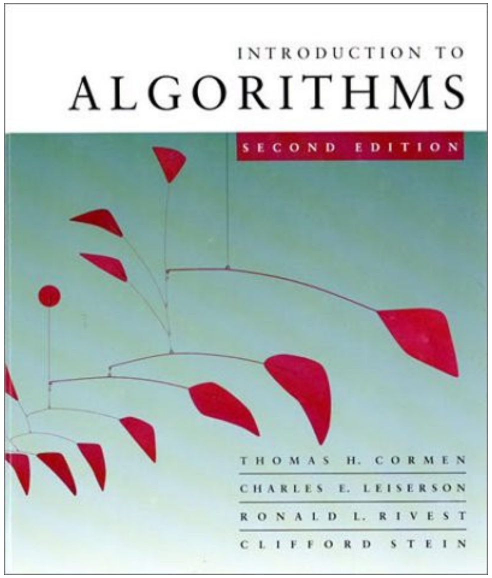
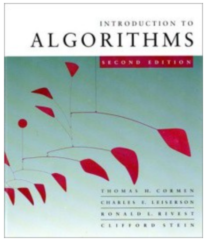
Introduction to Algorithms

6.046J/18.401J/SMA5503



Lecture 5

Based on slides by Prof. Charles E. Leiserson



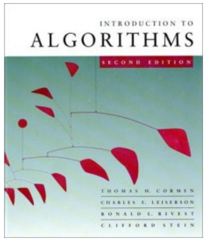
A weakness of hashing

Problem: For any hash function h, a set of keys exists that can cause the average access time of a hash table to skyrocket.

- An adversary can pick all keys from $\{k \in U : h(k) = i\}$ for some slot i.

IDEA: Choose the hash function at random, independently of the keys.

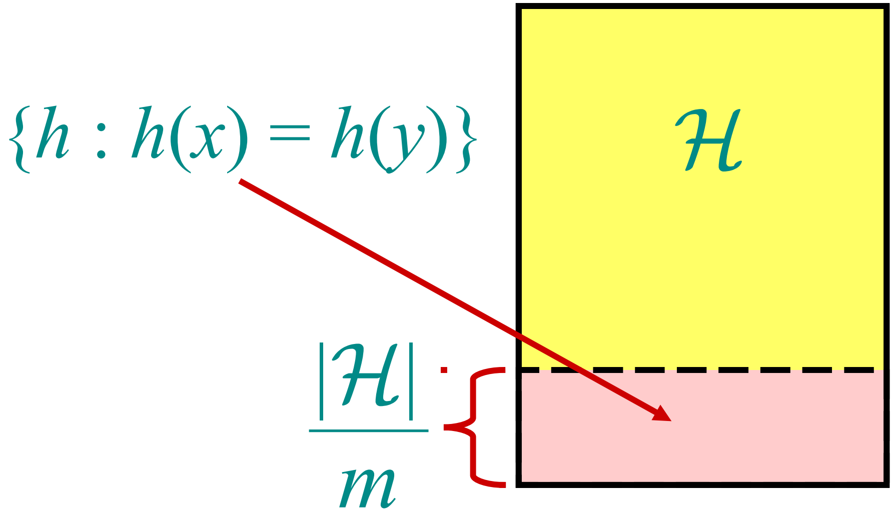
- Even if an adversary can see your code, he or she cannot find a bad set of keys, since he or she doesn't know exactly which hash function will be chosen.

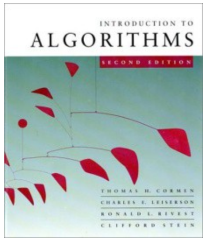


Universal hashing

Definition. Let U be a universe of keys, and let $\mathcal{H}$ be a finite collection of hash functions, each mapping U to $\{0, 1, \ldots, m-1\}$. We say $\mathcal{H}$ is ***universal*** if for all $x, y \in U$, where $x \neq y$, we have $|\{h \in \mathcal{H} : h(x) = h(y)\}| \leq |\mathcal{H}|/m$.

That is, the chance of a collision between x and y is $1/m$ if we choose h randomly from $\mathcal{H}$.

