11

Linear Programming

LEARNING OBJECTIVES :

After studying this unit you will be able to :

- Formulate a problem in LP

- Understand the circumstances in which LP technique can be used

- Calculate marginal rates of substitution and opportunity cost using the graphic method

- Construct the initial tableau using the Simplex Method

- Explain the meaning odf entries in each column of final tableau

- Describe hoe technique of Linear Programming can be used in decision making, planning and control

11.1 INTRODUCTION

Linear programming is a mathematical technique for determining the optimal allocation of resources and achieving the specified objective when there are alternative uses of the resources like money, manpower, materials, machines and other facilities. The objective in resource allocation may be either cost minimization or profit maximization. The technique of linear programming is applicable to all problems in which the total effectiveness about the use of resources can be expressed as a linear function of individual allocations, and the limitations on resources give rise to *linear* equalities or inequalities of individual allocations. The adjective linear, is to be particularly noted here. It implies that all the limitations or constraints and the objective must be expressed as linear functions.

Although the technique of linear programming is applicable to all such problems, there are some more easy methods for solving specific categories of problems. Special algorithms have been evolved to ease up computational task. The algorithms have acquired special names of their own. This has led to the following categories of the linear programming problems :

(i) General Linear Programming Problems.

(ii) Transportation Problems.

(iii) Assignment Problems.

We know that there is an opportunity cost for scarce resources that should be included in the relevant cost calculation for decision making and variance calculations. In practice several resources may be scarce. The opportunity cost of these scarce resources can be determined by the use of linear programming techniques. Our objective in this chapter is to examine linear programming techniques and to consider how they can be applied to some specific types of decisions that a firm may have to make.

11.1.1 General Linear Programming Problems : An examination of the following simple example should illustrate the basic concepts of linear programming problem (abbreviated as LPP).

Illustration

A small scale industry unit manufactures two products, X_1 and X_2 which are processed in the machine shop and the assembly shop. The time (in hours) required for each product in the shops are given in the matrix below. Profit per unit is also given along.

	Machine shop	*Assembly shop*	*Profit per unit*
Product X_1	2 hours	4 hours	Rs. 3
Product X_2	3 hours	2 hours	Rs. 4
Total time available (hours) (in a day)	16	16	

Assuring that there is unlimited demand for both products how many units of each should be produced every day to maximize total profit ?

Let x_1 and x_2 be the number of units of X_1 and X_2 respectively that maximize the profit.

To begin solving the problem, let us restate the information in mathematical form. To do so, we introduce a term "*objective function*". This refers to the expression which shows the relationship of output to profit.

$3x_1$ = total daily profit from the sale of x_1 units product X_1

$4x_2$ = total daily profit from the sale of x_2 units product X_2

Objective function Max Profit = $3x_1 + 4x_2$

Department time constraints

Time used in making x_1 and x_2 units of two products must not exceed the total daily time available in two shops. In other words, (the hours required to make 1 unit of product X_1) multiplied by (the number of units of product X_1) plus (the hours required to make 1 unit of product X_2) multiplied by (the number of units of product X_2) must be less than or equal to the daily time available in each shop. Mathematically it is stated as

$2x_1 + 3x_2 \leq 16$ Machine shop constraint

$4x_1 + 2x_2 \leq 16$ Assembles shop constraint

In order to obtain meaningful answers, the values calculated for x_1 and x_2 cannot be negatives; they must represent real product X_1 and product X_2. Thus, all elements of the solution to the formulated linear programming problem must be either greater than or equal to 0 ($x_1 \geq 0$, $x_2 \geq 0$). The machine and assembly shops contraints are called structural constraints.

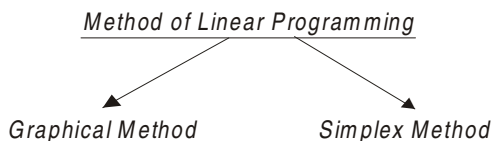The problem can be summarized in a mathematical form as below :

Maximize:

$$\text{Profit} = 3x_1 + 4x_2$$

Subject to the constraints

$$2x_1 + 3x_2 \leq 16$$
$$4x_1 + 2x_2 \leq 16$$
$$x_1 \geq 0, x_2 \geq 0.$$

Method of Linear Programming

*Graphical Method*   *Simplex Method*

11.2 GRAPHICAL METHOD

The graphical method discussed here provides necessary grounding in understanding the computational steps of the simplex algorithms, it itself being of little use in handling practical problems. It is possible to solve conveniently linear programming problems graphically as long as the number of variables (products, for example) are not more than two. Before we proceed to explain the graphical method, let us first know how linear inequalities are handled graphically.

11.2.1 Linear inequalities in one variable and the solution space

Any relation or function of degree one that involves an inequality sign is a *linear inequality*. It may be of one variable, or, may be of more than one variable. Simple examples of linear inequalities are those of one variable only; *viz.* $x > 0$, $x \leq 0$, etc.

$x > 0$ $x \leq 0$

-3   -2   -1   0   1   2   3   -3   -2   -1   0   1   2   3
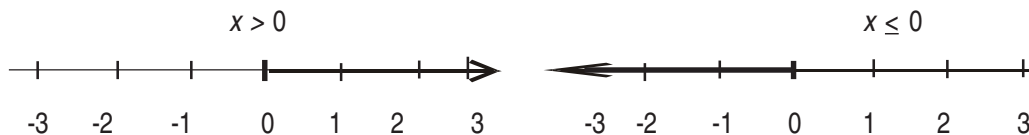
The values of the variables that satisfy an inequality are called the solution space, and is abbreviated as S.S. The solution spaces for (i) $x > 0$, (ii) $x \leq 0$ are shown by the dark portion in the above diagrams.

11.2.2 Linear inequalities in two variables : Now we turn to linear inequalities in two variables x, and y, and shade a few S.S.

$x > 0$

$x \geq 0$

$x > 0$
$y > 0$

$x \geq 0$
$y \geq 0$
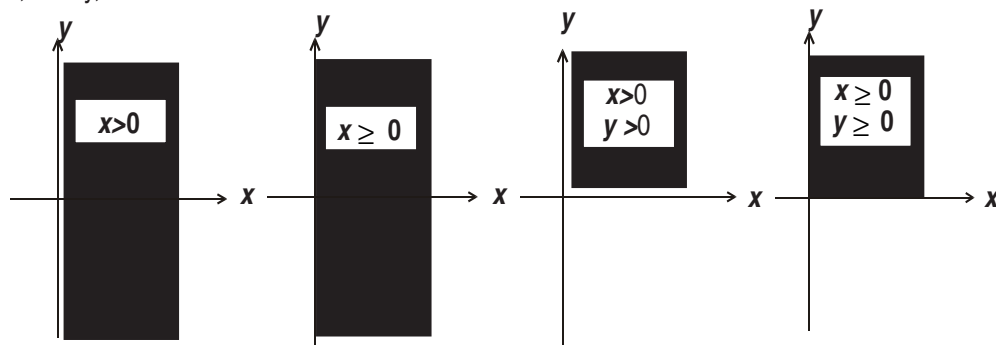
11.4   Advanced Management Accounting

Note :   The pair of inequalities $x \geqslant 0$, $y \geqslant 0$ play an important role in linear programming problems.

Let us now consider a linear inequality in two variables given by

$$3x + y < 6.$$

The inequality mentioned above is true for certain pairs of numbers, (x, y) that satisfy $3x + y < 6$. By trial, we may arbitrarily find such a pair to be (1,1) because $3 \times 1 + 1 = 4$, and $4 < 6$.

Linear inequalities in two variables may be solved easily by extending our knowledge of straight lines.

Figure 2

For this purpose, we replace the inequality by an equality and seek the pairs of number that satisfy $3x + y = 6$. We may write $3x + y = 6$ as $y = 6 - 3x$, and draw the graph of this linear function.

Let $x = 0$ so that $y = 6$. Let $y = 0$, so that $x = 2$

Any pair of numbers (x,y) that satisfies the equation $y = 6 - 3x$ falls on the line AB.

Therefore, if y is to be less than $6 - 3x$ for the same value of x, it must assume a value that is less than the ordinate of length $6 - 3x$.

All such points (x,y) for which the ordinate is less than $6 - 3x$ lie below the line AB.

The region where these points fall is indicated by an arrow and is shaded too in the diagram above.

Now we consider two inequalities

$3x + y < 6$ and $x - y < -2$

being satisfied simultaneously by x and y.

The pairs of numbers (x, y) that satisfy both the inequalities may be found by drawing the graphs of the two lines $y = 6 - 3x$ and $y = 2 + x$, and

determining the region where both the inequalities hold. It is convenient to express each equality with y on the left-side and the remaining terms in the right-side.

The first inequality $3x + y < 6$ is equivalent to

$$y < 6 - 3x$$

and requires the value of y for each x to be less than that of $y = 6 - 3x$. This inequality is therefore satisfied by all points lying below the line $y = 6 - 3x$. The region where these points fall has been shaded in the previous diagram.
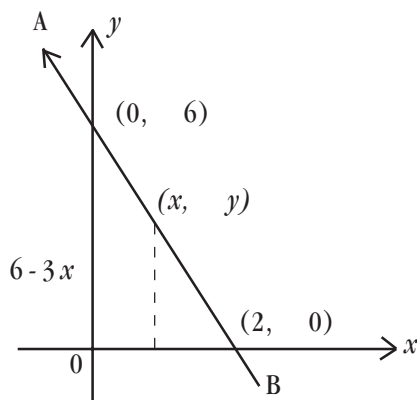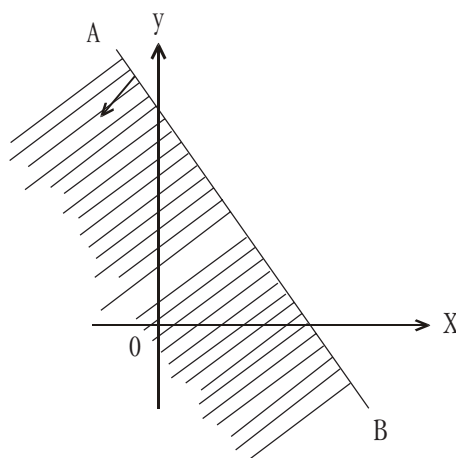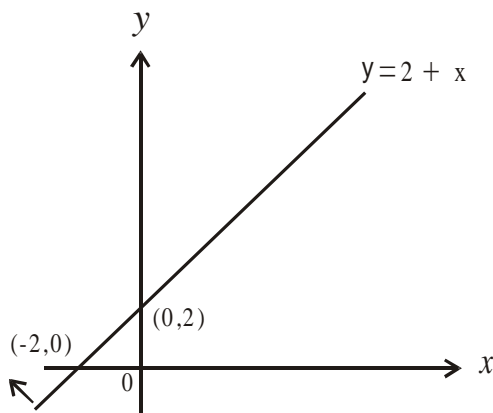
We consider the second inequality $x - y < -2$, and note that this is equivalent to $y > 2 + x$.
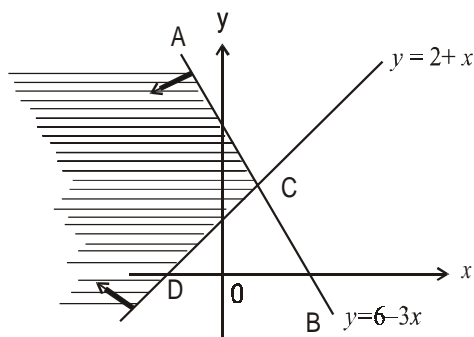
It requires the value of y for each x to be larger than that of $y = 2 + x$. This inequality is, therefore, satisfied by all points lying above the line $y = 2 + x$.

The region is indicated by an arrow on the line $y = 2 + x$

For $x = 0$, $y = 2 + 0 = 2$.   For $y = 0$, $0 = 2 + x$ \ $x = -2$.

By superimposing the above two graphs we determine the common region ACD in which the pairs (x,y) satisfy both the inequalities.

Note : [1]   The inequalities $3x + y \leq 6$ and $x - y \leq 2$ differ from the preceding ones in that these also include equality signs. It means that the points lying on the corresponding lines are also included in the region.

[2]   The procedure may be extended to any number of inequalities.

11.2.3 Graphical solution for linear programming problem : We consider the problem of maximizing a linear function $Z = x + 2y$ subject to the restrictions

$x \geq 0$, $y \geq 0$, $x \leq 6$, $y \leq 7$, $x + y \leq 12$

We note that the above inequalities may be grouped as follows:

$$\begin{Bmatrix} x \geq 0 \\ x \leq 6 \end{Bmatrix} \qquad \begin{Bmatrix} y \geq 0 \\ y \leq 7 \end{Bmatrix}, \qquad x + y \leq 12 .$$
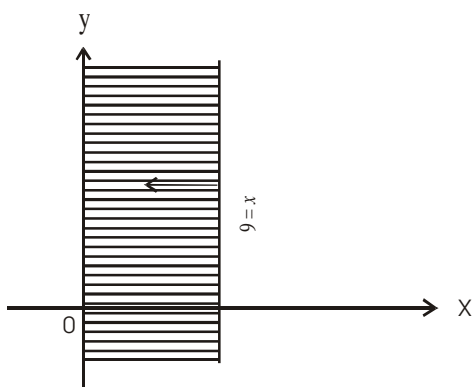
$x \geq 0, \ x \leq 6$
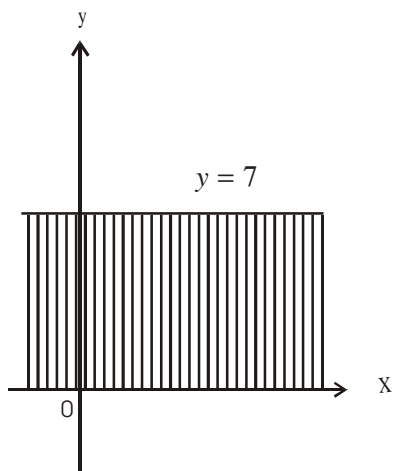
$y \geq 0, \ y \leq 7$
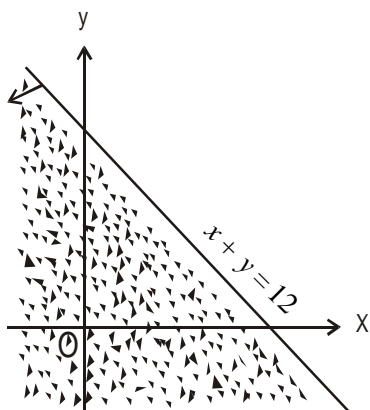
$x + y \leq 12$

By superimposing the above three diagrams, we determine the common region in the *xy* plane where all the five inequalities are simultaneously satisfied.
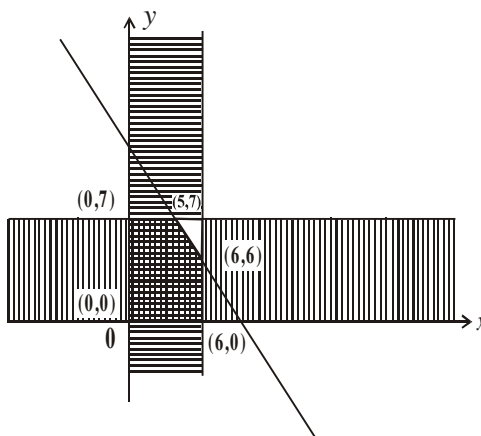
This common region is known as *feasible region* or the *solution set (*or the *polygonal convex sets).*

A region is said to be *bounded* if it can be totally included within a (very large) circle. The shaded region enclosed by deep lines in the previous diagram is bounded, since it can be included within a circle.

Corresponding to an arbitrary definite value for Z, $Z = x + 2y$ becomes an equation of a straight line. Let Z =2, 4, etc.

　　　For Z = 2, we have *x + 2 y = 2.*

　　　Let $x = 0,$ so that $y = 1$.

　　　Let $y = 0,$ so that $x = 2$.

　　　Next, let us suppose Z = 4.

　　　In this case Z = *x+2y* takes the form $x + 2 y = 4$.

　　　Let $x = 0,$ so that $y = 2$.

　　　Let $y = 0,$ so that $x = 4$.

On drawing the graphs for both the above lines we observe that each one of them intersects the feasible region. Here we are keen to determine a point in the feasible region which makes Z maximum (*i.e.,* as large as possible) we would prefer any point in the feasible region that lies on *x+ 2y =4* over any point that lies on *x + 2 y =2*.
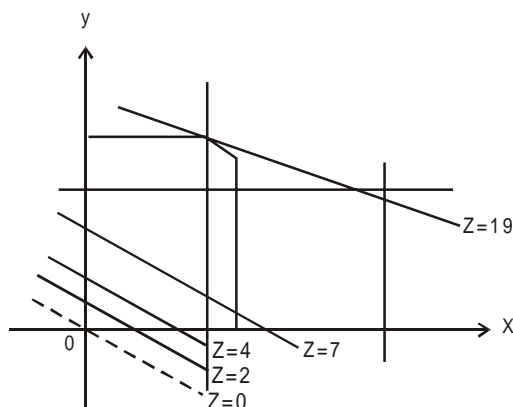
Following the same reasoning we find that we can get point in the feasible region for which Z is maximum. The maximum value for Z is achieved when the line $x + 2y = Z$ is moved parallel to itself upward as far as possible subject to the restriction that it must intersect the feasible region (shown by the deep line in the diagram on previous page). This is seen to occur when it passes through the point (5,7) the point of intersection of $x + y = 12$ and $y = 5$. It is noted that this point (5,7) is one of the corner points (or extreme points) of the solution set.

