Chapter 1 & 2

FORMULATION OF LINEAR PROGRAMMING PROBLEM

INTRODUCTION TO LINEAR PROGRAMMING

Linear Programming is a problem solving approach that has been developed to help managers to make decisions.

Linear Programming is a mathematical technique for determining the optimum allocation of resources and obtaining a particular objective when there are alternative uses of the resources, money, manpower, material, machine and other facilities.

THE NATURE OF LINEAR PROGRAMMING PROBLEM

Two of the most common are:

1. The product-mix problem
2. The blending Problem

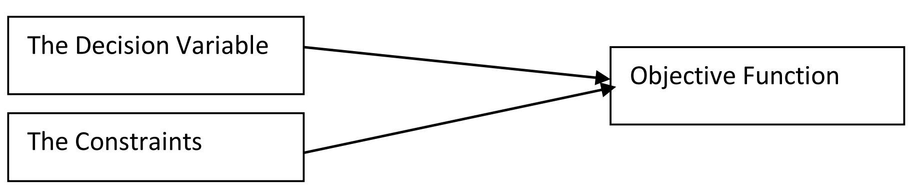
In the product- mix problem there are two or more product also called candidates or activities competing for limited resources. The problem is to find out which products to include in production plan and in what quantities these should be produced or sold in order to maximize profit, market share or sales revenue.

The blending problem involves the determination of the best blend of available ingredients to form a certain quantity of a product under strict specifications. The best blend means the least cost blend of the required inputs.

FORMULATION OF THE LINEAR PROGRAMMING MODEL

Three components are:

1. The decision variable
2. The environment (uncontrollable) parameters
3. The result (dependent) variable



Linear Programming Model is composed of the same components

TERMINOLOGY USED IN LINEAR PROGRAMMING PROBLEM

1. Components of LP Problem: Every LPP is composed of a. Decision Variable, b. Objective Function, c. Constraints.

2. Optimization: Linear Programming attempts to either maximise or minimize the values of the objective function.

3. Profit of Cost Coefficient: The coefficient of the variable in the objective function express the rate at which the value of the objective function increases or decreases by including in the solution one unit of each of the decision variable.

4. Constraints: The maximisation (or minimisation) is performed subject to a set of constraints. Therefore LP can be defined as a constrained optimisation problem. They reflect the limitations of the resources.

5. Input-Output coefficients: The coefficient of constraint variables are called the Input-Output Coefficients. They indicate the rate at which a given resource is unitized or depleted. They appear on the left side of the constraints.

6. Capacities: The capacities or availability of the various resources are given on the right hand side of the constraints.

THE MATHEMATICAL EXPRESSION OF THE LP MODEL

The general LP Model can be expressed in mathematical terms as shown below:

Let

O_{ij} = Input-Output Coefficient

C_j = Cost (Profit) Coefficient

b_i = Capacities (Right Hand Side)

X_j = Decision Variables

Find a vector (x_1, x_2, x_3 ....x_n) that minimise or maximise a linear objective function F(x) where $F(x) = c_1x_1 + c_2x_2 + \ldots\ldots\ldots + c_nx_n$

subject to linear constraints

$a_1x_1 + a_2x_2 + \ldots\ldots\ldots + a_nx_n = b_2$

$a_1x_1 + a_2x_2 + \ldots\ldots\ldots + a_nx_n \leq b_2$

....

....

....

....

$am_1x_1 + am_2x_2 + \ldots\ldots\ldots + am_nx_n \leq b_2$

and non-negativity constraints

$x_1 \geq 0, x_2 \geq 0, \ldots\ldots\ldots, x_n \geq 0$

FORMULATION OF LPP

STEPS

1. Identify decision variables
2. Write objective function
3. Formulate constraints

EXAMPLE 1. (PRODUCTION ALLOCATION PROBLEM)

A firm produces three products. These products are processed on three different machines. The time required to manufacture one unit of each of the three products and the daily capacity of the three machines are given in the table below:

Machine	Time per unit (Minutes)			Machine Capacity (minutes/day)
	Product 1	Product 2	Product 3	
M_1	2	3	2	440
M_2	4	-	3	470
M_3	2	5	-	430

It is required to determine the daily number of units to be manufactured for each product. The profit per unit for product 1, 2 and 3 is Rs. 4, Rs.3 and Rs.6 respectively. It is assumed that all the amounts produced are consumed in the market. Formulate the mathematical (L.P.) model that will maximise the daily profit.

Formulation of Linear Programming Model

Step 1

From the study of the situation find the key-decision to be made. In the given situation key decision is to decide the extent of products 1, 2 and 3, as the extents are permitted to vary.

Step 2

Assume symbols for variable quantities noticed in step 1. Let the extents (amounts) of products 1, 2 and 3 manufactured daily be x_1, x_2 and x_3 units respectively.

Step 3

Express the feasible alternatives mathematically in terms of variable. Feasible alternatives are those which are physically, economically and financially possible. In the given situation feasible alternatives are sets of values of x_1, x_2 and x_3 units respectively.

where $\quad x_1, x_2$ and $x_3 \geq 0$.

since negative production has no meaning and is not feasible.

Step 4

Mention the objective function quantitatively and express it as a linear function of variables. In the present situation, objective is to maximize the profit.

i.e., $\quad Z = 4x_1 + 3x_2 + 6x_3$

Step 5

Put into words the influencing factors or constraints. These occur generally because of constraints on availability (resources) or requirements (demands). Express these constraints also as linear equations/inequalities in terms of variables.

Here, constraints are on the machine capacities and can be mathematically expressed as

$$2x_1 + 3x_2 + 2x_3 \leq 440$$
$$4x_1 + 0x_2 + 3x_3 \leq 470$$
$$2x_1 + 5x_2 + 0x_3 \leq 430$$

EXAMPLE 2: PRODUCT MIX PROBLEM

A factory manufactures two products A and B. To manufacture one unit of A, 1.5 machine hours and 2.5 labour hours are required. To manufacture product B, 2.5 machine hours and 1.5 labour hours are required. In a month, 300 machine hours and 240 labour hours are available.

Profit per unit for A is Rs. 50 and for B is Rs. 40. Formulate as LPP.

Solution:

Products	**Resource/unit**	
	Machine	**Labour**
A	1.5	2.5
B	2.5	1.5
Availability	300 hrs	240 hrs

There will be two constraints. One for machine hours availability and for labour hours availability.

Decision variables

X_1 = Number of units of A manufactured per month.

X_2 = Number of units of B manufactured per month.

The objective function:

$$\text{Max } Z = 50x_1 + 40x_2$$

Subjective Constraints

For machine hours

$$1.5x_1 + 2.5x_2 \leq 300$$

For labour hours

$$2.5x_1 + 1.5x_2 \leq 240$$

Non negativity

$$x_1, x_2 \geq 0$$

EXAMPLE: 3

A company produces three products A, B, C.

For manufacturing three raw materials P, Q and R are used.

Profit per unit

A - Rs. 5, B - Rs. 3, C - Rs. 4

Resource requirements/unit

Raw Material / Product	P	Q	R
A	-	20	50
B	20	30	-
C	30	20	40

Maximum raw material availability:

P - 80 units; Q - 100 units; R - 150 units. Formulate LPP.

Solution:

Decision variables:

x_1 = Number of units of A

x_2 = Number of units of B

x_3 = Number of units of C

Objective Function

Since Profit per unit is given, objective function is maximisation

$$\text{Max } Z = 5x_1 + 3x_2 + 4x_3$$

Constraints:

For P:

$$0x_1 + 20x_2 + 30x_3 \leq 80$$

For Q:

$$20x_1 + 30x_2 + 20x_3 \leq 100$$

For R:

$$50x_1 + 0x_2 + 40x_3 \leq 150$$

(for B, R is not required)

$$X_1, X_2, X_3 \geq 0$$

EXAMPLE 4: PORTFOLIO SELECTION (INVESTMENT DECISIONS)

An investor is considering investing in two securities 'A' and 'B'. The risk and return associated with these securities is different.

Security 'A' gives a return of 9% and has a risk factor of 5 on a scale of zero to 10. Security 'B' gives return of 15% but has risk factor of 8.

Total amount to be invested is Rs. 5, 00, 000/- Total minimum returns on the investment should be 12%. Maximum combined risk should not be more than 6. Formulate as LPP.

Solution:

Decision Variables:

X_1 = Amount invested in Security A

X_2 = Amount invested in Security B

Objective Function:

The objective is to maximise the return on total investment.

$\therefore$ Max Z = 0.09 X_1 + 0.015 X_2 ((% = 0.09, 15% = 0.15)

Constraints:

1. Related to Total Investment:

$X_1 + X_2 = 5, 00, 000$

2. Related to Risk:

$5X_1 + 8X_2 = (6 \text{ X } 5, 00, 000)$

$5X_1 + 8X_2 = 30, 00, 000$

3. Related to Returns:

$0.09X_1 + 0.15X_2 = (0.12 \text{ X } 5, 00, 000)$

$\therefore 0.09X_1 + 0.15X_2 = 60, 000$

4. Non-negativity

$X_1, X_2 \geq 0$

EXAMPLE 5: INSPECTION PROBLEM

A company has two grades of inspectors, I and II to undertake quality control inspection. At least 1, 500 pieces must be inspected in an 8-hour day. Grade I inspector can check 20 pieces in an hour with an accuracy of 96%. Grade II inspector checks 14 pieces an hour with an accuracy of 92%.

Wages of grade I inspector are Rs. 5 per hour while those of grade II inspector are Rs. 4 per hour. Any error made by an inspector costs Rs. 3 to the company. If there are, in all, 10 grade I inspectors and 15 grade II inspectors in the company, find the optimal assignment of inspectors that minimise the daily inspection cost.

Solution:

Let x_1 and x_2 denote the number of grade I and grade II inspectors that may be assigned the job of quality control inspection.

The objective is to minimise the daily cost of inspection. Now the company has to incur two types of costs; wages paid to the inspectors and the cost of their inspection errors. The cost of grade I inspector/hour is

Rs. (5 + 3 X 0.04 X 20) = Rs. 7.40.

Similarly, cost of grade II inspector/hour is

Rs. (4 + 3 X 0.08 X 14) = Rs. 7.36.

$\therefore$ The objective function is

minimise $Z = 8(7.40x_1 + 7.36x_2) = 59.20\, x_1 + 58.88x_2.$

Constraints are

on the number of grade I inspectors: $x_1 \leq 10$, on the number of grade II inspectors: $x_2 \leq 15$

on the number of pieces to be inspected daily: $20 \times 8x_1 + 14 \times 8x_2 \geq 1500$

or $160x_1 + 112x_2 \geq 1500$

where, $x_1, x_2 \geq 0$.

EXAMPLE 6: TRIM LOSS PROBLEM

A manufacturer of cylindrical containers receives tin sheets in widths of 30 cm and 60 cm respectively. For these containers the sheets are to be cut to three different widths of 15 cm, 21 cm and 27 cm respectively. The number of containers to be manufactured from these three widths are 400, 200 and 300 respectively. The bottom plates and top covers of the containers are purchased directly from the market. There is no limit on the lengths of standard tin sheets. Formulate the LPP for the production schedule that minimises the trim losses.

Solution:

Key decision is to determine how each of the two standard widths of tin sheets be cut to the require widths so that trim losses are minimum.

From the available widths of 30 cm and 60 cm, several combinations of the three required widths of 15 cm, 21 cm and 27 cm are possible.

Let x_{ij} represent these combinations. Each combination results in certain trim loss.

Constraints can be formulated as follows:

The possible cutting combinations (plans) for both types of sheets are shown in the table below:

Width (cm)	i = I (30 cm)			i = II (60 cm)					
	X_{11}	X_{12}	X_{13}	X_{21}	X_{22}	X_{23}	X_{24}	X_{25}	X_{26}
15	2	0	0	4	2	2	1	0	0
21	0	1	0	0	1	0	2	1	0
27	0	0	1	0	0	1	0	1	2
Trim Loss (cm)	0	9	3	0	9	3	3	12	6

Thus, the constraints are

$2x_{11} + 4x_{21} + 2x_{22} + 2x_{23} + x_{24} \geq 400$

$x_{12} + x_{22} + 2x_{24} + x_{25} \geq 200$

$x_{13} + x_{23} + x_{25} + x_{26} \geq 300$

Objective is to maximise the trim losses.

i.e., minimise $Z = 9x_{12} + 3x_{13} + 9x_{22} + 3x_{23} + 3x_{24} + 12x_{25} + 6x_{26}$

where $x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{24}, x_{25}, x_{26} \geq 0$.

EXAMPLE 7: MEDIA SELECTION

An advertising agency is planning to launch an ad campaign. Media under consideration are T.V., Radio & Newspaper. Each medium has different reach potential and different cost.

Minimum 10, 000, 000 households are to be reached through T.V. Expenditure on newspapers should not be more than Rs. 10, 00, 000. Total advertising budget is Rs. 20 million.

Following data is available:

Medium	**Cost per Unit (Rs.)**	**Reach per unit (No. of households)**
Television	2, 00, 000	20, 00, 000

Radio	80, 000	10, 00, 000
Newspaper	40, 000	2, 00, 000

Solution:

Decision Variables:

x_1 = Number of units of T.V. ads,

x_2 = Number of units of Radio ads,

x_3 = Number of units of Newspaper ads.

Objective function: (Maximise reach)

$$\text{Max. } Z = 20,00,000\, x_1 + 10,00,000\, x_2 + 2,00,000 x_3$$

Subject to constraints:

$20,00,000\, x_1 \geq 10,000,000$ .... (for T.V.)

$40,000\, x_3 \leq 10,00,000$ .... (for Newspaper)

$2,00,000 x_1 + 80,000 x_2 + 40,000 x_3 \leq 20,000,000$ .... (Ad. budget)

$x_1, x_2, x_3 \geq 0$

$\therefore$ Simplifying constraints:

for T.V. $\quad 2\, x_1 \geq 10 \quad \therefore x_1 \geq 5$

for Newspaper $\quad 4\, x_3 \leq 100 \quad \therefore x_3 \leq 25$

Ad. Budget

$20\, x_1 + 8\, x_2 + 4\, x_3 \leq 2000$

$5\, x_1 + 2x_2 + x_3 \leq 500$

$x_1, x_2, x_3 \geq 0$

EXAMPLE 8: DIET PROBLEM

Vitamins B_1 and B_2 are found in two foods F_1 and F_2. 1 unit of F_1 contains 3 units of B_1 and 4 units of B_2. 1 unit of F_2 contains 5 units of B_1 and 3 units of B_2 respectively.

Minimum daily prescribed consumption of B_1 & B_2 is 50 and 60 units respectively. Cost per unit of F_1 & F_2 is Rs. 6 & Rs. 3 respectively.

Formulate as LPP.

Solution:

Vitamins	Foods		Minimum Consumption
	F_1	F_2	
B_1	3	5	30
B_2	5	7	40

Decision Variables:

x_1 = No. of units of P_1 per day.

x_2 = No. of units of P_2 per day.

Objective function:

$$\text{Min. } Z = 100\, x_1 + 150\, x_2$$

Subject to constraints:

$3x_1 + 5x_2 \geq 30$ (for N_1)

$5x_1 + 7x_2 \geq 40$ (for N_2)

$x_1, x_2 \geq 0$

EXAMPLE 9: BLENDING PROBLEM

A manager at an oil company wants to find optimal mix of two blending processes. Formulate LPP.

Data:

Process	Input (Crude Oil)		Output (Gasoline)	
	Grade A	Grade B	X	Y
P_1	6	4	6	9
P_2	5	6	5	5

Profit per operation: Process 1 (P_1) = Rs. 4, 000

Process 2 (P_2) = Rs. 5, 000

Maximum availability of crude oil: Grade A = 500 units

Grade B = 400 units

Minimum Demand for Gasoline: X = 300 units

Y = 200 units

Solution:

Decision Variables:

x_1 = No. of operations of P_1

x_2 = No. of operations of P_2

Objective Function:

Max. Z = 4000 x_1 + 5000 x_2

Subjective to constraints:

$6x_1 + 5x_2 \leq 500$

$4x_1 + 6x_2 \leq 400$

$6x_1 + 5x_2 \geq 300$

$9x_1 + 5x_2 \geq 200$

$x_1, x_2 \geq 0$

EXAMPLE 10: FARM PLANNING

A farmer has 200 acres of land. He produces three products X, Y & Z. Average yield per acre for X, Y & Z is 4000, 6000 and 2000 kg.

Selling price of X, Y & Z is Rs. 2, 1.5 & 4 per kg respectively. Each product needs fertilizers. Cost of fertilizer is Rs. 1 per kg. Per acre need for fertilizer for X, Y & Z is 200, 200 & 100 kg respectively. Labour requirements for X, Y & Z is 10, 12 & 10 man hours per acre. Cost of labour is Rs. 40 per man hour. Maximum availability of labour is 20, 000 man hours. Formulate as LPP to maximise profit.

Solution:

Decision variables:

The production/yield of three products X, Y & Z is given as per acre.

Hence,

x_1 = No. of acres allocated to X

x_2 = No. of acres allocated to Y

x_3 = No. of acres allocated to Z

Objective Function:

Profit = Revenue - Cost

Profit = Revenue - (Fertiliser Cost + Labour Cost)

Product	X	Y	Z
Revenue	2 (4000) x_1	1.5 (6000) x_2	4 (2000) x_3
(-) Less:			

Fertiliser Cost	1 (200) x_1	1 (200) x_2	1 (100) x_3
Labour Cost	40 (10) x_1	40 (12) x_2	40 (10) x_3
Profit	7400 x_1	8320 x_2	7500 x_3

∴ Objective function

$$\text{Max.} = 7400\, x_1 + 8320\, x_2 + 7500\, x_3$$

Subject to constraints:

$$x_1 + x_2 + x_3 = 200 \qquad \text{(Total Land)}$$

$$10\, x_1 + 12\, x_2 + 10\, x_3 \leq 20,000 \qquad \text{(Max Man hours)}$$

$$x_1, x_2, x_3 \geq 0$$

MERITS OF LPP

1. Helps management to make efficient use of resources.
2. Provides quality in decision making.
3. Excellent tools for adjusting to meet changing demands.
4. Fast determination of the solution if a computer is used.
5. Provides a natural sensitivity analysis.
6. Finds solution to problems with a very large or infinite number of possible solution.

DEMERITS OF LPP

1. Existence of non-linear equation: The primary requirements of Linear Programming is the objective function and constraint function should be linear. Practically linear relationship do not exist in all cases.

2. Interaction between variables: LP fails in a situation where non-linearity in the equation emerge due to joint interaction between some of the activities like total effectiveness.

3. Fractional Value: In LPP fractional values are permitted for the decision variable.

4. Knowledge of Coefficients of the equation: It may not be possible to state all coefficients in the objective function and constraints with certainty.

EXERCISES

1. Explain what is meant by decision variables, objective function and constraints in Linear Programming.
2. Give the mathematical formulation of the linear programming problems.
3. What are the components of LPP? What is the significance of non-negativity restriction?
4. State the limitations of LPP.
5. Give the assumptions and advantages of LPP.
6. An investor wants to identify how much to invest in two funds, one equity and one debt. Total amount available is Rs. 5, 00, 000. Not more than Rs. 3, 00, 000 should be invested in a single fund. Returns expected are 30% in equity and 8% in debt. Minimum return on total investment should be 15%. Formulate as LPP.
7. A company manufactures two products P_1 and P_2. Profit per unit for P_1 is Rs. 200 and for P_2 is Rs. 300. Three raw materials M_1, M_2 and M_3 are required. One unit of P_1 needs 5 units of M_1 and 10 units of M_2. One unit of P_2 needs 18 units of M_2 and 10 units of M_3. Availability is 50 units of M_1, 90 units of M_2 and 50 units of M_3. Formulate as LPP.
8. A firm produces two products X and Y. Minimum 50 units of X should be produced. There is no limit for producing Y. Profit per unit is Rs. 100 for X and Rs. 150 for Y.

Product	Resource Requirement	Resource Availability
X	20 Machine Hours	Machine Hours = 2500
	10 Labour Hours	Labour Hours = 3000
Y	10 Machine Hours	
	15 Labour Hours	

Formulate as LPP.

9. A patient has been recommended two nutrients N_1 and N_2 everyday. Minimum intake is 10g for N_1 and 15g for N_2 everyday.

These nutrients are available in two products P_1 and P_2. One unit of P_1 contains 2g of N_1 and 3g of N_2. One unit of P_2 contains 1g of N_1 and 2g of N_2. Cost per unit is Rs. 200 for P_1 and Rs. 150 for P_2.

Formulate as LPP such that nutrient requirement can be fulfilled at the lowest cost.

10. Two vitamins A and B are to be given as health supplements on daily basis to students. There are two products Alpha & Beta which contain vitamins A and B. One unit of Alpha contains 2g of A and 1g of B. One unit of Beta contains 1g of A and 2g of B. Daily requirements for A and B are atleast 10g each. Cost per unit of Alpha is Rs. 20 and of Beta is Rs. 30. Formulate as LPP to satisfy the requirements at minimum cost.

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UNIT 3

LINEAR PROGRAMMING SOLUTION - GRAPHICAL METHOD

INTRODUCTION

There are two methods available to find optimal solution to a Linear Programming Problem. One is graphical method and the other is simplex method.

Graphical method can be used only for a two variables problem i.e. a problem which involves two decision variables. The two axes of the graph (X & Y axis) represent the two decision variables X_1 & X_2.

METHODOLOGY OF GRAPHICAL METHOD

Step 1: Formulation of LPP (Linear Programming Problem)

Use the given data to formulate the LPP.

Maximisation

Example 1

A company manufactures two products A and B. Both products are processed on two machines M_1 & M_2.

	M_1	M_2
A B	6 Hrs/Unit 4 Hrs/Unit	2 Hrs/Unit 4 Hrs/Unit
Availability	7200 Hrs/month	4000 Hrs/month

Profit per unit for A is Rs. 100 and for B is Rs. 80. Find out the monthly production of A and B to maximise profit by graphical method.

Formulation of LPP

X_1 = No. of units of A/Month

X_2 = No. of units of B/Month

Max Z = $100 X_1 + 80 X_2$

Subject to constraints:

$6 X_1 + 4 X_2 \leq 7200$

$2 X_1 + 4 X_2 \leq 4000$

$X_1, X_2 \geq 0$

Step 2: Determination of each axis

Horizontal (X) axis: Product A (X_1)

Vertical (Y) axis: Product B (X_2)

Step 3: Finding co-ordinates of constraint lines to represent constraint lines on the graph.

The constraints are presently in the form of inequality ($\leq$). We should convert them into equality to obtain co-ordinates.

Constraint No. 1: $6 X_1 + 4 X_2 \leq 7200$

Converting into equality:

$6 X_1 + 4 X_2 \leq 7200$

X_1 is the intercept on X axis and X_2 is the intercept on Y axis.

To find X_1, let $X_2 = 0$

$6 X_1 = 7200$

$6X_1 = 7200$ $\therefore X_1 = 1200;\ X_2 = 0$ (1200, 0)

To find X_2, let $X_1 = 0$

$4X_2 = 7200$ $X_2 = 1800;\ X_1 = 0$ (0, 1800)

Hence the two points which make the constraint line are:

(1200, 0) and (0, 1800)

Note: When we write co-ordinates of any point, we always write (X_1, X_2). The value of X_1 is written first and then value of X_2. Hence, if for a point X_1 is 1200 and X_2 is zero, then its co-ordinates will be (1200, 0).

