Introduction to Algorithms

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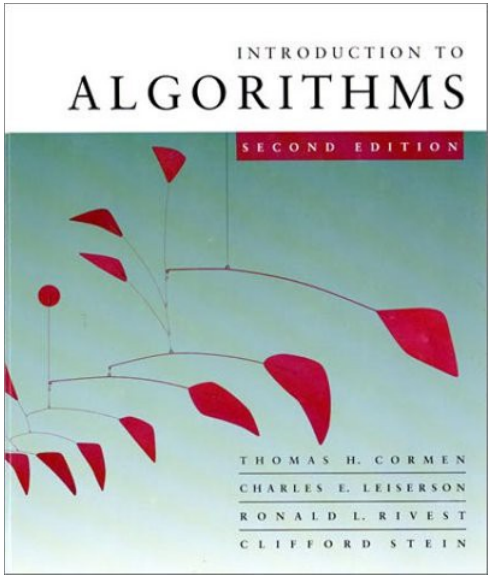


LECTURE 8

Amortized Analysis

- Dynamic tables
- Aggregate method
- Accounting method
- Potential method

Chapter17 in Text book

Based on slides by Prof. Charles E. Leiserson

How large should a hash table be?

Goal: Make the table as small as possible, but large enough so that it won't overflow (or otherwise become inefficient).

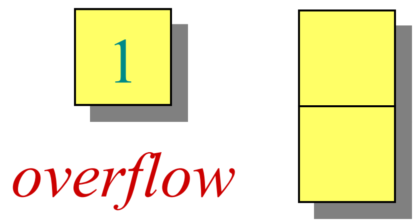
Problem: What if we don't know the proper size in advance?

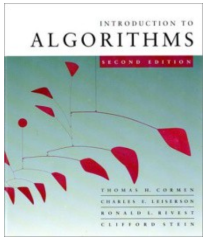
Solution: ***Dynamic tables.***

IDEA: Whenever the table overflows, "grow" it by allocating (via **`malloc`** or **`new`**) a new, larger table. Move all items from the old table into the new one, and free the storage for the old table.

Example of a dynamic table

1. INSERT
2. INSERT





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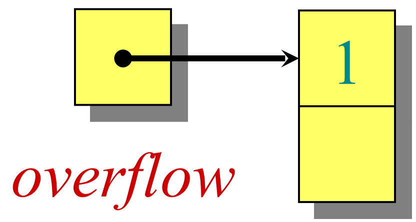

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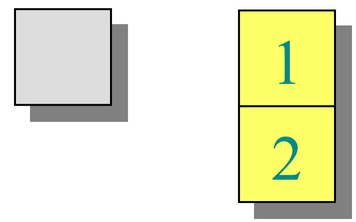

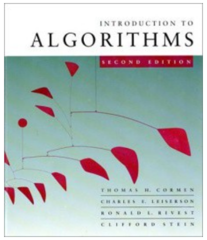

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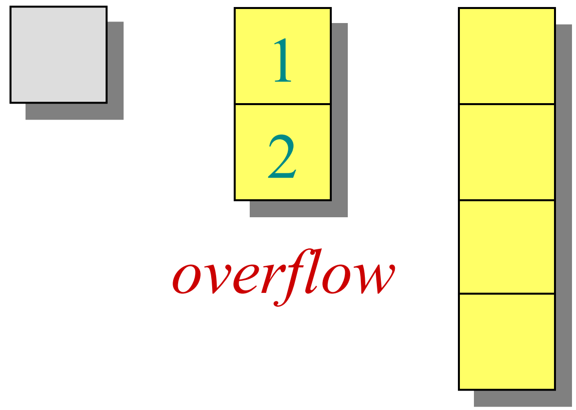

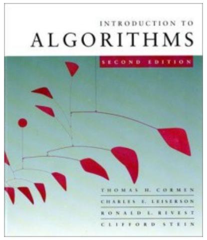

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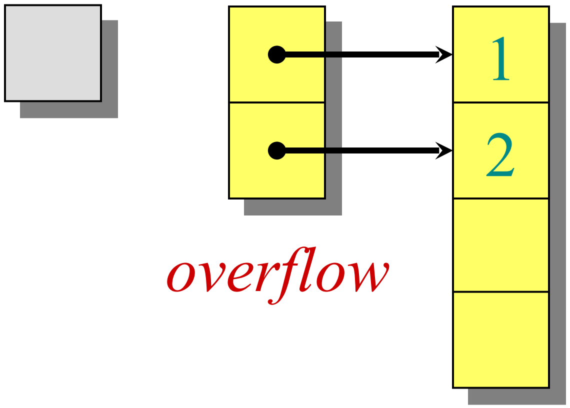

