\right).$$

Teorema. Sean $X_1, X_2, \ldots, X_n$ v.a. independientes t.q. su FGM existe $\forall |t| < h$, p.a. $h > 0$. Sea

$$Y = \sum_{i=1}^{n} X_i. \tag{642}$$

Entonces,

$$M_Y(t) = \mathbb{E}\Big[e^{\sum_{i=1}^{n} X_i t}\Big] = \prod_{i=1}^{n} M_{X_i}(t), \qquad |t| < h. \tag{643}$$

Demostración.

$$M_Y(t) = \mathbb{E}\left[\exp\left(\sum_{i=1}^{n} X_i t\right)\right] = \mathbb{E}\left[\prod_{i=1}^{n} e^{X_i t}\right] \tag{644}$$

$$= \prod_{i=1}^{n} \mathbb{E}\left[e^{X_i t}\right] = \prod_{i=1}^{n} M_{X_i}(t). \tag{645}$$

—

Ejemplo. Sean $X_1, X_2, \ldots, X_n$ v.a. independientes con distribución Ber(p).

$$M_X(t) = pe^t + q, \tag{646}$$

entonces,

$$M_{\sum X_i}(t) = \prod_{i=1}^{n} M_{X_i}(t) = (pe^t + q)^n, \qquad \text{i.e. } Y = \sum_{i=1}^{n} X_i. \tag{647}$$

Se tiene distribución

$$\text{Binomial}(n, p). \tag{648}$$

—

Ejemplo. Sean $X_1, X_2, \ldots, X_n$ v.a. independientes con

$$X_i \sim \text{Po}(\lambda_i). \tag{649}$$

$$M_{X_i}(t) = \mathbb{E}\left[e^{X_i t}\right] = \exp\left(\lambda_i(e^t - 1)\right). \tag{650}$$

$$M_{\sum X_i}(t) = \prod_{i=1}^{n} M_{X_i}(t) = \prod_{i=1}^{n} \exp\left(\lambda_i(e^t - 1)\right) \tag{651}$$

$$= \exp\left(\sum_{i=1}^{n} \lambda_i(e^t - 1)\right), \quad \text{i.e. } Y \sim \text{Po}\left(\sum_{i=1}^{n} \lambda_i\right).$$

Ejemplo. Sean $X_1, X_2, \ldots, X_n$ v.a. independientes con distribución Exp(λ). Entonces,

$$M_{X_i}(t) = \frac{\lambda}{\lambda - t}, \tag{653}$$

y por lo tanto,

$$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t) = \prod_{i=1}^{n} \left(\frac{\lambda}{\lambda - t}\right) = \left(\frac{\lambda}{\lambda - t}\right)^n. \tag{654}$$

i.e.

$$Y \sim \mathrm{Ga}(n, \lambda). \tag{655}$$

—

Ejemplo. Sean $X_1, X_2, \ldots, X_n$ v.a. independientes t.q.

$$X_i \sim \mathcal{N}(\mu_i, \sigma_i^2), \tag{656}$$

entonces

$$a_i X_i \sim \mathcal{N}(a_i\mu_i, a_i^2\sigma_i^2), \tag{657}$$

y

$$M_{a_iX_i}(t) = \exp\left(a_i\mu_i t + \frac{1}{2}a_i^2\sigma_i^2 t^2\right). \tag{658}$$

Entonces,

$$M_{\sum_{i=1}^n a_iX_i}(t) = \prod_{i=1}^n M_{a_iX_i}(t) = \exp\left(\sum_{i=1}^n a_i\mu_i t + \frac{1}{2}\sum_{i=1}^n a_i^2\sigma_i^2 t^2\right). \tag{659}$$

i.e.

$$Y \sim \mathcal{N}\left(\sum_{i=1}^n a_i\mu_i, \sum_{i=1}^n a_i^2\sigma_i^2\right). \tag{660}$$

—

Resultado. Si

$$X \sim \mathcal{N}(\mu_x, \sigma_x^2), \qquad Y \sim \mathcal{N}(\mu_y, \sigma_y^2), \tag{661}$$

y X, Y son independientes, entonces

$$X + Y \sim \mathcal{N}(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2), \tag{662}$$

$$X - Y \sim \mathcal{N}(\mu_x - \mu_y, \sigma_x^2 + \sigma_y^2). \tag{663}$$

Tema: (Teorema Central del Límite) (Tema límite central)
Sea $n \in \mathbb{N}$, $X_1, X_2, \ldots, X_n$ sean v.a. i.i.d. con media μ_X y varianza σ_X^2 finitas $\forall i$. Entonces,

$$F_{Z_n}(z) \longrightarrow \Phi(z) \quad \text{cuando } n \to \infty. \tag{664}$$

Donde

$$Z_n = \frac{\bar{X}_n - \mathbb{E}[\bar{X}_n]}{\sqrt{\mathrm{Var}(\bar{X}_n)}} = \frac{\bar{X}_n - \mu_X}{\sigma_X/\sqrt{n}}. \tag{665}$$

Para n fijo, el valor de la FDP de Z_n $(n = 1, 2, \ldots)$ converge al valor $\phi(z)$.

