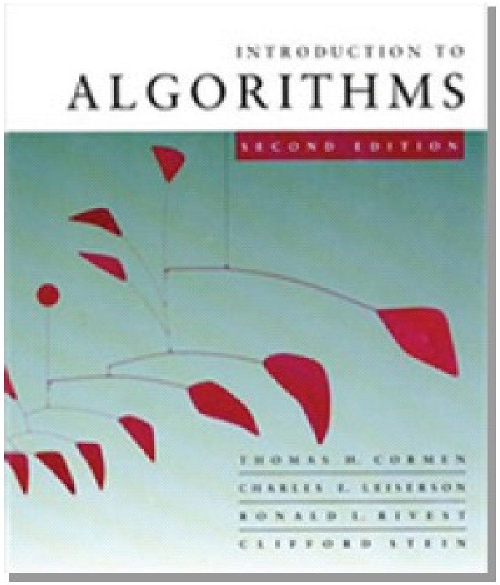
Introduction to Algorithms

6.046J/18.401J



LECTURE 3

Order Statistics

- Randomized divide and conquer
- Analysis of expected time
- Worse-case linear-time order statistics
- Analysis

Based on slides by Prof. Erik Demaine

Order statistics

Select the *i*th smallest of *n* elements (the element with ***rank*** ***i***).

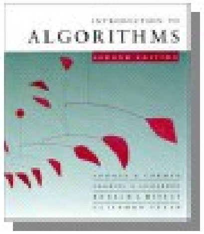
- $i = 1$: ***minimum***;
- $i = n$: ***maximum***;
- $i = \lfloor (n+1)/2 \rfloor$ or $\lceil (n+1)/2 \rceil$: ***median***.

Naive algorithm: Sort and index *i*th element.

Worst-case running time $= \Theta(n \lg n) + \Theta(1)$

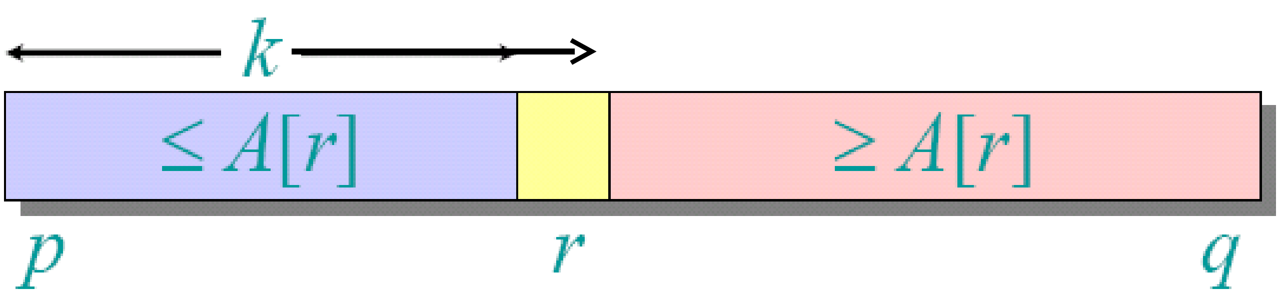
$$= \Theta(n \lg n),$$

using merge sort or heapsort (*not* quicksort).



Randomized divide-and-conquer algorithm

RAND-SELECT(A, p, q, i) $\rhd$ i th smallest of $A[p \,..\, q]$
  if $p = q$ **then return** $A[p]$
  $r \leftarrow$ RAND-PARTITION(A, p, q)
  $k \leftarrow r - p + 1$ $\rhd$ $k = \text{rank}(A[r])$
  if $i = k$ **then return** $A[r]$
  if $i < k$
    then return RAND-SELECT$(A, p, r - 1, i)$
    else return RAND-SELECT$(A, r + 1, q, i - k)$



Example

Select the $i = 7$th smallest:

6	10	13	5	8	3	2	11

$i = 7$

pivot

Partition:

2	5	3	6	8	13	10	11

$k = 4$

Select the $7 - 4 = 3$rd smallest recursively.

Intuition for analysis

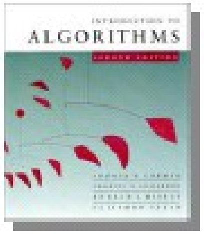
(All our analyses today assume that all elements are distinct.)