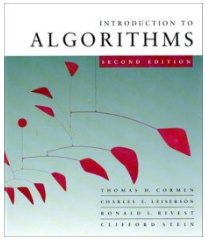

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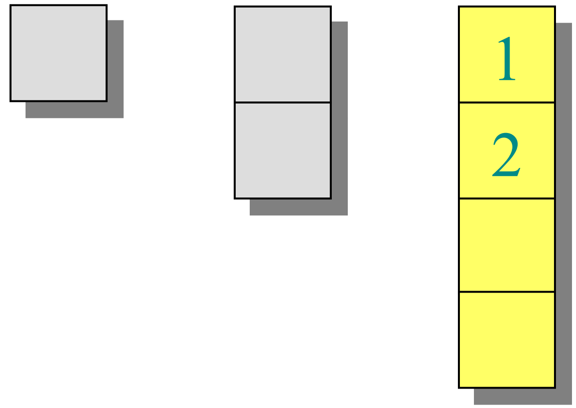

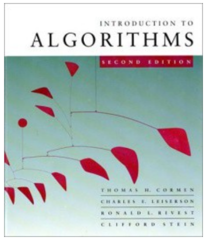

Example of a dynamic table

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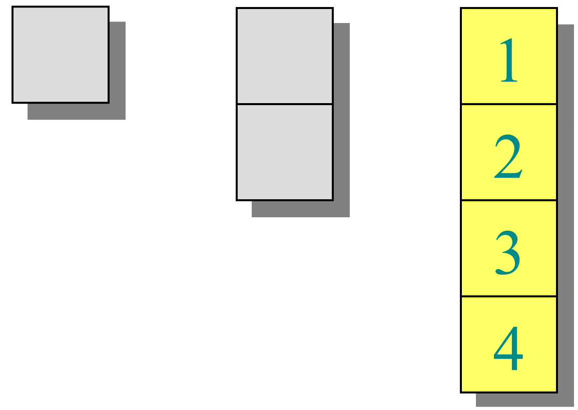
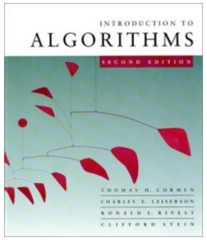
4. INSERT





