

# Foundation of Linear Programming: A Managerial Perspective from Solving System of Inequalities to Software Implementation

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## ABSTRACT

*The aim and scope of this paper are the infusion of purposeful action by decision makers with an explicit understanding of analytical linear programming (LP) tools in order that the managers will implement strategic results correctly. The advances in computing software have brought LP tools to the desktop for a variety of applications to support managerial decision-making. However, it is already recognized that current LP tools do not answer the managerial questions satisfactorily. For instance, there is a costly difference between the mathematical and managerial interpretations of sensitivity analysis. The LP software packages provide sensitivity results about the optimality of a basis and not about the optimality of the values of the decision variables, and the shadow prices that are of interest to the manager. Society has developed the largest sensitivity region based on "optimal solution," not that which "preserves the basis" that allows for simultaneous dependent/independent changes. The aim and results of this article are found further in the work.*

**Keywords:** *Algebraic Methods for Linear Programming Tools, Degeneracy, Dual Prices, Linear Programming (LP), Managerial Understanding, Modeler and Decision Maker, More-for-Less, Sensitivity Regions*

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## 1. INTRODUCTION

Linear programming has been a fundamental topic in the development of managerial decision-making. The subject has its origins in the early work of Fourier (1820s) on attempting to solve systems of linear inequalities. However, it has to wait for the invention of the Standard Simplex Method.

There are also many diverse applications, and therefore many specialized solution algorithms have been developed. Hatami-Marbini and Tavana (2011) introduced a new method called the bounded dual simplex method for bounded fuzzy number linear programming problems which is useful for these situations. This algorithm constructs a dual feasible basic solution after obtaining a working basic and then moves towards attaining primal feasibility while maintaining dual feasibility throughout.

DOI: 10.4018/jsds.2012070104

The use of graph-based LP model is reported by Diaby (2010), Sowlati, Assadi, and Paradi, (2010) extend the LP-based sensitivity analysis to multi-criteria decisionmaking. While Bor-gonovo and Percoco (2011) applied sensitivity analysis a portfolio problem with budget constraints that allows for simultaneous sensitivity analysis. Khurana and Arora (2012) modified the classical transportation problem to solve a three-dimensional linear trans-shipment problem. Sun (2010) uses linear programming as nonparametric statistical procedures for multiple-class discriminant and classification analysis.

### 1.1. Standard Simplex Method

Since World War II, linear programming (LP) has been used to solve small and large problems in almost all disciplines. The most popular solution algorithm is the Simplex method that is implemented in all LP software packages.

The simplex algorithm can be considered as a sub-gradient directional method, jumping from an initial feasible vertex to a neighboring vertex of feasible region until it arrives at an optimal vertex (if any). To start the algorithm, if it is needed, one must create an equivalent LP problem to satisfy its “Standard form requirement.” When the manager obtains the simplex optimal solution, then the manager has to interpret and to transform the final solution of the algorithm to obtain the optimal solution in terms of the original model, including the variables. This may not be an easy task. Moreover, solving LP problems in which some constraints are in ( $\geq$ ) or ( $=$ ) form with non-negative right-hand side (RHS) has raised difficulties. One version of the simplex, known as the two-phase method, introduces an artificial objective function, which is the sum of artificial variables (Arsham 1997). Another version adds the penalty terms, which are the sum of artificial variables with very large, positive coefficients. The latter approach is known as the Big-M method (Arsham, 2006, 2007). Understanding the intuitive notion of Standard-form, artificial variables, and Big-M, may require a greater mathematical sophistica-

tion from most students/managers. The simplex methods have to iterate through many infeasible vertices to reach an initial feasible vertex.

Using the dual simplex method has its own difficulties. For example, when some coefficients in the objective function are not dual feasible, one must introduce an artificial constraint. Handling equality ( $=$ ) constraints by the dual simplex method is tedious because of introduction of two new variables for each equality constraint: one extraneous slack variable and one surplus variable. Also, one may not be able to remove some equality ( $=$ ) constraints by elimination at the outset, as this may violate the non-negativity condition introduced when constructing the “standard form.”

### 1.2. Warnings on Using LP Software Packages

Unfortunately, the widely used LINDO LP software does not provide any direct warnings about the existence of multiple solutions nor degenerate optimal solution. Clearly, the final report on sensitivity analysis is not valid in these cases, Lin (2010).

Numerous recent laments that OR/MS has lost its problem solving origins are reviewed. This study verifies these claims by examining popular introductory OR/MS textbooks, and finds that they are overwhelmingly devoted to OR algorithms rather than the problem solving process.

Coverage of the complete development cycle from problem observation modeling through complete implementation must serve as the “backbone” of the course. In a broader context, even research reported in professional journals tends to report the solutions to problems, rather than the problem solving for the managers strategic decision making that he needs to understand.

Sometimes one gets a surprise result in implementing LINDO. For example solving:

$$\begin{aligned} &\text{Minimize } 18X_1 + 10X_2 \\ &\text{Subject to } 12X_1 + 10X_2 \geq 120,000, \quad 12X_1 + \\ &10X_2 \leq 150,000, \quad X_1 \geq 0, \quad X_2 \geq 0 \end{aligned}$$

in the final report the shadow prices are reported to be  $U_1 = -2.125$ , and  $U_2 = 0.75$ , while the correct one is  $U_1 = 2.125$ , and  $U_2 = -0.75$ .

These unfortunate cases are not limited to LINDO, for more examples see, Arsham (2008),

### 1.3. The Costly Difference between the Managerial and Modeler Interpretation

Koltai and Terlaky (2000) stated that managerial questions are not answered satisfactorily with the mathematical interpretation of sensitivity analysis, since software packages provide sensitivity results about the optimality of a basis and not about the optimality of the values of the decision variables. The misunderstanding of the shadow price and the validity range information provided by a simplex based computer program may lead to wrong decision with considerable financial losses and strategic consequences. For example note that the Phrases “shadow price” and “opportunity cost” have somewhat different meanings in LP and Economics literature. The “opportunity cost” of an action in economics can be interpreted as the “shadow price” of that action on the budget.

Johnson and Coyle (2010) ask for ethical decision-making, relies on models that describe the steps that in addition to the purposes that can be easily identified, can be objectively understood, modelled, and optimized.

Petrovic (2012) observes that the increasing methodologies, methods, techniques, and models that can be employed in problem situation structuring and solving, must be considered as relevant aspects of management process in contemporary circumstances. Davies (2010) points out the unfortunate situation namely the interdependent or no relationships between management academics and management practice. Managers may have framed their problems inappropriately, made inappropriate assumptions, tackled the wrong problems, attacked problems at the wrong levels, or just addressed them in poor fashion. Problems poorly addressed create more problems and take

longer to fix in the long-term. Those working within the domain of the decision or management sciences, or operations research, often find themselves confronted by a double-headed gloomy vision. Not only do management books paint a gloomy picture of the problem-solving and decision-making abilities of managers and organizational decision makers highlighting the decision traps faced by managers and the common failings of managers.

The LP software packages provide sensitivity results about the optimality of a basis and not about the optimality of the values of the decision variables, and the shadow prices that are of interest to the manager. We develop the development of the largest sensitivity region based of “optimal solution” not that which “Preserves the Basis” and allows for simultaneous dependent/independent changes.