Lucky:

$$T(n) = T(9n/10) + \Theta(n) \qquad n^{\log_{10/9} 1} = n^0 = 1$$
$$= \Theta(n) \qquad \text{Case 3}$$

Unlucky:

$$T(n) = T(n-1) + \Theta(n) \qquad \text{arithmetic series}$$
$$= \Theta(n^2)$$



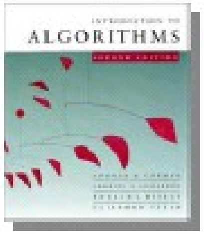
Analysis of expected time

The analysis follows that of randomized quicksort, but it's a little different.

Let $T(n)$ =the random variable for the running time of RAND-SELECTon an input of size n, assuming random numbers are independent.

For $k = 0, 1, \ldots, n-1$, define the ***indicator random variable***

$$X_k = \begin{cases} 1 & \text{if PARTITION generates a } k : n-k-1 \text{ split,} \\ 0 & \text{otherwise} \end{cases}$$



Analysis (continued)

To obtain an upper bound, assume that the ith element always falls in the larger side of the partition:

$$T(n) = \begin{cases} T(\max\{0, n-1\}) + \Theta(n) & \text{if } 0 : n-1 \text{ split,} \\ T(\max\{1, n-2\}) + \Theta(n) & \text{if } 1 : n-2 \text{ split,} \\ \quad\vdots & \\ T(\max\{n-1, 0\}) + \Theta(n) & \text{if } n-1 : 0 \text{ split,} \end{cases}$$

$$= \sum_{k=0}^{n-1} X_k \left(T(\max\{k, n-k-1\}) + \Theta(n)\right).$$

Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \big(T(\max\{k, n-k-1\}) + \Theta(n)\big)\right]$$

Take expectations of both sides.

Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k\big(T(\max\{k, n-k-1\}) + \Theta(n)\big)\right]$$

$$= \sum_{k=0}^{n-1} E\big[X_k\big(T(\max\{k, n-k-1\}) + \Theta(n)\big)\big]$$

Linearity of expectation.

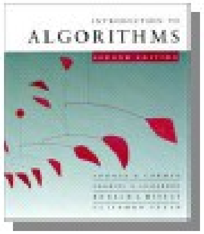
Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k\big(T(\max\{k, n-k-1\}) + \Theta(n)\big)\right]$$

$$= \sum_{k=0}^{n-1} E\big[X_k\big(T(\max\{k, n-k-1\}) + \Theta(n)\big)\big]$$

$$= \sum_{k=0}^{n-1} E[X_k] \cdot E\big[T(\max\{k, n-k-1\}) + \Theta(n)\big]$$

Independence of X_k from other random choices



Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k\bigl(T(\max\{k, n-k-1\}) + \Theta(n)\bigr)\right]$$

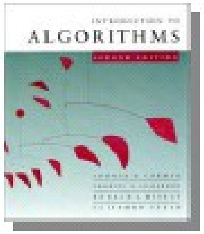
$$= \sum_{k=0}^{n-1} E\bigl[X_k\bigl(T(\max\{k, n-k-1\}) + \Theta(n)\bigr)\bigr]$$

$$= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(\max\{k, n-k-1\}) + \Theta(n)]$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} E[T(\max\{k, n-k-1\})] + \frac{1}{n}\sum_{k=0}^{n-1} \Theta(n)$$

Linearity of expectation;

$E[X_k] = 1/n$.



Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k\big(T(\max\{k, n-k-1\}) + \Theta(n)\big)\right]$$

$$= \sum_{k=0}^{n-1} E\big[X_k\big(T(\max\{k, n-k-1\}) + \Theta(n)\big)\big]$$

$$= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(\max\{k, n-k-1\}) + \Theta(n)]$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} E[T(\max\{k, n-k-1\})] + \frac{1}{n}\sum_{k=0}^{n-1} \Theta(n)$$

This part is $\Theta(n)$

Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k\left(T(\max\{k, n-k-1\}) + \Theta(n)\right)\right]$$

$$= \sum_{k=0}^{n-1} E\left[X_k\left(T(\max\{k, n-k-1\}) + \Theta(n)\right)\right]$$

$$= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(\max\{k, n-k-1\}) + \Theta(n)]$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} E[T(\max\{k, n-k-1\})] + \frac{1}{n}\sum_{k=0}^{n-1}\Theta(n)$$

$$\le \frac{2}{n}\sum_{k=\lfloor n/2 \rfloor}^{n-1} E[T(k)] + \Theta(n)$$

Upper terms appear twice.
E.g. $k = 1$ vs. $k = n - 2$
$max(1, n\text{-}2) = max(n\text{-}2, 1)$

Hairy recurrence

(But not quite as hairy as the quicksort one.)

$$E[T(n)] = \frac{2}{n} \sum_{k=\lfloor n/2 \rfloor}^{n-1} E[T(k)] + \Theta(n)$$

Prove: $E[T(n)] \le cn$ for constant $c > 0$.