Example of a dynamic table

1. INSERT
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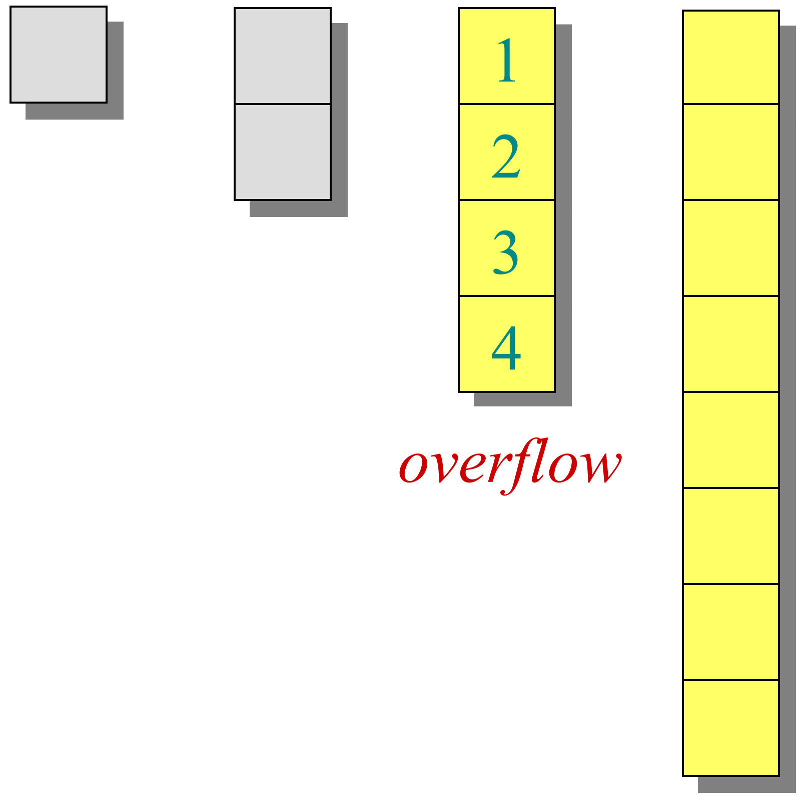

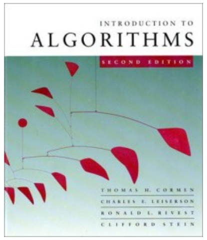

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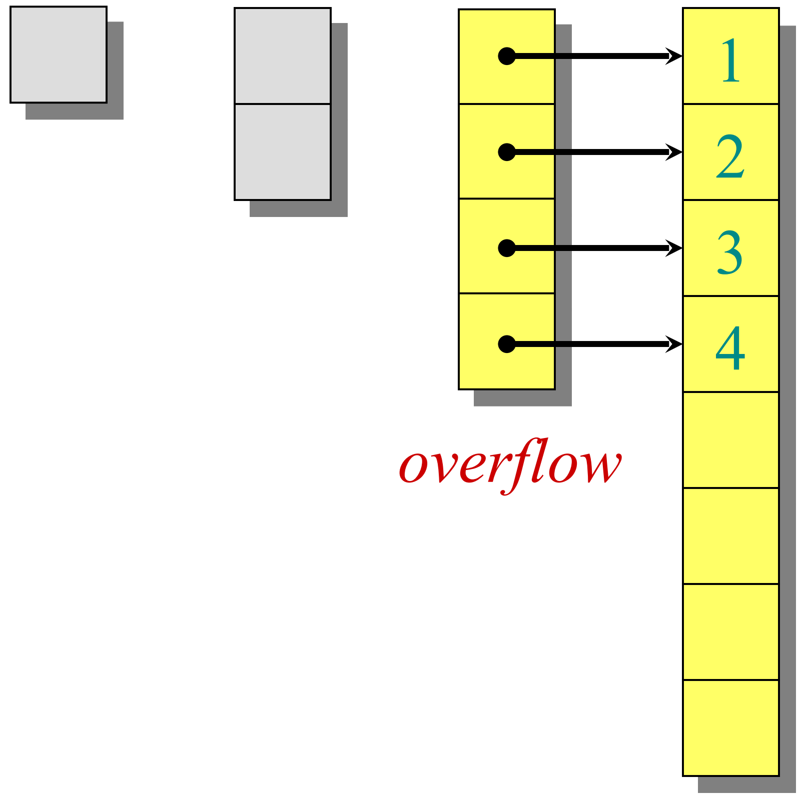

Example of a dynamic table

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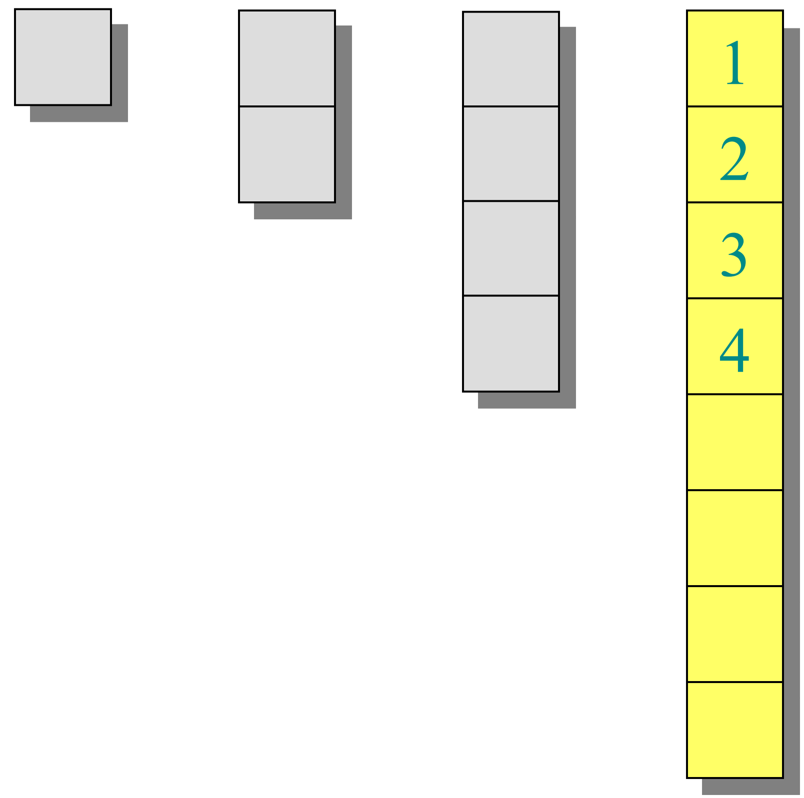

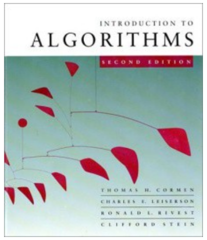

Example of a dynamic table

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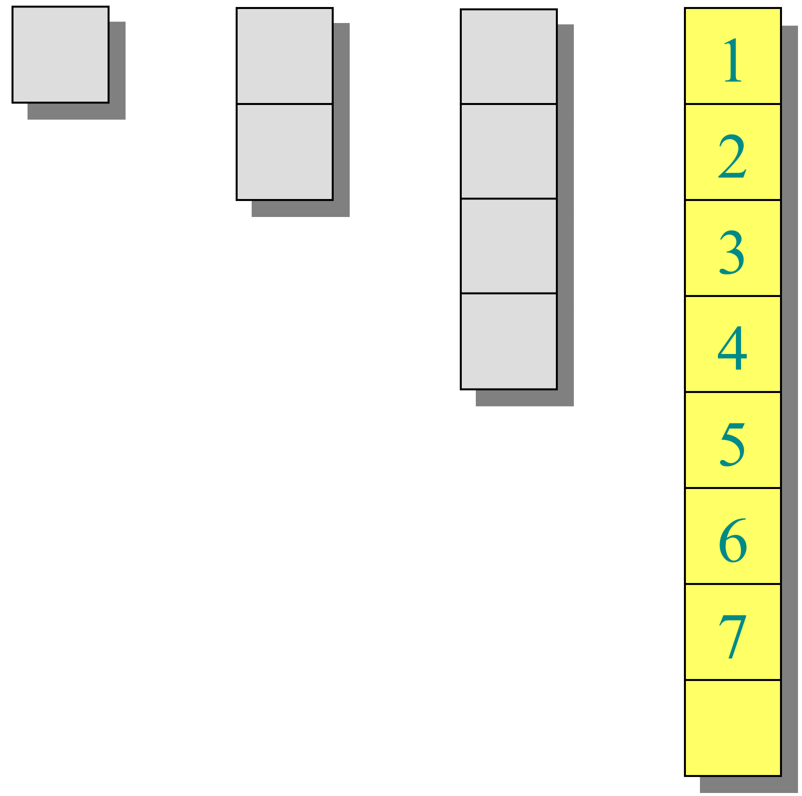
5. INSERT
6. INSERT
7. INSERT