### 1.4. The Confusion Between the Model and the Solution Algorithm

Naturally, a modeler, reflecting the modeler’s decision problem, creates the LP model. On the other hand, the solution algorithm is designed by, for example, a systems analyst to solve LP problem. Because of popularity of simplex method, there is confusion between model and the solution algorithm. For instance, LINDO software requires that all variables be non-negative, while some models do contain negative-sign or unrestricted variables. In this paper, the algorithm adjusts itself to the original model and solves it, instead of changing the model. The result is readily understandable by the modeler, because we did not change the structure of the model. For example the need for standard form changes the manager LP model to something that is not his and therefore optimal solution must be transfer back to his that certainly confuses the manager in making costly misinterpretations. One must understand the difference between the model and its solution algorithm; unfortunately the current state of arts of LP does not differentiate and causes confusions. This section follows by development of the largest sensitivity region based of “optimal

solution” not any simplex tableau that “Preserves the Basis.” This section is independent of all previous sections; in that one needs the optimal solution in any way it is obtained. No one expects the Manager to understand Basis, but the optimal value for his decision variable and the consequence of any change on them. The Dual generated from the “standard form” does not look like the Manager’s dual problem of his primal model.

This paper provides short critical reviews of these developments by looking for their limitations in order to expand their scopes. We propose tools that work directly in the original decision variables and constraints space. The topics presented in this paper are free from any extraneous artificial variables, artificial objective functions, artificial constraints, non-negativity condition on decision variables.

Numerical examples illustrate the proposed methods and their implications are presented.

The aim is the unifications of diverse topics in its natural states. The main objective is to present a diverse set of topics in a manner which is easy to understand, easy to implement, and provide useful information to the managers.

The remainder of this paper is organized in twelve sections. Almost every section can stand by itself, independent of previous sections. Section 2 introduces linear programming models from model builders and decision makers. Some oddities are included. Section 3 describes the decision maker’s environment explicitly in order to help to understand the decision problem. Section 4 explores the feasible region by using algebra that is understandable by the decision maker. The algebraic approach to solving linear programs is given in Section 5, which includes computation of slack and surplus variables. The methodology solves LP model as is, without any need of “standard form,” artificial variables, artificial constraints, the Big-M, and slack/surplus.

Section 6 constructs the largest sensitivity region for the right hand side of the constraints based on optimal solution, however it was obtained. The shadow prices are byproducts

of the sensitivity analysis. The dual problem formulation is included in Section 7, which is general and easy to apply. Section 8 considers the general sensitivity region for the cost coefficients of the objective function.

Throughout this paper, the LP problems could be higher than two-dimensional. We do not use the graphical method because it is limited to two-dimensional problems. We use the algebraic approach throughout this paper to overcome the limitations of the graphical method. Unfortunately, the textbooks using graphical methods for finding the cost sensitivity range are misleading. A counterexample is provided. The sensitivity analysis of the degenerate optimal vertex to remain optimal is developed in Section 9. Section 10 does sensitivity analysis of multiple optimal solutions. The existence of a curious property, known as more for less (MFL) or less for more (LFM) is covered in Section 11. Concluding remarks are summarized in Section 12.

## 2. LINEAR PROGRAMMING PROBLEM

**Problem P:** Max (or Min)  $cX$

Subject to  $AX \leq a,$

$BX \geq b,$

$DX = d,$

$X_i \geq 0, i = 1, \dots, j$

$X_i \leq 0, i = j+1, \dots, k$

$X_i$  unrestricted in sign,  $i = k+1, \dots, n$

Where matrices  $A, B,$  and  $D$  have  $p, q,$  and  $r$  rows respectively with  $n$  columns and vectors  $c, a, b,$  and  $d$  have appropriate dimensions. Therefore, there are  $m = (p + q + r + k)$  constraints and  $n$  decision variables. It is assumed that  $m \geq n$ . Note that the main constraints have been separated into three subgroups. Without loss of generality, we assume that all RHS elements,  $a, b,$  and  $d$  are non-negative. We do not deal with trivial cases, such as where  $A = B = D = 0$  (no constraints), or  $a = b = d = 0$  (all boundaries pass through the origin point), or having LP with

single feasible region. We assume the LP has a nonempty, bounded feasible region.

## 2.1. Odd LP Models as the Possible Sign for Performing Modeling Validation

The following are two examples of modeling validations tools.

### 2.1.1. Unbounded Solutions

**Identification:** An unbound optimal solution means the constraints do not limit the optimal solution, and the feasible region effectively extends to infinity.

**Resolution:** In real life, this is very rare. Check the formulation of the constraints. Possibly, one or more constraints are missing. Check also the constraints for any misspecification in the direction of inequality constraints, and numerical errors.

### 2.1.2. Infeasibility

**Identification:** An infeasible solution means the constraints are too limiting and have left no feasible region. That is, no solution satisfies all the constraints of a problem.

**Resolution:** Check the constraints for any misspecification in the direction of inequality constraints, and numerical errors. If no error exists, then there are conflicts of interests.

These and other odd situations, including multiple solutions, and in particular degeneracy, are rare (except in network models since they often have a redundant constraint, causing degeneracy), and often resulted at the modeling stage. For example, since the decision maker sets the objective function, while the constraints came from the decision maker environment, therefore, it is rare that the objective function had the coefficient proportion to those of at least one binding constraint. In this case, rounding the coefficient(s) could be the source of error.

For example, the need for Standard Form changes the manager's LP model to something

that is not his and therefore optimal solution must be transferred back to him. That certainly confuses the manager, causing him to make costly misinterpretations. One must understand the difference between the model and its solution algorithm; unfortunately the current state-of-art LP does not differentiate and causes confusion. This section follows by developing the largest sensitivity region based on "optimal solution" not on any simplex tableau that "Preserves the Basis." This section is independent of previous Sections in that one needs the optimal solution in any way it is obtained. No one expects the Manager to understand Basis, but one may expect the Manager to understand the optimal value for his decision variable and the consequence of any change on them. The Dual generated from the "standard form" does not look like the Manager's dual problem of his primal model.

## 2.2. Model-Builders and Decision-Makers

The operations research process must be considered from the viewpoint of General Systems Theory. The components of the OR process and the relations between them are critically examined. The five components are: (1) the "reality" of the problem situation, (2) the conceptual model of the problem situation, (3) the scientific model of the conceptual model, (4) the solution to the scientific model, and (5) the implementation of the solution. In most cases we have sub-optimized both our knowledge (study) and our application of the OR process. That is, there have been extremely few studies and applications of OR which have concerned themselves with OR from a whole systems point of view. Clearly, without a whole systems perspective, OR can neither be understood nor effectively applied.

The process of modeling and validation goes hand-in-hand. That is achieved by providing an interpretation for each part of the modeling-validation involved 'managerial situation,' 'conceptual model,' 'formal model,' and 'decision.' Such an approach to modeling validation process is also instrumental in posi-



tioning OR workers with respect to the nature of their work. The effective validation process makes terms like ‘confidence,’ ‘credibility and reliability,’ ‘model assessment and evaluation,’ ‘usefulness and usability of the model’ common.