- The constant c can be chosen large enough so that $E[T(n)] \le cn$ for the base cases.

Use fact: $\displaystyle\sum_{k=\lfloor n/2 \rfloor}^{n-1} k \le \frac{3}{8}n^2$ (exercise).

Substitution method

$$E[T(n)] \le \frac{2}{n} \sum_{k=\lfloor n/2 \rfloor}^{n-1} ck + \Theta(n)$$

Substitute inductive hypothesis.

Substitution method

$$E[T(n)] \le \frac{2}{n} \sum_{k=\lfloor n/2 \rfloor}^{n-1} ck + \Theta(n)$$

$$\le \frac{2c}{n}\left(\frac{3}{8}n^2\right) + \Theta(n)$$

Use
fact.

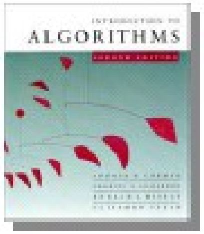
Substitution method

$$E[T(n)] \le \frac{2}{n} \sum_{k=\lfloor n/2 \rfloor}^{n-1} ck + \Theta(n)$$

$$\le \frac{2c}{n}\left(\frac{3}{8}n^2\right) + \Theta(n)$$

$$= cn - \left(\frac{cn}{4} - \Theta(n)\right)$$

Express as ***desired–residual***.



Substitution method

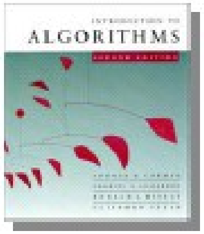
$$E[T(n)] \le \frac{2}{n} \sum_{k=\lfloor n/2 \rfloor}^{n-1} ck + \Theta(n)$$

$$\le \frac{2c}{n}\left(\frac{3}{8}n^2\right) + \Theta(n)$$

$$= cn - \left(\frac{cn}{4} - \Theta(n)\right)$$

$$\le cn\,,$$

if c is chosen large enough so that $cn/4$ dominates the $\Theta(n)$.



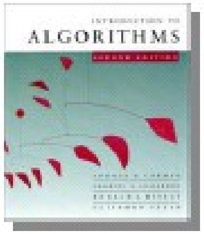
Summary of randomized order-statistic selection

- Works fast: linear expected time.
- Excellent algorithm in practice.
- But, the worst case is ***very*** bad: $\Theta(n^2)$.

Q. Is there an algorithm that runs in linear time in the worst case?

A. Yes, due to Blum, Floyd, Pratt, Rivest, and Tarjan [1973].

IDEA : Generate a good pivot recursively.