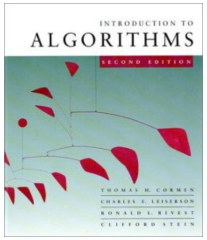


Worst-case analysis

Consider a sequence of n insertions. The worst-case time to execute one insertion is $\Theta(n)$. Therefore, the worst-case time for n insertions is $n \cdot \Theta(n) = \Theta(n^2)$.

WRONG! In fact, the worst-case cost for n insertions is only $\Theta(n) \lll \Theta(n^2)$.

Let's see why.

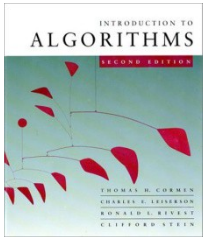


Tighter analysis

Let c_i = the cost of the i th insertion

$$= \begin{cases} i & \text{if } i - 1 \text{ is an exact power of } 2, \\ 1 & \text{otherwise.} \end{cases}$$

i	1	2	3	4	5	6	7	8	9	10
$size_i$	1	2	4	4	8	8	8	8	16	16
c_i	1	2	3	1	5	1	1	1	9	1



Tighter analysis

Let $c_i =$ the cost of the i th insertion

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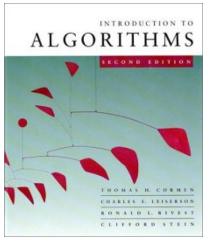
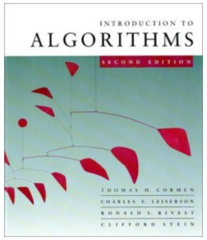
$size_i$	1	2	4	4	8	8	8	8	16	16
c_i	1	1	1	1	1	1	1	1	1	1
		1	2		4				8	



Tighter analysis (continued)

$$\begin{aligned}
\text{Cost of } n \text{ insertions} &= \sum_{i=1}^{n} c_i \\
&\le n + \sum_{j=0}^{\lfloor \lg(n-1) \rfloor} 2^j \\
&\le 3n \\
&= \Theta(n).
\end{aligned}$$

Thus, the average cost of each dynamic-table operation is $\Theta(n)/n = \Theta(1)$.

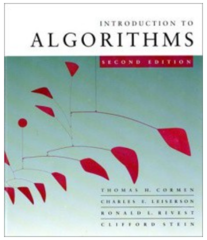


Amortized analysis

An ***amortized analysis*** is any strategy for analyzing a sequence of operations to show that the average cost per operation is small, even though a single operation within the sequence might be expensive.

Even though we're taking averages, however, probability is not involved!

- An amortized analysis guarantees the average performance of each operation in the *worst case*.



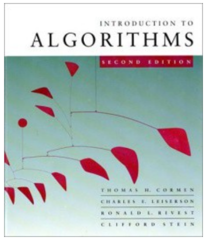
Types of amortized analyses

Three common amortization arguments:

- the ***aggregate*** method,
- the ***accounting*** method,
- the ***potential*** method.

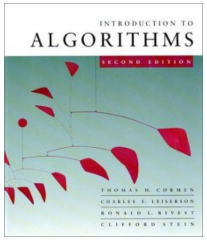
We've just seen an aggregate analysis.

The aggregate method, though simple, lacks the precision of the other two methods. In particular, the accounting and potential methods allow a specific ***amortized cost*** to be allocated to each operation.



Accounting method

- Charge i th operation a fictitious ***amortized cost*** $\hat{c}_i$, where \$1 pays for 1 unit of work (*i.e.*, time).
- This fee is consumed to perform the operation.
- Any amount not immediately consumed is stored in the ***bank*** for use by subsequent operations.
- The bank balance must not go negative! We must ensure that
$$\sum_{i=1}^{n} c_i \le \sum_{i=1}^{n} \hat{c}_i$$
for all n.
- Thus, the total amortized costs provide an upper bound on the total true costs.



Accounting analysis of dynamic tables

Charge an amortized cost of $\hat{c}_i = \$3$ for the i th insertion.