The model validation and related issues should put in a framework that will be of use both to model-builders and to decision-makers. Often off-the-shelf software is not the answer, for example Bagloee and Reddick (2011) consider the budget allocation problem formulated an optimization model with a tailor-made solution algorithm and then is tested numerically. Zeleny (2010) reminds us that *strategy is a purposeful action*, therefore, strategy is what company does, and what company does is its strategy.

Model validation has traditionally been treated from the perspective of efficiency (doing things right). The perspective of effectiveness (doing the right things) on the other hand, has been neglected or completely ignored. This Special Issue is more on the side of effectiveness. The effectiveness perspective addresses the basic question of knowing what a valid model is. The following are examples of tools for modeling validations.

What can go wrong in the process of building a Linear Programming (LP) model? Potential pitfalls exist which affect any LP application; therefore, the decision-maker and the analyst should be cognizant of deficiencies of LP at the modeling stage.

### 3. DECISION-MAKER'S ENVIRONMENT

Since a model of a decision problem system is a re-presentation of the problem that contains those elements that affect the objective of our decision, it is important to identify the most important elements and categorize them. The problem understanding requires criteria for grouping together entities of the decision model in a same category. By finding *fundamental determinants* Hammami and Boujelbene (2012) modeled successfully the linkages between

macroeconomic environment and stock market cycles by identifying stock market boom-bust cycles, is a good example. In reality there are often conflicts or even multiple objectives that the model should address explicitly. Das, Sarkar, and Ray (2012), proposed a new method for multi-objective optimization with discrete alternatives that is, multi-objective optimization on the basis of simple ratio analysis. Competitions, and ethics are also a fact of business, for example Nabelsi (2012) developed and validated a set of key performance indicators in supply chain of healthcare sector to improve operational efficiency and to gain a competitive advantage. Jajimoggala, Kesava Rao, and Satyanarayana (2011) provide an optimal maintenance strategy mix for increasing availability and reliability levels of production facilities without significantly increasing operational costs.

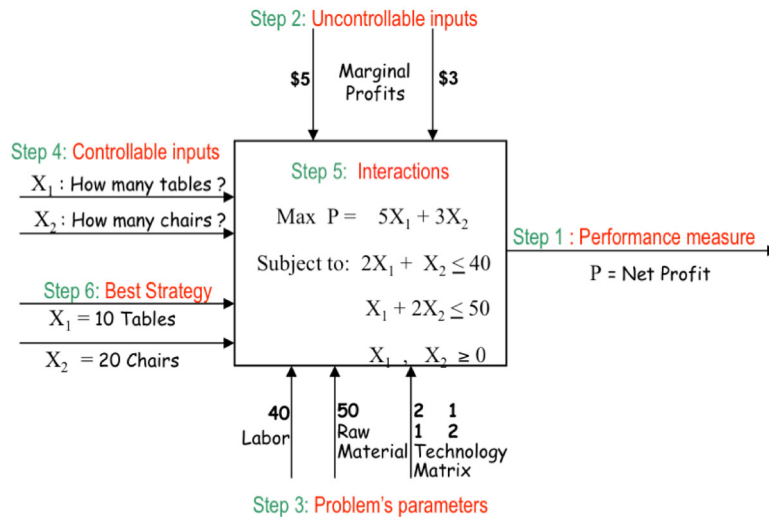
Performance measure (or indicators): Measuring business performance is at the top of the management decision-making. The development of effective performance measurement (or indicator) is seen as increasingly important by many organizations. However, the challenges of achieving this in the public sector and for non-profit organizations are, arguably, considerable. However, Burmaoglu and Kazancoglu (2012) proposed an effective formation of optimization criteria to the problem of *government agencies* website evaluation.

The model must provide the desirable but feasible level of outcome for the objective of decision-maker. Clearly, objective is important in identifying the problem. Other elements of decision-makers environment are classified as follows:

**Uncontrollable inputs:** These come from the decision maker's environment. Uncontrollable inputs often create the problem and constrain the actions.

**Parameters:** Parameters are the constant elements that do not change during the time horizon of the decision review. These are the factors partially defining the problem. Strategic decisions usually have longer time horizons than both the Tactical and the

Figure 1. Decision-maker's environment in a mixed product problem



Decision-Maker's Environment in a Mixed Product Problem

Operational decisions. Cigdem, Kadri, and Songwut (2011) developed an LP model for truck allocation which is both meets needs for short and long-term mine planning by taking into account economic parameters, multi time periods and the uncertainty of load and travel times and ore grades.

### 3.1. An Illustrative Numerical Example

Consider the following mixed products LP problem of a price-taker small manufacturer producing table and chairs his objective is to maximize his net profit, on weekly basis:

$$\begin{aligned} \text{Maximize } P(\mathbf{X}) &= 5X_1 + 3X_2 \\ \text{Subject to: } 2X_1 + X_2 &\leq 40, \\ X_1 + 2X_2 &\leq 50, \\ X_1 \geq 0, X_2 &\geq 0. \end{aligned}$$

Figure 1 analyses and then depicts his decision environment.

**Controllable Inputs:** The collection of all possible courses of action you might take.

**Interactions among These Components:** These are logical, mathematical functions representing the cause-and-effect relationships among inputs, parameters, and outcomes.

There is also a set of constraints that applies to each of these. Therefore, they do not need to be treated separately.

**Actions:** Action is the ultimate decision and is the best course of strategy to achieve the desirable goal. Decision-making involves the selection of a course of action (means) in the pursuit of one's objective (ends).

**Controlling the Problem:** Few problems in life, once solved, stay that way. Changing conditions tends to un-solve problems that were previously solved, and their solutions create new problems. One must identify and anticipate these new problems.

## 4. SOLVING THE SYSTEM OF LINEAR INEQUALITIES

The solution to an LP problem is strictly based on the theory and solution of a system of linear inequalities (Arsham, 2006). The basic solutions to a linear program are the solutions to the systems of equations consisting of constraints at binding position. Not all basic solutions satisfy all of the problem constraints. Those that do meet all of the constraint restrictions are called the basic (i.e., simple) feasible solutions. The basic feasible solutions correspond precisely to the vertices of the feasible region.

We present an Improved Algebraic Method of solving a System of Linear Inequalities (SLI) that does not require the formulation of an auxiliary LP problem and solution algorithms such as Simplex. We provide a simple methodology to extend the solution of SLI of one or two dimensions to systems of higher dimensions. We are interested in finding the vertices of the feasible region of Problem P, expressed as a system of linear equalities and inequalities

$$\begin{aligned}AX &\leq a, \\ BX &\geq b, \\ DX &= d,\end{aligned}$$

Where some  $X_i \geq 0$ , some  $X_i \leq 0$ , and some  $X_i$  are unrestricted in sign. Matrices A, B, and D, as well as vectors a, b, and d have appropriate dimensions. For the sake of convenience, we refer to this general system of equalities and inequalities as a "system" and its feasible region set as S. Therefore, the optimization problem can be expressed as:

**Problem P:** Max (or min)  $C(X)$   
Subject to  $X \in S$

### 4.1. Steps of the Improved Algebraic Method

*Step 1.* Convert all inequalities into equalities (including any variable restricted constraints).