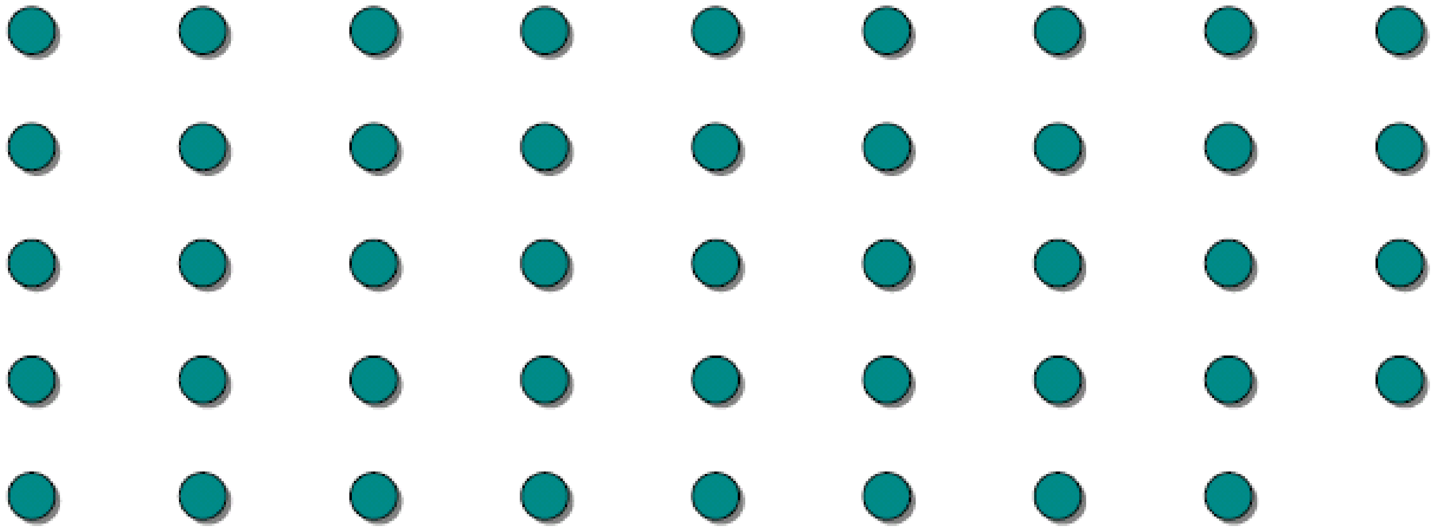
Worst-case linear-time order statistics

SELECT(i, n)

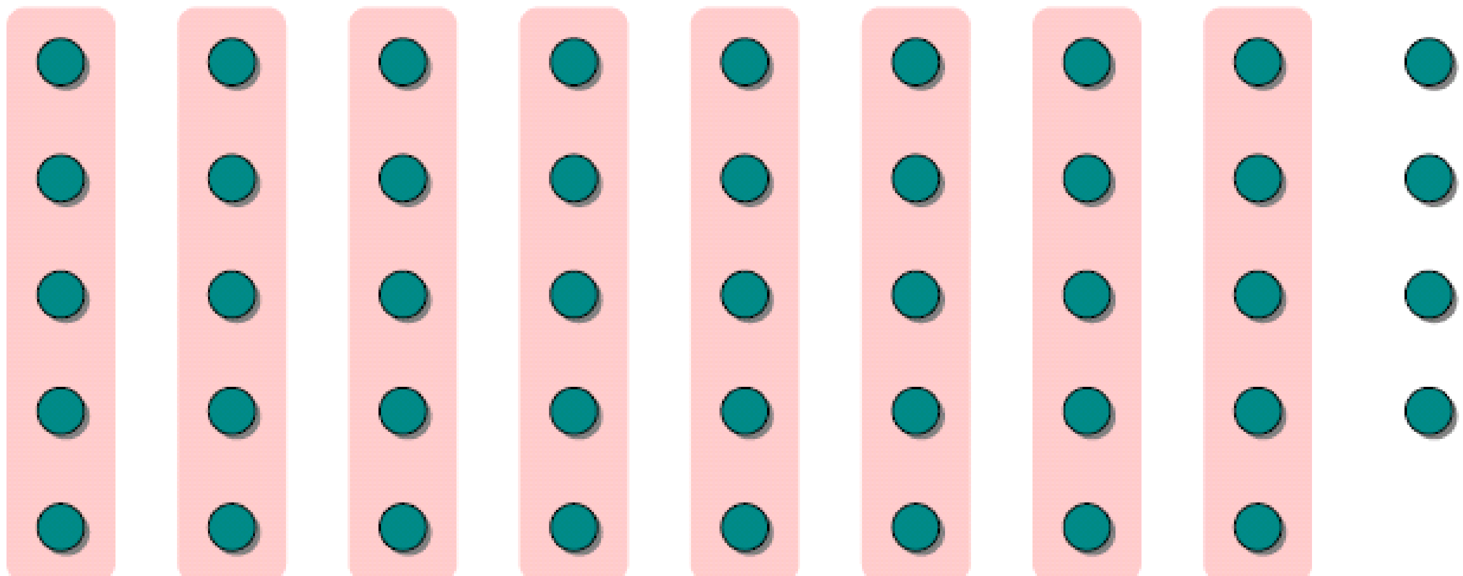
1. Divide the n elements into groups of 5. Find the median of each 5-element group by rote.
2. Recursively SELECT the median x of the $\lfloor n/5 \rfloor$ group medians to be the pivot.
3. Partition around the pivot x. Let $k = rank(x)$.
4. **If** $i = k$ **then return** x
   else if $i < k$
   then recursively SELECT the i th smallest element in the lower part
   else recursively SELECT the $(i\text{-}k)$ th smallest element in the upper part