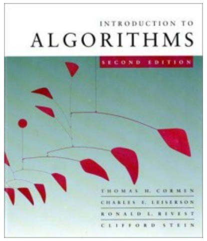




Universality is good

Theorem. Let h be a hash function chosen (uniformly) at random from a universal set $\mathcal{H}$ of hash functions. Suppose h is used to hash n arbitrary keys into the m slots of a table T. Then, for a given key x, we have

$$E[\#\text{collisions with } x] < n/m.$$

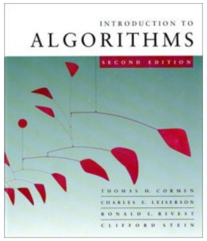


Proof of theorem

Proof. Let C_x be the random variable denoting the total number of collisions of keys in T with x, and let

$$c_{xy} = \begin{cases} 1 & \text{if } h(x) = h(y), \\ 0 & \text{otherwise.} \end{cases}$$

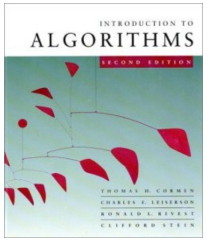
Note: $E[c_{xy}] \leq 1/m$ and $C_x = \sum_{y \in T - \{x\}} c_{xy}$.



Proof (continued)

$$E[C_x] = E\left[\sum_{y \in T-\{x\}} c_{xy}\right]$$

- Take expectation of both sides.



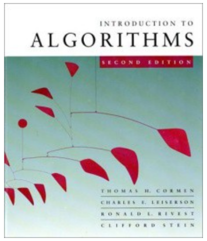
Proof (continued)

$$E[C_x] = E\left[\sum_{y \in T-\{x\}} c_{xy}\right]$$

- Take expectation of both sides.

$$= \sum_{y \in T-\{x\}} E[c_{xy}]$$

- Linearity of expectation.



Proof (continued)

$$E[C_x] = E\left[\sum_{y \in T - \{x\}} c_{xy}\right]$$

- Take expectation of both sides.

$$= \sum_{y \in T - \{x\}} E[c_{xy}]$$

- Linearity of expectation.

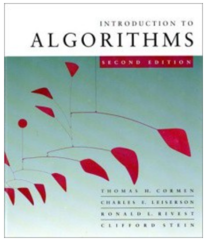
$$\leq \sum_{y \in T - \{x\}} 1/m$$

- $E[c_{xy}] \leq 1/m$.

Proof (continued)

$$\begin{aligned}
E[C_x] &= E\left[\sum_{y \in T - \{x\}} c_{xy}\right] \\
&= \sum_{y \in T - \{x\}} E[c_{xy}] \\
&= \sum_{y \in T - \{x\}} 1/m \\
&= \frac{n-1}{m} . \quad \square
\end{aligned}$$

- Take expectation of both sides.
- Linearity of expectation.
- $E[c_{xy}] \le 1/m$.
- Algebra.



Constructing a set of universal hash functions

Let m be prime. Decompose key k into $r + 1$ digits, each with value in the set $\{0, 1, \ldots, m–1\}$. That is, let $k = \langle k_0, k_1, \ldots, k_r \rangle$, where $0 \le k_i < m$.

Randomized strategy:

Pick $a = \langle a_0, a_1, \ldots, a_r \rangle$ where each a_i is chosen randomly from $\{0, 1, \ldots, m–1\}$.

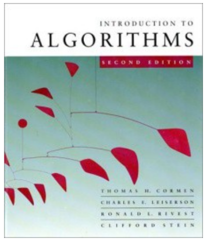
Define $h_a(k) = \sum_{i=0}^{r} a_i k_i \bmod m$.

Dot product, modulo m

How big is $\mathcal{H} = \{h_a\}$?

$$\boxed{|\mathcal{H}| = m^{r+1}.}$$

← **REMEMBER THIS!**



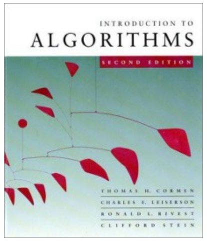
Universality of dot-product hash functions

Theorem. The set $\mathcal{H} = \{h_a\}$ is universal.

Proof. Suppose that $x = \langle x_0, x_1, \ldots, x_r \rangle$ and $y = \langle y_0, y_1, \ldots, y_r \rangle$ be distinct keys. Thus, they differ in at least one digit position, wlog position 0. For how many $h_a \in \mathcal{H}$ do x and y collide?