*Step 2.* Calculate the difference between the number of variables (n) and the number of equations (m).

*Step 3.* Determine solution to all square system of equations. The maximum number of systems of equation to be solved is:  $m!/[n!(m-n)!]$ .

*Step 4.* Check feasibility of each solution obtained in Step 3 by using the constraints of all other equations.

The coordinates of vertices are called the basic (i.e., simple) solutions (BS) of the systems of equations obtained by setting some of the constraints at binding (i.e., equality) position. For a bounded feasible region, the number of vertices is at most  $m!/[n!(m-n)!]$  where m is the number of constraints and n is the number of variables. Therefore, taking any set of n equations and solving them simultaneously obtain a BS. By plugging this BFS in the constraints of other equations, one can check for feasibility of the FS. If it is feasible, then this solution is a BF solution that provides the coordinates of a corner point of the feasible region.

### 4.2. Numerical Example for Solving System of Inequalities

We provide an example to explain the Improved Algebraic Method. Consider the following system of inequality:

$$\begin{aligned}12X_1 + 10X_2 &\geq 120000 \\ 10X_1 + 15X_2 &\leq 150000 \\ X_1 &\geq 0 \\ X_2 &\geq 0\end{aligned}$$

There will be four equations and two variables, yielding six possible combinations. The Improved Algebraic Method provides six vertices for the feasible region as in Table 1.

Notice that from second constraint and the non-negativity conditions; clearly the feasible region is bounded; since none of  $X_1$  nor  $X_2$  can be have positive large numbers. Therefore, the three *basic feasible solutions* (of the bounded



Table 1. The basic solutions of the numerical example

Binding	Solution	Is it Feasible?
The Constraints	(X1, X2)	Why?
1 and 2	(3750, 7500)	Yes, all satisfied
1 and 3	(0, 12000)	No, constraint 2 is not
1 and 4	(100000, 0)	Yes, all satisfied
2 and 3	(0, 10000)	No, constraint 2 is not
2 and 4	(15000, 0)	Yes, all satisfied
3 and 4	(0, 0)	No constraint 1 is not

feasible vertices) to the system of inequalities are the following three vertices:

$$\begin{aligned} X_1 &= 10000 \quad X_1 = 15000 \quad X_1 = 3750 \\ X_2 &= 0 \quad X_2 = 0 \quad X_2 = 7500 \end{aligned}$$

### 4.3. Algebraic Presentation of Feasible Region

Using the parameters  $\lambda_1, \lambda_2, \lambda_3$  for the three vertices, we obtain the following convex combination of vertices which is a parametric representation of the feasible region:

$$\begin{aligned} X_1 &= 10000\lambda_1 + 15000\lambda_2 + 3750\lambda_3 \\ X_2 &= 7500\lambda_3 \end{aligned}$$

For all parameters  $\lambda_1, \lambda_2, \lambda_3$  such that each is non-negative, and they all add to 1.

Note that the above representation is valid as the feasible region is bounded. For example, the feasible region  $X_1 + X_2 \leq 1, X_1 \geq 1, X_2 \geq 0$ , is unbounded and contains one vertex only.

Moreover, if the feasible region is bounded, it must have more than one vertex. The feasible region of  $X_1 + X_2 = 1, X_1 - X_2 = 1, X_1 \geq 0, X_2 \geq 0$ , has a feasible region with a single point, (1, 0).

## 5. SOLVING LINEAR PROGRAMS

Consider the following linear programs with bounded feasible region.

$$\begin{aligned} \text{Minimize } C(X) &= 18X_1 + 10X_2 \\ \text{Subject to: } 12X_1 + 10X_2 &\geq 120000 \\ 10X_1 + 15X_2 &\leq 150000 \\ \text{where } X_1, X_2 &\geq 0 \end{aligned}$$

Substituting the parametric version of the feasible region into the objective function, we obtain:

$$C(\lambda) = 18X_1 + 10X_2 = 180000\lambda_1 + 270000\lambda_2 + 142500\lambda_3 \quad (1)$$

The optimal solution occurs when  $\lambda_3 = 1$  and all other  $\lambda_i$ 's are set to 0, with a maximum value of 142500. The optimal solution is ( $X_1 = 3750, X_2 = 7500$ ), one of the vertices.

**Proposition 1:** The maximum (minimum) points of an LP with a bounded feasible region correspond to the maximization (minimization) of the parametric objective function  $f(\lambda)$ .

Let the terms with the largest (smallest) coefficients in  $f(\lambda)$  be denoted by  $\lambda_L$  and  $\lambda_S$  respectively. Since  $f(\lambda)$  is a (linear) convex combination of its coefficients, the optimal solution of  $C(\lambda)$  is obtained by setting  $\lambda_L$  or  $\lambda_S$  equal to 1 and all other  $\lambda_i = 0$ .

**Lemma 1:** The maximum and minimum points of an LP with a bounded feasible region correspond to  $\lambda_L = 1$  and  $\lambda_S = 1$ , respectively.

If a feasible region has a finite number of vertices, then this result suggests that the optimal solution to an LP problem can be found by enumerating all BFSs found by the Improved Algebraic Method. The optimum is associated with the BFS yielding the largest or smallest objective value, assuming the problem is of maximization or minimization type, respectively.

### 5.1. Computing Slack and Surplus

Given that the RHS of a constraint is non-negative, the slack is the leftover amount of a resource ( $\leq$ ) constraint, and surplus is the access over a requirement ( $\geq$ ) constraint. These quantities represent the absolute values of the difference between the RHS value and the LHS (Left Hand Side) evaluated at an optimal point. Having obtained an optimal solution, one can compute the slack and surplus for each constraint at optimality. Equality constraints are always binding with zero slack/surplus.

Since this numerical example is a two dimensional LP, one expects (at least) to have two binding constraints. The binding constraints at optimality are equations 1 and 2:

$$\begin{aligned} 12X_1 + 10X_2 &= 120000 \\ 10X_1 + 15X_2 &= 150000 \end{aligned}$$

With surplus and slack value of zero, respectively, i.e.,  $S_2 = 0$  and  $S_3 = 0$  which are the "Reduce Cost" of the dual problem.

## 6. CONSTRUCTION OF THE LARGEST SENSITIVITY REGIONS BASED ON OPTIMAL VERTEX NOT "BASIS"

We develop a novel approach to postoptimal-ity analysis for general LP problems, given a unique non-degenerate optimal solution that provides a simple framework for the analysis

of any single or simultaneous change of right hand side (RHS) or cost coefficients by solving the nominal LP problem with perturbed RHS terms. Assume that the LP is of size (mxn); i.e., it has n decision variables and m constraints including the non-negativity conditions (if any).

### 6.1. General Sensitivity Region for the RHS Values of Constraints: Maintaining the Validity of Current Shadow Prices

**Step 1:** Identify the n constraints that are binding at optimal solution. If there are more than n binding constraints, then the problem is degenerate. This case is discussed in Section 4.

**Step 2:** Construct the parametric RHS of the constraints, excluding the non-negativity conditions.

**Step 3:** Solve the parametric system of equations consisting of the binding constraints. This provides the parametric optimal solution.