Same as RAND-SELECT

Choosing the pivot

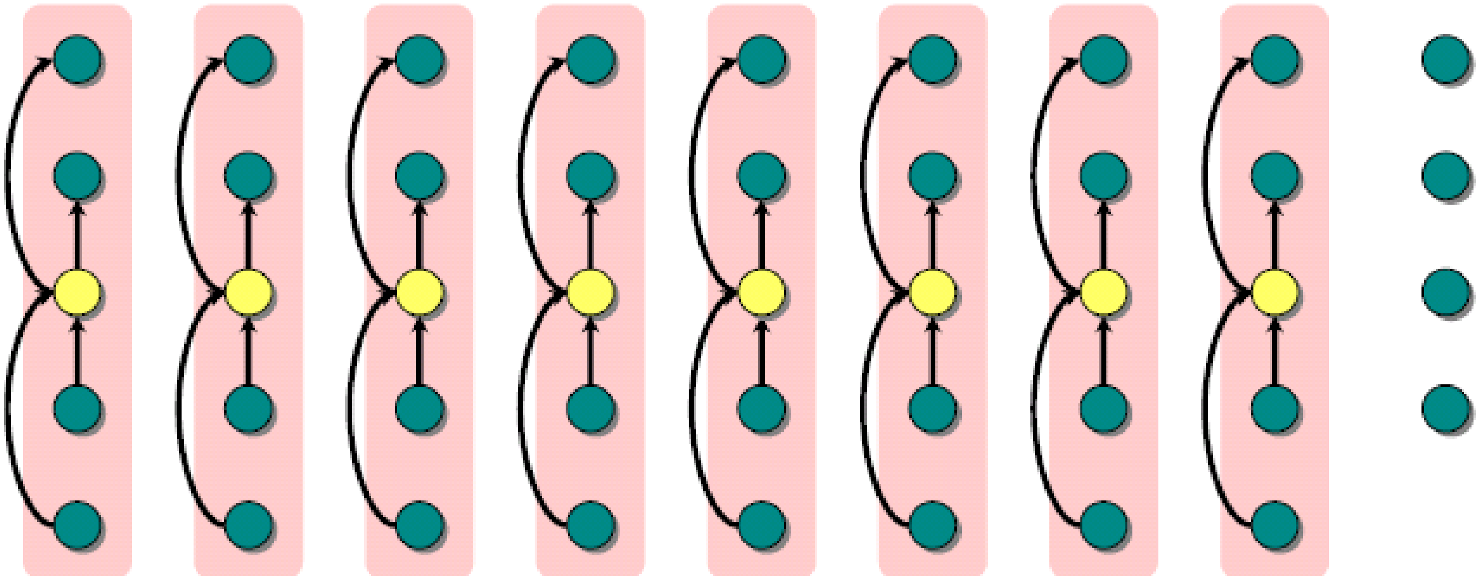


Choosing the pivot



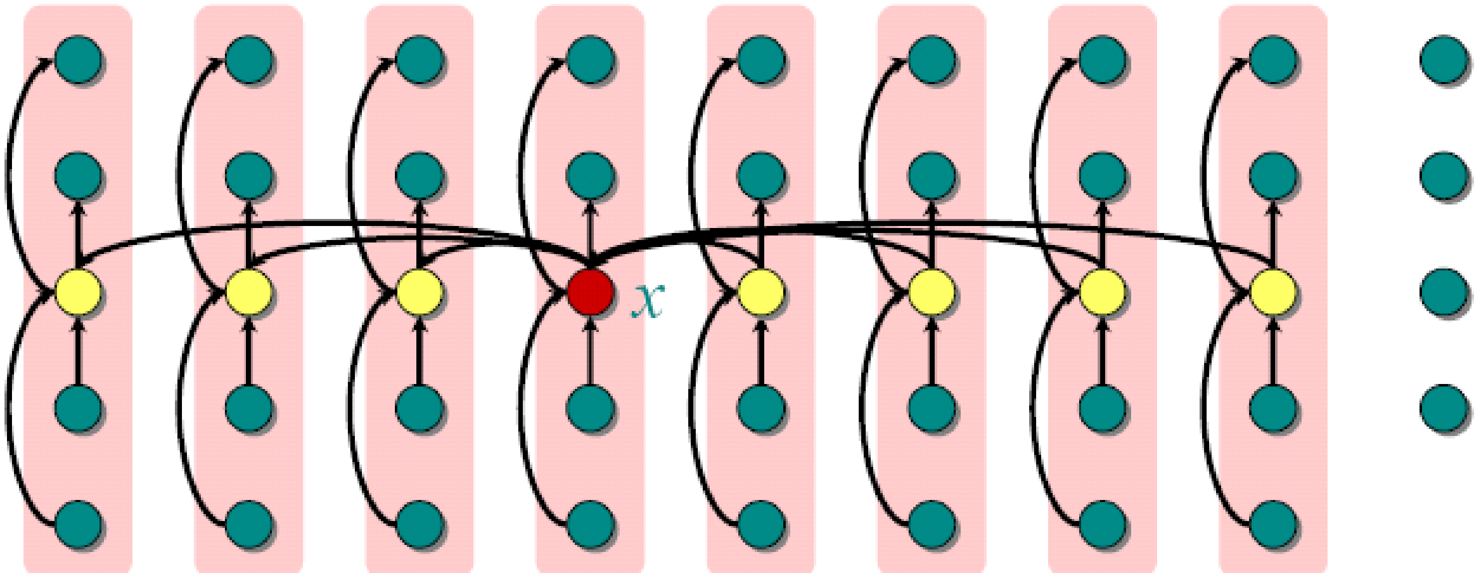
1. Divide the *n* elements into groups of 5.