We must have $h_a(x) = h_a(y)$, which implies that

$$\sum_{i=0}^{r} a_i x_i \equiv \sum_{i=0}^{r} a_i y_i \pmod{m} .$$



Proof (continued)

Equivalently, we have

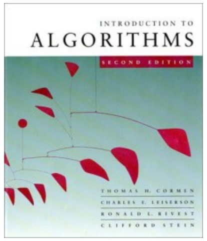
$$\sum_{i=0}^{r} a_i(x_i - y_i) \equiv 0 \pmod m$$

or

$$a_0(x_0 - y_0) + \sum_{i=1}^{r} a_i(x_i - y_i) \equiv 0 \pmod m ,$$

which implies that

$$a_0(x_0 - y_0) \equiv -\sum_{i=1}^{r} a_i(x_i - y_i) \pmod m .$$



Fact from number theory

Theorem. Let m be prime. For any $z \in \mathbb{Z}_m$ such that $z \neq 0$, there exists a unique $z^{-1} \in \mathbb{Z}_m$ such that

$$z \cdot z^{-1} \equiv 1 \quad (\text{mod } m).$$

Example: $m = 7$.

z	1	2	3	4	5	6
z^{-1}	1	4	5	2	3	6

Back to the proof

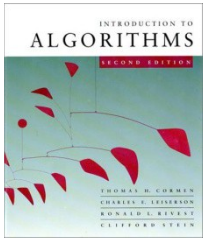
We have

$$a_0(x_0 - y_0) \equiv -\sum_{i=1}^{r} a_i(x_i - y_i) \pmod m,$$

and since $x_0 \neq y_0$, an inverse $(x_0 - y_0)^{-1}$ must exist, which implies that

$$a_0 \equiv \left(-\sum_{i=1}^{r} a_i(x_i - y_i)\right) \cdot (x_0 - y_0)^{-1} \pmod m.$$

Thus, for any choices of $a_1, a_2, \ldots, a_r$, exactly one choice of a_0 causes x and y to collide.



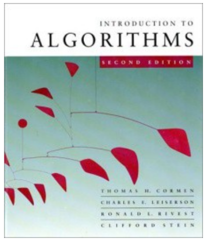
Proof (completed)

Q. How many h_a’s cause x and y to collide?

A. There are m choices for each of $a_1, a_2, \ldots, a_r$, but once these are chosen, exactly one choice for a_0 causes x and y to collide, namely

$$a_0 = \left(\left(-\sum_{i=1}^{r} a_i (x_i - y_i) \right) \cdot (x_0 - y_0)^{-1} \right) \bmod m .$$

Thus, the number of h_a’s that cause x and y to collide is $m^r \cdot 1 = m^r = |\mathcal{H}|/m$. ∎



Another universal class of hash functions

Let p be prime such that all keys lie in $\{0,1\ldots, p\text{-}1\}$ and m *be arbitrary*.

REMEMBER THIS!

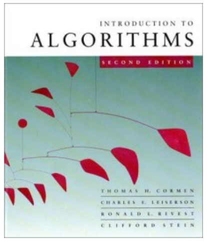
Randomized strategy:

Pick $a \in \{1,\ldots, p\text{-}1\}$, $b \in \{0,1,\ldots, p\text{-}1\}$ randomly.

Define $h_{a,b}(k) = ((ak + b) \bmod p) \bmod m$

How big is $\mathcal{H} = \{h_{a,b}\}$? $|\mathcal{H}| = p(p-1)$.

Theorem: the number of $h_{a,b}$'s that cause $x \neq y$ to collide is at most $p(p-1)/m = |\mathcal{H}|/m$.