We may similarly find the point in the feasible region for which Z is minimum. It is found to occur at (0,0) which is again a corner point of the solution set but situated opposite to the point for which Z is maximum. The values of x and y so determined constitutes the solution to the problem.

Thus the objective function attains a maximum or a minimum value at one of the corner points of the feasible solution known as extreme points of the solution set. Once these extreme points (the points of intersection of lines bounding the region) are known, a compact matrix representation of these points is possible. We shall denote the matrix of the extreme points by E.

The coefficients of the objective function may also be represented by a column vector. We shall represent this column vector by C.

The elements in the product matrix EC shows different values, which the objective function attains at the various extreme points. The largest and the smallest elements in matrix EC are respectively the maximum and the minimum values of the objective function. The row in matrix EC in which this happens is noted and the elements in that row indicate the appropriate pairing and is known as the optimal solution.

In the context of the problem under consideration.

$$E = \begin{pmatrix} 0 & 0 \\ 0 & 7 \\ 5 & 7 \\ 6 & 0 \\ 6 & 6 \end{pmatrix} \begin{matrix} x & y \end{matrix}, \quad C = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{matrix} x \\ y \end{matrix}$$

$$EC = \begin{pmatrix} 0 & 0 \\ 0 & 7 \\ 5 & 7 \\ 6 & 0 \\ 6 & 6 \end{pmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{pmatrix} 0 \times 1 + 0 \times 2 \\ 0 \times 1 + 7 \times 2 \\ 5 \times 1 + 7 \times 2 \\ 6 \times 1 + 0 \times 2 \\ 6 \times 1 + 6 \times 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 14 \\ 19 \\ 6 \\ 18 \end{pmatrix}$$

The given objective function viz. Z = x + 2y is maximum at the points (5, 7) present in the third row of the matrix E. Thus the optimal solution is x = 5, y = 7, and the maximum value of the objective function is 19.

We now list the steps to be followed under graphical solution to a linear programming problem.

Step1. Determine the region that satisfies the set of given inequalities.

Step 2. Ensure that the region is bounded*. If the region is not bounded, either there are additional hidden conditions which can be used to bound the region or there is no solution to the problem.

Step 3. Construct the matrix E of the extreme points, and the column vector C of the objective function.

Step 4. Find the matrix product EC. For maximization, determine the row in EC where the largest element appears; while for minimization, determine the row in EC where the smallest element appears.

Step 5. The objective function is optimized corresponding to the same row elements of the extreme point matrix E.

Note : If the slope of the objective function be same as that of one side of feasible region, there are multiple solutions to the problem. However, the optimized value of the objective function remains the same.

11.2.4 Application of Graphical Method

Illustration

A manufacturer produces two products A and B, and has his machines in operation for 24 hours a day. Production of A requires 2 hours of processing in machine M_1 and 6 hours in machine M_2. Production of B requires 6 hours of processing in machine M_1 and 2 hours in machine M_2. The manufacturer earns a profit of Rs. 5 on each unit of A and Rs. 2 on each unit of B. How many units of each product should be produced in a day in order to achieve maximum profit?

Solution

Let x_1 be the number of units of type A product to be produced, and x_2 is that of type B product to be produced. The formulation of the L.P.P. in this case is as below :

Maximize $Z = 5x_1 + 2x_2$

subject to the constraints.

$x_1 \geq 0, x_2 \geq 0,$

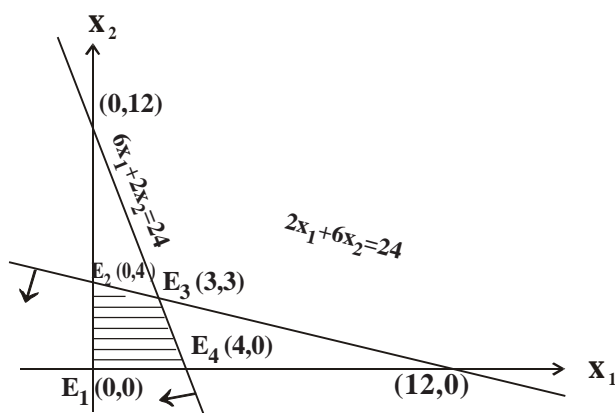$2x_1 + 6x_2 \leq 24,$

$6x_1 + 2x_2 \leq 24.$

For the line $2x_1 + 6x_2 = 24,$

let $x_1 = 0,$ so that $x_2 = 4;$

let $x_2 = 0,$ so that $x_1 = 12;$

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* It is inconceivable for a practical problem to have an unbounded solution.

For the line $6x_1 + 2x_2 = 24$;

let $x_1 = 0$, so that $x_2 = 12$;

let $x_2 = 0$, so that $x_1 = 4$.

The shaded portion in the diagram is the feasible region and the matrix of the extreme points E_1, E_2, E_3 and E_4 is

$$
E \;=\;
\begin{array}{cc}
x_1 & x_2 \\
\end{array}
\begin{pmatrix}
0 & 0 \\
0 & 4 \\
3 & 3 \\
4 & 0 \\
\end{pmatrix}
\begin{array}{l}
E_1 \\
E_2 \\
E_3 \\
E_4 \\
\end{array}
$$

The column vector for the objective function is $C = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \begin{matrix} x_1 \\ x_2 \end{matrix}$

The column vector the values of the objective function is given by

$$
EC =
\begin{pmatrix}
0 & 0 \\
0 & 4 \\
3 & 3 \\
4 & 0 \\
\end{pmatrix}
\begin{bmatrix} 5 \\ 2 \end{bmatrix}
=
\begin{pmatrix}
0\times5+0\times2 \\
0\times5+4\times2 \\
3\times5+3\times2 \\
4\times5+0\times2 \\
\end{pmatrix}
=
\begin{pmatrix}
0 \\
8 \\
21 \\
20 \\
\end{pmatrix}
\begin{matrix}
E_1 \\
E_2 \\
E_3 \\
E_4 \\
\end{matrix}
$$

Since 21 is the largest element in matrix EC, therefore the maximum value is reached at the extreme point E_3 whose coordinates are (3,3).

Thus, to achieve maximum profit the manufacturer should produce 3 units each of both the products A and B.

Illustration

A company produces two types of presentation goods A and B that require gold and silver. Each unit of type A requires 3 gms. of silver and 1 gm. of gold while that of B requires 1 gm. of silver and 2 gms. of gold. The company can procure 9 gms of silver and 8 gms. of gold. If each unit of type A brings a profit of Rs. 40 and that of types B Rs. 50, determine the number of units of each type that the company should produce to maximize the profit. What is the maximum profit?

Solution

Let x_1 be the number of units of type A of presentation goods to be produced and x_2 is that of type B. The formulation of L.P.P. based on the given data may be stated as follows:

Maximize $Z = 40\, x_1 + 50\, x_2$

Subject to the constraints

$$x_1 \geq 0,$$

$$x_2 \geq 0,$$
$$3x_1 + x_2 \leq 9,$$
$$x_1 + 2x_2 \leq 8.$$

For the line $3x_1 + x_2 = 9$,

Let $x_1 = 0$, so that $x_2 = 9$;

Let $x_2 = 0$, so that $x_1 = 3$;

For the line $x_1 + 2x_2 = 8$,

Let $x_1 = 0$, so that $x_2 = 4$;

Let $x_2 = 0$, so that $x_1 = 8$.
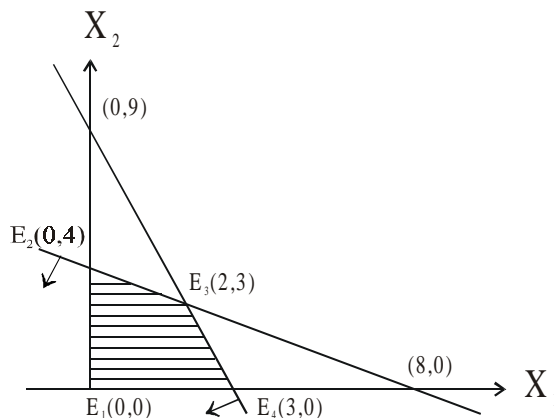
The shaded portion in the diagram represent the feasible region, and the matrix of the extreme points E_i, $i = 1, 2, 3, 4$ is

$$x_1 \qquad x_2$$

$$E = \begin{pmatrix} 0 & 0 \\ 0 & 4 \\ 2 & 3 \\ 3 & 0 \end{pmatrix} \begin{matrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{matrix}$$

[Since the dealer cannot buy negative quantities]
[Since he has space only for 20 items]
[Since the total investment cannot exceed the amount in hand]

The column vector for the objective function is $C = \begin{bmatrix} 40 \\ 50 \end{bmatrix} \begin{matrix} x_1 \\ x_2 \end{matrix}$

The column vector for the values of the objective function is given by

$$EC = \begin{pmatrix} 0 & 0 \\ 0 & 4 \\ 2 & 3 \\ 3 & 0 \end{pmatrix} \begin{bmatrix} 40 \\ 50 \end{bmatrix} = \begin{pmatrix} 0 \times 40 + 0 \times 50 \\ 0 \times 40 + 4 \times 50 \\ 2 \times 40 + 3 \times 50 \\ 3 \times 40 + 0 \times 50 \end{pmatrix} = \begin{pmatrix} 0 \\ 200 \\ 230 \\ 120 \end{pmatrix} \begin{matrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{matrix}$$

Since 230 is the maximum value in EC, this value is reached at the extreme point E_3 whose coordinates are (2,3).

Thus, for maximising profit, the company should produce 2 units of type A and 3 units of type B of presentation goods.

The maximum profit is Rs. 230.

Illustration

A dealer wishes to purchase a number of fans and sewing machines. He has only Rs. 5,760 to invest and has space utmost for 20 items. A fan costs him Rs. 360 and a sewing machine Rs.

240. His expectation is that he can sell a fan at a profit of Rs. 22 and a sewing machine at a profit of Rs. 18. Assuming that he can sell all the items that he can buy, how should he invest his money in order to maximize his profit? Formulate this problem as a linear programming problem and then use graphical method to solve it.

Solution

Let x_1 be the number of fans and x_2 be the number of sewing machines purchased by this dealer

The formulation of the problem as per given information is as under.

Maximize $Z = 22\,x_1 + 18\,x_2$

Subject to the constraints

$x_1 \geq 0, x_2 \geq 0$

$x_1 + x_2 \leq 20,$

$360x_1 + 240x_2 \leq 5760,$

For the line $x_1 + x_2 = 20,$

let $x_1 = 0$, so that $x_2 = 20;$

let $x_2 = 0$, so that $x_1 = 20.$

For the line $360x_1 + 240x_2 = 5760;$

i.e. $3x_1 + 2x_2 = 48,$

let $x_1 = 0,$ so that $x_2 = 24;$

let $x_2 = 0,$ so that $x_1 = 16.$
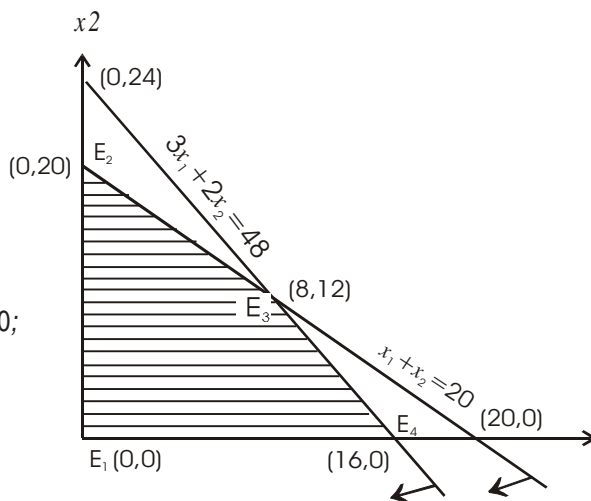
The shaded portion in the diagram is the feasible region, and the matrix E of the extreme points, E_i, $i=1, 2, 3, 4$ is given by

$$
E = \begin{array}{cc}
x_1 & x_2 \\
\left(\begin{array}{cc}
0 & 0 \\
0 & 20 \\
8 & 12 \\
16 & 0
\end{array}\right) &
\begin{array}{c}
E_1 \\
E_2 \\
E_3 \\
E_4
\end{array}
\end{array}
$$

The column vector for the objective function is

$$
C = \begin{array}{cc}
\left(\begin{array}{c}
22 \\
18
\end{array}\right) &
\begin{array}{c}
x_1 \\
x_2
\end{array}
\end{array}
$$

The column vector for the values of the objective function is given by

$$EC = \begin{pmatrix} 0 & 0 \\ 0 & 20 \\ 8 & 12 \\ 16 & 0 \end{pmatrix} \begin{pmatrix} 22 \\ 18 \end{pmatrix} = \begin{pmatrix} 0\times22+0\times18 \\ 0\times22+20\times18 \\ 8\times22+12\times18 \\ 16\times22+0\times18 \end{pmatrix} = \begin{pmatrix} 0 \\ 360 \\ 392 \\ 352 \end{pmatrix} \begin{matrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{matrix}$$

Since 392 is the largest element of matrix EC, therefore the maximum profit is reached at the extreme point E_3 whose coordinates are (8,12). Thus, to maximize his profit the dealer should buy 8 fans and 12 sewing machines.

Illustration

A company produces two products X and Y, each of which requires three types of processing. The length of time for processing each unit and the profit per unit are given in the following table :

	Product X (hr/unit)	Product Y (hr/unit)	Available capacity per day (hr.)
Process I	12	12	840
Process II	3	6	300
Process III	8	4	480
Process per unit (Rs.)	5	7	

How many units of each product should the company produce per day in order to maximize profit?

Solution

Let x be the number of units of type X and y be the number of units of type Y products to be produced. The formulation of the problem based on the given data is as follows:

Maximize $Z = 5x + 7y$

Subject to the constraints

$$x \geq 0, y \geq 0,$$
$$12x + 12y \leq 840,$$
$$3x + 6y \leq 300$$
$$8x + 4y \leq 480$$
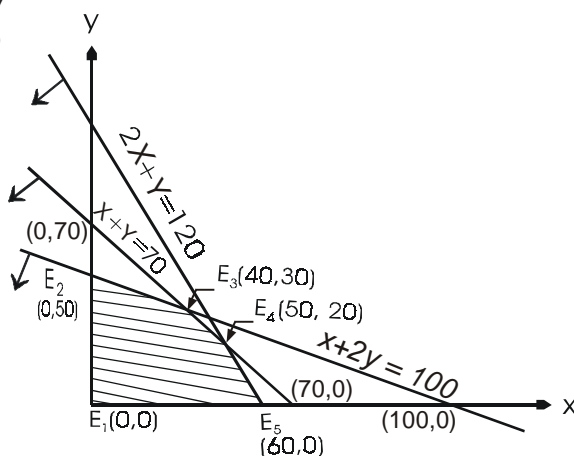
The line $12x + 12y = 840$ is equivalent to

$x + y = 70$	x	40	50
	y	30	20

The line $3x + 6y = 300$ is equivalent to

$x + 2y = 100$	x	0	50
	y	50	25

The line $8x + 4yx = 480$ is equivalent to

$2x + y = 120$	x	60	40
	y	0	40

The shaded portion in the diagram (on previous page) represents the feasible region, and the matrix of the extreme points

$E_j, i = 1, 2, 3, 4, 5$ is

The column vector for the values of the objective function is given by

$$EC = \begin{pmatrix} 0 & 0 \\ 0 & 50 \\ 40 & 30 \\ 50 & 20 \\ 60 & 0 \end{pmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{pmatrix} 0 \times 5 + 0 \times 7 \\ 0 \times 5 + 50 \times 7 \\ 40 \times 5 + 30 \times 7 \\ 50 \times 5 + 20 \times 7 \\ 60 \times 5 + 0 \times 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 350 \\ 410 \\ 390 \\ 300 \end{pmatrix} \begin{matrix} E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \end{matrix}$$

Since 410 is the largest element in EC, the maximum value is achieved at the extreme point E_3 whose coordinates are (40, 30).

Thus to maximize profit, the company should produce 40 units of X and 30 units of Y.

Illustration

A company that produces soft drinks has a contract that requires that a minimum of 80 units of the chemical A and 60 units of the chemical B into each bottle of the drink. The chemicals are available in a prepared mix from two different suppliers. Supplier X_1 has a mix of 4 units of A and 2 units of B that costs Rs. 10, and supplier X_2 has a mix of 1 unit of A and 1 unit of B that costs Rs. 4. How many mixes from company X_1 and company X_2 should the company purchase to honour contract requirement and yet minimize cost?

Solution

The problem may be summarized as follows:

	Chemical A	Chemical B	Cost per mix Rs.
Supplier X_1	4	2	10
Supplier X_2	1	1	4
Units required	80	60	

Let x_1 be the number of mixes to be purchased from supplier X_1 and x_2 be of those to be purchased from supplier X_2.

The conditions of the problem when symbolised, take the form:

Minimize $Z = 10\,x_1 + 4\,x_2$

Subject to the resrictions

$$x_1 \geq 0,\ x_2 \geq 0$$

$$4x_1 + x_2 \geq 80,$$

$$2x_1 + x_2 \geq 60.$$

For the line $4x_1 + x_2 = 80$,

let $x_1 = 0$, so that $x_2 = 80$;

let $x_2 = 0$, so that $x_1 = 20$.