Choosing the pivot

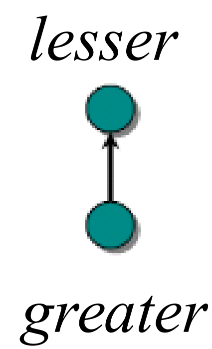


1. Divide the n elements into groups of 5. Find the median of each 5-element group by rote.

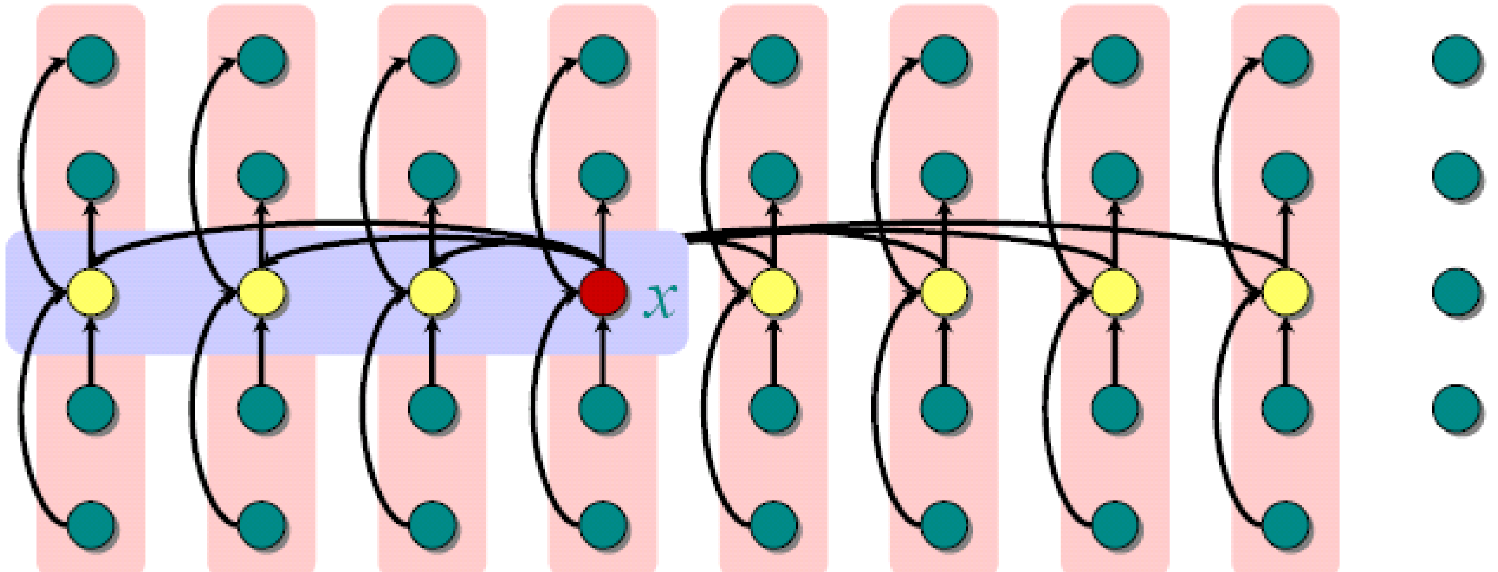
Choosing the pivot



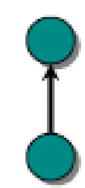
1. Divide the n elements into groups of 5. Find the median of each 5-element group by rote.
2. Recursively SELECT the median x of the $\lfloor n/5 \rfloor$ group medians to be the pivot.

