OVERVIEW LINEAR PROGRAMMING AND ITS APPLICATION

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Abstract. This report provide an overview on the international research literature on linear programming as applied to various problems in mathematics. This report summarizes recents developments as well as applications in recent years.

1. Introduction

A mathematical method for finding the optimal answer to a problem is the optimization strategy. It is a branch of science created to offer numerical tools to support decision-making processes. The use of optimization techniques is crucial in both tactical and strategic commercial decision-making. Managers use optimization techniques to choose the most cost-effective financial arrangements, the best times to start and complete projects, the projects to select to minimize the total net present cost of capital, the product mix that will maximize the total profit, and management-based portfolio section problems, among other things. The most popular optimization technique, linear programming, can be used to solve real-world issues where the objective function and constraints are represented as linear functions of the choice variables. It can be characterized as a mathematical method for figuring out how to use a company's limited resources most effectively to produce the best results. Resources can include people, things, machines, land, etc. When presented as an LP model, life problems contain more than two variables, necessitating a more effective approach in order to arrive at the best answer. For optimum results in linear programming, the simplex method is most frequently used in mathematics optimization. The simplex approach was published in 1947 by air force group member George B. Dantzig. Following that, many industries began utilizing it, and it was a crucial step in putting LP into more widespread use.

In actual life, there are numerous instances where it is applied.For instance, someone might seek to increase product profit or reduce labor costs in a business. Or, an airline corporation may wish to reduce the cost of fuel or the travel time of a plane. Or, in a hospital, nurse allocation is done based on LPP (A/c to patient care services). Simply put, to achieve the best results, amplify or decrease the difficulties. It is well-known techniques that can be used in many different fields. It is well-know tecniques that can be used in many different fields. Nowadays, the simplex approach is an effective method, which is why it is more often used in various sectors to achieve a workable outcome. This paper contributes to the study of various application techniques. Some significant applications of optimization techniques are selected from previous literatures and discussed.

2. Literature review of primal dual linear programming

Semi-infinite linear optimization (LSIO for short) solves linear optimization problems where the size of the decision space or the number of constraints (but not both) is infinite. A linear optimization problem is said to be ordinary (respectively infinite) when the size of the decision space and the number of constraints are both finite (respectively infinite). The first three known contributions to LSIO are credited to Dantzig, who developed the simplex approach for the dual LSIO problem in 1939, see Remez [21], and Haar [16]. In fact, Dantzig only wrote about his discoveries on LSIO within his professional memories, which were published many years later. These three early contributions were largely ignored until the 1960s due to either the low diffusion, within the mathematical community, of the journals where Haar and Remez published their discoveries, and the languages used (German and French, respectively) (see Dantzig [9]). More in detail, Haar's paper (1924) was focussed on the extension of the homogeneous Farkas lemma for linear systems from $\mathbb{R}^n$ to an Euclidean space equipped with a scalar product $\langle \cdot, \cdot \rangle$.Due to the finite number of linear inequalities they contain and the infinite dimensionality of the variable space, these systems are semi-infinite. For a class of LSIO problems originating in polynomial approximation, Remez's paper from 1934 offered an exchange numerical method. Last but not least, Dantzig reformulated a Neyman-Pearson-type statistical inference problem (posed by the same J. Neyman in a doctoral course that Dantzig attended) as a linear optimization problem with finitely many constraints and infinitely many variables; Dantzig observed that the feasible set of this LSIO problem was the convex hull of its extreme points and conceptualized a geometry of columns allowing to jump from a given extreme point to a better djacent one, which is a clear antecedent of the celebrated simplex method for linear optimization problems he proposed in 1947. The LSIO problems with finitely many variables were referred to be primal in Charnes et al.[8]. These problems can be expressed as

$$\begin{aligned} Primal: \quad & \inf_{x \in \mathbb{R}^n} \quad \langle c, x \rangle \\ & \text{subject to} \quad \langle a_t, x \rangle \geq b_t, \quad \forall t \in T, \end{aligned} \tag{1}$$

where the $\langle \cdot, \cdot \rangle$ denotes the dot product in $\mathbb{R}^n$. Here, c is the cost function, a_t are the coefficient vectors indexed by t, and b_t are the right-hand side values. The set T is possibly uncountably infinite, with no assumed topological structure. It is helpful to think of the coefficients and right-hand sides as functions, namely, $a : T \to \mathbb{R}^n$ with $a : t \mapsto a_t$ and $b : T \to \mathbb{R}$ with $b : t \mapsto b_t$. The first theoretical results on LSIO dealt with optimality conditions and duality, as in any optimization field, and demonstrated that LSIO is more similar to ordinary convex optimization than to LP because the existence of an optimal solution is not necessarily implied by the finiteness of the optimal value, and a positive duality gap can occur for the so-called Haar's dual problem of (1).