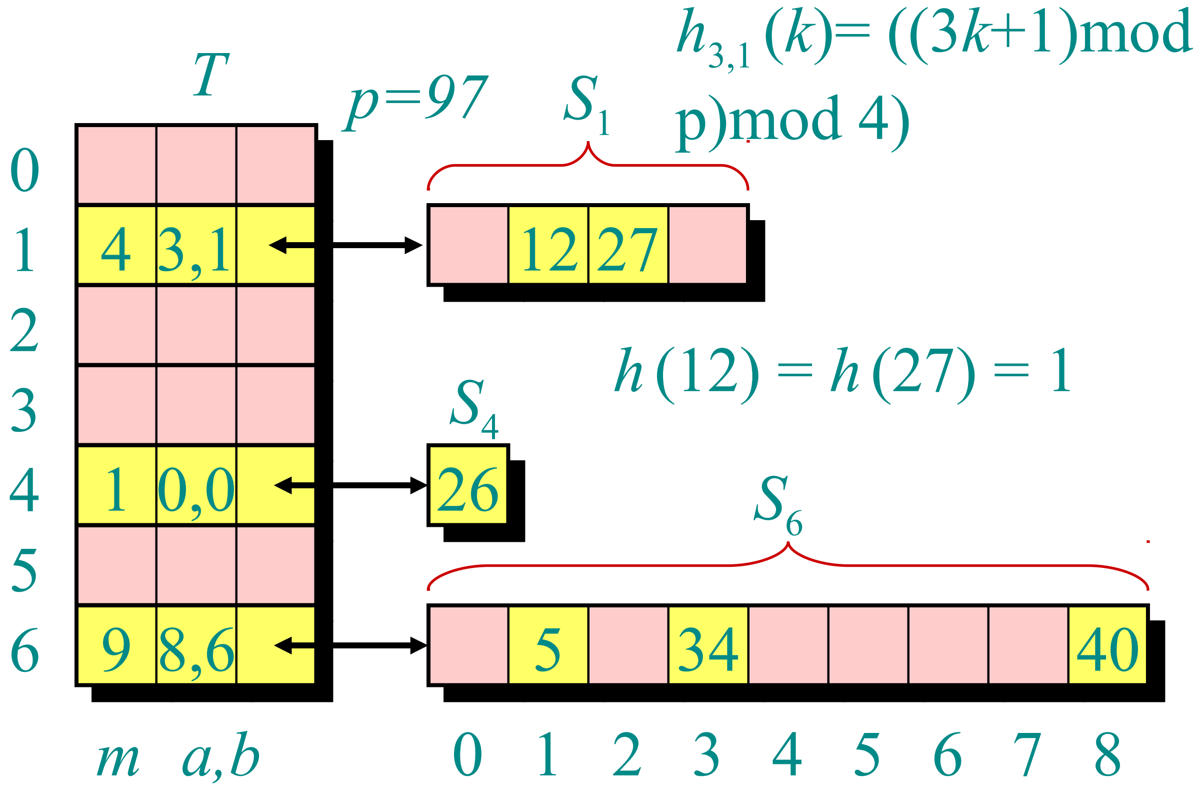


Perfect hashing

Given a set of n keys, construct a static hash table of size $m = O(n)$ such that SEARCH takes $\Theta(1)$ time in the *worst case*.

IDEA: Two-level scheme with universal hashing at both levels.

No collisions at level 2!



$h_{3,1}(k)$= ((3k+1)mod p)mod 4)

T

p=97

	m	a,b
0		
1	4	3,1
2		
3		
4	1	0,0
5		
6	9	8,6

S_1

0	1	2	3
	12	27	

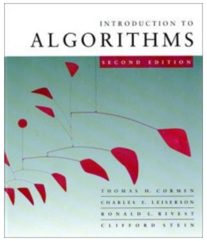
$h(12) = h(27) = 1$

S_4

0
26

S_6

0	1	2	3	4	5	6	7	8
	5		34					40

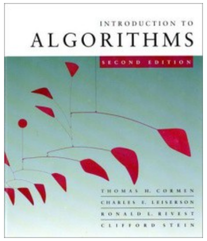


Collisions at level 2

Theorem. Let $\mathcal{H}$ be a class of universal hash functions for a table of size $m = n^2$. Then, if we use a random $h \in \mathcal{H}$ to hash n keys into the table, the expected number of collisions is at most $1/2$.

Proof. By the definition of universality, the probability that 2 given keys in the table collide under h is $1/m = 1/n^2$. Since there are $\binom{n}{2}$ pairs of keys that can possibly collide, the expected number of collisions is

$$\binom{n}{2} \cdot \frac{1}{n^2} = \frac{n(n-1)}{2} \cdot \frac{1}{n^2} < \frac{1}{2} \, . \quad \square$$



No collisions at level 2

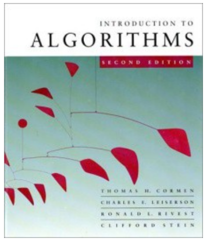
Corollary. The probability of no collisions is at least $1/2$.

Proof. ***Markov's inequality*** says that for any nonnegative random variable X, we have

$$\Pr\{X \geq t\} \leq E[X]/t.$$

Applying this inequality with $t = 1$, we find that the probability of 1 or more collisions is at most $1/2$. □

Thus, just by testing random hash functions in $\mathcal{H}$, we'll quickly find one that works.



Analysis of storage

For the level-1 hash table T, choose $m = n$, and let n_i be random variable for the number of keys that hash to slot i in T. By using n_i^2 slots for the level-2 hash table S_i, the expected total storage required for the two-level scheme is therefore

$$E\left[\sum_{i=0}^{m-1}\Theta\left(n_i^2\right)\right] = \Theta(n),$$

since the analysis is identical to the analysis from recitation of the expected running time of bucket sort. (For a probability bound, apply Markov.)