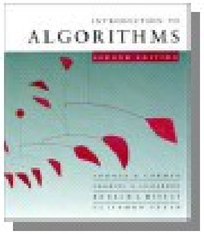


Analysis

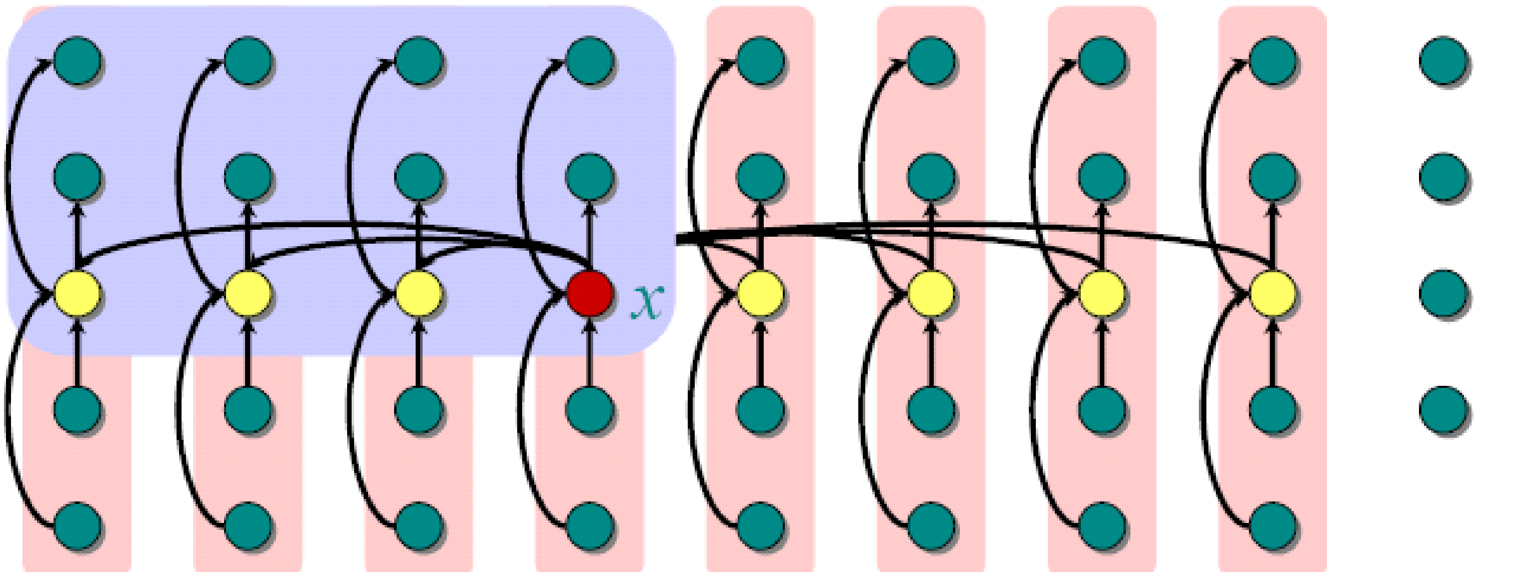


At least half the group medians are $\leq x$, which is at least $\lfloor \lfloor n/5 \rfloor /2 \rfloor = \lfloor n/10 \rfloor$ group medians.



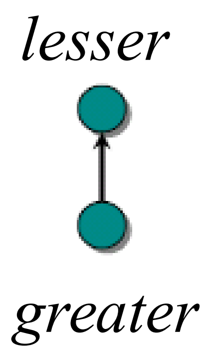


Analysis (Assume all elements are distinct.)