**Step 4:** Construct the simultaneous sensitivity analysis by plugging-in the solution obtained in Step 3, in all other remaining parametric constraints, including the non-negativity conditions (if any).

### 6.2. Computation of Shadow Prices

By definition, the shadow price for a non-binding constraint is always zero. To compute the shadow price of binding constraints, one must first solve the following RHS parametric system of equations,

$$\begin{aligned} 12X_1 + 10X_2 &= 120000 \\ 10X_1 + 15X_2 &= 150000 \end{aligned}$$

By setting all inequalities in binding positions and solving the system of equations, we get the following parametric solution:

$$\begin{aligned} 12X_1 + 10X_2 &= 120000 + r_1 \\ 10X_1 + 15X_2 &= 150000 + r_2 \end{aligned}$$

Solving this parametric system of equation for  $X_1$  and  $X_2$ , we have:

$$\begin{aligned} X_1 &= 3750 + 3/16r_1 - 1/8r_2 \\ X_2 &= 7500 - 1/8r_1 + 3/20r_2 \end{aligned}$$

The solution can be verified by substitution. For the higher dimension LP, the parametric solution can be obtained by using the JavaScript:

<http://www.mirror-service.org/sites/home.ubalt.edu/ntsbarsh/Business-stat/otherapplets/ParHSSyEqu.htm>

Plugging the parametric solution into objective function, we have:

$$18X_1 + 10X_2 = 142500 + 17/8r_1 - 3/4r_2$$

The shadow prices are the coefficients of the parametric optimal function, i.e.,  $U_1 = 17/8$  and  $U_2 = -3/4$ , for the RHS of constraints 1 and 2, respectively. They are the *rate of change* in optimal value with respect to changes in the RHS of each constraint.

### 6.3. Construction of the RHS Sensitivity Region: Maintaining the Validity of

#### 6.3.1. Current Shadow Prices

Notice that the parametric objective function  $1425 + 17/8r_1 - 3/4r_2$  is valid when the parametric solution satisfies all other constraints. In the current numerical example, the other constraints are:

$$X_1 \geq 0, \text{ and } X_2 \geq 0.$$

These produce the following largest set of sensitivity region for the RHS of all the constraints simultaneously:

$$\begin{aligned} 3750 + 3/16r_1 - 1/8r_2 &\geq 0, \\ \text{and } 7500 - 1/8r_1 + 3/20r_2 &\geq 0. \end{aligned} \quad (2)$$

Notice that this is the largest sensitivity region for RHS of binding constraints that allows for simultaneous, dependent/independent changes. This convex region is non-empty since it always containing the origin ( $r_1 = 0, r_2 = 0$ ), with a vertex at ( $r_1 = -3750, r_2 = -7500$ ) where both right-hand sides of both constraints vanish.

### 6.4. Sensitivity Range for the RHS of the Binding Constraints

The inequality set (2) can be used to find the ordinary sensitivity range (one change at time) for the RHS values of the constraints. The range for the RHS of the first constraint ( $RHS_1$ ) can be obtained by setting  $R_2 = 0$  in the inequalities set (2) at binding position. This provides  $r_1 = 60000$  and  $r_1 = -20000$ . Therefore, the allowable increase and decrease in the original value 12000 for  $RHS_1$  are 60000 and 20000, respectively i.e.,  $(10000 \leq RHS_1 \leq 180000)$ .

Similarly, the range for the RHS of the third constraint ( $RHS_2$ ) can be obtained by setting  $R_1 = 0$  in binding position of the inequality set (2). This implies that  $r_2 = 30000$  and  $r_2 = -50000$ . Therefore the allowable increase and decrease in the original value 150000 for  $RHS_2$  are 30000 and 50000, respectively; i.e.,  $(150000 \leq RHS_2 \leq 100000)$ .

Notice that, in a two-dimensional problem, one can construct the above general sensitivity region in the  $(r_1, r_2)$  space provided the ordinary sensitivity analysis is available; e.g., one can obtain from LINDO software. One of the boundary lines in perturbed space of  $(r_1, r_2)$  pass through the amount of (decrease in  $r_2$ , and increase in  $r_1$ ). The other boundary line passes through the amount of (decrease in  $r_1$ , and increase in  $r_2$ ). This reverse construction of the general sensitivity region, based on the available ordinary sensitivity range, will be much more complicated for higher dimensional LP problems.

Having constructed the general sensitivity region for both the RHS, and the Cost Coefficients, all other types of sensitivity analysis, such as Ordinary Sensitivity analysis, Parametric, Tolerance analysis can be carried out

easily. Arsham (1990) give detail treatments of the most popular sensitivity analysis with numerical examples including the following.

### 6.5. Other Applications of the Largest Sensitivity Region

One easily can produce results for the other kinds of sensitivity analysis. The most popular ones are:

**Perturbation Analysis:** Simultaneous and independent changes in any parameter in either direction (over or under estimation) for each parameter that maintains the optimal basis. This provides the largest set of perturbations.

**Tolerance Analysis:** Simultaneous and independent changes expressed as the maximum allowable percentage of the parameter's value in either direction (over or under estimation) for each parameter that maintains the optimal basis. This provides a range of values for each parameter.

**Individual Symmetric Tolerance Analysis:** Simultaneous and independent percentage changes of all parameters values in both directions (increase and decrease) for each parameter that maintains the optimal basis. This provides a range of values for each parameter with the current value at its center.

**Symmetric Tolerance Analysis:** Simultaneous and independent maximum *equal* percentage changes of the parameters value in both directions (over and under estimation) for all activity that maintain the optimal basis. This provides one single range of values of uncertainty for all parameters. If plural, the apostrophe goes after the s; if singular, the apostrophe goes before the s. Consider the context.

**Parametric Sensitivity Analysis:** Simultaneous changes of dependent parameter values from their nominal values that maintain the optimal basis. This provides the maximum magnitude of change for values of dependent parameters.

In performing the above various type of sensitivity analysis, the needed computations are some elementary algebraic manipulations using the largest sensitivity regions.

### 6.6. Sensitivity Range for the Non-Binding Constraints

In this numerical example, the non-binding constraints are the non-negativity conditions that are the non-negativity conditions that are not subject to sensitivity analysis. Let us pretend "as if" there is a non-binding constraint  $X_1 + 100X_2 \geq 20000$ . To find the sensitivity range, construct the parametric form and plug in the optimal solution. We get:

$$\begin{aligned} X_1 + 100X_2 &\geq 20000 + r_3 \\ 37500000 + 75000000 &\geq 20000 + r_3 \\ r_3 &\leq 553750 \end{aligned}$$

The amount of increase is 553750 and decreasing amount is unlimited; i.e.,  $(RHS_3 \leq 753750)$ .

The following proposition formalizes the shifting of parametric non-binding constraint.

**Proposition 2:** For any given point  $X^0 = (X_1^0, X_2^0, \dots, X_n^0)$  the parameter  $r$  value for any resource/production constraint is proportional to the (usual) distance between the point  $X^0$  and the hyper-plane of the constraint.