Similarly, for second point, X_1 is 0 and X_2 is 1800. Hence, its co-ordinates are (0, 1800).

Constraint No. 2:

$2X_1 + 4X_2 \leq 4000$

To find X_1, let $X_2 = 0$

$2X_1 = 4000$ $\therefore X_1 = 2000;\ X_2 = 0$ (2000, 0)

To find X_2, let $X_1 = 0$

$4X_2 = 4000$ $\therefore X_2 = 1000;\ X_1 = 0$ (0, 1000)

Each constraint will be represented by a single straight line on the graph. There are two constraints, hence there will be two straight lines.

The co-ordinates of points are:

1. Constraint No. 1: (1200, 0) and (0, 1800)
2. Constraint No. 2: (2000, 0) and (0, 1000)

Step 4: Representing constraint lines on graph

To mark the points on the graph, we need to select appropriate scale. Which scale to take will depend on maximum value of X_1 & X_2 from co-ordinates.

For X_1, we have 2 values $\longrightarrow$ 1200 and 2000

$\therefore$Max. value for $X_2 = 2000$

For X_2, we have 2 values $\longrightarrow$ 1800 and 1000

$\therefore$Max. value for $X_2 = 1800$

Assuming that we have a graph paper 20 X 30 cm. We need to accommodate our lines such that for X-axis, maximum value of 2000 contains in 20 cm.

$\therefore$ Scale 1 cm = 200 units

$\therefore$ 2000 units = 10 cm (X-axis)

1800 units = 9 cm (Y-axis)

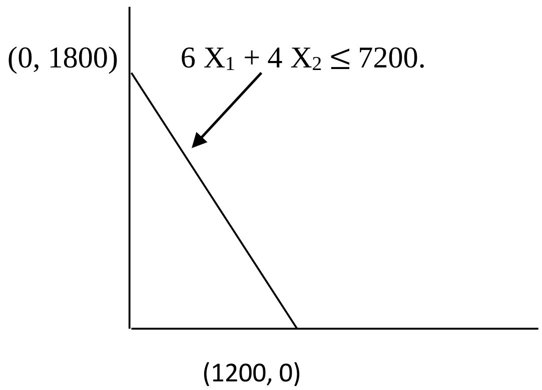
The scale should be such that the diagram should not be too small.

Constraint No. 1:

The line joining the two points (1200, 0) and (0, 1800) represents the constraint

$6X_1 + 4X_2 \leq 7200$.

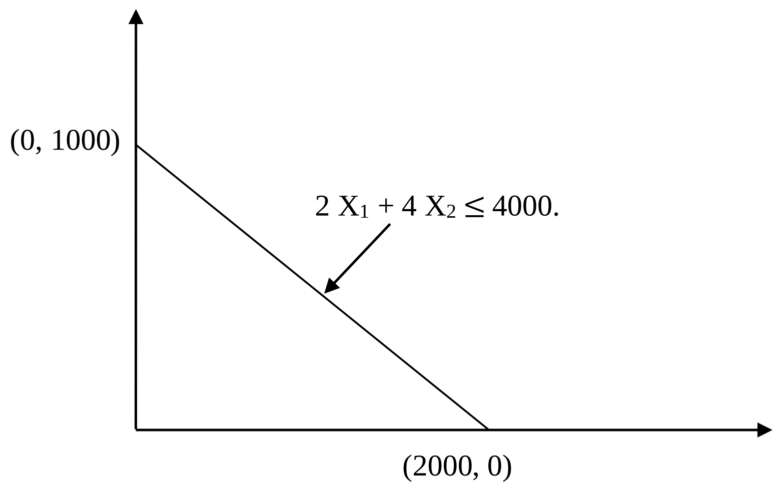
Fig 1.



Every point on the line will satisfy the equation (equality) $6 X_1 + 4 X_2 \leq 7200$.
Every point below the line will satisfy the inequality (less than) $6 X_1 + 4 X_2 \leq 7200$.

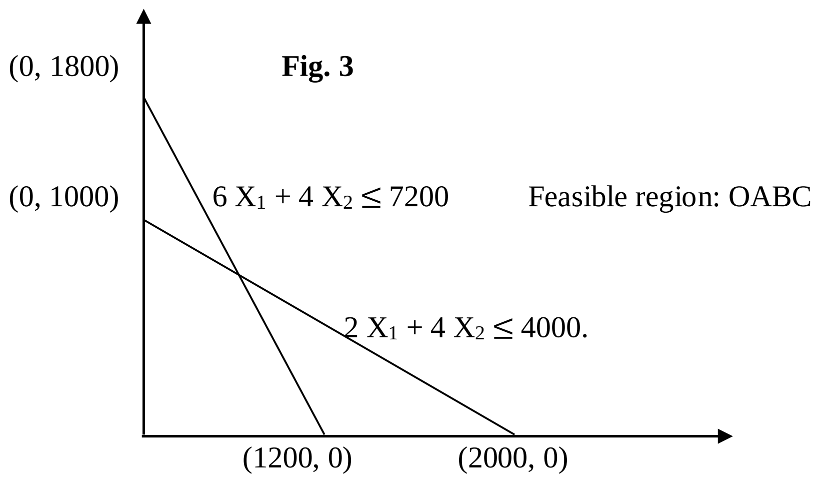
Constraint No. 2:
The line joining the two points (2000, 0) and (0, 1000) represents the constraint
$2 X_1 + 4 X_2 \leq 4000$
Every point on the line will satisfy the equation (equality) $2 X_1 + 4 X_2 \leq 4000$.
Every point below the line will satisfy the inequality (less than) $2 X_1 + 4 X_2 \leq 4000$.

Fig 2



Now the final graph will look like this:

Fig. 3



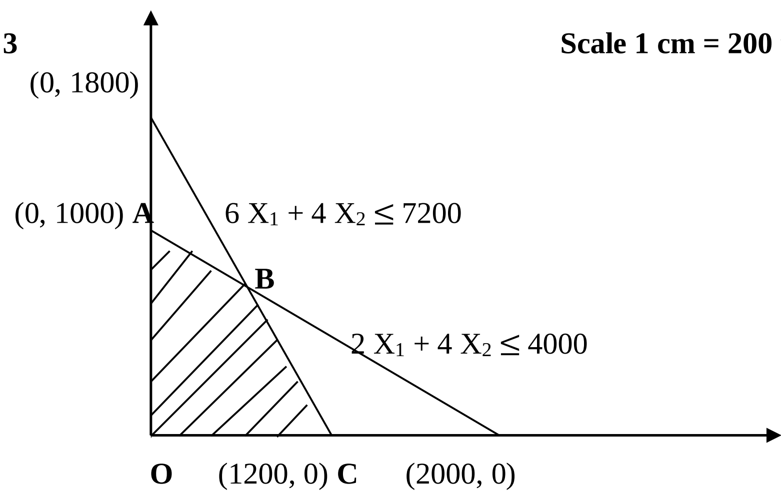
Step 5: Identification of Feasible Region

The feasible region is the region bounded by constraint lines. All points inside the feasible region or on the boundary of the feasible region or at the corner of the feasible region satisfy all constraints.

Both the constraints are 'less than or equal to' ($\leq$) type. Hence, the feasible region should be inside both constraint lines.

Hence, the feasible region is the polygon OABC. 'O' is the origin whose coordinates are (0, 0). O, A, B and C are called vertices of the feasible region.

Fig 3 **Scale 1 cm = 200 units**



Step 6: Finding the optimal Solution

The optimal solution always lies at one of the vertices or corners of the feasible region.

To find optimal solution:

We use corner point method. We find coordinates (X_1, X_2 Values) for each vertex or corner point. From this we fine 'Z' value for each corner point.

Vertex	**Co-ordinates**	**Z = 100 X_1 + 80 X_2**
O	$X_1 = 0, X_2 = 0$ From Graph	Z = 0
A	$X_1 = 0, X_2 = 1000$ From Graph	Z = Rs. 80, 000

B	$X_1 = 800$, $X_2 = 600$ From Simultaneous equations	Z = Rs. 1, 28, 000
C	$X_1 = 1200$, $X_2 = 0$ From Graph	Z = Rs. 1, 20, 000

Max. Z = Rs. 1, 28, 000 (At point B)

For B $\longrightarrow$ B is at the intersection of two constraint lines $6 X_1 + 4 X_2 \leq 7200$ and $2 X_1 + 4 X_2 \leq 4000$. Hence, values of X_1 and X_2 at B must satisfy both the equations.

We have two equations and two unknowns, X_1 and X_2. Solving simultaneously.

$6 X_1 + 4 X_2 \leq 7200$ (1)

$2 X_1 + 4 X_2 \leq 4000$ (2)

$4 X_1 = 3200$ Subtracting (2) from (1)

$X_1 = 800$

Substituting value of X_1 in equation (1), we get

$4 X_2 = 2400$ $\therefore X_2 = 600$

Solution

Optimal Profit = Max Z = Rs. 1, 28, 000

Product Mix:

X_1 = No. of units of A / Month = 800

X_2 = No. of units of A / Month = 600

ISO Profit line:

ISO profit line is the line which passes through the points of optimal solution (Maximum Profit). The slope of the iso-profit line depends on the objective function.

In the above example, the objective function is:

Max. $Z = 100 X_1 + 80 X_2$

How to find slope of iso-profit line:

Equation of a straight line: $y = mx + c$

where, m = slope of the straight line

In our case, y means ' X_2' and x means ' X_1'.

c means 'Z'.

$\therefore X_2 = m . X_1 + Z$

Converting original objective function in this format:

Max. $Z = 100 X_1 + 80 X_2$

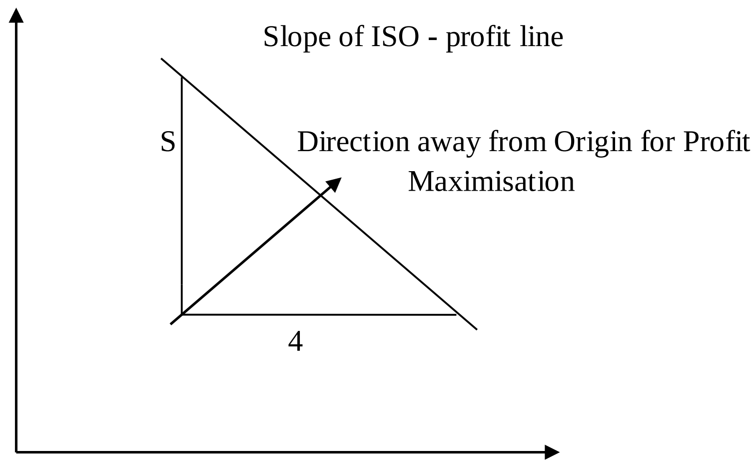
$\therefore 80 X_2 = Z - 100 X_1 = - 100 X_1 + Z$

$\therefore X_2 = \frac{-100}{80} X_1 + \frac{Z}{80}$

$\therefore X_2 = \frac{-5}{4} X_1 + \frac{Z}{80}$

$\therefore$ Slope of ISO profit line = $\frac{-5}{4}$

Negative sign indicates that the line will slope from left to right downwards. And slope will be 5/4. Every 4 units on X-axis for 5 units on Y-axis.



As we start from the origin and go away from origin to maximise profit, 'B' is the last point on the feasible region that is intersected by the iso-profit line. Hence, B is the optimal solution.

MINIMISATION

Example 2

A firm is engaged in animal breeding. The animals are to be given nutrition supplements everyday. There are two products A and B which contain the three required nutrients.

Nutrients	**Quantity/unit**		**Minimum Requirement**
	A	**B**	
1	72	12	216
2	6	24	72
3	40	20	200

Product cost per unit are: A: rs. 40; B: Rs. 80. Find out quantity of product A & B to be given to provide minimum nutritional requirement.

Step 1: Formulation as LPP

X_1 - Number of units of A

X_2 - Number of units of B

Z - Total Cost

Min. $Z = 40\ X_1 + 80\ X_2$

Subject to constraints:

$72\ X_1 + 12\ X_2 \geq 216$

$6\ X_1 + 24\ X_2 \geq 72$

$40\ X_1 + 20\ X_2 \geq 200$

$X_1, X_2 \geq 0.$

Step 2: Determination of each axis

Horizontal (X) axis: Product A (X_1)

Vertical (Y) axis: Product B (X_2)

Step 3: Finding co-ordinates of constraint lines to represent the graph

All constraints are 'greater than or equal to' type. We should convert them into equality:

1. Constraint No. 1: $72\ X_1 + 12\ X_2 \geq 216$

Converting into equality

$72\ X_1 + 12\ X_2 = 216$

To find X_1, let $X_2 = 0$

$72\ X_1 = 216$ $\therefore X_1 = 3, X_2 = 0$ (3, 0)

To find X_2, let $X_1 = 0$

$12\ X_2 = 216$ $\therefore X_1 = 0, X_2 = 18$ (0, 18)

2. Constraint No. 2:

$6\ X_1 + 24\ X_2 \geq 72$

To find X_1, let $X_2 = 0$

$6\ X_1 = 72$ $\therefore X_1 = 12, X_2 = 0$ (12, 0)

To find X_2, let $X_1 = 0$

$24\ X_2 = 72$ $\therefore X_1 = 0, X_2 = 3$ (0, 3)

3. Constraint No. 3:

$40\ X_1 + 20\ X_2 \geq 200$

To find X_1, let $X_2 = 0$

$40\ X_1 = 200$ $\therefore X_1 = 5, X_2 = 0$ (5, 0)

To find X_2, let $X_1 = 0$

$20\ X_2 = 200$ $\therefore X_1 = 0, X_2 = 10$ (0, 10)

The co-ordinates of points are:

1. Constraint No. 1: (3, 0) & (0, 18)

2. Constraint No. 2: (12, 0) & (0, 3)

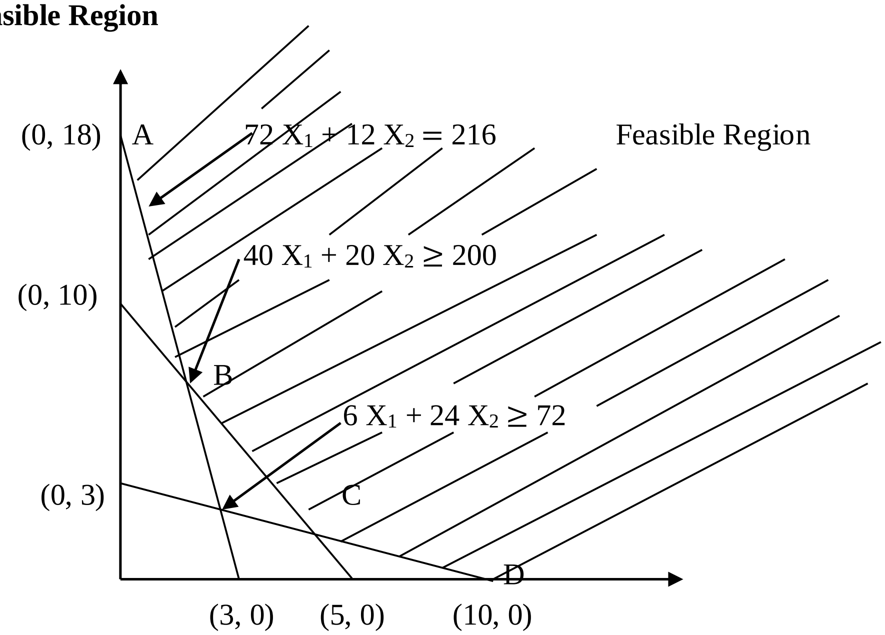
3. Constraint No. 3: (5, 0) & (0, 5)

Every point on the line will satisfy the equation (equality) $72\ X_1 + 12\ X_2 = 216$.

Every point above the line will satisfy the inequality (greater than) $72\ X_1 + 12\ X_2 = 216$.

Similarly, we can draw lines for other two constraints.

Step 5: Feasible Region



All constraints are greater than or equal to (≥) type. Hence, feasible region should be above (to the right of) all constraints.

The vertices of the feasible region are A, B, C & D.

Step 6: Finding the optimal solution

Corner Point Method

Vertex	Co-ordinates	$Z = 40\ X_1 + 80\ X_2$
A	$X_1 = 0$, $X_2 = 18$ From Graph	$\therefore Z = 1,440$
B	$X_1 = 2$, $X_2 = 6$ From Simultaneous Equations	$\therefore Z = 560$
C	$X_1 = 4$, $X_2 = 2$ From Simultaneous Equations	$\therefore Z = 320$
D	$X_1 = 12$, $X_2 = 0$ From graph	$\therefore Z = 480$

$\therefore$ Min. Z = Rs. 320 (At point 'C')

For B - Point B is at intersection of constraint lines '$72\ X_1 + 12\ X_2 \geq 216$' and '$40\ X_1 + 20\ X_2 \geq 200$'. Hence, point B should satisfy both the equations.

$72\ X_1 + 12\ X_2 = 216$ (1)

$40\ X_1 + 20\ X_2 = 200$ (2)

$\therefore 360\ X_1 + 60\ X_2 = 1080$ (1) x 5

$120\ X_1 + 60\ X_2 = 600$ (2) x 3

$\therefore 240\ X_1 = 480$

$X_1 = 2$

Substituting value of X_1 in equation (1), we get:

$12\ X_2 = 216 - 144 = 72$

$X_2 = 6$

For C - Point C is at intersection of constraint lines '$6\ X_1 + 24\ X_2 = 72$' and '$40\ X_1 + 20\ X_2 = 200$'. Hence, point C should satisfy both the equations.

$6\ X_1 + 24\ X_2 = 72$ (1)

$40\ X_1 + 20\ X_2 = 200$ (2)

$30\ X_1 + 120\ X_2 = 360$ (1) x 5

$240\ X_1 + 120\ X_2 = 1200$ (2) x 6

$210\ X_1 = 840$

$X_1 = 4$

Substituting value of X_1 in equation (1), we get

$24\ X_2 = 72 - 24 = 48$

$X_2 = 2$

Solution

Optimal Cost = Z min = Rs. 320

Optimal Product Mix:

X_1 = No. of units of product A = 4

X_2 = No. of units of product B = 2

ISO Cost Line

ISO Cost line passes through the point of optimal solution (Minimum cost)

Objective function: $Z = 40\ X_1 + 80\ X_2$

Equation of straight line: $y = mx + c$

where m = slope

In this case, y is X_2 & x is X_1

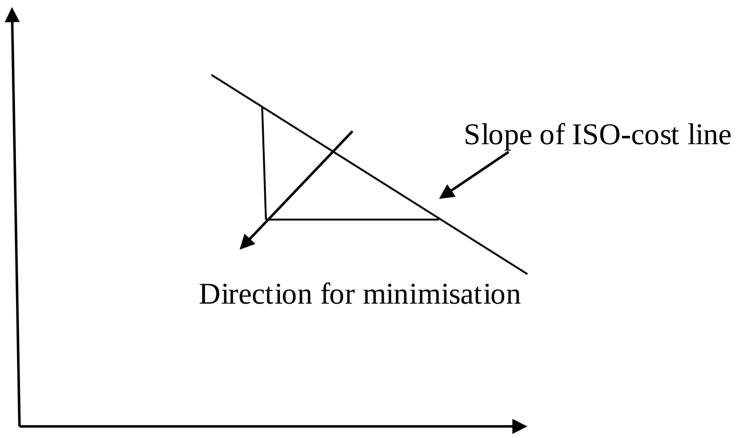
$Z = 40\ X_1 + 80\ X_2$

$\therefore 80\ X_2 = -\ 40\ X_1 + Z$

$\therefore\ X_2 = -1/2x_1 + Z/80$

$\therefore$ Slope of ISO-cost line = -1/2

Sloping from left to right downwards



$\therefore$ Point C is the nearest to the origin.

MAXIMISATION-MIXED CONSTRAINTS

Example 1

A firm makes two products P_1 & P_2 and has production capacity of 18 tonnes per day. P_1 & P_2 require same production capacity. The firm must supply at least 4 t of P_1 & 6 t of P_2 per day. Each tonne of P_1 & P_2 requires 60 hours of machine work each. Maximum machine hours available are 720. Profit per tonne for P_1 is Rs. 160 & P_2 is Rs. 240. Find optimal solution by graphical method.

LPP Formulation

X_1 = Tonnes of P_1 / Day

X_2 = Tonnes of P_2 / Day

Max. $Z = 160\ X_1 + 240\ X_2$

Subject to constraints

$X_1 \geq 4$

$X_2 \geq 6$

$X_1 + X_2 \leq 18$

$60\ X_1 + 60\ X_2 \leq 720$

$X_1,\ X_2 \geq 0$

Coordinates for constraint lines:

1. $X_1 \geq 4$ (4, 0) .... No value for X_2, $\therefore X_2 = 0$
2. $X_2 \geq 6$ (0, 6) .... No value for X_1, $\therefore X_1 = 0$
3. $X_1 + X_2 \leq 18$ (18, 0) (0, 18)
4. $60\ X_1 + 60\ X_2 \leq 720$ (12, 0) (0, 12)

If $X_1 = 0$, $60\ X_2 = 720$ $\therefore X_2 = 12$ (0, 12)

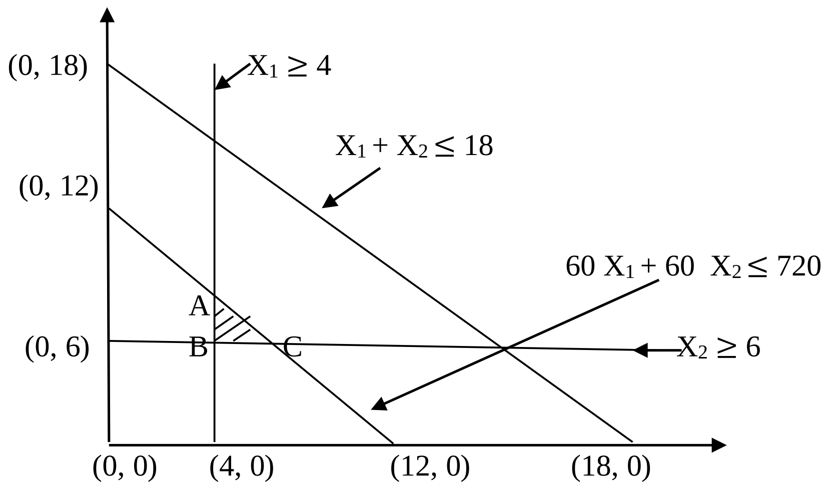
If $X_2 = 0$, $60\ X_1 = 720$ $\therefore X_1 = 12$ (12, 0)

Graph: X_1: X Axis

X_2: Y Axis

Scale:

Maximum value for $X_1 = 18$; Maximum value for $X_2 = 18$; $\therefore$ Scale: 1 cm = 2 Tonnes.



Two constraints are 'greater than or equal to' type. Hence, feasible region will be above or to the right of these constraint lines. Two constraints are 'less than or equal to' type. Hence, feasible region will be below or to the left of these constraint lines. Hence, feasible region is ABC.