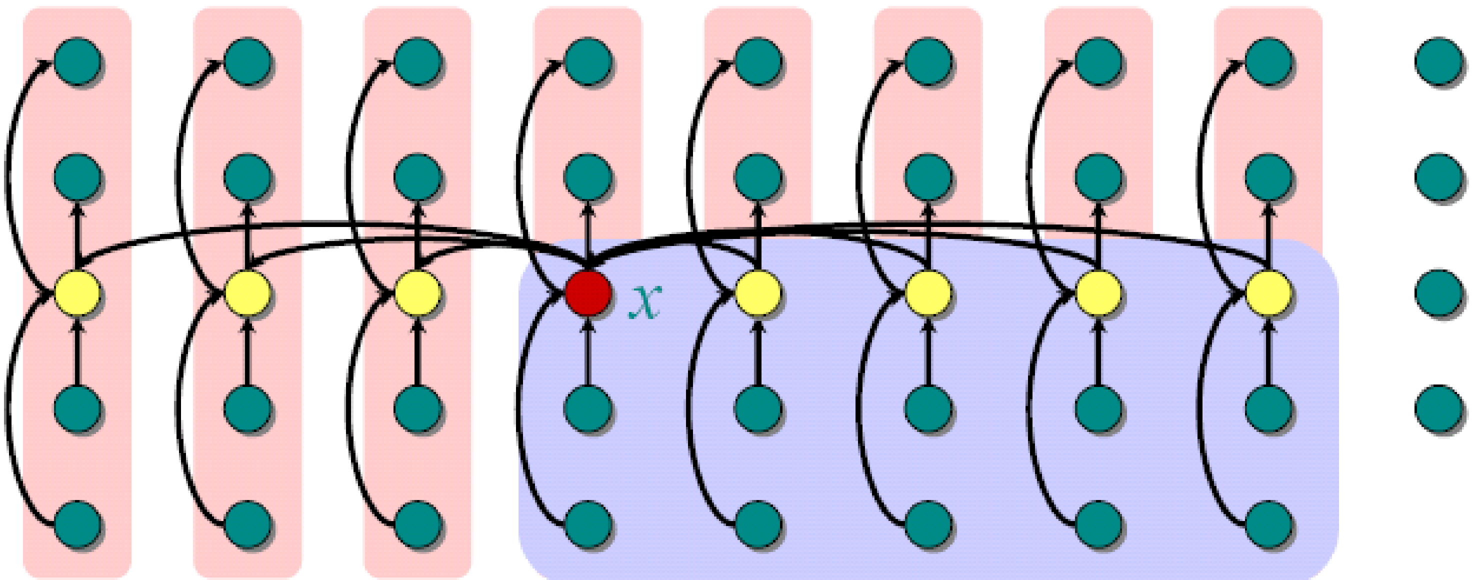


At least half the group medians are $\le x$, which is at least $\lfloor \lfloor n/5 \rfloor /2 \rfloor = \lfloor n/10 \rfloor$ group medians.

- Therefore, at least $3\lfloor n/10 \rfloor$ elements are $\le x$.

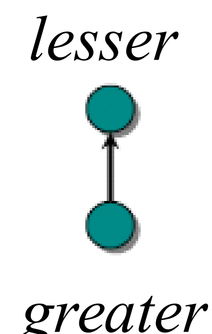


Analysis (Assume all elements are distinct.)



At least half the group medians are $\le x$, which is at least $\lfloor \lfloor n/5 \rfloor /2 \rfloor = \lfloor n/10 \rfloor$ group medians.

- Therefore, at least $3\lfloor n/10 \rfloor$ elements are $\le x$.
- Similarly, at least $3\lfloor n/10 \rfloor$ elements are $\ge x$.



Minor simplification

- For $n \geq 50$, we have $3\lfloor n/10 \rfloor \geq n/4$.
- Therefore, for $n \geq 50$ the recursive call to SELECT in Step 4 is executed recursively on $\leq 3n/4$ elements.
- Thus, the recurrence for running time can assume that Step 4 takes time $T(3n/4)$ in the worst case.
- For $n < 50$, we know that the worst-case time is $T(n) = \Theta(1)$.

Developing the recurrence

$T(n)$	SELECT(i, n)
$\Theta(n)$	1. Divide the n elements into groups of 5. Find the median of each 5-element group by rote.
$T(n/5)$	2. Recursively SELECT the median x of the $n/5$ group medians to be the pivot.
$\Theta(n)$	3. Partition around the pivot x. Let $k = rank(x)$.
$T(3n/4)$	4. **If** $i = k$ **then return** x **else if** $i < k$ **then** recursively SELECT the i th smallest element in the lower part **else** recursively SELECT the $(i\text{-}k)$ th smallest element in the upper part

Solving the recurrence

$$T(n) = T\left(\frac{1}{5}n\right) + T\left(\frac{3}{4}n\right) + \Theta(n)$$

Substitution:

$T(n) \le cn$

$$\begin{aligned}
T(n) &\le \frac{1}{5}cn + \frac{3}{4}cn + \Theta(n) \\
&= \frac{19}{20}cn + \Theta(n) \\
&= cn - \left(\frac{1}{20}cn - \Theta(n)\right) \\
&\le cn \quad ,
\end{aligned}$$

if c is chosen large enough to handle both the $\Theta(n)$ and the initial conditions.

Conclusions

- Since the work at each level of recursion is a constant fraction (19/20) smaller, the work per level is a geometric series dominated by the linear work at the root.
- In practice, this algorithm runs slowly, because the constant in front of n is large.
- The randomized algorithm is far more practical.

Exercise: *Why not divide into groups of 3?*