**Proof:** Proof follows from the fact that the distance from point  $X^0 = (X_1^0, X_2^0, \dots, X_n^0)$  to any nonbinding constraint in, i.e.

$$a_1 X_1^0 + a_2 X_2^0 + \dots + a_n X_n^0 = b + r$$

is

$$\text{Absolute } [a_1 X_1^0 + a_2 X_2^0 + \dots + a_n X_n^0 - b - r] / (a_1^2 + a_2^2 + \dots + a_n^2)^{1/2}$$

That reduces to:

$$\text{Absolute } R / (a_1^2 + a_2^2 + \dots + a_n^2)^{1/2}.$$

Table 2. Input and output of LINDO software package for the numerical example

INPUT and OUTPUT			
Minimize 18X1+10X2 Subject to 12X1+10X2>120000 10X1+15X2<150000 End			
LP OPTIMUM FOUND AT STEP 2			
OBJECTIVE FUNCTION VALUE 1) 142500.0			
VARIABLE		VALUE	REDUCED COST
X1		3750.0	0.0
X2		7500.0	0.0
ROW		SLACK OR SURPLUS	DUAL PRICES
2)		0.0	-2.125
3)		0.0	0.75
NO. ITERATIONS=2			
RANGES IN WHICH <i>THE BASIS</i> IS UNCHANGED			
OBJ COEFFICIENT RANGES			
VARIABLE	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
X1	18.0	INFINITY	6.0
X2	10.0	5.0	INFINITY
RIGHTHAND SIDE RANGES			
ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
2	120000.0	60000.0	20000.0
3	150000.0	30000.0	49999.9

Therefore the parameter R value is proportional to the distance with the constant proportionality, that is  $/ (a_1^2 + a_2^2 + \dots + a_n^2)^{1/2}$ . This is independent of point  $X^o$ . In the above example,  $X^o$  is the optimal vertex. This completes the proof.

## 6.7. Software Packages Sensitivity Analysis

Sometimes one gets surprise result in implementing LINDO. For example running our

numerical example, one gets the following output (Table 2).

In this LINDO final report the shadow prices are reported to be  $U_1 = -2.125$ , and  $U_2 = 0.75$ , while the correct ones are  $U_1 = 2.125$ , and  $U_2 = -0.75$ , as we found them.

## 7. DUAL PROBLEM CONSTRUCTION

Associated with each (primal) LP problem there is a companion problem called its dual problem.



*Table 3. A one-to-one correspondence between the constraint type and the variable type exists using this classification of constraints and variables for both the primal and the dual problems*

<b>The Dual Problem Construction</b>		
<b>Objective: Max (e.g., Profit)</b>	<b>Variables types</b>	<b>Objective: Min (e.g., Cost)</b>
Constraint types: $\leq$ a Sensible constraint $=$ a Restricted constraint $\geq$ an Unusual const	$\geq 0$ a Sensible condition ...un-Restricted in sign $\leq 0$ an Unusual condition	Constraint types: $\geq$ a Sensible constraint $=$ a Restricted constraint $\leq$ an Unusual const.

The following classification of the decision variable constraints is useful and easy to remember in construction of the dual (Table 3).

### 7.1. Dual Problem Construction for General Primal Problems

- If the primal is a maximization problem, then its dual is a minimization problem (and vice versa).
- Use the variable type of one problem to find the constraint type of the other problem.
- Use the constraint type of one problem to find the variable type of the other problem.
- The RHS elements of one problem become the objective function coefficients of the other problem (and vice versa).
- The matrix coefficients of the constraints of one problem are the transpose of the matrix coefficients of the constraints for the

other problem. That is, rows of the matrix become columns and vice versa.

## 8. GENERAL SENSITIVITY REGION FOR COST COEFFICIENTS: MAINTAINING THE VALIDITY OF CURRENT OPTIMAL SOLUTION NOT “BASIS”

Knowing the unique optimal solution for the dual problem by using any LP solver, one may construct the simultaneous sensitivity analysis for all coefficients of the objective function of the primal problem as follows. Assume that the dual problem has  $n$  decision variables and  $m$  constraints, including the non-negativity conditions (if any) (see Listing 1).

Suppose we wish to construct the dual of our numerical example:

### Listing 1. Sensitivity ranges for the objective function coefficients

**Step 0:** Construct and solve the dual problem.

**Step 1:** Identify the  $n$  constraints that are binding at optimal solution. If there are more than  $n$  constraints binding, then the primal problem may have multiple solutions. This case is discussed in Section 5.

**Step 2:** Construct the parametric RHS of the constraints, excluding the non-negativity conditions.

**Step 3:** Solve the parametric system of equations consisting of the binding constraints. This provides the parametric optimal solution.

**Step 4:** Construct simultaneous sensitivity analysis by plugging in the solution obtained in Step 3, in all other parametric constraints, including the non-negativity conditions (if any).

Minimize  $C(X) = 18X_1 + 10X_2$

Minimization Problem

Subject to:  $12X_1 + 10X_2 \geq 120000$

**Sensible Constraint**  $10X_1 + 15X_2 \leq 150000$

**Unusual Constraint:**  $X_1 \geq 0$

*Sensible Condition:*  $X_2 \geq 0$

The dual problem is:

Maximize  $120000U_1 + 150000U_2$

Maximization Problem

Subject to:  $2U_1 + 10U_2 \leq 18$

*Sensible Constraint:*  $10U_1 + 15U_2 \leq 10$

*Sensible Constraint:*  $U_1 \geq 0$

**Sensible Condition**

$U_2 \leq 0$  **Unusual Condition**

The optimal solution to the dual problem is the shadow prices to the original (primal) mode, and vice versa. The shadow prices of the primal, i.e., ( $U_1 = 17/8$ ,  $U_2 = -3/4$ ), as calculated earlier (or one can solve it afresh without referring to previous section). The optimal value for the dual  $120000(17/8) + 150000(-3/4) = 142500$  that is equal to the optimal value of the primal problem, as always expected. This property that the primal and optimal values are equal is known as equilibrium property. The slacks of the first two constraints are  $12(17/8) + 10(-3/4) - 18 = 0$ ,  $10(17/8) + 15(-3/4) - 10 = 0$ , which are the "Reduce Cost" of the primal problem,

To find the ranges for the objective function coefficients, one may use the RHS parametric version of the dual problem;

The parametric presentations of the RHS of the binding constraints are as follows:

$$\begin{aligned} 12U_1 + 10U_2 &= 18 + c_1 \\ 10U_1 + 15U_2 &= 10 + c_2, \end{aligned}$$

Solving these equations for  $U_1 = 17/8 + 3/16c_1 - 1/8c_2$ , and  $U_2 = -3/4 - 1/8c_1 + 3/20c_2$  and plugging into objective function, the parametric objective function is  $14250000 + 3750c_1 + 37500c_2$  with optimal value of  $14250000$ , same as for the primal problem, as expected. Again, this parametric optimal solution is subject to satisfying the unused constraints; namely,

$$U_1 \geq 0, \text{ and } U_2 \leq 0.$$

These produce the following largest sensitivity region for the objective function coefficients simultaneously:

$$\begin{aligned} 3/16c_1 - 1/8c_2 &\geq -17/8, \text{ and } -1/8c_1 \\ + 3/20c_2 &\leq 3/4. \end{aligned} \quad (3)$$

Notice that this is the largest sensitivity region for cost coefficient, allowing for simultaneous, dependent/ independent changes. This convex region is non-empty because it contains the origin ( $c_1 = 0$ ,  $c_2 = 0$ ), with a vertex at ( $c_1 = -17/8$ ,  $c_2 = 3/4$ ) where both cost coefficients vanish.