For the line $2x_1 + x_2 = 60$,

let $x_1 = 0$, so that $x_2 = 60$;

let $x_2 = 0$, so that $x_1 = 30$.
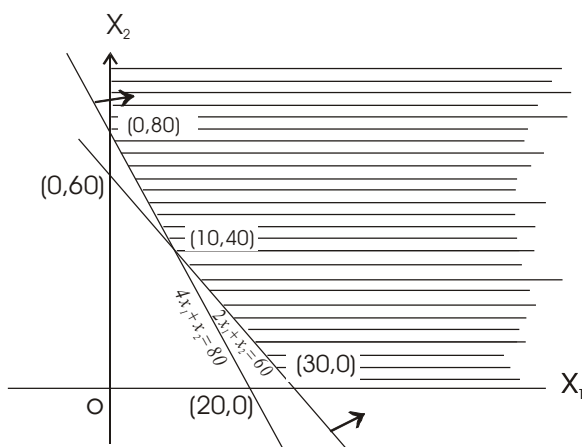
Feasible region is shaded in the diagram which appears to be unbounded. We now try to determine the additional hidden conditions in the problem for which the feasible region becomes bounded.

Each bottle of the drink contains 80 units of the chemical A and 60 units of the chemical B. If the drinks are made from the mixes of company X_1 only, each bottle would require 20 mixes to meet the chemical A requirement and 30 mixes to meet the chemical B requirement. Therefore, 30 mixes would meet all minimum requirements. Any lesser amount of mixes would not meet the minimum requirement for chemical B.

Similarly, if the drinks are made from the mixes of company X_2 only, each bottle would require 80 mixes to meet both the chemical requirements.

Since the company that manufactures drinks wants to make a blending of the mixes purchased from both the companies X_1 and X_2, it is possible that the company would need less than the maximum amount of both supplies to meet the guaranteed minimum.

Thus we add the two additional hidden conditions $x_1 \le 30$, $x_2 \le 80$.

With these two additional restrictions the above diagram takes the form :
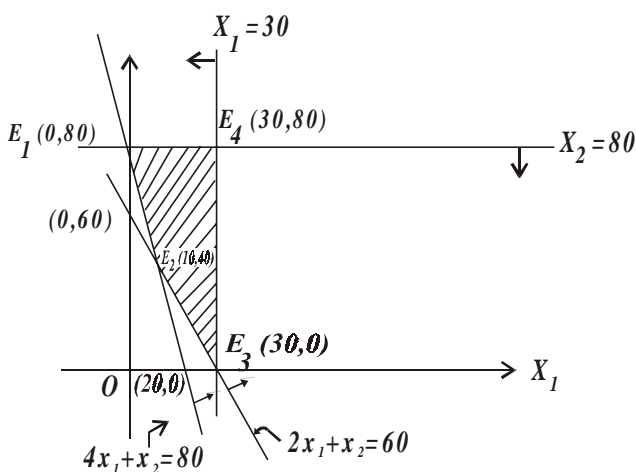
The matrix of the extreme points, in this case, is E =

$$
\begin{array}{cc}
x_1 & x_2 \\
\end{array}
$$

$$
E = \begin{pmatrix} 0 & 80 \\ 10 & 40 \\ 30 & 0 \\ 30 & 80 \end{pmatrix} \begin{array}{l} E_1 \\ E_2 \\ E_3 \\ E_4 \end{array}
$$

The column vector for the objective function is $C = \begin{pmatrix} 10 \\ 4 \end{pmatrix} \begin{array}{l} x_1 \\ x_2 \end{array}$

The column vector for the values of the objective function is given by

$$
EC = \begin{pmatrix} 0 & 80 \\ 10 & 40 \\ 30 & 0 \\ 30 & 80 \end{pmatrix} \begin{bmatrix} 10 \\ 4 \end{bmatrix} = \begin{pmatrix} 0\times10 + 80\times4 \\ 10\times10 + 40\times4 \\ 30\times10 + 0\times4 \\ 30\times10 + 80\times4 \end{pmatrix} = \begin{pmatrix} 320 \\ 260 \\ 300 \\ 620 \end{pmatrix} \begin{array}{l} E_1 \\ E_2 \\ E_3 \\ E_4 \end{array}
$$

Since 260 is the smallest element in EC, the minimum value is reached at the extreme point E_2, whose coordinates are (10,40).

Thus, to honour the contract and yet to minimize cost, the company should purchase 10 mixes from X_1 and 40 mixes from X_2.

Summary of Graphical Method

It involves:

(i)   Formulating the linear programming problem, i.e.,expressing the objective function and constraints in the standardised format.

(ii)   Plotting the capacity constraints on the graph paper. For this purpose normally two terminal points are required. This is done by presuming simultaneously that one of the constraints bis zero. When constraints concerns only one factor, then line will have only one origin point and it will run parallel to the other axis.

(iii) Identifying feasible region and coordinates of corner points. Mostly it is done by breading the graph, but a point can be identified by solving simultaneous equation relating to two lines which intersect to form a point on graph.

(iv)   Testing the corner point which gives maximum profit. For this purpose the coordinates relating to the corner point should put in objectives function and the optimal point should be ascertained.

(v)   For decision – making purpose, sometimes, it is required to know whether optimal point leaves some resources unutilized. For this purpose value of coordinates at the optimal point should be put with constraint to find out which constraints are not fully utilized.

11.3   TRIAL & ERROR METHOD OF SOLVING LINEAR PROGRAMMING PROBLEM

11.3.1 Method of Solving LPP: Trial & Error Method (or Algebraic Approach)

Graphical method cannot be used when there are more than 2 variables in a LPP., in such a case we use the simplex method which is highly efficient and versatile as also amenable to further mathematical treatment and offers interesting economic interpretations. However, its underlying concepts are rather lengthy to discuss and the student should patiently go through the following material on the trial and error method to gain a good grasp over the simplex technique.

Slack Variable

Consider the following example :

Example :   Maximize $Z = 3x_1 + 4x_2$

subject to $2x_1 + 3x_2 + \leq 16$

$4x_1 + 2x_2 \leq 16$

$x_1, x_2 \geq 0.$

The $\leq$ type inequalities can be transformed into equalities by the addition of non negative variables, say x_3 and x_4 (known as slack variables) as below. The slack variables represent idle resources, therefore, they are to be positive or non negative. Further the contribution per unit of a slack variable is always taken as zero in the objective function of a LPP. This is so because profits are not made on unused resources.

$$\left\{ \begin{array}{l} 2x_1 + 3x_2 + 1x_3 = 16 \\ 4x_1 + 2x_2 + 1x_4 = 16 \end{array} \right\} \qquad \text{...(A)}$$

And the objective function may be rewritten as below.

Maximize $Z = 3x_1 + 4x_2 + 0x_3 + 0x_4$

Surplus Variable

In ^{3} type inequalities, we subtract a variable (called the **surplus variable**) to make it an equality. The value of this variable can be interpreted as the excess amount of the resources utilized over and above the given level. The contribution per unit of a surplus variable is also taken as zero in the objective function.

11.3.2   The linear programming Theorems : The trial and error and simplex methods are based on the concept of slack variables and theorems described below:

Extreme point theorem : It states that an optimal solution to a LPP occurs at one of the vertices of the feasible region. This should be obvious from the discussion on the graphical method.

Now the vertices are defined by the intersection of equations. The first step of the method is, therefore, to convert the inequalities into equalities by the addition (or subtraction) of the slack (or surplus variables) depending on the direction of the inequality.

It is to be noted that the system of equations (A) above has more variables than the number of equations. Such a system of equations has an infinite number of solutions; yet it has finite and few vertices, the co-ordinates of which can be determined by applying the Basis theorem.

Basis theorem : It states that for a system of m equations in n variables (where $n > m$) has a solution in which at least $(n\text{-}m)$ of the variables have value of zero as a vertex. This solution is called a basic solution.

Extreme point theorem can be extended to state that the objective function is optimal at least at one of the basic solutions. Some of the vertices may be infeasible in that they have -ve co-ordinates and have to be dropped in view of the non-negativity condition on all variables including the slack and surplus variables.

Consider the following LPP for elucidating the basis theorem.

Maximize $\qquad$ $Z = 3x_1 + 4x_2$

Subject to $\qquad$ $2x_1 + 3x_2 \le 16$

$\qquad\qquad$ $4x_1 + 2x_2 \le 16$

$\qquad\qquad$ $x_1, x_2 \ge 0.$

Introducing slack variables x_3 and x_4, we have

Maximize $Z =$ $\qquad$ $3x_1 + 4x_2 + 0x_3 + 0x_4$

Subject to $\qquad$ $2x_1 + 3x_2 + 1x_3 + 0x_4 = 16$

$\qquad\qquad$ $4x_1 + 2x_2 + 0x_3 + 1x_4 = 16$ $\qquad\qquad\qquad\qquad$ ... (B)

$\qquad\qquad$ $x_1, x_2, x_3, x_4 \ge 0.$

Here n (number of variables) = 4 and m (number of equations) = 2. Thus n - m = 2. According to the Basis theorem we set 2 (= $n - m$) variables in (B) equal to zero at a time, solve the resulting system of equations and obtain a basic solution. Thus if we zeroise x_1 and x_2, the resulting system of equations would be

$$\left. \begin{cases} 1x_3 + 0x_4 = 16 \\ 0x_3 + 1x_4 = 16 \end{cases} \right\} \quad \text{......(C)}$$

Set 1 (x_1 = x_2 = 0)

These equations directly yield x_3 = 16 and x_4 = 16 as the basic solution *i.e.* the co-ordinate of a vertex.

The other sets of equations, upon zeroising two variables at a time in (B), would be as follows:

$$\begin{bmatrix} 2x_1 + 3x_2 = 16 \\ 4x_1 + 2x_2 = 16 \end{bmatrix}$$

Set 2 (x_3 = x_4 = 0)

$$\begin{bmatrix} 2x_1 + 0x_4 = 16 \\ 4x_1 + 1x_4 = 16 \end{bmatrix}$$

Set 3 (x_2 = x_3 = 0)

$$\begin{bmatrix} 2x_1 + 1x_3 = 16 \\ 4x_1 + 0x_3 = 16 \end{bmatrix}$$

Set 4 (x_2 = x_4 = 0)

$$\begin{bmatrix} 3x_2 + 1x_3 = 16 \\ 2x_2 + 0x_3 = 16 \end{bmatrix}$$

Set 5 (x_1 = x_4 = 0)

$$\begin{bmatrix} 3x_2 + 0x_4 = 16 \\ 2x_2 + 1x_4 = 16 \end{bmatrix}$$

Set 6 (x_1 = x_3 = 0)

It is a simple matter to solve these six sets of simultaneous equations and obtain the six basic solutions *i.e.* co-ordinates of the six vertices of the feasible region. The solutions are given below. The student may verify these by solving each of the sets of simultaneous equations.

Set	Solution
1	x_3 =16, x_4 = 16
2	x_1 = 2, x_2 = 4
3	x_1 = 8, x_4 = −16
4	x_1 = 4, x_3 = 8
5	x_2 = 8, x_3 = −8
6	x_2 = 16/3, x_4 = 16/3

Since solutions of set 3 and 5 yield a negative co-ordinate each, contradicting thereby the non-negativity of constraints, thus these are infeasible and are to be dropped from consideration.

Now according to the basis theorem the optimal solution lies at one of these vertices. By substituting these co-ordinates the values of objective function are derived below:

Set	Solution	Z (Profit)
1	$x_3 = 16$, $x_4 = 16$	0
2	$x_1 = 2$, $x_2 = 4$	22
3	Infeasible	
4	$x_1 = 4$, $x_3 = 8$	12
5	Infeasible	
6	$x_2 = 16/3$, $x_4 = 16/3$	$21\dfrac{1}{3}$

Thus solution of set 2 is optimal with a profit of 22. In this way we can solve a LPP simply by employing the theorems stated above; but the simplex method is a further improvement over the trial and error method. Following three are the inefficiencies in the trial and error method.

11.3.3 Inefficiencies of Trial and Error Method :

(a) In a LPP where m and n are larger, solving of numerous sets of simultaneous equations would be extremely cumbersome and time-consuming.

(b) Scanning the profit table we notice that we jump from profit 0 to 22 to $21^1/_3$ *i.e.* there are ups and downs. The simplex method ensures that successive solutions yield progressively higher profits, culminating into the optimal one.

(c) Some of the sets yield infeasible solutions. There should be means to detect such sets and not to solve them at all.

11.4 THE SIMPLEX METHOD

The simplex method is a computational procedure - *an algorithm* - for solving linear programming problems. It is an iterative optimizing technique. In the *simplex process, we must first find an initial basis solution (extreme point). We then proceed to an adjacent extreme point. We continue moving from point to point until we reach an optimal solution.* For a maximization problem, the simplex method always moves in the direction of steepest ascent, thus ensuring that the *value of the objective function improves with each solution.*

The process is illustrated with the help of the following example:

Illustration

Maximize $Z = 3x_1 + 4x_2$

Subject to $2x_1 + 3x_2 \leq 16$ (machining time)

                   $4x_1 + 2x_2 \leq 16$ (assembly time)

                   $x_1 \geq 0, x_2 \geq 0$

Solution

Step 1. *Obtaining an initial solution:* Obtain an initial solution that satisfies all the constraints of the problem. The simplex process requires that all constraints be expressed as equations. Therefore we must convert all inequalities into equalities.

Consider the constraint $\qquad\qquad 2x_1 + 3x_2 \leq 16$

Value on the left hand side of the inequality represents the amount of machine time a particular solution uses, while the quantity on right side of inequality sign represents the total amount of machine time available. Let x_3 be a variable which represents the unused machine time in this solution so that

$$2x_1 + 3x_2 + x_3 = 16$$

Similarly, let x_4 represent the amount of assembly time that is available but not used so that

$$4x_1 + 2x_2 + x_4 = 16$$

The variables x_3 and x_4 are referred to as slack variables.

Thus slack variables represent the quantity of a resource not used by a particular solution, and they are necessary to convert the constraint inequalities to equalities.

As we proceed with the *simplex method,* it will be helpful to place certain information in a table known as a *simplex tableau.*

The simplex solution is as follows.

	1	2	3	4	5				6
	Fixed Ratio	Program (basic variables)	Profit Per unit	Qty.	3 x_1	4 x_2	0 x_3	0 x_4	Replacement Ratio
		x_3	0	16	2	3	1	0	16/3 ⟵
I	2/3	x_4	0	16	4	2	0	1	8
		NER			3	4	0	0	
					↑				
	1/4	x_2	4	16/3	2/3	1	1/3	0	8 ⟵
II		x_4	0	16/3	8/3	0	–2/3	1	2
		NER			1/3	0	–4/3	0	
					↑				
III		x_2	4	4	0	1	1/2	–1/4	
		x_1	3	2	1	0	–1/4	3/8	
		NER			0	0	–5/4	–1/8	

The main simplex table, aside from headings, is seen divided into 3 horizontal sub-tables, I to III. There are 6 main headings 1 to 6 as explained below for sub-table I.

Col. 1.   Fixed ratio—explanation deferred.

Col. 2.   Program —it contains the basic variables that are included in the solution. Initially, in sub-table 1, there are slack variables, this means we do not produce x_1 and x_2 products, thus the profit is zero.

Col. 3.   In the column labeled 'Profit' are listed the objective co-efficients of the basic variables that are included in the specific program. Thus, the profit co-efficients of x_3 and x_4 are all zero.

Col. 4.   In the column labelled 'QTY' are listed the values of the basic variables included in the solution. Since our initial program consists of 16 units of x_3, and 16 units of x_4, these values are listed in 'QTY' column. Any variable which is not listed under the 'Program' column is known as the *non-basic variable,* the value of each non-basic variable is zero.

The total profit contribution resulting from a specific program can be calculated by multiplying corresponding entries in the "profit per unit" column and the "quantity" column and adding the product. Thus, total profit contribution in our first program is 16(0) + 16(0) i.e. zero.

Col. 5.   This heading is divided in as many sub-headings as the number of real and slack variables. It indicates each of the variables with profit/unit as read from the objective function at its top. The body of the sub-table carries the co-efficients of the variables as read from the constraints. *That the co-efficients of the basic variables constitute a unit matrix in this initial sub-table is to be noted.* The numbers under the non basic variables represent substitution ratio. For example, the number 2 gives the rate of substitution between x_1, and x_3. If we wish to produce 1 unit of x_1, 2 unit x_3, must be sacrificed. The number 4 has similar interpretation. To produce 1 unit of x_1, 4 units of x_4 must be sacrificed. By using the same explanation, to produce 1 unit of x_2, we must sacrifice 3 units of x_3 and 2 units of x_4.

Col. 6.   Replacement ratio as discussed earlier.