Optimal Solution

Corner Point Method

Vertex	Coordinates	$Z = 160\ X_1 + 240\ X_2$
A	$X_1 = 4$, $X_2 = 8$ Simultaneous Equation	$\therefore$ Z = Rs. 2, 560
B	$X_1 = 4$, $X_2 = 6$ From Graph	$\therefore$ Z = Rs. 2, 080
C	$X_1 = 6$, $X_2 = 6$ Simultaneous Equations	$\therefore$ Z = Rs. 2, 400

For A $\longrightarrow$ $X_1 = 4$ from graph

A is on the line $60\ X_1 + 60\ X_2 = 720$

$60\ X_2 = 720 - 60\ (4) = 480$ $\therefore X_2 = 8$

For C $\longrightarrow$ $X_2 = 6$ from graph

A is on the line $60\ X_1 + 60\ X_2 = 720$

$60\ X_1 = 720 - 60\ (6) = 360$ $\therefore X_1 = 6$

$\therefore$ Z = Rs. 2, 560 (At point 'A')

Solution

Optimal Profit Z. Max = Rs. 2, 560

X_1 = Tonnes of P_1 = 4 tonnes

X_2 = Tonnes of P_2 = 8 tonnes.

MINIMISATION MIXED CONSTRAINTS

Example 1:

A firm produces two products P and Q. Daily production upper limit is 600 units for total production. But at least 300 total units must be produced every day. Machine hours consumption per unit is 6 for P and 2 for Q. At least 1200 machine hours must be used daily. Manufacturing costs per unit are Rs. 50 for P and Rs. 20 for Q. Find optimal solution for the LPP graphically.

LPP formulation

X_1 = No. of Units of P / Day

X_2 = No. of Units of Q / Day

Min. $Z = 50 X_1 + 20 X_2$

Subject to constraints

$X_1 + X_2 \leq 600$

$X_1 + X_2 \geq 300$

$6 X_1 + 2 X_2 \geq 1200$

$X_1, X_2 \geq 0$

Coordinates for Constraint lines

1. $X_1 + X_2 = 600$

If $X_1 = 0$, $X_2 = 600$ $\quad\therefore$ (0, 600)

If $X_2 = 0$, 60 $X_1 = 600$ $\quad\therefore$ (600, 0)

2. $X_1 + X_2 = 300$

If $X_1 = 0$, $X_2 = 300$ $\quad\therefore$ (0, 300)

If $X_2 = 0$, 60 $X_1 = 300$ $\quad\therefore$ (300, 0)

3. $6 X_1 + 2 X_2 \geq 1200$

If $X_1 = 0$, $2X_2 = 1200$ $\quad\therefore X_2 = 600$ $\quad$ (0, 600)

If $X_2 = 0$, 6 $X_1 = 1200$ $\quad\therefore X_1 = 200$ $\quad$ (200, 0)

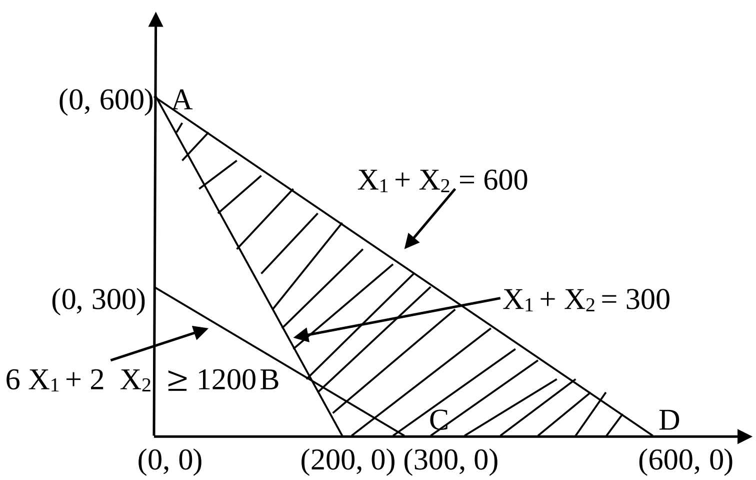
Graph: X_1: X Axis

X_2: Y Axis

Scale:

Maximum value for X_1 = 600; Maximum value for X_2 = 600; $\therefore$ Scale: 1 cm = 50 units.

Feasible region is ABCD.



Two constraints are 'greater than or equal to' type. Hence, feasible region will be above or to the right of these constraint lines. Two constraints are 'less than or equal to' type. Hence, feasible region will be below or to the left of these constraint lines. Hence, feasible region is ABCD.

Optimal Solution

Corner Point Method

Vertex	Coordinates	$Z = 160\ X_1 + 240\ X_2$
A	$X_1 = 0$, $X_2 = 600$ From Graph	$\therefore Z =$ Rs. 12, 000
B	$X_1 = 150$, $X_2 = 150$ Simultaneous Equations	$\therefore Z =$ Rs. 10, 500
C	$X_1 = 300$, $X_2 = 0$ From Graph	$\therefore Z =$ Rs. 15, 000
D	$X_1 = 600$, $X_2 = 0$ From Graph	$\therefore Z =$ Rs. 30, 000

Min. Z = Rs. 10, 500

For B - B is at intersection of two constraint lines '$6\ X_1 + 2\ X_2 \geq 1200$' and '$X_1 + X_2 = 300$'.

$6\ X_1 + 2\ X_2 \geq 1200$ (1)

$X_1 + X_2 = 300$ (2)

$2X_1 + 2X_2 = 600$ (2) X 2

$2X_1 = 600$

$X_1 = 150$

Substituting value in Equation (2), $X_2 = 150$.

Solution

Optimal Cost = Rs. 10, 500/-

X_1 = No. of Units of P = 150

X_2 = No. of Units of P = 150.

EXERCISES

1. What is meant by feasible region in graphical method.

2. What is meant by 'iso-profit' and 'iso-cost line' in graphical solution.

3. Mr. A. P. Ravi wants to invest Rs. 1, 00, 000 in two companies 'A' and 'B' so as not to exceed Rs. 75, 000 in either of the company. The company 'A' assures average return of 10% in whereas the average return for company 'B' is 20%. The risk factor rating of company 'A' is 4 on 0 to 10 scale whereas the risk factor rating for 'B' is 9 on similar scale. As Mr. Ravi wants to maximise his returns, he will not accept an average rate of return below 12% risk or a risk factor above 6.

Formulate this as LPP and solve it graphically.

4. Solve the following LPP graphically and interpret the result.

Max. $Z = 8X_1 + 16 X_2$

Subject to:

$X_1 + X_2 \leq 200$

$X_2 \leq 125$

$3X_1 + 6X_2 \leq 900$

$X_1, X_2 \geq 0$

5. A furniture manufacturer makes two products - tables and chairs.

Processing of these products is done on two types of machines A and B. A chair requires 2 hours on machine type A and 6 hours on machine type B. A table requires 5 hours on machine type A and no time on Machine type B. There are 16 hours/day available on machine type A and 30 hours/day on machine type B. Profits gained by the manufacturer from a chair and a table are Rs. 2 and Rs. 10 respectively. What should be the daily production of each of the two products? Use graphical method of LPP to find the solution.

Special Cases in Linear Programming

a. Infeasible Solution (Infeasibility)

Infeasible means not possible. Infeasible solution happens when the constraints have contradictory nature. It is not possible to find a solution which can satisfy all constraints.

In graphical method, infeasibility happens when we cannot find Feasible region.

Example 1:

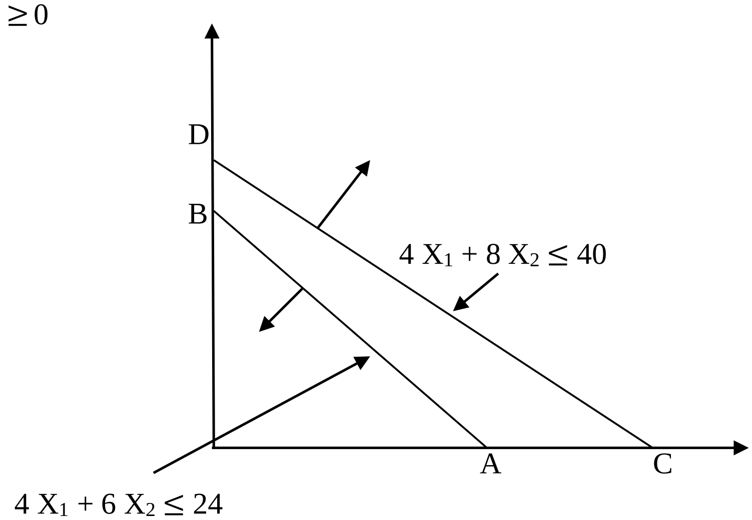
Max. $Z = 5 X_1 + 8 X_2$

Subject to constraints

$4 X_1 + 6 X_2 \leq 24$

$4 X_1 + 8 X_2 \leq 40$

$X_1, X_2 \geq 0$



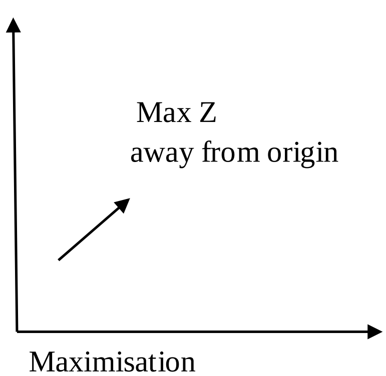
There is no common feasible region for line AB and CD.

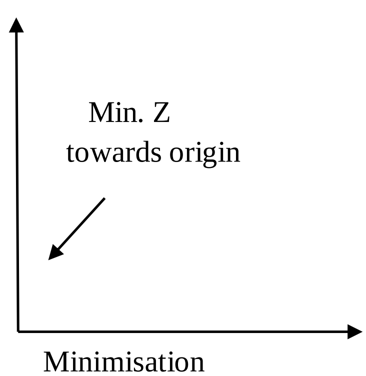
Hence, solution is infeasible.

b. Unbounded Solution (Unboundedness)

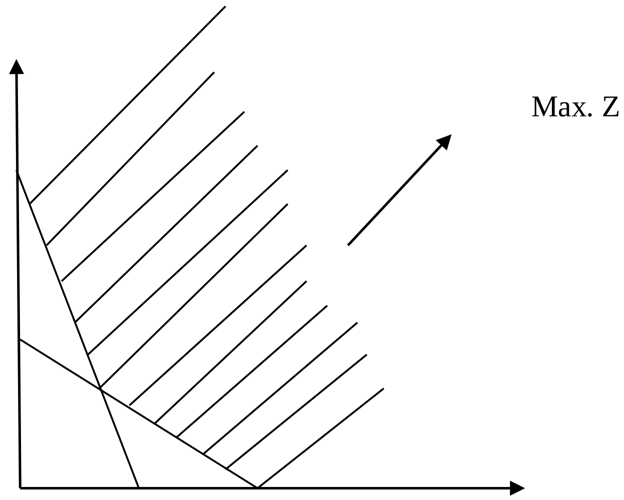
Unbounded mean infinite solution. A solution which has infinity answer is called unbounded solution.

In graphical solution, the direction with respect to origin is as follows:





Now, in a maximisation problem, if we have following feasible region:



There is no upper limit (away from origin), hence the answer will be infinity. This is called unbounded solution.

c. Redundant Constraint (Redundancy)

A constraint is called redundant when it does not affect the solution. The feasible region does not depend on that constraint.

Even if we remove the constraint from the solution, the optimal answer is not affected.

Example

Max. $Z = 5 X_1 + 8 X_2$

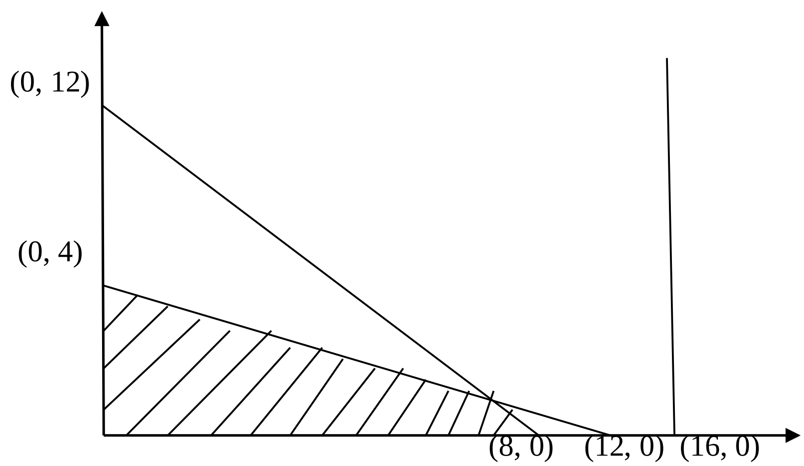
Subject to Constraints

$3 X_1 + 2 X_2 \leq 24$

$X_1 + 3 X_2 \leq 12$

$X_1 \leq 16$

$X_{1,} X_2 \geq 0$



The feasible region for the above problem is OABC. The 3rd constraint does not affect the feasible region.

Hence, the constraint $X_1 \leq 16$ is redundant constraint.

d. Alternate Optimal Solution: (Multiple Optimal Solution)

Alternate or multiple optimal solution means a problem has more than one solution which gives the optimal answer.

There are two or more sets of solution values which give maximum profit or minimum cost. In graphical method, we come to know that there is optimal solution which is alternative when:
The iso-profit or iso-cost line is parallel to one of the boundaries of feasible region (they have the same slope value).

SPECIAL CASES IN SIMPLEX

1. Unbounded Solution

Max $Z = 60 X_1 + 20X_2\Delta$
Subject to
$2 X_1 + 4X_2 \geq 120$
$8 X_1 + 6X_2 \geq 240$
$X_1, X_2 \geq 0$

Solution

Max $Z = 60 X_1 + 20X_2 + 0S_1 + 0S_2 - MA_1 - MA_2$
Subject to
$2 X_1 + 4X_2 - S_1 + A_1 = 120$
$8 X_1 + 6X_2 - S_2 + A_1 = 240$
$X_1, X_2, S_1, S_2, A_1, A_2, \geq 0$
When we solve this LPP by simplex method, we will get the following values in 4th Simplex Table.

C_j			60	20	0	0	- M	- M	R.R
C	X	B	X_1	X_2	S_1	S_2	A_1	A_2	
0	S_2	240	0	10	- 4	1			- 60
60	X_1	60	1	2	-1/2	0			- 120
$Z_j \longrightarrow$			60	120	- 30	0			
$\Delta = C_j - Z_j \longrightarrow$			0	- 100	30	0			
						$\uparrow$			

Max Positive $C_j - Z_j = 30$
Key Column = S_1
But there is no positive Replacement Ration R means there is an Entering variable, but there is no outgoing variable.
Hence, the solution is unbounded or infinity.
The value of Z (Profit) keeps on increasing infinitely.

2. Infeasible Region

Max $Z = 3 X_1 + 2 X_2$
Subject to:
$X_1 + X_2 \leq 4$
$2 X_1 + X_2 \geq 10$
$X_1, X_2 \geq 0$
Solution
Standard Form
Max $Z = 3 X_1 + 2 X_2 + 0S_1 + 0S_2 - MA_1$
Subject to
$X_1 + X_2 + S_1 = 4$

$2 X_1 + X_2 - S_2 + A_1 = 10$

$X_1, X_2, S_1, S_2, A_1 \geq 0$

When we solve this LPP by simplex method, we will get the following values in 2nd Simplex Table.

C_j			60	20	0	0	- M	R.R
C	X	B	X_1	X_2	S_1	S_2	A_1	
3	X_1	4	1	1	1	0	0	
- M	A_1	2	- 2	- 1	-1/2	- 1	1	
$Z_j \longrightarrow$			3	3 + M	3 + 2M	M	- M	
$\Delta = C_j - Z_j \longrightarrow$			0	- 1 - M	-3 -2M	- M	0	

No positive Δ value.

All $C_j - Z_j$ values are either zero or negative. Hence, test of optimality is satisfied. So, the solution appears to be optimal. But an artificial variable (A_1) is present in the basis, which has objective function coefficient of - M (infinity).

Hence, the solution is infeasible (Not feasible).

Infeasibility occurs when there is no solution which satisfies all the constraints of the LPP.

Concept of Shadow Price

Shadow price of resource means value of one extra unit of resource. It is the maximum price the company should pay for procuring extra resources from market. It also indicates profitability or profit contribution of each resource (per unit).

Shadow price = 'Z_i' value of slack variables.

S_1 - slack variable of resource 1 and S_2 - Slack Variable of resource 2. A slack variable represents unutilised capacity of a resource. Slack Variable is represented by 'S'.

Concept of Duality

Every linear programming problem has a mirror image associated with it. If the original problem is maximisation, the mirror image is minimisation and vice versa.

The original problem is called 'primal' and the mirror image is called 'dual'.

The format of simplex method is such that when we obtain optimal solution of any one out of primal or dual, we automatically get optimal solution of the other.

For example, if we solve dual by simplex method, we also get optimal solution of primal.

Characteristics of Dual Problem

1. Dual of the dual is primal.
2. If either the primal or dual has a solution, then the other also has a solution. The optimal value of both the solutions is equal.
3. If any of the primal or dual is infeasible then the other has an unbounded solution.

Advantages of Duality

1. If primal problem contains a large number of rows and a smaller number of columns we can reduce the computational procedure by converting into dual.
2. Solution of the dual helps in checking computational accuracy of the primal.
3. Economic interpretation of the dual helps the management in decision making.
For example,
Minimisation LPP can be solved by two methods:
1. Simplex of Dual Method and
2. Artificial Variable Method
Method:1 Simplex of Dual Method
The original problem is called 'Primal'. We convert the problem in its 'Dual'.

Primal	Dual
1. Minimisation Problem Min. Z	Maximisation Problem Max. Z^*
2. Constraints are of ' $\geq$ ' type	Constraints are of ' $\leq$ ' type.
3. Decision variables are X_1, X_2 etc.	Decision variables are Y_1, Y_2 etc.

4. Objective function coefficients of primal (4, 3) become RHS of constraints in Dual.
5. RHS of Constraints of Primal (4000, 50, 1400) become objective function coefficients of Dual.
6. In the LHS (left side) of constraints, all vertical values are written horizontally in Dual.
7. No. of Decision Variables in Primal = No. of constraints in Dual
8. No. of Constraints in Primal = No. of Decision Variables in Dual
For example,

Primal	Dual
Min $Z = 4 X_1 + 3 X_2$ Subject to: $200 X_1 + 100 X_2 \geq 4000$ $1 X_1 + 2 X_2 \geq 50$ $40 X_1 + 40 X_2 \geq 1400$	Max $Z = 4000 Y_1 + 50 Y_2 + 1400 Y_3$ Subject to: $200 y_1 + 1y_2 + 40y_3 \leq 4$ $100 y_1 + 2y_2 + 40y_3 \leq 3$

The numerical is calculated as shown in Simplex Method.

EXERCISES

1. Why an optimal solution to an unbounded maximisation LPP cannot be found in Simplex Method?
2. What is meant by shadow price of a resource?

UNIT 6
SENSITIVITY ANALYSIS

The solution to a LPP is determined by simplex method is a static solution, It means that solution corresponds to:

1. Value of profit coefficients in the objective function, and
2. Availability of resources (i.e R.H.S of constraints)

But in reality, profit coefficients of variables may increase or decrease. Similarly, availability of resources may also increase or decrease. In that case, the optimal profit and optimal quantity (b values) of variables calculated as per Simplex solution will change.

The objective of sensitivity analysis is to determine the new values of solution. If possible, from the given simplex solution. This will be possible only if the changes (increase/decrease) in the objective function or constraint capacities is in certain limits. These will be two limits, lower limit (max. possible decrease) and upper limit (max. possible increase). These two limits provide the range within which the present simplex table remains optimal.

Hence, if the change in profit or change in capacity constraints is in the range, we can find new values of the solution. If it is not in the range, we cannot find the new values of the solution. Because the present simplex table will not remain optimal any more.

EXAMPLE 1: An electronics firm manufactures three products - transistors, resistors and capacitors which give profit of Rs. 100, 60 and 40 per unit respectively.
The firm uses three resources - Engineering, Direct Labour and Admin. capacities are 100, 600 and 300 hrs. respectively.
Following simplex solution is obtained.

C_j			100	60	40	0	0	0
C	X	B	X_1	X_2	X_3	S_1	S_2	S_3
60	X_2	200/3	0	1	5/6	1	- 1/6	0
100	X_1	100/3	1	0	1/6	0	1/6	0
0	S_3	100	0	0	4		0	1
Z_j $\longrightarrow$			100	60	200/3	0	20/3	0
$\Delta = C_j - Z_j$ $\longrightarrow$			0	0	-80/3	0	- 20/3	0

Note: The solution is optimal as there is no positive $C_j - Z_j$.

Optimal Product Mix is:
X_1 = 100/3 = No. of units of transistors.
X_2 = 200/3 = No. of units of resistors
X_3 (capacitors) is not produced.

Optimal Profit is:
Max. Z = [60 X 200/3] + [100 X 100/3] = [4000 + 10, 000/3]
Max. Z = Rs. 22, 000/3

A. Range for profit coefficients of products:

Formula = $\frac{\Delta j}{Xn}$

B. Range for profit coefficients of X_1:

Formula = $\frac{\Delta j}{Xi}$

We take ration of 'Δ' (C_j - Z_j) row and ' X_1' row:

The ratios will be:

$\frac{-80/3}{1/6} = \frac{-80}{3}$ x $\frac{6}{1}$ = - 160

$\frac{-\frac{100}{3}}{-2/3} = \frac{-100}{3}$ x $-\frac{3}{2}$ = 50

$\frac{-20/3}{1/6} = \frac{-20}{3}$ x $\frac{6}{1}$ = - 40

'-' sign indicates decrease in profit & '+' sign indicates increase in profit. It means possible decrease is 160 or 40. Hence, we can decrease profit by only 40. Possible increase is 50.

∴ Range for profit coefficient of X_1 is:

Original Profit $\pm$ (increase or decrease)

100 + 50 = 150

100 - 40 = 60

∴ Rs. 60 to Rs. 150

It means profit of X_1 (transistors) can fluctuate within the range of Rs. 60 to Rs. 150. The simplex solution will remain optimal in this range.

2. Range for profit coefficient of X_2

Formula = $\frac{\Delta 1}{X2}$

We take ration of 'Δ' (C_j - Z_j) row and 'X_2' row:

$\frac{-80/3}{5/6} = \frac{-80}{3}$ x $\frac{6}{5}$ = - 32

$\frac{-\frac{100}{3}}{5/3} = \frac{-100}{3}$ x $-\frac{3}{5}$ = - 20

$\frac{-20/3}{1/6} = \frac{-20}{3}$ x $\frac{6}{1}$ = 40

Hence, possible decrease is Rs. 20 and possible increase is Rs. 40. Range for profit coefficient of X_2 is

60 - 20 = 40

60 + 40 = 100

Rs. 40 to Rs. 100.

It means the present simplex solution will remain optimal even if profit of X_2 (resistors)fluctuates in the range of Rs. 40 to Rs. 100.

3. Range for profit coefficient of X_3

X_3 capacitors is not produced.

Δ of X_3 = -80/3

It means if we produce $X_{3,}$ we will incur a loss of Rs. 80/3 per unit of X_3.So, X_3 will be produced only if present profit of X_3 is increased by Rs. 80/3.

∴ X_3 will be produced if its profit becomes [40 + 80/3] = Rs. 200/3 or more than that.

Hence, the present simplex solution remains optimal till the profit value of 200/3 for X_3.

∴ Range for profit coefficient of X_3 is

Rs. zero to 200/3.

B. Range for capacity of resources

OR Range for validity of shadow prices of Resources

Formula = - $[\frac{b}{Sn}]$

1. Range for capacity or availability of Engineering hours

Engineering hours is represented in simplex table by slack variable s_{1}:

Formula = - $[\frac{b}{S1}]$

We will take ratio of 'b' column and 'S_1' column.