$$\begin{aligned} Dual: \quad & \sup_{x \in \mathbb{R}^T_+} \sum_{t \in T} \lambda_t b_t, \\ & \text{subject to} \quad \sum_{t \in T} \lambda_t a_t = c, \quad \forall t \in T, \end{aligned} \tag{2}$$

where $\mathbb{R}^T_+$ is the positive cone in the linear space of generalized finite sequences $\mathbb{R}^T$. Although there are many intriguing results, a satisfactory framework for infinite dimensional programming has not yet been fully developed. As it has substantial applications to continuous transportation, piecewise continuous assignments, time-continuous network flows, space-continuous flow optimization, optimal structure design, and other areas, this is a crucial topic of research; R. Bellman [6] was the first to think about this kind of issue back in 1957. His problem involved continuous functions of time and was connected to a linear optimum control model utilized in manufacturing systems. Some key findings in the theory of infinite dimensional linear programming were made by R.J. Duffin [12] in 1956. D. Gale, L. Hurwicz, K.O. Kortanek, K.S. Kretschmer, J.M. Borwein, A. Shapiro, C. Zalinescu, and many more authors have added to this hypothesis further.

Some details regarding current developments in continuous linear programming and infinite dimensional linear programming in general may be found in [1], [3], [2] [4],[5] [25], [13], [14], [26], [15] and and the references therein.

3. Some applications

In all industries, including agriculture, linear programming is applied. In order to lower the risk and uncertainty associated with unsealing the products, farms in agriculture can diversify their farming structures greatly. In order to achieve the best results in terms of maximal production per hectare of land, the major goal of this research was to identify the ideal structure of crops, taking into account the revenue and expenditure of crops per hectare. Using some pertinent resources, such as land resources, technical facilities, etc., Montazemi and Wright (1982) implemented the mathematical programming approach in agriculture. During this economic activity, the research question raised was whether after applying the econometric model the returns of the economic activity is higher or not. By altering crop structure, cultivation profit increased significantly, and farm profit was maximized. According to the paper assessment, the data showed that profit increased by 143% while costs decreased by 81%.

4. conclusion

Numerous industries, including manufacturing, transportation, the service industry, the healthcare industry, and even agriculture, have implemented optimization strategies. Though the linear programming problem first appeared in the military, its widespread use in resource allocation, portfolio management, money allocation, product mix, advertising mix, and media mix have made it one of the most widely used tools for making decisions. We tried to discuss a few situations from the literature in our study. The aforementioned study has gathered several examples of the implementation of optimization approach in business from various regions of the world, regardless of their economic standing, whether Nigeria, India, or the United States. It was found that an overwhelming majority of problems in the actual world are solved using optimization techniques. Small family businesses and huge corporations alike should embrace optimization techniques to improve decision-making, firm performance, and efficiency and effectiveness. Better decisions will ultimately lead to profit maximization by using the best resources.

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References

[1] Aliprantis, C., and Border, K. (2005). *Infinite dimensional analysis: A Hitchhiker's guide (3rd ed.).* Berlin: Springer.

[2] Anderson, E. J., and Lewis, A. S. (1989). An extension of the simplex algorithm for semi-infinite linear programming. Mathematical Programming, 44A, 247–269.

[3] E. J. Anderson and P. Nash. *Linear programming in infinite-dimensional spaces.* Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons, Ltd., Chichester, 1987.

[4] Basu, A., Martin, K., and Ryan, C. T. (2014). On the sufficiency of finite support duals in semi-infinite linear programming. Operations Research Letters, 42, 16–20.