The numbers in the *net evaluation* row (NER) $C_j - Z_j$ give the net effect of exchange between one unit of each variable and basic variable. They are always zero under the basic variables. Under the non-basic variables, they can be positive, negative or zero. *Thus numbers in NER, under each column represent the opportunity cost of not having one unit of the respective column variable in the solution, in other words, the number represent the potential improvement in the objective function that will result by introducing into the program one unit of the respective column variable.*

11.4.1   Test for Optimality : In so far as the total profit contribution (obtained by multiplying corresponding entries in the 'Profit' column and 'QTY' column and adding the product) resulting from the initial program is zero, it can be improved and hence is not optimal program. **In maximization problem, the program is optimal if each C_j-Z_j is either zero or negative.**

Computational aspects for the derivation of improved solutions are explained step by step below.

Step 1. Select the incoming variable in sub-table I in the NER. The NER entries in the first sub-table are simply the profit unit figures as read from the objective function or copied from headings 5. *However there is a more rigorous method of making these entries which alone has to be followed in making entries in the subsequent NER's and explanation on which is deferred.* Selection of the incoming variable is simple indeed. **The positive $(C_j{-}Z_j)$ in NER indicate the magnitude of opportunity cost of not including 1 unit of variables x_1 and x_2 respectively at this stage. We select the one with highest entry in NER as incoming variable** and this happens to be x_2, with a value of 4. We note this selection by putting an arrow below 4 in the NER of sub-table I. The column with the arrow is known as the **key column.**

Step 2. Select the outgoing variable, for this we have to calculate how many units of x_2 can be brought in without exceeding the existing capacity of any one of the resources? Thus, we must calculate the maximum allowable number of units of x_2, that can be brought into the program without violating the non-negative constraints. For this, we compute the replacement ratio of col. 6. This is done by dividing col. 4 by the key column and we get 16/3 and 8 and as the ratio. The variable x_3, against the *least ratio* of 16/3 is selected as the outgoing variable and the fact is noted by putting an arrow against the least ratio of 16/3. This is maximum quantity of x_2 that can be produced at this stage without violating the non negative constraint. This row with the arrow is called the **key row.** The element at the intersection of the key row and the key column is known as the **pivot** or key element and it is encircled.

Step 3. Having noted the incoming variable (x_2) and the outgoing variable (x_3) we are ready to perform the row operations on sub-table-I and fill in columns 4 and 5 of sub-table II.

Before that however, we fill in cols. 2 and 3 of sub-table II which is straight forward; x_3 is replaced by x_2 in col. 2, x_4 stays in it and in col. 3 the profit/unit figure corresponding to x_2 is copied from heading while other figures remain the same.

11.4.2 Transformation of key row : The rule for transforming the key row is: *Divide all the numbers in the key row by the key number. The resulting numbers form the corresponding row in the next table.*

The key row of sub-table-I (under col. 4 and 5) is divided by the pivot element and this becomes the corresponding row of the sub-table-II. It reads 16/3, 2/3, 1, 1/3, 0

11.4.3 Transformation of the non-key rows : *The rule for transforming a non-key row is: Subtract from the old row number (in each column) the product of the corresponding key-row number and the corresponding fixed ratio* formed by dividing the old row number in the key column by the key number. The result will give the corresponding new row number.

This rule can be placed in the following equation form:

$$\text{Fixed ration} = \frac{\text{Old row number in key column}}{\text{key number}}$$

These are entered in the non key row under column I in the sub-table-I itself.

New row number = old row number

– (corresponding number in key row × corresponding fixed ratio)

The key row in sub-table-I is multiplied by the fixed ratio of the non key row. This leads to

$$16 \times \frac{2}{3} = \frac{32}{3}, \quad 2 \times \frac{2}{3} = \frac{4}{3}, \quad 3 \times \frac{2}{3} = 2, \quad 1 \times \frac{2}{3} = \frac{2}{3}, \quad 0 \times \frac{2}{3} = 0$$

This, however, is a rough work. Entries are not made any where. The result of this multiplication are subtracted from the non-key row of sub-table-I to yield the non key row of the sub-table-II as below:

$$16 - \frac{32}{3} = \frac{16}{3}, \quad 4 - \frac{4}{3} = \frac{8}{3}, \quad 2 - 2 = 0, \quad 0 - \frac{2}{3} = -\frac{2}{3}, \quad 1 - 0 = 1$$

These entries are made for the non-key row of sub-table-II.

Step IV. This consists of deriving the NER of sub-table-II. Each of its elements is derived by multiplying col. 3 of sub-table-II with col. 5, summing these up and subtracting the sum from the profit/unit is heading 5. Computations for each element in the 2nd NER are shown below:

$$3 - \left(4 \times \frac{2}{3} + 0 \times \frac{8}{3} \right) = \frac{1}{3}$$

$$4 - (4 \times 1 + 0 \times 0) = 0$$

$$0 - \left(4 \times \frac{1}{3} + 0 \times \frac{-2}{3} \right) = -\frac{4}{3}$$

$$0 - (4 \times 0 + 0 \times 1) = 0$$

But what is the underlying logic? The constraints in the sub-table – II have become

$$\frac{2}{3} x_1 + 1 x_2 + \frac{1}{3} x_3 + 0 x_4 = \frac{16}{3}$$

$$\frac{8}{3} x_1 + 0 x_2 - \frac{2}{3} x_3 + 1 x_4 = \frac{16}{3}$$

i.e.

$$x_2 = \frac{16}{3} - \frac{2}{3} x_1 - \frac{1}{3} x_3 \qquad (i)$$

$$x_4 = \frac{16}{3} - \frac{8}{3} x_1 + \frac{2}{3} x_3 \qquad (ii)$$

Substituting these values of x_2 and x_4 in the original objective function

$$Z = 3x_1 + 4x_2 + 0x_3 + 0x_4$$

We get the following modified function,

$$Z = 3x_1 + 4 \left\{ \frac{16}{3} - \frac{2}{3}x_1 - \frac{1}{3}x_3 \right\} + 0x_3 + 0 \left\{ \frac{16}{3} - \frac{8}{3}x_1 + \frac{2}{3}x_3 \right\}$$

$$= x_1 \left\{ 3 - \frac{2}{3} \times 4 - \frac{8}{3} \times 0 \right\} + 0x_2 + x_3 \left\{ 0 - 4 \times \frac{1}{3} + 0 \times \frac{2}{3} \right\} + 0x_4 + 4 \times \frac{16}{3}$$

i.e., $$Z = \frac{1}{3} x_1 + 0x_2 - \frac{4}{3} x_3 + 0x_4 + \frac{64}{3}$$

These co-efficients have been entered in the 2nd NER

[This might appear to be mathematical jugglery but there is a sound interpretation to this.]

The reason for this becomes clear if we keep in mind that $\frac{1}{3} x_1 + 0x_2 - \frac{4}{3} x_3 + 0x_4 + \frac{64}{3}$ becomes the modified objective function corresponding to the solution stage given by the program 2 i.e. , sub-table-II, that has fully utilised the machining capacity. Thus the production of 1 unit of X_1 at this stage will mean (according to the 1st original constraint $2x_1 + 3x_2 \leq 16$) that 2 units of machining capacity (required for producing 1 unit of X_1) must be reallocated from X_2. Since each unit of X_2 requires 3 units of machining capacity, current level of production will thus be reduced by $\frac{2}{3}$ unit which will mean reduction of Rs. $\frac{8}{3}$ ($= 4 \times \frac{2}{3}$) in the contribution. Thus, introduction of 1 unit of X_1 at this stage results in a decrease of Rs. $\frac{8}{3}$ in the profit contribution of X_2. Hence the introduction of 1 unit X_1 at this stage will give a net profit i.e., of $3 - \frac{8}{3}$ = Rs. $\frac{1}{3}$ Similar interpretation can be made for the co-efficients in the modified objective function].

By applying these 4 steps on sub-table-II we derive sub-table-III. Since its NER does not have any positive element i.e. opportunity cost, we stop here. This is an indication of the optimal solution with $x_1=2$ and $x_2 = 4$ (and Z = 22), for if we derive another sub-table by bringing in x_3 or x_4 into the solution we shall decrease the profit. To sum up, when, in a maximization problem, all the elements in an NER are negative or zero we stop further iterations. The student may himself derive sub-table-III as an exercise.

It is obvious that the student needs a plenty of practice to be thorough with the computationally tedious though otherwise very elegant and versatile simplex method. *As a tip*, the *fractions in the simplex tables must be retained and by trying to decimalise them the student may land himself in trouble.* Numerous examples and exercises follow to give understanding and practice to the students.

Illustration

An electronics firm is undecided as to the most profitable mix for its products. The products now manufactured are transistors, resistors and carbon tubes with a profit (per 100 unit) of Rs. 10, Rs.

6 and Rs. 4 respectively. To produce a shipment of transistors containing 100 units requires 1 hour of engineering, 10 hours of direct labours and 2 hours of administrative service. To produce 100 resistors are required 1 hour, 4 hours and 2 hours of engineering, direct labour and administrative time respectively. To produce on shipment of the tubes (100 units) requires 1 hour of engineering, 5 hours of direct labour and 6 hours of administration. There are 100 hours of engineering services available, 600 hours of direct labour and 300 hours of administration. What is the most profitable mix?

Solution

Let us tabulate the data in convenient manner.

	Products			
	Transistors	Resistors	Carbon Tubes	Availability
Engineering	1	1	1	100
Labour	10	4	5	600
Admnstraton	2	2	6	300
Profit (per 100 units) (Rs.)	10	6	4	

Symbolically, Maximize $10x + 6y + 4z$

here x, y, z represents units of Transistors, Resistors and Carbon tubes (in 100 units)

Subject to $1x + 1y + 1z \leq 100$ and $x \geq 0$

$10x + 4x + 5z \leq 600$ and $y \geq 0$

$2x + 2y + 6z \leq 300$ and $z \geq 0$

Introducing slack variables

Maximize $10x + 6y + 4z$

Subject to $x + y + z + S_1 = 100$

$10x + 4y + 5z + S_2 = 600$

$2x + 2y + 6z + S_3 = 300$

$x \geq 0, y \geq 0, z \geq 0, S_1 \geq 0, S_2 \geq 0, S_3 \geq 0$

	Program	Profit/ Unit	Qty.	10 x	6 y	4 z	0 S_1	0 S_2	0 S_3	Replacement Ratio
F.R = 1/10	S_1	0	100	1	1	1	1	0	0	100
	S_2	0	600	10	4	5	0	1	0	60←
1/5	S_3	0	300	2	2	6	0	0	1	150
	Net Evaluation Row			10	6	4	0	0	0	

↑

					X_j				
S_1	0	40	0	6/10	5/10	1	–1/10	0	400/6=67
F.R =2/3 x	10	60	1	4/10	5/10	0	1/10	0	600/4=150
F.R.=2 S_3	0	180	0	12/10	5	0	–2/10	1	150
Net Evaluation Row		0	2	–1	0	–1	0		

↑

y	6	400/6	0	1	5/6	10/6	–1/6	0
x	10	100/3	1	0	1/6	–2/3	1/6	0
S_3	0	100	0	0	4	–2	0	1
Net Evaluation Row		0	0	– ve	–ve	–ve	0	

Hence the most profitable mix is y = $\frac{400}{6}$ Resistors and x = $\frac{100}{3}$ Transistors. The maximum profit is $400 + \frac{1000}{3} = 733\frac{1}{3}$

11.5 SIMPLEX METHOD FOR MINIMIZATION PROBLEMS

The simplex algorithm applies to both maximization and minimization problems. The only difference in the algorithm involves the selection of the incoming variable. In the maximization problem it is the one with highest +ve NER element. Conversely, it is the most – ve variable that is selected as the incoming variable in a minimization problem. And if all elements in the NER are either positive or zero, it is the indication for the optimal solution and we stop there.

Let us take an example straightway to explain the solution procedure.

Illustration

A small township of 15,000 people requires, on the average, 300,000 gallons of water daily. The city is supplied water from a central water-works where the water is purified by such conventional methods as filteration and chlorination. In addition, two different chemical compounds : (i) softening chemical and (ii) health chemical, are needed for softening the water and for health purposes. The waterworks plans to purchase two popular brands that contain these chemicals. One unit of Chemico Corporation's product gives 8 pounds of softening chemical and 3 pounds of health chemical. One unit of Indian Chemical's product contains 4 pounds and 9 pounds per unit, respectively, for the same purposes.

To maintain the water at a minimum level of softness and to meet a minimum programme of health protection, experts have decided that 150 and 100 pounds of the two chemicals that make up each product must be added to water daily. At a cost of Rs. 8 and Rs. 10 per   unit respectively for Chemico's and Indian Chemical's products, what is the optimal quantity of each product that should be used to meet the minimum level of softness and minimum health stand-ard?

Solution

The relevant data may be tabulated as below :

	Brand		
Chemical	Chemico	Indian	Daily Requirement
(i) Softening	8	4	150
(ii) Health	3	9	100
Cost/Unit of each brand (Rs.)	8	10	

Let us formulate the LPP as below

Minimize (cost), $Z = 8x + 10y$

Subject to

$$8x + 4y \geq 150$$

$$3x + 9y \geq 100$$

and $\qquad x \geq 0, y \geq 0$

Here x and y represents units of softening and health chemical units respectively.

Subtracting surplus variables to convert inequalities into equalities.

$$8x + 4y - S_1 = 150$$

$$3x + 9y - S_2 = 100$$

11.5.1 Physical Interpretation of Surplus Variables : Let S_1 and S_2 represent the extra units (if any) of the two chemicals over 150 and 100 units respectively. Their magnitude being such as to just convert the inequalities into equalities, thereby restricting them to non-negative value. The problem may then be restated as below:

Minimise (cost) Z = $\qquad$ $8x + 10y + 0S_1 + 0S_2$

Subject to $\qquad$ $8x + 4y - S_1 = 150$

$\qquad$ $3x + 9y - S_2 = 100$

and $\qquad$ $S_1 \geq 0, S_2 \geq 0, x \geq 0, y \geq 0.$

11.5.2 Artificial Slack Variables : If x and y are set equal to zero S_1 and S_2 turn out to be negative, violating the non-negativity restriction. Therefore, to circumvent this, we introduce another similar device of artificial slack variables. Let us represent these by A_1 and A_2 respectively. The problem then becomes :

Minimize cost Z = $8x + 10y + 0S_1 + 0S_2 + MA_1 + MA_2$

Subject to $\qquad$ $\begin{cases} 8x + 4y - S_1 + A_1 = 150 \\ 3x + 9y - S_2 + A_2 = 100 \end{cases}$ $\qquad$ All variables ≥ 0

Physical Interpretation of the Artificial Variables.

These are imaginary brands, each unit containing 1 unit of the pertinent chemical. Both are restricted to non-negatives. Whereas surplus variables have zeros as their cost co-efficient, each artificial variable is assigned an infinitely large cost co-efficient (usually denoted by M).

Uses of artificial variables

1. It is our basic assumption that none of the basic variables in the L.P. problem can have a negative value. Thus, artificial variable is added to act as basic variable in a particular equation and hence it avoids the possibility of getting negative values for basic variables.

2. These artificial variables are such that their objective function co-efficients impose a huge and hence unacceptable penalty. In the case of maximization problem, the objective function is modified by subtracting a quantity (MA_1) where M is arbitrarily large value and A_1 is the artificial variable. For minimization problem, the objective function is modified by adding the quantity MA_1. Thus artificial variables enable us to make a convenient and correct start in obtaining an initial solution.

3. The artificial variables with the high penalty once replaced by a real variable will never enter the optimal program. Hence the solution to the modified problem will give the optimal solution to the original problem.

11.5.3 Designing the Initial Program : It is obtained by letting each of the variables x, y, S_1 and S_2 assume value of zero, *i.e.,* retaining artificial variables in the initial solution.

Program	Cost per Unit	Qty.	8	10	0	0	M	M	Replacement Ratio
			x	y	S_1	S_2	A_1	A_2	
A_1	M	150	8	4	-1	0	1	0	150/4
A_2	M	100	3	9	0	-1	0	1	100/9 ←
Net Evaluation Row			8–11M	10–13M	M	M	0	0	

↑

11.5.4 Initial Solution : It is to be noted that in the NER we do not copy the co-efficients from the cost objective but derive them by step IV of Section 14.4.3 as below :

$$
\begin{array}{ll}
x & 8{-}8M - 3M = 8{-}11M \\
y & 10{-}4M - 9M = 10{-}13M \\
S_1 & 0 + M - 0 = M \\
S_2 & 0 - 0 + M = M \qquad\qquad (A) \\
A_1 & M - M - 0 = 0 \\
A_2 & M - 0 - M = 0
\end{array}
$$

The reason for this is explained below. The problem is rewritten below:

Minimize $Z = 8x + 10y + 0S_1 + 0S_2 + MA_1 + MA_2$

Subject to $8x + 4y - S_1 + A_1 = 150$ ... (i)

                          $3x + 9y - S_2 + A_2 = 100$ ... (ii)

                          All variables ≥ 0.

From (i) $A_1 = 150 - 8x - 4y + S_1$

From (ii) $A_2 = 100 - 3x - 9y + S_2$

Substituting these in the objective function,

Z $= 8x + 10y + 0S_1 + 0S_2 + M(150 - 8x - 4y + S_1) + M(100 - 3x - 9y + S_2)$

              $= (8 - 11M)x + (10 - 13M)y + MS_1 + MS_2 + 0A_1 + 0A_2$.

These co-efficients correspond to set A above.

Now consider the initial solution with an initial cost of $(150 + 100)M$ for further inspiration.