—

Corolario: Si $X_1, X_2, \ldots, X_n$ son v.a. i.i.d. con media μ_X y varianza σ_X^2 finitas, entonces

$$\mathbb{P}\left(a < \frac{\bar{X}_n - \mu_X}{\sigma_X/\sqrt{n}} < b\right) \approx \Phi(b) - \Phi(a). \tag{666}$$

—

Nota: Recordemos que para $Y = g(X)$,

$$f_Y(y) = \left|\frac{d}{dy} g^{-1}(y)\right| f_X\big(g^{-1}(y)\big)\, \mathbf{1}_D(y). \tag{667}$$

Ej: Sup. $X \sim \mathrm{Be}(\alpha, \beta)$, Â¿cómo se distribuye

$$Y = -\log_e X\,? \tag{668}$$

Se tiene

$$y = -\ln x \;\;\Rightarrow\;\; x = e^{-y}, \qquad y = \ln x^{-1} \;\Rightarrow\; e^y = x^{-1}. \tag{669}$$

Luego,

$$x = g^{-1}(y) = e^{-y}, \qquad \frac{dx}{dy} = -e^{-y}. \tag{670}$$

Por la fórmula de transformación,

$$f_Y(y) = \left|\frac{d}{dy} g^{-1}(y)\right| f_X\big(g^{-1}(y)\big). \tag{671}$$

Como

$$f_X(x) = \frac{1}{\mathrm{Be}(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1}, \tag{672}$$

entonces

$$f_Y(y) = e^{-y} \frac{1}{\mathrm{Be}(\alpha, \beta)} (e^{-y})^{\alpha-1}(1-e^{-y})^{\beta-1}. \tag{673}$$

Simplificando,

$$f_Y(y) = \frac{1}{\mathrm{Be}(\alpha, \beta)} e^{-\alpha y}(1-e^{-y})^{\beta-1}. \tag{674}$$

Si $\beta = 1$, se tiene

$$f_Y(y) = \frac{1}{\mathrm{Be}(\alpha, 1)} e^{-\alpha y}. \tag{675}$$

Además,

$$\mathrm{Be}(\alpha, 1) = \frac{\Gamma(\alpha)\Gamma(1)}{\Gamma(\alpha+1)} = \frac{1}{\alpha}. \tag{676}$$

Por tanto,

$$f_Y(y) = \alpha e^{-\alpha y}, \qquad Y \sim \mathrm{Exp}(\alpha). \tag{677}$$

—

Ej: Sea X v.a. con densidad Pareto, i.e.

$$f_X(x) = \theta x^{-\theta-1}\, \mathbf{1}_{(1,\infty)}(x), \tag{678}$$

y

$$Y = \log_e X = \ln X. \tag{679}$$

Entonces

$$y = \ln x, \qquad x = e^y, \qquad dx = e^y dy. \tag{680}$$

Luego,

$$f_Y(y) = e^y\, \theta (e^y)^{-\theta-1}\, \mathbf{1}_{(0,\infty)}(y) = \theta e^{-\theta y}\, \mathbf{1}_{(0,\infty)}(y). \tag{681}$$

Ej: Sean X_1, X_2 v.a. independientes,

$$X_1, X_2 \sim N(0,1). \tag{682}$$

Definimos

$$Y_1 = X_1 + X_2, \qquad Y_2 = \frac{X_1}{X_2}. \tag{683}$$

Entonces,

$$x_1 = g_1^{-1}(y_1, y_2) = \frac{y_1 y_2}{1+y_2}, \qquad x_2 = g_2^{-1}(y_1, y_2) = \frac{y_1}{1+y_2}. \tag{684}$$

Las transformaciones directas son

$$g_1(x_1, x_2) = x_1 + x_2, \qquad g_2(x_1, x_2) = \frac{x_1}{x_2}. \tag{685}$$

Derivadas parciales:

$$\frac{\partial g_1}{\partial x_1} = 1, \qquad \frac{\partial g_1}{\partial x_2} = 1, \tag{686}$$

$$\frac{\partial g_2}{\partial x_1} = \frac{1}{x_2}, \qquad \frac{\partial g_2}{\partial x_2} = -\frac{x_1}{x_2^2}. \tag{687}$$

El jacobiano es

$$|J| = \begin{vmatrix} 1 & 1 \\ \frac{1}{x_2} & -\frac{x_1}{x_2^2} \end{vmatrix} = \left| -\frac{x_1}{x_2^2} - \frac{1}{x_2} \right| = \left| -\frac{x_1 + x_2}{x_2^2} \right|. \tag{688}$$

Por lo tanto,

$$|J| = \frac{x_1 + x_2}{x_2^2}. \tag{689}$$

Luego,

$$|J^{-1}| = \frac{x_2^2}{x_1 + x_2}. \tag{690}$$

Sustituyendo

$$x_2 = \frac{y_1}{1 + y_2}, \qquad x_1 + x_2 = y_1, \tag{691}$$

se obtiene

$$|J^{-1}| = \frac{\left(\frac{y_1}{1+y_2}\right)^2}{y_1} = \frac{y_1}{(1 + y_2)^2}. \tag{692}$$