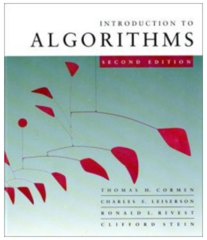
- \$1 pays for the immediate insertion.
- \$2 is stored for later table doubling.

When the table doubles, \$1 pays to move a recent item, and \$1 pays to move an old item.

Example:

\$0	\$0	\$0	\$0	\$2	\$2	\$2	\$2

overflow



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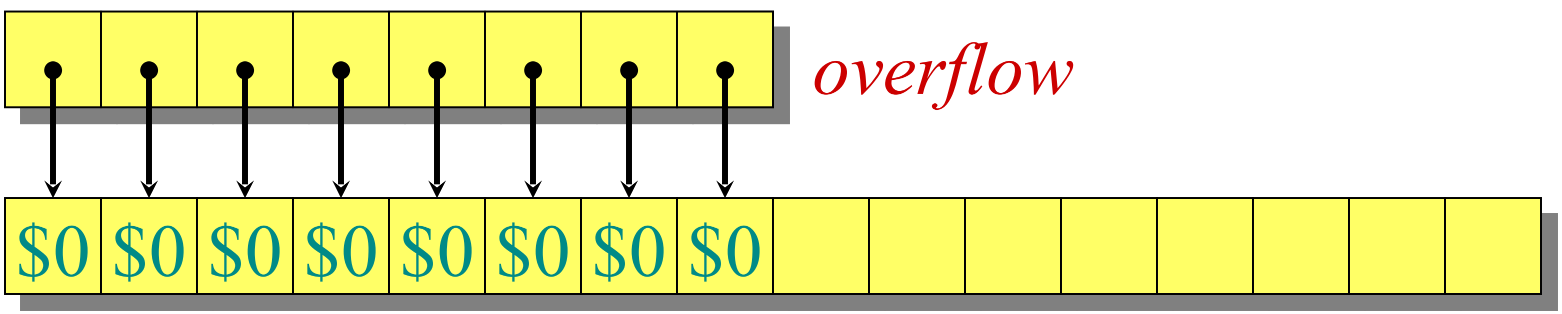

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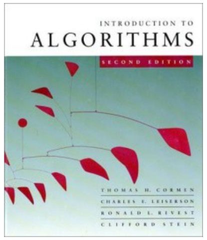
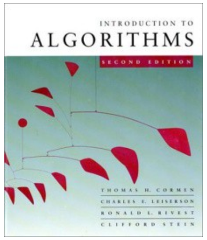


Accounting analysis (continued)

Key invariant: Bank balance never drops below 0. Thus, the sum of the amortized costs provides an upper bound on the sum of the true costs.

i	1	2	3	4	5	6	7	8	9	10
$size_i$	1	2	4	4	8	8	8	8	16	16
c_i	1*	2	3	1	5	1	1	1	9	1
$\hat{c}_i$	2	3	3	3	3	3	3	3	3	3
$bank_i$	1	2	2	4	2	4	6	8	2	4

*The first operation costs only $2, not $3.

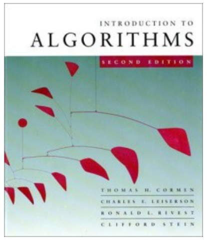


Potential method

IDEA: View the bank account as the potential energy (*à la* physics) of the dynamic set.

Framework:

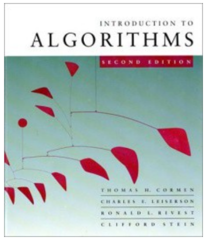
- Start with an initial data structure D_0.
- Operation i transforms D_{i-1} to D_i.
- The cost of operation i is c_i.
- Define a ***potential function*** $\Phi : \{D_i\} \rightarrow \mathbb{R}$, such that $\Phi(D_0) = 0$ and $\Phi(D_i) \geq 0$ for all i.
- The ***amortized cost*** $\hat{c}_i$ with respect to Φ is



Understanding potentials

$$\hat{c}_i = c_i + \underbrace{\Phi(D_i) - \Phi(D_{i-1})}_{\textit{potential difference } \Delta\Phi_i}$$

- If $\Delta\Phi_i > 0$, then $\hat{c}_i > c_i$. Operation i stores work in the data structure for later use.
- If $\Delta\Phi_i < 0$, then $\hat{c}_i < c_i$. The data structure delivers up stored work to help pay for operation i.

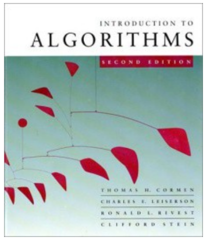


The amortized costs bound the true costs

The total amortized cost of n operations is

$$\sum_{i=1}^{n} \hat{c}_i = \sum_{i=1}^{n} \left(c_i + \Phi(D_i) - \Phi(D_{i-1}) \right)$$

Summing both sides.