$10 - 13M$ is the most – ve and, therefore, the column, under 'y' is the key column. Also, the ratio $100/9$ is less than $150/4$ (on extreme right). The pivot element, then, is 9. The outgoing variable is A_2 being replaced by y. The cost of the existing solution, $250M$ is forbiddingly high. The stage is set for revising the initial program.

The key row is revised by dividing it through by 9, the pivot element and the results are as follows:

$$\frac{100}{3}, \frac{3}{9}, 1, 0, \frac{-1}{9}, 0, \frac{1}{9}.$$

The fixed ratio for revising the non-key row is 4/9. The revised figures for this row are computed below:

$$150 - 100 \times \frac{4}{9} = \frac{1350 - 400}{9} = \frac{950}{9}$$

$$8 - 3 \times \frac{4}{9} = \frac{72 - 12}{9} = \frac{6}{9}$$

$$4 - 9 \times \frac{4}{9} = 0$$

$$-1 - 0 = -1$$

$$0 + \frac{4}{9} = \frac{4}{9}$$

$$1 - 0 = 1$$

$$0 - 1 \times \frac{4}{9} = \frac{-4}{9}$$

The new table is shown below:

Prog ram	Cost/ Unit	Qty.	8 x	10 y	0 S_1	0 S_2	M A_1	M A_2	
A_1	M	950/9	60/9	0	−1	4/9	1	−4/9	950/60←
y	10	100/9	3/9	1	0	−1/9	0	1/9	100/3
Net Evaluation Row			42/9− 60M/9	0	M	10/9− 4M/9	0	13M−10 9	

↑

Revising the key row by dividing it through by $\dfrac{60}{9}$ *i.e.* multiply by $\dfrac{9}{60}$, we get

$$\frac{950}{60},\ 1,\ 0,\ \frac{-9}{60},\ \frac{4}{60},\ \frac{9}{60},\ \frac{-4}{60}$$

For the non-key row the fixed ratio is 3/9 ¸ 60/9 = 3/60. Revising the non-key row.

$$\frac{100}{9} - \frac{950}{9} \times \frac{3}{60} = \frac{35}{6}$$

$$\frac{3}{9} - \frac{60}{9} \times \frac{3}{60} = 0$$

$$1 - \frac{3}{60} \times 0 = 1$$

$$0 + 1 \times \frac{3}{60} = \frac{1}{20}$$

$$-\frac{1}{9} - \frac{4}{9} \times \frac{3}{60} = \frac{-2}{15}$$

$$0 - \frac{3}{60} = -\frac{1}{20}$$

$$\frac{1}{9} + \frac{4}{9} \times \frac{3}{60} = \frac{2}{15}$$

The result are tabulated below:

Prog ram	Cost/ Unit	Qty.	8 x	10 y	0 S_1	0 S_2	M A_1	M A_2
x	8	95/6	1	0	–3/20	1/15	3/20	–1/15
y	10	35/6	0	1	1/20	–2/15	–1/20	2/15
NER (all non-negative)			0	0	7/10	4/5	M– 7/10	M–4/5

Hence the optimal cost is $\dfrac{95}{6} \times 8 + \dfrac{35}{6} \times 10$

$$= \frac{760 + 350}{3}$$

$$= \frac{1110}{6} = 185 \ \textbf{Answer.}$$

11.6   MARGINAL VALUE OF A RESOURCE

Let us consider the first example under section 15.4 once again to explain this concept. In the said example there are two resources machining and assembly. The optimal tableau of that example is reproduced below. We see in its NER -5/4 and –1/8 below x_3 and x_4. The marginal value of the machining and assembly resources are then respectively 5/4 and 1/8. The implication of this is that if now (at the optimal stage) we wish to bring in x_3 in the solution, the total profit will be reduced from 22 (the optimal) by 5/4 times the no. of units of x_3 brought in the program. This is demonstrated below where we bring x_3 in this program.

Optimal table of first Example in Section 15.4 follows (rest omitted). x_3 is brought in the next to optimal table.

				x_1	x_2	x_3	x_4	
	x_2	4	4	0	1	1/2	–1/4	8 ←
1/2	x_1	3	2	1	0	–1/4	3/8	–8
Optimal	NER	22*		0	0	–5/4	–1/8	Bring x_3 in solution
	x_3	0	8	0	2	1	–1/2	
	x_1	3	4	1	1/2	0	1/4	
		12*	0	0	5/2	0	–3/4	

*Profit has been reduced from 22 to 12 because 8 units of x_3 have been brought in the new solution, thereby the net reduction of profit 10 naturally equals 8 × 5/4.

Please note that marginal value of resource is synonymous with *opportunity cost or shadow price.*

11.7 SOME REMARKS

1.   It may be desired to convert a maximization problem into a minimization one and *vice versa.* Mathematically, this can be accomplished by reversing signs though of just the objective function.

2.   *Inequalities in the wrong direction:* Consider the problem:

Maximize $Z = x_1 + 5x_2$

Subject to $3x_1 + 4x_2 \leq 6$ ... (i)

$x_1 + 3x_2 \geq 2$ ... (ii)

$x_1, x_2 \geq 0$

[Whether to introduce slack or surplus or artificial variables depends   on the type of inequality and has nothing to do with type of the problem i.e., maximization or minimization].

The 2nd inequality is in the *wrong* direction. Upon introducing the "surplus" variable.

$x_1 + 3x_2 - S_2 = 2$

If S_2 is taken in the initial solution it would be – ve when x_1 and x_2 are zero. To circumvent this, an artificial variable is also introduced in this inequality. The problem becomes:

Maximize $Z = x_1 + 5x_2 - MA_2$

Subject to $3x_1 + 4x_2 + S_1 = 6$

$x_1 + 3x_2 - S_2 + A_2 = 2$

$x_1, x_2, S_1, S_2, A_2 \geq 0$

(Note that in maximization problems M always has -ve sign and in minimization problems M always has a + ve sign in the objective function).

The initial solution consists of S_1 and A_2. Several examples on inequalities in the wrong direction follow. Surplus variables can never come in initial solution.

3.   Any linear programming problem can be re-formulated into what is known as its dual. Any of the primal (the original) or the dual may be selected for iterating by the simplex method. The selection is made on the basis of computational burden. Also the dual provides interesting insights into the methodology of the LP solution. This matter is discussed in a following section at a greater length.

4.   If two or more variables share the maximum positive co-efficient in the net evaluation row any one may be chosen for introduction for the new solution arbitrarily,   *viz.,* in $Z = 2x_1 + 2x_2 + x_3$ it matters little if x_1 or x_2 is chosen.

5.   Lower bounds may be specified in an LPP. For example, over and above to the three usual constraints, it may be stipulated that x_1 cannot be less than 25 or 40 or l_1.i.e., $x_1 \geq l_1$. This can

be handled quite easily by introducing a variable y_1 such that $x_1 = l_1 + y_1$. Substitute $x_1 = l_1 + y_1$ wherever it occurs and solve the LPP. Computations would be greatly reduced. Please see illustration on page no. 15.32 in this connection.

6. In all the simplex tables there is bound to be a unit matrix of size $p \times p$ where p is the no. of rows (excluding net evaluation row). The columns that constitute such a unit matrix need not be adjacent.

7. In view of the tediousness of computational aspects it is useful to make a check at each iteration. This can be done by deriving the net evaluation row in two ways. (i) just like any other row in the simplex tableau by deriving its fixed ratio (ii) by summing the product of the quantities column with the profit/cost column and subtracting this sum from the original profit contribution or cost co-efficient of variable. These should tally. Also, having obtained the optimal solution it is desirable to verify it if it obeys the given constraints.

8. The simplex method, the graphical and trial and error methods, the dual approach provide several ways of doing an LPP. The student may want to do each LPP in more than one way for the sake of verification of the answer and practice.

Illustration

A manufacturer of three products tries to follow a policy of producing those which contribute most to fixed cost and profit. However, there is also a policy of recognising certain minimum sales requirements. Currently these are:

Product	Unit per week
X	20
Y	30
Z	60

There are three producing departments. The product time in hour per unit in each department and the total times available for each week in each department are:

		Time required per product, in hours			*Total hours available*
Product		X	Y	Z	
Department	1	0.25	0.20	0.15	420
	2	0.30	0.40	0.50	1,048
	3	0.25	0.30	0.25	529

The contribution per unit of product X,Y, Z is Rs. 10.50, Rs. 9.00 and Rs. 8.00 respectively. The company has scheduled 20 units of X, 30 units of Y and 60 units of Z for production in the following week.

You are required to state:

(a) Whether the present schedule is an optimum one from a profit point of view and if it is not, what it should be;

(b) The recommendations that should be made to the firm about their production facilities (following the answer to (a) above).

Solution

Let us formulate the L.P.P.

Objective: Maximize $10.50X + 9.00Y + 8.00Z$

Subject to $0.25X + 0.20Y + 0.15Z \leq 420$

$0.30X + 0.40Y + 0.50Y \leq 1048$

$0.25X + 0.30Y + 0.25Z \leq 529$

$0 \leq X \geq 20$ $0 \leq X$ means

$0 \leq Y \geq 30$ $X \ {}^3 \ 0$, etc.

$0 \leq Z \geq 60$

Since the company is already producing minimum of Y and Z it should, at least, produce maximum of X limited by the first constraint. It can be arithmetically verified that there is a scope for improvement.

Substituting (Please refer to point 5 in Section 15.7) :

$X = x_1 + 20$

$Y = x_2 + 30$

$Z = x_3 + 60$

This problem   becomes

Maximize $10.50x_1 + 9.00\ x_2 + 8.00\ x_3 +$ a constant

Subject   to $0.25\ x_1 + 0.20x_2 + 0.15x_3 \leq 400$

$0.30\ x_1 + 0.40x_2 + 0.50x_3 \leq 1000$

$0.25\ x_1 + 0.30x_2 + 0.25x_3 \leq 500$

$x_1 \geq 0, \ x_2 \geq 0, \ x_3 \geq 0$

The student may now derive the optimal solution, as an exercise.

◆ **Miscellaneous Illustrations**

Illustration

The owner of Fancy Goods Shop is interested to determine, how many advertisements to release in the selected three magazines A,B and C. His main purpose is to advertise in such a way that exposure to principal buyers of his goods is maximized. Percentages of readers for each magazine are known. Exposure in any particular magazine is the number of advertisements released multiplied by the number of principal buyers. The following data are available:

	Magazines		
Particulars	*A*	*B*	*C*
Readers	1.0 lakh	0.6 lakh	0.4 lakh
Principal Buyers	20%	15%	8%
Cost per Advertisement	8,000	6,000	5,000

The budgeted amount is at the most 1.0 lakh for the advertisement. The owner has already decided that magazine A should have no more than 15 advertisements and that B and C each gets at least 8 advertisements. Formulate a Linear Programming Model for this problem.

Solution

Let x_1, x_2 and x_3 denote the number of advertisements to be released in three magazines A, B and C respectively. Let Z denote the total exposure to the principal buyers of the goods.

Since the exposure in any magazine is the number of advertisements multiplied by the number of principal buyers, therefore, the value of Z is given by :

$Z = (0.20 \times 1,00,000) \, x_1 + (0.15 \times 60,000) \, x_2 + (.08 \times 40,000) \, x_3$

$= 20,000 \, x_1 + 9000 \, x_2 + 3,200 \, x_3$

The owner of Fancy Goods shop wishes to maximise the total exposure to principal buyers, hence the objective function is given by

Maximize $Z = 20,000 \, x_1 + 9,000 \, x_2 + 3,200 \, x_3$

subject to the following constraints :

(i)   The budgeted amount for the advertisement is at the most Rs. 1,00,000.

Hence, $8,000 \, x_1 + 6,000 \, x_2 + 5,000 \, x_3 \leq 1,00,000$.

(ii) Also, the magazine A should have no more than 15 advertisements, B and C each should get at least 8 advertisements.

Hence, $x_1 \leq 15$

and $x_2 \geq 8, \quad x_3 \geq 8$

Thus, the linear programming model for the problem is

Maximise $Z = 20,000 \, x_1 + 9,000 \, x_2 + 3,200 \, x_3$

subject to

$8,000 \, x_1 + 6,000 \, x_2 + 5,000 \, x_3 \leq 1,00,000$

$x_1 \leq 15, \quad x_2 \geq 8, \quad x_3 \geq 8$

Where x_1, x_2, and $x_3 \geq 0$.

Illustration

A Mutual Fund Company has Rs. 20 lakhs available for investment   in Government Bonds, blue chip stocks, speculative stocks and short-term deposits. The annual expected return and risk factor are given below:

Type of investment	Annual Expected Return (%)	Risk Factor (0 to 100)
Government Bonds	14	12
Blue Chip Stocks	19	24
Speculative Stocks	23	48
Short-term Deposits	12	6

Mutual fund is required to keep at least Rs. 2 lakhs in short-term deposits and not to exceed average risk factor of 42. Speculative stocks must be at most 20 percent of the total amount invested. How should mutual fund invest the funds so as to maximize its total expected annual return? Formulate this as a Linear Programming Problem. Do not solve it.

Solution

Let x_1, x_2, x_3 and x_4 denote the amount of funds to be invested in government bonds, blue chip stocks, speculative stocks and short term deposits respectively. Let Z denote the total expected return.

Since the Mutual Fund Company has Rs. 20 lakhs available for investment,

$$x_1 + x_2 + x_3 + x_4 \leq 20,00,000 \qquad \text{... (i)}$$

Also, Mutual fund is required to keep at least Rs. 2 lakhs in short-term deposits,

$$\text{Hence, } x_4 \geq 2,00,000 \qquad \text{...(ii)}$$

The average risk factor is given by

$$\frac{12x_1 + 24x_2 + 48x_3 + 6x_4}{x_1 + x_2 + x_3 + x_4}$$

Since the average risk factor Mutual Fund should not exceed 42, we get the following constraint.

$$\frac{12x_1 + 24x_2 + 48x_3 + 6x_4}{x_1 + x_2 + x_3 + x_4} \leq 42$$

or $12x_1 + 24x_2 + 48x_3 + 6x_4 \leq 42 (x_1 + x_2 + x_3 + x_4)$

or $-30x_1 - 18x_2 + 6x_3 - 36x_4 \leq 0$ \qquad ...(iii)

Further, speculative stock must be at most 20 per cent of the total amount invested, hence

$$x_3 \leq 0.20 \, (x_1 + x_2 + x_3 + x_4)$$

or $-0.2x_1 - 0.2 \, x_2 - 0.8x_3 - 0.2x_4 \leq 0$ ...(iv)

Finally, the objective is to maximise the total expected annual return, the objective function for Mutual Fund can be expressed as

Maximise $Z = 0.14x_1 + 0.19x_2 + 0.23x_3 + 0.12x_4$ ...(v)

Summarising equations (i) to (v), the linear programming model for the Mutual Fund company is formulated as below:

Objective function :

Maximize $Z = 0.14x_1 + 0.19x_2 + 0.23x_3 + 0.12x_4$

Subject to the constraints

$$x_1 + x_2 + x_3 + x_4 \leq 20,00,000$$

$$x_4 \geq 2,00,000$$

$$-30x_1 - 18x_2 + 6x_3 - 36x_4 \leq 0$$

$$-0.2x_1 - 0.2x_2 + 0.8x_3 - 0.2x_4 \leq 0$$

where $x_1 \geq 0, \; x_2 \geq 0, \; x_3 \geq 0$ and $x_4 \geq 0$

Illustration

A firm produces three products A, B and C. It uses two types of raw materials I and II of which 5,000 and 7,500 units respectively are available. The raw material requirements per unit of the products are given below:

Raw Material	Requirements per unit of Product		
	A	B	C
I	3	4	5
II	5	3	5

The labour time for each unit of product A is twice that of product B and three times that of product C. The entire labour force of the firm can produce the equivalent of 3,000 units. The (marks) minimum demand of the three products is 600, 650 and 500 units respectively. Also, the ratios of the number of units produced must be equal to 2 : 3 : 4. Assuming the profits per unit of A,B and C as Rs. 50, 50 and 80 respectively.

Formulate the problem as a linear programming model in order to determine the number of units of each product which will maximize the profit.

Solution

Let the firm produce x_1 units of product A, x_2 units of products B and x_3 units of product C.

The profit per unit of products A, B and C is Rs. 50, and Rs. 80 respectively. Since the objective of the firm is to maximize the profit, therefore, the objective function is given by

Maximise $Z = 50x_1 + 50x_2 + 80x_3$

The firm uses two types of raw materials I and II of which 5,000 and 7,500 units respectively are available. As per the given data, the raw material constraints can be formulated as given below:-

$$3x_1 + 4x_2 + 5x_3 \le 5{,}000 \qquad\qquad\qquad ...(i)$$

and $\quad 5x_1 + 3x_2 + 5x_3 \le 7{,}500 \qquad\qquad\qquad ...(ii)$

The labour time for each unit of product A is twice that of product B and three times that of product C. Also the entire labour force can produce the equivalent of 3000 units.

$$\therefore \;\; x_1 + \frac{x_2}{2} + \frac{x_3}{3} \le 3000$$

or $\quad 6x_1 + 3x_2 + 2x_3 \le 18000 \qquad\qquad\qquad ...(iii)$

The minimum demand of the three products is 600, 650 and 500 units respectively.