The ratios will be

- $[\frac{200/3}{5/6}] = -[\frac{200}{3} \text{ x } \frac{3}{5}] = -40$

- $[\frac{100/3}{-2/3}] = -[\frac{100}{3} \text{ x } \frac{-3}{2}] = 50$

- $[\frac{100/3}{-2/3}] = -[-50] = 50$

Hence, possible decrease in capacity is 40 hrs and possible increase in capacity is 50 hrs.

Range for resource capacity of Engineering is:

Original Capacity $\pm$ (increase or decrease)

100 + 950) = 150

100 - 40 = 60

∴ Range is 60 hrs to 150 hrs.

It means the present simplex solution will remain optimal even if availability of Engineering resource fluctuates between 60 hrs to 150 hrs.

Note: In other words the shadow price of Engineering resource [which is Rs. 100/3] will remain valid even if the resource availability fluctuates between 60 hrs. to 150 hrs.

2. Range for availability of Direct Labour

Formula = - $[\frac{b}{S2}]$

The ratios will be

- $[\frac{200/3}{-1/6}] = -[\frac{200}{3} \text{ x} \frac{-6}{1}] = 400$

- $[\frac{100/3}{-1/6}] = -[\frac{100}{3} \text{ x } \frac{6}{1}] = -200$

- $[\frac{100}{0}]$ = Infinity.

∴ Range for resource capacity of Direct Labour is:

600 + 400 = 1000

600 - 200 = 400

∴ Range is 400 hrs to 1000 hrs.

It means the present simplex solution will remain optimal even if availability of Direct Labour resource fluctuates between 400 hrs to 1000 hrs.

Note: In other words, the shadow price of Direct Labour resource (which is Rs. 20/3) will remain valid even if the resource availability fluctuates between 400 hrs to 1000 hrs.

3. Range for availability of Admin:

Formula = - $[\frac{b}{S3}]$

The ratios will be

$- [\frac{200/3}{0}] = -$ [Infinity] = Infinity

$- [\frac{100/3}{0}] = -$ [Infinity]= Infinity

$- [\frac{100}{1}] = - 100.$

It means the capacity can increase upto infinity.

Possible decrease = 100 hrs.

Range for resource capacity of Admin is

300 + Infinity = Infinity

300 - 100 = 200

∴ Range is 200 hrs to Infinity.

It means the present simplex solution will remain optimal even if availability of Admin resource fluctuates between 200 hrs to Infinity.

Note: In other words, the shadow price of Admin resource (which is Rs. zero) will remain valid even if the resource availability fluctuates between 200 hrs to infinity.

c. Effect on the solution due to increase or decrease in the availability of resources

1. What will be the effect on solution if capacity of Engineering is increased by 30%?

Answer:

Original capacity = 100 hrs.

Increase = 30%

New capacity = 130 hrs.

Note:

To find the effect on solution we need to find the range of resource capacity. We can find the effect on solution only if the new capacity is in the range.

If the new capacity goes out of the range, then we cannot find effect on solution.

Because in that case, the present simplex solution does not remain optimal any more.

From earlier calculation, we know that, Range for resource capacity of Engineering is 60 hrs to 150 hrs.

∴ New capacity is in the range.

Change in capacity = 130 - 100 = + 30 hrs.

Hence, we multiply column S_1 by + 30.

From that we will get change in 'b' column.

	S_1 X (+30) = Change in 'b' column
X_2 ⟶	5/3 X + 30 = + 50
X_1 ⟶	-2/3 X + 30 = - 20

Now we can find new basis values.

New Basis:

c	X	New b
60	X_2	200/3 + 50 = 350/3
100	X_1	100/3 - 20 = 40/3

New Z = (60 x 350/3) + (100 X 40/3)

New Z = 25, 000/3 Rs.

Increase in optimal profit = 25, 000 - 22, 000/3

= Rs. 1, 000

New Optimal Product Mix:

X_1 = 40/3 units

X_2 = 350/3 units.

2. Can you find out effect on optimal solution if excess capacity of abundant resource is transferred to Direct Labour?

Answer:

In the optional solution, S_3 is present in the basis.

S_3 = 100

S_3 represents slack value of Admin.

Hence, Admin is abundant resource and its excess (unused) capacity is 100 hrs.

Capacity of Direct Labour = 600 hrs.

If 100 hrs are transferred,

New capacity of Direct labour = 700 hrs.

We, know that range of Direct Labour capacity is 400 hrs to 1000 hrs.

New capacity is in the range.

	S_1 X (+30) = Change in 'b' column
$X_2 \longrightarrow$	-1/6 X + 100 = - 100/6 - -50/3
$X_1 \longrightarrow$	1/6 X + 100 = 100/6 = 50/3

New Basis:

c	X	New b
60	X_2	200/3 - 50/3 = 150/3 = 50
100	X_1	100/3 + 50/3 = 150/3 = 50

New Z = (60 x 50) + (100 + 50) = Rs. 8, 000

Increase in optimal profit = 8, 000 - 22, 000/3

= Rs. 2, 000/3

New Optimal Product Mix:

X_1 = 50 units

X_2 = 50 units.

3. What will be the effect on optimal solution if capacity of Admin is reduced to 175 hrs?

Answer: Range for capacity of Admin = 200 hrs to Infinity.

Since, 175 hrs is out of the range, if Admin capacity is reduced to 175 hr. solution will not remain optimal.

EXERCISES

1. An engineering company BMS Ltd. produces three products A, B and C using three machines M_1, M_2 and M_3. The resource constraints on M_1, M_2 and M_3 are 96, 40 and 60 hours respectively. The profits earned by the products A, B and C are Rs. 2, Rs.5 and Rs. 8 per unit respectively. A simplex optimal solution to maximize the profit is given below where X_1, X_2 and X_3 are quantities of products A, B and C produced by the company s_1, s_2 and s_3 represent the slack in the resources M_1, M_2 and M_3. Study the solution given below and answer the following questions:

C	X Variables in the basis	X_1	X_2	X_3	S_1	S_2	S_3	B Solution Values
5 8 0	X_2 X_3 S_3	1/3 5/6 7/3	1 0 0	0 1 0	1/6 - 1/12 - 1/13	- 1/3 2/3 - 1/3	0 0 1	8/3 56/3 44/3
	$\Delta = C - Z$	- 19/3	0	0	- 1/6	- 11/3	0	

1. Indicate the shadow price of each resource. Which of the resources are abundant and which are scarce?

2. What profit margin for product A do you expect the marketing department to secure if it is to be produced, and justify your advice?

3. Within what range, the profit of product B can change for the above solution to remain optimal?

4. How would an increase of 10 hours in the resource M_2 affect the optimality?

5. If the company BMS Ltd. wishes to raise production which of the three resources should be given priority for enhancement?

2. A business problem is formulated and expressed below as an LPP. (Profit is in Rs. and Resources are in units).

Objective function

Maximise $Z = 80 X_1 + 100 X_2$

Subject to resource constraints,

$X_1 + 2X_2 \leq 720$ (Resource 1)

$5X_1 + 4X_2 \leq 1800$ (Resource 2)

$3X_1 + X_2 \leq 900$ (Resource 3)

$X_1, X_2 \geq 0$

Simplex algorithm of LPP, applied to the above problem yielded following solution:

Basis							B_1
C_b	X_b	X_1	X_2	S_1	S_2	S_3	
100	X_2	0	1	5/6	- 1/6	0	300
80	X_2	1	0	- 2/3	1/3	0	120
0	S_3	0	0	7/6	- 5/6	1	240
	C_j	80	100	0	0	0	
	$\Delta = C_j - Z_j$	0	0	- 30	- 10	0	

1. Answer the following questions with justification:

a. Is the solution optimal and unique?

b. Is the above solution infeasible?

c. What is the maximum profit as per optimal solution?

d. Which resources are abundant and which are scarce as per optimal solution?

2. Find out the range of coefficient of X_1 in the objective function for which the above solution remains optimal.

3. Can you obtain the solution values of basic variables form the optimal solution when resource constraint (a) Changes to 750 units? If yes, find the new values of the basic variables.

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9.3 THE SIMPLEX METHOD: MAXIMIZATION

For linear programming problems involving two variables, the graphical solution method introduced in Section 9.2 is convenient. However, for problems involving more than two variables or problems involving a large number of constraints, it is better to use solution methods that are adaptable to computers. One such method is called the **simplex method,** developed by George Dantzig in 1946. It provides us with a systematic way of examining the vertices of the feasible region to determine the optimal value of the objective function. We introduce this method with an example.

Suppose we want to find the maximum value of $z = 4x_1 + 6x_2$, where $x_1 \geq 0$ and $x_2 \geq 0$, subject to the following constraints.

$$\begin{aligned} -x_1 + x_2 &\leq 11 \\ x_1 + x_2 &\leq 27 \\ 2x_1 + 5x_2 &\leq 90 \end{aligned}$$

Since the left-hand side of each *inequality* is less than or equal to the right-hand side, there must exist nonnegative numbers s_1, s_2 and s_3 that can be added to the left side of each equation to produce the following system of linear *equations*.

$$\begin{aligned} -x_1 + x_2 + s_1 \phantom{{}+ s_2 + s_3} &= 11 \\ x_1 + x_2 \phantom{{}+ s_1} + s_2 \phantom{{}+ s_3} &= 27 \\ 2x_1 + 5x_2 \phantom{{}+ s_1 + s_2} + s_3 &= 90 \end{aligned}$$

The numbers s_1, s_2 and s_3 are called **slack variables** because they take up the "slack" in each inequality.

Standard Form of a Linear Programming Problem

A linear programming problem is in **standard form** if it seeks to *maximize* the objective function $z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\leq b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\leq b_m \end{aligned}$$

where $x_i \geq 0$ and $b_i \geq 0$. After adding slack variables, the corresponding system of **constraint equations** is

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + s_1 \phantom{{}+ s_2 + s_m} &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \phantom{{}+ s_1} + s_2 \phantom{{}+ s_m} &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \phantom{{}+ s_1 + s_2} + s_m &= b_m \end{aligned}$$

where $s_i \geq 0$.

REMARK: Note that for a linear programming problem in standard form, the objective function is to be maximized, not minimized. (Minimization problems will be discussed in Sections 9.4 and 9.5.)

A **basic solution** of a linear programming problem in standard form is a solution $(x_1, x_2, \ldots, x_n, s_1, s_2, \ldots, s_m)$ of the constraint equations in which *at most m* variables are nonzero—the variables that are nonzero are called **basic variables.** A basic solution for which all variables are nonnegative is called a **basic feasible solution.**

The Simplex Tableau

The simplex method is carried out by performing elementary row operations on a matrix that we call the **simplex tableau.** This tableau consists of the augmented matrix corresponding to the constraint equations together with the coefficients of the objective function written in the form

$$-c_1x_1 - c_2x_2 - \cdots - c_nx_n + (0)s_1 + (0)s_2 + \cdots + (0)\,s_m + z = 0.$$

In the tableau, it is customary to omit the coefficient of z. For instance, the simplex tableau for the linear programming problem

$$z = 4x_1 + 6x_2 \qquad \text{Objective function}$$

$$\left.\begin{aligned} -x_1 + x_2 + s_1 \qquad\qquad &= 11 \\ x_1 + x_2 \qquad + s_2 \qquad &= 27 \\ 2x_1 + 5x_2 \qquad\qquad + s_3 &= 90 \end{aligned}\right\} \qquad \text{Constraints}$$

is as follows.

$$\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & b & \text{Basic Variables} \\ \hline -1 & 1 & 1 & 0 & 0 & 11 & s_1 \\ 1 & 1 & 0 & 1 & 0 & 27 & s_2 \\ 2 & 5 & 0 & 0 & 1 & 90 & s_3 \\ \hline -4 & -6 & 0 & 0 & 0 & 0 & \\ & & & & & \uparrow & \\ & & & & & \text{Current } z\text{–value} & \end{array}$$

For this **initial simplex tableau,** the **basic variables** are s_1, s_2, and s_3, and the **nonbasic variables** (which have a value of zero) are x_1 and x_2. Hence, from the two columns that are farthest to the right, we see that the current solution is

$$x_1 = 0, \quad x_2 = 0, \quad s_1 = 11, \quad s_2 = 27, \quad \text{and} \quad s_3 = 90.$$

This solution is a basic feasible solution and is often written as

$$(x_1, x_2, s_1, s_2, s_3) = (0, 0, 11, 27, 90).$$

The entry in the lower–right corner of the simplex tableau is the current value of z. Note that the bottom–row entries under x_1 and x_2 are the negatives of the coefficients of x_1 and x_2 in the objective function

$$z = 4x_1 + 6x_2.$$

To perform an **optimality check** for a solution represented by a simplex tableau, we look at the entries in the bottom row of the tableau. If any of these entries are negative (as above), then the current solution is *not* optimal.

Pivoting

Once we have set up the initial simplex tableau for a linear programming problem, the simplex method consists of checking for optimality and then, if the current solution is not optimal, improving the current solution. (An improved solution is one that has a larger z-value than the current solution.) To improve the current solution, we bring a new basic variable into the solution—we call this variable the **entering variable.** This implies that one of the current basic variables must leave, otherwise we would have too many variables for a basic solution—we call this variable the **departing variable.** We choose the entering and departing variables as follows.

1. The **entering variable** corresponds to the smallest (the most negative) entry in the bottom row of the tableau.
2. The **departing variable** corresponds to the smallest nonnegative ratio of b_i/a_{ij}, in the column determined by the entering variable.
3. The entry in the simplex tableau in the entering variable's column and the departing variable's row is called the **pivot.**

Finally, to form the improved solution, we apply Gauss-Jordan elimination to the column that contains the pivot, as illustrated in the following example. (This process is called **pivoting.**)

EXAMPLE 1 *Pivoting to Find an Improved Solution*

Use the simplex method to find an improved solution for the linear programming problem represented by the following tableau.

x_1	x_2	s_1	s_2	s_3	b	*Basic Variables*
-1	1	1	0	0	11	s_1
1	1	0	1	0	27	s_2
2	5	0	0	1	90	s_3
-4	-6	0	0	0	0	

The objective function for this problem is $z = 4x_1 + 6x_2$.

Solution Note that the current solution ($x_1 = 0, x_2 = 0, s_1 = 11, s_2 = 27, s_3 = 90$) corresponds to a z–value of 0. To improve this solution, we determine that x_2 is the entering variable, because -6 is the smallest entry in the bottom row.

x_1	x_2	s_1	s_2	s_3	b	Basic Variables
-1	1	1	0	0	11	s_1
1	1	0	1	0	27	s_2
2	5	0	0	1	90	s_3
-4	-6	0	0	0	0	
	↑ Entering					

To see *why* we choose x_2 as the entering variable, remember that $z = 4x_1 + 6x_2$. Hence, it appears that a unit change in x_2 produces a change of 6 in z, whereas a unit change in x_1 produces a change of only 4 in z.

To find the departing variable, we locate the b_i's that have corresponding positive elements in the entering variables column and form the following ratios.

$$\frac{11}{1} = 11, \quad \frac{27}{1} = 27, \quad \frac{90}{5} = 18$$

Here the smallest positive ratio is 11, so we choose s_1 as the departing variable.

x_1	x_2	s_1	s_2	s_3	b	Basic Variables	
-1	1	1	0	0	11	s_1	← Departing
1	1	0	1	0	27	s_2	
2	5	0	0	1	90	s_3	
-4	-6	0	0	0	0		
	↑ Entering						

Note that the pivot is the entry in the first row and second column. Now, we use Gauss-Jordan elimination to obtain the following improved solution.

Before Pivoting *After Pivoting*

$$\begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 11 \\ 1 & 1 & 0 & 1 & 0 & 27 \\ 2 & 5 & 0 & 0 & 1 & 90 \\ -4 & -6 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 11 \\ 2 & 0 & -1 & 1 & 0 & 16 \\ 7 & 0 & -5 & 0 & 1 & 35 \\ -10 & 0 & 6 & 0 & 0 & 66 \end{bmatrix}$$

The new tableau now appears as follows.

x_1	x_2	s_1	s_2	s_3	b	Basic Variables
-1	1	1	0	0	11	x_2
2	0	-1	1	0	16	s_2
7	0	-5	0	1	35	s_3
-10	0	6	0	0	66	

Note that x_2 has replaced s_1 in the basis column and the improved solution

$$(x_1, x_2, s_1, s_2, s_3) = (0, 11, 0, 16, 35)$$

has a z-value of

$$z = 4x_1 + 6x_2 = 4(0) + 6(11) = 66.$$

In Example 1 the improved solution is not yet optimal since the bottom row still has a negative entry. Thus, we can apply another iteration of the simplex method to further improve our solution as follows. We choose x_1 as the entering variable. Moreover, the smallest nonnegative ratio of $11/(-1)$, $16/2 = 8$, and $35/7 = 5$ is 5, so s_3 is the departing variable. Gauss-Jordan elimination produces the following.

$$\begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 11 \\ 2 & 0 & -1 & 1 & 0 & 16 \\ 7 & 0 & -5 & 0 & 1 & 35 \\ -10 & 0 & 6 & 0 & 0 & 66 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 11 \\ 2 & 0 & -1 & 1 & 0 & 16 \\ 1 & 0 & -\frac{5}{7} & 0 & \frac{1}{7} & 5 \\ -10 & 0 & 6 & 0 & 0 & 66 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 0 & 1 & \frac{2}{7} & 0 & \frac{1}{7} & 16 \\ 0 & 0 & \frac{3}{7} & 1 & -\frac{2}{7} & 6 \\ 1 & 0 & -\frac{5}{7} & 0 & \frac{1}{7} & 5 \\ 0 & 0 & -\frac{8}{7} & 0 & \frac{10}{7} & 116 \end{bmatrix}$$

Thus, the new simplex tableau is as follows.

x_1	x_2	s_1	s_2	s_3	b	Basic Variables
0	1	$\frac{2}{7}$	0	$\frac{1}{7}$	16	x_2
0	0	$\frac{3}{7}$	1	$-\frac{2}{7}$	6	s_2
1	0	$-\frac{5}{7}$	0	$\frac{1}{7}$	5	x_1
0	0	$-\frac{8}{7}$	0	$\frac{10}{7}$	116	

In this tableau, there is still a negative entry in the bottom row. Thus, we choose s_1 as the entering variable and s_2 as the departing variable, as shown in the following tableau.

x_1	x_2	s_1	s_2	s_3	b	Basic Variables	
0	1	$\frac{2}{7}$	0	$\frac{1}{7}$	16	x_2	
0	0	$\frac{3}{7}$	1	$-\frac{2}{7}$	6	s_2	← *Departing*
1	0	$-\frac{5}{7}$	0	$\frac{1}{7}$	5	x_1	
0	0	$-\frac{8}{7}$	0	$\frac{10}{7}$	116		
		↑ *Entering*					

By performing one more iteration of the simplex method, we obtain the following tableau. (Try checking this.)

x_1	x_2	s_1	s_2	s_3	b	Basic Variables
0	1	0	$-\frac{2}{3}$	$\frac{1}{3}$	12	x_2
0	0	1	$\frac{7}{3}$	$-\frac{2}{3}$	14	s_1
1	0	0	$\frac{5}{3}$	$-\frac{1}{3}$	15	x_1
0	0	0	$\frac{8}{3}$	$\frac{2}{3}$	132	← *Maximum z-value*

In this tableau, there are no negative elements in the bottom row. We have therefore determined the optimal solution to be

$$(x_1, x_2, s_1, s_2, s_3) = (15, 12, 14, 0, 0)$$

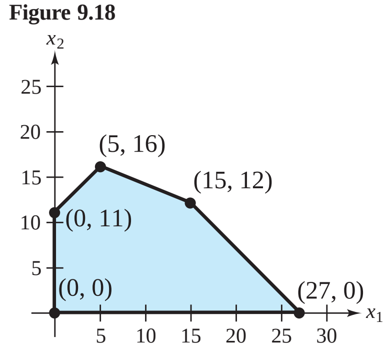
with

$$z = 4x_1 + 6x_2 = 4(15) + 6(12) = 132.$$

REMARK: Ties may occur in choosing entering and/or departing variables. Should this happen, any choice among the tied variables may be made.

Because the linear programming problem in Example 1 involved only two decision variables, we could have used a graphical solution technique, as we did in Example 2, Section 9.2. Notice in Figure 9.18 that each iteration in the simplex method corresponds to moving from a given vertex to an adjacent vertex with an improved z-value.

Figure 9.18



(0, 0)	→	(0, 11)	→	(5, 16)	→	(15, 12)
$z = 0$		$z = 66$		$z = 116$		$z = 132$

The Simplex Method

We summarize the steps involved in the simplex method as follows.

The Simplex Method (Standard Form)

To solve a linear programming problem in standard form, use the following steps.

1. Convert each inequality in the set of constraints to an equation by adding slack variables.
2. Create the initial simplex tableau.
3. Locate the most negative entry in the bottom row. The column for this entry is called the **entering column.** (If ties occur, any of the tied entries can be used to determine the entering column.)
4. Form the ratios of the entries in the "b-column" with their corresponding positive entries in the entering column. The **departing row** corresponds to the smallest nonnegative ratio b_i/a_{ij}. (If all entries in the entering column are 0 or negative, then there is no maximum solution. For ties, choose either entry.) The entry in the departing row and the entering column is called the **pivot.**
5. Use elementary row operations so that the pivot is 1, and all other entries in the entering column are 0. This process is called **pivoting.**
6. If all entries in the bottom row are zero or positive, this is the final tableau. If not, go back to Step 3.
7. If you obtain a final tableau, then the linear programming problem has a maximum solution, which is given by the entry in the lower-right corner of the tableau.

Note that the basic feasible solution of an initial simplex tableau is

$$(x_1, x_2, \ldots, x_n, s_1, s_2, \ldots, s_m) = (0, 0, \ldots, 0, b_1, b_2, \ldots, b_m).$$

This solution is basic because at most m variables are nonzero (namely the slack variables). It is feasible because each variable is nonnegative.

In the next two examples, we illustrate the use of the simplex method to solve a problem involving three decision variables.

EXAMPLE 2 ***The Simplex Method with Three Decision Variables***

Use the simplex method to find the maximum value of

$$z = 2x_1 - x_2 + 2x_3 \qquad \text{Objective function}$$

subject to the constraints

$$\begin{aligned} 2x_1 + x_2 \phantom{{}- 2x_3} &\leq 10 \\ x_1 + 2x_2 - 2x_3 &\leq 20 \\ x_2 + 2x_3 &\leq 5 \end{aligned}$$

where $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$.

Solution Using the basic feasible solution

$$(x_1, x_2, x_3, s_1, s_2, s_3) = (0, 0, 0, 10, 20, 5)$$

the initial simplex tableau for this problem is as follows. (Try checking these computations, and note the "tie" that occurs when choosing the first entering variable.)

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables	
2	1	0	1	0	0	10	s_1	
1	2	-2	0	1	0	20	s_2	
0	1	2	0	0	1	5	s_3	← *Departing*
-2	1	-2	0	0	0	0		
		↑ *Entering*						

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables	
2	1	0	1	0	0	10	s_1	← *Departing*
1	3	0	0	1	1	25	s_2	
0	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	$\frac{5}{2}$	x_3	
-2	2	0	0	0	1	5		
↑ *Entering*								

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	5	x_1
0	$\frac{5}{2}$	0	$-\frac{1}{2}$	1	1	20	s_2
0	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	$\frac{5}{2}$	x_3
0	3	0	1	0	1	15	

This implies that the optimal solution is

$$(x_1, x_2, x_3, s_1, s_2, s_3) = (5, 0, \tfrac{5}{2}, 0, 20, 0)$$

and the maximum value of z is 15.