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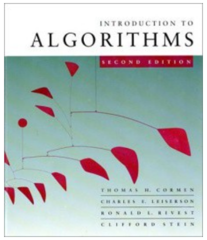
$$= \sum_{i=1}^{n} c_i + \Phi(D_n) - \Phi(D_0)$$

The series telescopes (*each term added and subtracted once, except first and last terms*)



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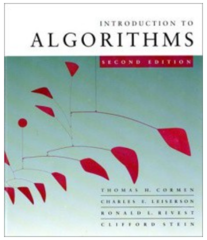
The total amortized cost of n operations is

$$\sum_{i=1}^{n} \hat{c}_i = \sum_{i=1}^{n} \left(c_i + \Phi(D_i) - \Phi(D_{i-1}) \right)$$

$$= \sum_{i=1}^{n} c_i + \Phi(D_n) - \Phi(D_0)$$

$$\geq \sum_{i=1}^{n} c_i$$

since $\Phi(D_n) \geq 0$ and $\Phi(D_0) = 0$.



Potential analysis of table doubling

Define the potential of the table after the ith insertion by $\Phi(D_i) = 2i - 2^{\lceil \lg i \rceil}$. (Assume that $2^{\lceil \lg 0 \rceil} = 0$.)

The second term is just the current table size!

Note:

- $\Phi(D_0) = 0$,
- $\Phi(D_i) \geq 0$ for all i.

Example:

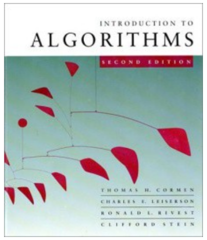
•	•	•	•	•	•		

$\Phi = 2 \cdot 6 - 2^3 = 4$

(

\$0	\$0	\$0	\$0	\$2	\$2		

accounting method)

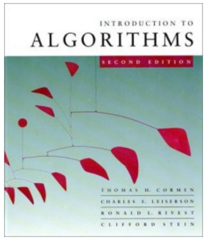


Calculation of amortized costs

The amortized cost of the i th insertion is

$$\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$$

$$= \begin{cases} i + \left(2i - 2^{\lceil \lg i \rceil}\right) - \left(2(i-1) - 2^{\lceil \lg (i-1) \rceil}\right) \\ \qquad \text{if } i - 1 \text{ is an exact power of } 2, \\ 1 + \left(2i - 2^{\lceil \lg i \rceil}\right) - \left(2(i-1) - 2^{\lceil \lg (i-1) \rceil}\right) \\ \qquad \text{otherwise.} \end{cases}$$



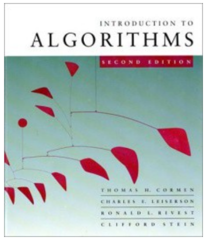
Calculation (Case 1)

Case 1: $i - 1$ is an exact power of 2.

$$\begin{aligned}
\hat{c}_i &= i + \left(2i - 2^{\lceil \lg i \rceil}\right) - \left(2(i-1) - 2^{\lceil \lg (i-1) \rceil}\right) \\
&= i + 2 - \left(2^{\lceil \lg i \rceil} - 2^{\lceil \lg (i-1) \rceil}\right) \\
&= i + 2 - \left(2(i-1) - (i-1)\right) \\
&= i + 2 - 2i + 2 + i - 1 \\
&= 3
\end{aligned}$$

If $(i\text{-}1)$ *is exact power of* 2, $\lg(i\text{-}1)$ *is integer, therefore:* $2^{\lceil \lg (i-1) \rceil} = 2^{\lg (i-1)} = (i\text{-}1)$

If $\lg(i-1)$ *is integer, then* $\lceil \lg i \rceil$ *is the next integer up, hence* $2^{\lceil \lg i \rceil} = 2 \,.\, 2^{\lceil \lg (i-1) \rceil}$



Calculation (Case 2)

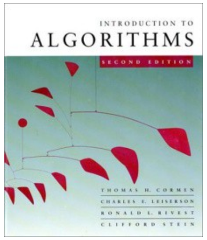
Case 2: $i - 1$ is not an exact power of 2.

$$\begin{aligned}\hat{c}_i &= 1 + \left(2i - 2^{\lceil \lg i \rceil}\right) - \left(2(i-1) - 2^{\lceil \lg (i-1) \rceil}\right) \\ &= 1 + 2 - \left(2^{\lceil \lg i \rceil} - 2^{\lceil \lg (i-1) \rceil}\right) \\ &= 3\end{aligned}$$

If $(i\text{-}1)$ *is not exact power of* 2,, *then* $2^{\lceil \lg i \rceil} = 2^{\lceil \lg (i-1) \rceil}$

Therefore, n insertions cost $\Theta(n)$ in the worst case.

Exercise: Fix the bug in this analysis to show that the amortized cost of the first insertion is only 2.



Deletion

Suppose we allow deletions.

IDEA: Whenever the table becomes less than half-full, shrink it to half its current size by allocating (via **`malloc`** or **`new`**) a new, smaller table. Move all items from the old table into the new one, and free the storage for the old table.

Does this work?

Bad example:

$\Theta(n^2)$

- As sequence of $n/2$ insertions
- insert, delete, delete, insert, insert, delete, delete, insert, insert,....

Deletion

Suppose we allow deletions.