Hence, $x_1 \ge 600$, $x_2 \ge 650$ and $x_3 \ge 500 \qquad\qquad\qquad ...(iv)$

Since the ratios of the number of units produced must be equal to 2 : 3 : 4, therefore,

$\frac{1}{2} x_1 = \frac{1}{3} x_2$, and $\frac{1}{3} x_2 = \frac{1}{4} x_3$

or $3x_1 = 2x_2$ and $4x_2 = 3x_3 \qquad\qquad\qquad ...(v)$

The linear programming model can be formulated as follows:

Maximise $Z = 50x_1 + 50x_2 + 80x_3$

Subject to the constraints :

$$3x_1 + 4x_2 + 5x_3 \le 5000$$

$$5x_1 + 3x_2 + 5x_3 \le 7500$$

$$6x_1 + 3x_2 + 2x_3 \le 18000$$

$$3x_1 = 2x_2 \text{ and } 4x_2 = 3x_3$$

$$x_1 \ge 600, \; x_2 \ge 650 \text{ and } x_3 \ge 500.$$

11.8 PRACTICAL APPLICATIONS OF LINEAR PROGRAMMING

In addition to its wide use in industrial and administrative applications, linear programming has extensive application to agricultural, aircraft and several military problems. These are briefly discussed below. The student need not memories the numerical examples. They are given merely to enhance understanding of applications. However, he is advised to keenly study the formulation of various LPPs given as examples.

Illustration

For a company engaged in the manufacture of three products viz., X, Y and Z, the available data are give in Tables 1, 2 and 3 below:

Table 1 : Minimum sales requirements

Product	Minimum Sales Requirements per month
X	10
Y	20
Z	30

Table 2 : Operations, Required Processing Times and Capacity

Operations	Time (Hrs) required per item of			Total available hours per month
	X	Y	Z	
1	1	2	2	200
2	2	1	1	220
3	3	1	2	180

Table 3 :   Profit   (Rs.) per unit

Product	Profit (Rs.)/Unit
X	10
Y	15
Z	8

Find out the product   mix to maximize profit.

Solution

Let x, y and z denote the number of units produced per month for the products X, Y and Z respectively.

Minimum sales requirements give following constraints :

$$x \geq 10$$
$$y \geq 20 \qquad \text{where x, y, z} \geq 0$$
$$z \geq 30$$

Operations, processing times and capacity lead to following constraints:

$$x + 2y + 2z \leq 200$$
$$2x + y + z \leq 220$$
$$3x + y + 2z \leq 180$$

The objective function is

Maximize $10x + 15y + 8z$

Thus, we have to solve the following problem

Maximize $P = 10x + 15y + 8z$

Subject to $x + 2y + 2z \leq 200$ .... (i)

$\qquad\qquad 2x + y + z \leq 220$ .... (ii)

$\qquad\qquad 3x + y + 2z \leq 180$ .... (iii)

$\qquad\qquad x \geq 10$ .... (iv)

$\qquad\qquad y \geq 20$ .... (v)

$\qquad\qquad z \geq 30$ .... (vi)

$\qquad\qquad x, y, z \geq 0.$

Let us make following substitutions

$$x = a + 10$$
$$y = b + 20$$
$$z = c + 30$$

Where a, b, c $\geq$ 0.

Substituting these values in the objective function and constraints (i), (ii), & (iii), the problem becomes

Maximize $P = 10a + 15b + 8c + 640$

subject to $(a + 10) + 2(b + 20) + 2(c + 30) \leq 200$

$\qquad\qquad\qquad 2(a + 10) + (b + 20) + (c + 30) \leq 220$

$\qquad\qquad\qquad 3(a + 10) + (b + 20) + 2(c + 30) \leq 180$

where a,b,c $\geq$ 0

or   Maximize P = $10a + 15b + 8c + 640$

subject to $\quad a + 2b + 2c \leq 90$

$\qquad 2a + b + c \leq 150$

$\qquad 3a + b + 2c \leq 70$

where $a,b,c \geq 0$.

Adding slack variables S_1, S_2, and S_3, we get

Maximize $P = 10a + 15b + 8c + 640 + 0S_1 + 0S_2 + 0S_3$

Subject to $\qquad a + 2b + 2c + S_1 = 90$

$\qquad\qquad 2a + b + c + S_2 = 150$

$\qquad\qquad 3a + b + 2c + S_3 = 70$

The initial feasible solution is obtained by setting $a = b = c = 0$. The solution is $S_1 = 90$, $S_2 = 150$, $S_3 = 70$, $P = 640$. This solution and further improved solutions are presented in following matrix form:

Prog.	Profit	Qty	10	15	8	0	0	0	Replacement
			a	b	c	S_1	S_2	S_3	ratios
S_1	0	90	1	2	2	1	0	0	45
S_2	0	150	2	1	1	0	1	0	150
S_3	0	70	3	1	2	0	0	1	70
			10	15 ↑	8	0	0	0	
b	15	45	1/2	1	1	1/2	0	0	90
S_2	0	105	3/2	0	0	−1/2	1	0	70
S_3	0	25	5/2	0	1	−1/2	0	1	10
			5/2 ↑	0	−7	−15/2	0	0	
b	15	40	0	1	4/5	3/5	0	−1/5	
S_2	0	90	0	0	−3/5	−1/5	1	−3/5	
a	10	10	1	0	2/5	−1/5	0	2/5	
			0	0	−8	−7	0	−1	

Since all NER elements are less than or equal to zero, the previous table gives the optimal solution. The solution is

$\qquad a = 10$, $b = 40$ and $c = 0$

Substituting these values

$\qquad x = 10 + 10 = 20$, $\qquad y = 40 + 20 = 60$,

$\qquad z = 0 + 30 = 30$

and the value of objective function is given by

$\qquad P = $ Rs. $10 \times 20 + $ Rs. $15 \times 60 + $ Rs. $8 \times 30 = $ Rs. $200 + $ Rs. $900 + $ Rs. 240

$\qquad\qquad\qquad\qquad = $ Rs. 1340

The optimal product mix is to produce 20 units of X, 60 units of Y and 30 units of Z to get a profit of Rs. 1340.

Illustration

The Delhi Florist Company is planning to make up floral arrangements for the upcoming festival. The company has available the following supply of flowers at the costs shown:

Type	Number available	Cost per flower
Red roses	800	Rs. 0.20
Gardenias	456	Rs. 0.25
Carnations	4,000	Rs. 0.15
White roses	920	Rs. 0.20
Yellow roses	422	Rs. 0.22

These flowers can be used in any of the four popular arrangements whose makeup and selling prices are as follows:

Arrangement	Requirements	Selling price
Economy	4 red roses	Rs. 6
	2 gardenias	
	8 carnations	
May time	8 white roses	Rs. 8
	5 gardenias	
	10 carnations	
	4 yellow roses	
Spring colour	9 red roses	Rs. 10
	10 carnations	
	9 white roses	
	6 yellow roses	
Deluxe rose	12 red roses	Rs. 12
	12 white roses	
	12 yelow roses	

Formulate a linear programming problem which allows the florist company to determine how many units of each arrangement should be made up in order to maximize profits assuming all arrangements can be sold.

The formulated LP problem is not required to be solved.

Solution

The profits for each arrangement are:

Economy	= 6.00 – 4 (0.20) – 2(0.25) – 8 (0.15)	= Rs. 3.50
May time	= 8.00 – 8 (0.20) – 5 (0.25) – 10 (0.15) – 4 (0.22)	= Rs. 2.77
Spring colour	= 10.00 – 9 (0.20) – 10 (0.15) – 9 (0.20) – 6 (0.22)	= Rs. 3.58
Deluxe rose	= 12.00 – 12 (0.20) – 12 (0.20) – 12 (0.22)	= Rs. 4.56

Let x_1, x_2, x_3, x_4 be number of units arrangements of type Economy, May time, Spring colour & Deluxe rose.

Then the objective is Maximise $Z = 3.5x_1 + 2.77x_2 + 3.58x_3 + 4.56x_4$

$$\text{subject to} \qquad 4x_1 + 9x_3 + 12x_4 \le 800$$
$$2x_1 + 5x_2 \le 456$$
$$8x_1 + 10x_2 + 10x_3 \le 4000$$
$$8x_2 + 9x_3 + 12x_4 \le 920$$
$$4x_2 + 6x_3 + 12x_4 \le 422$$
$$\text{All } x_i\text{'s } \ge 0.$$

Illustration

A local travel agent is planning a charter trip to a major sea resort. The eight day/seven-night package includes the fare for round-trip travel, surface transportation, board and lodging and selected tour options. The charter trip is restricted to 200 persons and past experience indicates that there will not be any problem for getting 200 persons. The problem for the travel agent is to determine the number of Deluxe, Standard, and Economy tour packages to offer for this charter. These three plans each differ according to seating and service for the flight, quality of accomodation , meal plans and tour options. The following table summarises the estimated prices for the three packages and the corresponding expenses for the travel agent. The travel agent has hired an aircraft for the flat fee of Rs. 2,00,000 for the entire trip.

Price and costs for tour packages per person

Tour Plan	Price	Hotel Costs	Meals & other Expenses
	(Rs.)	(Rs.)	(Rs.)
Deluxe	10,000	3,000	4,750
Standard	7,000	2,200	2,500
Economy	6,500	1,900	2,200

In planning the trip, the following considerations must be taken into account :

(i)   At least 10 percent of the packages must be of the deluxe type.

(ii)   At least 35 percent but not more than 70 percent must be of the standard type.

(iii) At least 30 percent must be of the economy type.

(iv) The maximum number of deluxe packages available in any aircraft is restricted to 60.

(v) The hotel desires that at least 120 of the tourists should be on the deluxe and standard packages together.

The travel agent wishes to determine the numer of packages to offer in each type so as to maximize the total profit.

(a) Formulate the above as a linear programming problem.

(b) Restate the above linear programming problem in terms of two decision variables, taking advantage of the fact that 200 packages will be sold.

(c) Find the optimum solution using graphical methods for the restated linear programming problem and interpret your results.

Solution

Let x_1, x_2, x_3 denote the number of Deluxe, Standard & Economy tour packages to be offered to 200 persons that will maximize the profit. In other words, the total number of tours of three types offered by the concern is restricted to 200 only to maximize its profits.

The contribution (per person) arising out of each type of tour package offered is as follows:

Packages offered	Price (Rs.) (1)	Hotel Costs (Rs.) (2)	Meals & other Expenses (Rs.) (3)	Net Profit (Rs.) (4)=(1)–[(2)+(3)]
Deluxe	10,000	3,000	4,750	2,250
Standard	7,000	2,200	2,500	2,300
Economy	6,500	1,900	2,200	2,400

The travel agent has to pay the flat fee of Rs. 2,00,000 for the chartered aircraft for the entire trip. Consequently the objective function (profit function) will be:

Max Z = Rs. 2,250 x_1 + Rs. 2,300x_2 + Rs. 2,400x_3 – Rs. 2,00,000

The objective of the profit function is to maximize the profit.

The constraints based on the given conditions (i) to (v) are as folllow:

$x_1 \geq 20$ from condition (i) $x_3 \geq 60$ from condition (iii)

$x_2 \geq 70$ from condition (ii) $x_1 \leq 60$ from conditions (iv)

$x_2 \leq 140$ from condition (ii) $x_1 + x_2 \geq 120$ from condition (v)

Also $x_1 + x_2 + x_3 = 200$, $x_1, x_2, x_3 \geq 0$

On combining suitably, the above constraints reduce to the following constraints:

$$20 \le x_1 \le 60$$
$$70 \le x_2 \le 140$$
$$x_3 \ge 60$$
$$x_1 + x_2 \ge 120$$
$$x_1 + x_2 + x_3 = 200 \quad \& \quad x_1, x_2, x_3 \ge 0$$

(a) The linear programming problem formulated from the given data is :

Max Z = Rs. 2,250 x_1 + Rs. 2,300x_2 + Rs. 2,400x_3 – Rs. 2,00,000

Subject to constraints

$$20 \le x_1 \le 60$$
$$70 \le x_2 \le 140$$
$$x_3 \ge 60$$
$$x_1 + x_2 \ge 120$$
$$x_1 + x_2 + x_3 = 200$$
$$x_1, x_2, x_3 \ge 0$$

(b) Since $x_1 + x_2 + x_3 = 200$ or $x_3 = 200 - (x_1 + x_2)$

Substitute the value of x_3 in the relations mentioned under (a), the linear programming problem reduces to the following :
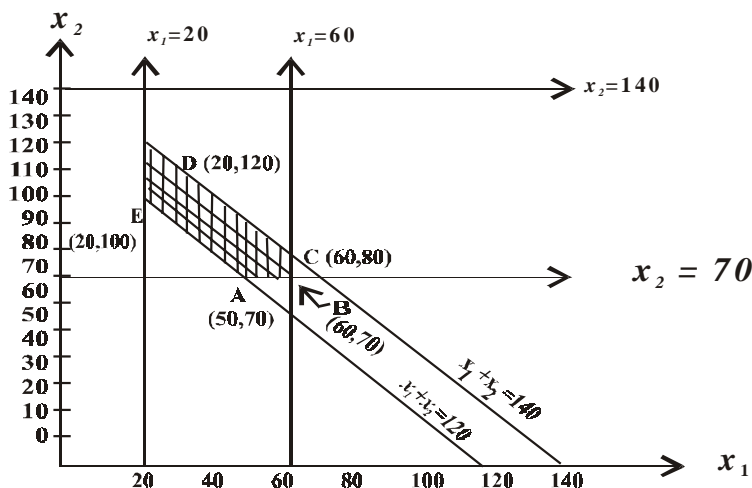
Maximize Z = $- 150x_1 + 100x_2 + 2,800,000$

Subject to $20 \le x_1 \le 60$

$$70 \le x_2 \le 140$$
$$120 \le x_1 + x_2 \le 140$$
$$\& \ x_1, x_2 \ge 0$$

(c) Refer to the figure drawn for the optimum solution of the restated L.P. problem under (b).

Point	Co-ordinates of the corner points of the feasible region (values of x_1 and x_2)	Values of the objective function $Z = -150x_1 - 100x_2 + 2{,}80{,}000$
A	(50,70)	Rs. 2,65,500
B	(60,70)	Rs. 2,64,000
C	(60,80)	Rs. 2,63,000
D	(20,120)	Rs. 2,65,000
E	(20,100)	Rs. 2,67,000

The maximum profit is attained at the corner point E whose co-ordinates are (20,100)

Interpretation: The profit of the objective function stated under (a) becomes maximum ie. Rs. 2,67,000 when

$$x_1 = 20; \; x_2 = 100 \text{ and } \; x_3 = 80 \left\{ \begin{array}{l} \because x_3 = 200 - (x_1 + x_2) \\ = 200 - (20 + 100) \end{array} \right\}$$

Illustration

WELL TYPE Manufacturing Company produces three types of typewriters; Manual type-writer, Electronic typewriters, and Deluxe Electronic typewriters. All the three models are required to be machined first and then assembled. The time required for the various models are as follows:

Type	Machine Time (in hour)	Assembly Time (in hour)
Manual Typewriter	15	4
Electronic Typewriter	12	3
Deluxe Electronic Typewriter	14	5

The total available machine time and assembly time are 3,000 hours and 1,200   hours respectively. The data regarding the selling price and variable costs for the three types are :

	Manual	Electronic	Deluxe Electronic
Selling Price (Rs.)	4,100	7,500	14,600
Labour, Material and other variable costs (Rs.)	2,500	4,500	9,000

The company sells all the three types on credit basis, but will collect the amounts on the first of next month. The labour, material and other variable expenses will have to be paid in cash. This company has taken a loan of Rs. 40,000 from a co-operative bank and this company will have to repay it to the bank on 1st April, 1999. The TNC bank from whom this company has borrowed Rs. 60,000 has expressed its approval to renew the loan.

The Balance Sheet of this Company as on 31.3.99 is as follows:

Liabilities	Rs.	Assets	Rs.
Equity Share capital	1,50,000	Land	90,000
Capital Reserve	15,000	Building	70,000
General Reserve	1,10,000	Plant & Machinery	1,00,000
Profit & Loss a/c	25,000	Furniture & Fixtures	15,000
Long term loan	1,00,000	Vehicles	30,000
Loan from TNC Bank	60,000	Inventory	5,000
Loan from Co-op. Bank	40,000	Receivables	50,000
		Cash	1,40,000
Total	5,00,000	Total	5,00,000

The company will have to pay a sum of Rs. 10,000 towards the salary from top management executives and other fixed overheads for the month. Interest on long term loans is to be paid every month at 24% per annum. Interest on loans from TNC and Co-operative Banks may be taken to be Rs. 1,200 for the month. Also this company has promised to deliver 2 Manual typewriters and 8 Deluxe Electronic typewriters to one of its valued costomers next month.

Also make sure that the level of operations in this company is subject to the availability of cash next month. This company will also be able to sell all their types of typewriter in the market. The Senior Manager of this company desires to know as to how many units of each typewriter must be manufactured in the factory next month so as to maximize the profits of the company. Formulate this as a linear programming problem. The formulated problem need not be solved.

Solution

Let X_1, X_2 and X_3 denote the number of Manual, Electronic and Deluxe Electronic typewriters respectively to be manufactured in the factory next month. We are given the following data.