Occasionally, the constraints in a linear programming problem will include an equation. In such cases, we still add a "slack variable" called an **artificial variable** to form the initial simplex tableau. Technically, this new variable is not a slack variable (because there is no slack to be taken). Once you have determined an optimal solution in such a problem, you should check to see that any equations given in the original constraints are satisfied. Example 3 illustrates such a case.

EXAMPLE 3 *The Simplex Method with Three Decision Variables*

Use the simplex method to find the maximum value of

$$z = 3x_1 + 2x_2 + x_3 \qquad \textbf{Objective function}$$

subject to the constraints

$$\begin{aligned} 4x_1 + x_2 + x_3 &= 30 \\ 2x_1 + 3x_2 + x_3 &\le 60 \\ x_1 + 2x_2 + 3x_3 &\le 40 \end{aligned}$$

where $x_1 \ge 0$, $x_2 \ge 0$, and $x_3 \ge 0$.

Solution Using the basic feasible solution

$$(x_1, x_2, x_3, s_1, s_2, s_3) = (0, 0, 0, 30, 60, 40)$$

the initial simplex tableau for this problem is as follows. (Note that s_1 is an artificial variable, rather than a slack variable.)

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables	
4	1	1	1	0	0	30	s_1	← Departing
2	3	1	0	1	0	60	s_2	
1	2	3	0	0	1	40	s_3	
−3	−2	−1	0	0	0	0		
↑ Entering								

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables	
1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	0	$\frac{15}{2}$	x_1	
0	$\frac{5}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	1	0	45	s_2	← Departing
0	$\frac{7}{4}$	$\frac{11}{4}$	$-\frac{1}{4}$	0	1	$\frac{65}{2}$	s_3	
0	$-\frac{5}{4}$	$-\frac{1}{4}$	$\frac{3}{4}$	0	0	$\frac{45}{2}$		
	↑ Entering							

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
1	0	$\frac{1}{5}$	$\frac{3}{10}$	$-\frac{1}{10}$	0	3	x_1
0	1	$\frac{1}{5}$	$-\frac{1}{5}$	$\frac{2}{5}$	0	18	x_2
0	0	$\frac{12}{5}$	$\frac{1}{10}$	$-\frac{7}{10}$	1	1	s_3
0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	45	

This implies that the optimal solution is

$$(x_1, x_2, x_3, s_1, s_2, s_3) = (3, 18, 0, 0, 0, 1)$$

and the maximum value of z is 45. (This solution satisfies the equation given in the constraints because $4(3) + 1(18) + 1(0) = 30$.)

Applications

EXAMPLE 4 ***A Business Application: Maximum Profit***

A manufacturer produces three types of plastic fixtures. The time required for molding, trimming, and packaging is given in Table 9.1. (Times are given in hours per dozen fixtures.)

TABLE 9.1

Process	*Type A*	*Type B*	*Type C*	*Total time available*
Molding	1	2	$\frac{3}{2}$	12,000
Trimming	$\frac{2}{3}$	$\frac{2}{3}$	1	4,600
Packaging	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	2,400
Profit	\$11	\$16	\$15	—

How many dozen of each type of fixture should be produced to obtain a maximum profit?

Solution Letting x_1, x_2, and x_3 represent the number of dozen units of Types A, B, and C, respectively, the objective function is given by

$$\text{Profit} = P = 11x_1 + 16x_2 + 15x_3.$$

Moreover, using the information in the table, we construct the following constraints.

$$\begin{aligned} x_1 + 2x_2 + \tfrac{3}{2}x_3 &\leq 12{,}000 \\ \tfrac{2}{3}x_1 + \tfrac{2}{3}x_2 + x_3 &\leq 4{,}600 \\ \tfrac{1}{2}x_1 + \tfrac{1}{3}x_2 + \tfrac{1}{2}x_3 &\leq 2{,}400 \end{aligned}$$

(We also assume that $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$.) Now, applying the simplex method with the basic feasible solution

$$(x_1, x_2, x_3, s_1, s_2, s_3) = (0, 0, 0, 12{,}000, 4{,}600, 2{,}400)$$

we obtain the following tableaus.

x_1	x_2	x_3	s_1	s_2	s_3	b	*Basic Variables*	
1	2	$\frac{3}{2}$	1	0	0	12,000	s_1	← *Departing*
$\frac{2}{3}$	$\frac{2}{3}$	1	0	1	0	4,600	s_2	
$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	0	0	1	2,400	s_3	
−11	−16	−15	0	0	0	0		
	↑ *Entering*							

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables	
$\frac{1}{2}$	1	$\frac{3}{4}$	$\frac{1}{2}$	0	0	6,000	x_2	
$\frac{1}{3}$	0	$\frac{1}{2}$	$-\frac{1}{3}$	1	0	600	s_2	
$\frac{1}{3}$	0	$\frac{1}{4}$	$-\frac{1}{6}$	0	1	400	s_3	← *Departing*
-3	0	-3	8	0	0	96,000		
↑ *Entering*								

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables	
0	1	$\frac{3}{8}$	$\frac{3}{4}$	0	$-\frac{3}{2}$	5,400	x_2	
0	0	$\frac{1}{4}$	$-\frac{1}{6}$	1	-1	200	s_2	← *Departing*
1	0	$\frac{3}{4}$	$-\frac{1}{2}$	0	3	1,200	x_1	
0	0	$-\frac{3}{4}$	$\frac{13}{2}$	0	9	99,600		
		↑ *Entering*						

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
0	1	0	1	$-\frac{3}{2}$	0	5,100	x_2
0	0	1	$-\frac{2}{3}$	4	-4	800	x_3
1	0	0	0	-3	6	600	x_1
0	0	0	6	3	6	100,200	

From this final simplex tableau, we see that the maximum profit is \$100,200, and this is obtained by the following production levels.

Type A:	600 dozen units
Type B:	5,100 dozen units
Type C:	800 dozen units

REMARK: In Example 4, note that the second simplex tableau contains a "tie" for the minimum entry in the bottom row. (Both the first and third entries in the bottom row are -3.) Although we chose the first column to represent the departing variable, we could have chosen the third column. Try reworking the problem with this choice to see that you obtain the same solution.

EXAMPLE 5 *A Business Application: Media Selection*

The advertising alternatives for a company include television, radio, and newspaper advertisements. The costs and estimates for audience coverage are given in Table 9.2

TABLE 9.2

	Television	*Newspaper*	*Radio*
Cost per advertisement	\$ 2,000	\$ 600	\$ 300
Audience per advertisement	100,000	40,000	18,000

The local newspaper limits the number of weekly advertisements from a single company to ten. Moreover, in order to balance the advertising among the three types of media, no more than half of the total number of advertisements should occur on the radio, and at least 10% should occur on television. The weekly advertising budget is \$18,200. How many advertisements should be run in each of the three types of media to maximize the total audience?

Solution To begin, we let x_1, x_2, and x_3 represent the number of advertisements in television, newspaper, and radio, respectively. The objective function (to be maximized) is therefore

$$z = 100{,}000x_1 + 40{,}000x_2 + 18{,}000x_3 \qquad \textbf{Objective function}$$

where $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$. The constraints for this problem are as follows.

$$\begin{aligned}
2000x_1 + 600x_2 + 300x_3 &\leq 18{,}200 \\
x_2 &\leq 10 \\
x_3 &\leq 0.5(x_1 + x_2 + x_3) \\
x_1 &\geq 0.1(x_1 + x_2 + x_3)
\end{aligned}$$

A more manageable form of this system of constraints is as follows.

$$\left.\begin{aligned}
20x_1 + 6x_2 + 3x_3 &\leq 182 \\
x_2 &\leq 10 \\
-x_1 - x_2 + x_3 &\leq 0 \\
-9x_1 + x_2 + x_3 &\leq 0
\end{aligned}\right\} \qquad \textbf{Constraints}$$

Thus, the initial simplex tableau is as follows.

x_1	x_2	x_3	s_1	s_2	s_3	s_4	b	*Basic Variables*	
20	6	3	1	0	0	0	182	s_1	← *Departing*
0	1	0	0	1	0	0	10	s_2	
−1	−1	1	0	0	1	0	0	s_3	
−9	1	1	0	0	0	1	0	s_4	
−100,000	−40,000	−18,000	0	0	0	0	0		
↑ *Entering*									

Now, to this initial tableau, we apply the simplex method as follows.

x_1	x_2	x_3	s_1	s_2	s_3	s_4	b	Basic Variables
1	$\frac{3}{10}$	$\frac{3}{20}$	$\frac{1}{20}$	0	0	0	$\frac{91}{10}$	x_1
0	1	0	0	1	0	0	10	s_2 ← Departing
0	$-\frac{7}{10}$	$\frac{23}{20}$	$\frac{1}{20}$	0	1	0	$\frac{91}{10}$	s_3
0	$\frac{37}{10}$	$\frac{47}{20}$	$\frac{9}{20}$	0	0	1	$\frac{819}{10}$	s_4
0	$-10{,}000$	$-3{,}000$	5,000	0	0	0	910,000	
	↑ Entering							

x_1	x_2	x_3	s_1	s_2	s_3	s_4	b	Basic Variables
1	0	$\frac{3}{20}$	$\frac{1}{20}$	$-\frac{3}{10}$	0	0	$\frac{61}{10}$	x_1
0	1	0	0	1	0	0	10	x_2
0	0	$\frac{23}{20}$	$\frac{1}{20}$	$\frac{7}{10}$	1	0	$\frac{161}{10}$	s_3 ← Departing
0	0	$\frac{47}{20}$	$\frac{9}{20}$	$-\frac{37}{10}$	0	1	$\frac{449}{10}$	s_4
0	0	$-3{,}000$	5,000	10,000	0	0	1,010,000	
		↑ Entering						

x_1	x_2	x_3	s_1	s_2	s_3	s_4	b	Basic Variables
1	0	0	$\frac{1}{23}$	$-\frac{9}{23}$	$-\frac{3}{23}$	0	4	x_1
0	1	0	0	1	0	0	10	x_2
0	0	1	$\frac{1}{23}$	$\frac{14}{23}$	$\frac{20}{23}$	0	14	x_3
0	0	0	$\frac{8}{23}$	$-\frac{118}{23}$	$-\frac{47}{23}$	1	12	s_4
0	0	0	$\frac{118{,}000}{23}$	$\frac{272{,}000}{23}$	$\frac{60{,}000}{23}$	0	1,052,000	

From this tableau, we see that the maximum weekly audience for an advertising budget of \$18,200 is

$$z = 1{,}052{,}000 \qquad \textbf{Maximum weekly audience}$$

and this occurs when $x_1 = 4$, $x_2 = 10$, and $x_3 = 14$. We sum up the results here.

Media	Number of Advertisements	Cost	Audience
Television	4	\$ 8,000	400,000
Newspaper	10	\$ 6,000	400,000
Radio	14	\$ 4,200	252,000
Total	28	\$18,200	1,052,000

SECTION 9.3 ❑ EXERCISES

In Exercises 1–4, write the simplex tableau for the given linear programming problem. You do not need to solve the problem. (In each case the objective function is to be maximized.)

1. Objective function:
$z = x_1 + 2x_2$
Constraints:
$$\begin{aligned} 2x_1 + x_2 &\le 8 \\ x_1 + x_2 &\le 5 \\ x_1, x_2 &\ge 0 \end{aligned}$$

2. Objective function:
$z = x_1 + 3x_2$
Constraints:
$$\begin{aligned} x_1 + x_2 &\le 4 \\ x_1 - x_2 &\le 1 \\ x_1, x_2 &\ge 0 \end{aligned}$$

3. Objective function:
$z = 2x_1 + 3x_2 + 4x_3$
Constraints:
$$\begin{aligned} x_1 + 2x_2 &\le 12 \\ x_1 + x_3 &\le 8 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

4. Objective function:
$z = 6x_1 - 9x_2$
Constraints:
$$\begin{aligned} 2x_1 - 3x_2 &\le 6 \\ x_1 + x_2 &\le 20 \\ x_1, x_2 &\ge 0 \end{aligned}$$

In Exercises 5–8, explain why the linear programming problem is *not* in standard form as given.

5. (Minimize)
Objective function:
$z = x_1 + x_2$
Constraints:
$$\begin{aligned} x_1 + 2x_2 &\le 4 \\ x_1, x_2 &\ge 0 \end{aligned}$$

6. (Maximize)
Objective function:
$z = x_1 + x_2$
Constraints:
$$\begin{aligned} x_1 + 2x_2 &\le 6 \\ 2x_1 - x_2 &\le -1 \\ x_1, x_2 &\ge 0 \end{aligned}$$

7. (Maximize)
Objective function:
$z = x_1 + x_2$
Constraints:
$$\begin{aligned} x_1 + x_2 + 3x_3 &\le 5 \\ 2x_1 - 2x_3 &\ge 1 \\ x_2 + x_3 &\le 0 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

8. (Maximize)
Objective function:
$z = x_1 + x_2$
Constraints:
$$\begin{aligned} x_1 + x_2 &\ge 4 \\ 2x_1 + x_2 &\ge 6 \\ x_1, x_2 &\ge 0 \end{aligned}$$

In Exercises 9–20, use the simplex method to solve the given linear programming problem. (In each case the objective function is to be maximized.)

9. Objective function:
$z = x_1 + 2x_2$
Constraints:
$$\begin{aligned} x_1 + 4x_2 &\le 8 \\ x_1 + x_2 &\le 12 \\ x_1, x_2 &\ge 0 \end{aligned}$$

10. Objective function:
$z = x_1 + x_2$
Constraints:
$$\begin{aligned} x_1 + 2x_2 &\le 6 \\ 3x_1 + 2x_2 &\le 12 \\ x_1, x_2 &\ge 0 \end{aligned}$$

11. Objective function:
$z = 5x_1 + 2x_2 + 8x_3$
Constraints:
$$\begin{aligned} 2x_1 - 4x_2 + x_3 &\le 42 \\ 2x_1 + 3x_2 - x_3 &\le 42 \\ 6x_1 - x_2 + 3x_3 &\le 42 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

12. Objective function:
$z = x_1 - x_2 + 2x_3$
Constraints:
$$\begin{aligned} 2x_1 + 2x_2 &\le 8 \\ x_3 &\le 5 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

13. Objective function:
$z = 4x_1 + 5x_2$
Constraints:
$$\begin{aligned} x_1 + x_2 &\le 10 \\ 3x_1 + 7x_2 &\le 42 \\ x_1, x_2 &\ge 0 \end{aligned}$$

14. Objective function:
$z = x_1 + 2x_2$
Constraints:
$$\begin{aligned} x_1 + 3x_2 &\le 15 \\ 2x_1 - x_2 &\le 12 \\ x_1, x_2 &\ge 0 \end{aligned}$$

15. Objective function:
$z = 3x_1 + 4x_2 + x_3 + 7x_4$
Constraints:
$$\begin{aligned} 8x_1 + 3x_2 + 4x_3 + x_4 &\le 7 \\ 2x_1 + 6x_2 + x_3 + 5x_4 &\le 3 \\ x_1 + 4x_2 + 5x_3 + 2x_4 &\le 8 \\ x_1, x_2, x_3, x_4 &\ge 0 \end{aligned}$$

16. Objective function:
$z = x_1$
Constraints:
$$\begin{aligned} 3x_1 + 2x_2 &\le 60 \\ x_1 + 2x_2 &\le 28 \\ x_1 + 4x_2 &\le 48 \\ x_1, x_2 &\ge 0 \end{aligned}$$

17. Objective function:
$z = x_1 - x_2 + x_3$
Constraints:
$$\begin{aligned} 2x_1 + x_2 - 3x_3 &\le 40 \\ x_1 + x_3 &\le 25 \\ 2x_2 + 3x_3 &\le 32 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

18. Objective function:
$z = 2x_1 + x_2 + 3x_3$
Constraints:
$$\begin{aligned} x_1 + x_2 + x_3 &\le 59 \\ 2x_1 + 3x_3 &\le 75 \\ x_2 + 6x_3 &\le 54 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

19. Objective function:
$z = x_1 + 2x_2 - x_4$
Constraints:
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &\le 24 \\ 3x_2 + 7x_3 + x_4 &\le 42 \\ x_1, x_2, x_3, x_4 &\ge 0 \end{aligned}$$

20. Objective function:
$z = x_1 + 2x_2 + x_3 - x_4$
Constraints:
$$\begin{aligned} x_1 + x_2 + 3x_3 + 4x_4 &\le 60 \\ x_2 + 2x_3 + 5x_4 &\le 50 \\ 2x_1 + 3x_2 + 6x_4 &\le 72 \\ x_1, x_2, x_3, x_4 &\ge 0 \end{aligned}$$

21. A merchant plans to sell two models of home computers at costs of \$250 and \$400, respectively. The \$250 model yields a profit of \$45 and the \$400 model yields a profit of \$50. The merchant estimates that the total monthly demand will not exceed 250 units. Find the number of units of each model that should be stocked in order to maximize profit. Assume that the merchant does not want to invest more than \$70,000 in computer inventory. (See Exercise 21 in Section 9.2.)

22. A fruit grower has 150 acres of land available to raise two crops, A and B. It takes one day to trim an acre of crop A and two days to trim an acre of crop B, and there are 240 days per year available for trimming. It takes 0.3 day to pick an acre of crop A and 0.1 day to pick an acre of crop B, and there are 30 days per year available for picking. Find the number of acres of each fruit that should be planted to maximize profit, assuming that the profit is \$140 per acre for crop A and \$235 per acre for B. (See Exercise 22 in Section 9.2.)

23. A grower has 50 acres of land for which she plans to raise three crops. It costs \$200 to produce an acre of carrots and the profit is \$60 per acre. It costs \$80 to produce an acre of celery and the profit is \$20 per acre. Finally, it costs \$140 to produce an acre of lettuce and the profit is \$30 per acre. Use the simplex method to find the number of acres of each crop she should plant in order to maximize her profit. Assume that her cost cannot exceed \$10,000.

24. A fruit juice company makes two special drinks by blending apple and pineapple juices. The first drink uses 30% apple juice and 70% pineapple, while the second drink uses 60% apple and 40% pineapple. There are 1000 liters of apple and 1500 liters of pineapple juice available. If the profit for the first drink is \$0.60 per liter and that for the second drink is \$0.50, use the simplex method to find the number of liters of each drink that should be produced in order to maximize the profit.

25. A manufacturer produces three models of bicycles. The time (in hours) required for assembling, painting, and packaging each model is as follows.

	Model A	*Model B*	*Model C*
Assembling	2	2.5	3
Painting	1.5	2	1
Packaging	1	0.75	1.25

The total time available for assembling, painting, and packaging is 4006 hours, 2495 hours and 1500 hours, respectively. The profit per unit for each model is \$45 (Model A), \$50 (Model B), and \$55 (Model C). How many of each type should be produced to obtain a maximum profit?

26. Suppose in Exercise 25 the total time available for assembling, painting, and packaging is 4000 hours, 2500 hours, and 1500 hours, respectively, and that the profit per unit is \$48 (Model A), \$50 (Model B), and \$52 (Model C). How many of each type should be produced to obtain a maximum profit?

27. A company has budgeted a maximum of \$600,000 for advertising a certain product nationally. Each minute of television time costs \$60,000 and each one-page newspaper ad costs \$15,000. Each television ad is expected to be viewed by 15 million viewers, and each newspaper ad is expected to be seen by 3 million readers. The company's market research department advises the company to use at most 90% of the advertising budget on television ads. How should the advertising budget be allocated to maximize the total audience?

28. Rework Exercise 27 assuming that each one-page newspaper ad costs \$30,000.

29. An investor has up to \$250,000 to invest in three types of investments. Type A pays 8% annually and has a risk factor of 0. Type B pays 10% annually and has a risk factor of 0.06. Type C pays 14% annually and has a risk factor of 0.10. To have a well-balanced portfolio, the investor imposes the following conditions. The average risk factor should be no greater than 0.05. Moreover, at least one-fourth of the total portfolio is to be allocated to Type A investments and at least one-fourth of the portfolio is to be allocated to Type B investments. How much should be allocated to each type of investment to obtain a maximum return?

30. An investor has up to \$450,000 to invest in three types of investments. Type A pays 6% annually and has a risk factor of 0. Type B pays 10% annually and has a risk factor of 0.06. Type C pays 12% annually and has a risk factor of 0.08. To have a well-balanced portfolio, the investor imposes the following conditions. The average risk factor should be no greater than 0.05. Moreover, at least one-half of the total portfolio is to be allocated to Type A investments and at least one-fourth of the portfolio is to be allocated to Type B investments. How much should be allocated to each type of investment to obtain a maximum return?

31. An accounting firm has 900 hours of staff time and 100 hours of reviewing time available each week. The firm charges \$2000 for an audit and \$300 for a tax return. Each audit requires 100 hours of staff time and 10 hours of review time, and each tax return requires 12.5 hours of staff time and 2.5 hours of review time. What number of audits and tax returns will bring in a maximum revenue?

32. The accounting firm in Exercise 31 raises its charge for an audit to $2500. What number of audits and tax returns will bring in a maximum revenue?

In the simplex method, it may happen that in selecting the departing variable all the calculated ratios are negative. This indicates an *unbounded solution*. Demonstrate this in Exercises 33 and 34.

33. (Maximize)
Objective function:
$z = x_1 + 2x_2$
Constraints:
$$\begin{aligned} x_1 - 3x_2 &\leq 1 \\ -x_1 + 2x_2 &\leq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

34. (Maximize)
Objective function:
$z = x_1 + 3x_2$
Constraints:
$$\begin{aligned} -x_1 + x_2 &\leq 20 \\ -2x_1 + x_2 &\leq 50 \\ x_1, x_2 &\geq 0 \end{aligned}$$

If the simplex method terminates and one or more variables *not in the final basis* have bottom-row entries of zero, bringing these variables into the basis will determine other optimal solutions. Demonstrate this in Exercises 35 and 36.

35. (Maximize)
Objective function:
$z = 2.5x_1 + x_2$
Constraints:
$$\begin{aligned} 3x_1 + 5x_2 &\leq 15 \\ 5x_1 + 2x_2 &\leq 10 \\ x_1, x_2 &\geq 0 \end{aligned}$$

36. (Maximize)
Objective function:
$z = x_1 + \frac{1}{2}x_2$
Constraints:
$$\begin{aligned} 2x_1 + x_2 &\leq 20 \\ x_1 + 3x_2 &\leq 35 \\ x_1, x_2 &\geq 0 \end{aligned}$$

C **37.** Use a computer to maximize the objective function
$z = 2x_1 + 7x_2 + 6x_3 + 4x_4$
subject to the constraints
$$\begin{aligned} x_1 + x_2 + 0.83x_3 + 0.5x_4 &\leq 65 \\ 1.2x_1 + x_2 + x_3 + 1.2x_4 &\leq 96 \\ 0.5x_1 + 0.7x_2 + 1.2x_3 + 0.4x_4 &\leq 80 \end{aligned}$$
where $x_1, x_2, x_3, x_4 \geq 0$.

C **38.** Use a computer to maximize the objective function
$z = 1.2x_1 + x_2 + x_3 + x_4$
subject to the same set of constraints given in Exercise 37.