The ordinary sensitivity range (one change at a time) of the coefficient of first decision variable  $X_1$ , currently at 18, can be found by setting  $c_2 = 0$  in the inequality set (3) at binding position, yielding  $c_1 = -6$ . Therefore is the allowable decrease for the coefficient of  $X_1$ , is 6 with unbounded allowable increase.

Similarly, the sensitivity range of the coefficient of first decision variable  $X_2$ , currently at 10, can be found by setting  $c_1 = 0$  in the inequality set (3) yielding  $c_2 = 5$ . Therefore, the allowable increase for the coefficient of  $X_1$  is 5 with unbounded allowable decrease.

## 8.1. Finding the Cost Sensitivity Range by the Graphical Method Could be Misleading

It is a commonly held belief that one can compute the cost sensitivity range by bracketing the

slope of the (iso-value) objective function by the slopes of the two lines resulting from the binding constraints. This graphical slope-based method to compute the sensitivity ranges is described in popular textbooks, such as Anderson et al. (2007), Lawrence and Pasternack (2002), and Taylor (2010). The following is a counterexample.

Counterexample:

Maximize  $5X_1 + 3X_2$   
 Subject to:  $X_1 + X_2 \leq 2$ ,  $X_1 - X_2 \leq 0$ ,  $X_1 \geq 0$ ,  
 $X_2 \geq 0$

Gives  $-1 \leq -C_1 / C_2 \leq 1$  which is wrong. The correct ranges are  $C_1 \geq 3$ ,  $-5 \leq C_2 \leq 5$ , respectively.

## 9. MAINTAINING A DEGENERATE OPTIMAL VERTEX (PRIMAL DEGENERATE)

Almost all research work in LP literature starts with an assumption of a non-degenerate primal/dual LP. Degeneracy is the case of having multiple vertices at the same point.

*The necessary condition for the existence of LP degeneracy:* If the numbers of binding constraints at optimal point are more than  $n$ , the dimension of the problem, then the solution to an LP *might be* degenerate.

*Resolution:* Double-check the coefficients of all constraints, including the RHS values. There could have been rounding error.

### 9.1. False Degenerate Optimal Solution

Consider the following  $n=2$  dimensional LP problem:

Maximize  $X_2$   
 Subject to:  
 $X_1 + X_2 = 5$   
 $-X_1 + X_2 \leq 1$

$X_1 \geq 0$ ,  $X_2 \geq 0$

This problem has a unique, non-degenerate optimal solution at  $(X_1 = 2, X_2 = 3)$ . However, if one rewrites the equality constraint in the form of two inequalities, the equivalent problem is:

This is an issue of spacing around the + signs.

Maximize  $X_2$

Subject to:

$X_1 + X_2 \leq 5$

$X_1 + X_2 \geq 5$

$-X_1 + X_2 \leq 1$

$X_1 \geq 0$ ,  $X_2 \geq 0$

Clearly, the optimal solution  $(2, 3)$  now makes three constraints binding. Three is more than the dimension  $n=2$  of this problem; therefore one may incorrectly conclude that this solution is a degenerate optimal solution. This simple numerical example serves the purpose of the necessary (but not a sufficient) condition for a degenerate optimal solution.

An occurrence of degeneracy can have a significant impact on the shadow prices and may cause oddities in the output provided by LP solvers. Moreover, the usual sensitivity analyses do not provide complete information in the degenerate case; that is, the information one obtains from most LP packages are a subset of the true sensitivity intervals. There are more effective approaches to cope with this problem; however, they are computationally much more involved, Lin (2010). The popular software package LINDO does state if a problem is degenerate.

#### Numerical Example:

Maximize  $C(X) = X_1 + X_2$

Subject to  $X_1 \leq 1$

$X_2 \leq 1$

$X_1 + X_2 \leq 2$

$X_1 \geq 0$

$X_2 \geq 0$

The optimal solution is  $X_1 = 1$ , and  $X_2 = 1$ , at which all three constraints are binding. We

are interested in finding out how far the RHS of binding constraints can change while maintaining the degenerate optimal solution. Because of degeneracy, there is dependency among these constraints at the binding position, that is:

$$\begin{aligned} X_1 &= 1 \\ X_2 &= 1 \\ X_1 + X_2 &= 2 \end{aligned}$$

In order to maintain the degenerate vertex while changing the RHS values, the RHS proportionately must be changed to the coefficients of the decision variables. The resulting parametric RHS system of equations is as follows:

$$\begin{aligned} X_1 &= 1 + 1r_1 + 0r_2 \\ X_2 &= 1 + 0r_1 + 1r_2 \\ X_1 + X_2 &= 2 + 1r_1 + 1r_2 \end{aligned}$$

Since there are two decision variables, any two of the parametric equations can be used to find the parametric degenerate optimal solution. For example, the first two equations provide ( $X_1 = 1 + r_1$ ,  $X_2 = 1 + r_2$ ), which clearly satisfies the third one. As mentioned earlier, for larger problems one may use JavaScript software:

<http://home.ubalt.edu/ntsbarsh/Business-stat/otherapplets/ParHSSyEqu.htm>

The perturbed optimal value is  $C(X) = X_1 + X_2 = 2 + r_1 + r_2$ . In order for this vertex to remain optimal, it must satisfy all other constraints that have not been used. For this numerical example, the non-negativity conditions are the remaining constraints; therefore, we obtain the following conditions for  $r_1$  and  $r_2$ :

$$\{r_1 \text{ and } r_2 \mid r_1 \geq -1, r_2 \geq -1\}.$$

Notice that while the changes satisfy these conditions the degenerate solution remain degenerate, for any change outside of this set ensures the manager that the solution is not any longer degenerate therefore the sensitivity and shadow prices are valid.

## 10. MAINTAINING THE MULTIPLE OPTIMAL VERTICES (DUAL DEGENERATE)

*The necessary condition for the existence of LP multiple solutions:* If in the “software final report” the total number of zeros in the Reduced Cost, together with number of zeros in the Shadow Price columns, exceeds the number of constraints, then you might have multiple solutions. Sensitivity analysis is not applicable. That is, the sensitivity analysis based on one optimal solution may not be valid for the others, Lin (2010). The popular software package LINDO does state if a problem has multiple solutions.

*Resolution:* Check the coefficients in the objective function and the constraint. There could have been rounding error.