	Manual Typewriter	Electronic Typewriter	Deluxe Electronic Typewriter
Selling Price p.u. (Rs.)	4,100	7,500	14,600
Labour, Material and other variable costs p.u. (Rs.)	2,500	4,500	9,000
Profit contribution p.u. (Rs.)	1,600	3,000	5,600

The objective of the company is to maximize the profit hence the objective function is given by

Maximize $Z = 1600 X_1 + 3000 X_2 + 5600 X_3 -$ (Rs. 10,000 + Rs. 2,000 + Rs. 1,200)

From the data given for time required for various models and the total number of hours available for machine time and assembly time, we get the following constraints:

$15X_1 + 12 X_2 + 14X_3 \leq 3,000$ (machine time restriction)

and $4X_1 + 3X_2 + 5X_3 \leq 1,200$ (assembly time restriction)

The level of operations in the company is subject to the availability of cash next month. That is, the cash required for manufacturing various models should not exceed the cash available for the next month.

The cash requirements for X_1 units of Manual, X_2 units of Electronic and X_3 units of Deluxe Electronic typewriters are;

$$2{,}500\ x_1 + 4{,}500\ x_2 + 9{,}000\ x_3 \qquad \qquad \text{...(i)}$$

The cash availability for the next month from the balance sheet is as below:

Cash availability (Rs.)   =   Cash balance (Rs. 1,40,000) + Receivables (Rs. 50,000)

–   Loan to repay to co-operative bank (Rs. 40,000)

–   Interest on loan from TNC & co-operative banks (Rs. 1200)

–   Interest on long term loans $\left(\dfrac{0.24 \times Rs.\,1{,}00{,}000}{12} \right)$

–   (Top management salary and fixed overheads) (Rs. 10,000)

Or cash availability   = Rs. 1,40,000 + Rs.50,000 – (Rs.40,000 + Rs.1,200 + Rs.2,000 + Rs.10,000)

= Rs. 1,90,000 – Rs. 53,200

= Rs. 1,36,800   ...(ii)

From (i) & (ii), we get

$$2{,}500\ X_1 + 4{,}500\ X_2 + 9{,}000\ X_3 \le \text{Rs. 1,36,800}$$

Further, the company has promised to deliver 2 Manual typewriters and 8 Deluxe Electronic Typewriters to one of its customers.

Hence, $x_1 \ge 2$, $x_2 \ge 0$, and $x_2 \ge 8$

The linear programming formulation of the given problem is as follows:

Maximize Z = 1,600 x_1 + 3,000 x_2 + 5,600 x_3 – (Rs. 10,000 + Rs. 2000 + Rs. 1,200)

Subject to the constraints :

$$15X_1 + 12X_2 + 14X_3 \le 3{,}000$$

$$4X_1 + 3X_2 + 5X_3 \le 1{,}200$$

$$2500X_1 + 4{,}500X_2 + 9{,}000\ X_3 \le 1{,}36{,}800$$

$$X_1 \ge 2,\ X_2 \ge 0,\ X_3 \ge 8,$$

X_1, X_2 and X_3 can take only positive integral values.

Illustration

Consider a company that must produce two products over a production period of three months of duration. The company can pay for materials and labour from two sources: company funds and borrowed funds.

The firm faces three decisions :

(1) How many units should it produce of Product 1?

(2) How many units should it produce of Product 2?

(3) How much money should it borrow to support the production of the two products?

In making these decisions, the firm wishes to maximize the profit contribution subject to the conditions stated below:

(i) Since the company's products are enjoying a seller's market, it can sell as many units as it can produce. The company would therefore like to produce as many units as possible subject to production capacity and financial constraints. The capacity constraints, together with cost and price data, are given in Table-1.

TABLE-1 Capacity, Price and cost data

Product	Selling Price (Rs. per unit)	Cost of Production (Rs. per unit)	Required Hours per unit in Department		
			A	B	C
1	14	10	0.5	0.3	0.2
2	11	8	0.3	0.4	0.1
Available hours per production period of three months			500	400	200

(ii) The available company funds during the production period will be Rs. 3 lakhs.

(iii) A bank will give loans upto Rs. 2 lakhs per production period at an interest rate of 20 percent per annum provided the company's acid (quick) test ratio is at least 1 to 1 while the loan is outstanding. Take a simplified acid-test ratio given by

$$\frac{\text{Surplus cash on hand after production } + \text{Accounts receivable}}{\text{Bank Borrowing} + \text{Interest accrued thereon}}$$

(iv) Also make sure that the needed funds are made available for meeting the production costs.

Formulate the above as a Linear Programming Problem.

Solution

Let x_1 = No. of units of product 1 produced

x_2 = No of units of product 2 produced

x_3 = Amount of money borrowed

The profit contribution per unit of each product is given by the selling price minus the variable cost of production. Total profit may be computed by summing up the profit from producing the two products minus the cost associated with borrowed funds (if any):

The objective function is thus stated as

Maximize Z $= (14-10) x_1 + (11-8) x_2 - 0.05x_3$ (Note that the interest rate is 20%

$= 4x_1 + 3x_2 - 0.05 x_3$ per annum, hence 5% for a period

subject to the following constraints :   of three months)

The production capacity constraints for each department as given by table 1 are:

$0.5x_1 + 0.3x_2 \leq 500$ ...(1)

$0.3x_1 + 0.4x_2 \leq 400$ ...(2)

$0.2x_1 + 0.1x_2 \leq 200$ ...(3)

The funds available for production include both Rs. 3,00,000 cash that the firm possesses and any borrowed funds maximum up to Rs. 2,00,000. Consequently production is limited to the extent that funds are available to pay for production costs. The constraint expressing this relation-ship is

Funds required for production £ Funds available

i.e. $10x_1 + 8x_2 \leq$ Rs. 3,00,000 + x_3

or $10x_1 + 8x_2 - x_3 \leq$ Rs. 3,00,000   ...(4)

The borrowed funds constraint (from condition (iii) of the Question) is

     $x_3 \leq$ Rs. 2,00,000   ....(5)

The constraint based on the acid – test condition is developed as follows :

$$\frac{\text{Surplus cash on hand after production} + \text{Accounts receivable}}{\text{Bank Borrowing} + \text{Interest accrued thereon}} \geq 1$$

$$\frac{(3,00,000 + x_3 - 10x_1 - 8x_2) + 14x_1 + 11x_2}{(x_3 + 0.05x_3)} \geq 1$$

or $3,00,000 + x_3 + 4x_1 + 3x_2 \geq (x_3 + 0.05x_3)$

or $-4x_1 - 3x_2 + 0.05x_3 \geq 3,00,000$ .....(6)

Thus, the linear programming problem is given by

Maximize Z = $4x_1 + 0.3x_2 - 0.05x_3$

subject to $0.5x_1 + 0.3x_2$ ≤ 500 .....(1)

           $0.3x_1 + 0.4x_2$ ≤ 400 .....(2)

           $0.2x_1 + 0.1x_2$ ≤ 200 .....(3)

           $10x_1 + 8x_2 - x_3$ $\leq$ Rs.3,00,000   .....(4)

                      x_3 $\leq$ Rs.2,00,000   .....(5)

         $-4x_1 - 3x_2 + 0.05x_3$ $\leq$ Rs.3,00,000   .....(6)

    where $x_1, x_2, x_3 \geq 0$.

Industrial Applications

These are basically product-mix problems in which the general objective is to derive the optimal production and procurement plan for the time period under consideration. The measure of effectiveness is either a defined return that is sought to be maximized or a defined cost that is to be minimized. Some examples follow:

Production Planning - product - mix problem :

An industrial concern has available a certain production capacity on various manufacturing processes and has the opportunity to utilize this capacity to manufacture various products. Typically, different products will have different selling prices, will require different amounts of production capacity at the several processes, and therefore will have different unit profits; there may also be stipulations on maximum and/or minimum production levels. The problem is to determine the optimal mix so that the total profit is maximized.

Illustration

The Voltex Company produces an air conditioner/heating unit. The company currently has firm orders for 6 months into the future. The company can schedule its production over the next 6 months to meet orders on either a regular or on overtime basis. Consider orders and the associated production costs for the next 6 months as follows:

Month	Jan.	Feb.	Mar.	Apr.	May	Jun.
Orders	590	610	650	700	500	700
Cost/unit (Rs.)						
Regular Production	50	52	51	55	47	50
Cost/Unit (Rs.)						
Overtime Production	62	58	63	60	55	52

Maximum no. of units which can be produced on regular and overtime basis are 500 and 300 respectively.

With 75 airconditioners in stock at the beginning of January, the company wishes to have at least 100 air conditioners in stock at the end of June. The inventory-carrying costs for air-conditioners is Rs. 10 per unit per month.

Solution

Let x_{ij} = number of units produced in month i (i = 1, 2 ......, 6),

on a regular or overtime basis (j = 1, 2)

y_i – number of units of ending inventory in month i (i = 1,2...., 6)

Minimize (cost) $Z = 50x_{11} + 62x_{12} + 52x_{21} + 58x_{22} +$

$$51x_{31} + 63x_{32} + 55x_{41} + 60x_{42} +$$
$$47x_{51} + 55x_{52} + 50x_{61} + 52x_{62} +$$
$$10(y_1 + y_2 + y_3 + y_4 + y_5 + y_6)$$

Subject to (January)

$$75 + x_{11} + x_{12} - 590 = y_1$$
$$y_1 + x_{21} + x_{22} - 610 = y_2$$
$$y_2 + x_{31} + x_{32} - 650 = y_3 \quad \text{Monthly}$$
$$y_3 + x_{41} + x_{42} - 700 = y_4 \quad \text{Inventory constraints}$$
$$y_4 + x_{51} + x_{52} - 500 = y_5$$
$$y_5 + x_{61} + x_{62} - 700 = y_6$$
$$y_6 \geq 100 \text{ (Ending Inventory Constraints)}$$

$x_{11} \leq 500;\ x_{12} \leq 300$

$x_{21} \leq 500;\ x_{22} \leq 300$ $x_{ij} \geq 0$, all i and j

$x_{31} \leq 500;\ x_{32} \leq 300$ $y_i \geq 0$, all i

$x_{41} \leq 500;\ x_{42} \leq 300$

$x_{51} \leq 500;\ x_{52} \leq 300$

$x_{61} \leq 500;\ x_{62} \leq 300.$

Blending Problems : These problems are likely to arise when a product can be made from a variety of available raw materials of various compositions and prices. The manufacturing process involves blending (mixing) some of these materials in varying quantities to make a product conforming to given specifications. The supply of raw materials and speicifications serve as constraints in obtaining the minimum cost material blend. The solution would state the number of units of each raw material which are to be blended to make one unit of product.

Illustration

A refinery makes 3 grades of petrol (A, B, C) from 3 crude oils *(d, e, f)*. Crude can be used in any grade but the others statisfy the following specifications.

Grade	Specifications	Selling Price per litre
A	Not less than 50% crude *d*	8.0
	Not more than 25% crude *e*	
B	Not less than 25% crude *d*	6.5
	Not more than 50% crude *e*	
C	No specifications	5.5

There are capacity limitations on the amount of the three crude elements that can be used;

Crude	Capacity	Price per litre
d	500	9.5
e	500	5.5
f	300	6.5

It is required to produce the maximum profit.

Formulation

Let there be x_1 litres of d in A

x_2 litres of e in A

x_3 litres of f in A

y_1 litres of d in B

y_2 litres of e in B

y_3 litres of f in B

z_1 litres of d in C

z_2 litres of e in C

z_3 litres of f in C

Then $\dfrac{x_1}{x_1 + x_2 + x_3} \geq \dfrac{1}{2}$ i.e. $-\ x_1 + x_2 + x_3 \leq 0$

$\dfrac{x_2}{x_1 + x_2 + x_3} \geq \dfrac{1}{4}$ i.e. $-\ x_1 + 3x_2 - x_3 \leq 0$

$\dfrac{y_1}{y_1 + y_2 + y_3} \geq \dfrac{1}{4}$ i.e. $-\ 3y_1 + y_2 + y_3 \leq 0$

$\dfrac{y_2}{y_1 + y_2 + y_3} \geq \dfrac{1}{2}$ i.e. $-\ y_1 + y_2 + y_3 \leq 0$

Also $x_1 + y_1 + z_1 \leq 500{,}000$

$x_2 + y_2 + z_2 \leq 500{,}000$

$x_3 + y_3 + z_3 \leq 300{,}000$

Profit $= -\ 1.5x_1 + 2.5x_2 + 1.5x_3 - 3.0y_1 + 1.0\ y_2 + 0.y_3 - 4.0z_1 + 0.z_2 + 1.0z_3$

Diet Problems are not much different from the blending problem as brought out by the following example.

Illustration

The vitamins V and W are found in two different foods, F_1 and F_2. The amount of vitamin in each of the two foods, respective prices per unit of each food, and the daily vitamin requirements are given in the following table. The data indicate that one unit of F_1 contains 2 units of vitamin V and 3 units of vitamin W. Similarly one unit of F_2 contains 4 units of vitamin V and 2 units vitamin W. Daily requirements of vitamin V is at least 40 units and vitamin W of at least 50 units.

The problem is to determine optimal quantities of foods F_1 and F_2 to be bought so that the daily vitamin requirements are met and, simultaneously the cost of buying the goods is minimized.

Vitamin	Food		Daily
	F_1	F_2	Requirements
V	2	4	40
W	3	2	50
Cost/Unit of food	3	2.5	

Formulation :

Let x_1 and x_2 be quantities of F_1 and F_2 respectively.

Minimize $3x_1 + 2.5x_2$

subject to $2x_1 + 4x_2 \geq 40$

$3x_1 + 2x_2 \geq 50$

$x_1 \geq 0$

$x_2 \geq 0$

Trim Problems are applicable to paper industry where paper of standard width has to be cut into smaller width as per customer requirements with the objective of minimizing the waste   produced.

Illustration

The Fine Paper Company   produces rolls of paper used in cash registers. Each roll of paper   is 500 ft. in length and can be produced in widths of 1, 2, 3 and 5 inch. The company's production process results in 500' rolls that are 12 inches in width. Thus the company must cut its 12 inch roll to the desired width. It has six basic cutting alternatives as follows:

Cutting	No. of Rolls				Waste
Alternative	1"	2"	3"	5"	(Inches)
1	6	3	0	0	0
2	0	3	2	0	0
3	1	1	1	1	1
4	0	0	2	1	1
5	0	4	1	0	1
6	4	2	1	0	1

The maximum demand requirements for the four rolls are as follows:

Roll Width (inches)	Demand Requirements (Rolls)
1	3000
2	2000
3	1500
5	1000

The company wishes to minimize the waste generated by its production meeting its demand requirements. Formulate the LP model.

Formulation. Let x_j be the number of times cutting alternative i (j = 1,2, ...6) is employed.

Minimise (waste produced) $Z = 1x_3 + 1x_4 + 1x_5 + 1x_6$ subject to

$$6x_1 + 1x_3 + 4x_6 \qquad\qquad \leq 3000$$
$$3x_1 + 3x_2 + 1x_3 + 4x_5 + 2x_6 \qquad \leq 2000$$
$$2x_2 + 1x_3 + 2x_4 + 1x_5 + 1x_6 \qquad \leq 1500$$
$$1x_3 + 1x_4 \qquad \leq 1000$$
$$x_j \qquad \geq 0, \text{ for all } j$$

Distribution Problems

Illustration

At three factories, a company has a total of 2000 bags of a particular food to be shipped to five warehouses. The demands of the warehouses, the supplies available at different factories, and the transportation costs per bag are given in the following matrix.

		Factory 1	2	3	Demand
	1	.50	.65	.54	320
	2	.48	.55	.60	470
Warehouse	3	.72	.60	.58	440
	4	.53	.51	.57	350
	5	.45	.49	.59	420
Supply		900	500	600	

Formulate the LP model. (as an exercise for you)

Production Distribution Problems : These problems occur when the products needed by the various destinations in a transportation problem do not exist in finished form but rather must be manufactured at the sources before shipment.

Illustration

A decision is to be made how many units of a product to manufacture at each of the 4 factories to be transported to five markets. The following table lists the availabilities and demands and profit per unit for the distribution form each market. These profits take into account the sale price in each market, the variable cost in each factory and the transportation costs from each factory to each market.

		\multicolumn{4}{c	}{Factory}			Requirements
		1	2	3	4	Requirements
	1	5	6	3	8	420
	2	7	8	5	4	490
Market	3	3	6	7	7	510
	4	4	9	2	7	530
	5	4	7	5	3	550
Availability		700	500	800	1000	

Formulate the LP Model. (as an exercise for you)

Marketing Applications

Advertising-mix problem is analogous to the product-mix problem as brought out by an example in the earlier section.

Financial Applications : The investment portfolio selection problems can be satisfactorily handled by linear programming as brought out by the following example although integer programming is more generally used in this function :

Illustration

The Agro Promotion Bank is trying to select investment portfolio for a cotton farmer. The bank has chosen a set of five investment alternatives, with subjective estimates of rates of return and risk, as follows:

Investment	Annual Rate of Retrun	Risk
Tax-free municipal bonds	6.0	1.3
Corporate bonds	8.0	1.5
High grade common stock	5.0	1.9
Mutual fund	7.0	1.7
Real estate	15.0	2.7

The bank officer in charge of the portfolio would like to maximize the average annual rate of return on the portfolio. However, the wealthy investor has specified that the average risk of the

portfolio should not exceed 2.0; and does not want more than 20% of the investment to be put into real estate. Formulate an LP model for the problem.