9.4 THE SIMPLEX METHOD: MINIMIZATION

In Section 9.3, we applied the simplex method only to linear programming problems in standard form where the objective function was to be *maximized*. In this section, we extend this procedure to linear programming problems in which the objective function is to be *minimized*.

A minimization problem is in **standard form** if the objective function $w = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ is to be minimized, subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\geq b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\geq b_m \end{aligned}$$

where $x_i \geq 0$ and $b_i \geq 0$. The basic procedure used to solve such a problem is to convert it to a *maximization problem* in standard form, and then apply the simplex method as discussed in Section 9.3.

In Example 5 in Section 9.2, we used geometric methods to solve the following minimization problem.

32. The accounting firm in Exercise 31 raises its charge for an audit to \$2500. What number of audits and tax returns will bring in a maximum revenue?

In the simplex method, it may happen that in selecting the departing variable all the calculated ratios are negative. This indicates an *unbounded solution*. Demonstrate this in Exercises 33 and 34.

33. (Maximize)

Objective function:

$z = x_1 + 2x_2$

Constraints:

$$\begin{aligned} x_1 - 3x_2 &\leq 1 \\ -x_1 + 2x_2 &\leq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

34. (Maximize)

Objective function:

$z = x_1 + 3x_2$

Constraints:

$$\begin{aligned} -x_1 + x_2 &\leq 20 \\ -2x_1 + x_2 &\leq 50 \\ x_1, x_2 &\geq 0 \end{aligned}$$

If the simplex method terminates and one or more variables *not in the final basis* have bottom-row entries of zero, bringing these variables into the basis will determine other optimal solutions. Demonstrate this in Exercises 35 and 36.

35. (Maximize)

Objective function:

$z = 2.5x_1 + x_2$

Constraints:

$$\begin{aligned} 3x_1 + 5x_2 &\leq 15 \\ 5x_1 + 2x_2 &\leq 10 \\ x_1, x_2 &\geq 0 \end{aligned}$$

36. (Maximize)

Objective function:

$z = x_1 + \frac{1}{2}x_2$

Constraints:

$$\begin{aligned} 2x_1 + x_2 &\leq 20 \\ x_1 + 3x_2 &\leq 35 \\ x_1, x_2 &\geq 0 \end{aligned}$$

C **37.** Use a computer to maximize the objective function

$z = 2x_1 + 7x_2 + 6x_3 + 4x_4$

subject to the constraints

$$\begin{aligned} x_1 + x_2 + 0.83x_3 + 0.5x_4 &\leq 65 \\ 1.2x_1 + x_2 + x_3 + 1.2x_4 &\leq 96 \\ 0.5x_1 + 0.7x_2 + 1.2x_3 + 0.4x_4 &\leq 80 \end{aligned}$$

where $x_1, x_2, x_3, x_4 \geq 0$.

C **38.** Use a computer to maximize the objective function

$z = 1.2x_1 + x_2 + x_3 + x_4$

subject to the same set of constraints given in Exercise 37.

9.4 THE SIMPLEX METHOD: MINIMIZATION

In Section 9.3, we applied the simplex method only to linear programming problems in standard form where the objective function was to be *maximized*. In this section, we extend this procedure to linear programming problems in which the objective function is to be *minimized*.

A minimization problem is in **standard form** if the objective function $w = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ is to be minimized, subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\geq b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\geq b_m \end{aligned}$$

where $x_i \geq 0$ and $b_i \geq 0$. The basic procedure used to solve such a problem is to convert it to a *maximization problem* in standard form, and then apply the simplex method as discussed in Section 9.3.

In Example 5 in Section 9.2, we used geometric methods to solve the following minimization problem.

Minimization Problem: Find the minimum value of

$$w = 0.12x_1 + 0.15x_2 \qquad \text{Objective function}$$

subject to the following constraints

$$\left.\begin{aligned} 60x_1 + 60x_2 &\geq 300 \\ 12x_1 + 6x_2 &\geq 36 \\ 10x_1 + 30x_2 &\geq 90 \end{aligned}\right\} \quad \text{Constraints}$$

where $x_1 \geq 0$ and $x_2 \geq 0$. The first step in converting this problem to a maximization problem is to form the augmented matrix for this system of inequalities. To this augmented matrix we add a last row that represents the coefficients of the objective function, as follows.

$$\begin{bmatrix} 60 & 60 & \vdots & 300 \\ 12 & 6 & \vdots & 36 \\ 10 & 30 & \vdots & 90 \\ \cdots & \cdots & \cdots & \cdots \\ 0.12 & 0.15 & \vdots & 0 \end{bmatrix}$$

Next, we form the **transpose** of this matrix by interchanging its rows and columns.

$$\begin{bmatrix} 60 & 12 & 10 & \vdots & 0.12 \\ 60 & 6 & 30 & \vdots & 0.15 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 300 & 36 & 90 & \vdots & 0 \end{bmatrix}$$

Note that the rows of this matrix are the columns of the first matrix, and vice versa. Finally, we interpret the new matrix as a *maximization* problem as follows. (To do this, we introduce new variables, y_1, y_2, and y_3.) We call this corresponding maximization problem the **dual** of the original minimization problem.

Dual Maximization Problem: Find the maximum value of

$$z = 300y_1 + 36y_2 + 90y_3 \qquad \text{Dual objective function}$$

subject to the constraints

$$\left.\begin{aligned} 60y_1 + 12y_2 + 10y_3 &\leq 0.12 \\ 60y_1 + 6y_2 + 30y_3 &\leq 0.15 \end{aligned}\right\} \quad \text{Dual constraints}$$

where $y_1 \geq 0$, $y_2 \geq 0$, and $y_3 \geq 0$.

As it turns out, the solution of the original minimization problem can be found by applying the simplex method to the new dual problem, as follows.

y_1	y_2	y_3	s_1	s_2	b	*Basic Variables*	
60	12	10	1	0	0.12	s_1	← *Departing*
60	6	30	0	1	0.15	s_2	
−300	−36	−90	0	0	0		
↑ *Entering*							

y_1	y_2	y_3	s_1	s_2	b	*Basic Variables*	
1	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{60}$	0	$\frac{1}{500}$	y_1	
0	−6	20	−1	1	$\frac{3}{100}$	s_2	← *Departing*
0	24	−40	5	0	$\frac{3}{5}$		
		↑ *Entering*					

y_1	y_2	y_3	s_1	s_2	b	*Basic Variables*
1	$\frac{1}{4}$	0	$\frac{1}{40}$	$-\frac{1}{120}$	$\frac{7}{4000}$	y_1
0	$-\frac{3}{10}$	1	$-\frac{1}{20}$	$\frac{1}{20}$	$\frac{3}{2000}$	y_3
0	12	0	3	2	$\frac{33}{50}$	
			↑ x_1	↑ x_2		

Thus, the solution of the dual maximization problem is $z = \frac{33}{50} = 0.66$. This is the same value we obtained in the minimization problem given in Example 5, in Section 9.2. The x-values corresponding to this optimal solution are obtained from the entries in the bottom row corresponding to slack variable columns. In other words, the optimal solution occurs when $x_1 = 3$ and $x_2 = 2$.

The fact that a dual maximization problem has the same solution as its original minimization problem is stated formally in a result called the **von Neumann Duality Principle,** after the American mathematician John von Neumann (1903–1957).

Theorem 9.2

The von Neumann Duality Principle

The objective value w of a minimization problem in standard form has a minimum value if and only if the objective value z of the dual maximization problem has a maximum value. Moreover, the minimum value of w is equal to the maximum value of z.

Solving a Minimization Problem

We summarize the steps used to solve a minimization problem as follows.

Solving a Minimization Problem

A minimization problem is in standard form if the objective function $w = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ is to be minimized, subject to the constraints

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\geq b_1\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\geq b_2\\
&\vdots\\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\geq b_m
\end{aligned}$$

where $x_i \geq 0$ and $b_i \geq 0$. To solve this problem we use the following steps.

1. Form the **augmented matrix** for the given system of inequalities, and add a bottom row consisting of the coefficients of the objective function.

$$\left[\begin{array}{cccc:c}
a_{11} & a_{12} & \cdots & a_{1n} & b_1\\
a_{21} & a_{22} & \cdots & a_{2n} & b_2\\
 & & & & \\
a_{m1} & a_{m2} & \cdots & a_{mn} & b_m\\
\cdots & \cdots & \cdots & \cdots & \cdots\\
c_1 & c_2 & \cdots & c_n & 0
\end{array}\right]$$

2. Form the **transpose** of this matrix.

$$\left[\begin{array}{cccc:c}
a_{11} & a_{21} & \cdots & a_{m1} & c_1\\
a_{12} & a_{22} & \cdots & a_{m2} & c_2\\
 & & & & \\
a_{1n} & a_{2n} & \cdots & a_{mn} & c_n\\
\cdots & \cdots & \cdots & \cdots & \cdots\\
b_1 & b_2 & \cdots & b_m & 0
\end{array}\right]$$

3. Form the **dual maximization problem** corresponding to this transposed matrix. That is, find the maximum of the objective function given by $z = b_1y_1 + b_2y_2 + \cdots + b_my_m$ subject to the constraints

$$\begin{aligned}
a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m &\leq c_1\\
a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m &\leq c_2\\
&\vdots\\
a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m &\leq c_n
\end{aligned}$$

where $y_1 \geq 0, y_2 \geq 0, \ldots,$ and $y_m \geq 0$.

4. Apply the **simplex method** to the dual maximization problem. The maximum value of z will be the minimum value of w. Moreover, the values of $x_1, x_2, \ldots,$ and x_n will occur in the bottom row of the final simplex tableau, in the columns corresponding to the slack variables.

We illustrate the steps used to solve a minimization problem in Examples 1 and 2.

EXAMPLE 1 *Solving a Minimization Problem*

Find the minimum value of

$$w = 3x_1 + 2x_2 \qquad \text{Objective function}$$

subject to the constraints

$$\left.\begin{aligned} 2x_1 + x_2 &\geq 6 \\ x_1 + x_2 &\geq 4 \end{aligned}\right\} \qquad \text{Constraints}$$

where $x_1 \geq 0$ and $x_2 \geq 0$.

Solution The augmented matrix corresponding to this minimization problem is

$$\left[\begin{array}{cc|c} 2 & 1 & 6 \\ 1 & 1 & 4 \\ \hline 3 & 2 & 0 \end{array}\right].$$

Thus, the matrix corresponding to the dual maximization problem is given by the following transpose.

$$\left[\begin{array}{cc|c} 2 & 1 & 3 \\ 1 & 1 & 2 \\ \hline 6 & 4 & 0 \end{array}\right].$$

This implies that the dual maximization problem is as follows.

Dual Maximization Problem: Find the maximum value of

$$z = 6y_1 + 4y_2 \qquad \text{Dual objective function}$$

subject to the constraints

$$\left.\begin{aligned} 2y_1 + y_2 &\leq 3 \\ y_1 + y_2 &\leq 2 \end{aligned}\right\} \qquad \text{Dual constraints}$$

where $y_1 \geq 0$ and $y_2 \geq 0$. We now apply the simplex method to the dual problem as follows.

y_1	y_2	s_1	s_2	b	*Basic Variables*	
2	1	1	0	3	s_1	← *Departing*
1	1	0	1	2	s_2	
−6	−4	0	0	0		
↑ *Entering*						

y_1	y_2	s_1	s_2	b	Basic Variables	
1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{3}{2}$	y_1	
0	$\frac{1}{2}$	$-\frac{1}{2}$	1	$\frac{1}{2}$	s_2	← *Departing*
0	-1	3	0	9		
	↑ *Entering*					

y_1	y_2	s_1	s_2	b	Basic Variables
1	0	1	-1	1	y_1
0	1	-1	2	1	y_2
0	0	2	2	10	
		↑ x_1	↑ x_2		

From this final simplex tableau, we see that the maximum value of z is 10. Therefore, the solution of the original minimization problem is

$$w = 10 \qquad \textbf{Minimum Value}$$

and this occurs when

$$x_1 = 2 \text{ and } x_2 = 2.$$

Both the minimization and the maximization linear programming problems in Example 1 could have been solved with a graphical method, as indicated in Figure 9.19. Note in Figure 9.19 (a) that the maximum value of $z = 6y_1 - 4y_2$ is the same as the minimum value of $w = 3x_1 + 2x_2$, as shown in Figure 9.19 (b). (See page 515.)

EXAMPLE 2 *Solving a Minimization Problem*

Find the minimum value of

$$w = 2x_1 + 10x_2 + 8x_3 \qquad \textbf{Objective function}$$

subject to the constraints

$$\left.\begin{aligned} x_1 + x_2 + x_3 &\geq 6 \\ x_2 + 2x_3 &\geq 8 \\ -x_1 + 2x_2 + 2x_3 &\geq 4 \end{aligned}\right\} \qquad \textbf{Constraints}$$

where $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$.

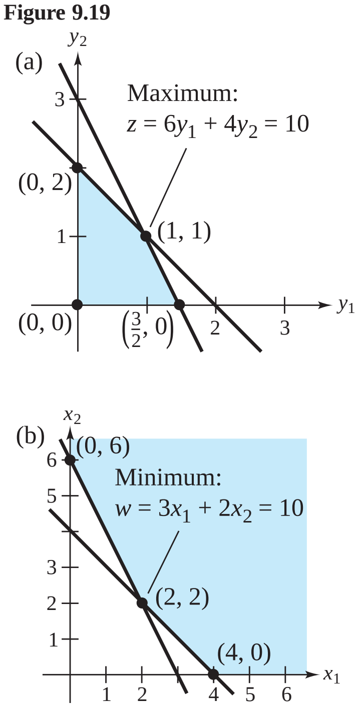
Solution The augmented matrix corresponding to this minimization problem is

$$\begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 0 & 1 & 2 & \vdots & 8 \\ -1 & 2 & 2 & \vdots & 4 \\ \cdots & \cdots & \cdots & \vdots & \cdots \\ 2 & 10 & 8 & \vdots & 0 \end{bmatrix}.$$

Thus, the matrix corresponding to the dual maximization problem is given by the following transpose.

$$\begin{bmatrix} 1 & 0 & -1 & \vdots & 2 \\ 1 & 1 & 2 & \vdots & 10 \\ 1 & 2 & 2 & \vdots & 8 \\ \cdots & \cdots & \cdots & \vdots & \cdots \\ 6 & 8 & 4 & \vdots & 0 \end{bmatrix}$$

Figure 9.19



This implies that the dual maximization problem is as follows.

Dual Maximization Problem: Find the maximum value of

$$z = 6y_1 + 8y_2 + 4y_3 \qquad \textbf{Dual objective function}$$

subject to the constraints

$$\left.\begin{aligned} y_1 \qquad\quad - \ y_3 &\le 2 \\ y_1 + \ y_2 + 2y_3 &\le 10 \\ y_1 + 2y_2 + 2y_3 &\le 8 \end{aligned}\right\} \qquad \textbf{Dual constraints}$$

where $y_1 \ge 0$, $y_2 \ge 0$, and $y_3 \ge 0$. We now apply the simplex method to the dual problem as follows.

y_1	y_2	y_3	s_1	s_2	s_3	b	*Basic Variables*	
1	0	-1	1	0	0	2	s_1	
1	1	2	0	1	0	10	s_2	
1	2	2	0	0	1	8	s_3	← *Departing*
-6	-8	-4	0	0	0	0		
	↑ *Entering*							

y_1	y_2	y_3	s_1	s_2	s_3	b	*Basic Variables*	
1	0	-1	1	0	0	2	s_1	← *Departing*
$\frac{1}{2}$	0	1	0	1	$-\frac{1}{2}$	6	s_2	
$\frac{1}{2}$	1	1	0	0	$\frac{1}{2}$	4	y_2	
-2	0	4	0	0	4	32		
↑ *Entering*								

$$\begin{array}{ccccccc|c} y_1 & y_2 & y_3 & s_1 & s_2 & s_3 & b & \text{Basic Variables} \\ \hline 1 & 0 & -1 & 1 & 0 & 0 & 2 & y_1 \\ 0 & 0 & \frac{3}{2} & -\frac{1}{2} & 1 & -\frac{1}{2} & 5 & s_2 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & 3 & y_2 \\ \hline 0 & 0 & 2 & 2 & 0 & 4 & 36 & \\ & & & \uparrow & \uparrow & \uparrow & & \\ & & & x_1 & x_2 & x_3 & & \end{array}$$

From this final simplex tableau, we see that the maximum value of z is 36. Therefore, the solution of the original minimization problem is

$$w = 36 \qquad \textbf{Minimum Value}$$

and this occurs when

$$x_1 = 2, \ x_2 = 0, \text{ and } \ x_3 = 4.$$

Applications

EXAMPLE 3 ***A Business Application: Minimum Cost***

A small petroleum company owns two refineries. Refinery 1 costs $20,000 per day to operate, and it can produce 400 barrels of high-grade oil, 300 barrels of medium-grade oil, and 200 barrels of low-grade oil each day. Refinery 2 is newer and more modern. It costs $25,000 per day to operate, and it can produce 300 barrels of high-grade oil, 400 barrels of medium-grade oil, and 500 barrels of low-grade oil each day.

The company has orders totaling 25,000 barrels of high-grade oil, 27,000 barrels of medium-grade oil, and 30,000 barrels of low-grade oil. How many days should it run each refinery to minimize its costs and still refine enough oil to meet its orders?

Solution To begin, we let x_1 and x_2 represent the number of days the two refineries are operated. Then the total cost is given by

$$C = 20{,}000x_1 + 25{,}000x_2. \qquad \textbf{Objective function}$$

The constraints are given by

$$\left.\begin{array}{rl} \text{(High-grade)} & 400x_1 + 300x_2 \geq 25{,}000 \\ \text{(Medium-grade)} & 300x_1 + 400x_2 \geq 27{,}000 \\ \text{(Low-grade)} & 200x_1 + 500x_2 \geq 30{,}000 \end{array}\right\} \qquad \textbf{Constraints}$$

where $x_1 \geq 0$ and $x_2 \geq 0$. The augmented matrix corresponding to this minimization problem is

$$\begin{bmatrix} 400 & 300 & \vdots & 25{,}000 \\ 300 & 400 & \vdots & 27{,}000 \\ 200 & 500 & \vdots & 30{,}000 \\ \dots & \dots & \vdots & \dots \\ 20{,}000 & 25{,}000 & \vdots & 0 \end{bmatrix}.$$

The matrix corresponding to the dual maximization problem is

$$\begin{bmatrix} 400 & 300 & 200 & \vdots & 20{,}000 \\ 300 & 400 & 500 & \vdots & 25{,}000 \\ \dots & \dots & \dots & \vdots & \dots \\ 25{,}000 & 27{,}000 & 30{,}000 & \vdots & 0 \end{bmatrix}.$$

We now apply the simplex method to the dual problem as follows.

y_1	y_2	y_3	s_1	s_2	b	Basic Variables
400	300	200	1	0	20,000	s_1
300	400	500	0	1	25,000	s_2 ← *Departing*
−25,000	−27,000	−30,000	0	0	0	
		↑ *Entering*				

y_1	y_2	y_3	s_1	s_2	b	Basic Variables
280	140	0	1	$-\frac{2}{5}$	10,000	s_1 ← *Departing*
$\frac{3}{5}$	$\frac{4}{5}$	1	0	$\frac{1}{500}$	50	y_3
−7,000	−3,000	0	0	60	1,500,000	
↑ *Entering*						

y_1	y_2	y_3	s_1	s_2	b	Basic Variables
1	$\frac{1}{2}$	0	$\frac{1}{280}$	$-\frac{1}{700}$	$\frac{250}{7}$	y_1
0	$\frac{1}{2}$	1	$-\frac{3}{1400}$	$\frac{1}{350}$	$\frac{200}{7}$	y_3
0	500	0	25	50	1,750,000	
			↑ x_1	↑ x_2		

From the third simplex tableau, we see that the solution to the original minimization problem is

$C = \$1{,}750{,}000$ **Minimum cost**

and this occurs when $x_1 = 25$ and $x_2 = 50$. Thus, the two refineries should be operated for the following number of days.

Refinery 1: 25 days

Refinery 2: 50 days

Note that by operating the two refineries for this number of days, the company will have produced the following amounts of oil.

$$\begin{aligned}\text{High-grade oil:}\quad & 25(400) + 50(300) = 25{,}000 \text{ barrels}\\ \text{Medium-grade oil:}\quad & 25(300) + 50(400) = 27{,}500 \text{ barrels}\\ \text{Low-grade oil:}\quad & 25(200) + 50(500) = 30{,}000 \text{ barrels}\end{aligned}$$

Thus, the original production level has been met (with a surplus of 500 barrels of medium-grade oil).

SECTION 9.4 ❑ EXERCISES

In Exercises 1–6, determine the dual of the given minimization problem.

1. Objective function:
$w = 3x_1 + 3x_2$
Constraints:
$2x_1 + x_2 \ge 4$
$x_1 + 2x_2 \ge 4$
$x_1, x_2 \ge 0$

2. Objective function:
$w = 2x_1 + x_2$
Constraints:
$5x_1 + x_2 \ge 9$
$2x_1 + 2x_2 \ge 10$
$x_1, x_2 \ge 0$

3. Objective function:
$w = 4x_1 + x_2 + x_3$
Constraints:
$3x_1 + 2x_2 + x_3 \ge 23$
$x_1 + x_3 \ge 10$
$8x_1 + x_2 + 2x_3 \ge 40$
$x_1, x_2, x_3 \ge 0$

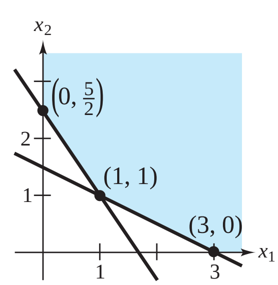
4. Objective function:
$w = 9x_1 + 6x_2$
Constraints:
$x_1 + 2x_2 \ge 5$
$2x_1 + 2x_2 \ge 8$
$2x_1 + x_2 \ge 6$
$x_1, x_2 \ge 0$

5. Objective function:
$w = 14x_1 + 20x_2 + 24x_3$
Constraints:
$x_1 + x_2 + 2x_3 \ge 7$
$x_1 + 2x_2 + x_3 \ge 4$
$x_1, x_2, x_3 \ge 0$

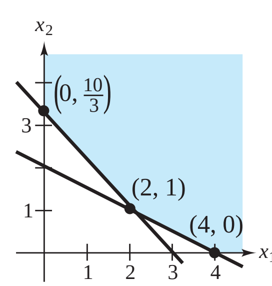
6. Objective function:
$w = 9x_1 + 4x_2 + 10x_3$
Constraints:
$2x_1 + x_2 + 3x_3 \ge 6$
$6x_1 + x_2 + x_3 \ge 9$
$x_1, x_2, x_3 \ge 0$

In Exercises 7–12, (a) solve the given minimization problem by the graphical method, (b) formulate the dual problem, and (c) solve the dual problem by the graphical method.