Solution: allow for some operations to be done before rebuilding the table; this way we will have done enough work to pay for the cost of rebuilding.

Method that works:

- If the load factor α becomes 1, double the table size
- If the load factor α drops below 1/4 shrink the table to half its current size

Potential analysis of table doubling/shrinking

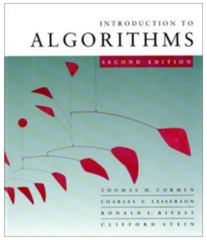
Recall: We defined the potential of the table after the ith insertion by

$$\Phi(D_i) = 2i - 2^{\lceil \lg i \rceil} = 2^*\mathrm{num}(D_i) - \mathrm{size}(D_i).$$

New potential function: We define the potential of the table after the ith operation by

$$\Phi(D_i) = 2^*\mathrm{num}(D_i) - \mathrm{size}(D_i) \quad \text{if } \alpha(D_i) \geq 1/2$$

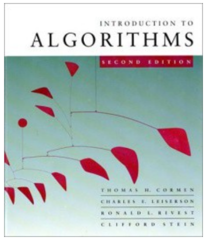
$$\Phi(D_i) = \mathrm{size}(D_i)/2 - \mathrm{num}(D_i) \text{ if } \alpha(D_i) < 1/2.$$



Potential analysis of table doubling/shrinking

Case analysis: ith operation is

INSERT	DELETE
$\alpha(D_i) \geq 1/2,\ \alpha(D_{i-1}) \geq 1/2$	$\alpha(D_{i-1}) = 1/4$
$\alpha(D_i) < 1/2,\ \alpha(D_{i-1}) < 1/2$	$1/4 < \alpha(D_{i-1}) < 1/2$
$\alpha(D_i) \geq 1/2,\ \alpha(D_{i-1}) < 1/2$	$\alpha(D_{i-1}) \geq 1/2$



Case: INSERT, $\alpha(D_i) < 1/2$, $\alpha(D_{i-1}) < 1/2$

$\Phi(D_i) = \text{size}(D_i)/2 - \text{num}(D_i)$ *no expansion*

$\Phi(D_{i-1}) = \text{size}(D_{i-1})/2 - \text{num}(D_{i-1})$

$\text{size}(D_{i-1}) = \text{size}(D_i)$

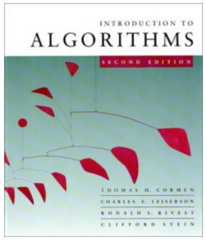
$\text{num}(D_{i-1}) = \text{num}(D_i) - 1,\qquad c_i = 1$

$$\hat{c}_i = 1 + \Phi(D_i) - \Phi(D_{i-1})$$
$$= 1 + (\text{size}(D_i)/2 - \text{num}(D_i)) - (\text{size}(D_{i-1})/2 - \text{num}(D_{i-1}))$$
$$= 0.$$

Potential analysis of table doubling/shrinking

Case analysis: *i*th operation is

INSERT	DELETE
$\alpha(D_i) \geq 1/2,\ \alpha(D_{i-1}) \geq 1/2$	$\alpha(D_{i-1}) = 1/4$
$\alpha(D_i) < 1/2,\ \alpha(D_{i-1}) < 1/2$	$1/4 < \alpha(D_{i-1}) < 1/2$
$\alpha(D_i) \geq 1/2,\ \alpha(D_{i-1}) < 1/2$	$\alpha(D_{i-1}) \geq 1/2$



Case: INSERT, $\alpha(D_i) \geq 1/2$, $\alpha(D_{i-1}) < 1/2$

no expansion

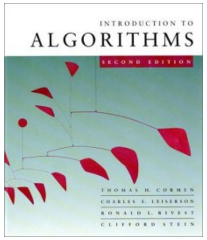
$$\Phi(D_i) = 2*\text{num}(D_i) - \text{size}(D_i)$$

$$\Phi(D_{i-1}) = \text{size}(D_{i-1})/2 - \text{num}(D_{i-1})$$

$$\text{size}(D_{i-1}) = \text{size}(D_i)$$

$$\text{num}(D_{i-1}) = \text{num}(D_i) - 1, \qquad c_i = 1$$

$$\begin{aligned}\hat{c}_i &= 1 + \Phi(D_i) - \Phi(D_{i-1}) \\ &= 1 + (2*\text{num}(D_i) - \text{size}(D_i)) - (\text{size}(D_{i-1})/2 - \text{num}(D_{i-1})) \\ &= 1 + (2*(\text{num}(D_{i-1}) + 1) - \text{size}(D_i)) \\ &\quad - (\text{size}(D_{i-1})/2 - \text{num}(D_{i-1}))\end{aligned}$$