Formulation

Let x_{ij} be the percent of portfolio allocated to investment j (j = 1,2,3,4,5).

Maximize (average annual rate of return on the portfolio) $Z = 6\%x_1 + 8\%x_2 + 5\% x_3 + 7\%x_4 +$
$$15\% x_5$$

subject to (Average risk constraint) $1.3x_1 + 1.5x_2 + 1.9x_3 + 1.7x_4 + 2.7x_5 \leq 2.0$

(Real estate constraint) $x_5 \leq 0.2$

(Total investment constraint) $x_1 + x_2 + x_3 + x_4 + x_5 = 1.0$

$x_1, x_2, x_3, x_4, x_5 \geq 0.$

Administrative Applications

The use of linear programming as a tool of administrative problem-solving is well-documented in both academic circles and the area of business operations. Included in these areas of application are the following:

Personnel Assignment Problems

Bidding Problems as illustrated by following example:

Illustration

A company is to subcontract work on four assemblies. The five sub–contractors have agreed to submit a bid price on each assembly type and a limit on the total number of assemblies (in any combination) for which they are willing to contract. These bids, the contract times, and the requirements for assemblies are given in the following matrix.

Sub–contractor		A	B	C	D	E	Assemblies Required
	1	3.45	3.80	3.00	3.10	3.70	500
	2	3.40	3.13	3.50	3.10	3.20	300
Assembly	3	3.35	3.50	3.50	3.20	3.40	300
	4	3.60	3.00	3.00	3.30	3.90	400
Contract Limits		250	280	330	360	380	

Formulate the LP model. (Hint : Treat the problem as a transportation problem).

Balancing Production, Inventories and Work-force

The sales program cannot usually be accepted as the production program since it is likely to impose fluctuating load on the work facilities and equipment. Anyhow, in theory at least, the sales program is a possible production plan on one extreme out of infinite possible plans. The sales programs in the production plan would keep the inventories at a minimum level (zero) but the fluctuations in the rates of production and workforce would be at their highest cost. The other extreme production plan would be to have a constant production rate (at the average sales requirements) so the cost of changing the production rate and work-force are minimal, but the inventory-carrying costs are at their highest. In between, these two extremes are infinite number of plans involving varying amounts of inventory-carrying costs on the one hand and costs associated with changing production rate and work-force level on the other hand.

Example on the production smoothing problem would serve as an illustration of their problem too. Overtime production premiums constitute the "costs associated with the change of the work-force level".

Agriculture Applications

These deal with somewhat different resource, but the objectives are nevertheless same - to maximize the return from the allocation activity or to minimize some defined cost. Two typical problems are illustrated by the following couple of examples.

Illustration

A farmer has 1,000 acres of land on which he can grow corn, wheat or soyabeans. Each acre of corn costs Rs. 100 for preparation, requires 7 man-days of work and yields a profit of Rs. 30. An acre of wheat costs Rs. 120 for preparation, requires 10 man-days of work and yields a profit of Rs. 40. An acre of soyabeans costs Rs. 70 to prepare, requires 8 man days of work and yields a profit of Rs. 20. If the farmer has Rs. 100,000 for preparation and can count on 8000 man-days of work, how many acres should be allocated to each crop to maximize profits? Formulate an LP model.

Formulation: Let x_1, x_2 and x_3 designate the average of corn, wheat and soyabeans respectively.

$$\text{Maximize} \qquad Z = 30x_1 + 40x_2 + 20x_3$$

subject to (Money)	$100x_1 + 120x_2 + 70x_3$	£ 100,000
(Man-day)	$7x_1 + 10x_2 + 8x_3$	£ 8000
(Acreage)	$x_1 + x_2 + x_3$	£ 1000
	x_1, x_2, x_3	≥ 0

Illustration

The managers of several cattle feed lots are interested in determining how many of each of several types of livestock feed to purchase in order to satisfy the nutritional requirements for their livestock. They wish to purchase these foods in a manner that minimizes the cost of feeding their livestock. Relevant costs and nutritional data is as below.

Required Nutrient	Units of Nutritional Elements				Minimum Nutrient Requirement
	Alfa	*Corn*	*Soyabeans*	*Sorghum*	
Nutrient A	40	50	30	60	500
Nutrient B	30	60	35	40	750
Nutrient C	25	30	25	50	600
Cost per unit	Rs.1.00	Rs. 1.25	Rs.0.95	Rs.1.35	

Formulation

Let x_j to be number of units of food type $j = (1,2,3,4)$ used

Minimize (Cost) = $1.00x_1 + 1.25x_2 + 0.95x_3 + 1.35x_4$

Subject to   (Nutrient A) $40x_1 + 50x_2 + 30x_3 + 60x_4 \geq 500$

                 (Nutrient B) $30x_1 + 60x_2 + 35x_3 + 40x_4 \geq 750$

                 (Nutrient C) $25x_1 + 30x_2 + 25x_3 + 50x_4 \geq 600$

                                     $x_1, x_2, x_3, x_4 \geq 0$

Flight Scheduling Applications

Linear programming has been effectively applied to a variety of operational scheduling problems. The most common example of this type is the flight scheduling problem.

Illustration

North-East Aircraft company, which operates out of a central terminal has 8 aircraft of Type I, 15 aircraft of Type II, and 12 aircraft of Type III available for to-day's flights. The tonnage capacities (in thousands of tons) are 4.5 for Type, 1,7 for type II and 4 for Type III.

The company dispatches its planes to cities A and B. Tonnage requirements in thousands of tons are 20 at city A and 30 at city B: excess tonnage capacity supplied to a city has no value. A plane can fly once only during the day.

The cost of sending a plane from the terminal to each city is given by the following table:

	Type I	Type II	Type III
City A	23	5	1.4
City B	58	10	3.8

Formulate the LPP model to minimize the air-transportation cost

Formulation

Aircraft Type

	1	II	III	
	8	15	12	
	4.5	7	4	(Nos.)
				(Capacity in tons)

Tons	x_{11}		x_{12}		x_{13}	Cost Matrix
A(20)	23		5	1.4		
Requirement a	x_{21}		x_{22}		x_{23}	(Cost of sending
B(30)	58		10	3.8		plane)

Let x_{11} be the I plane coming to A, x_{12} the II planes coming to A; and so on.

Minimize $Z = 23x_{11} + 5x_{12} + 1.4x_{13} + 58x_{21} + 10x_{22} + 3.8x_{23}$

$$4.5x_{11} + 7x_{12} + 4x_{12} = 20$$
$$4.5x_{21} + 7x_{22} + 4x_{23} = 30$$
$$x_{21} + x_{11} \leq 8$$
$$x_{22} + x_{12} \leq 15$$
$$x_{23} + x_{13} \leq 12$$
$$x_{ij} \geq 0.$$

Other applications of linear programming include structural design, scheduling military tanker fleet, minimizing the number of carriers to meet a fixed schedule, the least ballast shipping required to meet a specific shipping program, cost cutting in business, fabrication scheduling, computations of maximum flows in network, steel production scheduling, stocks and flows the balancing of assembly lines, etc.

As such linear programming has contributed greatly to the enhancement of productivity at the firm, industry and national levels and continues to be promising in this regard.

11.9 LIMITATIONS OF LINEAR PROGRAMMING

Important limitations of linear programming problems are as follows :

(i) A primary requirement of linear programming is that the objective function and every constraint function must be linear. This requires that the measure of effectiveness and resource usage must be proportional to the level of each activity conducted individually. However, programming problems which are non-linear airse quite frequently. It is occasionally possible to reformulate a non-linear programming problem into the linear programming format so that the simplex method can be used. This is, however, the fortunate exception rather than the rule.

(ii) It may not be possible to solve those problems using linear programming, in which non-linerarity arises because of joint interactions between some of the activities regarding the total measure of effectiveness or total usage of some resource. Hence linear programming problem requires that the total measure of effectiveness and total resource usage resulting from the joint performance of the activities must equal the respective sums of these quantities resulting from each activity being performed individually.

In some situations, it may not be true. For example, consider a situation where by a product is produced with the scrap material from the primary product. The material would still have to be procured if only one of the two products were produced. However, the total material requirements if both products are produced is less than the sum of requirements is each were produced individually. It may not be possible to handle such situation with linear programming problems.

(iii) In linear programming problem, fractional values are permitted for the decisions variables. However, many decision problems require that the solution for decision variable should be obtained in non-fractional values. Rounding-off the values obtained by linear programming techniques may not result into an optimal solution in such cases.

(iv) In linear programming problem, coefficients in the objective function and the constraint equations must be completely known and they should not change during the period of study i.e. they should be known constraints. In practical situation, it may not be possible to state all coefficients in the objective function and constraints with certainity. Furthermore, these coefficients may actually be random variables, each with an underlying probablility distribution for the values. Such problems cannot be solved using linear programming.

SUMMARY

Linear Programming: The method of maximising (or minimising) a linear function of several variables (called objective function) subject to the condition that the variables are non negative and satisfy a set of linear equations and / or in equations (called linear constraints) hence called linear programming.

It involves three steps:

1.   To identify the objective function as a linear function of its variables and state all the limitation on resources as linear equation and / or in equations (constraints) – **This is called a linear programming problem.**

2.   To use mathematical techniques to find all possible set or values of the variables (unknowns), satisfying the constraints. This is called a set of feasible solution.

3.   To select the particular set of values of the variables obtained in step 2 above that lead to our objective pr maximum profit or least cost etc – This is called Optimum solution of the Linear programming problem.

SELF-EXAMINATION QUESTIONS

1. The following is the production chart for a firm that produces both commercial use and scientific use calculators. The firm uses two departments in the assembling process.

	Department D_1	Department D_2
Commercial-use	1 hour/unit	5 hours/unit
Scientific use	1 hour/unit	1 hour/unit
Available hours	12 hours	12 hours

   Profit for each unit of commercial use calculators is Rs. 30, and that for each unit of scientific use calculators is Rs. 60. Determine the number of calculators of each variety the firm should produce in order to achieve the maximum profit. Also determine the maximum profit.

2. A tonic feed for cattle is to be prepared using two types of grain X_1 and X_2. Each unit of grain X_1 weights 4 gms, and contain 2 gms of protein. Each unit of grain X_2 weights 5 gms. and contains 1 gm. of protein. The feed is to weight at most 40 gms. and must contain at least 10 gms. of protein. If the grain X_1 costs 30 paise a unit and that X_2 costs 10 paise a unit, determine by graphical method the amount of two types of grain to be used in the preparation of the feed to minimize the cost yet meeting the requirements. Also determine the minimum cost.

3. A manufacturer makes two products P_1 and P_2 using two machines M_1 and M_2. Product P_1 requires 2 hours on machine M_1 and 6 hours on machine M_2. Product P_2 requires 5 hours on machine M_1 and no time on machine M_2. There are 16 hours of time per day available on machine M_1 and 30 hours on M_2. Profit margin from P_1 and P_2 is Rs. 2 and Rs. 10 per unit respectively. What should be the daily production mix to optimise profit? Also determine the maximum daily profit.

4. A firm manufactures two products A and B on which profits earned per unit are Rs. 3 and Rs. 4 respectively. Each product is processed on two machines, M_1 and M_2. The product A requires one minute of processing time on M_1 and two minutes on M_2 while B requires one minute on M_1 and one minute on M_2. Machine M_1 is available for use for not more than 7 hours 30 minutes, while M_2 is available for 10 hours during any working day. Find the number of units of products A_2 and B to be manufactured to get the maximum profit.

5. A company produces two products X and Y, each of which requires processing in three machines. The first machine can be used at most 70 hours; the second machine at most 40 hours; and the third machine at most 90 hours. The product X requires 2 hours on machine 1,1 hour on machine 2, and 1 hour on machine 3; the product Y requires 1 hour each on machines 1 and 2, and 3 hours on the third machine. The profit is Rs. 50 per unit of X and Rs. 30 per unit of Y. How many units of each product should the company produce to maximize profit?

6. A carpenter has 90,80 and 50 running feet respectively of teak, plywood and rosewood. The product A requires 2,1 and 1 running feet and the product B requires 1,2, and 1 running feet of teak, plywood and rosewood respectively. If A would sell for Rs. 48 and B for Rs. 40 per unit, how much of each should he make and sell in order to obtain maximum gross income out of his stock of wood?

7. A machine producing either product A or B can produce A by using 2 units of chemicals and 1 unit of a compound and can produce B by using 1 unit of chemicals and 2 units of the compound. Only 800 units of chemicals and 1,000 units of the compound are available. The profits available per unit of A and B are respectively Rs. 30 and Rs. 20. Draw a suitable diagram to show the feasible region. Also, find the optimum allocation of units between A and B to maximize the total profit. Find the maximum profit.

8. A media specialist has decided on the allocation of advertisement in three media vehicles. Let x_1 be the number of messages carried on the i-th media, $i =1,2,3$. the unit costs of a message in the 3 media are Rs. 1000, Rs. 750 and Rs. 500. The total budget available is Rs. 20,000 for the campaign period of a year. The first medium is a monthly magazine and it is desired to advertise not more than one insertion in one issue. At least six messages should appear in the 2nd medium. The number of messages in the third medium should appear in between 4 and 8. The expected effective audience for unit message in the media vehicles is shown below:

Vehicle	Expected effective audience
1	80,000
2	60,000
3	45,000

It is desired to build the linear programming model to maximise the total effective audience.

9. A fertilizer company produces two types of fertilizer called grade I and II. Each of these types is processed through two critical chemical plants. Plant A has maximum of 120 hours available in a week and plant B has a maximum of 180 hours available. Manufacturing 1,000 kg of grade I fertilizer requires approximately 6 hours in Plant A and 4 hours in Plant B: manufacturing 1,000 kg of grade II fertilizer requires 3 hours on plant A and 10 hours in plant B. The maximum that the company can sell of grade I fertilizer is 18,000 kg. If profit is Rs. 450 per 1,000 kg of grade I fertilizer and Rs. 550 per 1,000 kg of grade II fertilizer, it is required to build a mathematical model for amounts of the fertilizers that maximize the profit.

    Also solve the problem graphically.

    (**Answer :**   I   =   13.75 units

                   II   =   12.50 units

                   Z   =   Rs. 13062.50)

10. The Manager of Sohan Oil Co., wishes to find the optimal mix of two possible blending processes. For process 1, an input of 1 unit of crude oil A and three units crude oil B produces an out put of 5 units of gasoline X and two units of gasoline Y. For process 2, an input of 4 units of crude oil A and 2 unit of crude oil B produces an output of 3 units of gasoline X and 8 units of gasoline Y. Let x_1 and x_2 be the number of units the company decides to use of process 1 and process 2, respectively. The maximum amount of crude oil A available is 100 units and that of crude oil B is 150 units. Sales commitments require that at least 200 units of gasoline X and 75 units of gasoline Y are   produced. The unit profits of process 1 and process 2 are p_1 and p_2 respectively. Formulate the blending problem as a linear programming model.

ANSWER TO SELF-EXAMINATION QUESTIONS

1. Commercial use calculator 0, Scientific use calculator 12; Max. profit = Rs. 720.

2. 5/3 units of X_1 and 20/3 units of X_2; Min. Cost = 350/3 paise

3. 0 unit of P_1 and 3.2 unit of P_2; Max Profit = Rs. 32.

4. 0 unit of A and 450 units of B.

5. 30 units of X and 10 units of Y.

6. 40 units of A and 10 units of B.

7. In figure 16, the shaded region is the feasible region.

   Optimum allocation : 200 units of A and 400 units of B;

   Max. Profit = Rs. 14,000.
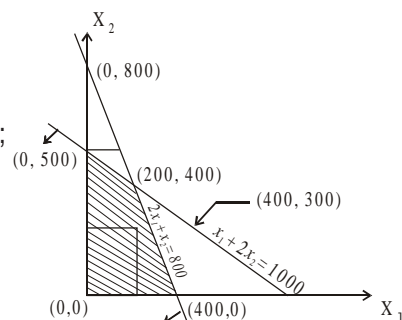
8. Maximize $Z = 80{,}000\, x_1 + 60{,}000\, x_2 + 45{,}000\, x_3$

   Subject to $1{,}000 x_1 + 750 x_2 + 500 x_3 \le 20{,}000$ (budget)

   $x_1 \le 12$

   $x_2 \ge 6$

   $x_3 \ge 4$

   $x_3 \le 8$

   $x_1, x_{2,} \ge 0$

9. Let x_1 be the amount in 1,000 kg. of grade I fertilizer produced and x_2 the amount in 1,000 kg of grade II produced. The total profit

   $Z = 450\, x_1 + 550 x_2$ is to be maximized

   subject to constraints $6x_1 + 3x_2 \le 120$ Plant A Capacity

   $2x_1 + 10x_2 \le 180$ Plant B Capacity

   $x_1 \le 18$

   $x_1, x_2 \ge 0.$

10. Maximize $Z = P_1 x_1 + P_2 x_2$

    Subject to $x_1 + 4x_2 \le 100$ ⎤ Availability
    $3x_1 + 2x_2 \le 150$ ⎦

    $5x_1 + 3x_2 \ge 200$ ⎤ Demand
    $2x_1 + 8x_2 \ge 75$ ⎦

    $x_1, x_2 \ge 0$