7. Objective function:
$w = 2x_1 + 2x_2$
Constraints:
$x_1 + 2x_2 \ge 3$
$3x_1 + 2x_2 \ge 5$
$x_1, x_2 \ge 0$



8. Objective function:
$w = 14x_1 + 20x_2$
Constraints:
$x_1 + 2x_2 \ge 4$
$7x_1 + 6x_2 \ge 20$
$x_1, x_2 \ge 0$

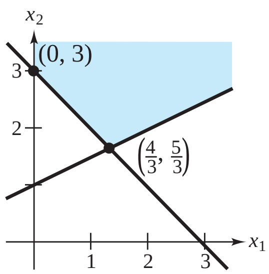


9. Objective function:

$w = x_1 + 4x_2$

Constraints:

$$\begin{aligned} x_1 + x_2 &\geq 3 \\ -x_1 + 2x_2 &\geq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

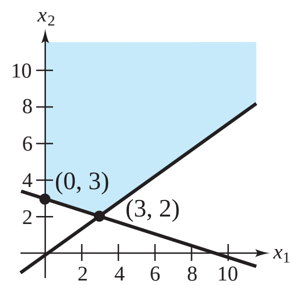


10. Objective function:

$w = 2x_1 + 6x_2$

Constraints:

$$\begin{aligned} -2x_1 + 3x_2 &\geq 0 \\ x_1 + 3x_2 &\geq 9 \\ x_1, x_2 &\geq 0 \end{aligned}$$

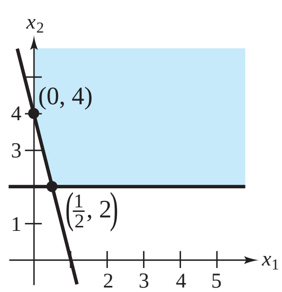


11. Objective function:

$w = 6x_1 + 3x_2$

Constraints:

$$\begin{aligned} 4x_1 + x_2 &\geq 4 \\ x_2 &\geq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

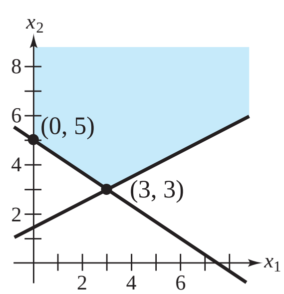


12. Objective function:

$w = x_1 + 6x_2$

Constraints:

$$\begin{aligned} 2x_1 + 3x_2 &\geq 15 \\ -x_1 + 2x_2 &\geq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$



In Exercises 13–20, solve the given minimization problem by solving the dual maximization problem with the simplex method.

13. Objective function:

$w = x_2$

Constraints:

$$\begin{aligned} x_1 + 5x_2 &\geq 10 \\ -6x_1 + 5x_2 &\geq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

14. Objective function:

$w = 3x_1 + 8x_2$

Constraints:

$$\begin{aligned} 2x_1 + 7x_2 &\geq 9 \\ x_1 + 2x_2 &\geq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

15. Objective function:

$w = 2x_1 + x_2$

Constraints:

$$\begin{aligned} 5x_1 + x_2 &\geq 9 \\ 2x_1 + 2x_2 &\geq 10 \\ x_1, x_2 &\geq 0 \end{aligned}$$

16. Objective function:

$w = 2x_1 + 2x_2$

Constraints:

$$\begin{aligned} 3x_1 + x_2 &\geq 6 \\ -4x_1 + 2x_2 &\geq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

17. Objective function:

$w = 8x_1 + 4x_2 + 6x_3$

Constraints:

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &\geq 6 \\ 4x_1 + x_2 + 3x_3 &\geq 7 \\ 2x_1 + x_2 + 4x_3 &\geq 8 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

18. Objective function:

$w = 8x_1 + 16x_2 + 18x_3$

Constraints:

$$\begin{aligned} 2x_1 + 2x_2 - 2x_3 &\geq 4 \\ -4x_1 + 3x_2 - x_3 &\geq 1 \\ x_1 - x_2 + 3x_3 &\geq 8 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

19. Objective function:

$w = 6x_1 + 2x_2 + 3x_3$

Constraints:

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &\geq 28 \\ 6x_1 + x_3 &\geq 24 \\ 3x_1 + x_2 + 2x_3 &\geq 40 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

20. Objective function:

$w = 42x_1 + 5x_2 + 17x_3$

Constraints:

$$\begin{aligned} 3x_1 - x_2 + 7x_3 &\geq 5 \\ -3x_1 - x_2 + 3x_3 &\geq 8 \\ 6x_1 + x_2 + x_3 &\geq 16 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

In Exercises 21–24, two dietary drinks are used to supply protein and carbohydrates. The first drink provides 1 unit of protein and 3 units of carbohydrates in each liter. The second drink supplies 2 units of protein and 2 units of carbohydrates in each liter. An athlete requires 3 units of protein and 5 units of carbohydrates. Find the amount of each drink the athlete should consume to minimize the cost and still meet the minimum dietary requirements.

21. The first drink costs \$2 per liter and the second costs \$3 per liter.

22. The first drink costs \$4 per liter and the second costs \$2 per liter.

23. The first drink costs \$1 per liter and the second costs \$3 per liter.

24. The first drink costs \$1 per liter and the second costs \$2 per liter.

In Exercises 25–28, an athlete uses two dietary drinks that provide the nutritional elements listed in the following table.

Drink	*Protein*	*Carbohydrates*	*Vitamin D*
I	4	2	1
II	1	5	1

Find the combination of drinks of minimum cost that will meet the minimum requirements of 4 units of protein, 10 units of carbohydrates, and 3 units of vitamin D.

25. Drink I costs \$5 per liter and drink II costs \$8 per liter.

26. Drink I costs \$7 per liter and drink II costs \$4 per liter.

27. Drink I costs \$1 per liter and drink II costs \$5 per liter.

28. Drink I costs \$8 per liter and drink II costs \$1 per liter.

29. A company has three production plants, each of which produces three different models of a particular product. The daily capacities (in thousands of units) of the three plants are as follows.

	Model 1	*Model 2*	*Model 3*
Plant 1	8	4	8
Plant 2	6	6	3
Plant 3	12	4	8

The total demand for Model 1 is 300,000 units, for Model 2 is 172,000 units, and for Model 3 is 249,500 units. Moreover, the daily operating cost for Plant 1 is \$55,000, for Plant 2 is \$60,000, and for Plant 3 is \$60,000. How many days should each plant be operated in order to fill the total demand, and keep the operating cost at a minimum?

30. The company in Exercise 29 has lowered the daily operating cost for Plant 3 to \$50,000. How many days should each plant be operated in order to fill the total demand, and keep the operating cost at a minimum?

31. A small petroleum company owns two refineries. Refinery 1 costs \$25,000 per day to operate, and it can produce 300 barrels of high-grade oil, 200 barrels of medium-grade oil, and 150 barrels of low-grade oil each day. Refinery 2 is newer and more modern. It costs \$30,000 per day to operate, and it can produce 300 barrels of high-grade oil, 250 barrels of medium-grade oil, and 400 barrels of low-grade oil each day. The company has orders totaling 35,000 barrels of high-grade oil, 30,000 barrels of medium-grade oil, and 40,000 barrels of low-grade oil. How many days should the company run each refinery to minimize its costs and still meet its orders?

32. A steel company has two mills. Mill 1 costs \$70,000 per day to operate, and it can produce 400 tons of high-grade steel, 500 tons of medium-grade steel, and 450 tons of low-grade steel each day. Mill 2 costs \$60,000 per day to operate, and it can produce 350 tons of high-grade steel, 600 tons of medium-grade steel, and 400 tons of low-grade steel each day. The company has orders totaling 100,000 tons of high-grade steel, 150,000 tons of medium-grade steel, and 124,500 tons of low-grade steel. How many days should the company run each mill to minimize its costs and still fill the orders?

C **33.** Use a computer to minimize the objective function

$$w = x_1 + 0.5x_2 + 2.5x_3 + 3x_4$$

subject to the constraints

$$\begin{aligned} 1.5x_1 + x_2 + 2x_4 &\geq 35 \\ 2x_2 + 6x_3 + 4x_4 &\geq 120 \\ x_1 + x_2 + x_3 + x_4 &\geq 50 \\ 0.5x_1 + 2.5x_3 + 1.5x_4 &\geq 75 \end{aligned}$$

where $x_1, x_2, x_3, x_4 \geq 0$.

C **34.** Use a computer to minimize the objective function

$$w = 1.5x_1 + x_2 + 0.5x_3 + 2x_4$$

subject to the same set of constraints given in Exercise 33.

4 UNIT FOUR: Transportation and Assignment problems

4.1 Objectives

By the end of this unit you will be able to:

- formulate special linear programming problems using the transportation model.
- define a balanced transportation problem
- develop an initial solution of a transportation problem using the Northwest Corner Rule
- use the Stepping Stone method to find an optimal solution of a transportation problem
- formulate special linear programming problems using the assignment model
- solve assignment problems with the Hungarian method.

4.2 Introduction

In this unit we extend the theory of linear programming to two special linear programming problems, the **Transportation** and **Assignment Problems**. Both of these problems can be solved by the simplex algorithm, but the process would result in very large simplex tableaux and numerous simplex iterations.

Because of the special characteristics of each problem, however, alternative solution methods requiring significantly less mathematical manipulation have been developed.

4.3 The Transportation problem

The general transportation problem is concerned with determining an optimal strategy for distributing a commodity from a group of supply centres,such as factories, called *sources*, to various receiving centers, such as warehouses, called *destinations*, in such a way as to minimise total distribution costs.

Each source is able to supply a fixed number of units of the product, usually called the *capacity* or *availability*, and each destination has a fixed demand, often called the *requirement*.

Transportation models can also be used when a firm is trying to decide where to locate a new facility. Good financial decisions concerning facility location also attempt to minimize total transportation and production costs for the entire system.

4.3.1 Setting up a Transportation problem

To illustrate how to set up a transportation problem we consider the following example;

Example 4.1
A concrete company transports concrete from three plants, 1, 2 and 3, to three construction sites, A, B and C.

The plants are able to supply the following numbers of tons per week:

Plant	*Supply (capacity)*
1	*300*
2	*300*
3	*100*

The requirements of the sites, in number of tons per week, are:

Construction site	*Demand (requirement)*
A	*200*
B	*200*
C	*300*

The cost of transporting 1 ton of concrete from each plant to each site is shown in the figure 8 in Emalangeni per ton.

For computational purposes it is convenient to put all the above information into a table, as in the simplex method. In this table each row represents a source *and each column represents a* destination.

Sites

	From \ *To*	*A*	*B*	*C*	*Supply (Availability)*
	1	*4*	*3*	*8*	*300*
Plants	***2***	*7*	*5*	*9*	*300*
	3	*4*	*5*	*5*	*100*
	Demand (requirement)	*200*	*200*	*300*	

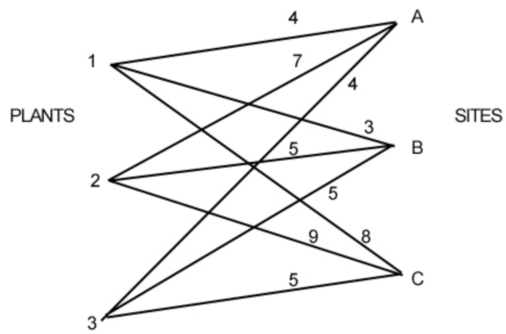


Figure 8: Constructing a transportation problem

4.3.2 Mathematical model of a transportation problem

Before we discuss the solution of transportation problems we will introduce the notation used to describe the transportation problem and show that it can be formulated as a linear programming problem.

We use the following notation;

$x_{ij} =$	the number of units to be distributed from source i to destination j $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$;
$s_i =$	supply from source i;
$d_j =$	demand at destination j;
$c_{ij} =$	cost per unit distributed from source i to destination j

With respect to Example 4.1 the decision variables x_{ij} are the numbers of tons transported from plant i (where $i = 1, 2, 3$) to each site j (where $j =$ A, B, C)

A basic assumption is that the distribution costs of units from source i to destination j is directly proportional to the number of units distributed. A typical **cost and requirements table** has the form shown on Table 4.

Let Z be total distribution costs from all the m sources to the n destinations. In example 4.1 each term in the objective function Z represents the total cost of tonnage transported on one route. For example, in the route $2 \longrightarrow C$, the term in $9x_{2C}$, that is:

$$(\text{Cost per ton} = 9) \quad \times \quad (\text{number of tons transported} = x_{2C})$$

	Destination				
	1	2	...	n	*Supply*
1	c_{11}	c_{12}	...	c_{1n}	s_1
2	c_{21}	c_{22}	...	c_{2n}	s_2
Source $\vdots$	$\vdots$	$\vdots$	...	$\vdots$	$\vdots$
m	c_{m1}	c_{m2}	...	c_{mn}	s_m
Demand	d_1	d_2	...	d_n	

Table 4: *Cost and requirements table*

Hence the objective function is:

$$\begin{aligned} Z &= 4x_{1A} + 3x_{1B} + 8x_{1C} \\ &+ 7x_{2A} + 5x_{2B} + 9x_{2C} \\ &+ 4x_{3A} + 5x_{3B} + 5x_{3C} \end{aligned}$$

Notice that in this problem the total supply is 300 + 300 + 200 = 700 and the total demand is 200 + 200 + 300 = 700. Thus

Total supply = total demand.

In mathematical form this expressed as

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j \tag{47}$$

This is called a **balanced problem** . In this unit our discussion shall be restricted to the balanced problems.

In a balanced problem all the products that can be supplied are used to meet the demand. There are no slacks and so all constraints are *equalities* rather than *inequalities* as was the case in the previous unit.

The formulation of this problem as a linear programming problem is presented as

$$\text{Minimise} \quad Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\, x_{ij}, \tag{48}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = s_i, \qquad \text{for} \quad i = 1, 2, \ldots, m \tag{49}$$

$$\sum_{i=1}^{n} x_{ij} = d_j, \qquad \text{for} \quad j = 1, 2, \ldots, n \tag{50}$$

and

$$x_{ij} \geq 0, \text{ for all } i \text{ and } j.$$

Any linear programming problem that fits this special formulation is of the transportation type, regardless of its physical context. For many applications, the supply and demand quantities in the model will have integer values and implementation will require that the distribution quantities also be integers. Fortunately, the unit coefficients of the unknown variables in the constraints guarantee an optimal solution with only integer values.

4.3.3 Initial solution - Northwest Corner Rule

The initial basic feasible solution can be obtained by using one of several methods. We will consider only the **North West corner rule** of developing an initial solution. Other methods can be found in standard texts on linear programming.

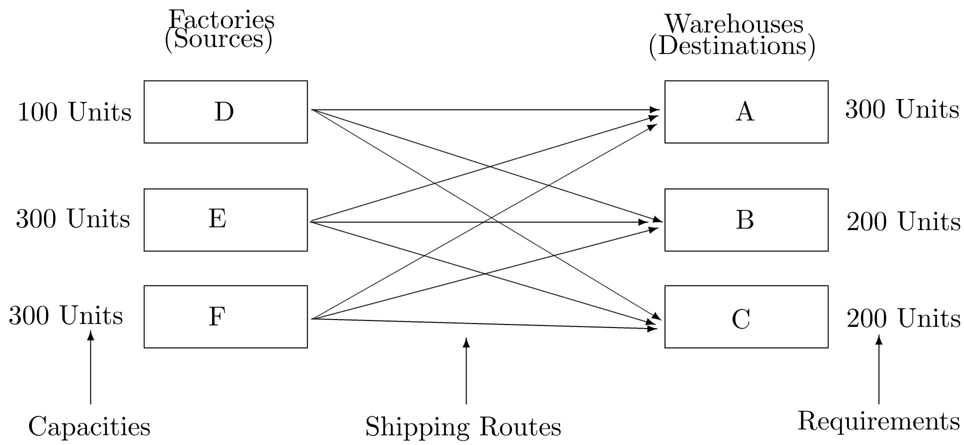
The procedure for constructing an initial basic feasible solution selects the basic variables one at a time. The North West corner rule begins with an allocation at the top left-hand corner of the tableau and proceeds systematically along either a row or a column and make allocations to subsequent cells until the bottom right-hand corner is reached, by which time enough allocations will have been made to constitute an initial solution.

The procedure for constructing an initial solution using the North West Corner rule is as follows:

NORTH WEST CORNER RULE

1. Start by selecting the cell in the most "North-West" corner of the table.
2. Assign the maximum amount to this cell that is allowable based on the requirements and the capacity constraints.
3. Exhaust the capacity from each row before moving down to another row.
4. Exhaust the requirement from each column before moving right to another column.
5. Check to make sure that the capacity and requirements are met.

Let us begin with an example dealing with Executive Furniture corporation, which manufactures office desks at three locations: D, E and F. The firm distributes the desks through regional warehouses located in A, B and C (see the Network format diagram below)



It is assumed that the production costs per desk are identical at each factory. The only relevant costs are those of shipping from each source to each destination. The costs are shown in Table 5

From \ To	A	B	C
D	\$5	\$4	\$3
E	\$8	\$4	\$3
F	\$9	\$7	\$5

Table 5: Transportation Costs per desk for Executive Furniture Corp.

We proceed to construct a transportation table and label its various components as show in Table 6.

We can now use the Northwest corner rule to find an initial feasible solution to the problem. We start in the upper left hand cell and allocate units to shipping routes as follows:

To / From	A	B	C	Capacity
D	5	4	3	**100**
E	8	4	3	**300**
F	9	7	5	**300**
Requirements	**300**	**200**	**200**	**700**

Table 6: Transportation Table for Executive Furniture Corporation

1. Exhaust the supply (factory capacity) of each row before moving down to the next row.
2. Exhaust the demand (warehouse) requirements of each column before moving to the next column to the right.
3. Check that all supply and demand requirements are met.

The initial shipping assignments are given in Table 7

To / From	A	B	C	Factory Capacity
D	100			100
E	200	100		300
F		100	200	300
Warehouse Requirements	300	200	200	700

Table 7: Initial Solution of the North West corner Rule

This initial solution can also be presented together with the costs per unit as shown in the Table 8.

We can compute the cost of this shipping assignment as follows;

Therefore, the initial feasible solution for this problem is $4200.

Example 4.2
Consider a transportation problem in which the cost, supply and demand values are presented in Table 10.

(a) *Is this a balanced problem? Why?*

To / From	A	B	C	Capacity
D	5 100	4	3	**100**
E	8 200	4 100	3	**300**
F	9	7 100	5 200	**300**
Requirements	**300**	**200**	**200**	**700**

Table 8: Representing the initial feasible solution with costs

ROUTE FROM	ROUTE TO	UNITS SHIPPED	×	PER UNIT COST ($)	=	TOTAL COST ($)
D	A	100		5		500
E	A	200		8		1600
E	B	100		4		400
F	B	100		7		700
F	C	200		5		1000
						Total 4200

Table 9: Calculation of costs of initial shipping assignments

(b) *Obtain the initial feasible solution using the North-West Corner rule.*

Solution:

(a) *We calculate the total supply and total demand.*
Total supply = 14 + 10 + 15 + 13 = 52
Total demand = 10 + 15 + 12 + 15 = 52
Since the total supply is equal to the total demand we conclude that the problem is balanced.

(b) *The allocations according to the North-West corner rule are shown in Table 11 The initial feasible solution is*

$$Total\ Cost = 10 \cdot 10 + 4 \cdot 30 + 10 \cdot 15 + 1 \cdot 30 + 12 \cdot 20 + 2 \cdot 20 + 13 \cdot 45 = \$1265$$

Note that this is not necessarily equal to the optimal solution.

		Destination				
		1	2	3	4	Supply
	1	10	30	25	15	14
Source	2	20	15	20	10	10
	3	10	30	20	20	15
	4	30	40	35	45	13
	Demand	10	15	12	15	

Table 10: Supply and Demand values for Transportation problem

	1	2	3	4	Supply
1	10	4			14
2		10			10
3		1	12	2	15
4				13	13
Demand	10	15	12	15	

Table 11: Initial feasible solution

4.4 Exercises 4.1: Northwest Corner rule

In each of the following problems check whether the solution is balanced or not then use the North West Corner rule to find the basic feasible solution.

1.

TO / FROM	1	2	3	Supply
1	3	2	0	45
2	1	5	0	60
3	5	4	0	35
Demand	50	60	30	

2.

TO / FROM	1	2	3	Supply
1	5	4	3	100
2	8	4	3	300
3	9	7	5	300
Demand	300	200	200	

3.

TO / FROM	1	2	3	4	Supply
A	12	13	4	6	500
B	6	4	10	11	700
C	10	9	12	4	800
Demand	400	900	200	500	

4.

TO / FROM	1	2	3	4	Supply
1	10	30	25	15	14
2	20	15	20	10	10
3	10	30	20	20	15
4	30	40	35	45	13
Demand	10	15	12	15	

4.4.1 Optimality test - the Stepping Stone method

The next step is to determine whether the current allocation at any stage of the solution process is optimal. We will present one of the methods used to determine optimality of and improve a current solution. The method derives its name from the analogy of crossing a pond using stepping stones. The occupied cells are analogous to the stepping stones, which are used in making certain movements in this method.

The five steps of the **Stepping-Stone Method** are as follows:

STEPPING-STONE METHOD

1. Select an unused square to be evaluated.
2. Beginning at this square, trace a closed path back to the original square via squares that are currently being used (only horizontal or vertical moves allowed). You can only change directions at occupied cells!.
3. Beginning with a plus (+) sign at the unused square, place alternative minus (-) signs and plus signs on each corner square of the closed path just traced.
4. Calculate an **improvement index**, I_{ij} by adding together the unit cost figures found in each square containing a plus sign and then subtracting the unit costs in each square containing a minus sign.
5. Repeat steps 1 to 4 until an improvement index has been calculated for all unused squares.
    - If all indices computed are greater than or equal to zero, an optimal solution has been reached.
    - If not, it is possible to improve the current solution and decrease total shipping costs.

4.4.2 The optimality criterion

If all the cost index values obtained for all the currently unoccupied cells are nonnegative, then the current solution is optimal. If there are negative values the solution has to be improved. This means that an allocation is made to one of the empty cells (unused routes) and the necessary adjustments in the supply and demand effected accordingly.

To see how the Stepping-Stone method works we apply these steps to the Furniture Corporation example to evaluate the shipping routes.

Steps 1-3 Beginning with the D-B route, we first trace a closed path using only currently occupied squared (see Table 12) and then place alternate plus signs and minus signs in the corners of this path.

Step 4 An improvement index I_{ij} for the D-B route in now computed by adding unit costs in squares with plus signs and subtracting costs in squares with minus signs. Thus

$$I_{DB} = +4 - 5 + 8 - 4 = +3$$

This means that for every desk shipped via the D-B route, total transportation costs will increase by \$3 over their current level.

From \ To	A	B	C	Capacity
D	[5] 100 - ←	[4] Start +	[3]	**100**
E	[8] ↓ 200 + →	[4] ↑ - 100	[3]	**300**
F	[9]	[7] 100	[5] 200	**300**
Requirements	**300**	**200**	**200**	**700**

Table 12: Evaluating the D-B route

Step 5 Next we consider the D-C unused route. The closed path we use is (see Table 13)

$$+DC - DA + EA - EB + FB - FC$$

The D-C improvement index is

$$I_{DC} = +3 - 5 + 8 - 4 + 7 - 5 = +4$$

From \ To	A	B	C	Capacity
D	[5] 100 - ←	[4] ⟵ ⟵ ←	[3] Start +	**100**
E	[8] ↓ 200 + →	[4] - 100	[3] ↑ ↑	**300**
F	[9]	[7] ↓ + ⟶ → 100	[5] ↑ - 200	**300**
Requirements	**300**	**200**	**200**	**700**

Table 13: Evaluating the D-C route

The other two routes may be evaluated in a similar fashion

E-C route: closed path = +EC - EB + FB - FC

$$I_{EC} = +3 - 4 + 7 - 5 = +1$$

FA route: closed path = +FA - FB + EB - EA

$$I_{FA} = +9 - 7 + 4 - 8 = -2$$

Because the I_{FA} index is negative, a cost saving may be attained by making use of the FA route i.e the FA cell can be improved. The Stepping-Stone path used to evaluate the route FA is shown in Table 14

To / From	A		B		C		Capacity
D	100	5		4		3	**100**
E	200 - ←	8	+ 100	4		3	**300**
F	Start + →	↓ 9	- 100	↑ 7	200	5	**300**
Requirements	**300**		**200**		**200**		**700**

Table 14: Stepping-Stone Path used to evaluate FA route

The next step, then is to ship the maximum allowable number of units on the new route (FA route). What is the maximum quantity that can be shipped on the money-saving route? The quantity is found by referring to the closed path of plus signs and minus signs drawn for the route and selecting the *smallest number* found in those squares containing minus signs. To obtain a new solution, that number is added to all squares on the closed path with plus signs and subtracted from all squares on the path assigned minus signs. All other squares are left unchanged. The new solution is shown in Table 15.