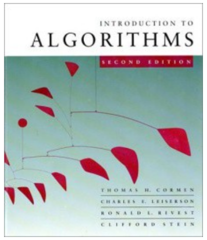
Case: INSERT, $\alpha(D_i) \geq 1/2$, $\alpha(D_{i-1}) < 1/2$

$$
\begin{aligned}
\hat{c}_i &= 1 + \Phi(D_i) - \Phi(D_{i-1}) \\
&= 1 + (2*\text{num}(D_i) - \text{size}(D_i)) - (\text{size}(D_{i-1})/2 - \text{num}(D_{i-1})) \\
&= 1 + (2*(\text{num}(D_{i-1}) + 1) - \text{size}(D_i)) \\
&\qquad - (\text{size}(D_{i-1})/2 - \text{num}(D_{i-1})) \\
&= 3 + 3*\text{num}(D_{i-1}) - 3\ \text{size}(D_{i-1})/2 \\
&= 3 + 3\,\alpha(D_{i-1})\ \text{size}(D_{i-1}) - 3\ \text{size}(D_{i-1})/2 \\
&< 3 + \text{size}(D_{i-1})/2 - 3\ \text{size}(D_{i-1})/2 \\
&\qquad (\text{since } \alpha(D_{i-1}) < 1/2) \\
&= 3
\end{aligned}
$$

Potential analysis of table doubling/shrinking

Case analysis: ith operation is

INSERT	DELETE
$\alpha(D_i) \geq 1/2,\ \alpha(D_{i-1}) \geq 1/2$	$\alpha(D_{i-1}) = 1/4$
$\alpha(D_i) < 1/2,\ \alpha(D_{i-1}) < 1/2$	$1/4 < \alpha(D_{i-1}) < 1/2$
$\alpha(D_i) \geq 1/2,\ \alpha(D_{i-1}) < 1/2$	$\alpha(D_{i-1}) \geq 1/2$



Case: DELETE, $\alpha(D_{i-1}) = 1/4$

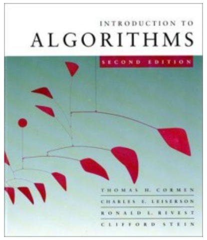
$\Phi(D_i) = \text{size}(D_i)/2 - \text{num}(D_i)$ *contract*

$\Phi(D_{i-1}) = \text{size}(D_{i-1})/2 - \text{num}(D_{i-1})$

$\text{size}(D_{i-1}) = 2 * \text{size}(D_i)$

$\text{num}(D_{i-1}) = \text{num}(D_i) + 1,$ $c_i = \text{num}(D_i) + 1$

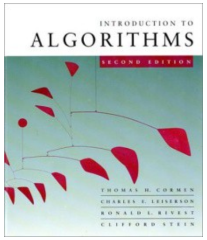
$$\begin{aligned}
\hat{c}_i &= (\text{num}(D_i)+1) + \Phi(D_i) - \Phi(D_{i-1}) \\
&= (\text{num}(D_i)+1) + (\text{size}(D_i)/2 - \text{num}(D_i)) \\
&\quad - (\text{size}(D_{i-1})/2 - \text{num}(D_{i-1})) \\
&= 1 + \text{size}(D_i)/2 - \text{size}(D_{i-1})/2 + \text{size}(D_{i-1})/4
\end{aligned}$$



Potential analysis of table doubling/shrinking

Case analysis: ith operation is

INSERT	DELETE
$\alpha(D_i) \geq 1/2,\ \alpha(D_{i-1}) \geq 1/2$	$\alpha(D_{i-1}) = 1/4$
$\alpha(D_i) < 1/2,\ \alpha(D_{i-1}) < 1/2$	$1/4 < \alpha(D_{i-1}) < 1/2$
$\alpha(D_i) \geq 1/2,\ \alpha(D_{i-1}) < 1/2$	$\alpha(D_{i-1}) \geq 1/2$



Case: DELETE, $1/4 < \alpha(D_{i-1}) < 1/2$

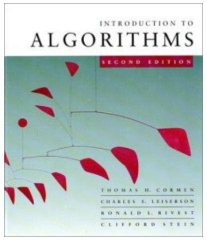
$\Phi(D_i) = \text{size}(D_i)/2 - \text{num}(D_i)$ *no contraction*

$\Phi(D_{i-1}) = \text{size}(D_{i-1})/2 - \text{num}(D_{i-1})$

$\text{size}(D_{i-1}) = \text{size}(D_i)$

$\text{num}(D_{i-1}) = \text{num}(D_i) + 1$, $c_i = 1$

$$
\begin{aligned}
\hat{c}_i &= 1 + \Phi(D_i) - \Phi(D_{i-1}) \\
&= 1 + (\text{size}(D_i)/2 - \text{num}(D_i)) - (\text{size}(D_{i-1})/2 - \text{num}(D_{i-1})) \\
&= 1 + \text{num}(D_{i-1}) - \text{num}(D_i) \\
&= 2.
\end{aligned}
$$



Conclusions

- Amortized costs can provide a clean abstraction of data-structure performance.
- Any of the analysis methods can be used when an amortized analysis is called for, but each method has some situations where it is arguably the simplest.
- Different schemes may work for assigning amortized costs in the accounting method, or potentials in the potential method, sometimes yielding radically different bounds.