[5] Basu, A., Martin, K., and Ryan, C. T. (2017). Strong duality and sensitivity analysis in semi-infinite linear programming. Mathematical Programming, 161A, 451–485.

[6] R. Bellman Dynamic Programming. Princeton University Press, Princeton, NJ (1957).

[7] J. F. Bonnans and A. Shapiro. *Perturbation analysis of optimization problems.* Springer Series in Operations Research. Springer-Verlag, New York, 2000.

[8] Charnes, A., Cooper, W. W., and Kortanek, K. O. (1963). Duality in semi-infinite programs and some works of Haar and Carathéodory. Management Science, 9, 209–228.

[9] Dantzig, G. B. (1991). *Linear programming.* In J. K. Lenstra, et al. (Eds.), History of mathematical programming: A collection of personal reminiscences (pp. 19–31). Amsterdam: North-Holland.

[10] J. Dieudonné. Sur la séparation des ensembles convexes. *Math. Ann.*, 163:1–3, 1966.

[11] R. J. Duffin and L. A. Karlovitz. An infinite linear program with a duality gap. *Management Sci.*, 12:122–134, 1965.

[12] R.J. Duffin Infinite programs H.W. Kuhn, A.W. Tucker (Eds.), Linear Inequalities and Related Systems, Princeton University Press, Princeton, NJ (1956).

[13] Goberna, M. A., and López, M. A. (1998b). *Linear semi-infinite optimization.* Chichester: Wiley.

[14] Goberna, M. A., and López, M. A. (2018). Recent contributions to linear semi-infinite optimization: an update Annals of Operations Research volume 271, pages 237–278 (2018).

[15] N. E. Gretsky, J. M. Ostroy, and W. R. Zame. Subdifferentiability and the duality gap. *Positivity*, 6(3):261–274, 2002. Special issue of the mathematical economics.

[16] Haar, A. (1924). Über lineare ungleichungen (in German). Acta Scientiarum Mathematicarum, 2, 1–14.

[17] R. B. Holmes. *Geometric functional analysis and its applications.* Graduate Texts in Mathematics, No. 24. Springer-Verlag, New York-Heidelberg, 1975.

[18] A. A. Khan, C. Tammer, and C. Zălinescu. *Set-valued optimization.* Vector Optimization. Springer, Heidelberg, 2015.

[19] K. S. Kretschmer. Programmes in paired spaces. *Canadian J. Math.*, 13:221–238, 1961.

[20] M. Pintér. Algebraic duality theorems for infinite LP problems. *Linear Algebra Appl.*, 434(3):688–693, 2011.

[21] Remez, E. (1934). Sur la détermination des polynômes d'approximation de degré donné (in French). Commun Soc Math Kharkoff and Inst Sci Math et Mecan, 10, 41–63.

[22] R. T. Rockafellar. *Conjugate duality and optimization*. Conference Board of the Mathematical Sciences Regional Conference Series in Applied Mathematics, No. 16. Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1974.

[23] A. Shapiro. On duality theory of conic linear problems. In *Semi-infinite programming (Alicante, 1999)*, volume 57 of *Nonconvex Optim. Appl.*, pages 135–165. Kluwer Acad. Publ., Dordrecht, 2001.

[24] R. M. Van Slyke and R. J.-B. Wets. A duality theory for abstract mathematical programs with applications to optimal control theory. *J. Math. Anal. Appl.*, 22:679–706, 1968.

[25] N. T. Vinh, D. S. Kim, N. N. Tam, and N. D. Yen. Duality gap function in infinite dimensional linear programming. *J. Math. Anal. Appl.*, 437(1):1–15, 2016.

[26] H.-C. Wu Continuous-time linear programming problems revisited: a perturbation approach Optimization, 62 (2013), pp. 33-70

[27] C. Zălinescu. A generalization of the Farkas lemma and applications to convex programming. *J. Math. Anal. Appl.*, 66(3):651–678, 1978.

[28] C. Zălinescu. *Convex analysis in general vector spaces*. World Scientific Publishing Co. Inc., River Edge, NJ, 2002.

[29] C. Zălinescu. On the lower semicontinuity and subdifferentiability of the value function for conic linear programming problems. *arXiv:2205.09561* (2022).

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