To / From	A		B		C		Capacity
D	100	5		4		3	**100**
E	100	8	200	4		3	**300**
F	100	9		7	200	5	**300**
Requirements	**300**		**200**		**200**		**700**

Table 15: Improved solution: Second solution

The shipping cost for this new solution is

$$100 \cdot 5 + 100 \cdot 8 + 200 \cdot 4 + 100 \cdot 9 + 200 \cdot 4 = \$4000$$

This solution may or may not be optimal. To determine whether further improvement is possible, we return to the first five steps to test each square that is now unused. The four improvement indices - each representing an available shipping route are as follows:

$$\begin{aligned} \text{D to B} \quad &= \quad I_{DB} = 4 - 5 + 8 - 4 = +\$3 \\ & \quad\quad (\text{Closed path} : +DB - DA + EA - EB) \end{aligned}$$

$$
\begin{aligned}
\text{D to C} &= I_{DC} = 3 - 5 + 9 - 5 = +\$2 \\
&\quad (\text{Closed path} : +DC - DA + FA - FC) \\
\text{E to C} &= I_{EC} = 3 - 8 + 9 - 5 = -\$1 \\
&\quad (\text{Closed path} : +EC - EA + FA - FC) \\
\text{F to B} &= I_{FB} = 7 - 4 + 8 - 9 = +\$2 \\
&\quad (\text{Closed path} : +FB - EB + EA - FA)
\end{aligned}
$$

Hence, an improvement can be made by shipping the maximum allowable number of units from E to C (see Table 16).

From \ To	A	B	C	Capacity
D	5 100	4	3	**100**
E	- 8 100 ↓	← ← 4 200	← + 3 Start	**300**
F	↓ 9 100 + →	7 → → →	↑ 5 → - 200	**300**
Requirements	**300**	**200**	**200**	**700**

Table 16: Path to evaluate the E-C route

The improved solution is shown in Table 17. The total cost for the third solution is

$$100 \cdot 5 + 200 \cdot 4 + 100 \cdot 3 + 200 \cdot 9 + 100 \cdot 5 = \$3900$$

To determine if the current solution is optimal we calculate the improvement indices - each

From \ To	A	B	C	Capacity
D	5 100	4	3	**100**
E	8	4 200	3 100	**300**
F	9 200	7	5 100	**300**
Requirements	**300**	**200**	**200**	**700**

Table 17: Improved solution: Third solution

representing an available shipping route - as follows:

$$\text{D to B} = I_{DB} = 4 - 5 + 9 - 5 + 3 - 4 = +\$2$$

$$
\begin{aligned}
& \text{(Closed path: } + DB - DA + FA - FC + EC - EB) \\
\text{D to C} \quad = \quad & I_{DC} = 3 - 5 + 9 - 5 = +\$2 \\
& \text{(Closed path: } + DC - DA + FA - FC) \\
\text{E to A} \quad = \quad & I_{EA} = 8 - 9 + 5 - 3 = +\$1 \\
& \text{(Closed path: } + EA - FA + FC - EC) \\
\text{F to B} \quad = \quad & I_{FB} = 7 - 5 + 3 - 4 = +\$1 \\
& \text{(Closed path: } + FB - FC + EC - EB)
\end{aligned}
$$

Table 17 contains the optimal solution because each improvement index for the Table is greater than or equal to zero.

4.5 Summary

In this section we discussed the formulation of transportation problems and their methods of solution. We used the North West corner rule to obtain the initial feasible solution and the Stepping-Stone method to find the optimal solution. We restricted focus to balanced transportation problems where it is assumed that the total supply is equal to total demand.

4.6 Exercises 4.2: Transportation problems

1. A company has factories at A, B and C which supply warehouses at D, E and F. Weekly factory capacities are 200, 160 and 90 units respectively. Weekly warehouse requirements (demands) are 180, 120 and 150 units respectively. Unit shipping costs (in Emalangeni) are as follows:

Factory	D	E	F	Capacity
A	16	20	12	**200**
B	14	8	18	**160**
C	26	24	16	**90**
Demand	**180**	**120**	**150**	**450**

   Determine the optimum distribution for this company to minimize shipping costs. [E5920]

2. A Timber company ships pine flooring to three building supply houses from its mills in Bhunya, Mondi and Pigg's Peak. Determine the best transportation schedule for the data given below using the Northwest corner rule and the Stepping Stone method. [E230]

TO / FROM	*Supply House 1*	*Supply House 2*	*Supply House 3*	*Mill Capacity (tons)*
Bhunya	3	3	2	**25**
Mondi	4	2	3	**40**
Pigg's Peak	3	2	3	**30**
Supply House Demand (tons)	**30**	**30**	**35**	**95**

3. A trucking company has a contract to move 115 truckloads of sand per week between three sand-washing plants W,X and Y, and three destinations, A,B and C. Cost and volume information is given below. Compute the optimal transportation cost.

To / From	Project A	Project B	Project C	Supply
Plant W	5	10	10	35
Plant X	20	30	20	40
Plant Y	5	8	12	40
Demand	45	50	20	

[C=1345]

4. In each of the following cases write down the North West corner solution and use the Stepping Stone method to find the minimal cost.

(a)

TO / FROM	D	E	F	Capacity
A	8	6	9	**20**
B	6	3	8	**30**
C	10	7	9	**70**
Demand	**90**	**20**	**10**	**120**

[E970]

(b)

TO / FROM	D	E	F	Capacity
A	2	2	3	**4**
B	2	1	6	**6**
C	1	3	4	**8**
Demand	**2**	**5**	**11**	**18**

[E48]

(c)

FROM \ TO	D	E	F	Capacity
A	5	2	2	**7**
B	7	3	4	**5**
C	6	4	3	**3**
Demand	**4**	**5**	**6**	

4.7 Assignment Problem

The **assignment problem** refers to the class of linear programming problems that involve determining the most efficient assignment of

- people to projects
- salespeople to territories
- contracts to bidders
- jobs to machines, etc.

The objective is most often to minimize total costs or total time of performing the tasks at hand.

One important characteristic of assignment problems is that only one job or worker is assigned to one machine or project. An example is the problem of a taxi company with 4 taxis and 4 passengers. Which taxi should collect which passenger in order to minimize costs?

Each assignment problem has associated with it a table, or matrix. Generally, the rows contain the objects or people we wish to assign, and the columns comprise the tasks or things we want them assigned to. The numbers in the table are the costs associated with each particular assignment.

An assignment problem can be viewed as a transportation problem in which

- the capacity from each source (or person to be assigned) is 1 and
- the demand at each destination (or job to be done) is 1.

As an illustration of the assignment problem, let us consider the case of a Fix-It-Shop, which has just received three new rush projects to repair: (1) a radio, (2) a toaster oven, and (3) a broken coffee table. Three repair persons, each with different talents and abilities, are available to do the jobs. The owner of the shop estimates what it will cost in wages to assign each of the workers to each of the three projects. The costs which are shown in Table 18 differ because the owner believes that each worker will differ in speed and skill on these quite varied jobs.

Table 19 summarizes all six assignment options. The table also shows that the least-cost solution would be to assign Cooper to project 1, Brown to project 2, and Adams to project 3, at a total cost of $25.

The owner's objective is to assign the three projects to the workers in a way that will result in the lowest cost to the shop. Note that the assignment of people to projects must be on a one-to-one basis; each project will be assigned exclusively to one worker only.

	PROJECT		
PERSON	1	2	3
Adams	\$11	\$14	\$6
Brown	8	10	11
Cooper	9	12	7

Table 18: Repair costs of the Fix-It-Shop assignment problem

PROJECT ASSIGNMENT				
1	2	3	LABOUR COSTS (\$)	TOTAL COSTS (\$)
Adams	Brown	Cooper	11 + 10 +7	28
Adams	Cooper	Brown	11 + 12 +11	34
Brown	Adams	Cooper	8 + 14 + 7	29
Brown	Cooper	Adams	8 + 12 + 6	26
Cooper	Adams	Brown	9 + 14 + 11	34
Cooper	Brown	Adams	9 + 10 + 6	25

Table 19: Assignment alternatives and Costs of Fix-It-Shop assignment problem

Special algorithms exist to solve assignment problems. The most common is probably the **Hungarian** solution method. The Hungarian method of assignment provides us with an efficient means of finding the optimal solution without having to make a direct comparison of every assignment option. It operates on a principle of matrix reduction, which means that by subtracting and adding appropriate numbers in the cost table or matrix, we can reduce the problem to a matrix of *opportunity costs*. Opportunity costs show the relative penalties associated with assigning any person to a project as opposed to making the best or least-cost assignment. We would like to make assignments such that the opportunity cost for each assignment is zero.

The steps involved in the Hungarian method are outlined below.

THE HUNGARIAN METHOD

1. *Find the opportunity cost table by*
   (a) Subtracting the smallest number in each row of the original cost table or matrix from every number in that row.
   (b) Then subtracting the smallest number in each column of the table obtained in part (a) from every number in that column.
2. *Test the table resulting from step 1 to see whether an optimal assignment can be made.* The procedure is to draw the minimum number of vertical and horizontal straight lines necessary to cover all zeros in the table. If the number of lines equals either the number of rows or columns, an optimal assignment can be made. If the number of lines is less than the number of rows or columns, we proceed to step 3.
3. *Revise the present opportunity cost table.* This is done by subtracting the smallest number not covered by a line from every other uncovered number. This same smallest number is also added to any number(s) lying at the intersection of the horizontal and vertical lines. We then return to step 2 and continue the cycle until an optimal assignment is possible.

Let us now apply the three steps to the Fix-It-Shop assignment example.

The original cost table for the problem is given in Table 20

	PROJECT		
PERSON	1	2	3
Adams	11	14	6
Brown	8	10	11
Cooper	9	12	7

Table 20: Initial Table

	PROJECT		
PERSON	1	2	3
Adams	5	8	0
Brown	0	2	3
Cooper	2	5	0

Table 21: Row reduction (part a)

After the row reduction (Step 1 part a) we get the cost Table 21.

Taking the costs in Table 21 and subtracting the the smallest number in each column from each number in that column results in the total opportunity costs given in Table 22. This step is the column reduction of Step 1 part (b)

If we draw vertical and horizontal straight lines (Step 2) to cover all the zeros in Table 22 we get Table 23. Since the number of lines is less than the number of rows or columns an optimal assignment cannot be made.

Since Table 23 doesn't give an optimal solution we revise the table. This is accomplished by subtracting the smallest number not covered by a line from all numbers not covered by

	PROJECT		
PERSON	1	2	3
Adams	5	6	0
Brown	0	0	3
Cooper	2	3	0

Table 22: Column Reduction (Step 1 part b)

	PROJECT		
PERSON	1	2	3
Adams	5	6	0
Brown	0	0	3
Cooper	2	3	0

Table 23: Testing for an optimal solution

a straight line. This same smallest number is then added to every number (including zeros) *lying in the intersection* on any two lines. The smallest uncovered number in Table 23 is 2, so this value is subtracted from each of the four uncovered numbers. A 2 is also added to the number that is covered by the intersecting horizontal and vertical lines. The results of this step are shown in Table 24

To test now for an optimal assignment, we return to Step 2 and find the minimum number of lines necessary to cover all zeros in the revised opportunity cost table. Because it requires three lines to cover the zeros (see Table 25), an optimal assignment can be made.

	PROJECT		
PERSON	1	2	3
Adams	3	4	0
Brown	0	0	5
Cooper	0	1	0

Table 24: Revised opportunity cost table

	PROJECT		
PERSON	1	2	3
Adams	3	4	0
Brown	0	0	5
Cooper	0	1	0

Table 25: Optimality test on the revised table

Finally, we make the allocation. Note that only one assignment will be made from each row or column. We use this fact to proceed to making the final allocation as follows:

(a) Find a row or column with only one zero cell.

(b) Make the assignment corresponding to that zero cell.

(c) Eliminate that row and column from the table.

(d) Continue until all the assignments have been made.

For our Fix-It-Shop problem these steps are summarized in Table 26.

	FIRST ASSIGNMENT				SECOND ASSIGNMENT				THIRD ASSIGNMENT		
	1	2	3		1	2	3		1	2	3
Adams	3	4	$\boxed{0}$	Adams	3	4	$\boxed{0}$	Adams	3	4	$\boxed{0}$
Brown	0	0	5	Brown	0	0	5	Brown	0	$\boxed{0}$	5
Cooper	0	1	0	Cooper	$\boxed{0}$	1	0	Cooper	$\boxed{0}$	1	0

Table 26: Making the final assignment

To interpret the table we recall that our objective was to minimize costs, there is only one assignment that Adams can go to where the opportunity costs are \$0. That is to assign Adams Project 3. If Adams gets assigned to Project 3, then there is only one project left where the opportunity cost is \$0 for Cooper. Therefore Cooper gets assigned to Project 1. This leaves Brown being assigned to Project 2, where the opportunity costs are \$0.

The optimal allocation is to assign Adams to Project 3, Brown to Project 2, and Cooper to Project 1. The total labour cost of this assignment are computed from the original cost table (see Table 18). They are as follows:

ASSIGNMENT	COST (\$)
Adams to Project 3	6
Brown to Project 2	10
Cooper to Project 1	9
Total cost	25

Example 4.3 *Suppose we have to allocate 4 tasks (1,2,3,4) between 4 people (W,X,Y,Z). The costs are set out in the following table:*

	Task			
Person	*1*	*2*	*3*	*4*
W	*8*	*20*	*15*	*17*
X	*15*	*16*	*12*	*10*
Y	*22*	*19*	*16*	*30*
Z	*25*	*15*	*12*	*9*

The entries in the table denote the costs of assigning a task to a particular person.

Solution: Step 1 of the Hungarian method involves the following parts:

(a) subtract the minimum value from each column (see Table 27)

(b) subtract the minimum value from each column (see Table 28)

	Task			
Person	1	2	3	4
W	0	12	7	9
X	5	6	2	0
Y	6	3	0	14
Z	16	6	3	0

Table 27: Subtract the minimum value from each row

	Task			
Person	1	2	3	4
W	0	9	7	9
X	5	3	2	0
Y	6	0	0	14
Z	16	3	3	0

Table 28: subtract the minimum value from each column

The next step is to check whether optimal assignment can be made. This is done by finding the minimum *number of lines necessary to cross-out all the zero cells in the table. If this is equal to n (the number of people/tasks) then the solution has been found. The minimum number of lines necessary to cross through all the zeros (see Table 29)is 3 ¡ n = 4 so that an optimal allocation has not been found.*

*(*Note that there may be more than one way to draw the lines through the zero cells. It does not matter which way you choose as long as there is no alternative way involving fewer lines*)*

	Task			
Person	1	2	3	4
W	0	9	7	9
X	5	3	$\boxed{2}$	0
Y	6	0	0	14
Z	16	3	3	0

Table 29: Checking if an optimal assignment can been made

Next we revise the table by

(a) *Finding the minimum uncovered cell. Table 29 shows that the minimum uncovered cell has a value of 2*

(b) *Subtracting the value obtained in (a) (i.e subtract 2) from all the uncovered cells.*

(c) *Adding to all the cells at the intersection of the two lines.*

The result of the above steps is given in Table 30.

We then check if the revised allocation is optimal. This is done by finding the minimum number of lines required to cover all zeros (see Table 31).

This time the minimum number of lines necessary to cross through all the zeros is n = 4 so that an optimal allocation has been found.

To make the final allocation we use the following steps.

	Task			
Person	1	2	3	4
W	0	7	5	9
X	5	1	0	0
Y	8	0	0	16
Z	16	1	1	0

Table 30: Revising the Table

	Task			
Person	1	2	3	4
W	0	7	5	9
X	5	1	0	0
Y	8	0	0	16
Z	16	1	1	0

Table 31: Checking for optimality

- *Find a row or column with only one zero cell.*
- *Make the assignment corresponding to that zero cell.*
- *eliminate that row and column from the table.*
- *Continue until all assignments have been found.*

	Task			
Person	*1*	*2*	*3*	*4*
W	*0*	*7*	*5*	*9*
X	*5*	*1*	*0*	*0*
Y	*8*	*0*	*0*	*16*
Z	*16*	*1*	*1*	*0*

- *Assign person W to task 1 and eliminate row W and column 1.*
- *Assign person Y to task 2 and eliminate row Y and column 2.*
- *Assign person Z to task 4 and eliminate row Z and column 4.*
- *This leaves final person X assigned to remaining task 3.*

From the original cost table, we can determine the costs associated with the optimal assignment:

$$Total \quad Cost = 48$$

4.8 Maximization Assignment Problems

Some assignment problems are phrased in terms of **maximizing** the payoff, profit, or effectiveness of an assignment instead of *minimization* costs. It is easy to obtain an equivalent minimization problem by converting all numbers in the table to opportunity costs; efficiencies to inefficiencies,etc. This is achieved through *subtracting every number in the original payoff table from the largest single number in the number.* The transformed entries represent opportunity costs; it turns out that minimizing the opportunity costs produces the same assignment as the original maximization problem. Once the optimal assignment for this transformed problem has been computed, the total payoff or profit is found by adding the original payoffs of those cells that are in the original assignment.

Example. The British Navy wishes to assign four ships to patrol four sectors of the North Sea. In some areas ships are to be on the outlook for illegal fishing boats, and in other sectors to watch for enemy submarines, so the commander rates each ship in terms of its profitable efficiency in each sector. These relative efficiencies are illustrated in Tables 32. On the basis of the ratings shown, the commander wants to determine the patrol assignments producing the greatest overall efficiencies.

	SECTOR			
SHIP	A	B	C	D
1	20	60	50	55
2	60	30	80	75
3	80	100	90	80
4	65	80	75	70

Table 32: Efficiencies of British Ships in Patrol sectors

	SECTOR			
SHIP	A	B	C	D
1	80	40	50	45
2	40	70	20	25
3	20	0	10	20
4	35	20	25	30

Table 33: Opportunity Costs of British Ships

We start by converting the maximizing efficiency table into a minimization opportunity cost table. This is done by subtracting each rating from 100, the largest rating in the whole table. The resulting opportunity costs are given in Table 33.

Next, we follow steps 1 and 2 of the assignment algorithm. The smallest number is subtracted from every number in that row to give Table 34; and then the smallest number in each column is subtracted from every number in that column as shown in Table 35.

	SECTOR			
SHIP	A	B	C	D
1	40	0	10	5
2	20	50	0	5
3	20	0	10	20
4	15	0	5	10

Table 34: Row opportunity costs for the British Navy Problem

	SECTOR			
SHIP	A	B	C	D
1	25	0	10	0
2	5	50	0	0
3	5	0	10	15
4	0	0	5	5

Table 35: Total opportunity costs for the British Navy Problem

The minimum number of straight lines needed to cover all zeros in this total opportunity cost table is four. Hence an optimal assignment can be made. The optimal assignment is ship 1 to sector D, ship 2 to sector C, ship 3 to sector B, and ship 4 to sector A.

The overall efficiency, computed from the original efficiency data Table 32, can now be shown:

ASSIGNMENT	EFFICIENCY
Ship 1 to Sector D	55
Ship 2 to Sector C	80
Ship 3 to Sector B	100
Ship 4 to Sector A	65
Total Efficiency	300

4.9 Summary

In this section we discussed the Hungarian method for solving both maximization and minimization assignment problems.

4.10 Exercises 4.3: Minimization Assignment Problems

1. Three accountants, Phindile, Rachel and Sibongile, are to be assigned to three projects, 1, 2 and 3. The assignment costs in units of E1000 are given in the table below.

		Project		
		1	**2**	**3**
Accountant	**P**	15	9	12
	R	7	5	10
	S	13	4	6

2. Joy Taxi has four taxis, 1,2,3 and 4, and there are four customers, P, Q, R and S requiring taxis. The distance between the taxis and the customers are given in the table below, in Kilometres. The Taxi company wishes to assign the taxis to customers so that the distance traveled is a minimum.

		Customers			
		P	Q	R	S
Taxis	**1**	10	8	4	6
	2	6	4	12	8
	3	14	10	8	2
	4	4	14	10	8

3. Four precision components are to be shaped using four machine tools, one tool being assigned to each component. The machining times, in minutes, are given in the table below.

		Component			
		1	**2**	**3**	**4**
	A	21	20	39	36
Machine Tool	**B**	25	22	24	25
	C	36	22	36	26
	D	34	21	25	39

4. In a job shop operation, four jobs may be performed on any of four machines. The hours required for each job on each machine are presented in the following table. The plant supervisor would like to assign jobs so that total time in minimized. Use the assignment method to find the best solution.

	MACHINE			
JOB	W	X	Y	Z
A12	10	14	16	13
A15	12	13	15	12
B2	9	12	12	11
B9	14	16	18	16

Answer: A12 to W, A15 to Z, B2 to Y, B9 to Z, 50 hours.

4.11 Exercises 4.4: Maximization Assignment Problems

1. A head of department has four lecturers to assign to pure maths (1), mechanics (2), statistics (3) and Quantitative techniques (4). All of the teachers have taught the courses in the past and have been evaluated with a score from 0 to 100. The scores are shown in the table below.

	1	**2**	**3**	**4**
Peters	80	55	45	45
Radebe	58	35	70	50
Tsabedze	70	50	80	65
Williams	90	70	40	80

The head of department wishes to know the optimal assignment of teachers to courses that will maximize the overall total score. Use the Hungarian algorithm to solve this problem. [$P \to 1$, $R \to 3$, $T \to 4$, $W \to 2$ Max Score = 285]

2. A department store has leased a new store and wishes to decide how to place four departments in four locations so as to maximize total profits. The table below gives the profits, in thousands of emalangeni, when the departments are allocated to the various locations. Find the assignment that maximizes total profits.

		Location			
		1	**2**	**3**	**4**
Department	Shoes	20	16	22	18
	Toys	25	28	15	21
	Auto	27	20	23	26
	Housewares	24	22	23	22

3. The head of the business department, has decided to apply the Hungarian method in assigning lecturers to courses next semester. As a criterion for judging who should teach each course, the head of department reviews the past two years' teaching evaluations. All the four lecturers have taught each of the courses at one time or another during the two year period. The ratings are shown in the table below.
   Find the best assignment of lecturers to courses to maximize the overall teaching rating. Total Rating = 335

	COURSE			
LECTURER	STATISTICS	MANAGEMENT	FINANCE	ECONOMICS
Dlamini	90	65	95	40
Khumalo	70	60	80	75
Masuku	85	40	80	60
Nxumalo	55	80	65	55