### Numerical Example:

Consider the following LP:

$$\begin{aligned} \text{Maximize } C(X) &= 6X_1 + 4X_2 \\ \text{Subject to } X_1 + 2X_2 &\leq 16 \\ 3X_1 + 2X_2 &\leq 24 \\ X_1 &\geq 0 \\ X_2 &\geq 0 \end{aligned}$$

This LP has two optimal vertices; namely, ( $X_1 = 8$ ,  $X_2 = 0$ ) and ( $X_1 = 4$ ,  $X_2 = 6$ ). Notice that the existence of two solutions means we have innumerable optimal solutions. For example, for the above problem, for all  $0 \leq a \leq 1$ , the following solutions are also optimal:

$$\begin{aligned} X_1 &= 8a + 4(1 - a) = 4 + 4a, \\ X_2 &= 0a + 6(1 - a) = 6 - a. \end{aligned}$$

We apply the modified Dual Sensitivity Algorithm in this case. The Dual LP is:

$$\begin{aligned} \text{Minimize } 16U_1 + 24U_2 \\ \text{Subject to } U_1 + 3U_2 &\geq 6 \\ 2U_1 + 2U_2 &\geq 4 \\ U_1 &\geq 0 \\ U_2 &\geq 0 \end{aligned}$$

This LP has a degenerate optimal solution ( $U_1 = 0, U_1 = 2$ ). In order to maintain the degenerate vertex while changing the RHS, the RHS values must be changed in proportion to the coefficients of the decision variables. The parametric RHS system of equations is:

$$\begin{aligned}U_1 + 3U_2 &= 6 + 1c_1 + 3c_2 \\2U_1 + 2U_2 &= 4 + 2c_1 + 2c_2 \\U_1 &= 0 + 1c_1\end{aligned}$$

As before, any two of the parametric equations can be used to find the parametric degenerate optimal solution. For example, the first two equations provide  $U_1 = c_1$  and  $U_2 = 2 + c_2$ , which clearly satisfies the third equation. For this vertex to remain optimal, it must satisfy all other constraints that have not been used. For this example, the non-negativity condition,  $U_2 \geq 0$ , is the only remaining constraint. Therefore, we obtain the following conditions for  $c_1$ , and  $c_2$ :

$$\{c_1, c_2 \mid c_2 \geq -2\}.$$

Moreover, the perturbed cost coefficients  $(6 + c_1)X_1 + (4 + c_2)X_2$  must be proportional to its parallel constraint  $3X_1 + 2X_2 \leq 24$ , i.e.,  $(6 + c_1)/3 = (4 + c_2)/2$ . This simplifies to  $2c_1 = 3c_2$ . Putting together all these conditions, we obtain the largest sensitivity set as follows:

$$\{c_1, c_2 \mid c_2 \geq -2, 2c_1 = 3c_2\}.$$

Notice that while the changes satisfy these conditions the problem has multiple solutions, for any change outside of this set ensures the manager that the solution is not any longer has multiple solutions therefore the sensitivity and shadow prices are valid.

## 11. THE MORE-FOR-LESS OR LESS-FOR-MORE SITUATIONS

A curious property called the more-for-less (MFL), or less-for-more (LFM) phenomenon is associated with some linear programs (LP). The existing literature has demonstrated the practicality and value of identifying cases where the paradoxical situation exists, Arsham

(1997b). For example, consider the following production LP.

### Numerical Example:

$$\begin{aligned}\text{Maximize } C(X) &= X_1 + 3X_2 + 2X_3 \\ \text{Subject to } X_1 + 2X_2 + X_3 &= 4 \\ 3X_1 + 2X_3 &= 9 \\ X_1 \geq 0, X_2 \geq 0, X_3 &\geq 0\end{aligned}$$

The optimal solution for this LP occurs at vertex  $X_1 = 1, X_2 = 0, X_3 = 3$ , with the maximum value of objective function as \$7.

Note that if the second available labor resource is increased to 12, the optimal value changes to \$4; i.e., a decrease in profit while working more hours. This situation arises frequently in LP models and is known as the "more-for-less" paradox where further analysis could bring significant reduction in costs. We apply the primal sensitivity algorithm to solve the RHS perturbed equation constraints, with  $X_2 = 0$ ,

$$\begin{aligned}X_1 + X_3 &= 4 + r_1 \\ 3X_1 + 2X_3 &= 9 + r_2\end{aligned}$$

The parametric solution is  $X_1 = 1 - 2r_1 + r_2$ ,  $X_2 = 0$ , and  $X_3 = 3 + 3r_1 - r_2$ , with the optimal parametric solution as  $f(X) = 7 + 4r_1 - r_2$ . Plugging the parametric solution into other non-binding constraints, we obtain

$$\begin{aligned}X_1 = 1 - 2r_1 + r_2 &\geq 0 \\ X_3 = 3 + 3r_1 - r_2 &\geq 0\end{aligned}$$

Looking at the parametric optimal function, notice that the shadow price of the second constraint, coefficient of  $r_2$ , is negative. To find out the best number of hours, one must work to maximize the profit function  $7 + 4r_1 - r_2$ , by setting  $r_1 = 0$  and finding the largest negative value for  $r_2$ . Therefore, the constraints reduce to:

$$\begin{aligned}X_1 = 1 + r_2 &\geq 0, \\ X_3 = 3 - r_2 &\geq 0.\end{aligned}$$

The largest negative value is  $r_2 = -1$ . This gives the optimal solution of  $(X_1 = 0, X_2 = 0,$



$X_3 = 4$ ) with the optimal value of  $C(X) = 8$ . Therefore, the optimal strategy is to work 8 hours instead of 9 hours.

**Proposition 3:** A necessary and sufficient condition for the existence of a less-for-more (more-for-less) solution to a maximization (minimization) problem is the existence of an equality constraint with a negative shadow price.

**Theorem 1:** We are in LFM (nothing) [MFL (nothing)] situation iff at least one of  $r_i$ 's in the above parametric LP formulation is negative [positive]

## 12. CONCLUSION

A business environment is dynamic. A problem solution is valid in a limited time window only and is subject to revision in the next time window. Generally, a constraint set is less subject to change as compared to the objective function. For example, in production and transportation problems, the capacity constraints may remain rather stable over a period of time. On the other hand, profit coefficients of the objective function are inversely related to the price, which may fluctuate, being determined by the market conditions and competition. Therefore, sensitivity analysis is a vital component of LP. The parametric representation introduced in this paper, quickly provides all dual prices to carry out analysis for desirability of obtaining additional resources. The parametric representation also allows one to study versatility of the coefficients of the objective functions.

We provide a comprehensive managerial coverage of linear programming. The collections of presented tools are easy to understand, easy to implement, and provide useful information to the manager. Therefore they could prove valuable to involve the manager throughout the decision-making process to understand therefore to implement.

It is the algebraic approach to solve a system of inequalities that provides the bridge for the

manager between the graphical method, the simplex method that is an efficient computer implementation version of algebraic method, and almost all LP software the use it.

All the LP tools are covered within the manager's decision variable space; no additional variables, such as slack/surplus/artificial variables, or constraint are added. Therefore, the manager does not require formulation of an auxiliary model such as "standard form." The proposed approach provides all useful information for the manager such as shadow prices and expands the sensitivity analysis scope by proving sensitivity regions rather than ranges that LP software provides.

Under the proposed methods, the elaborate fundamental LP theorems fall out as by-products naturally. The proposed tools fill the gap between the graphical method, and the simplex method, and software packages when teaching linear programming to the future modelers/managers.

## ACKNOWLEDGMENTS

I am most appreciative to the reviewers' for their useful comments and suggestions. The National Science Foundation Grant CCR-9505732 supports this work.

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